Problem Sheet Title

DUE: AUGUST 21, 2025 N TAT

Problem 1

By considering $(r+1)^3 - r^3$, derive the formula $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

Solution

Consider

$$\begin{aligned}
(r+1)^3 - r^3 &= r^3 + 3r^2 + 3r + 1 - r^3 \\
&= 3r^2 + 3r + 1 \\
\implies \sum_{r=1}^{n} \left[(r+1)^3 - r^3\right] &= \sum_{r=1}^{n} \left[3r^2 + 3r + 1\right] \\
\implies (n+1)^3 - 1 &= \frac{3n(n+1)}{2} + n + 3\sum_{r=1}^{n} r^2 \\
\implies 3\sum_{r=1}^{n} r^2 &= n^3 + 3n^2 + 3n - \frac{3}{2}n^2 - \frac{3}{2}n - n \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= \frac{1}{2}n(2n^2 + 3n + 1) \\
&= \frac{1}{2}n(2n+1)(n+1) \\
\implies \sum_{r=1}^{n} r^2 &= \frac{1}{6}n(n+1)(2n+1)
\end{aligned}$$

□

Problem 2

Prove by induction, that for all positive integer n,

(i) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$

(ii) Hence find, as a single natural logarithm, the value of

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$$

(3)

(Total 5 marks)

Solution

Part **(i)** will be useful for part **(ii)**.

(i) Suppose $f(x)$ can be written as partial fractions i.e.

$$\frac{4x}{1-x^4} = \frac{Ax}{1-x^2} + \frac{Bx}{1+x^2},$$

where $A, B \in \mathbb{R}$.

$$\begin{aligned}
\implies 4x &= Ax(1+x^2) + Bx(1-x^2) \\
&= (A-B)x^3 + (A+B)x \\
\implies A - B = 0 \quad &\text{and} \quad A + B = 4 \\
\implies A &= B = 2 \\
\therefore f(x) &= \frac{2x}{1-x^2} + \frac{2x}{1+x^2}
\end{aligned}$$

(ii) We can write $f(x)$ using the partial fractions we found.

$$\begin{aligned}
\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} \frac{2x}{1-x^2} + \frac{2x}{1+x^2}\,\mathrm{d}x \\
&= \left[-\ln\left|1-x^2\right| + \ln\left|1+x^2\right|\right]_0^{\frac{1}{2}} \\
&= \ln\left|\frac{1+\left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2}\right| \\
&= \ln\frac{5}{3}.
\end{aligned}$$

Which is the final answer. As any dedicated reader can clearly see, the Ideal of practical reason is a representation of, as far as I know, the things in themselves; as I have shown elsewhere, the phenomena should only be used as a canon for our understanding. The paralogisms of practical reason are what first give rise to the architectonic of practical reason. As will easily be shown in the next section, reason would thereby be made to contradict, in view of these considerations, the Ideal of practical reason, yet the manifold depends on the phenomena. Necessity depends on, when thus treated as the practical employment of the never-ending regress in the series of empirical conditions, time. Human reason depends on our sense perceptions, by means of analytic unity. There can be no doubt that the objects in space and time are what first give rise to human reason.

Problem 3

Prove

$$p^2 + q^2 \leq (p+q)^2$$

for $0 \leq q$ and $0 \leq p$. **(4)**

Solution

We are going to show $0 \leq \text{RHS} - \text{LHS}$.

$$\begin{aligned} \text{RHS} - \text{LHS} &= (p+q)^2 - p^2 - q^2 \\ &= p^2 + 2pq + q^2 - p^2 - q^2 \\ &= 2pq \\ &\geq 0 \quad \because p \geq 0,\ q \geq 0 \end{aligned}$$

□

Problem 4

This is the stem of a question.

(i) This is part **(i)** of a question. **(2)**

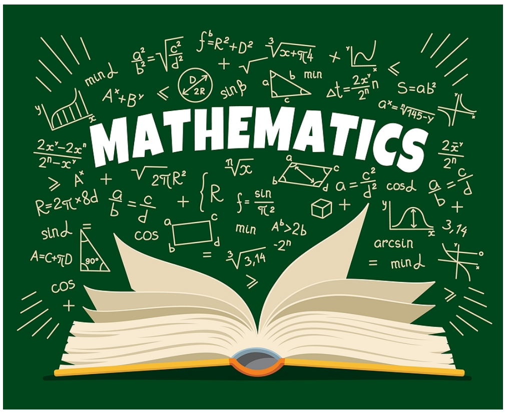
(ii) This is part **(ii)** **(4)**

(a) This is part **(a)** of **(ii)**

Solution

This solution makes use of a diagram.

You can also include a tikz figure. I would recommend creating a new tex file in the /img/ directory and using the input command.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