Chapter - 8

# Z-transform

$$\mathcal{Z}\{x[n]\} = X(z)$$

$$\mathcal{Z}^{-1}\{X(z)\} = x[n]$$

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

Formula:

$\mathcal{Z}^{-1}\left\{\frac{1}{1-z^{-1}}\right\} = u[n]$ / $-u[-n-1]$

1. $\mathcal{Z}\{\delta[n]\} = 1$
2. $\mathcal{Z}\{u[n]\} = \dfrac{1}{1-z^{-1}}$ ; $\boxed{|z| > 1}$
3. $\mathcal{Z}\{-u[-n-1]\} = \dfrac{1}{1-z^{-1}}$ ; $\boxed{|z| < 1}$
4. $\mathcal{Z}\{\delta[n-m]\} = z^{-m}$ → $\mathcal{Z}\{\delta[n+2]\} = z^{2}$, $\mathcal{Z}\{\delta[n-5]\} = z^{-5}$
5. $\mathcal{Z}\{a^n u[n]\} = \dfrac{1}{1-az^{-1}}$ ; $|z| > |a|$

6. $\mathcal{Z}\{-a^n u[-n-1]\} = \dfrac{1}{1-az^{-1}}$ ; $\boxed{|z| < |a|}$

7. $\mathcal{Z}\{n a^n u[n]\} = \dfrac{az^{-1}}{(1-az^{-1})^2}$ ; $|z| > |a|$

8. $\mathcal{Z}\{-n a^n u[-n-1]\} = \dfrac{az^{-1}}{(1-az^{-1})^2}$ ; $|z| < |a|$

# $\delta[n] = \begin{cases} 1 ; & n = 0 \\ 0 ; & n \neq 0 \end{cases}$

# $u[n] = \begin{cases} 0 ; & n < 0 \\ 1 ; & n \geq 0 \end{cases}$

Main Formula:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Formula proof: $\rightarrow$ H.W

$\delta[n] = \begin{cases} 1 ; & n = 0 \\ 0 ; & n \neq 0 \end{cases}$

1. $\mathcal{Z}\{\boxed{\delta[n]}\} = 1$

$$\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$

$$\rightarrow = \cdots + \delta[-2] z^{2} + \delta[-1] z^{1} + \boxed{\delta[0] z^{0}} + \delta[1] z^{-1} + \delta[2] z^{-2} + \cdots$$

$$= \cdots + 1 + \cdots = 1$$

2. $Z\{u[n]\} = \dfrac{1}{1-z^{-1}}$

$u[n] = \begin{cases} 0 & ; n<0 \\ 1 & ; n \geq 0 \end{cases}$

$$Z\{u[n]\} = \sum_{n=-\infty}^{\infty} u[n]\, z^{-n}$$

$$= \cdots + u[-2]\, z^{2} + u[-1]\, z^{1} + u[0]\, z^{0} + u[1]\, z^{-1} + u[2]\, z^{-2} + \cdots$$

$$= 1 + z^{-1} + z^{-2} + \cdots$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} ; \left|\frac{1}{z}\right| < 1$$

$|z| < 1$

$(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$

$$= \frac{1}{1 - \frac{1}{z}} ; \frac{1}{|z|} < 1$$

$$= \frac{1}{1 - z^{-1}} ; |z| > 1$$

3. $\mathcal{Z}\{-u[n-1]\} = \dfrac{1}{1-z^{-1}}$ ; $|z| < 1$

$u[n] = \begin{cases} 0 ; & n < 0 \\ 1 ; & n \geq 0 \end{cases}$

$\Rightarrow u[-n-1] = \begin{cases} 0 ; & -n-1 < 0 \\ 1 ; & -n-1 \geq 0 \end{cases}$

$\Rightarrow \boxed{-} u[-n-1] = \begin{cases} 0 ; & -n-1 < 0 \\ -1 ; & -n-1 \geq 0 \end{cases}$

$= \begin{cases} 0 ; & -n < 1 \\ -1 ; & -n \geq 1 \end{cases}$

$\Rightarrow -u[-n-1] = \begin{cases} 0 ; & n > -1 \\ -1 ; & n \leq -1 \end{cases}$

$\mathcal{Z}\{-u[-n-1]\} = \sum_{n=-\infty}^{\infty} -u[-n-1] z^{-n}$

$= \ldots - z^3 - z^2 - z + 0 + 0 + \ldots$

$= \ldots - z^3 - z^2 - z = -z - z^2 - z^3 - \ldots$

$= -(z + z^2 + z^3 + \ldots)$

$= -z(1 + z + z^2 + \ldots)$

$= -z(1-z)^{-1}$ ; $\boxed{|z| < 1}$

$= \dfrac{-z}{1-z}$

$= \dfrac{-z}{-z(1-\frac{1}{z})}$

$= \dfrac{1}{1-z^{-1}}$ ; $\boxed{|z| < 1}$

$n = -3$

$-1$

$n = -2$

$-1 \cdot z^2$

$n = -1$

$-1 \cdot z^1$

$n = 0$

4. $\mathcal{Z}\{\delta[n-m]\} = z^{-m}$

$\delta[n] = \begin{cases} 1 ; & n = 0 \\ 0 ; & n \neq 0 \end{cases}$

$\Rightarrow \delta[n-m] = \begin{cases} 1 ; & n - m = 0 \\ 0 ; & n - m \neq 0 \end{cases}$

$\Rightarrow \delta[n-m] = \begin{cases} 1 ; & n = m \\ 0 ; & n \neq m \end{cases}$

$\ldots, m-2, m-1, m, m+1, m+2, \ldots$

$\mathcal{Z}\{\delta[n-m]\} = \sum_{n=-\infty}^{\infty} \delta[n-m] z^{-\boxed{n}}$

$= \ldots 0 + 0 + 0 + \boxed{1} \cdot z^{-\boxed{m}} + \boxed{0} + \boxed{0} + \ldots$

$= z^{-m}$

8. $\mathcal{Z}\{-na^n u[-n-1]\} = \dfrac{az^{-1}}{(1-az^{-1})^2}; \quad |z| < |a|$

$$u[n] = \begin{cases} 0; & n<0 \\ 1; & n \geq 0 \end{cases}$$

$$u[-n-1] = \begin{cases} 0; & -n-1<0 \\ 1; & -n-1 \geq 0 \end{cases}$$

$$\Rightarrow na^n u[-n-1] = \begin{cases} 0; & -n-1<0 \\ na^n; & -n-1 \geq 0 \end{cases}$$

$$\Rightarrow -na^n u[-n-1] = \begin{cases} 0; & -n-1<0 \\ -na^n; & -n-1 \geq 0 \end{cases}$$

$$= \begin{cases} 0; & -n<1 \\ -na^n; & -n \geq 1 \end{cases}$$

$$\Rightarrow -na^n u[-n-1] = \begin{cases} 0; & n>-1 \\ -na^n; & n \leq -1 \end{cases}$$

$$\mathcal{Z}\{-na^n u[-n-1]\} = \sum_{n=-\infty}^{\infty} -na^n u[-n-1] \cdot z^{-n}$$

$$= \cdots + 3a^{-3} \cdot z^3 + 2a^{-2} \cdot z^2 + a^{-1}z + 0 + 0 + \cdots$$

$$= a^{-1}z + 2a^{-2}z^2 + 3a^{-3}z^3 + \cdots$$

$$= \frac{z}{a} + 2\left(\frac{z}{a}\right)^2 + 3\left(\frac{z}{a}\right)^3 + \cdots$$

$$= \frac{z}{a}\left[1 + 2\frac{z}{a} + 3\left(\frac{z}{a}\right)^2 + \cdots\right]$$

$1+2z+3z^2+\cdots = (1-z)^{-2}; \; |z|<1$

$$= \frac{z}{a}\left(1-\frac{z}{a}\right)^{-2}; \quad \left|\frac{z}{a}\right| < 1$$

$$= \frac{z}{a} \frac{1}{\frac{(a-z)^2}{a^2}}; \quad \frac{|z|}{|a|} < 1$$

$$= \frac{z}{a} \frac{a^2}{(a-z)^2}; \quad |z| < |a|$$

$$= \frac{az}{(a-z)^2}$$

$$= \frac{az}{\{-z(1-\frac{a}{z})\}^2}$$

$$= \frac{az}{z^2(1-az^{-1})^2}$$

$$= \frac{a}{z(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}; \quad |z| < |a|$$

Formula proof: 5, 6, 7 (H.W) ← do it by yourself

1-8 (H.W)

# Exercise 8.1

1. H.W: c. Find $\mathcal{Z}\{n^2 a^n u[n]\}$ → proof

   d. Find $\mathcal{Z}\{(n+1)\,u[n]\}$ → proof

2. $X(z) = (1+2z)(1+3z^{-1})(1-z^{-1})$ ; Find $x[n] = ?$

$\Rightarrow X(z) = (1+3z^{-1}+2z+6)(1-z^{-1})$

$\Rightarrow X(z) = 1 - z^{-1} + 3z^{-1} - 3z^{-2} + 2z - 2 + 6 - 6z^{-1}$

$\Rightarrow X(z) = 5 - 4z^{-1} - 3z^{-2} + 2z$

$\mathcal{Z}\{\delta[n-m]\} = z^{-m}$

$\Rightarrow \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\{5\} - 4\,\mathcal{Z}^{-1}\{z^{-1}\} - 3\,\mathcal{Z}^{-1}\{z^{-2}\} + 2\,\mathcal{Z}^{-1}\{z^{1}\}$

$\mathcal{Z}^{-1}\{z^{-m}\} = \delta[n-m]$

$\Rightarrow x[n] = 5\,\delta[n] - 4\,\delta[n-1] - 3\,\delta[n-2] + 2\,\delta[n+1]$

Ans.