AMERICAN INTERNATIONAL UNIVERSITY–BANGLADESH (AIUB)

FACULTY OF SCIENCE & TECHNOLOGY

DEPARTMENT OF MATH

COMPLEX VARIABLE LAPLACE & Z-TRANSFORMATION

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Section: A

MID ASSIGNMENT

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Chapter-1

Proved that, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

Ans:-

$$\mathcal{L}\{e^{at}\} = \int_0^{\alpha} e^{at} . e^{-st} \, dt$$

$$= \int_0^{\alpha} e^{at-st} \, dt$$

$$= \int_0^{\alpha} e^{-(s-a)t} \, dt$$

$$= -\left[\frac{e^{-(s-a)t}}{s-a}\right]_0^{\alpha}$$

$$= -\left[0 - \frac{1}{s-a}\right]$$

$$= \frac{1}{s-a} \quad [\text{proved}]$$

Prove that, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Sign	diff	int
$+$	$\sin at$	e^{-st}
$-$	$a\cos at$	$-\frac{1}{s}e^{-st}$
$+$	$-a^2 \sin at$	$\frac{1}{s^2} e^{-st}$

Ans:-

$$\mathcal{L}\{\sin at\} = \int_0^\infty \sin at . e^{-st} \, dt$$

$$\Rightarrow I = \int_0^\infty \sin at \, e^{-st} \, dt$$

$$\Rightarrow I = -\frac{1}{s}\sin at \, e^{-st} - \frac{a}{s^2}\cos at . e^{-st}$$

$$- \frac{a^2}{s^2}\int_0^\infty \sin at . e^{-st} \, dt$$

$$\Rightarrow I = \left[-\frac{1}{s}\sin at \, e^{-st} - \frac{a}{s^2}\cos at . e^{-st}\right]_0^\infty$$

$$-\frac{a^2}{s^2} \times I$$

$$\Rightarrow \left(1+\frac{a^2}{s^2}\right) I = \left[0 - \left(-\frac{a}{s^2}\right)\right]$$

$$\Rightarrow \left(\frac{s^2+a^2}{s^2}\right) I = \frac{a}{s^2}$$

$$\therefore I = \frac{a}{s^2+a^2}$$

$$\therefore \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \text{ [proved]}$$

Prove that,

$$\mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2}$$

Ans:

$$\mathcal{L}\{\sinh(at)\} = \int_0^\infty \sinh(at)e^{-st}\,dt$$

Sign	Diff	Int
+	$\sinh(at)$	e^{-st}
−	$a\sinh(at)$	$-\frac{1}{s}e^{-st}$
+	$a^2\sinh(at)$	$\frac{1}{s^2}e^{-st}$

$$\Rightarrow I = \int_0^\infty \sinh(at)\cdot e^{-st}\,dt$$

$$\Rightarrow I = \left[-\frac{1}{s}\sinh(at)\cdot e^{-st} - \frac{a}{s^2}\cosh(at)\cdot e^{-st}\right]_0^\infty + \frac{a^2}{s^2}\int_0^\infty \sinh(at)e^{-st}\,dt$$

$$\Rightarrow I = \left[0 - \left(-\frac{a}{s^2}\right)\right] + \frac{a^2}{s^2}\times I$$

$$\Rightarrow \left(\frac{s^2-a^2}{s^2}\right)I = \frac{a}{s^2}$$

$$\therefore I = \frac{a}{s^2-a^2}$$

$$\therefore \mathcal{L}\{\sinh(at)\} = \frac{a}{s^2-a^2} \quad \text{[proved]}$$

Prove that,

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2}$$

Ans:-

$$\mathcal{L}\{\cosh(at)\} = \int_0^\infty \cosh(at) \cdot e^{-st}\, dt$$

Sign	Diff.	Int.
$+$	$\cosh(at)$	e^{-st}
$-$	$a\sinh(at)$	$-\frac{1}{s} e^{-st}$
$+$	$a^2 \cosh(at)$	$\frac{1}{s^2} e^{-st}$

$$\Rightarrow I = \int_0^\infty \cosh(at)\, e^{-st}\, dt$$

$$\Rightarrow I = \left[-\frac{1}{s}\cosh(at) e^{-st} - \frac{a}{s^2}\sinh(at) e^{-st}\right]_0^\infty + \frac{a^2}{s^2}\int_0^\infty \cosh(at) e^{-st}\, dt$$

$$\Rightarrow I = \left[0 - \left(-\frac{1}{s}\right)\right] + \frac{a^2}{s^2} \times I$$

$$\Rightarrow I = \frac{s}{s^2 - a^2}$$

$$\therefore \mathcal{L}\{\cosh(at)\} = \frac{s}{s^2 - a^2} \text{ [proved]}$$

1.1

1. $f(t) = 3t + 12$

$$\therefore \mathcal{L}\{f(t)\} = 3\frac{1!}{s^{1+1}} + 12\frac{0!}{s^{0+1}}$$

$$= \frac{3}{s^2} + \frac{12}{s}.$$

2. $f(t) = e^{5t}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s-5} \quad \left[\because \mathcal{L}\{e^{at}\} = \frac{1}{s-a}\right]$$

3. $f(t) = e^{-2t}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{1}{s+2}$$

4. $f(t) = (a-bt)^2$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{a^2 - 2abt + b^2t^2\}$$

$$= \mathcal{L}\{a^2\} - 2ab\mathcal{L}\{t\} + b^2\mathcal{L}\{t^2\}$$

$$= \frac{a^2}{s} - \frac{2ab}{s^2} + \frac{2b^2}{s^3}$$

$\therefore F(s) = \dfrac{a^2s^2 - 2abs + 2b^2}{s^3}$ Ans

5. $f(t) = \cos \pi t$

$\therefore F(s) = \mathcal{L}\{f(t)\} = \dfrac{s}{s^2+\pi^2}$

6. $f(t) = \cos^2 \omega t$

$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2} 2\cos^2 \omega t\right\}$

$= \frac{1}{2} \mathcal{L}\{1 + \cos 2\omega t\}$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4\omega^2}\right]$

7. $f(t) = \sin(\omega t + \theta)$

$= \mathcal{L}\{\sin(\omega t + \theta)\}$

$= \mathcal{L}\{(\sin \omega t \cdot \cos\theta + \cos \omega t \cdot \sin\theta)\}$

$= \cos\theta \, \mathcal{L}\{\sin \omega t\} + \sin\theta \{\cos \omega t\}$

$= \cos\theta \left(\frac{\omega}{s^2+\omega^2}\right) + \sin\theta \left(\frac{s}{s^2+\omega^2}\right)$

$= \dfrac{\omega\cos\theta + \sin\theta}{s^2+\omega^2}$

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8. $\mathcal{L}\left\{1.5\sin\left(3t-\frac{\pi}{2}\right)\right\}$

$= 1.5\,\mathcal{L}\left\{\sin\left(3t-\pi/2\right)\right\}$

$= 1.5\,\mathcal{L}\left[\sin 3t.\cos \pi/2 - \cos 3t.\sin \pi/2\right]$

$= 1.5\,\mathcal{L}\left[\sin 3t.0 - \cos 3t.1\right]$

$= \cdot 1.5\,\mathcal{L}\left\{\cos 3t\right\}$

$= -1.5 \times \frac{s}{s^2+9}$

$\therefore -\frac{1.5s}{s^2-9}$

9. $f(t) = e^{2t}\sinh t$

$\therefore \mathcal{L}\{f(t)\} = \frac{1}{(s-2)^2-1}$ $\left[\begin{array}{l} \because \mathcal{L}\{e^{at}f(t)\} = F(s-a) \\ \because \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2} \end{array}\right]$

10. $f(t) = e^{-t}\sinh 4t$

$\therefore \mathcal{L}\{f(t)\} = \frac{4}{(s+1)^2-16}$ Ans

11. $f(t) = e^{2t}\cos 3t$

$$\left[\because \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$\therefore \mathcal{L}\{f(t)\} = \frac{s-2}{(s-2)^2+9} \text{ Ans}$$

12. $f(t) = t^{10}e^{-7t}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{10!}{(s+7)^{10+1}}$$

$$= \frac{10!}{(s+7)^{11}}$$

13. $f(t) = \cosh 5t \, e^{3t}$

$$\therefore \mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2-25} \quad \left[\because \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}\right]$$

14. $f(t) = t\sin 2t$

$$\therefore \mathcal{L}\{f(t)\} = (-1)^1 \frac{d}{ds}\mathcal{L}\{\sin 2t\}$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\left\{\frac{(s^2+4)\cdot 0 - 2\cdot 2s}{(s^2+4)^2}\right\}$$

$$= \frac{4s}{(s^2+4)^2} \text{ Ans}$$

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15. $f(t) = t\cos bt$

$$\therefore \mathcal{L}\{f(t)\} = (-1)^1 \frac{d}{ds} \mathcal{L}\{\cos bt\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+b^2}\right)$$

$$= -\left\{\frac{(s^2+b^2)\cdot 1 - s\cdot 2s}{(s^2+b^2)^2}\right\}$$

$$= \frac{-(s^2+b^2)+2s^2}{(s^2+b^2)^2}$$

$$= \frac{2s^2}{(s^2+b^2)^2} - \frac{1}{(s^2+b^2)}$$

16. $f(t) = \begin{cases} 1-t; & 0 < t < 1 \\ 0; & t > 1 \end{cases}$

$$f(t) = (1-t)\,[u(t) - u(t-1)]$$

$$f(t) = (1-t)\,u(t) - (1-t)\,u(t-1)$$

$$F(s) = \mathcal{L}\,(1-t)\,u(t) - \mathcal{L}\,(1-t)\,u(t-1)$$

$$= e^{0}\mathcal{L}\,(1-t) - e^{-s}\mathcal{L}\,(1-t)$$

$$= \frac{1}{s} - \frac{1}{s^2} - e^{-s}\cdot\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

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$$= \frac{1}{s} - \frac{1}{s^2} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}$$

$$\therefore \quad \frac{s - 1 - se^{-s} + e^{-s}}{s^2}$$

17. $f(t) = \begin{cases} k; & 0 < t < c \\ 0; & t > c \end{cases}$

$$= k\,[u(t) - u(t-c)]$$

$$F(s) = k\,[\mathcal{L}u(t) - \mathcal{L}u(t-c)]$$

$$= k\left[\frac{e^{-0}}{s} - \frac{e^{-cs}}{s}\right]$$

$$= k\left[\frac{1}{s} - \frac{e^{-cs}}{s}\right]$$

$$= \frac{k(1 - e^{-cs})}{s}$$ Ans

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18. $f(t) = \begin{cases} b-t & ; 0<t<b \\ 0 & ; t>b \end{cases}$

$= (b-t)[u(t) - u(t-b)]$

$= (b-t)u(t) - \cancel{b(t)}(b-t)u(t-b)$

Now,

$\mathcal{L}f(t) = \mathcal{L}\{(b-t)u(t)\} - \mathcal{L}\{(b-t)u(t-b)\}$

$= e^{0}\mathcal{L}(b-t) - e^{-bs}\mathcal{L}\{b-(t+b)\}$

$= \mathcal{L}(b-t) - e^{-bs}\mathcal{L}(-t)$

$= \dfrac{b}{s} - \dfrac{1}{s^2} + \dfrac{e^{-bs}}{s^2}$

$= \dfrac{bs - 1 + e^{-bs}}{s^2}$

$= \dfrac{e^{-bs} + bs - 1}{s^2}$ Ans

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19. $f(t) = \begin{cases} 1 \,; & 0 < t < 1 \\ -1 \,; & 1 < t < 2 \\ 0 \,; & t > 2 \end{cases}$

$$= [u(t) - u(t-1)] - 1[u(t-1) - u(t-2)]$$

$$= u(t) - 2u(t-1) + u(t-2)$$

Now

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t) - 2u(t-1) + u(t-2)\}$$

$$\Rightarrow F(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$= \frac{1 - 2e^{-s} + e^{-2s}}{s} \quad \underline{\text{Ans}}$$

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20. $f(t) = \begin{cases} t ; & 0 \le t \le 4 \\ 1 ; & t > 4 \end{cases}$

$$= t\,[u(t) - u(t-4)] + u(t-4)$$

$$= t\,u(t) - t\,u(t-4) + u(t-4)$$

$$= t\,u(t) - u(t-4)\,(t-1)$$

$$F(s) = \mathcal{L}\, t\,u(t) - \mathcal{L}\,(t-1)\,u(t-4)$$

$$= \frac{1}{s}\mathcal{L}\, t - \frac{e^{-4s}}{s}\mathcal{L}\, t-1+4$$

$$= \frac{1}{s}\left(\frac{1}{s^2}\right) - \frac{e^{-4s}}{s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

$$= \frac{1}{s^3} - \frac{e^{-4s}}{s^3} - \frac{3e^{-4s}}{s^2}$$

$$= \frac{1 - e^{-4s} - 3se^{-4s}}{s^3}$$

$$= \frac{e^{-4s} + 3se^{-4s} - 1}{s^3} \quad \text{Ans}$$

21. $f(t) = tu(t-1)$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{tu(t-1)\}$

$= e^{-s}\mathcal{L}\{f(t+1)\}$

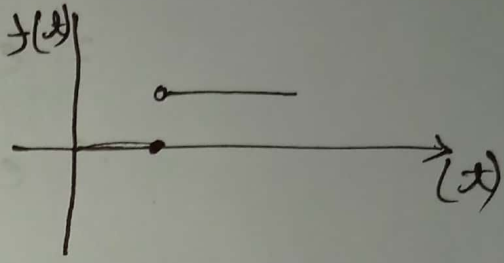
$= e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$

$= \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s}$

Now,

$f(t) = tu(t-1)$

$= \begin{cases} 0 ; & t<1 \\ 1 ; & t>1 \end{cases}$



22. $f(t) = (t-1)u(t-1)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)u(t-1)\}$

$= e^{-s}\mathcal{L}\{(t-1+1)\}$

$= e^{-s}\mathcal{L}\{t\}$

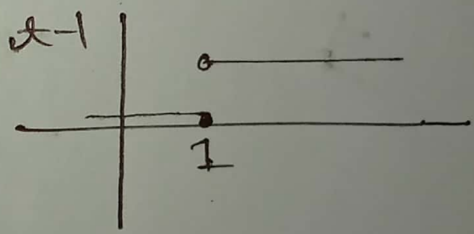
$= e^{-s} \times \frac{1}{s^2}$

$= \frac{e^{-s}}{s^2}$

Now,

$f(t) = (t-1)u(t-1)$

$= \begin{cases} t-1 ; & t>1 \\ 0 ; & t<1 \end{cases}$



23. $f(t) = (t-1)^2 u(t-1)$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)^2 u(t-1)\}$

$= e^{-s}\mathcal{L}\{(t-1+1)^2\}$

$= e^{-s}\mathcal{L}\{t^2\}$

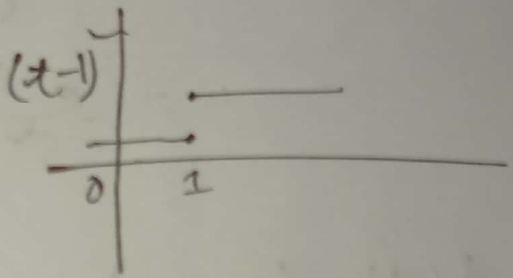
$= e^{-s} \times \frac{2}{s^3}$

$= \frac{2e^{-s}}{s^3}$

Now,

$f(t) = (t-1)^2 u(t-1)$

$= \begin{cases} (t-1)^2 ; & t>1 \\ 0 ; & t<1 \end{cases}$



24. $f(t) = e^{-2t} u(t-3)$

$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-2t} u(t-3)\}$

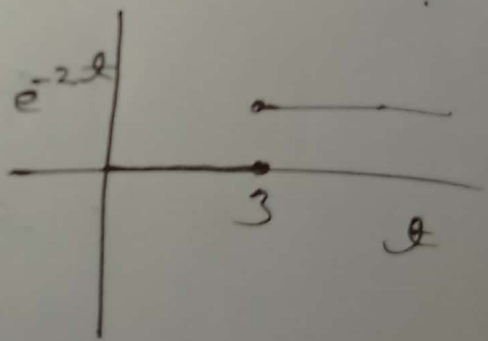
$\Rightarrow = e^{-3s}\mathcal{L}\{e^{-2(t+3)}\}$

$= e^{-3s} \cdot e^{-6} \cdot \frac{1}{s+2}$

$= \frac{e^{-6} \cdot e^{-3s}}{s+2}$

Now, $f(t) = e^{-2t} u(t-3)$

$= \begin{cases} 0 ; & t<3 \\ e^{-2t} ; & t>3 \end{cases}$



25. $f(t) = 4\cos t\, u(t-\pi)$

$F(s) = 4\,\mathcal{L}\, \cos t\, u(t-\pi)$

$= 4e^{-\pi s} \cdot \mathcal{L} \cos(t+\pi)$

$= 4e^{-\pi s} \cdot \mathcal{L}(\cos t \cos\pi - \sin t \sin\pi)$

$= 4e^{-\pi s} \cdot \mathcal{L}(-\cos t - 0)$

$= 4e^{-\pi s} \cdot \dfrac{-s}{s^2+1}$

$= \dfrac{-4se^{-\pi s}}{s^2+1}$ Ans

26. $f(t) = \begin{cases} t \,; & 0 < t < 1 \\ 2 \,; & t > 1 \end{cases}$

$f(t) = t\,[u(t) - u(t-1)] + 2\,[u(t-1)]$

$= t\,u(t) - t\,u(t-1) + 2u(t-1)$

$= t\,u(t) + (2-t)\,u(t-1)$

$F(s) = \mathcal{L}\, t\,u(t) + \mathcal{L}\,(2-t)\,u(t-1)$

$= \mathcal{L}\, t + e^{-s}\,\mathcal{L}\, 2-(t+1)$

$= \mathcal{L}\, t + e^{-s}\,\mathcal{L}(1-t)$

$= \dfrac{1}{s^2} + e^{-s}\left(\dfrac{1}{s} - \dfrac{1}{s^2}\right)$ Ans.

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27. $f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ (t-3), & t \ge 1 \end{cases}$

$$f(t) = t^2 [u(t) - u(t-1)] + (t-3)[u(t-1)]$$

$$= t^2 u(t) - t^2 u(t-1) + (t-3) u(t-1)$$

$$= t^2 u(t) + (t-3-t^2) u(t-1)$$

$$F(s) = \mathcal{L}\, t^2 u(t) + \mathcal{L}\,(t-3-t^2)\, u(t-1)$$

$$= \frac{2}{s^3} + e^{-s} \mathcal{L}\,(t+1) - 3 - (t+1)^2$$

$$= \frac{2}{s^3} + e^{-s} \mathcal{L}\, t + 1 - 3 - t^2 - 2t - 1$$

$$= \frac{2}{s^3} + e^{-s} \mathcal{L} - t - 3 - t^2$$

$$= \frac{2}{s^3} - e^{-s} \left(\frac{1}{s^2} + \frac{3}{s} + \frac{2}{s^3} \right)$$

Ans

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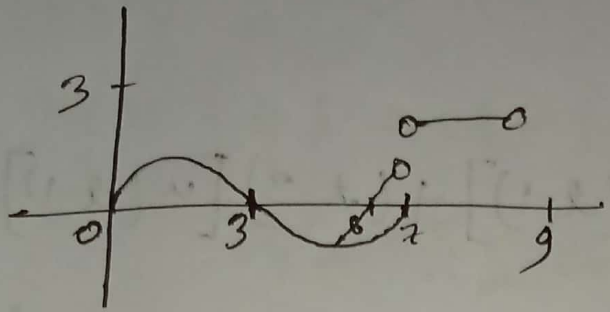
28.



$f(t) = \sin t$

$f(t) = 3$

$\Rightarrow \sin t\,[u(t) - u(t-7)] + 3\,[u(t-7) - u(t-9)]$

$\Rightarrow \sin t \cdot u(t) + (3 - \sin t)\,u(t-7) - 3u(t-9)$

$F(s) = \mathcal{L}\{\sin t\, u(t)\} + \mathcal{L}\{(3-\sin t)\,u(t-7)\}$

$\qquad - 3\mathcal{L}\{u(t-9)\}$

$= \mathcal{L}\{\sin t\} + e^{-7s}\,\mathcal{L}\{3 - \sin t\cos 7 - \cos t\sin 7\}$

$\qquad - 3\,\frac{e^{-9s}}{s}$

$= \frac{1}{s^2+1} + e^{-7}\left[\frac{3}{s} - \cos 7\,\frac{1}{s^2+1} - \sin 7\,\frac{s}{s^2+1}\right]$

$\qquad - 3\,\frac{e^{-9s}}{s}$

$= e^{-7s}\left(\frac{3}{s} - \frac{\cos 7 + s\sin 7}{s^2+1}\right) - \frac{3e^{-9s}}{s} + \frac{1}{s^2+1}$

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Chapter-2

1. $F(s) = \frac{1}{s-5}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}$ Ans

2. $F(s) = \frac{1}{s^5}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^5}\right) = \frac{t^4}{4!} = \frac{t^4}{24}$ Ans

3. $F(s) = \frac{s^3 - 5s^2 + 6}{s^4}$

$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{5}{s^2} + \frac{6}{s^4}\right\}$

$= 1 - 5t + t^3$ Ans.

4. $F(s) = \frac{2+4s}{s^2+25}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{2 \cdot \frac{1+2s}{s^2+25}\right\}$

$= 2\left\{\frac{1}{5}\mathcal{L}^{-1}\frac{5}{s^2+25} + 2\mathcal{L}^{-1}\frac{s}{s^2+25}\right\}$

$= 4\cos 5t + \frac{2}{5}\sin 5t$ Ans

5. $F(s) = \frac{3}{s^2+4}$

$\Rightarrow \frac{3}{2} \mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2} \sin 2t$ Ans.

6. $F(s) = \frac{3}{s^2-4}$

$\Rightarrow \frac{3}{2} \mathcal{L}^{-1}\left(\frac{3}{s^2-4}\right)$

$\Rightarrow \frac{3}{2}\left(\frac{e^{2t} - e^{-2t}}{2}\right)$

$= \frac{3}{4} e^{2t} - \frac{3}{4} e^{-2t}$ Ans

7. $F(s) = \frac{1}{(s-3)^4}$

$\Rightarrow \mathcal{L}^{-1} \frac{1}{(s-3)^4} = e^{3t} \frac{t^3}{6}$ Ans

8. $F(s) = \frac{3}{(s+3)^2+9}$

$\Rightarrow \mathcal{L}^{-1}\left(\frac{3}{(s+2)^2+9}\right) = e^{-2t} \sin 3t.$

9. $F(s) = \frac{s-2}{(s-2)^2-16}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2-16}\right\}$

$= e^{2t}\cos 4t$

$= e^{2t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)$

$= \frac{e^{6t}}{2} + \frac{e^{-2t}}{2}$ Ans

10. $F(s) = \frac{s}{s^2+4s-9}$

$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+4-13}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2-13}\right\}$

$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-13}\right\}$

$= e^{-2t}\left\{\cosh(\sqrt{13})t - \frac{2}{\sqrt{13}}\sinh(\sqrt{3}t)\right\}$

$= e^{-2t}\left\{\cosh(\sqrt{3}t) - \frac{2\sqrt{13}\sinh(\sqrt{13}t)}{13}\right\}$ Ans

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11. $F(s) = \dfrac{5s-7}{s^2-6s+25}$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{5s-7}{s^2-6s+25}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{5s-7}{s^2-6s+9+16}\right\}$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{5(s-3)+8}{(s-3)^2+16}\right\}$

$= \mathcal{L}^{-1}\left\{\dfrac{5(s-3)}{(s-3)^2+16}\right\} + \mathcal{L}^{-1}\left\{\dfrac{8}{(s-3)^2+16}\right\}$

$= 5e^{3t}\mathcal{L}^{-1}\left(\dfrac{s}{s^2+16}\right) + 2e^{3t}\mathcal{L}^{-1}\left(\dfrac{4}{s^2+16}\right)$

$= 5e^{3t}\cos 4t + 2e^{3t}\sin 4t$ Ans

12. $F(s) = \dfrac{s}{s^2-6s+10}$

$\Rightarrow \mathcal{L}^{-1}\left\{\dfrac{s-3}{(s-3)^2+1-3}\right\}$

$= e^{3t}\left\{\mathcal{L}^{-1}\left(\dfrac{s}{s^2-2}\right)\right.$

$= e^{3t}\cosh(\sqrt{2})t$ Ans

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13. $F(s) = \dfrac{s+1}{s(s-2)(s+3)}$

Let,

$$\frac{s+1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$\Rightarrow s+1 = A(s+3)(s-2) + B(s+3)s + C(s-2)s \quad \text{—(1)}$$

Let,

$s = -3$

$$\Rightarrow -3+1 = 0+0+C(-3-2)(-3)$$

$$\Rightarrow -2 = 15C$$

$$\therefore C = \frac{-2}{15}$$

Let,

$s = 2$,

$$\Rightarrow 2+1 = 0+B(2+3)2+0$$

$$\Rightarrow 3 = 10B$$

$$\therefore B = \frac{3}{10}$$

Let,

$s = 0$

$\Rightarrow 0+1 = A(0-2)(0+3)+0+0$

$\Rightarrow 1 = -6A$

$\therefore A = \frac{-1}{6}$

Now,

$\mathcal{L}^{-1} \frac{s+1}{s(s-2)(s+3)}$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-1}{6s} + \frac{3}{10(s-2)} - \frac{2}{15(s+2)}\right\}$

$= \frac{-1}{6} + \frac{3}{10} e^{2t} - \frac{2}{15} e^{-3t}$ Ans.

14. $F(s) = \dfrac{6}{(s+2)(s-4)}$

Let,

$$\frac{6}{(s+2)(s-4)} = \frac{A}{s+2} + \frac{B}{(s-4)}$$

$$\Rightarrow 6 = A(s-4) + B(s+2)$$

Let,

$s=4$

$$\Rightarrow 6 = 0 + B(4+2) \Rightarrow 6 = 6B \quad \therefore B = 1.$$

Let,

$s = -2,$

$$\Rightarrow 6 = A(-2-4) + 0 \Rightarrow 6 = -6A \quad \therefore A = -1$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{6}{(s+2)(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s+2} + \frac{1}{s-4}\right\}$$

$$= -e^{-2t} + e^{4t}$$

Ans

15. $F(s) = \dfrac{6s-17}{s^2-5s+6}$

$= \dfrac{6s-17}{s^2-2s-3s+6}$

$= \dfrac{6s-17}{(s-2)(s-3)}$

Partial fraction:-

$$\frac{6s-17}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

$\Rightarrow 6s-17 = A(s-3) + B(s-2) \quad - (i)$

(i) $\Rightarrow s = 3$

$\Rightarrow 18-17 = 0+B \quad \therefore B = 1$

(i) $\Rightarrow s = 2$,

$\Rightarrow 12-17 = A(-1) \quad \therefore A = 5$

Now, $f(t) = \mathcal{L}^{-1}\left\{\dfrac{5}{s-2}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\}$

$= 5e^{2t} + e^{3t}$ Ans

16. $F(s) = \dfrac{s}{(s+1)^2}$

Let, $\dfrac{s}{(s+1)^2} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+1)^2}$

$\Rightarrow s = A(s+1) + B$

Let,

$A = -1$

$\Rightarrow -1 = 0 + B \quad \therefore B = -1$

Let,

$s = 0$

$\Rightarrow 0 = A + B$

$\therefore A = 1$

Now,

$\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+1} - \dfrac{1}{(s+1)^2}\right\}$

$\Rightarrow e^{-t} - te^{-t}$.

Ans

17. $F(s) = \dfrac{7s^2+14s-9}{(s-1)^2(s-2)}$

Let,

$$\frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}$$

$\Rightarrow 7s^2+14s-9 = A(s-1)(s-2) + B(s-2) + C(s-1)^2$

Let,

$s=1,$

$\Rightarrow 7+14-9 = 0+B(1-2)+0$

$\Rightarrow 12 = -B$

$\therefore B = -12.$

Let, $s=2$

$\Rightarrow 28+28-9 = 0+0+C(2-1)^2$

$\Rightarrow 47 = C$

$\therefore C = 47.$

Let,

$s = 0$

$\Rightarrow 0 + 0 - 9 = A(-1)(-2) + (-12)(-2) + 47(-1)^2$

$\Rightarrow -9 = 2A + 24 + 47$

$\Rightarrow 2A = -80$

$\therefore A = -40$

$\mathcal{L}^{-1}\left\{\frac{7s^2 + 14s - 9}{(s-1)^2(s-2)}\right\} =$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{-40}{s-1} + \frac{-12}{(s-1)^2} + \frac{47}{s-1}\right\}$

$= -40e^{t} - 12e^{t}t + 47e^{2t}$

Ans

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18. $F(s) = \dfrac{20}{(s^2+4s+1)(s+1)}$

Let,

$$\frac{20}{(s^2+4s+1)(s+1)} = \frac{As+B}{s^2+4s+1} = \frac{B}{s+1}$$

$$\Rightarrow 20 = (As+B)(s+1) + C(s^2+4s+1)$$

Let,

$s = -1$,

$$\Rightarrow 20 = 0 + C(1-4+1)$$

$\therefore C = -10.$

Let, $s = 0$,

$$\Rightarrow 20 = (0+B)(0+1) + C(0+0+1)$$

$\therefore B = 30.$

Let, $s = 1$

$$\Rightarrow 20 = (A+30)(1+1) - 10(1+4+1)$$

$= 20 = (A+30)2 - 60 \quad \therefore A = 10$

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NOW,

$$\mathcal{L}^{-1}\left\{\frac{20}{(s^2+4s+1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{10s+30}{s^2+4s+1} + \frac{-10}{s+1}\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{10(s+2)+10}{s^2+4s+4-3} - \frac{10}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{10(s+2)+0}{(s+2)^2-3} - \frac{10}{s+1}\right\}$$

$$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{10s}{s^2-3} + \frac{1}{\sqrt{3}}\,\frac{10\cdot\sqrt{3}}{s^2-3}\right\} - \mathcal{L}^{-1}\left\{\frac{10}{s+1}\right\}$$

$$= e^{-2t}\left\{10\cosh(\sqrt{3})t + \frac{10}{3}\sinh(\sqrt{3})t\right\} - 10e^{-t}$$

$$= e^{-2t}\left\{\cosh(\sqrt{3}t)\right\} + \frac{\sinh(\sqrt{3}t)}{\sqrt{3}} - 10e^{-t}.$$

19. $F(s) = \dfrac{s}{(s^2+4)(s-1)}$

Let,

$$\frac{s}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}$$

$\Rightarrow s = (As+B)(s-1) + C(s^2+4)$

Let,

$s=1,$

$\Rightarrow 1 = 0 + C(1+4) \quad \therefore C = \dfrac{1}{5}$

Let,

$s=0,$

$\Rightarrow 0 = (0+B)(-1) + C(0+4)$

$\Rightarrow 0 = -B + 4C \quad \therefore B = \dfrac{4}{5}$

Let,

$s=2$

$\Rightarrow 2 = \left(2A + \dfrac{4}{5}\right)(2-1) + C(4+4)$

$\Rightarrow 2 = \left(2A + \dfrac{4}{5}\right) + \dfrac{8}{5}$

$\Rightarrow 2A = 2 - \dfrac{4}{5} - \dfrac{8}{5}$

$\therefore A = -\dfrac{1}{5}$

Now,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{\left(-\frac{1}{5}\right)s+\frac{4}{5}}{s^2+4} + \frac{1/5}{s-1}\right\}$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-4}{s^2+4} - \frac{1}{s-1}\right\}$$

$$= -\frac{1}{5}\left\{\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \mathcal{L}^{-1}\left(\frac{4}{s^2+4}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\right\}$$

$$= -\frac{1}{5}\left\{\cos 2t - 2\sin 2t - e^{t}\right\}$$

$$= \frac{e^{t}}{5} + \frac{2\sin 2t}{5} - \frac{\cos 2t}{5}$$

Ans

24. $F(s) = 3\left(\frac{e^{-5s}}{s}\right)$

$$\Rightarrow \mathcal{L}^{-1}\left\{3\left(\frac{e^{-5s}}{s}\right)\right\}$$

$$= 3u(t-5)$$

Ans

25. $F(s) = 4\left(\frac{e^{-3s}}{s^2}\right)$

$f(t) = \mathcal{L}^{-1}\left\{4\left(\frac{e^{-3s}}{s^2}\right)\right\}$

Let,

$F(s) = \frac{1}{s^2}$ and $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$

We got,

$\mathcal{L}^{-1}\left\{e^{-as} F(t)\right\} = 4(t-3)u(t-3)$

$= 4(t-3)\, u_3(t)$

$= \begin{cases} 0 \,; & 0 < t < 3 \\ 4(t-3) \,; & t > 3 \end{cases}$

26. $F(s) = \dfrac{se^{-\pi s}}{s^2+2s}$

Let,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s}\right\} = \cos 5t$$

So, $f(t) = \mathcal{L}^{-1}\left\{\dfrac{se^{-\pi s}}{s^2+2s}\right\}$

$$= \cos 5(t-\pi)\, u_\pi(t)$$

$$= -\cos(5t)\, u_\pi(t)$$

$$= \begin{cases} 0 \;; & 0 < t < \pi \\ -\cos(5t)\;; & t > \pi \end{cases}$$ Ans

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27. $F(s) = \frac{2(e^{-3s} - 3e^{-4s})}{s} = \frac{2e^{-3s}}{s} - \frac{6e^{-4s}}{s}$

Now,

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\left\{\frac{2e^{-3s}}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{6e^{-4s}}{s}\right\}$$

$$\Rightarrow f(t) = 2u(t-3) - 6u(t-4)$$

$$= 2u_3(t) - 6u_4(t-4)$$

$$= \begin{cases} 0, & 0 < t < 3 \\ 2, & 3 < t < 4 \\ -4, & t > 4 \end{cases}$$

Ans.

28. $F(s) = \frac{5(e^{-\pi s} + e^{-2\pi s})}{s^2+25}$

$$= \frac{5e^{-\pi s}}{s^2+25} + \frac{5e^{-2\pi s}}{s^2+25}$$

Now,

$$\mathcal{L}^{-1}F(s) = \mathcal{L}^{-1}\frac{5e^{-\pi s}}{s^2+25} + \mathcal{L}^{-1}\frac{5e^{-2\pi s}}{s^2+25}$$

$$= \sin 5(t+\pi)u(t-\pi) + u(t-2\pi)\sin 5(t+2\pi)$$

$$= (-\sin 5t)u(t-\pi) + (\sin 5t)u(t-2\pi)$$

$$= \begin{cases} 0; & 0 < t < \pi \\ -\sin 5t; & \pi < t < 2\pi \\ 0; & t > 2\pi \end{cases}$$

Ans.

29. $F(s) = 1$

$\Rightarrow \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1}(1)$

$= \delta(t)$. Ans

30. $F(s) = e^{-3s}$

$\Rightarrow \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} e^{-3s}$

$= \delta(t-3)$ Ans

31. $F(s) = 25\, e^{-2s}$

$\Rightarrow \mathcal{L}^{-1} F(s) = 25\, \mathcal{L}^{-1} e^{-2s}$

$= 25\, \delta(t-2)$ Ans

Chapter 3

3.1

1. $\dot{y}(t) = 3 \; ; \; y(0) = 2$

$$\mathcal{L}\,\dot{y}(t) = \mathcal{L}\,3$$

$$\Rightarrow sY(s) - y(0) = \frac{3}{s}$$

$$\Rightarrow sY(s) - 2 = \frac{3}{s}$$

$$\Rightarrow sY(s) = \frac{3+2s}{s^2}$$

Now

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{3+2s}{s^2}\right\}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2s}{s^2}\right\}$$

$$\Rightarrow y(t) = 3t+2$$

Ans

2. $\dot{y}(t) = 4t$, $y(0) = 1$

$$\Rightarrow \mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\,4t$$

$$\Rightarrow sY(s) - y(0) = \frac{4}{s^2}$$

$$\Rightarrow sY(s) - 1 = \frac{4}{s^2}$$

$$\Rightarrow Y(s) = \frac{4+s^2}{s^3}$$

Now,

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^3}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

$$\Rightarrow y(t) = 4\,\frac{t^2}{2!} + 1$$

$$\therefore y(t) = 2t^2 + 1$$

Ans

3. $\dot{y}(t) = 2t - 1;\quad y(0) = 3$

$\Rightarrow \mathcal{L}\,\dot{y}(t) = \mathcal{L}\,2t - \mathcal{L}\,1$

$\Rightarrow s\,Y(s) - y(0) = \dfrac{2}{s^2} - \dfrac{1}{s}$

$\Rightarrow s\,Y(s) - 3 = \dfrac{2 - s}{s^2}$

$\Rightarrow Y(s) = \dfrac{2 - s + 3s^2}{s^3}$

Now

$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}\dfrac{2}{s^3} - \mathcal{L}\dfrac{1}{s^2} + \mathcal{L}\dfrac{3}{s}$

$\Rightarrow y(t) = 2\,\dfrac{t^2}{2} - t + 3$

$\therefore y(t) = t^2 - t + 3$

$= t(t-1) + 3$ Ans

4. $\dot{y}(t) = t^2;\ y(0) = 4$

$\Rightarrow \mathcal{L}\,\dot{y}(t) = \mathcal{L}\,t^2$

$\Rightarrow sY(s) - y(0) = \frac{2!}{s^3}$

$\Rightarrow sY(s) - 4 = \frac{2}{s^3}$

$\Rightarrow Y(s) = \frac{2+4s^3}{s^4}$

Now,

$\mathcal{L}^{-1}\,Y(s) = \mathcal{L}^{-1}\left\{\frac{2}{s^4}\right\} + \mathcal{L}^{-1}\left\{\frac{4s^3}{s^4}\right\}$

$\Rightarrow y(t) = 2 \cdot \frac{t^3}{3!} + 4$

$\therefore y(t) = \frac{t^3}{3} + 4.$

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5. $\dot{y}(t) = e^{2t}$; $y(0) = 2$

$\Rightarrow \mathcal{L}\,\dot{y}(t) = \mathcal{L}\,e^{2t}$

$\Rightarrow sY(s) - y(0) = \dfrac{1}{s-2}$

$\Rightarrow sY(s) - 2 = \dfrac{1}{s-2}$

$\Rightarrow Y(s) = \dfrac{1+2s-4}{s^2-2s} = \dfrac{2s-3}{s(s-2)}$

Now,

$$\frac{2s-3}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$\Rightarrow 2s-3 = A(s-2) + Bs$ —— (i)

(i) $\Rightarrow s = 2$,

$\Rightarrow 4-3 = 0-2B$

$\therefore B = \dfrac{1}{2}$

(ii) $\Rightarrow s = 0$,

$\Rightarrow -3 = -2A$

$\therefore A = \dfrac{3}{2}$

$\therefore \dfrac{2s-3}{s(s+2)} = \dfrac{3}{2s} + \dfrac{1}{2(s-2)}$

So,

$$y(t) = \mathcal{L}\left\{\frac{3}{2s} + \frac{7}{2(s-2)}\right\}$$

$$= \frac{3}{2} + \frac{1}{2} e^{2t}$$

$$= \frac{e^{2t}}{2} + \frac{3}{2} \quad \text{Ans}$$

6. $\dot{y}(t) + y(t) = 2; \quad y(0) = 0$

$\Rightarrow \mathcal{L}\{\dot{y}(t) + y(t)\} = \mathcal{L}\,2$

$\Rightarrow sY(s) - y(0) + Y(s) = \dfrac{2}{s}$

$\Rightarrow Y(s).(s+1) = \dfrac{2}{s} + 0$

$\Rightarrow Y(s) = \dfrac{2}{s(s+1)}$

So,

$$\frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$\Rightarrow 2 = A(s+1) + Bs$

Let,

$s = 0$

$\Rightarrow 2 = A$

Let,

$s = 1$

$\Rightarrow 2 = A(1+1) + B$

$\Rightarrow 2 = 4 + B$

$\therefore B = -2$

Now,

$$Y(s) = \frac{2}{s} - \frac{2}{s+1}$$

$$\mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1} \frac{2}{s} - \mathcal{L}^{-1} \frac{2}{s+1}$$

$$\Rightarrow y(t) = 2 - 2e^{-t}$$

$7.\ \ddot{y}(t) = 5\ ;\ y(0) = 1\ ;\ \dot{y}(0) = 2$

$\Rightarrow \mathcal{L}\{\ddot{y}(t) = \mathcal{L}\,5$

$\Rightarrow s^2 Y(s) - s\,y(0) - \dot{y}(0) = \frac{5}{s}$

$\Rightarrow s^2 Y(s) = s - 2 = \frac{5}{s}$

$\Rightarrow Y(s) = \frac{s^2 + 2s + 5}{s^3}$

Now,

$\mathcal{L}^{-1}\,Y(s) = \mathcal{L}^{-1}\,\frac{s^2 + 2s + 5}{s^3}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\,\frac{1}{s} + \mathcal{L}^{-1}\,\frac{2}{s^2} + \mathcal{L}^{-1}\,\frac{5}{s^3}$

$\Rightarrow y(t) = 1 + 2t + \frac{5t^2}{2}$ Ans

8. $\ddot{y}(t) - 2\dot{y}(t) = \cos t$; $y(0) = 0, \dot{y}(0) = 1$

$\Rightarrow \mathcal{L}^{-1}\{\ddot{y}(t) - 2\dot{y}(t)\} = \mathcal{L}^{-1}\cos t$

$\Rightarrow s^2Y(s) - sy(0) - \dot{y}(0) - 2sY(s) + 2y(0) = \frac{s}{s^2+1}$

$\Rightarrow Y(s)(s^2 - 2s) - 1 = \frac{s}{s^2+1}$

$\Rightarrow Y(s)(s^2-2s) = \frac{s}{s^2+1} + 1$

$\Rightarrow Y(s) = \frac{s+s^2+1}{(s^2+1)(s^2-2s)} = \frac{s^2+s+1}{s(s-2)(s^2+1)}$

Now,

$$\frac{s^2+s+1}{s(s-2)(s^2+1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$$

$\Rightarrow s^2+s+1 = A(s-2)(s^2+1) + B(s^2+1)s + (Cs+D)s(s-2)$

Let, $s = 2$

$\Rightarrow 4+2+1 = 0 + B(4+1)2 + 0$

$\Rightarrow 7 = 10B \quad \therefore B = \frac{7}{10}$

Let, $s = 0$

$\Rightarrow 0+0+1 = A(-2)(1) + 0 + 0$

$\therefore A = -\frac{1}{2}$

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Let, $s = 1$,

$\Rightarrow 1+1+1 = \frac{-1}{2}(1-2)(1+1) + \frac{7}{10}(1+1) + (C+D)(1-2)$

$\Rightarrow 3 = 1 + \frac{7}{5} - C - D$

$\Rightarrow C + D = \frac{7}{5} + 1 - 3 = -\frac{3}{5}$ — (i)

Let
$s = -1$

$\Rightarrow 1-1+1 = \frac{-1}{2}(-1-2)(1+1) + \frac{7}{10}(1+1)(-1) + (-C+D)(-1)(-1-2)$

$\Rightarrow 1 = 3 - \frac{7}{5} + (-C+D)3$

$\Rightarrow C - D = 1 - \frac{7}{15} - \frac{1}{3}$

$\Rightarrow C - D = \frac{1}{5}$ — (ii)

From, (i) & (ii)

$-\frac{3}{5} - D - D = \frac{1}{5}$

$\Rightarrow 2D = \frac{-3}{5} - \frac{1}{5}$

$\therefore D = \frac{-2}{5}$

$C = \frac{-3}{5} + \frac{2}{5} = \frac{-1}{5}$

$\therefore A = \frac{-1}{2}, B = \frac{7}{10}$

$C = \frac{-1}{5}, D = \frac{-2}{5}$

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$$\frac{s^2+s+1}{s(s-2)(s^2+1)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{Cs+D}{s^2+1}$$

$$= \frac{-1}{2s} + \frac{7}{10(s-2)} + \frac{-s}{5(s^2+1)} \quad \frac{-2}{5(s^2+1)}$$

Now,

$$\mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1} \frac{-1}{2s} + \mathcal{L}^{-1} \frac{7}{10(s-2)} - \mathcal{L}^{-1} \frac{s}{5(s^2+1)} - \mathcal{L}^{-1} \frac{2}{5(s^2+1)}$$

$$\Rightarrow y(t) = \frac{-1}{2} + \frac{7e^{2t}}{10} + \frac{\cos t}{5} - \frac{2\sin t}{5}$$

$$\Rightarrow y(t) = \frac{7}{10} e^{2t} - \frac{2}{5}\left(\sin t - \frac{1}{2}\cos t\right) - \frac{1}{2}$$

Ans

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9. $\ddot{y}(t) + 3\dot{y}(t) - y(t) = e^{t}$; $y(0) = \dot{y}(0) = 0$

$\Rightarrow \mathcal{L}\{\ddot{y}(t) + 3\dot{y}(t) - y(t)\} = \mathcal{L}e^{t}$

$\Rightarrow s^2 Y(s) - s y(0) - \dot{y}(0) + 3s Y(s) - 3y(0) - Y(s) = \frac{1}{s-1}$

$\Rightarrow Y(s) \cdot (s^2 + 3s - 1) = \frac{1}{s-1}$

$\Rightarrow Y(s) = \frac{1}{(s-1)(s^2+3s-1)}$

So, $\frac{1}{(s-1)(s^2+3s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+3s-1}$

$\Rightarrow 1 = A(s^2+3s-1) + (Bs+C)(s-1)$

Let, $s = 1$

$\Rightarrow 1 = A(1+3-1) + 0$

$\therefore A = \frac{1}{3}$

Let, $s = 0$,

$\Rightarrow 1 = \frac{1}{3}(0+0-1) + (0+C)(-1)$

$\Rightarrow C = \frac{-1}{3} - 1 \quad \therefore C = -\frac{4}{3}$

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Let, $s=2$,

$$\Rightarrow 1 = \frac{1}{3}(4+6-2) + \left(2B+\frac{4}{3}\right)(2-1)$$

$$\Rightarrow 2B = 1-3-\frac{4}{3}$$

$$\therefore B = \frac{-5}{3}$$

Now,

$$\frac{1}{(s-1)(s^2+3s-1)} = \frac{1}{3(s-1)} + \frac{-5s}{3(s^2+3s-1)} + \frac{4}{3(s^2+3s-1)}$$

$$\Rightarrow \mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1}\frac{1}{3(s-1)} + \mathcal{L}^{-1}\left\{\frac{-5s}{3(s^2+3s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{3(s^2+3s-1)}\right\}$$

$$\Rightarrow y(t) = \frac{e^t}{3} - \frac{5}{3}\mathcal{L}^{-1}\frac{s}{s^2+3s+\frac{9}{4}-\frac{13}{4}} + \frac{3}{4}\mathcal{L}^{-1}\frac{1}{s^2+3s+\frac{9}{4}-\frac{13}{4}}$$

$$\Rightarrow y(t) = \frac{e^t}{3} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{(s+\frac{3}{2})-\frac{3}{2}}{(s+\frac{3}{2})^2-\frac{13}{4}}\right\} + \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s+\frac{3}{2})^2-\frac{13}{4}}\right\}$$

$$\Rightarrow \frac{e^t}{3} - \frac{5}{3}e^{-\frac{3}{2}t}\mathcal{L}^{-1}\frac{s-3/2}{s^2-13/4} + \frac{4}{3}e^{-3/2 t}\mathcal{L}^{-1}\left\{\frac{1}{s^2-\frac{13}{4}}\right\}$$

$$= \frac{e^{t}}{3} - \frac{5}{3} e^{-3/2\, t} \left\{ \mathcal{L}^{-1} \frac{s}{s^2 - \frac{13}{4}} - \frac{3}{\sqrt{3}} \mathcal{L}^{-1} \frac{\sqrt{13}/2}{s^2 - \frac{13}{4}} \right\}$$

$$+ \frac{4 \cdot 2 - e^{3/2\, t}}{3 \cdot \sqrt{13}} \mathcal{L}^{-1} \left\{ \frac{\frac{\sqrt{13}}{2}}{s^2 - 13/4} \right\}$$

$$\Rightarrow y(t) = \frac{e^{t}}{3} - \frac{5}{3} e^{-\frac{3t}{2}} \left\{ \cosh \frac{\sqrt{13}}{2} t - \frac{23}{5\sqrt{13}} \sinh \frac{\sqrt{13}}{2} t \right\}$$

Ans

13. $\dot{x} = y$

$\dot{y} = 16x;\ x(0) = 0,\ y(0) = 4$

Now,

$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{y(t)\}$

$\Rightarrow sX(s) - x(0) = Y(s)$

$\Rightarrow sX(s) - Y(s) = 0 \quad \text{--- (i)}$

And

$\mathcal{L}\{\dot{y}(t)\} = 16\,\mathcal{L}\{x(t)\}$

$\Rightarrow sY(s) - y(0) = 16X(s)$

$\Rightarrow sY(s) - 16X(s) = 4 \quad \text{--- (ii)}$

(i) $\times s \Rightarrow$

$$s^2X(s) - sY(s) = 0$$
$$-16X(s) + sY(s) = 4$$
$$\overline{X(s)\cdot(s^2-16) = 4}$$

$$\therefore X(s) = \frac{4}{s^2-16} \quad \text{--- (iii)}$$

(i) × 16 & (ii) × s ⇒

$$16sX(s) - 16Y(s) = 0$$

$$-16sX(s) + s^2Y(s) = 4s$$

$$Y(s) \cdot (-16 + s^2) = 4s$$

$$\therefore Y(s) = \frac{4s}{s^2+16} \quad \text{— (iv)}$$

Again, (iii)

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2-16}\right\}$$

$$\Rightarrow x(t) = \sinh 4t$$

And, iv,

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{4s}{s^2-16}\right\}$$

$$\Rightarrow y(t) = 4\cosh 4t.$$

Ans.

14. $\dot{x} = -4y$

$\dot{y} = x$; $x(0) = 2$, $y(0) = 0$

Now,

$$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{-4y(t)\}$$

$$\Rightarrow sX(s) - x(0) = -4Y(s)$$

$$\Rightarrow sX(s) + 4Y(s) = 2 \quad \text{—— (i)}$$

And,

$$\mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{x(t)\}$$

$$\Rightarrow sY(s) - y(0) = X(s)$$

$$\therefore sY(s) - X(s) = 0 \quad \text{—— (ii)}$$

(ii) $\times s \Rightarrow$

$$sX(s) + 4Y(s) = 2$$

$$-sX(s) + s^2Y(s) = 0$$

$$Y(s)(s^2+4) = 2$$

$$\therefore Y(s) = \frac{2}{s^2+4} \quad \text{— (iii)}$$

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(i) $\times s$ & (ii) $\times(-4)$ $\Rightarrow$

$$s^2X(s) + 4sY(s) = 2s$$

$$4X(s) - 4sY(s) = 0$$

$$X(s)(s^2+4) = 2s$$

$$\therefore X(s) = \frac{2s}{s^2+4} \quad \text{— (iv)}$$

(iii) $\Rightarrow$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$\Rightarrow y(t) = \sin 2t$$

(iv) $\Rightarrow$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2s}{s^2+4}\right\}$$

$$\Rightarrow x(t) = 2\cos 2t.$$

Ans

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15. $\dot{x} = 2x + y$

$\dot{y} = 4x + 2y$; $x(0) = 1, y(0) = 6$

Now,

$$\mathcal{L}^{-1}\{\dot{x}(t)\} = \mathcal{L}^{-1}\{2x(t)\} + \mathcal{L}\{y(t)\}$$

$$\Rightarrow sX(s) - x(0) = 2X(s) + Y(s)$$

$$\Rightarrow X(s)(s-2) - Y(s) = 1 \quad \text{— (i)}$$

And,

$$\mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{4x(t)\} + \mathcal{L}\{2y(t)\}$$

$$\Rightarrow sY(s) - y(0) = 4X(s) + 2Y(s)$$

$$\therefore Y(s)(s-2) - 4X(s) = 6 \quad \text{— (ii)}$$

(i) $\times (s-2) \Rightarrow$

$$(s-2)^2 X(s) - (s-2)Y(s) = s-2$$

$$-4X(s) + (s-2)Y(s) = 6$$

$$X(s)\{(s-2)^2 - 4\} = s+4$$

$$\therefore X(s) = \frac{s+4}{(s-2)^2-4} = \frac{(s-4)+6}{(s-2)^2-4} \quad \text{--- (iii)}$$

(i) $\times 4$ & (ii) $\times (s-2) \Rightarrow$

$$4(s-2)X(s) - 4Y(s) = 4$$

$$-4(s-2)X(s) + (s-2)^2 Y(s) = 6(s-2)$$

$$Y(s)\{(s-2)^2-4\} = 6s-8$$

$$\therefore Y(s) = \frac{6(s-2)+4}{(s-4)^2-4} \quad \text{--- (iv)}$$

(iii) $\Rightarrow$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2-4}\right\} + \mathcal{L}^{-1}\frac{6}{(s-2)^2-4}$$

$$\Rightarrow x(t) = e^{2t}\cosh 2t + 3\sinh 2t$$

(iv) $\Rightarrow$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6(s-2)}{(s-2)^2-4}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2-4}\right\}$$

$$\Rightarrow y(t) = 6e^{2t}\cosh 2t + 2\sinh 2t.$$

16. $\dot{x} = 3x + y$

$\dot{y} = 4x + 3y$; $x(0) = 3$, $y(0) = 2$.

Now,

$\mathcal{L}\{\dot{x}(t)\} = \mathcal{L}\{3x(t)\} + \mathcal{L}\, y(t)$

$\Rightarrow sX(s) - x(0) = 3X(s) + Y(s)$

$\Rightarrow (s-3)X(s) - Y(s) = 3$ —— (i)

$\mathcal{L}\{\dot{y}(t)\} = \mathcal{L}\{4x(t)\} + \mathcal{L}\{3y(t)\}$

$\Rightarrow sY(s) - y(0) = 4X(s) + 3Y(s)$

$\Rightarrow (s-3)Y(s) - 4X(s) = 2$ —— (ii)

(i) $\times (s-3) \Rightarrow$

$$(s-3)^2 X(s) - (s-3)Y(s) = 3(s-3)$$
$$-4X(s) + (s-3)Y(s) = 2$$

$$X(s)\{(s-3)^2 - 4\} = 3(s-3) + 2$$

$$\therefore X(s) = \frac{3(s-3)+2}{(s-3)^2 - 4} \quad \text{—— (iii)}$$

(i) $\times 4$ & (ii) $(s-3) \Rightarrow$

$$4(s-3)X(s) - 4Y(s) = 12$$

$$-4(s-3)X(s) + (s-3)^2 Y(s) = 2(s-3)$$

$$Y(s)\left\{(s-3)^2 - 4\right\} = 12 + 2(s-3)$$

$$\therefore Y(s) = \frac{2(s-3)+12}{(s-3)-4} \quad \text{— (iv)}$$

(iii) $\rightarrow$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{3(s-3)}{(s-3)^2-4}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-3)^2-4}\right\}$$

$$\Rightarrow x(t) = 3e^{3t}\cosh 2t + e^{3t}\sinh 2t$$

(iv) $\rightarrow$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2(s-3)}{(s-3)-4}\right\} + \mathcal{L}^{-1}\left\{\frac{12}{(s-3)^2-4}\right\}$$

$$\Rightarrow y(t) = 2e^{3t}\cosh 2t + 6e^{3t}\sin 2t.$$

Ans

Chapter-4

1. Express $\frac{(1+i)^2}{1-i}$ in the from $a+ib$

$$\Rightarrow \frac{(1+i)^2(1+i)}{(1-i)(1+i)}$$

$$= \frac{(1+i)^3}{1-i^2} = \frac{1+3i+3i^2-1}{1+1}$$

$$= \frac{1+3i-3-i}{2}$$

$$= \frac{2i-2}{2} = (i-1) \text{ Ans}$$

2. a) $Re\left\{\frac{1+\sqrt{3}i}{1-i}\right\}$

$$= \frac{1+\sqrt{3}i}{1-i} = \frac{(1-\sqrt{3}i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{1+i+\sqrt{3}i+\sqrt{3}i^2}{1+1}$$

$$= \frac{1-\sqrt{3}+i(1+\sqrt{3})}{2}$$

$$= \frac{1-\sqrt{3}}{2} + i\,\frac{1+\sqrt{3}}{2}$$

$$\therefore Re\left\{\frac{1+\sqrt{3}i}{1-i}\right\} = \frac{1-\sqrt{3}}{2} \text{ Ans}$$

3. d) $z = \left(\frac{1-i}{1+i}\right)^{18}$

$$= \left\{\frac{(1-i)(1-i)}{(1+i)(1-i)}\right\}^{18} = \left\{\frac{(1-i)^2}{1+1}\right\}^{18}$$

$$= \left\{\frac{1-2i+i^2}{2}\right\}^{18}$$

$$= \left\{\frac{1-2i-1}{2}\right\}^{18}$$

$$= (-i)^{18} = i^2 = -1$$

Polar form

$$r = \sqrt{(-1)^2 + 0^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

Ans.

4. a) $z = \sqrt{3}\, e^{i\frac{\pi}{3}}$

$\theta = \frac{\pi}{3}$

$\tan\frac{\pi}{3} = \frac{b}{a}$ $\therefore \frac{b}{a} = \sqrt{3}$

$\Rightarrow b = a\sqrt{3}$ — (i)

$\sqrt{a^2+b^2} = \sqrt{3}$

$\Rightarrow a^2+b^2 = 3$

$\Rightarrow a^2 + (a\sqrt{3})^2 = 3$

$\Rightarrow a^2 + a^2 3 = 3$

$\Rightarrow 4a^2 = 3$

$\Rightarrow a^2 = \frac{3}{4}$

$\therefore a = \frac{\sqrt{3}}{2}$ — (ii)

(i) & (ii)

$b = \frac{\sqrt{3}}{2} \times \sqrt{3}$

$\therefore b = \frac{3}{2}$

$\therefore a = \frac{\sqrt{3}}{2}, b = \frac{3}{2}$

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5. c) $z = \frac{(1+i)^3}{1-i}$

$$z = \frac{(1+i)^4}{(1-i)(1+i)}$$

$$= \frac{(1+i)^4}{2}$$

$$= \frac{(1+i)^2 (1+i)^2}{2} \quad \text{—(i)}$$

Now,

$$(1+i)^2 = 1 + 2i + i^2 = 2i$$

From, —(i)

$$\frac{(2i)(2i)}{2} = \frac{(2i)^2}{2} = \frac{4i^2}{2} = -2$$

$$\arg z = \tan^{-1}\left(\frac{0}{-2}\right)$$

$$= 0 + \pi = \pi$$

Finding principal argument:-

$$\arg z = \operatorname{Arg} z + 2n\pi$$

$$\operatorname{Arg} z = \arg z - 2n\pi.$$

$$\therefore \operatorname{Arg} z = \pi - 2n\pi \quad \text{Ans}$$

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6. a) $z^2+9=0$

$r=\sqrt{(-9)^2+0}$

$\Rightarrow z^2=-9$

$r=9$

$\Rightarrow z^2=9e^{i\pi}$

$\theta=\tan^{-1}\left(\frac{0}{-9}\right)=0+\pi=\pi$

$\Rightarrow z^2=9e^{i(\pi+2n\pi)}$

$\Rightarrow z^2=9e^{i(2n+1)\pi}$

$\Rightarrow z=9^{1/2}e^{i(2n+1)\pi/2}$

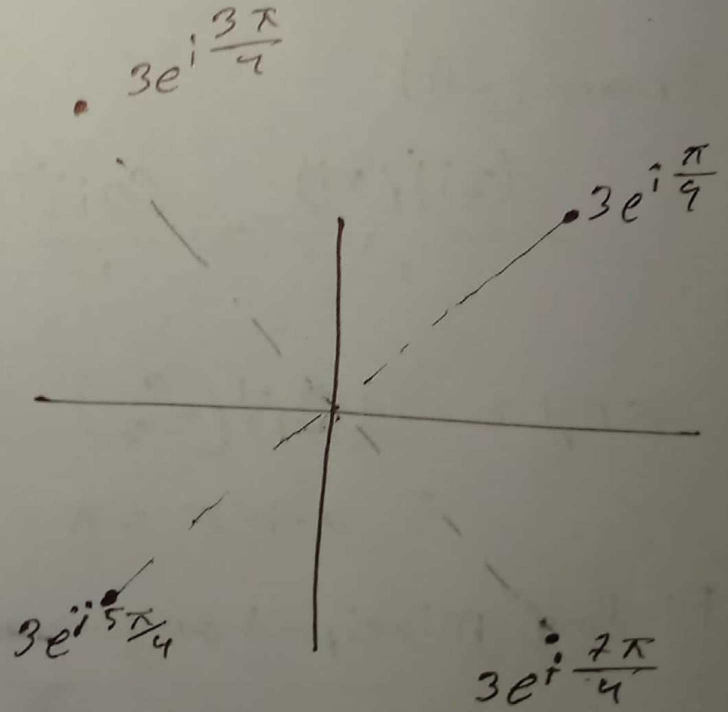
$\Rightarrow z=3e^{i(2n+1)\pi/2}$

$n=0,\ z_0=3e^{i\pi/2}$

$n=1,\ z_1=3e^{i\frac{3\pi}{2}}$

$n=2,\ z_2=3e^{i\frac{5\pi}{4}}$

$n=3,\ z_3=3e^{i\frac{7\pi}{4}}$



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b) $z^3 - \sqrt{3} - i = 0$

$r = \sqrt{(\sqrt{3})^2 + 1} = 2$

$\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\Rightarrow z^3 = \sqrt{3} + i$

$\Rightarrow z^3 = 2e^{i\frac{\pi}{6}}$

$\Rightarrow z_n^3 = 2e^{i(\pi/6 + 2n\pi)}$

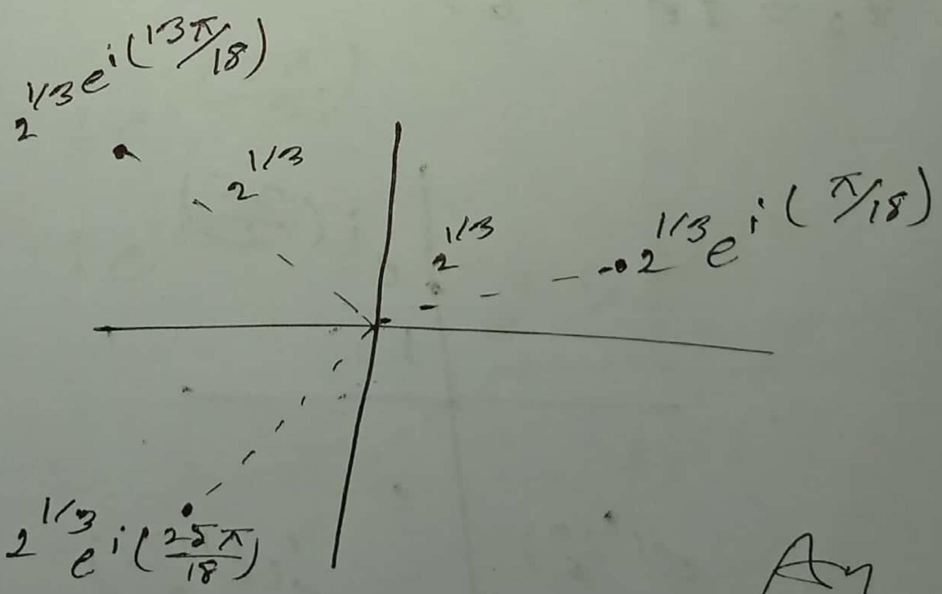
$\Rightarrow z_n^3 = 2e^{i\left(\frac{12n+1}{6}\right)\pi}$

$\Rightarrow z_n^3 = 2^{1/3} e^{i\left(\frac{12n+1}{18}\right)\pi}$

$n = 0,\ z_0 = 2^{1/3} e^{i(1/18)\pi}$

$n = 1,\ z_0 = 2^{1/3} e^{i(13/18)\pi}$

$n = 2,\ z_0 = 2^{1/3} e^{i\left(\frac{25}{18}\right)\pi}$



c) $z^3 = -i$

$r = \sqrt{(-1)^2} = 1 \quad (0, -1)$

$z^3 = 1 e^{i\left(\frac{3\pi}{2}\right)}$

$\theta = \tan^{-1}\left(\frac{-1}{0}\right)$

$= \frac{\pi}{2} + \pi$

$= \frac{3\pi}{2}$

$\Rightarrow z_n^3 = e^{i\left(\frac{3\pi}{2} + 2n\pi\right)}$

$\Rightarrow z_n^3 = e^{i\left(\frac{4n+3}{2}\right)\pi}$

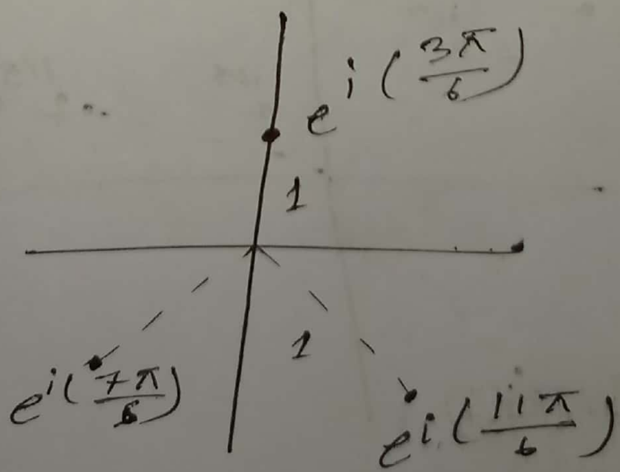
$\Rightarrow z_n = 1^{1/3} e^{i\left(\frac{4n+3}{6}\right)\pi}$

$\Rightarrow z_n = 1 e^{i\left(\frac{4\pi+3}{6}\right)\pi}$

$n = 0, \; z_0 = 1 e^{i\left(3\pi/6\right)}$

$n = 1, \; z_1 = 1 e^{i\left(\frac{7\pi}{6}\right)}$

$n = 2, \; z_2 = 1 e^{i\left(\frac{11\pi}{6}\right)}$



d) $z^4 - 1 = 0$

$r = 1$ $(1, 0)$

$\theta = \tan^{-1}\left(\frac{0}{1}\right) = 0$

$\Rightarrow z^4 = 1$

$\Rightarrow z^4 = 1e^{i(0)}$

$\Rightarrow z_n^4 = 1e^{i(0+2n\pi)}$

$\Rightarrow z_n^4 = 1e^{i(2n)\pi}$

$\Rightarrow z_n = 1e^{i(2n)\pi/4}$

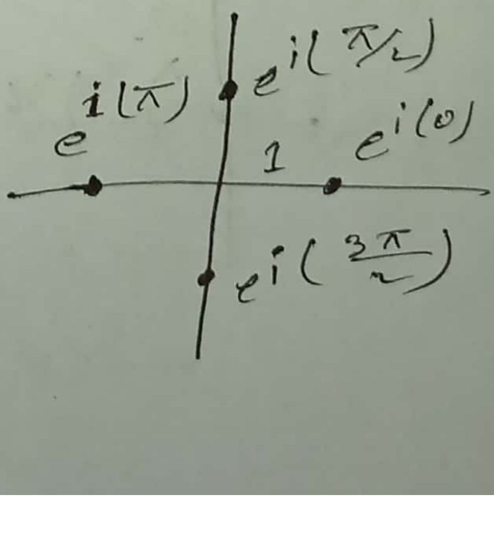
$n=0, \; z_0 = 1e^{i(0)}$

$n=1, \; z_1 = 1e^{i(\pi/2)}$

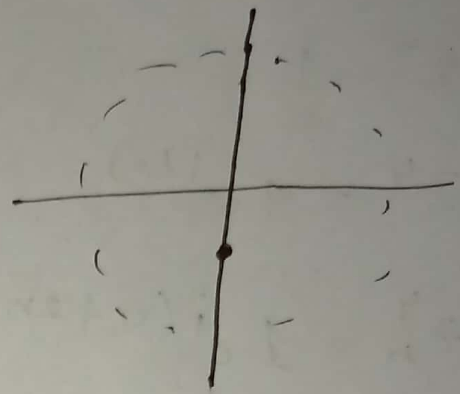
$n=2, \; z_2 = 1e^{i(\pi)}$

$n=3, \; z_3 = 1e^{i\left(\frac{3\pi}{2}\right)}$

$n=4, \; z_4 = 1e^{i(2\pi)}$



7\. a) $|z+2i| > 4$



center $(0,-2)$

radius $= 4$

Ans

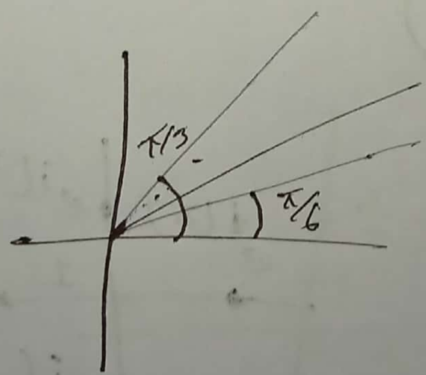
b) $1 < |z-2+i| \leq 3$

center $(2,-1)$

radius $= 3$

Ans

f) $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$



1. Let the rectangular region R in z plane which is bounded by the line $x=2$, $y=0$, $x=5$, and $y=4$ determine the region R′ of the w plane into which R is mapped under the following transformation :-

(iii) $w = \sqrt{2}\, e^{\frac{\pi}{4}i} z - (1-i)$

Solution :-

$$w = \sqrt{2}\, e^{\pi/4\, i} z - (1-i)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) z - (1-i)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)(x+iy) - (1-i)$$

$$= (1+i)(x+iy) - (1-i)$$

$$= x + iy + ix + i^2 y - 1 + i$$

$$= x - y - 1 + iy + ix + i$$

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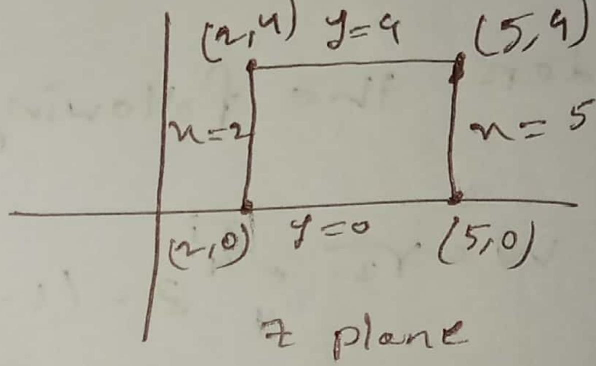
Or,

$$u + iv = (x - y - 1) + i(1 + x + y)$$

Hence,

$$u = x - y - 1, \quad v = 1 + x + y$$

When,



$x = 2,$

$x = 5,$

$y = 0,$

$y = 4,$

The points of the rectangle are:

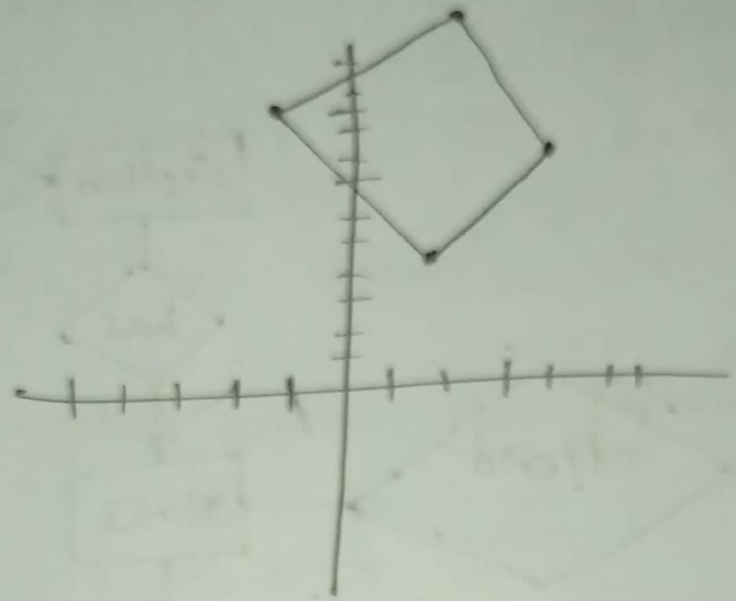
$$(2,0), (2,4), (5,0), (5,4)$$

At, $(2,0)$: $u = 2 - 0 - 1 = 1$, $v = 1 + 2 + 0 = 3$

At, $(2,4)$: $u = 2 - 4 - 1 = -3$, $v = 1 + 2 + 4 = 7$

At, $(5,0)$: $u = 5 - 0 - 1 = 4$, $v = 5 + 0 + 1 = 6$

At $(5,4)$: $u = 5 - 4 - 1 = 0$, $v = 1 + 5 + 4 = 10$



W-plane

2. Given triangle T in the z plane with vertices at 1, 1-3i, 3-i. Determine the tringle T' of the w-plane into which T is mapped under the following transformation

(iv) $w = \frac{1}{2} e^{\frac{\pi}{2}i} z - 4$

Solution:-

$$w = \frac{1}{2} e^{\frac{\pi}{2}i} z - 4$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) z - 4$$

$$= \frac{1}{2}(0+i)\cdot(x+iy) - 4$$

$$= \frac{xi}{2} + \frac{i^2 y}{2} - 4$$

$$= \frac{xi}{2} - \frac{y}{2} - 4$$

or,

$$u + iv = \left(-\frac{y}{2} - 4\right) + i\left(\frac{x}{2}\right)$$

Hence,

$$u = -\frac{y}{2} - 4, \quad v = \frac{x}{2}$$

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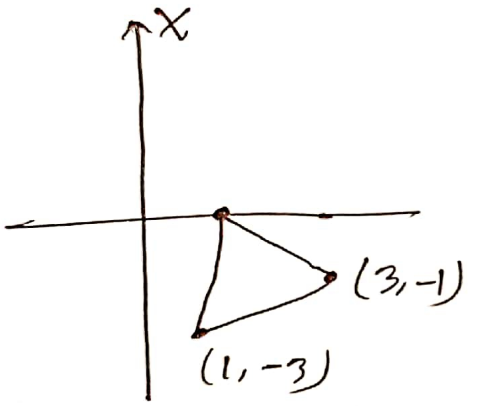
The vertices of the triangle are: $1, 1-3i, 3-i$

vertices can also be written as: $(1,0), (1,-3), (3,-1)$

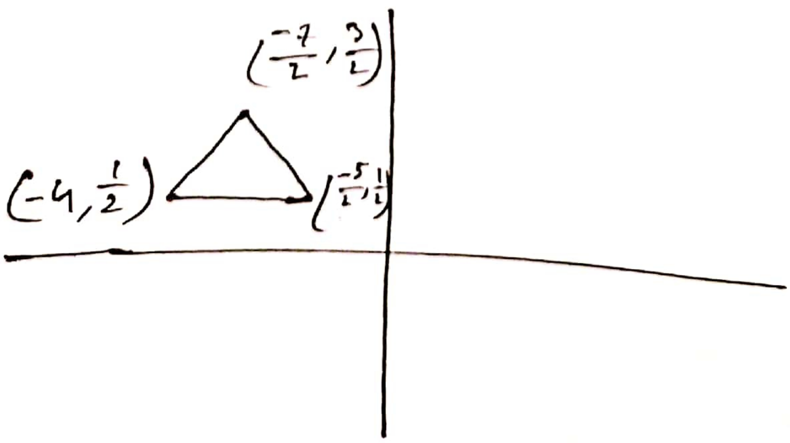
At, 1: $u = \frac{-0}{2} - 4 = -4$, $v = \frac{1}{2}$

At, $(1,-3)$; $u = \frac{3}{2} - 4 = \frac{-5}{2}$, $v = \frac{1}{2}$

At, $(3,-1)$: $u = \frac{1}{2} - 4 = \frac{-7}{2}$, $v = \frac{3}{2}$



z-plane



w-plane