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Annotation

We examined a natural interpolation of the factorial function and expanded it from natural numbers to real numbers, also known as the Gamma Function, its connection with the Harmonic function and the history behind application of the Euler's functions in high-energy physics. The methods of approximation and logarithmation were used to derive the formulae that eased the proof of product and integral forms of the Gamma function and the values of some factorials of real numbers were calculated with their use. As a result, the factorial function extention was found in several forms and the properties of the Gamma function were proved. Its application was found in quantum physics, where it helped scientists to describe the strong interaction between sub-atomic particles.

Аннотация

Мы изучили естественную интерполяцию функции факториала из натуральных чисел в вещественные числа, также известную как Гамма-функция, ее связь с Гармонической Функцией и историю применения функции Эйлера в физике высоких энергий. Методы аппроксимации и логарифмирования использовались для получения формул, облегчающих доказательство формы бесконечного произведения и интегральной формы Гамма-функции, и значения некоторых факториалов вещественных чисел рассчитывались с их использованием. В результате, расширение функции факториала было найдено в нескольких формах и свойства Гамма-функции были доказаны. Применение Гамма-функции было найдено в квантовой физике, где она помогла ученым описать сильное взаимодействие между субатомными частицами.

Keywords

Harmonic function, Gamma function, natural numbers, real numbers, interpolation.

1. Literature review

In 2020 Alexander Aycock published a paper "Euler and the Gamma Function", where he gives a definition and the derivation process of the Gamma Function made by Euler and with the different approach and aim *"to review and discuss some of the properties of the* Γ*-function"*. Besides that the paper gives several problems with solutions and brings up the connection with the Beta Function as well as derivation of different forms of the special functions. Although the paper discussed various forms of representation of the function, the derivation of initial integral of the Gamma Function was taken for granted but it was stated that one of the mentioned Euler's papers contained it and *"This paper can be considered as an addendum to De fractionibus continuis observationes"*

In the articles by Zhongfeng Sun and Huizeng Qin, 2017 and Jamal Y. Salah, 2015 they use the integral definition of the Gamma Function to prove the properties without derivation of the function beforehand. The natural derivation of the infinite product form and, subsequently, integral form in this paper shows the simpleness of the unpopular derivation.

2. Historic background

Factorials were invented by different cultures in different times between 200 and 500 A.D.. During the late 15th century factorials had become popular among western mathematicians. Gradually, people found it hard to find the factorials of large numbers, even those that go after 20, and came up with an approximation. But the obvious question was still open at that time: Is there any possibility to find a factorial of any number?

It is not hard to find the n!, where n is a natural number, but finding the factorial of a fraction can seem to be an unsolvable problem. Thankfully, there is an answer which swiss mathematician Leonhard Euler came up with more than 200 years ago.

3. Extending the Harmonic numbers to the Reals

3.1. **Some motivating examples.** If we look at the following sums

$$\begin{array}{ccc} 1 & = & 1 \\ 1+2 & = & 3 \\ 1+2+3 & = & 6 \\ \cdots & & \cdots \end{array}$$

and set $f(n) := 1 + 2 + \cdots + n$ we then one may wonder what about, say, $f(\frac{1}{2})$? In other words, can we extend this function $f : \mathbb{N} \to \mathbb{N}$ to the $\mathbb{R}$? More formally we want to define $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{N} \arrow[r, "f"] \arrow[d, hook] & \mathbb{N} \arrow[d, hook] \\ \mathbb{R} \arrow[r, "\widetilde{f}"'] & \mathbb{R} \end{tikzcd}$$

where $\mathbb{N} \hookrightarrow \mathbb{R}$ is a natural inclusion.

To do so we recall that for any n, $f(n) = \frac{n(n+1)}{2}$, thus we can set

$$f(x) := \frac{x(x+1)}{2}$$

for all $x \in \mathbb{R}$, and we obtain

$$\begin{aligned} f(0.5) &= 0.375 \\ f(1.5) &= 1.875 \\ f(\pi) &\approx 6.506 \\ f(e) &\approx 5.054 \end{aligned}$$

and so on and so forth.

It is well known that the following functions

$$\begin{aligned} g(n) &= 1^2 + 2^2 + \cdots + n^2, \\ h(n) &= 1^3 + 2^3 + \cdots + n^3, \\ y(n) &= 2^0 + 2^1 + \cdots + 2^n, \end{aligned}$$

also can be extended to the reals, by setting

$$\begin{aligned} g(x) &= \frac{x(x+1)(2x+1)}{6}, \\ h(x) &= \frac{x^2(x+1)^2}{4}, \\ y(x) &= 2(2^x - 1), \end{aligned}$$

for $x \in \mathbb{R}$.

The following question is naturally arising; *what about other sums can we extend in this way?*

3.2. **The sum of reciprocals (=the Harmonic numbers).** Let us consider the following function $H : \mathbb{N} \to \mathbb{R}$,

$$H(n) := 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

We aim to extend it to the reals, *i.e.,* the following diagram must be commutative

$$\begin{tikzcd} \mathbb{N} \arrow[r, "H"] \arrow[d, hook] & \mathbb{R} \\ \mathbb{R} \arrow[ur, "\widetilde{H}"'] & \end{tikzcd}$$

To do so we firstly note the following obvious formulas;

$$H(n-1) = H(n) - \frac{1}{n}, \tag{3.1}$$

$$H(n) = H(n-1) + \frac{1}{n}, \tag{3.2}$$

$$H(n+1) = H(n) + \frac{1}{n+1}, \tag{3.3}$$

indeed we have

$$H(n) = \underbrace{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}_{H(n-1)} + \frac{1}{n}.$$

Thus it follows the following

Key Point 3.1. *Formulas (3.1)–(3.3) allow us to rediscover all the harmonic numbers that we started with, based on just these formulas and starting point!*

and

Key Point 3.2. *Since (3.1)–(3.3) use only division and addition we then may plug any real numbers in them!*

We immediately get

$$H(0) = H(1) - \frac{1}{1} = 1 - 1 = 0.$$

But, we cannot obtain $H(-1)$ because of

$$H(-1) = H(0) - \frac{1}{0}$$

and we get undefined value.

From now we consider $H(x)$, where $x \in \mathbb{R}$.

Proposition 3.3. *Assume that H is defined on $[0,1]$ then there is a unique extension $\widetilde{H} : \mathbb{R} \to \mathbb{R}$ of H.*

Proof. By (3.1), (3.3),

$$\begin{aligned} H(x+1) &= H(x) + \frac{1}{x+1}, \\ H(x+2) &= H(x+1) + \frac{1}{x+2}, \end{aligned}$$

hence

$$H(x+2) = H(x) + \frac{1}{x+1} + \frac{1}{x+2},$$

therefore we obtain for any $n \in \mathbb{N}$,

$$H(x+n) = H(x) + \frac{1}{x+1} + \cdots + \frac{1}{x+n} = H(x) + \sum_{k=1}^{n} \frac{1}{x+k}. \tag{3.4}$$

It follows that if $H(x)$ is defined for any $x \in [0,1]$ then by (3.4), $H([n, n+1])$ is defined uniquely for any $n \in \mathbb{N}$, as claimed. □

Thus it is enough to define $H([0,1])$ in a "natural way". To do so we note that for large enough N we have $H(N+n) \approx H(N)$ for any $n \in \mathbb{N}$. For instance, we have

$$\begin{aligned} H(1000) &= 7.48547\ldots, \\ H(1001) &= 7.48646\ldots, \\ H(10^6) &= 14.392725\ldots, \\ H(10^6+1) &= 14.392726\ldots. \end{aligned}$$

We then may assume that a "natural way" to define $H(x)$ is following:

(1) take a large enough $N \in \mathbb{N}$ and set $H(N+x) \approx H(N)$,

(2) $H(x) := H(N+x) - \sum_{k=1}^{N} \dfrac{1}{x+k}$.

Example 3.4. Let us define $H(\frac{1}{2})$ in the such way.

By (3.4) and (2) from Proposition 2.3,

$H(\frac{1}{2}) = H(10^6 + \frac{1}{2}) - \sum_{k=1}^{10^6} \dfrac{1}{\frac{1}{2}+k} \approx H(10^6) - \sum_{k=1}^{10^6} \dfrac{1}{\frac{1}{2}+k} = 0.613705\ldots.$

After sketching the idea how $H(x)$ can be defined, we prove it in a formal way.

Theorem 3.5. *For any $x \in \mathbb{R}$,*

$$\boxed{H(x) := \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{x+k} \right).} \tag{3.5}$$

Proof. For large enough n, it is possible to say: $H_{n+x} - H_x \approx 0$

In other words: $\lim_{x\to\infty}(H_{n+x} - H_x) = 0$

Let's use formula **(3.4)** for the left harmonic number and definition of a harmonic number to the right one to see how n-harmonic number can be expressed as a limit:

$$\begin{aligned} \lim_{x\to\infty} \left(H_{n+x} - \sum_{k=1}^{x} \frac{1}{k} \right) &= 0 \\ \lim_{x\to\infty} (H_n + \sum_{k=1}^{x} \frac{1}{n+k} - \sum_{k=1}^{x} \frac{1}{k}) &= 0 \end{aligned}$$

Since n-th harmonic number does not depend on x, we can take it out of the brackets:

$$H_n + \lim_{x\to\infty} (\sum_{k=1}^{x} \frac{1}{n+k} - \sum_{k=1}^{x} \frac{1}{k}) = 0$$

We can subtract the limit from both sides of the equation and rearrange the summands inside the limit. The summations have the same number of summands and can be united:

$$\begin{aligned} H_n &= \lim_{x\to\infty} \left(\sum_{k=1}^{x} \frac{1}{k} - \sum_{k=1}^{x} \frac{1}{n+k} \right) \\ &= \lim_{x\to\infty} \left(\sum_{k=1}^{x} \frac{1}{k} - \frac{1}{n+k} \right) \end{aligned}$$

□

4. Euler's definition as an infinite product

From the definition of factorial of a number n, we multiply all natural numbers from 1 to n.

Definition 4.1. For any $n \in \mathbb{N}$,

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \tag{4.1}$$

Key Point 4.2. *By universal convention* $0! = 1$

Key Point 4.3. *Interesting observation can be made: the first $n-1$ terms are actually the $(n-1)!$, so*

$$n! = (n-1)! \cdot n \tag{4.2}$$

*Now, we can always find neighbouring factorials of any natural number, but it could be a lot easier to use one formula to find a factorial that we need instead of using recursive formula **4.2** several times.*

Proposition 4.4.

$$(x+n)! = x! \prod_{k=1}^{n} (x+k) \tag{4.3}$$

Proof. Let's look at some simple factorials and apply formula **(4.2)**:

$$\begin{aligned} 1! &= (1-1)! \cdot 1 = 0! \cdot 1 \\ 2! &= (2-1)! \cdot 2 = 1! \cdot 2 = 0! \cdot 1 \cdot 2 \\ 3! &= (3-1)! \cdot 3 = 2! \cdot 3 = 1! \cdot 2 \cdot 3 = 0! \cdot 1 \cdot 2 \cdot 3 \end{aligned}$$

Taking the induction base 1! and induction step 1, let's assume that $(x+n-1)! = x! \prod_{k=1}^{n-1}(x+k)$ is true and prove it for $(x+n)!$:

$$\begin{aligned} (x+n)! &= (x+n-1)! \cdot (x+n) \\ (x+n)! &= (x+n) \cdot x! \cdot \prod_{k=1}^{n-1}(x+k) \\ (x+n)! &= x! \cdot \prod_{k=1}^{n}(x+k) \end{aligned}$$

□

Now, that we have shown the connection between factorials of distinct numbers, we can move to real numbers, starting with the question: If we have managed to find the factorial of any natural number, is there a way to extend this definition to any $x \in \mathbb{R}$? If we graph the function that takes as an input number x and maps it to the factorial of this number, we can observe something:

Key Point 4.5. *Factorial function grows faster than exponential function (see. Fig.1)*

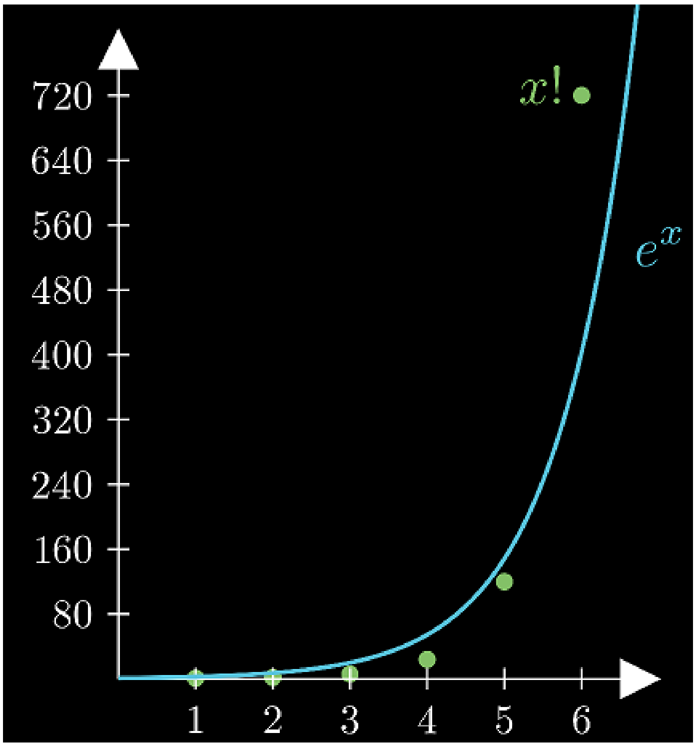

FIGURE 1

As we have previously figured it out, factorial function multiplies previous factorial by the next number and as a result the multiplications spiral out of control.

Example 4.6.

$$\begin{aligned} 4! &= 24 \\ 5! &= 4! \cdot 5 = 120 \end{aligned}$$

5! is 5 times bigger than 4!, 6! is 6 times greater than 5! and 70! is already larger than googol (10^{100}).

So, if only we could change the multiply signs between numbers into add signs, we could expect much more comfortable numbers to work with. But factorials are defined as products and that's that. And since we can draw any function that passes through the points of factorials of natural numbers we can draw any function we want (see Fig. 2) but how do we extend the factorial function to real numbers in a natural way?

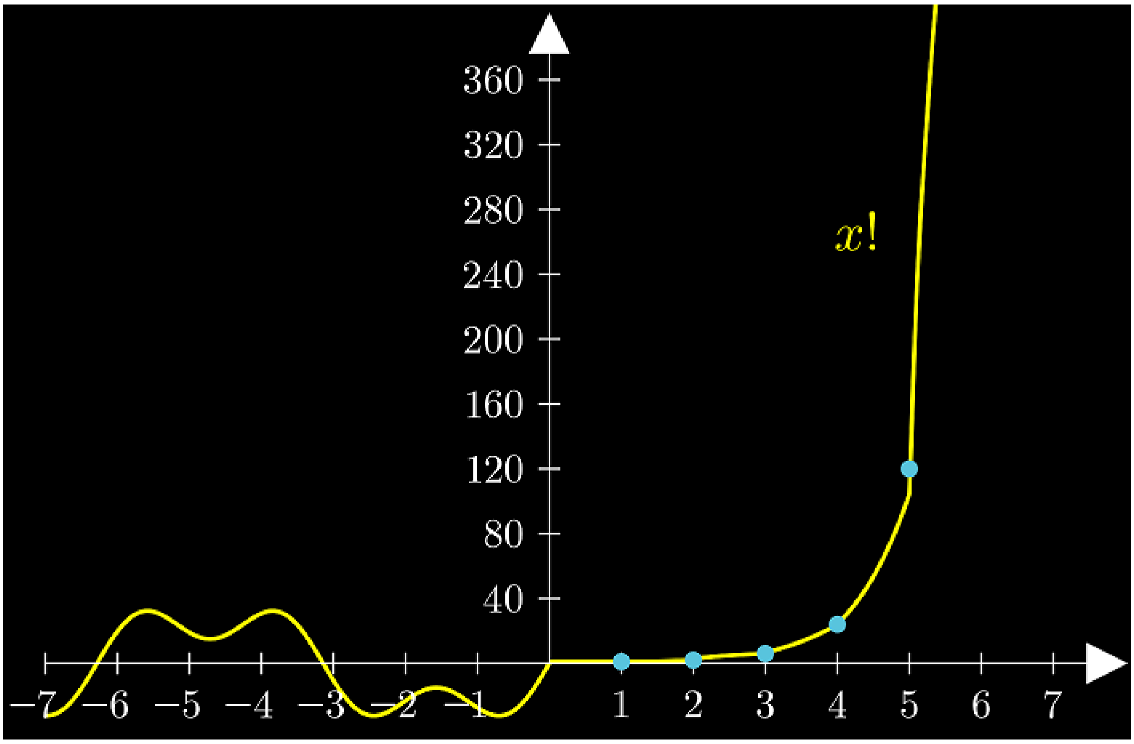

FIGURE 2

Actually, we can apply natural logarithm to the factorial function to use its property: $ln(a \cdot b) = ln(a) + ln(b)$

Key Point 4.7.

$$\ln(n!) = \sum_{k=1}^{n} \ln k \tag{4.4}$$

Indeed, let's take the natural logarithm of both sides of **(4.1)** to convert product of numbers into sum, so in the future when we exponentiate the equation, all what is left will be factorials:

$$\begin{aligned} \ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) \\ &= \ln(1) + \ln(2) + \ln(3) + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{k=1}^{n} \ln k \end{aligned}$$

Theorem 4.8.

$$\boxed{\Gamma(n) = (n-1)! = \lim_{x\to\infty} x^{n-1} \cdot \prod_{k=1}^{x} \frac{k}{n-1+k}} \tag{4.5}$$

Proof. Let's find the factorial of $n + x$, using formula **(4.4)**:

$$\ln\left((n+x)!\right) = \sum_{k=1}^{n+x} \ln k$$

We can decompose the summation into two summations, one for the first x components and the second one for the rest n summands:

$$\ln\left((n+x)!\right) = \sum_{k=1}^{x} \ln\left(k\right) + \sum_{k=1}^{n} \ln\left(x+k\right)$$

Using formula **(4.4)** for the first summation on the right-hand side of the equation:

$$\ln\left((n+x)!\right) = \ln(x!) + \sum_{k=1}^{n} \ln\left(x+k\right) \tag{4.6}$$

Applying approximation for large enough integrals:

$$\ln(n+x) \approx \ln(x) \text{ as } x \to \infty$$

Hence,

$$\ln((n+x)!) \approx \ln(x!) + \sum_{k=1}^{n} \ln x$$

In the sum, that is left, the expression does not depend on k, so it is the summation of n logarithms of x:

$$\ln((n+x)!) \approx \ln(x!) + n \cdot \ln x$$

Now we substitute the left part of the equation with formula **(4.6)** and $\ln\left(x!\right)$ with formula **(4.4)** on the right part of the equation.

$$\ln(n!) + \sum_{k=1}^{x} \ln\left(n+k\right) \approx \sum_{k=1}^{x} \ln(k) + n \cdot \ln x$$

Let's move all sums to the right part to unite them, since the number of summands is the same in both sums:

$$\ln(n!) \approx \sum_{k=1}^{x} \ln(k) - \sum_{k=1}^{x} \ln\left(n+k\right) + n \cdot \ln x$$

$$\ln(n!) \approx \sum_{k=1}^{x} \left(\ln(k) - \ln\left(n+k\right)\right) + n \cdot \ln x$$

Using the logarithms properties, yields us:

$$\ln(n!) \approx \sum_{k=1}^{x} \ln \frac{k}{n+k} + n \cdot \ln x$$

We did our approximation, saying that x tends to infinity, so taking the limit of right part of the approximation gives us the true equation for $\ln(n!)$:

$$\ln(n!) = \lim_{x\to\infty}\left(\sum_{k=1}^{x}\ln\frac{k}{n+k} + n\cdot\ln x\right)$$

Now, we exponentiate both sides of the equation, to undo the logarithm:

$$n! = \lim_{x\to\infty}\exp\left(\sum_{k=1}^{x}\ln\frac{k}{n+k} + n\cdot\ln x\right)$$

Using the logarithm property, we get:

$$n! = \lim_{x\to\infty} x^n\cdot\exp\left(\sum_{k=1}^{x}\ln\frac{k}{n+k}\right)$$

Recall that the logarithm turned our product into a sum of logarithms, exponentiation turns this sum into a product:

$$n! = \lim_{x\to\infty} x^n\cdot\prod_{k=1}^{x}\frac{k}{n+k}$$

After all the steps we get the generalized formula for $n!$ and the Gamma Function is defined as:

$$\Gamma(n) = (n-1)! = \lim_{x\to\infty} x^{n-1}\cdot\prod_{k=1}^{x}\frac{k}{n-1+k}$$

$\square$

Another definition of the Gamma Function was introduced and derived by Weierstrass:

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}(1+\frac{x}{n})e^{-x/n} \tag{4.7}$$

Example 4.9. With the Gamma function we can calculate factorial of any positive real number using generalized formula **(4.5)** or function from Python programming language (Fig. 3):

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \lim_{x\to\infty} x^{\frac{1}{2}-1}\cdot\prod_{k=1}^{x}\frac{k}{\frac{1}{2}-1+k} = \lim_{x\to\infty} x^{-\frac{1}{2}}\cdot\prod_{k=1}^{x}\frac{k}{-\frac{1}{2}+k} \approx 1.77 \\
\Gamma\left(\frac{5}{2}\right) &\approx 1.33 \\
\Gamma\left(-\frac{7}{2}\right) &\approx 2.36
\end{aligned}$$

```
import math

n = float(input())
print(math.gamma(n))
```

FIGURE 3

5. Connection with Harmonic Series

Remember we were deriving all those formulae with the Harmonic number function? As it turned out there is a beautiful connection between the Gamma Function and the Harmonic Series, when we take the logarithmic derivative of the factorial function.

Theorem 5.1.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}n}\big(\ln\Gamma(n+1)\big) = H_n - \gamma} \tag{5.1}$$

Proof. Let's take the logarithmic derivative of $n!$ and define it as a limit we have derived previously:

$$\frac{\mathrm{d}}{\mathrm{d}n}\big(\ln(n!)\big) = \frac{\mathrm{d}}{\mathrm{d}n}\Big(\lim_{x\to\infty}\left(\sum_{k=1}^{x}\ln\frac{k}{n+k} + n\cdot\ln x\right)\Big)$$

We can put the derivative operation inside the limit and find the derivative of the expression inside the sum:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}n}\big(\ln(n!)\big) &= \lim_{x\to\infty}\left(\sum_{k=1}^{x}\frac{\mathrm{d}}{\mathrm{d}n}\ln\frac{k}{n+k} + \frac{\mathrm{d}}{\mathrm{d}n}n\cdot\ln x\right)\\ &= \lim_{x\to\infty}\left(\sum_{k=1}^{x}\frac{\mathrm{d}}{\mathrm{d}n}\ln\frac{k}{n+k} + \ln x\right)\end{aligned}$$

Let's expand the natural logarithm of fraction. Since k does not depend on n, the derivative of $\ln(k)$ with respect to n is 0:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}n}\big(\ln(n!)\big) &= \lim_{x\to\infty}\left(\sum_{k=1}^{x}\frac{\mathrm{d}}{\mathrm{d}n}\ln k - \frac{\mathrm{d}}{\mathrm{d}n}\ln(n+k) + \ln x\right)\\ &= \lim_{x\to\infty}\left(\sum_{k=1}^{x} -\frac{\mathrm{d}}{\mathrm{d}n}\ln(n+k) + \ln x\right)\\ &= \lim_{x\to\infty}\left(\sum_{k=1}^{x} -\frac{1}{n+k} + \ln x\right)\end{aligned}$$

All we need is to add and subtract the x-th harmonic number inside the limit to apply formula **(3.4)**:

$$\begin{aligned}&= \lim_{x\to\infty}\left(\sum_{k=1}^{x}\frac{1}{k} + \sum_{k=1}^{x} -\frac{1}{n+k} - \sum_{k=1}^{x}\frac{1}{k} + \ln x\right)\\ &= \lim_{x\to\infty}\left(\sum_{k=1}^{x}\left(\frac{1}{k} - \frac{1}{n+k}\right) - \sum_{k=1}^{x}\frac{1}{k} + \ln x\right)\end{aligned}$$

Apply formula **(3.4)** to the first sum and definition of a harmonic number to the second sum:

$$= \lim_{x\to\infty}\left(H_n - H_x + \ln x\right)$$

Since the n-th harmonic number does not depend on x, we can take it out the limit:

$$= H_n - \lim_{x\to\infty}\left(H_x - \ln x\right)$$

, where the limit is defined as:

$\lim_{x\to\infty}(H_x - \ln x) = \gamma$ - Euler–Mascheroni constant
So,

$$\frac{\mathrm{d}}{\mathrm{d}n}\big(\ln(n!)\big) = H_n - \gamma$$

In the end, we get an equation, which advocates the connection between the Gamma Function and the Harmonic Series:

$$\frac{\mathrm{d}}{\mathrm{d}n}\big(\Gamma(n+1)\big) = H_n - \gamma$$

□

6. Euler's Definition as an Improper Integral

Theorem 6.1. *For any $x > 0$, we can define the Gamma Function with the improper integral:*

$$\boxed{\Gamma(x) = \int_0^\infty e^{-t}\cdot t^{x-1}\,dt} \tag{6.1}$$

Proof. First of all, we need to check whether this integral converges or not.
Let's split it into two improper integrals:

$$\int_0^1 e^{-t}\cdot t^{x-1}\,dt + \int_1^\infty e^{-t}\cdot t^{x-1}\,dt$$

The first integral converges since it is the integral of the second kind and is bounded from above and below: $0 < e^{-t}\cdot t^{x-1} \leq t^{x-1}$, if $t \geq 0$. To show the convergence of the second integral we can compare it to converging integral $\int_1^\infty t^{-2}\,dt$.

First, let's find the limit $\lim_{t\to+\infty} e^{-t}t^{x+1}$:

Since both enumerator and denominator tend to ∞ as t tends to ∞, we can apply L'Hôpital's rule $x+1$ times and get:

$$\lim_{t\to+\infty}(-1)^{x+1}\cdot\frac{(x+1)!}{e^{-t}}$$

which converges to 0.

Hence, $e^{-t}t^{x+1} \leq C$, where C is a constant. Dividing both sides by t^2 yields: $e^{-t}t^{x-1} \leq Ct^{-2}$. So, it is bounded above by Ct^{-2} and below by 0 meaning that the second integral also converges, thus the initial improper integral converges.

6.1. Finding the exact value of the integral.

The substitution $t = n\tau$ yields:

$$\begin{aligned}\int_0^n\left(1-\frac{t}{n}\right)^n t^{x-1}\,dt &= \int_0^1\left(1-\frac{n\tau}{n}\right)^n (n\tau)^{x-1}\,dn\tau \\ &= n^x\int_0^1 (1-\tau)^n\tau^{x-1}\,d\tau\end{aligned}$$

Let's integrate by parts:

$$\begin{aligned}
\int_0^1 (1-\tau)^n \tau^{x-1}\,d\tau &= \left[\frac{\tau^x}{x}(1-\tau)^n\right]\Bigg|_0^1 + \frac{n}{x}\int_0^1 (1-\tau)^{n-1}\tau^x\,d\tau \\
&= \frac{n}{x}\int_0^1 (1-\tau)^{n-1}\tau^x\,d\tau \\
&= \cdots \\
&= \frac{n(n-1)\cdots 1}{x(x+1)\cdots(x+n-1)}\int_0^1 \tau^{x+n-1}\,d\tau \\
&= \frac{n!}{x(x+1)\cdots(x+n)}
\end{aligned}$$

Hence,

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = n^x \frac{n!}{x(x+1)\cdots(x+n)}.$$

By looking at the formula of the Gamma Function we derived in the end of the second chapter, we can conclude:

$$\Gamma(x) = \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt$$

□

7. Properties of the Gamma Function

Proposition 7.1. ***Recursive formula***

$$\Gamma(x+1) = x\cdot\Gamma(x) \tag{7.1}$$

Proof. Indeed, applying the integral form will yield the result: $\Gamma(x+1) = \int_0^\infty e^{-t}\cdot t^x\,dt = -\int_0^\infty t^x\,de^{-t} = x\cdot\int_0^\infty e^{-t}\cdot t^{x-1}\,dt = x\cdot\Gamma(x)$

□

Proposition 7.2. ***Complement formula***

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \qquad x\in(0,1) \tag{7.2}$$

Proof. Let's start deriving using Weierstrass formula (4.7) for inverted Gamma function:

$$\begin{aligned}
\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(-x)} &= -x^2 e^{\gamma x}e^{-\gamma x}\prod_{n=1}^{\infty}(1+\frac{x}{n})e^{-x/n}\prod_{n=1}^{\infty}(1-\frac{x}{n})e^{x/n} \\
\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(-x)} &= -x^2\prod_{n=1}^{\infty}(1-\frac{x^2}{n^2})
\end{aligned}$$

Using recursive formula for $\Gamma(1-x)$ yields:

$$\Gamma(1-x) = -x\cdot\Gamma(-x)$$

$$\Gamma(-x) = -\frac{\Gamma(1-x)}{x}$$

$$\frac{1}{\Gamma(x)}\cdot\frac{1}{\Gamma(1-x)} = x\prod_{n=1}^{\infty}(1-\frac{x^2}{n^2})$$

Sin has an infinite product representation:

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2})$$

Substituting it into the upper equation yields:

$$\begin{aligned} \frac{1}{\Gamma(x)} \cdot \frac{1}{\Gamma(1-x)} &= x \cdot \frac{\sin \pi x}{\pi x} \\ \Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin \pi x} \end{aligned}$$

□

Example 7.3. This formula can yield us the exact value of $\Gamma(\frac{1}{2})$ calculated earlier as well as some other values of the Gamma function:

$$\begin{aligned} \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) &= \frac{\pi}{\sin \frac{\pi}{2}} \\ \Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\ \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) &= \frac{\pi}{\sin \frac{\pi}{3}} \\ \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) &= \frac{2\pi}{\sqrt{3}} \end{aligned}$$

8. Connection with Quantum Mechanics and String Theory

The Gamma Function has many applications in various spheres of science: astrophysics, fluid dynamics and the study of earthquakes. Moreover, the Gamma function, especially it's "twin-brother", Beta function, had found a way to describe strong force in physics.

In 1968 an Italian theoretical physicist Gabriele Veneziano observed that Euler's functions are useful at representing scattering amplitude for Regge trajectories and later in scattering amplitude in string theory. However, there was no explanation why Euler's function could manage to describe properties of strongly interacting particles.

Soon after, in 1970 three scientists: Leonard Susskind, Yoichiro Nambu, and Holger Nielsen demonstrated that if elementary particles are deemed as tiny, vibrating strings, then Beta function could easily describe their most fundamental nucleus interactions and trio showed that Veneziano's dual resonance model is based on quantum mechanics of the vibrating strings.

Although the theory was seeming to be correct and appropriate, during the 1970s, high-energy experiments in the subatomic world and their results were conflicting with the calculated predictions and the attention of scientists moved to other promising and developing fields such as quantum chromodynamics.

In 1974, two physicists, John Schwarz and Joel Scherk, proposed that if only the bosons are represented as strings then the theory could be consistent with quantum mechanics and special relativity but one also needed to assume that spacetime dimension to be 26, which was hard to believe and work with.

9. Conclusion

The set goal was achieved — the natural interpolation of the factorial function from the natural numbers to the real numbers yielded the expected result, the Gamma Function. The properties and special cases of the special function were described and proved as well as integral and infinite product forms of the function, the history of application in physics was also described. Even though this research has not given any new and unseen ways to solve mathematical problems or precisely describe the mystery behind connection of the special function with subatomic force, the research paper can be used as a introduction to the Gamma Function from the ground up.

10. References

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