Chapter - 1

Laplace Transformation

$f(t)$ be defined for all positive values of t.

$$\boxed{\int_0^\infty f(t)\cdot e^{-st}\,dt} = F(s)$$

Laplace Transformation.

$$\int_0^2 p\,x\,dx = p\left[\frac{x^2}{2}\right]_0^2 = 2p$$

given function

$$f(t) \xrightarrow{\text{Laplace}} \boxed{F(s)}$$

$$\mathcal{L}\{f(t)\} = F(s) \rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$$

$$\int e^{mx}\,dx = \frac{1}{m}e^{mx}$$

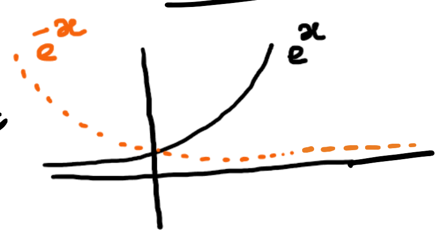
Important Formulae:

$$\mathcal{L}\{c\} = \boxed{\frac{c}{s}}$$

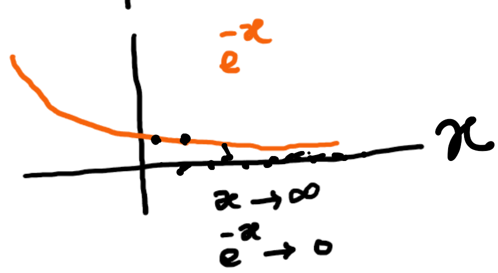
$$f(t) = c$$

$$\int_0^\infty f(t)\cdot e^{-st}\,dt = \int_0^\infty c\,e^{-st}\,dt = c\int_0^\infty e^{-st}\,dt = -\frac{c}{s}\left[e^{-st}\right]_0^\infty = -\frac{c}{s}\left[0-1\right] = \frac{c}{s}$$

$$\int e^{-st}\,dt = \frac{-1}{s}e^{-st}$$



$$\int e^{2x}\,dx = \frac{1}{2}e^{2x}$$



$$\int e^{5t}\,dt = \frac{1}{5}e^{5t}$$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$t^0 = 1$

$\mathcal{L}\{t^n\}$ $\quad n=0, \quad f(t) = 1$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$

$n=1, \quad f(t) = t$

$\mathcal{L}\{t\} = \int_0^\infty t \cdot e^{-st}\,dt$

$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^\infty$

$= \left[0 - \left(-\frac{1}{s^2}\right)\right]$

$= \frac{1}{s^2}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

sign	diff.	int.
+	t	e^{-st}
−	1	$-\frac{1}{s}e^{-st}$
+	0	$\frac{1}{s^2}e^{-st}$

LIATE

logarithmic

Inverse

Algebra

Trigonometric

exponential

$n=2, \quad f(t) = t^2$

$\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$

$= \left[-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right]_0^\infty$

$= \left[0 - \left(-\frac{2}{s^3}\right)\right] = \frac{2}{s^3}$

sign	diff.	int.
+	t^2	e^{-st}
−	$2t$	$-\frac{1}{s}e^{-st}$
+	2	$\frac{1}{s^2}e^{-st}$
−	0	$-\frac{1}{s^3}e^{-st}$

$\mathcal{L}\{t^0\} = \frac{1}{s} = \frac{0!}{s^{0+1}}$

$\mathcal{L}\{t^1\} = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$

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Similarly, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^6\} = \frac{6!}{s^7}$

$\mathcal{L}\{t^4\} = \frac{4!}{s^5}$

3. $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ H.W

4. $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

5. $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ H.W

$\mathcal{L}\{\cos 3t\} = \dfrac{s}{s^2+9}$

$\mathcal{L}\{\sin 4t\} = \dfrac{4}{s^2+16}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\boxed{\text{LIATE}}$$

f(t)

$$\mathcal{L}\{\cos at\} = \int_0^\infty \boxed{\cos at \cdot e^{-st}}\, dt$$

$$I = \int_0^\infty \cos at \cdot e^{-st}\, dt$$

sign	diff.	int.
+	$\cos at$	e^{-st}
−	$-a\sin at$	$-\frac{1}{s} e^{-st}$
+	$-a^2 \cos at$	$\frac{1}{s^2} e^{-st}$

$$- \frac{a^2}{s^2}\int_0^\infty \cos at \cdot e^{-st}\, dt$$

$$\Rightarrow I = -\frac{1}{s}\cos at \cdot e^{-st} + \frac{a}{s^2}\sin at\, e^{-st} - \frac{a^2}{s^2}\int_0^\infty \cos at \cdot e^{-st}$$

$$\Rightarrow I = \left[-\frac{1}{s}\cos at \cdot e^{-st} + \frac{a^2}{s^2}\sin at\, e^{-st}\right] - \frac{a^2}{s^2}\boxed{I}$$

$$\Rightarrow \left(1+\frac{a^2}{s^2}\right) I = \left[\left(-\frac{1}{s}\right)\cos at \cdot e^{-st} + \frac{a^2}{s^2}\sin at\, e^{-st}\right]_0^\infty$$

$$\Rightarrow \left(\frac{s^2+a^2}{s^2}\right) \cdot I = \left[0 \boxed{-} \left(-\frac{1}{s}\right)\right]$$

$$\Rightarrow I = \frac{1}{s} \cdot \frac{s^2}{s^2+a^2} \Rightarrow I = \frac{s}{s^2+a^2} \Rightarrow \int_0^\infty \cos at \cdot e^{-st}\, dt = \frac{s}{s^2+a^2}$$

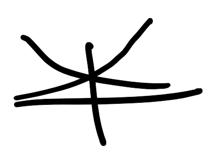
$$\Rightarrow \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

6. $\mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2-a^2}$ H.W

7. $\mathcal{L}\{\cosh(at)\} = \dfrac{s}{s^2-a^2}$ H.W

$$\left\{ \sinh(x) = \frac{e^x - e^{-x}}{2} \right\}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$



$$\mathcal{L}\{\sinh(at)\} = \int_0^\infty \sinh(at)\cdot e^{-st}\,dt$$

$$= \int_0^\infty \frac{e^{at}-e^{-at}}{2}\cdot e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{(a-s)t}\,dt - \frac{1}{2}\int_0^\infty e^{-(a+s)t}\,dt$$

$$= \frac{1}{2(a-s)}\left[e^{-(s-a)t}\right]_0^\infty + \frac{1}{2(a+s)}\left[e^{-(a+s)t}\right]_0^\infty$$

$$= -\frac{1}{2(a-s)} - \frac{1}{2(a+s)} = -\frac{1}{2}\left[\frac{1}{a-s}+\frac{1}{a+s}\right]$$

$$= -\frac{1}{2}\left[\frac{a+s+a-s}{(a-s)(a+s)}\right] = \frac{a}{s^2-a^2}$$

$F(s) = \mathcal{L}\{f(t)\}$

$F(s-2) = \mathcal{L}\{t\} = \frac{1}{s^2}$

Properties of Laplace Transformation

Linearity: $\mathcal{L}\{f_1(t) \pm f_2(t)\} = \mathcal{L}\{f_1(t)\} \pm \mathcal{L}\{f_2(t)\}$

First Shifting/translation: $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{f(t)\}$

$F(s)$

$s = s-a$

$F(s-a)$

$F(s-2)$

$F(s) = ?$

$\to \mathcal{L}\{f(t)\}$

$\mathcal{L}\{e^{2t} \cdot t\} = F(s-2) = \boxed{\frac{1}{(s-2)^2}}$

$f(t) = t, \quad a = 2$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{t\} = \boxed{\frac{1}{s^2}}$

$\boxed{(s-2)}$ $(s-2)$

Multiplication by t^n: $\mathcal{L}\{t^n f(t)\} = (-1)^{\boxed{n}} \frac{d^n}{ds^n}[F(s)]$

$\mathcal{L}\{t^1 \sin at\} = (-1)^1 \frac{d}{ds}[F(s)] = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right] = -\frac{(s^2+a^2).0 - a.2s}{(s^2+a^2)^2} = \frac{2sa}{(s^2+a^2)^2}$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

problem Set 1.1 : (1 - 15) H.W

$\mathcal{L}\{t^2 e^{2t}\} = (-1)^2 \frac{d^2}{ds^2}[F(s)]$

$\frac{d}{ds}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{d}{ds}(u) - u \cdot \frac{d}{ds}(v)}{v^2}$

$F(s) = \mathcal{L}\{e^{2t}\}$

$= \frac{1}{s-2}$

$= \frac{d^2}{ds^2}\left[\frac{1}{s-2}\right] = \frac{d}{ds}\left[-\frac{1}{(s-2)^2}\right]$

$= 2\frac{1}{(s-2)^3}$ [Using multiplication by t^n property]

$\mathcal{L}\{t^2 \cdot e^{2t}\} = F(s-2) \to \frac{2}{(s-2)^3}$

$F(s) = \mathcal{L}\{t^2\} = \frac{2}{s^3}$

Problem set 1.1

6. $\mathcal{L}\{\cos^2\omega t\} = \frac{1}{2}\mathcal{L}\{2\cos^2\omega t\}$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \frac{1}{2}\mathcal{L}\{\cos 2\omega t + 1\}$

$= \frac{1}{2}\left[\mathcal{L}\{\cos \underline{2\omega}\, t\} + \mathcal{L}\{1\}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+4\omega^2} + \frac{1}{s}\right]$

8. $\mathcal{L}\{1.5\sin(3t - \frac{\pi}{2})\} = 1.5\,\mathcal{L}\left[\sin 3t\cos\frac{\pi}{2} - \cos 3t\sin\frac{\pi}{2}\right]$

$= 1.5\,\mathcal{L}\{-\cos 3t\}$

$= -1.5\,\mathcal{L}\{\cos 3t\}$

$= -1.5\,\frac{s}{s^2+9}$

14. $\mathcal{L}\{t\sin 2t\} = (-1)^1\frac{d}{ds}[F(s)] = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right]$

$F(s) = \mathcal{L}\{f(t)\}$

$= \mathcal{L}\{\sin 2t\}$

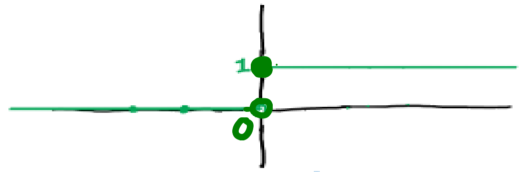
$= \frac{2}{s^2+4}$

$= -\frac{(s^2+4).0 - 2.2s}{(s^2+4)^2}$

$= \frac{4s}{(s^2+4)^2}$

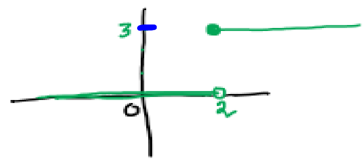
Unit step (Heaviside) Function:

$u(t) = \begin{cases} 0 & ; t<0 \\ 1 & ; t \geq 0 \end{cases}$ ← piece-wise format

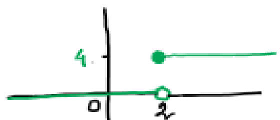


← graphical format

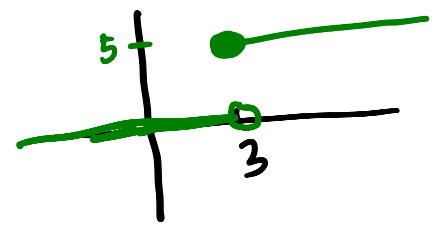
$3\,u(t-2) = \begin{cases} 0; & t-2<0 \\ 3; & t-2 \geq 0 \end{cases} = \begin{cases} 0; & t<2 \\ 3; & t \geq 2 \end{cases}$



$5u(t-3) = \begin{cases} 0; & t<3 \\ 5; & t \geq 3 \end{cases}$



$4u(t-2) = \begin{cases} 0; & t<2 \\ 4; & t \geq 2 \end{cases}$



Rectangular pulse:

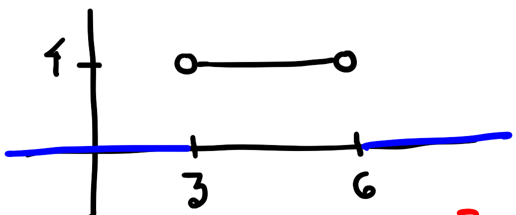
$v(t)$

$t=a \rightarrow$ applied

$t=b \rightarrow$ removed

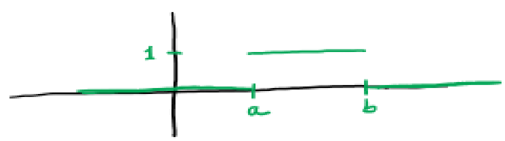
$v(t) = [u(t-a) - u(t-b)]$

$= \begin{cases} 1; & a<t<b \\ 0; & t<a \text{ or } t>b \end{cases}$



$4\,[u(t-3) - u(t-6)]$

$= \begin{cases} 0 & ; t<3 \\ 4 & ; 3<t<6 \\ 0 & ; t>6 \end{cases}$

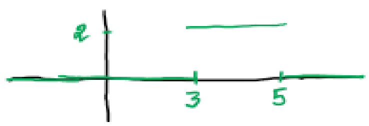


$3[u(t-2) - u(t-4)]$

$2<t<4$

$v(t) = 2[u(t-3) - u(t-5)]$

$= \begin{cases} 2; & 3<t<5 \\ 0; & t<3 \text{ or } t>5 \end{cases}$



$2 \cdot u(t-3)$

$= \begin{cases} 0; & t<3 \\ 2; & t \geq 3 \end{cases}$

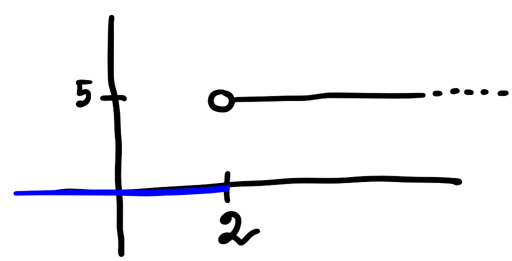
at $t=3$ s, 2 unit voltage is applied

$2\,u(t-3)$

→ $v(t) = 3\,u(t-4) - 5\,u(t-6)$

$= 3[u(t-4) - u(t-6)] + (-2)\,u(t-6)$

$= \begin{cases} 3 & ; 4<t<6 \\ -2 & ; t>6 \end{cases}$

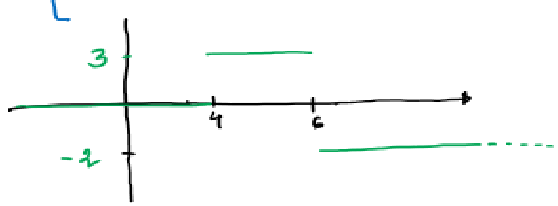


$5u(t-2)$

$= \begin{cases} 0 & ; t<2 \\ 5 & ; t \geq 2 \end{cases}$

Advanced Engineering Mathematics

– Erwin Kreszyg



→ $f(t) = \begin{cases} 0; & t<3 \\ 4; & 3<t<7 \\ 2; & t>7 \end{cases}$ Express in terms of unit step/rectangular pulse form:

$= 4\,[u(t-3) - u(t-7)] + 2\,u(t-7)$

$= 4\,u(t-3) - 2\,u(t-7)$

Laplace transform of Unit Step function:

- $\mathcal{L}\{f(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t+a)\}$
- $\mathcal{L}\{u(t-a)\} = \boxed{\dfrac{e^{-as}}{s}}$

$$\mathcal{L}\{\underbrace{e^{-2t}}_{f(t)}\cdot u(t-\underbrace{5}_{a})\} = e^{-5s}\,\mathcal{L}\{f(t+5)\}$$

$$= e^{-5s}\,\mathcal{L}\{e^{-2t}\cdot e^{-10}\}$$

$$= e^{-5s}\cdot e^{-10}\,\mathcal{L}\{e^{-2t}\}$$

$$= e^{-5s-10}\,\frac{1}{s+2}$$

$f(t) = e^{-2t}$

$f(t+5) = e^{-2\boxed{(t+5)}}$

$= e^{-2t-10}$

$= e^{-2t}\cdot e^{-10}$

$\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$

Find the Laplace transform of

$$f(t) = \begin{cases} 2; & 3 < t < 5 \\ 3; & t > 5 \end{cases}$$

$$= 2[u(t-3) - u(t-5)] + 3u(t-5)$$

$$f(t) = 2\,u(t-3) + u(t-5)$$

$$F(s) = 2\,\mathcal{L}\{u(t-3)\} + \mathcal{L}\{u(t-5)\}$$

$$= 2\,\frac{e^{-3s}}{s} + \frac{e^{-5s}}{s}$$ Ans.

H.W. exercise 16-20

exercise 21-29