Application of Laplace transformation

Solving differential equation

→ Applying Laplace trans. on given differential equation.

→ Using initial values.

→ Solve for $\boxed{Y(s)}$

→ Apply inverse Laplace on $\boxed{Y(s)} \to y(t)$

$\frac{d^2y}{dx^2} \,/\, y^2(x) \,/\, \ddot{y}(x)$

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

$\mathcal{L}^{-1}\{Y(s)\} = y(t)$

$$\boxed{\mathcal{L}\{y^n(t)\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2}\dot{y}(0) - s^{n-3}\ddot{y}(0) - s^{n-4}\dddot{y}(0) - \ldots}$$

$\frac{dy}{dx} = x$

$\boxed{y(x)} = \frac{x^2}{2} + c$

$\dot{y}(x)$

$\frac{dy}{dt} = t$

$\Rightarrow y(t)$

$\mathcal{L}\{y(t)\} = Y(s)$

$$\mathcal{L}\{y^3(t)\} = s^3 Y(s) - s^2 y(0) - s\,\dot{y}(0) - \ddot{y}(0)$$

$$\mathcal{L}\{y^2(t)\} = s^2 Y(s) - s\, y(0) - \dot{y}(0)$$

$$\mathcal{L}\{y^1(t)\} = s Y(s) - y(0)$$

$\boxed{\frac{dy}{dt}} = t$

$\Rightarrow y(t) = \frac{t^2}{2} + 1$

$y(0) = 1$

$\dot{y}(0) = 0$

$$\frac{d^3y}{dt^3} - y^2 t = ty + t \quad ; \quad y(0) = a,\ \dot{y}(0) = b,\ \ddot{y}(0) = c$$

$$\mathcal{L}\{y^4(t)\} = s^4 Y(s) - s^3 y(0) - s^2 \dot{y}(0) - s\,\ddot{y}(0) - \dddot{y}(0)$$

$$2\ddot{y}(x) - [\dddot{y}(x)] + 3 = 0$$

$\mathcal{L}\{f(t)\} = F(s)$
$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\mathcal{L}^{-1}\{Y(s)\} = y(t)$

$\ddot{y}(t) + y(t) = e^t$; $y(0) = 1$, $\dot{y}(0) = -2$

$$\ddot{y}(t) + y(t) = e^t$$

$$\Rightarrow \mathcal{L}\{\ddot{y}(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{e^t\}$$

step 1: $\Rightarrow s^2 Y(s) - s\,y(0) - \dot{y}(0) + Y(s) = \frac{1}{s-1}$ ✓

step 2: $\Rightarrow s^2 Y(s) - s + 2 + Y(s) = \frac{1}{s-1}$ ✓

step 3:

$$\Rightarrow (s^2+1)Y(s) = \frac{1}{s-1} + s - 2$$

$$\Rightarrow Y(s) = \frac{1 + s(s-1) - 2(s-1)}{(s-1)(s^2+1)}$$

$$\Rightarrow Y(s) = \frac{s^2 - 3s + 3}{(s-1)(s^2+1)}$$

$y^2 + xy = 0$
$\Rightarrow x = \boxed{}$
solve for x.

step 4: $\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \boxed{\mathcal{L}^{-1}\left\{\frac{s^2-3s+3}{(s-1)(s^2+1)}\right\}}$ —— ①

$$\mathcal{L}^{-1}\left\{\frac{s^2-3s+3}{(s-1)(s^2+1)}\right\}$$

$$\frac{s^2-3s+3}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow s^2 - 3s + 3 = A(s^2+1) + (Bs+C)(s-1)$$

$$\Rightarrow s^2 - 3s + 3 = (A+B)s^2 + (-B+C)s + (A-C)$$

$$\left.\begin{aligned} A + B &= 1 \\ -B + C &= -3 \\ A - C &= 3 \end{aligned}\right\} \quad A = \frac{1}{2},\ B = \frac{1}{2},\ C = -\frac{5}{2}$$

$$\frac{s^2-3s+3}{(s-1)(s^2+1)} = \frac{1}{2}\,\frac{1}{s-1} + \frac{\frac{1}{2}s - \frac{5}{2}}{s^2+1}$$

From equation ① $\Rightarrow$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow y(t) = \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{5}{2}\sin t$$

problem set 3.1: (1–9)

7. $\ddot{y}(t) = 5$; $y(0) = 1$; $\dot{y}(0) = 2$

$\Rightarrow \mathcal{L}\{\ddot{y}(t)\} = \mathcal{L}\{5\}$

$\Rightarrow s^2 Y(s) - s\,y(0) - \dot{y}(0) = \frac{5}{s}$

$\Rightarrow s^2 Y(s) - s - 2 = \frac{5}{s}$

$\Rightarrow s^2 Y(s) = s + 2 + \frac{5}{s}$

$\Rightarrow s^2 Y(s) = \frac{s^2+2s+5}{s}$

$\Rightarrow Y(s) = \frac{s^2+2s+5}{s^3}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$\Rightarrow y(t) = 1 + 2t + 5\frac{t^2}{2}$

$\ddot{y}(t) - 2\dot{y}(t) = \cos t$; $y(0) = 0$, $\dot{y}(0) = 1$

$\Rightarrow s^2 Y(s) - s\,y(0) - \dot{y}(0) - 2[sY(s) - y(0)] = \frac{s}{s^2+1}$

$\Rightarrow s^2 Y(s) - 1 - 2sY(s) = \frac{s}{s^2+1}$

$\Rightarrow Y(s)\left[s^2 - 2s\right] = \frac{s}{s^2+1} + 1$

$\Rightarrow (s^2 - 2s)\,Y(s) = \frac{s + s^2 + 1}{s^2+1}$

$\Rightarrow Y(s) = \frac{s + s^2 + 1}{s(s-2)(s^2+1)}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s(s-2)(s^2+1)}\right\}$

$\frac{s^2+s+1}{s(s-2)(s^2+1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1}$

$\Rightarrow s^2 + s + 1 = A(s-2)(s^2+1) + Bs(s^2+1) + (Cs+D)\,s(s-2)$

Solving Simultaneous Ordinary differential equation

Solve following system of differential equation:

$$\dot{x}(t) = 3x(t) + y(t)$$
$$\dot{y}(t) = 4x(t) + 3y(t); \quad x(0) = 3, \; y(0) = 2$$

$x(t) = ?$ $y(t) = ?$

$\dot{x}(t) = 3x(t) + y(t)$

$\Rightarrow \mathcal{L}\{\dot{x}(t)\} = 3\mathcal{L}\{x(t)\} + \mathcal{L}\{y(t)\}$

$\Rightarrow sX(s) - x(0) = 3X(s) + Y(s)$

$\Rightarrow sX(s) - 3 = 3X(s) + Y(s)$

$\Rightarrow (s-3)X(s) - Y(s) = 3$ —— (i)

$\dot{y}(t) = 4x(t) + 3y(t)$

$\Rightarrow \mathcal{L}\{\dot{y}(t)\} = 4\mathcal{L}\{x(t)\} + 3\mathcal{L}\{y(t)\}$

$\Rightarrow sY(s) - y(0) = 4X(s) + 3Y(s)$

$\Rightarrow sY(s) - 2 = 4X(s) + 3Y(s)$

$\Rightarrow -4X(s) + (s-3)Y(s) = 2$ —— (ii)

$$\left.\begin{aligned}(s-3)X(s) - Y(s) &= 3\\ -4X(s) + (s-3)Y(s) &= 2\end{aligned}\right\}$$

Multiplying equation (i) by $(s-3)$ we get.

$$(s-3)^2X(s) - (s-3)Y(s) = 3(s-3)$$
$$-4X(s) + (s-3)Y(s) = 2$$
$(+)$
$$(s^2-6s+5)X(s) = 3s-7$$
$$\Rightarrow X(s) = \frac{3s-7}{s^2-6s+5}$$

Multiplying equation (i) by -4 and equ^n (ii) by $(s-3)$ we get.

$$-4(s-3)X(s) + 4Y(s) = -12$$
$$\underset{+}{-}4(s-3)X(s) \underset{-}{+} (s-3)^2Y(s) = \underset{-}{2}(s-3)$$
$(-)$
$$(4 - s^2 + 6s - 9)Y(s) = -12 - 2s + 6$$
$$\Rightarrow Y(s) = \frac{2s+6}{s^2-6s+5}$$

s^2-6s+5
$= s^2-5s-s+5$
$= s(s-5) - 1(s-5)$
$= (s-5)(s-1)$

$X(s) = \dfrac{3s-7}{s^2-6s+5}$

$\Rightarrow \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\dfrac{3s-7}{s^2-6s+5}\right\}$

$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\dfrac{3s-7}{(s-5)(s-1)}\right\}$ —— (iii)

$\dfrac{3s-7}{(s-5)(s-1)} = \dfrac{A}{s-5} + \dfrac{B}{s-1}$

$A = \dfrac{8}{4} = 2$

$B = \dfrac{-4}{-4} = 1$

from equ^n (iii) $\Rightarrow$

$x(t) = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s-5}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$

$= 2e^{5t} + e^t$

$Y(s) = \dfrac{2s+6}{s^2-6s+5}$

$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{2s+6}{s^2-6s+5}\right\}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\dfrac{2s+6}{(s-5)(s-1)}\right\}$ —— (iv)

$\dfrac{2s+6}{(s-5)(s-1)} = \dfrac{A}{s-5} + \dfrac{B}{s-1}$

$A = 4, \quad B = -2$

from equation (iv) $\Rightarrow$

$y(t) = 4\mathcal{L}^{-1}\left\{\dfrac{1}{s-5}\right\} - 2\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$

$= 4e^{5t} - 2e^t$

problem set-3.2