AMERICAN INTERNATIONAL UNIVERSITY–BANGLADESH (AIUB)

FACULTY OF SCIENCE & TECHNOLOGY

DEPARTMENT OF CS

COMPLEX VARIABLE LAPLACE & Z-TRANSFORMATION

Fall 2021-2022

Section: A.

Supervised By

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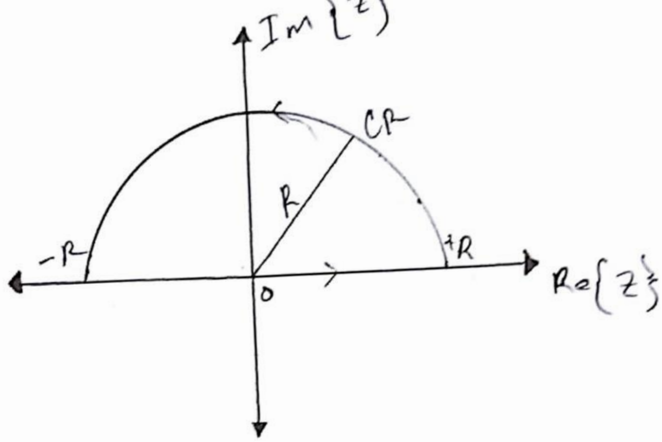
7.

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$



Consider $\displaystyle\oint_c \frac{dz}{(z^2-2z+2)^2}$ where C is the ~~ext~~ closed contour cosisting of the semi circle C_R of radius R togather with the part of real axic $-R$ to $+R$.

$$\oint_c \frac{dz}{(z^2-2z+2)^2} = \int_{-R}^{R} \frac{dx}{(x^2-2x+2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2} \quad \text{—(i)}$$

Now, The first integral has singularities or pole at,

$(z^2-2z+2)^2=0$

$\therefore z = 1 \pm i$ order of 2

But, the only pole $z = 1+i$ is inside the contour c.

Now,

$$Res\left(z = 1+i\right) = \lim_{z \to 1+i} \frac{1}{1!} \frac{d}{dz} \left\{ \frac{(z-1-i)^2}{(z^2-2z+2)^2} \right\}$$

$$= \lim_{z \to 1+i} \frac{d}{dz} \left\{ \frac{(z-1-i)^2}{\{(z-1-i)(z-1+i)\}^2} \right\}$$

$$= \lim_{z \to 1+i} \frac{d}{dz} \left\{ (z-1+i)^{-2} \right\}$$

$$= \lim_{z \to 1+i} (-2)(z-1+i)^{-3}$$

$$= (-2)(1+i-1+i)^{-3}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{1}{4i}$$

$$\therefore \oint_C \frac{dz}{(z^2-2z+2)^2} = 2\pi i \left[\frac{1}{4i} \right]$$

$$= \frac{\pi}{2}$$

from, equestion $-$ (i) $\Rightarrow$

$$\int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \frac{\pi}{2}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \lim_{R\to\infty} \frac{\pi}{2}$$

using Jondan's lemma method,

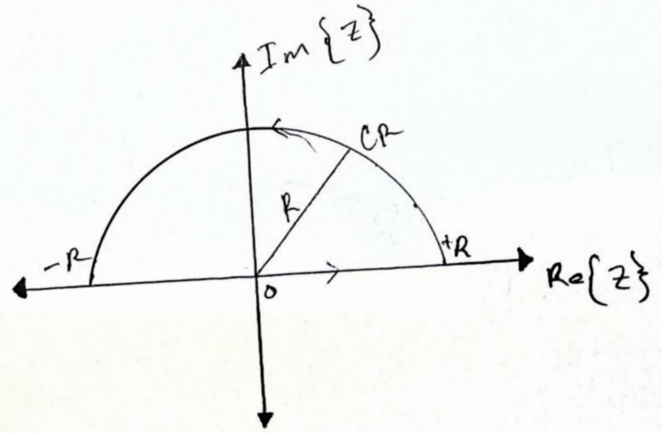
$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2} + 0 = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{dx}{(x^2-2x+2)^2} = \frac{\pi}{2}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(x^2-2x+2)^2} = \frac{\pi}{2}$$

Ans

(v) $\int_0^\infty \frac{x^2}{x^6+1}\,dx$



We consider $\oint_C \frac{z^2}{z^6+1}\,dz$, where C is the closed contour consisting of the semi circle C_R of radius R together withe the part of the real axis $-R$ to $+R$.

$$\oint_C \frac{z^2}{z^6+1}\,dz = \int_{-R}^{R} \frac{x^2\,dx}{x^6+1} + \int_{C_R} \frac{z^2\,dz}{z^6+1} \quad \text{--- (i)}$$

Now the first integral ha singularities on pole at $x^6+1=0$; order of 1.

Now,

$$z^6+1=0$$

$$z^6=-1$$

$$z_n^6=e^{i(\pi+2\pi n)}$$

$$\Rightarrow z_n = e^{i(2n+1)\frac{\pi}{6}}$$

where,

$n=0,1,2,\ ---.$

$n=0 \quad z_0=e^{i\frac{\pi}{6}}$

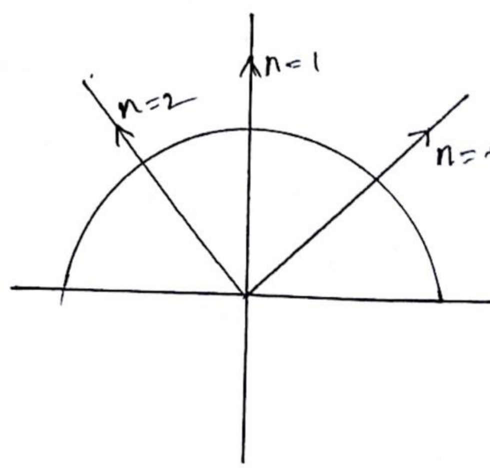
$n=1 \quad z_1=e^{i\frac{3\pi}{6}}=e^{i\frac{\pi}{2}}$

$n=2 \quad z_2=e^{i\frac{5\pi}{6}}$

$n=3 \quad z_3=e^{i\frac{7\pi}{6}}$

$n=4 \quad z_4=e^{i\frac{9\pi}{6}}$

$n=5 \quad z_5=e^{i\frac{11\pi}{6}}$



So, interior singular points are,

$$z_n = e^{i(2n+1)\frac{\pi}{6}} \;;\; n = 0, 1, 2$$

Agin,

$$\text{Rez}(z = z_n) = \lim_{z \to z_n} \frac{1}{0!} \frac{d^0}{dz^0} \left[\frac{z^2}{z^6+1} (z - z_n) \right]$$

$$= \lim_{z \to z_n} \left[\frac{z^2(z - z_n)}{z_6 + 1} \right] \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{z \to z_n} \left[\frac{2z}{6z^5} \right] \left[\text{Applying L'Hopital rule} \right]$$

$$= \lim_{z \to z_n} \left[\frac{1}{3z^4} \right]$$

$$= \frac{1}{3z_n^4}$$

Where,

$$z_n = e^{i(2\quad +1)\frac{\pi}{6}} \; ; \; n = 0, 1, 2$$

$$z_0^4 = e^{i\frac{4\pi}{6}}$$

$$= \cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right)$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_1^4 = e^{i\frac{4\pi}{2}} = 1$$

$$z_2^4 = e^{i\frac{20\pi}{6}}$$

$$= -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

So, $$\oint \frac{z^2}{z^6+1}\,dz = 2\pi i\left[\frac{1}{3z_0^4} + \frac{1}{3z_1^4} + \frac{1}{3z_2^4}\right]$$

$$= \frac{2\pi i}{3}\left[\frac{1}{-1/2 + i\sqrt{3}/2} + \frac{1}{1} + \frac{1}{-1/2 - i\sqrt{3}/2}\right]$$

$$= \frac{2\pi i}{3}\left[\frac{2}{-1+i\sqrt{3}} + 1 - \frac{2}{1+i\sqrt{3}}\right]$$

from, equestion ①,

$$\int_{-R}^{R}\frac{x^2}{x^6+1}dx+\int_{C_R}\frac{z^2}{z^6+1}dz=\frac{2\pi i}{3}\left[\frac{2}{-1+i\sqrt{3}}+1-\frac{2}{1+i\sqrt{3}}\right]$$

using Jordan Lemma leting $R\to\infty$

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}dx+\lim_{R\to\infty}\int\frac{z^2}{z^6+1}dz=\frac{2\pi i}{3}\left[\frac{2}{-1+i\sqrt{3}}+1-\frac{2}{1+i\sqrt{3}}\right]$$

$$\Rightarrow\int_{-\infty}^{\infty}\frac{x^2}{x^6+1}+0=\frac{2\pi i}{3}\left[\frac{2}{-1+i\sqrt{3}}+1-\frac{2}{-1+i\sqrt{3}}\right]$$

$$\Rightarrow\int_{0}^{\infty}\frac{x^2}{x^6+1}=\frac{\pi i}{3}\left[\frac{2}{-1+i\sqrt{3}}+1-\frac{2}{1+i\sqrt{3}}\right]$$