chapter - 6

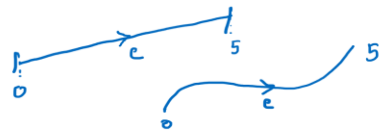
Complex Integration

Complex definite integral / Line integral:

$$\int_c f(z)\,dz$$

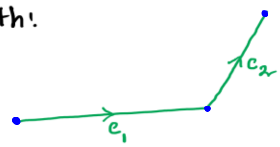
$\int x^2 dx$

$\int_0^2 x^2 dx$



$$\oint_c f(z)\,dz$$

Partioning path:



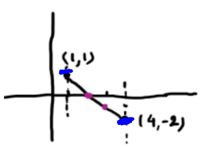
$\boxed{c} = c_1 + c_2$

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz$$

Sketch and represent the line segment from $1+i$ to $4-2i$ parametrically.

$(x_1, y_1) \leftarrow (1,1)$ $\quad$ $(4,-2) \rightarrow (x_2, y_2)$

$x = \sin t$
$y = t^2+1$
$z = e^{t^2} - 2$



$$y - 1 = \frac{-2-1}{4-1}(x-1)$$

$\Rightarrow y - 1 = -x + 1$

$\Rightarrow y = -x + 2$

$x = t$

$y = -t+2$; $1 \leq t \leq 4$

$-2 \leq -t+2 \leq 1$
$\Rightarrow -4 \leq -t \leq -1$
$\Rightarrow 4 \geq t \geq 1$
$\Rightarrow 1 \leq t \leq 4$

$t = 2$, $x = 2$, $y = 0$, $(2, 0)$
$t = 3$, $x = 3$, $y = -1$, $(3, -1)$

Sketch and represent unit circle (counterclockwise) parametrically.

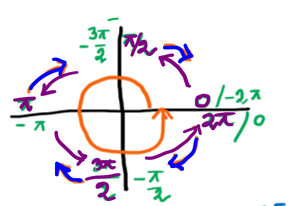
$x^2 + y^2 = r^2$
$\boxed{|z| = r}$

$x^2 + y^2 = 1$
$\boxed{|z| = 1}$

$|z|$
$= |x+iy|$
$= \sqrt{x^2+y^2}$

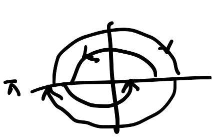
$|z| = 1$
$\Rightarrow \sqrt{x^2+y^2} = 1$
$\Rightarrow x^2+y^2 = 1$
$x = \cos\theta$
$y = \sin\theta$

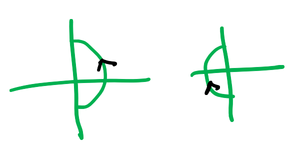
$\boxed{0 < \theta < 2\pi}$



counter: $0 < \theta < 2\pi$
clockwise: $2\pi < \theta < 0$

Clockwise direction (-ve direction)





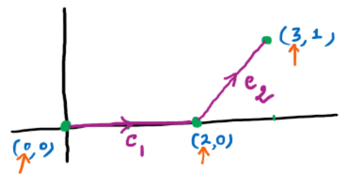
Right half circle of clockwise!

$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ or,

$-\frac{3\pi}{2} < \theta < -\frac{\pi}{2}$

Sketch the path C consisting of two line segments, one from $\underset{(0,0)}{z=0}$ to $\underset{(2,0)}{z=2}$ and other from $\underset{(2,0)}{z=2}$ to $\underset{(3,1)}{z=3+i}$, hence evaluate $\int_c f(z)\,dz$, if $\boxed{f(z)=z^2}$

$z=0$
$\Rightarrow x+iy=0+i\cdot 0$
$(x,y)=(0,0)$



$c=c_1+c_2$

For Path c_1: $y=0$

$f(z)=z^2=(x+iy)^2=x^2$

$z=x+iy$
$\Rightarrow z=x$
$\Rightarrow \frac{dz}{dx}=1 \Rightarrow dz=dx$

$$\int_{c_1} f(z)\,dz=\int_0^2 x^2\,dx=\frac{8}{3}$$

For Path c_2: $y-0=\frac{1-0}{3-2}(x-2) \Rightarrow y=x-2$

$f(z)=z^2=(x+iy)^2=\{x+i(x-2)\}^2$

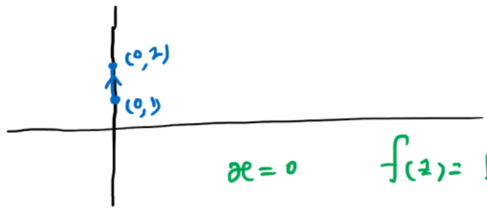
$z=x+i(x-2)$
$\Rightarrow z=x+ix-2i$
$\Rightarrow \frac{dz}{dx}=1+i$
$\Rightarrow dz=(1+i)\,dx$

$$\int_{c_2} f(z)\,dz=\int_2^3\{x+i(x-2)\}^2(1+i)\,dx$$
$= - - - -$
$= - - - -$
$=\frac{10}{3}+\frac{26}{3}i$

$$\int_c f(z)\,dz=\int_{c_1} f(z)\,dz+\int_{c_2} f(z)\,dz=\frac{8}{3}+\frac{10}{3}+i\cdot\frac{26}{3}$$
$$=6+i\cdot\frac{26}{3}\quad \text{Ans.}$$

Exercise: 2, 3, 4, 5, 6, 8, 11

6. $\int_c \ln z \, dz$; c: shortest path from i to $2i$
$(0,1)$ $(0,2)$



$x = 0$ $\quad f(z) = \ln z = \ln(x+iy) = \ln(iy)$

$z = x+iy \Rightarrow z = iy \Rightarrow \frac{dz}{dy} = i\frac{dy}{dy} \Rightarrow dz = i\,dy$

$$\int_1^2 \ln(iy)\, i\,dy = i\int_1^2 \ln(iy)\,dy = \frac{i}{i}\left[(iy)\ln(iy) - (iy)\right]_1^2$$

$\int \ln x \, dx$

$$= i\left[y\ln(iy) - y\right]_1^2$$

$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$

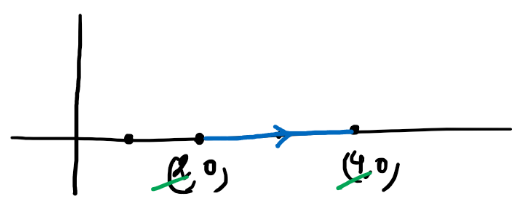
$$= i\left[2\ln(2i) - 2 - \ln(i) + 1\right]$$

$$= i\left[2\ln(2i) - \ln(i) - 1\right]$$

$$= i\left[\ln\frac{(2i)^2}{i} - 1\right]$$

$$= i\left[\ln(4i) - 1\right]$$

8. $\int_c \left(e^{2z} + \cos z\right) dz$; c: $z = 2$ to $z = 4$



$y = 0$

$$f(z) = e^{2(x+iy)} + \cos(x+iy)$$
$$= e^{2x} + \cos x$$

$z = x+iy$
$\Rightarrow z = x$
$\Rightarrow dz = dx$

$$\int_2^4 \left(e^{2x} + \cos x\right) dx = \frac{1}{2}\left[e^{2x}\right]_2^4 + \left[\sin x\right]_2^4$$
$$= \frac{1}{2}\left(e^8 - e^4\right) + \left(\sin 4 - \sin 2\right)$$

$|z+3i-5|=4$
$(5,-3)$

$|z|=r$

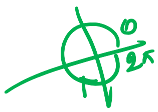
$|(z-p)+2i|=5$
$\Rightarrow |x+iy-p+2i|=5$
$\Rightarrow |(x-p)+i(y+2)|=5$
$\Rightarrow \sqrt{(x-p)^2+(y+2)^2}=5$
$\Rightarrow (x-p)^2+(y+2)^2=5^2$
$\downarrow$
$(p,-2)$, radius: 5

10. $\int_c \left(\frac{1}{z-i} - \frac{2}{(z-i)^2}\right) dz$; c: $|z-i|=3$; clockwise.

$|z|=r$
$\Rightarrow z = re^{i\theta}$

$|z-i| = 3$
$\Rightarrow \boxed{z-i = 3e^{i\theta}}$
$\Rightarrow z = 3e^{i\theta}+i$
$\Rightarrow \frac{dz}{d\theta} = 3ie^{i\theta}$
$\Rightarrow \boxed{dz = 3ie^{i\theta}\, d\theta}$

$|z-2+6i|=4$
$(2,-6)$



$$\int_{2\pi}^{0}\left[\frac{1}{3e^{i\theta}} - \frac{2}{9e^{2i\theta}}\right]3ie^{i\theta}\,d\theta$$

$$= \int_{2\pi}^{0}\left[i - \frac{2}{3}i\,e^{-i\theta}\right]d\theta$$

$$= i\left[\theta\right]_{2\pi}^{0} + \frac{2}{3}i\frac{1}{i}\left[e^{-i\theta}\right]_{2\pi}^{0}$$

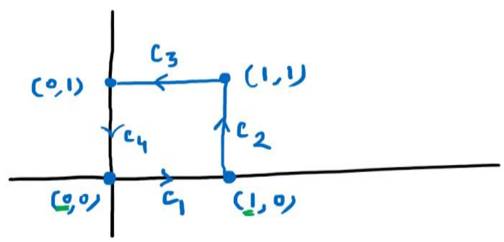
$$= -i2\pi + \frac{2}{3}\left[1 - e^{-i2\pi}\right]$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\rightarrow 1 - (\cos 2\pi - i\sin 2\pi)$
$= 1-1$
$= 0$

$$= -2\pi i$$

9. $\int_c (z.\bar{z})\,dz$; c is the path around the square with vertices 0, 1, 1+i, i



path c_1: $y=0$ $\quad f(z) = z.\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2 = x^2+y^2$
$= x^2$

$z = x+iy \Rightarrow z = x \Rightarrow dz = dx$

$$\int_{c_1} f(z)\,dz = \int_0^1 x^2\,dx = \frac{1}{3}$$

path c_2: path c_3: path c_4:

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$$

$$= \frac{1}{3} + \underline{\quad\quad} + \underline{\quad\quad} + \underline{\quad\quad}$$

z(t)=2 sin ⟦(t)+i 3 cos ⟦(t)+3+2i,⟧
(0≤t≤2π); (6,5)⟧

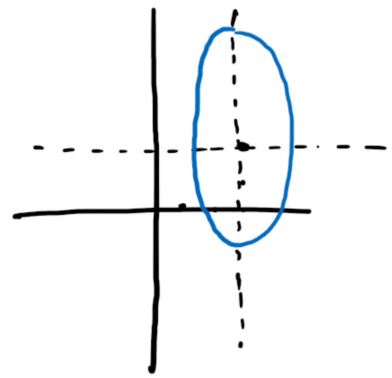
$$z(t) = 2\sin t + i\,3\cos t + 3+2i \quad ; \quad 0 \le t \le 2\pi \quad ; \quad (6,5)$$

$$\Rightarrow x+iy = (2\sin t + 3) + i(3\cos t + 2)$$

$$\Rightarrow x = 2\sin t + 3, \quad y = 3\cos t + 2$$

$$\Rightarrow \frac{x-3}{2} = \sin t, \quad \frac{y-2}{3} = \cos t$$

$$\frac{(x-3)^2}{2^2} + \frac{(y-2)^2}{3^2} = 1$$



$$\frac{(6-3)^2}{4} + \frac{(5-2)^2}{9} = 3.25 > 1.$$

(6,5) is exterior.