Chapter - 7

Integration Using Cauchy's Residue theorem

Cauchy-Residue theorem (CRT):

If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, ---, a_n$ inside C, then

$$\oint_c f(z)\,dz = 2\pi i\,[\underbrace{Res(a_1)}_{\text{Residue}} + Res(a_2) + --- + Res(a_n)]$$

Residue finding method:

If $f(z)$ is analytic inside and on a simple closed curve C except at pole or singularity at '$z = a$' of order 'm' then

$$Res(z=a) = \lim_{z\to a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

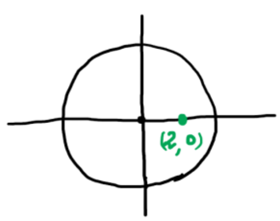
$$f(z) = \frac{1}{(z+1)^2 (z-3)^4}$$ poles : -1 of order 2; 3 of order 4.

cauchy's Residue theorem

Evaluate by $\boxed{CRT}$ $\oint_c \frac{\sin \pi z}{(z-2)^2}dz$; $c: |z| = 3$

$(z-2)^2 = 0$

$\Rightarrow z = 2$ singular point of order 2.



$$Res(z=2) = \lim_{z\to 2}\frac{1}{1!}\frac{d}{dz}\left[(z-2)^2 \cdot \frac{\sin(\pi z)}{(z-2)^2}\right]$$

$$= \lim_{z\to 2}\frac{d}{dz}(\sin \pi z)$$

$$= \lim_{z\to 2} \pi\cos(\pi z)$$

$$= \pi\cos(2\pi)$$

$$= \pi$$

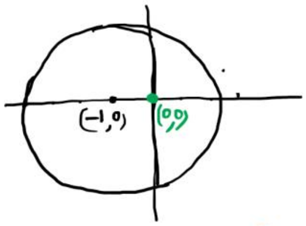
$$\oint_C \frac{\sin \pi z}{(z-2)^2}dz = 2\pi i\,[Res(z=2)]$$

$$= 2\pi i \times \pi = 2\pi^2 i$$

Evaluate the contour integral $\oint_c \frac{dz}{z^3}$ by CRT,

$c: |z+1| = 3$

singular point $z=0$ of order 3.



$$\text{Res}(z=0) = \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-0)^3 \frac{1}{z^3}\right]$$

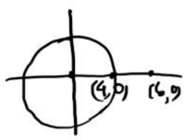
$$= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}(1)$$

$$= 0$$

$$\oint_c \frac{dz}{z^3} = 2\pi i \times \left[\text{Res}(z=0)\right] = 0$$

$$\int \frac{dz}{(z-6)^{10}} \quad ; \quad c: |z| = 4$$

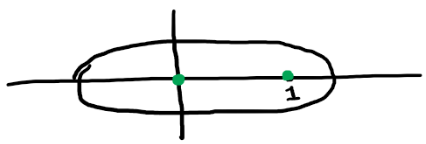
$z=6$ singular point of order 10 which is exterior of c.



$$\text{Res}(z=6) = 0$$

$$\oint_c \frac{dz}{(z-6)^{10}} = 0$$

$$\oint_c \frac{dz}{(z^2-z)} dz \quad ;$$



$z^2 - z = 0$

$\Rightarrow z(z-1) = 0 \Rightarrow z = 0, 1$ both are of order 1.

$$\text{Res}(z=0) = \lim_{z\to 0} \frac{1}{0!}\left[(z-0)\frac{1}{z^2-z}\right]$$

$$= \lim_{z\to 0} \frac{1}{z-1}$$

$$= -1$$

$$\text{Res}(z=1) = \lim_{z\to 1} \frac{1}{0!}\left[(z-1)\frac{1}{z^2-z}\right]$$

$$= \lim_{z\to 1} \frac{1}{z}$$

$$= 1$$

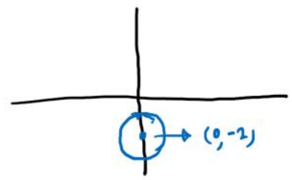
$$\oint_c \frac{dz}{z^2-z} dz = 2\pi i\left[\text{Res}(z=0) + \text{Res}(z=1)\right]$$

$$= 2\pi i\left[-1+1\right] = 0$$

7.1: exercise 3, & 4

4. (a) $\oint_c \frac{dz}{z^2+4}$; $c : |z+2i| = 1$

$z^2+4=0$
$\Rightarrow z^2 = -4$
$\Rightarrow z^2 = 4i^2$
$\Rightarrow z = \boxed{\pm 2i}$



So, $z=-2i$ is interior singular point of order 1.

$$Res(z=-2i) = \lim_{z\to -2i} \frac{1}{0!}\left[(z+2i)\cdot\frac{1}{z^2+4}\right]$$
$$= \lim_{z\to -2i} (z+2i)\frac{1}{(z-2i)(z+2i)}$$
$$= \lim_{z\to -2i} \frac{1}{z-2i}$$
$$= -\frac{1}{4i}$$

$$\therefore \oint \frac{dz}{z^2+4} = 2\pi i\left[Res(z=-2i)\right]$$
$$= 2\pi i \times \left(-\frac{1}{4i}\right)$$
$$= -\frac{\pi}{2}$$

7.1

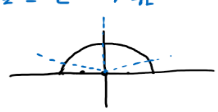
1. Find all the singular points from the following functions:

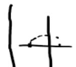
- $f(z) = \cot z = \frac{\cos z}{\sin z}$

$\sin z = 0$
$\Rightarrow z = n\pi$; $n \in \mathbb{Z}$

Evaluate by CRT $\oint_c \frac{dz}{(z^6+1)}$; $c : |z| = 2$; upper half circle (counter clockwise)

$z^6+1=0$
$\Rightarrow z^6 = -1$
$\Rightarrow z^6 = e^{i\pi}$ $\Rightarrow z_n^6 = e^{i(\pi+2n\pi)}$ $\Rightarrow z_n = e^{i(2n+1)\cdot\frac{\pi}{6}}$; where, $n = 0, 1, \dots, 5$





$\theta = \tan^{-1}\left(\frac{0}{-1}\right)$
$= 0 + \pi$
$= \pi$
$n = 1$

so, interior singular points are
$z_n = e^{i(2n+1)\frac{\pi}{6}}$; $n = 0, 1, 2$

$$Res(z=z_n) = \lim_{z\to z_n} \frac{1}{0!}\left[(z-z_n)\frac{1}{z^6+1}\right]$$
$$= \lim_{z\to z_n} \frac{z-z_n}{z^6+1} \left[\frac{0}{0} \text{ form}\right]$$
$$= \lim_{z\to z_n} \frac{1}{6z^5} \quad \text{[Applying L'Hôspital rule]}$$
$$= \frac{1}{6z_n^5} \text{ ; where } z_n = e^{i(2n+1)\cdot\frac{\pi}{6}} \text{ ; } n = 0, 1, 2$$

$\boxed{z^6+1} = 0$
$\Rightarrow z = z_n$

$\boxed{x^2 - 5x + 6} = 0$
$\Rightarrow x = 3, 2$

$$\oint_c \frac{dz}{z^6+1} = 2\pi i\left[Res(z=z_n)\right]$$
$$= 2\pi i\left[Res(z=z_0) + Res(z=z_1) + Res(z=z_2)\right]$$
$$= 2\pi i\left[\frac{1}{6z_0^5} + \frac{1}{6z_1^5} + \frac{1}{6z_2^5}\right]$$
$$= 2\pi i\left[\frac{1}{6\left(-\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)} + \frac{1}{6i} + \frac{1}{6\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)}\right]$$

Ans.

$z_0^5 = e^{i\frac{5\pi}{6}}$
$= \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$
$= -\frac{\sqrt{3}}{2} + i\frac{1}{2}$

$z_1^5 = e^{i\frac{5\pi}{2}}$
$= i$

$z_2^5 = \frac{\sqrt{3}}{2} + i\frac{1}{2}$