Application of Residue theorem

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2}$$

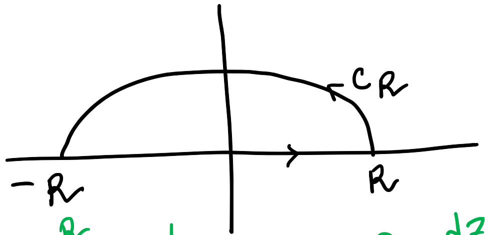
$x^2-2x+2=0$

$\Rightarrow x = \dfrac{2 \pm \sqrt{4-8}}{2\times 1}$

$= \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$

$\boxed{(1,1)}$, $(1,-1)$



$$\oint_C \frac{dz}{(z^2-2z+2)^2} = \int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2} \quad \text{—— (i)}$$

$$\oint_C \frac{dz}{(z^2-2z+2)^2} = ?$$

interior singular point is $z = 1+i$ of order 2.

$$\text{Res}(z=1+i) = \lim_{z\to 1+i} \frac{1}{1!} \frac{d}{dz}\left[\{z-(1+i)\}^2 \cdot \frac{1}{(z^2-2z+2)^2}\right]$$

$$= \lim_{z\to 1+i} \frac{d}{dz}\left[\{z-\cancel{(1+i)}\}^2 \frac{1}{\{z-\cancel{(1+i)}\}^2\{z-(1-i)\}^2}\right]$$

$$= \lim_{z\to 1+i} \frac{d}{dz}\left[\frac{1}{\{z-(1-i)\}^2}\right]$$

$$= \lim_{z\to 1+i} -2 \frac{1}{\{z-(1-i)\}^3}$$

$$= \frac{-2}{(1+i-1+i)^3}$$

$$= \frac{-2}{8i^3}$$

$$= \frac{1}{4i}$$

$$\oint_C \frac{dz}{(z^2-2z+2)^2} = 2\pi i \times \text{Res}(z=1+i)$$

$$= 2\pi i \times \frac{1}{4i}$$

$$= \frac{\pi}{2}$$

From equation (i) $\Rightarrow$

$$\int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \frac{\pi}{2}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2-2x+2)^2} + \lim_{R\to\infty} \int_{C_R} \frac{dz}{(z^2-2z+2)^2} = \lim_{R\to\infty} \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2} + 0 = \frac{\pi}{2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2-2x+2)^2} = \frac{\pi}{2}$$

Ans

Laurent series expansion 7.3

\# Obtain the Laurent Series expansion of

$f(z)=\frac{1}{(1+z^2)(z+2)}$ when (i) $1<|z|<2$ (ii) $|z|>2$

$f(z)=\frac{1}{(1+z^2)(z+2)}=\frac{Az+B}{1+z^2}+\frac{C}{z+2}$

$\Rightarrow 1=(Az+B)(z+2)+C(1+z^2)$

$\Rightarrow 1=Az^2+2Az+Bz+2B+C+Cz^2$

$=(A+C)z^2+(2A+B)z+(2B+C)$

$$\left.\begin{aligned} A+C&=0\\ 2A+B&=0\\ 2B+C&=1\end{aligned}\right\} \quad A=-\frac{1}{5},\ B=\frac{2}{5},\ C=\frac{1}{5}$$

$\therefore f(z)=\frac{-\frac{1}{5}z+\frac{2}{5}}{1+z^2}+\frac{1}{5}\frac{1}{z+2}$

$\Rightarrow f(z)=-\frac{1}{5}\frac{z}{1+z^2}+\frac{2}{5}\frac{1}{1+z^2}+\frac{1}{5}\frac{1}{z+2}$

i. $1<|z|<2$

$1<|z|$ and $\boxed{|z|<2}$

$\Rightarrow \frac{1}{|z|}<1$ $\qquad \Rightarrow \frac{|z|}{2}<1$

$\Rightarrow \frac{1}{|z|^2}<1 \Rightarrow \boxed{\left|\frac{1}{z^2}\right|<1}$ $\qquad \Rightarrow \left|\frac{z}{2}\right|<1$

$f(z)=-\frac{1}{5}\frac{z}{1+z^2}+\frac{2}{5}\frac{1}{1+z^2}+\frac{1}{5}\frac{1}{z+2}$

$=-\frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})}+\frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})}+\frac{1}{5}\frac{1}{2(1+\frac{z}{2})}$

$=-\frac{1}{5z}\left(1+\frac{1}{z^2}\right)^{-1}+\frac{2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1}+\frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$; $\left|\frac{1}{z^2}\right|<1$, $\left|\frac{z}{2}\right|<1$

$=-\frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)+\frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)$

$+\frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\cdots\right)$

$| \quad | < \square$

$\boxed{|z|<a}$

if $\boxed{|z|<1}$

$(1-z)^{-1}=1+z+z^2+\cdots$

$(1+z)^{-1}=1-z+z^2-z^3+\cdots$

$(1-z)^{-2}=1+2z+3z^2+4z^3+\cdots$

$(1+z)^{-2}=1-2z+3z^2-4z^3+\cdots$

$1+z^2$

$=z^2\left(1+\frac{1}{z^2}\right)$

$\boxed{|z|<1}$

$(1-z)^{-1}=1+z+z^2+\cdots$

$\left(1+\frac{1}{z^2}\right)^{-1}$

$\boxed{\left|\frac{1}{z^2}\right|<1}$

(ii) $|z|>2$

$$\frac{1}{(1+z^2)(z+2)} = -\frac{1}{5}\frac{z}{1+z^2} + \frac{2}{5}\frac{1}{1+z^2} + \frac{1}{5}\frac{1}{z+2}$$

$|z|>2 \Rightarrow \frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$

$\Rightarrow \left|\frac{2}{z}\right| < 1$

$\left|\frac{2}{z}\right| < 1$

$2(x) < 1$
$\Rightarrow x < 1$

$|z|>2$
$\Rightarrow \frac{|z|}{2} > 1$
$\Rightarrow \frac{2}{|z|} < 1$
$\Rightarrow \frac{1}{|z|} < 1$
$\Rightarrow \frac{1}{|z|^2} < 1$
$\Rightarrow \left|\frac{1}{z^2}\right| < 1$

$$\frac{1}{(1+z^2)(z+2)} = -\frac{1}{5}\frac{z}{z^2(1+\frac{1}{z^2})} + \frac{2}{5}\frac{1}{z^2(1+\frac{1}{z^2})} + \frac{1}{5}\frac{1}{z(1+\frac{2}{z})}$$

$$= -\frac{1}{5z}\left(1+\left(\frac{1}{z^2}\right)\right)^{-1} + \frac{2}{5z^2}\left(1+\frac{1}{z^2}\right)^{-1} + \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{5z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right) + \frac{2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)$$

$\left|\frac{1}{z^2}\right| < 1$

$\left|\frac{2}{z}\right| < 1$

$$+\frac{1}{5z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\cdots\right)$$ Ans.

7.3

1. (e) Expand $f(z) = \dfrac{3z}{(z-1)(2-z)}$ in a Laurent series for $\boxed{0 < |z-1| < 1}$

$|z-1| < 1$

$\Rightarrow |u| < 1$; Letting $z-1 = u$

$$f(u) = \frac{3(u+1)}{(u+1-1)(2-u-1)}$$

$$\Rightarrow f(u) = \frac{3(u+1)}{u(1-u)}$$

now, $\dfrac{3(u+1)}{u(1-u)} = \dfrac{A}{u} + \dfrac{B}{1-u}$

$\Rightarrow 3(u+1) = A(1-u) + Bu$

$\Rightarrow 3u+3 = A - Au + Bu$

$\Rightarrow 3u+3 = (B-A)u + A$

$-A+B = 3$

$A = 3 \Rightarrow B = 6$

$$\therefore \frac{3(u+1)}{u(1-u)} = \frac{3}{u} + \frac{6}{1-u}$$

$$= \frac{3}{u} + 6(1-u)^{-1}$$

$|u| < 1$

$$= \frac{3}{u} + 6(1+u+u^2+u^3+\cdots)$$

$$= \frac{3}{z-1} + 6\left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$$

H.W: 7.3 (1-3)

4. Find $f(z)$ and region of convergence (ROC) for the following series:

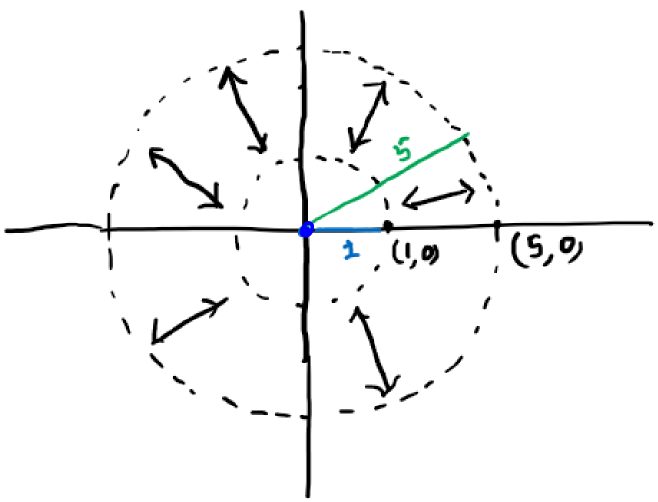
a. $1+z+z^2+z^3+\cdots$

$f(z) = (1-z)^{-1}$; $\boxed{|z|<1}$ ROC

d. $1-2z+3z^2-4z^3+\cdots$

$f(z) = (1+z)^{-2}$; $|z|<1$

5. Obtain Laurent series expansion of $f(z) = \dfrac{z}{(z-1)(z-2)}$



$1<|z|<5$

7.1, 7.2, 7.3 (second quiz Ans.)