$\boxed{x^2+4}$ $\rightarrow 2i, -2i$

Application of Residue theorem

$$\int_{-\infty}^{\infty} \frac{f_1(x)}{\boxed{f_2(x)}}\, dx \quad \text{or} \quad \int_{0}^{\infty} \frac{f_1(x)}{f_2(x)}\, dx$$

Improper Integral

- $f_2(x)$ has no real roots.
- degree of $f_2(x)$ is greater that of $f_1(x)$ by at least 2.

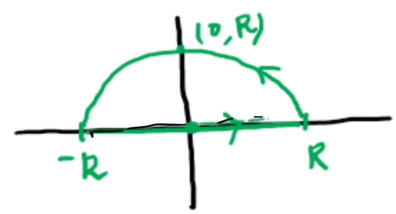
Procedure to solve:

To evaluate such integrals we consider the contour integrals

$$\oint_C \frac{f_1(z)}{f_2(z)}\, dz$$

where C is the closed contour, consisting the real axis from $-R$ to R and the upper half circle

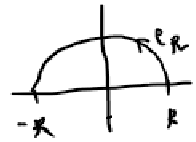
$C_R: |z| = R$



$$\oint_C \frac{f_1(z)}{f_2(z)}\, dz = \int_{-R}^{R} \frac{f_1(x)}{f_2(x)}\, dx + \int_{C_R} \frac{f_1(z)}{f_2(z)}\, dz$$

Solve $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$

$$\oint_C \frac{dz}{(z^2+4)^2} = \int_{-R}^{R} \frac{dx}{(x^2+4)^2} + \int_{C_R} \frac{dz}{(z^2+4)^2} \quad \text{— (i)}$$



$$\oint \frac{dz}{(z^2+4)^2} \; ; \qquad z^2+4=0 \Rightarrow z = \pm 2i$$

interior singular point $z = 2i$ of order 2.

$$\text{Res}(z=2i) = \lim_{z\to 2i} \frac{1}{1!} \frac{d}{dz}\left[(z-2i)^2 \frac{1}{(z^2+4)^2}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left[\cancel{(z-2i)^2} \frac{1}{\cancel{(z-2i)^2}(z+2i)^2}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]$$

$$= \lim_{z\to 2i} \frac{-2}{(z+2i)^3}$$

$$= \frac{1}{32i}$$

$z^2+4=0$
$\Rightarrow z = 2i, -2i$

$x^2-5x+6=0$
$\Rightarrow x = 3, 2$

z^2+4
$= (z-2i)(z+2i)$

x^2-5x+6
$= (x-3)(x-2)$

$$\frac{d}{dx}\frac{1}{x^2} = -\frac{2}{x^3}$$

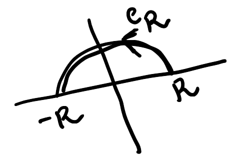
$$\therefore \oint_C \frac{dz}{(z^2+4)^2} = 2\pi i\,[\text{Res}(z=2i)] = 2\pi i \times \frac{1}{32i} = \frac{\pi}{16}$$

from equ^n (i) $\Rightarrow$

$$\oint_C \frac{dz}{(z^2+4)^2} = \frac{\pi}{16} \qquad \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \,?$$

$$\int_{-R}^{R} \frac{dx}{(x^2+4)^2} + \int_{C_R} \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+4)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2+4)^2} = \lim_{R\to\infty} \boxed{\frac{\pi}{16}}$$



$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} + \boxed{0} = \boxed{\frac{\pi}{16}}$$

using Jordan's Lemma

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}$$

$$\int_{0}^{\infty} \frac{dx}{(x^2+4)^2}$$

$$2\int_{0}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{16}$$

Exercise 7.2 [1 – iv, v]

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2+4)^2} = \frac{\pi}{32}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$$

$$x^2-2x+2=0$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4-8}}{2\times 1}$$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$$= 1 \pm i$$