Lecture 16: Damped Simple Harmonic Motion

The liquid exerts a damping force, $F_d \propto$ velocity, v of vane and liquid [if vane moves slowly]

$F_d \propto v$ [Let rod and vane = massless]

$F_d = -bv$ [b = damping constant]

$F_s = -kx$

Newton's second law for components along the x axis $F_{net,x} = ma_x$

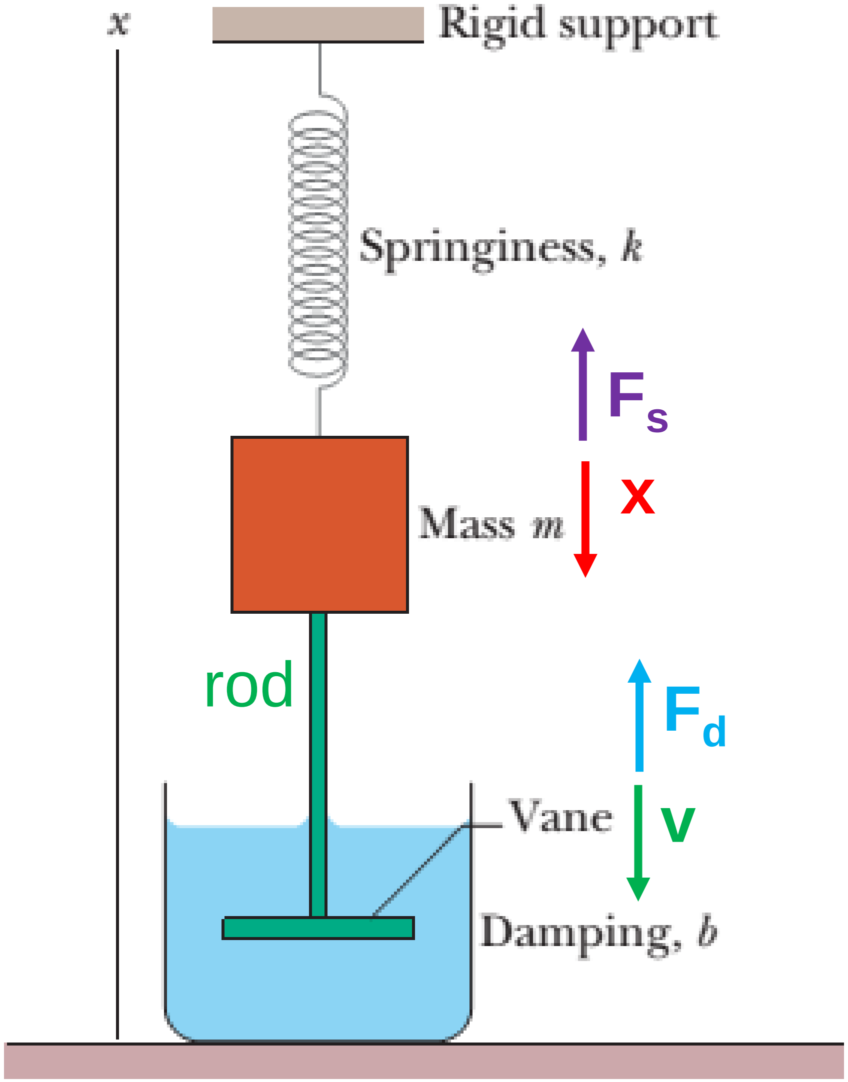
$$F_d + F_s = ma$$

$$-bv - kx = ma$$

$$-b\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + \left(\frac{b}{m}\right)\frac{dx}{dt} + \left(\frac{k}{m}\right)x = 0$$



The displacement of damped simple harmonic oscillator:

$$x(t) = \left[x_m e^{-\left(\frac{b}{2m}\right)t}\right] \cos(\omega' t + \varphi)$$

The amplitude, $x_m e^{-\left(\frac{b}{2m}\right)t}$ decreases exponentially with time.

$$\omega' = \sqrt{\omega^2 - \gamma^2}$$

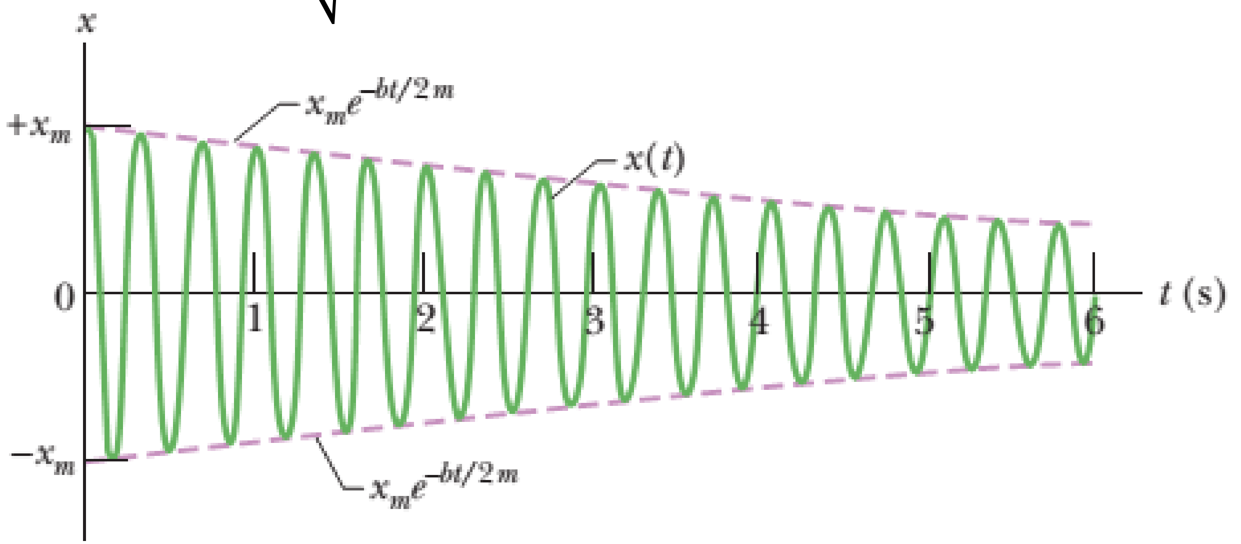
[ω' = angular frequency of the damped oscillator and

ω = angular frequency of the undamped oscillator]

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

[$\omega = \sqrt{\frac{k}{m}}$ $\gamma = \frac{b}{2m}$]

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



If there is no damping, b = 0 :

$$\omega' = \sqrt{\frac{k}{m} - \frac{0^2}{4m^2}}$$

$$\omega' = \sqrt{\frac{k}{m}} = \omega$$

[angular frequency of an undamped oscillator]

The displacement of undamped simple harmonic oscillator becomes $x(t) = x_m \cos(\omega t + \varphi)$.

For the undamped simple harmonic motion, the amplitude x_m does not change with time.

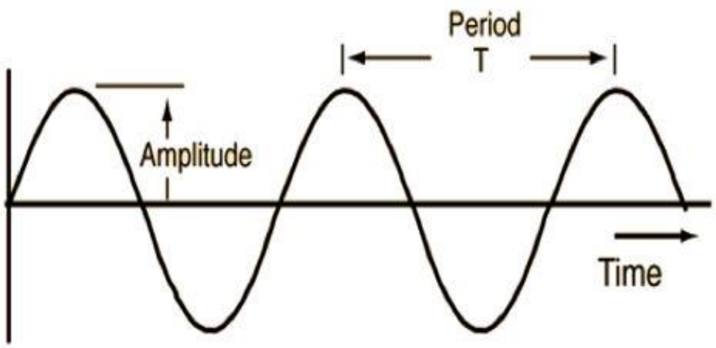
Damped *mechanical energy*:

$The\ mechanical\ energy\ for\ an$ $undamped$ $oscillator$ is constant, $E = \frac{1}{2}kx_m^2$

If damping (b) is very small, $x_m \approx x_m \mathrm{e}^{-\left(\frac{b}{2m}\right)t}$

$The\ mechanical\ energy\ for\ a$ $damped$ $oscillator\ decreases$

$$as\ a\ function\ of\ time, E \approx \frac{1}{2}k\left\{x_m \mathrm{e}^{-\left(\frac{b}{2m}\right)t}\right\}^2$$



$$E \approx \frac{1}{2}kx_m^2 e^{-\left(\frac{b}{m}\right)t}$$

58. For the damped oscillator system shown in Fig. 15-16, with m = 250 g, k = 85 N/m, and b = 70 g/s, T' = 0.34 s, what is the ratio of the oscillation amplitude at the end of 20 cycles to the initial oscillation amplitude?

Here, m = 250 g= 0.250 kg

k = 85 N/m

b = 70 g/s = 0.070 kg/s

T'= 0.34 s

The displacement of the damped oscillation is

$$x(t) = \left[x_m e^{-\left(\frac{b}{2m}\right)t}\right] \cos(\omega' t + \varphi)$$

Time for 20 cycles, t = 20 T'

$$\textbf{Amplitude} = x_m e^{-\left(\frac{b}{2m}\right)t} = x_m e^{-\left(\frac{b}{2m}\right)20T'} = x_m e^{-\left(\frac{b}{m}\right)10T'}$$

t = 0,

$$\text{Amplitude} = x_m e^{-\left(\frac{b}{2m}\right)t} = x_m e^{-\left(\frac{b}{2m}\right)0} = x_m e^{-0} = x_m(1) = x_m$$

$$\text{Ratio of amplitudes} = \frac{x_m e^{-\left(\frac{b}{m}\right)10T'}}{x_m} = e^{-\left(\frac{b}{m}\right)10T'}$$

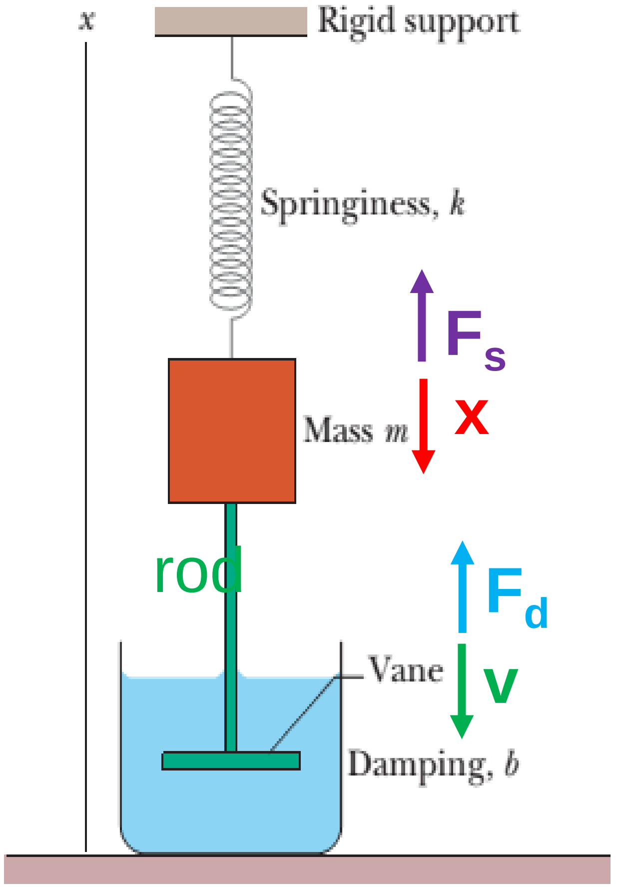
$$\left[\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\frac{85}{0.250} - \frac{(0.070)^2}{4(0.250)^2}}\right.$$

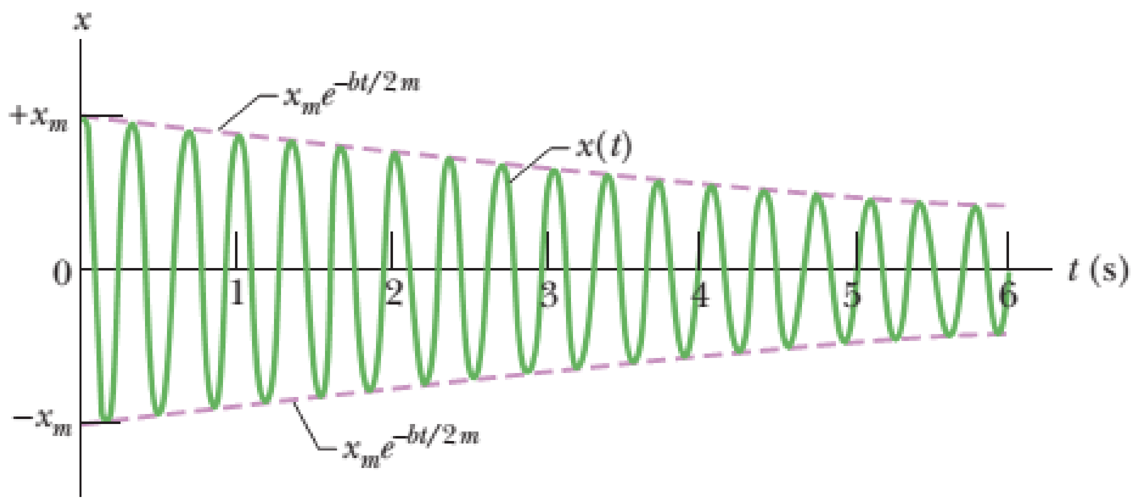
$$\omega' = 18.44\, rad/s$$

$$\left. T' = \frac{2\pi}{\omega'} = 0.34 \text{ s}\right]$$

$$\text{Ratio of amplitudes} = e^{-\left(\frac{0.070}{0.250}\right)10(0.34)}$$

$$= e^{-0.952} = 0.39$$





60. The suspension system of a 2000 kg automobile "sags" 10 cm when the chassis is placed on it. Also, the oscillation amplitude decreases by 50% each cycle. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming each wheel supports 500 kg.

(a) Hooke's law, $F = kx$ [magnitude only]

$$mg = kx$$

One quarter of the vehicle mass, $m = 2000/4\ \text{kg} = 500\ \text{kg}$

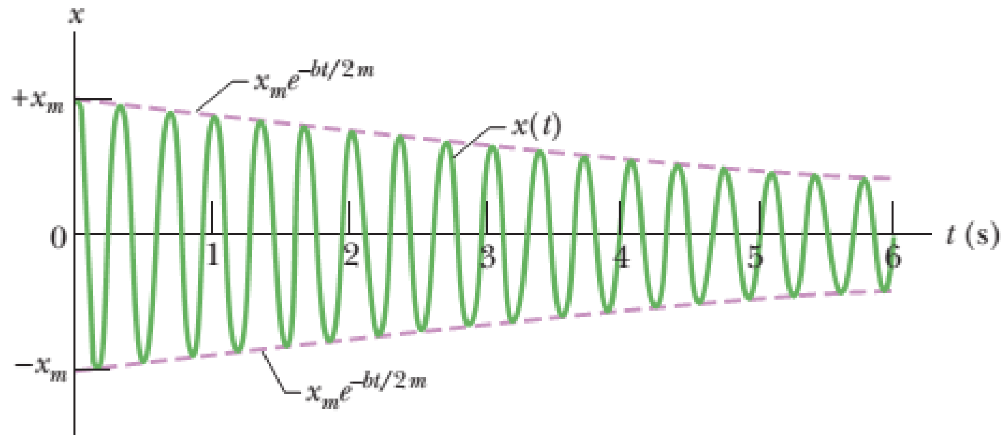
$$x = 10\ \text{cm} = 0.10\ \text{m}$$

$$k = mg/x = 500(9.8)/0.10 = 49000\ \text{N/m}$$

[The displacement is downward and the spring force is upward]

For one cycle, t = T′ s

$$(b)\ x_m e^{-\frac{bT'}{2m}} = 50\%\ x_m$$



$$e^{-\frac{bT'}{2m}} = 0.50$$

$$ln\ [e^{-\frac{bT'}{2m}}] = ln\ (0.50)$$

$$-\frac{bT'}{2m} = -\ 0.69$$

$$\frac{bT'}{2m} = 0.69$$

Time period of a damped SHM, T′ = $\frac{2\pi}{\omega'}$

$$T' = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

$$T'^2 = \frac{4\pi^2}{\frac{k}{m} - \frac{b^2}{4m^2}} = \frac{4\pi^2}{\frac{4mk - b^2}{4m^2}}$$

$$T'^2 = \frac{16\pi^2 m^2}{4mk - b^2}$$

$$\frac{\boldsymbol{bT'}}{\boldsymbol{2m}} = \mathbf{0.69}$$

$$\frac{\boldsymbol{b^2 T'^2}}{\boldsymbol{4m^2}} = \mathbf{0.48}$$

$$\frac{\boldsymbol{b^2}}{\boldsymbol{4m^2}}\left(\frac{16\pi^2 m^2}{4mk - b^2}\right) = \mathbf{0.48}$$

$$\boldsymbol{b^2}\left(\frac{4\pi^2}{4mk - b^2}\right) = \mathbf{0.48}$$

$$4\pi^2 b^2 = 0.48(4mk - b^2)$$

$$4\pi^2 b^2 = 1.92mk - 0.48b^2$$

$$4\pi^2 b^2 + 0.48b^2 = 1.92mk$$

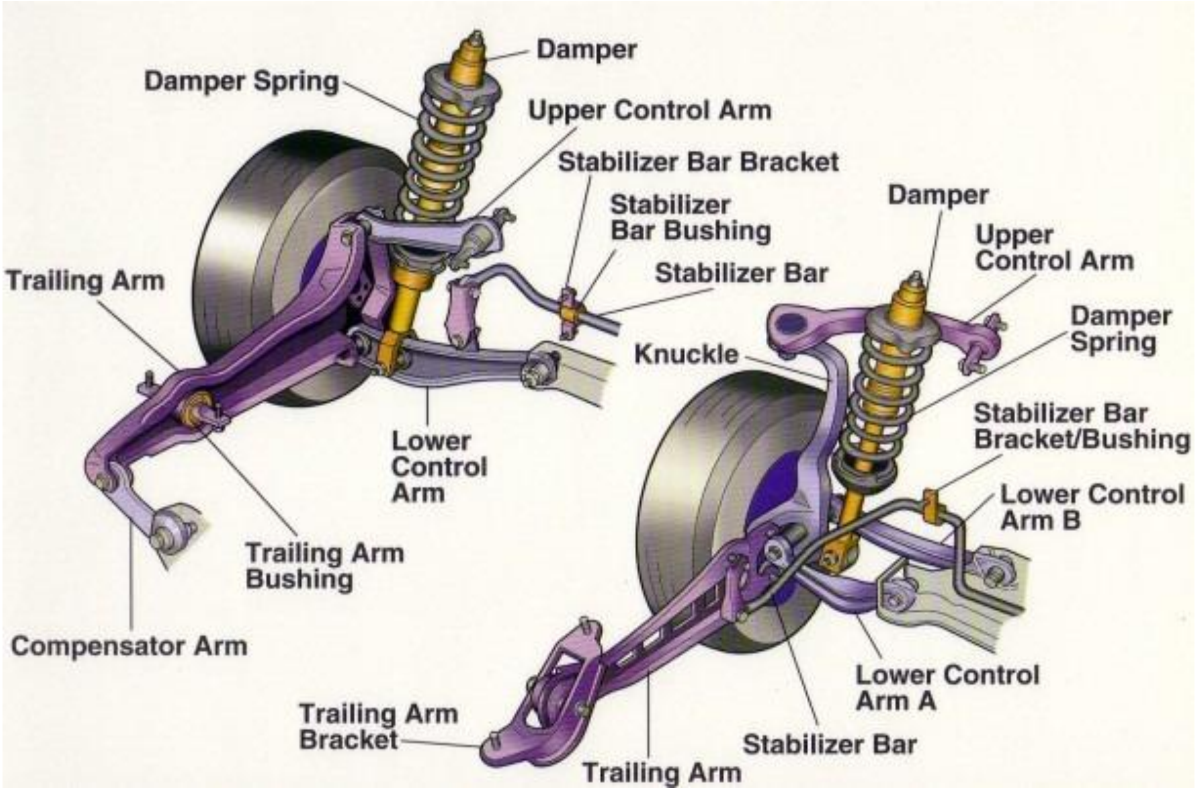
$$b^2(4\pi^2 + 0.48) = 1.92(500)(49000)$$

$$39.96b^2 = 47040000$$

$$b^2 = 1177177.18$$

$$b = \sqrt{1177177.18}$$

b = 1084.98 kg/s



Additional problem:

Sample problem 15:06, page: 432