Lecture 20: Standing Waves

If two sinusoidal waves of the same amplitude and wavelength travel in opposite directions along a stretched string, their interference with each other produces a standing wave.

$$\left.\begin{aligned} y_1(x, t) &= y_m \sin(kx - \omega t) \\ y_2(x, t) &= y_m \sin(kx + \omega t) \end{aligned}\right\} \text{traveling waves}$$

Superposition principle, $y'(x, t) = y_1(x, t) + y_2(x, t)$

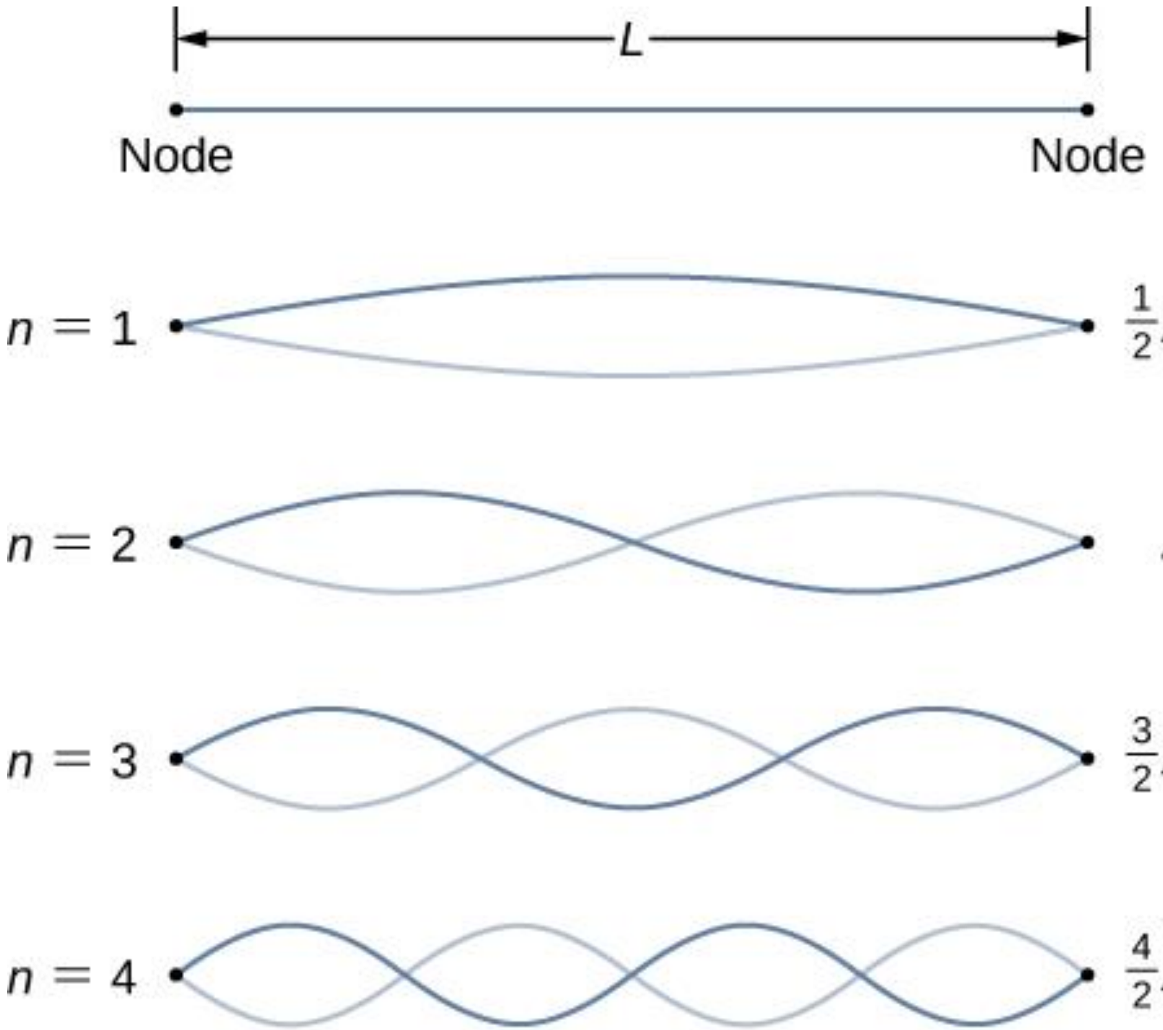
$$\begin{aligned} y'(x, t) &= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) \\ &= y_m \{\sin(kx - \omega t) + \sin(kx + \omega t)\} \\ &= y_m \{2 \sin\left(\frac{kx - \omega t + kx + \omega t}{2}\right) \cos\left(\frac{kx - \omega t - kx - \omega t}{2}\right)\} \\ &= 2y_m \sin\left(\frac{2kx}{2}\right) \cos\left(\frac{-2\omega t}{2}\right) \end{aligned}$$

$y'(x, t) = [2y_m \sin kx]\cos\omega t$ [standing wave]

Resultant displacement = $y'(x, t)$

Amplitude at position x = $[2y_m \sin kx]$

Oscillating term = $\cos\omega t$



$\frac{1}{2}\lambda_1 = L \qquad \lambda_1 = \frac{2}{1}L$

$\lambda_2 = L \qquad \lambda_2 = \frac{2}{2}L$

$\frac{3}{2}\lambda_3 = L \qquad \lambda_3 = \frac{2}{3}L$

$\frac{4}{2}\lambda_4 = L \qquad \lambda_4 = \frac{2}{4}L$

$\lambda_n = \frac{2}{n}L \qquad n = 1, 2, 3, \ldots$

Nodes: **The string never moves. The amplitude of the resultant wave will be zero.** *The fully destructive interference occurs.*

$$\text{Amplitude} = 2y_m (\sin kx) = 2y_m (0) = 0$$

If $\sin kx = 0$

$$\sin kx = \sin (0, \pi, 2\pi, 3\pi, \ldots\ldots\ldots\ldots)$$

$$\sin kx = \sin n\pi \qquad \text{for } n = 0, 1, 2, 3, \ldots \ldots$$

$$kx = n\pi$$

$$\frac{2\pi}{\lambda} x = n\pi$$

$$x = n\frac{\lambda}{2} \qquad \text{for } n = 0, 1, 2, 3 \ldots\ldots\ldots$$

Distance between adjacent nodes, $\Delta x = \frac{\lambda}{2}$

Antinodes: **The halfway between nodes are called antinodes, where the amplitude of the resultant wave will be maximum.** *The fully constructive interference occurs.*

Amplitude = $2y_m$ (sin kx) = $2y_m$ (1) = $2y_m$

If sin kx = 1

$\sin kx = \sin\left(1\frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}\ldots\ldots\ldots\ldots\right)$

$\sin kx = \sin\left(n+\frac{1}{2}\right)\pi \qquad \text{for } n = 0, 1, 2, 3, \ldots\ldots$

$kx = \left(n+\frac{1}{2}\right)\pi$

$\frac{2\pi}{\lambda}x = \left(n+\frac{1}{2}\right)\pi$

$x = \left(n+\frac{1}{2}\right)\frac{\lambda}{2} \qquad \text{for } n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots$

Distance between adjacent antinodes, $\Delta x = \frac{\lambda}{2}$

https://www.youtube.com/watch?v=eu1PC4botbM

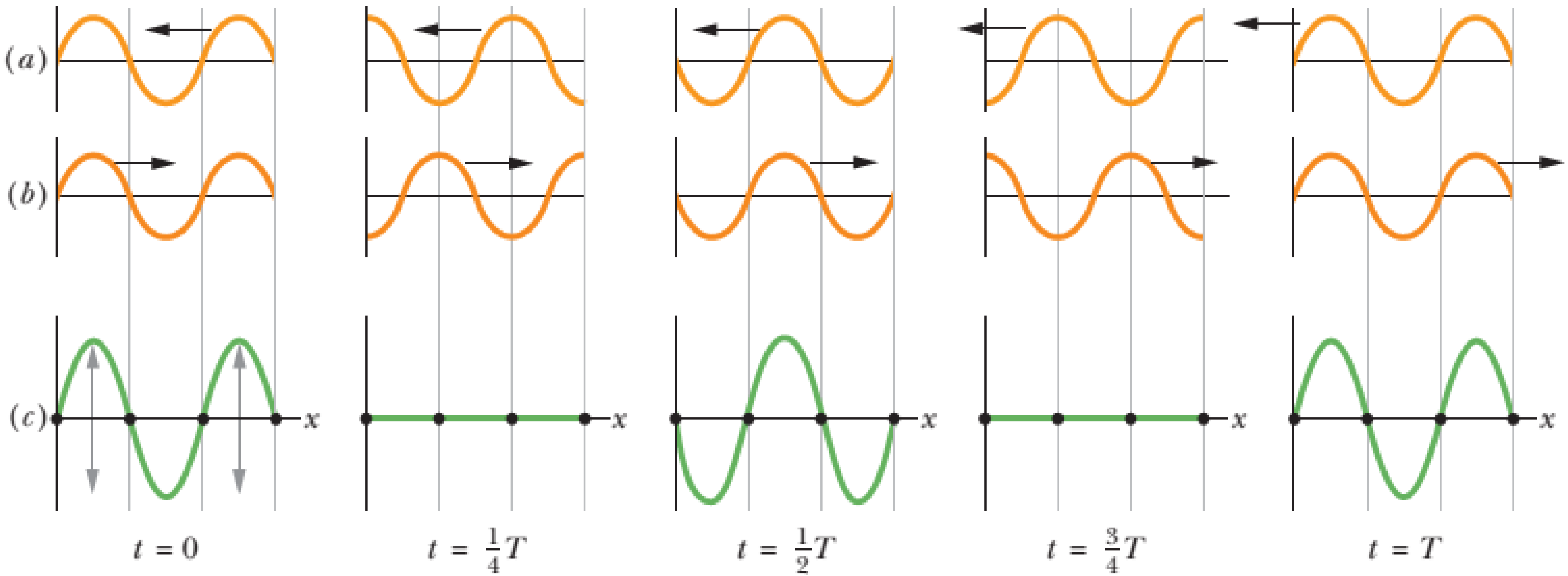


Fig. In phase: at t = 0, $\frac{1}{2}T, T$ *the fully constructive interference occurs because of the alignment at peaks with peaks and valleys with valleys*.

Fig. out of phase: at t = $\frac{1}{4}T, \frac{3}{4}T$ *the fully destructive interference occurs because of the alignment of peaks and valleys*.

Nodes: some points never oscillate.

Antinodes: some points oscillate the most.

53. A string oscillate according to the equation $y' = (0.50 \text{ cm}) \sin[(\pi/3 \text{ cm}^{-1})x] \cos[(40\pi \text{ s}^{-1})t]$. What are (a) the amplitude and (b) the speed of the two waves (identical except for direction of travel) whose superposition gives this oscillation? (c) What is the distance between nodes? (d) What is the speed of a particle of the string at the position x = 1.5 cm when t = 9/8 s?

$$y' = [(0.50 \text{ cm}) \sin\{(\pi/3 \text{ cm}^{-1})x\}] \cos[(40\pi \text{ s}^{-1})t]$$

$$y' = [2y_m \sin kx] \cos\omega t$$

Given, $k = \frac{\pi}{3}$ rad/cm $= \frac{\pi}{0.03}$ rad/m; $\omega = 40\pi$ rad/s

(a) $2y_m = 0.50$ cm

$2y_m = 0.0050$ m

$y_m = 0.0050/2$ m

$y_m = 0.0025$ m Ans.

(b) $v = \frac{\omega}{k} = \frac{40\pi}{\frac{\pi}{0.03}} = 0.03(40) = 1.20$ m/s Ans.

(c) Adjacent distance between nodes,

$$\Delta x = \frac{\lambda}{2} = \frac{\frac{2\pi}{k}}{2} = \frac{\pi}{k} = \frac{\pi}{\frac{\pi}{0.03}} = 0.03 \text{ m} \quad \text{Ans.}$$

(d) $y'(x, t) = [2y_m \sin kx] \cos\omega t$

The speed of a particle of a string is given by

$$u = \frac{\partial y'}{\partial t} = \frac{\partial}{\partial t}(2y_m \sin kx \cos\omega t)$$

$$= 2y_m \sin kx \frac{\partial}{\partial t}(\cos\omega t)$$

$$= 2y_m \sin kx \,(-\sin\omega t)\frac{\partial}{\partial t}(\omega t)$$

$$= 2y_m \sin kx \,(-\sin\omega t)(\omega)$$

$$= -\omega\, 2y_m \sin kx \,(\sin\omega t)$$

$$= -40\pi\, 2(0.0025) \sin\{(\tfrac{\pi}{0.03})(0.015)\} \sin\{40\pi(9/8)\}$$

$$= -40\pi\, 2(0.0025) \sin(\tfrac{\pi}{2}) \sin(45\pi)$$

$$= \{-40\pi\, 2(0.0025) \sin(\tfrac{\pi}{2})\}(0)$$

$$= 0$$

76. A standing wave results from the sum of two transverse traveling waves given by $y_1 = 0.050 \cos(\pi x - 4\pi t)$ and $y_2 = 0.050 \cos(\pi x + 4\pi t)$, where x, y_1, and y_2 are in meters and t is in seconds. (a) What is the smallest positive value of x that corresponds to a node? Beginning at t = 0, what is the value of the (b) first, (c) second, and (d) third time the particle at x = 0 has zero velocity?

$$y_1 = 0.050 \cos(\pi x - 4\pi t)$$

$$y_2 = 0.050 \cos(\pi x + 4\pi t)$$

$$y'(x, t) = y_1(x, t) + y_2(x, t)$$

$$y' = 0.050 \cos(\pi x - 4\pi t) + 0.050 \cos(\pi x + 4\pi t)$$

$$y' = 0.050 \{\cos(\pi x - 4\pi t) + \cos(\pi x + 4\pi t)\}$$

$$y' = 0.050 \{2\cos\left(\frac{\pi x - 4\pi t + \pi x + 4\pi t}{2}\right) \cos\left(\frac{\pi x - 4\pi t - \pi x - 4\pi t}{2}\right)$$

$$y' = 2(0.050) \cos\left(\frac{2\pi x}{2}\right) \cos\left\{\frac{-2(4\pi t)}{2}\right\}$$

$y' = [0.1 \cos \pi x] \cos 4\pi t$ [resultant wave is a standing wave]

Amplitude of the resultant wave = $0.1 \cos \pi x$

Node: Amplitude of the resultant wave = $0.1 \cos \pi x = 0.1\ (0) = 0$

If $\cos \pi x = 0$

$$\cos \pi x = \cos \left(1\frac{\pi}{2}, 3\frac{\pi}{2}, 5\frac{\pi}{2}\ldots\ldots\ldots\ldots\right)$$

$$\cos \pi x = \cos \left(n+\frac{1}{2}\right) \pi \qquad \text{for } n = 0, 1, 2, 3 \ldots\ldots\ldots\ldots\ldots$$

$$\pi x = \left(n+\frac{1}{2}\right) \pi$$

$$x = \left(n+\frac{1}{2}\right) \qquad \text{for } n = 0, 1, 2, 3 \ldots\ldots\ldots\ldots\ldots$$

(a) For the smallest value of x: $n = 0$, $\quad x = \left(0+\frac{1}{2}\right) = \frac{1}{2}$ m

The speed of a particle of a string is given by

$$u = \frac{\partial y'}{\partial t} = \frac{\partial}{\partial t}(0.1 \cos \pi x \cos 4\pi t)$$

$$u = 0.1 \cos \pi x \frac{\partial}{\partial t}(\cos 4\pi t)$$

$$u = 0.1 \cos \pi x (- \sin 4\pi t) \frac{\partial}{\partial t}(4\pi t)$$

$$u = 0.1 \cos \pi x (- \sin 4\pi t) (4\pi)$$

$u = -\,0.4\pi \cos \pi x \sin 4\pi t$

At $x = 0$, $u = 0$

$0 = -\,0.4\pi \cos \pi(0) \sin 4\pi t$

$-\,0.4\pi \cos 0 \sin 4\pi t = 0$

$-\,0.4\pi\,(1) \sin 4\pi t = 0$

$-\,0.4\pi \sin 4\pi t = 0$

$\sin 4\pi t = 0$

$\sin 4\pi t = \sin(0, \pi, 2\pi, 3\pi, \ldots\ldots\ldots.)$

$\sin 4\pi t = \sin n\pi$ for $n = 0, 1, 2, 3, \ldots\ldots\ldots.)$

$4\pi t = n\pi$

$4t = n$

$t = \frac{n}{4}$

(b) for n = 0, $t = \frac{0}{4}$ = 0 s, first time

(c) for n = 1, $t = \frac{1}{4}$ s, second times

(d) for n = 2, $t = \frac{2}{4} = \frac{1}{2}$ s, third times

So the particle velocity, $u = 0$ at $t = 0, \frac{1}{4}, \frac{1}{2}$ s.