Mappings

Lecture 12

OBJECTIVE

Using conformal mappings

OUTCOME

Will be able to construct conformal mappings between many kinds of domain

Geometrical Representation:

To draw curve of complex variable (x,y)we take two axes i.e., one real axis and the other imaginary axis. Several points (x,y) are plotted on z-plane, by taking different value of z (different value of x and y). The curve C is drawn by joining the plotted points. The diagram obtained is called **Argand diagram.**

Transformation:

For every point (x,y) in the z-plane, the relation $w=f(z)$ defines a corresponding point (u,v) in the w-plane. We call this “transformation or mapping of z-plane into w-plane”. If a point z_0 maps into the point w_0, w_0 is known as the image of z_0.

If the point $P(x,y)$ moves along a curve C in z-plane, the point $P'(u,v)$ will move along a corresponding curve C_1 in the w-plane. We, then, say that a curve C in the z-plane is mapped into the corresponding curve C_1 in the w- plane by the relation $w=f(z)$.

Translation, Rotation and reflection are the standard transformations. Terms such as **translation, rotation** and **reflection** are used to convey dominant geometric characteristics of certain mappings.

Translation

$w = z + C,$

where,
$C = a + ib$
$z = x + iy$
$w = u + iv$

Hence, $u + iv = x + iy + a + ib$
So, $u = x + a$ and $v = y + b$
$x = u - a$ and $y = v - b$

On substituting the values of x and y in the equation of the curve to be transformed we get the equation of the image in the w $-$pane.
As an example the mapping $w = z + 1$ where $z = x + iy$, can be thought of as a translation of each point of z one unit to the right.

Example of Translation

Let the rectangular region R in z-plane which is bounded by the lines $x = 0, y = 0, x = 2, y = 1$. Determine the region R' of the w-plane into which R is mapped under the transformation $\boxed{w = z + 1}$.

Solution:

Given $w = z + 1 = x + iy + 1$

or, $u + iv = \boxed{(x + 1)} + iy$.

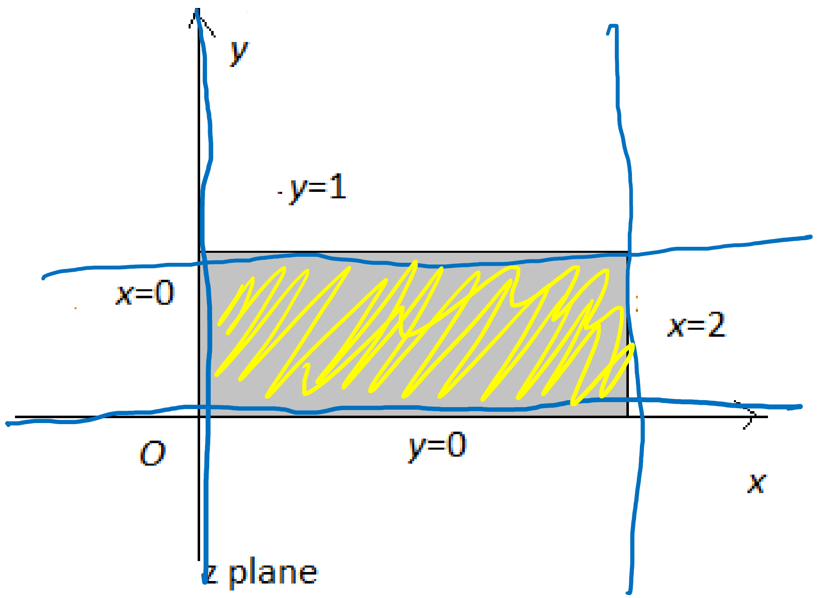
Hence, $u = x + 1$, and $v = y$

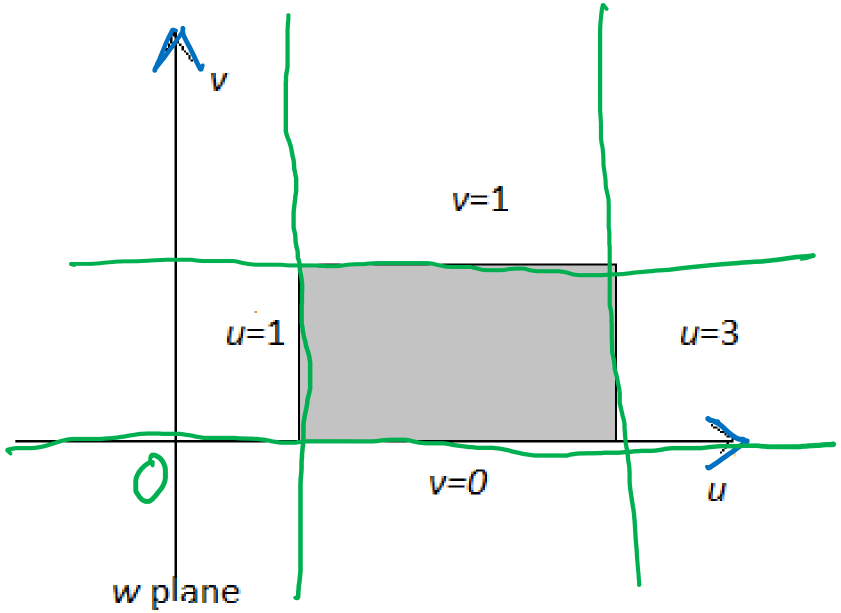
when $x = 0 \Rightarrow u = 1,$
$y = 0 \Rightarrow v = 0,$
$x = 2 \Rightarrow u = 3,$
$y = 1 \Rightarrow v = 1.$

$w = z + \boxed{(2+3i)}$

$z - (1-i)$

$z - 2$





Rotation:

The mapping $w = iz$ where $z = re^{i\theta}$ and $i = e^{i\frac{\pi}{2}}$, can be thought of as a rotation of the radius vector for each non-zero-point z through a right angle about the origin in the counterclockwise direction.

Example of Rotation:

Let the rectangular region R in z-plane which is bounded by the lines $x = 0, y = 0, x = 2, y = 1$. Determine the region R' of the w-plane into which R is mapped under the transformation $w = iz$.

Solution:

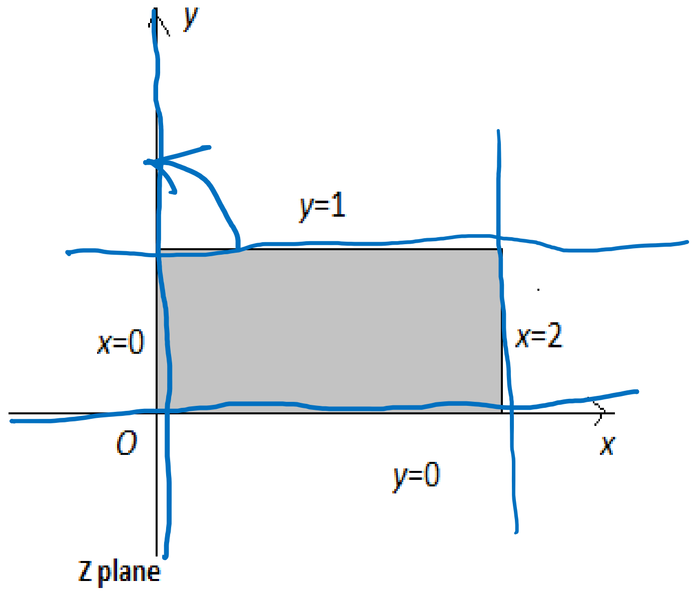
Given $w = iz = i(x + iy)$

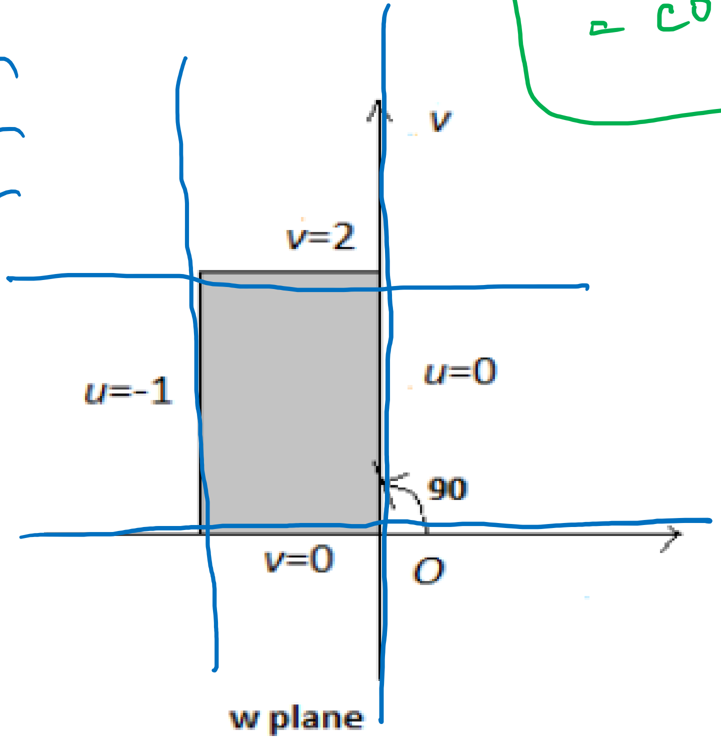
$i = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

or, $u + iv = -y + ix$.

Hence $u = -y$ and $v = x$.

when $x = 0 \Rightarrow v = 0,$
$y = 0 \Rightarrow u = 0,$
$x = 2 \Rightarrow v = 2,$
$y = 1 \Rightarrow u = -1.$





Reflection:

The mapping $w = \bar{z}$ transforms each point of $z = x + iy$ into its reflection in the real axis.

Example of Reflection:

Let the rectangular region R in z-plane which is bounded by the lines $x = 0, y = 0, x = 2, y = 1$. Determine the region R' of the w-plane into which R is mapped under the transformation $w = \bar{z}$.

Solution:

Given $w = \bar{z}$

or, $u + iv = x - iy$.

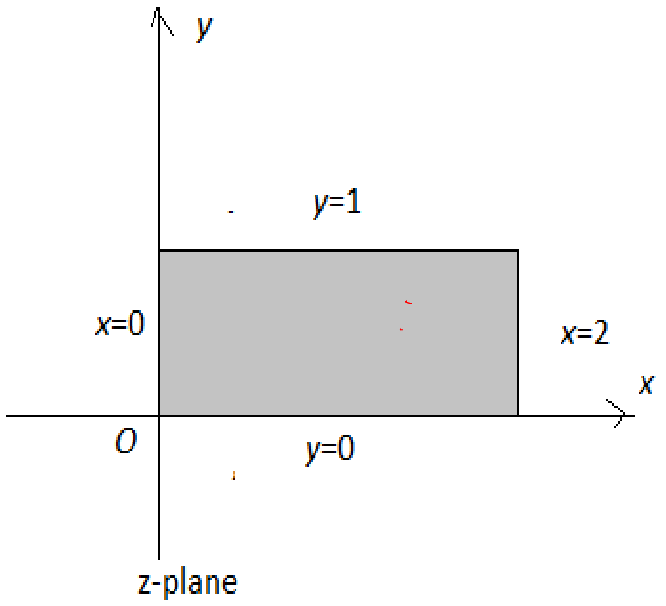
Hence $u = x$ and $v = -y$.

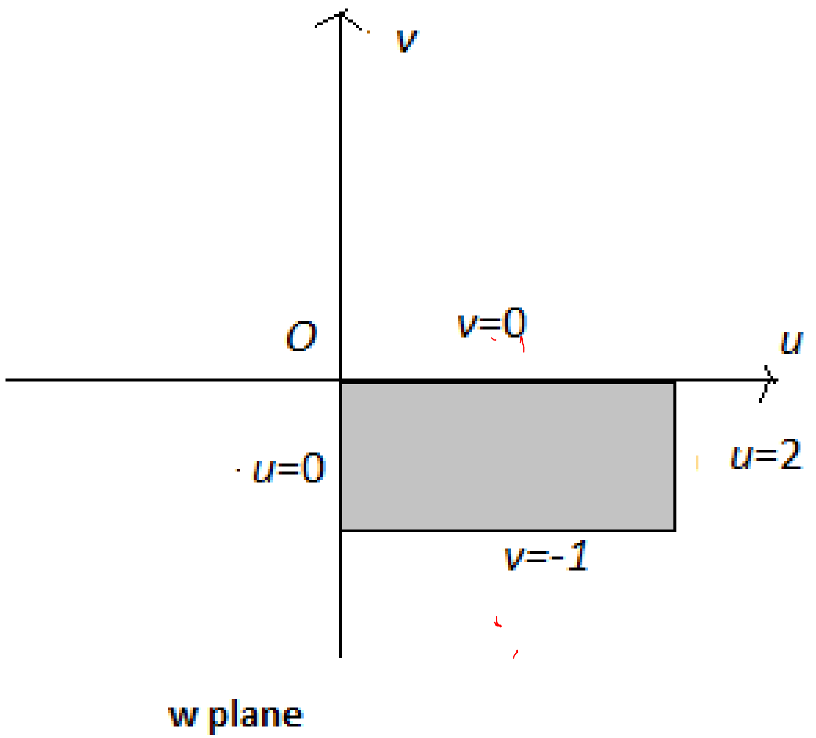
when $x = 0 \Rightarrow u = 0,$

$y = 0 \Rightarrow v = 0,$

$x = 2 \Rightarrow u = 2,$

$y = 1 \Rightarrow v = -1.$





Example:

(-1,2) (1,-2) (1,2)

Given triangle T in the z-plane with vertices at $-1+2i, 1-2i$ and $1+2i$. Determine the triangle T' of the w −plane into which T is mapped under the transformation $w = \sqrt{2}\, e^{\frac{\pi i}{4}} z$.

Solution:

Given $w = w = \sqrt{2}\, e^{\frac{\pi i}{4}} z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) z = (1+i)(x+iy) = x + ix + iy - y$

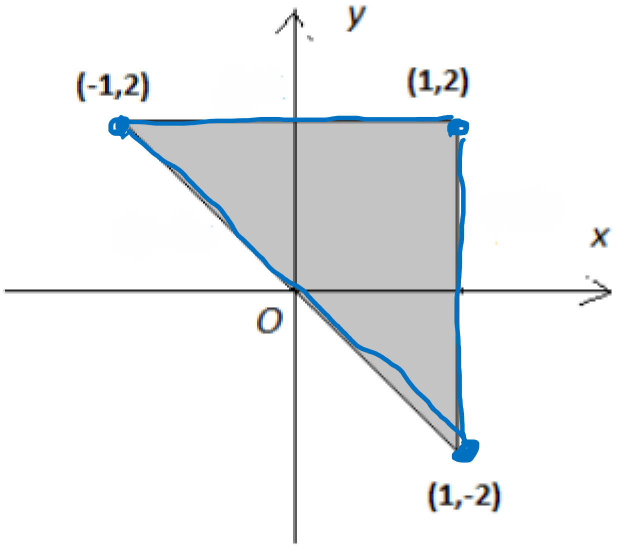
or, $u + iv = (x-y) + i(x+y)$. Hence $u = x - y$ and $v = x + y$.

The vertices of the triangle are $-1+2i, 1-2i, 1+2i$. Vertices can also be written as: $(-1,2)$, $(1,-2)$ and $(1,2)$.

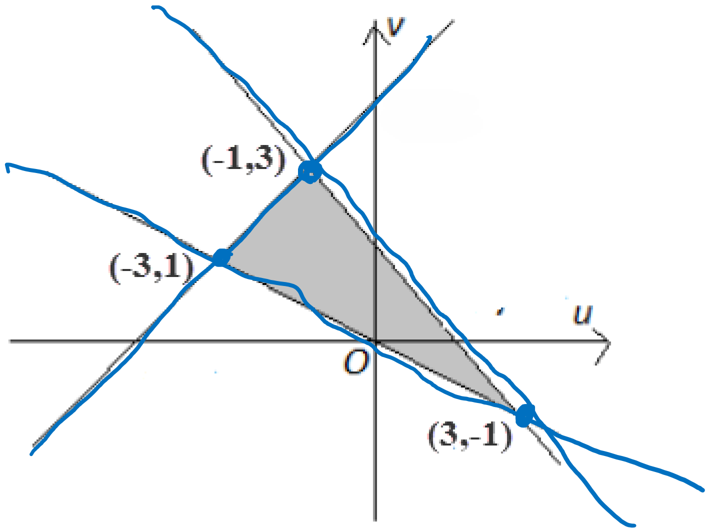
At $(-1,2)$: $u = -1-2 = -3$; $v = -1+2 = 1 \Rightarrow (\boldsymbol{u},\boldsymbol{v}) = (\mathbf{-3,1})$

At $(1,-2)$: $u = 1+2 = 3$; $v = 1-2 = -1 \Rightarrow (\boldsymbol{u},\boldsymbol{v}) = (\mathbf{3,-1})$

At $(1,2)$: $u = 1-2 = -1$; $v = 1+2 = 3 \Rightarrow (\boldsymbol{u},\boldsymbol{v}) = (\mathbf{-1,3})$.



z plane



w plane

Exercise Set

1. Let the rectangular region R in z-plane which is bounded by the lines $x = 2, y = 0, x = 5$ and $y = 4$. Determine the region R' of the w-plane into which R is mapped under the following transformations:

(i) $w = 2z - (2 + 3i)$,

(ii) $w = \frac{1}{2}\, e^{\frac{\pi i}{2}} z + 2i$,

(iii) $w = \sqrt{2}\, e^{\frac{\pi i}{4}} z - (1 - i)$,

(iv) $w = e^{i\pi} z + 3 + i$,

(v) $w = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{4}} z + 1 - 3i$.

2. Given triangle T in the z -plane with vertices at $1, 1 - 3i$ and $3 - i$. Determine the triangle T' of the w-plane into which T is mapped under the following transformations:

(i) $w = 3z + 1 - 3i$,

(ii) $w = iz + 3 + 2i$,

(iii) $w = (1 + 2i)z - i$,

(iv) $w = \frac{1}{2}\, e^{\frac{\pi i}{2}} z - 4$.

Sample MCQ

1. Which of the following is the center of the region for the image of $|z| = 2$ under the transformation of $w = z + 1 + i$?

(a) $(1,0)$ (b) $(0,0)$ (c) $(1,1)$ (d) $(2,1)$

2. Which is the image of the rectangular region of $z(x, y)$ plane bounded by the line $x = 1$ under the transformation $w = z + (2 - i)$ in $w(u, v)$ plane?

(a) $u = 2$ (b) $u = 3$ (c) $u = 1$ (d) $u = 0$

3. What is the image of triangular region of $z(x, y)$ plane bounded by the line $x + y = 1$ under the transformation $w = z + (1 + i)$ in $w(u, v)$ plane?

(a) $u = 0$ (b) $u + v = 1$ (c) $u + v = 3$ (d) $v = 1$

4. Which of the following expression gives us the transformation as rotation from $z \to w$ plane?

(a) $w = z + 2 + i$ (b) $w = \sqrt{2}\, e^{i\frac{\pi}{4}} z$ (c) $w = 2z$ (d) $w = \bar{z}$

5. Which of the following transformation happens with the expression $w = i\,\bar{z}$ from z → w plane?

(a) Rotation & reflection

(b) Rotation & magnification

(c) Reflection & magnification

(d) Reflection & translation