Mathematical Statistics

Homework 2

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C & B 7.14

Let X and Y be independent exponential random variables, with

$$f(x|\lambda) = \frac{1}{\lambda}\exp\{-x/\lambda\}, \;\; x > 0, \qquad f(y|\mu) = \frac{1}{\mu}\exp\{-y/\mu\}, \;\; y > 0.$$

We observe Z and W with

$$Z = \min(X, Y) \text{ and } W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$$

In Exercise 4.26, the joint distribution of Z and W was obtained. Now assume that $(Z_i, W_i), i = 1, \ldots, n$, are n iid observations. Find the MLEs of λ and μ.

Solution

To find the joint density, $f(z, w|\lambda, \mu)$, of Z and W, we first find $P(Z \le z, W = w)$. For $W = 1$, we have

$$\begin{aligned}
P(Z \le z, W = w) &= P(X \le z, X < Y) && (1)\\
&= \int_0^z \int_x^\infty \frac{1}{\lambda}\exp\{-x/\lambda\}\frac{1}{\mu}\exp\{-y/\mu\}\; dydx && (2)\\
&= \int_0^z \frac{1}{\lambda}\exp\{-x/\lambda\}\exp\{-x/\mu\}\; dx && (3)\\
&= \int_0^z \frac{1}{\lambda}\exp\left\{-x\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\}\; dx && (4)\\
&= -\frac{\mu}{\mu+\lambda}\left(\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\} - 1\right). && (5)
\end{aligned}$$

By symmetry, we have

$$P(Z \le z, W = 0) = -\frac{\lambda}{\mu+\lambda}\left(\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\}\right).$$

Thus, for $W = 1$, the joint density of Z and W is

$$\frac{d}{dz}\left[-\frac{\mu}{\mu+\lambda}\left(\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\}-1\right)\right] = \frac{1}{\mu}\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\},$$

and, for $W = 0$, it is

$$\frac{d}{dz}\left[-\frac{\lambda}{\mu+\lambda}\left(\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\}-1\right)\right] = \frac{1}{\lambda}\exp\left\{-z\left[\frac{\mu+\lambda}{\mu\lambda}\right]\right\}.$$

Now, the likelihood function for (Z_i, W_i) is

$$L = \prod_{i=1}^{n} f(z_i, w_i|\lambda, \mu).$$

Suppose that for m of the observations, the W_i take the value 1. Then the W_i take the value 0 for $n-m$ observations. Thus, we can write

$$L = \prod_{i=1}^{n} f(z_i, w_i|\lambda, \mu) \tag{6}$$

$$= \frac{1}{\mu^m}\frac{1}{\lambda^{n-m}}\exp\left\{-\left[\frac{\mu+\lambda}{\mu\lambda}\right]\sum_{i=1}^{n} z_i\right\}. \tag{7}$$

Therefore, the log-likelihood function is

$$\mathscr{L} = -m\log(\mu) - (n-m)\log(\lambda) - \frac{\mu+\lambda}{\mu\lambda}\sum_{i=1}^{n} z_i.$$

To find the MLEs of λ and μ, we solve $\frac{\partial}{\partial\lambda}\mathscr{L} = 0$ and $\frac{\partial}{\partial\mu}\mathscr{L} = 0$. But the solution to

$$\frac{\partial}{\partial\lambda}\mathscr{L} = -(n-m)\frac{1}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n} z_i \tag{8}$$

$$= 0 \tag{9}$$

is

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} z_i}{n-m}.$$

Similarly, the solution to $\frac{\partial}{\partial\mu}\mathscr{L} = 0$ is

$$\hat{\mu} = \frac{\sum_{i=1}^{n} z_i}{m}.$$

Using the second-derivative test, we can see that these points maximize $\mathscr{L}$. Thus, $(\hat{\lambda}, \hat{\mu})$ is the MLE of (λ, μ).

$\blacksquare$

C & B 7.19

Suppose that the random variables $Y_1, Y_2, \ldots, Y_n$ satisfy

$$Y_i = \beta x_i + \epsilon_i, \;\; i = 1, \ldots, n,$$

where $x_1, x_2, \ldots, x_n$ are fixed constants, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are iid $n(0, \sigma^2)$, with σ^2 unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2).

(b) Find the MLE of β, and show that it is an unbiased estimator of β.

(c) Find the distribution of the MLE of β.

Solution

(a) We begin by finding the distribution of each Y_i. This is easily achieved using the method of moment-generating functions. We have

$$\begin{aligned} M_{Y_i}(t) &= M_{\beta x_i + \epsilon_i}(t) && (10)\\ &= \exp\{\beta x_i t\} M_{\epsilon_i}(t) && (11)\\ &= \exp\{\beta x_i t\} \exp\left\{\sigma^2 t^2/2\right\} && (12)\\ &= \exp\left\{\beta x_i t + \sigma^2 t^2/2\right\}, && (13) \end{aligned}$$

which indicates that each $Y_i \sim n(\beta x_i, \sigma^2)$. We also note that the Y_i are independent, since the ϵ_i are independent. Thus, the joint density of $Y_1, Y_2, \ldots, Y_n$ is

$$\begin{aligned} f(\mathbf{y}|\beta, \sigma^2) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{\frac{-(y_i - \beta x_i)^2}{2\sigma^2}\right\} && (14)\\ &= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right\}, && (15) \end{aligned}$$

where $-\infty < y_i < \infty$. Next, observe that this density can be rewritten as

$$f(\mathbf{y}|\beta, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{\frac{-1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 - 2\beta\sum_{i=1}^{n} y_i x_i + \beta^2 \sum_{i=1}^{n} x_i^2\right)\right\}. \quad (16)$$

Now, define $T_1(\mathbf{y}) = \sum_{i=1}^{n} y_i^2$ and $T_2(\mathbf{y}) = \sum_{i=1}^{n} y_i x_i$. Also, let $h(\mathbf{y}) = 1$ and

$$g(T_1(\mathbf{y}), T_2(\mathbf{y})|\beta, \sigma^2) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{\frac{-1}{2\sigma^2}\left(T_1(\mathbf{y}) - 2\beta T_2(\mathbf{y}) + \beta^2 \sum_{i=1}^{n} x_i^2\right)\right\}.$$

Since we can write

$$f(\mathbf{y}|\beta,\sigma^2) = g(T_1(\mathbf{y}), T_2(\mathbf{y})|\beta,\sigma^2)h(\mathbf{y})$$

the Factorization Theorem guarantees that $(T_1(\mathbf{y}), T_2(\mathbf{y}))$ is a sufficient statistic for (β, σ^2).

(b) The log-likelihood function for the Y_i is

$$\mathscr{L}(\beta,\sigma^2|\mathbf{y}) = \log\left(\left[\frac{1}{\sqrt{2\pi}\sigma}\right]^n \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta x_i)^2\right\}\right) \tag{17}$$

$$= n\log\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta x_i)^2. \tag{18}$$

The equation $\frac{\partial}{\partial\beta}\mathscr{L}(\beta,\sigma^2|\mathbf{y}) = 0$ reduces to

$$-\sum_{i=1}^{n} y_i x_i + \beta\sum_{i=1}^{n} x_i^2 = 0 \tag{19}$$

which has the one solution

$$\hat{\beta} = \frac{\sum\limits_{i=1}^{n} y_i x_i}{\sum\limits_{i=1}^{n} x_i^2}.$$

To show that this is, in fact, the MLE of β, we note that

$$\frac{\partial^2}{\partial\beta^2}\mathscr{L}(\hat{\beta},\sigma^2|\mathbf{y}) = -\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2 < 0,$$

which indicates that, for a given value of σ^2, the log-likelihood function has a global *maximum* at $\beta = \hat{\beta}$. Therefore, $\hat{\beta}$ is the MLE of β.

Next, observe that

$$E[\hat{\beta}] = E\left[\frac{\sum\limits_{i=1}^{n} y_i x_i}{\sum\limits_{i=1}^{n} x_i^2}\right] \tag{20}$$

$$= \frac{1}{\sum\limits_{i=1}^{n} x_i^2}\sum_{i=1}^{n} x_i E[Y_i] \tag{21}$$

$$= \frac{1}{\sum\limits_{i=1}^{n} x_i^2}\sum_{i=1}^{n} x_i E[x_i\beta + \epsilon_i] \tag{22}$$

$$= \frac{\beta}{\sum\limits_{i=1}^{n} x_i^2}\sum_{i=1}^{n} x_i^2 \tag{23}$$

$$= \beta. \tag{24}$$

Thus, $\hat{\beta}$ is an unbiased estimator of β.

(c) From part (a), we know that each $Y_i \sim n(\beta x_i, \sigma^2)$. Multiplying a normal random variable by a constant k scales the mean by k and the variance by k^2. Thus, the distribution of $x_i Y_i$ is $n(\beta x_i^2, \sigma^2 x_i^2)$, and so

$$\sum_{i=1}^{n} x_i Y_i \sim n\left(\beta \sum_{i=1}^{n} x_i^2, \sigma^2 \sum_{i=1}^{n} x_i^2\right).$$

Multiplying this by $\frac{1}{\sum_{i=1}^{n} x_i^2}$ scales the mean by $\frac{1}{\sum_{i=1}^{n} x_i^2}$ and the variance by $\left(\frac{1}{\sum_{i=1}^{n} x_i^2}\right)^2$. Therefore

$$\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2} = \hat{\beta} \sim n\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right).$$

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C & B 7.20

Consider $Y_1, Y_2, \ldots, Y_n$ as defined in Exercise 7.19.

(a) Show that $\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i$ is an unbiased estimator of β.

(b) Calculate the exact variance of $\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i$ and compare it to the variance of the MLE.

Solution

(a) Let $\widetilde{\beta}_1 = \sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i$. We have

$$
\begin{aligned}
E[\widetilde{\beta}_1] &= E\left[\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i\right] && (25)\\
&= \frac{1}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} E[Y_i] && (26)\\
&= \frac{1}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} E[\beta x_i + \epsilon_i] && (27)\\
&= \frac{\beta}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} x_i && (28)\\
&= \beta. && (29)
\end{aligned}
$$

Therefore, $\widetilde{\beta}_1$ is an unbiased estimator of β.

(b) Since the Y_i are independent, the variance of $\widetilde{\beta}_1$ is

$$
\begin{aligned}
\mathrm{Var}[\widetilde{\beta}_1] &= \mathrm{Var}\left[\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i\right] && (30)\\
&= \left(\frac{1}{\sum_{i=1}^{n} x_i}\right)^2 \sum_{i=1}^{n} \mathrm{Var}[Y_i] && (31)\\
&= \frac{n\sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2}. && (32)
\end{aligned}
$$

We will show that this is greater than the variance of the $\hat{\beta}$ (the MLE of β). By Hölder's Inequality,

$$
\sum_{i=1}^{n} |x_i| \leq \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} 1^2\right)^{1/2}.
$$

So

$$\left(\sum_{i=1}^{n} |x_i|\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} 1\right) = n\sum_{i=1}^{n} x_i^2.$$

But clearly,

$$\left(\sum_{i=1}^{n} x_i\right)^2 \le \left(\sum_{i=1}^{n} |x_i|\right)^2.$$

Therefore,

$$\left(\sum_{i=1}^{n} x_i\right)^2 \le n\sum_{i=1}^{n} x_i^2.$$

Thus,

$$\mathrm{Var}[\widetilde{\beta}_1] = \frac{n\sigma^2}{\left(\sum\limits_{i=1}^{n} x_i\right)^2} \ge \frac{n\sigma^2}{n\sum\limits_{i=1}^{n} x_i^2} = \frac{\sigma^2}{\sum\limits_{i=1}^{n} x_i^2} = \mathrm{Var}[\hat{\beta}].$$

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C & B 7.21

Again, let $Y_1, Y_2, \ldots, Y_n$ be as defined in Exercise 7.19.

(a) Show that $\left[\sum_{i=1}^{n}(Y_i/x_i)\right]/n$ is an unbiased estimator of β.

(b) Calculate the exact variance of $\left[\sum_{i=1}^{n}(Y_i/x_i)\right]/n$ and compare it to the variances of the estimators in the previous two exercises.

Solution

(a) Let $\widetilde{\beta}_2 = \left[\sum_{i=1}^{n}(Y_i/x_i)\right]/n$. We have

$$
\begin{aligned}
E[\widetilde{\beta}_2] &= E\left[\left(\sum_{i=1}^{n}(Y_i/x_i)\right)/n\right] && (33)\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}E[Y_i] && (34)\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\beta x_i && (35)\\
&= \frac{1}{n}\sum_{i=1}^{n}\beta && (36)\\
&= \frac{1}{n}n\beta && (37)\\
&= \beta. && (38)
\end{aligned}
$$

Therefore, $\widetilde{\beta}_2$ is an unbiased estimator of β.

(b) The variance of $\widetilde{\beta}_2$ is

$$
\begin{aligned}
\mathrm{Var}[\widetilde{\beta}_2] &= \mathrm{Var}\left[\left(\sum_{i=1}^{n}(Y_i/x_i)\right)/n\right] && (39)\\
&= \frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{x_i^2}\mathrm{Var}[Y_i] && (40)\\
&= \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\frac{1}{x_i^2}. && (41)
\end{aligned}
$$

It can be shown that $\mathrm{Var}[\widetilde{\beta}_2] \geq \mathrm{Var}[\hat{\beta}]$. By Hölder's Inequality, we have

$$
n = \sum_{i=1}^{n}\left|\frac{x_i}{x_i}\right| \leq \left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right)^{1/2}\left(\sum_{i=1}^{n}x_i^2\right)^{1/2}
$$

This can be rewritten as

$$\frac{1}{n^2}\sum_{i=1}^{n}\frac{1}{x_i^2} \geq \frac{1}{\sum\limits_{i=1}^{n} x_i^2}.$$

Multiplication by σ^2 on both sides of the above inequality yields

$$\frac{\sigma^2}{n^2}\sum_{i=1}^{n}\frac{1}{x_i^2} = \operatorname{Var}[\widetilde{\beta}_2] \geq \frac{\sigma^2}{\sum\limits_{i=1}^{n} x_i^2} = \operatorname{Var}[\hat{\beta}].$$

We can also show that $\operatorname{Var}[\widetilde{\beta}_2] \geq \operatorname{Var}[\widetilde{\beta}_1]$ by the method of Lagrange Multipliers. Define a constraint function

$$g(\mathbf{x}) = \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i^2}} - c = 0,$$

where $c \in \mathbb{R}$. Furthermore, define

$$f(\mathbf{x}) = \bar{x}^2.$$

Solving the equation

$$\nabla f(\mathbf{x}) = \left\langle \frac{2\bar{x}}{n}, \frac{2\bar{x}}{n}, \ldots, \frac{2\bar{x}}{n} \right\rangle = \lambda \nabla g(\mathbf{x}) = \left\langle \frac{2\lambda}{x_1\left(\sum\frac{1}{x_i^2}\right)^2}, \frac{2\lambda}{x_2\left(\sum\frac{1}{x_i^2}\right)^2}, \ldots, \frac{2\lambda}{x_n\left(\sum\frac{1}{x_i^2}\right)^2} \right\rangle$$

for $\mathbf{x}$ yields

$$x_1 = x_2 = \ldots = x_n.$$

Thus, subject to the constraint g, the function f is minimized when $x_1 = x_2 = \ldots = x_n$. We crucially note that when $x_1 = x_2 = \ldots = x_n$ (i.e., when f is *minimized*), we have

$$\bar{x}^2 = \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i^2}}.$$

Put another way, it is always the case that

$$\bar{x}^2 \geq \frac{n}{\sum\limits_{i=1}^{n}\frac{1}{x_i^2}}.$$

But this implies that

$$\frac{1}{\bar{x}^2} = \frac{n^2}{\left(\sum\limits_{i=1}^{n} x_i\right)^2} \leq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i^2}.$$

Dividing by n and multiplying by σ^2 finally yields

$$\frac{n\sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2} = \mathrm{Var}\left[\sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} x_i\right] \leq \frac{\sigma^2}{n^2} \sum_{i=1}^{n} \frac{1}{x_i^2} = \mathrm{Var}\left[\left(\sum_{i=1}^{n} (Y_i/x_i)\right)/n\right].$$

We can now conclude that the estimator of β given in this problem is the least desirable of the three estimators studied in this homework (since it has the highest variance).

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C & B 7.44

Let $X_1, X_2, \ldots, X_n$ be iid $n(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\bar{X} - \frac{1}{n}$. Calculate its variance (use Stein's Identity from Section 3.6), and show that it is greater than the Crámer-Rao Lower Bound.

Solution

We begin by showing that $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ. Let $t(x) = x$. Then we can write $T(\mathbf{X}) = \sum_{i=1}^{n} t(X_i)$. Now, the probability density function of each X_i is

$$f(x|\theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-(x-\theta)^2/2\right\} \tag{42}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2} + \theta x - \frac{\theta^2}{2}\right\} \tag{43}$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2} + \theta t(x) - \frac{\theta^2}{2}\right\} \tag{44}$$

for $-\infty < x < \infty$. Since $f(x|\theta)$ belongs to an exponential family of probability density functions, Theorems 6.2.10 and 6.2.25 (Casella and Berger section 6.2) guarantee that $\sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} X_i = T(\mathbf{X})$ is a complete sufficient statistic for θ.

Now, let $\phi(T(\mathbf{X})) = \frac{T(\mathbf{X})^2}{n^2} - \frac{1}{n} = \bar{X}^2 - \frac{1}{n}$. If we can show that $E[\phi(T(\mathbf{X}))] = \theta^2$, we can use Theorem 7.3.23 (Casella and Berger section 7.3) to conclude that $\phi(T(\mathbf{X}))$ is the unique best unbiased estimator of θ^2. We have

$$E[\phi(T(\mathbf{X}))] = E\left[\bar{X}^2 - \frac{1}{n}\right] \tag{45}$$

$$= E[\bar{X}^2] - \frac{1}{n}. \tag{46}$$

By Theorem 5.3.1 (Casella and Berger section 5.3), the distribution of $\bar{X}$ is $n\left(\theta, \frac{1}{n}\right)$. Thus,

$$E[\bar{X}^2] = \mathrm{Var}[\bar{X}] + E^2[\bar{X}] \tag{47}$$

$$= \frac{1}{n} + \theta^2. \tag{48}$$

Therefore,

$$E[\phi(T(\mathbf{X}))] = \frac{1}{n} + \theta^2 - \frac{1}{n} = \theta^2.$$

Hence, $\phi(T(\mathbf{X})) = \bar{X}^2 - \frac{1}{n}$ is the best unbiased estimator of θ^2.

The variance of $\phi(T(\mathbf{X}))$ is

$$\begin{aligned}
\mathrm{Var}[\phi(T(\mathbf{X}))] &= E\left[\left(\bar{X}^2 - \frac{1}{n}\right)^2\right] - E^2\left[\bar{X}^2 - \frac{1}{n}\right] && (49)\\
&= E\left[\bar{X}^4 - \frac{2}{n}\bar{X}^2 + \frac{1}{n^2}\right] - \theta^4 && (50)\\
&= E\left[\bar{X}^4\right] - \frac{2}{n}E\left[\bar{X}^2\right] + \frac{1}{n^2} - \theta^4. && (51)
\end{aligned}$$

By Stein's Lemma,

$$\begin{aligned}
E[\bar{X}^4] &= E[\bar{X}^3(\bar{X} - \theta + \theta)] && (52)\\
&= E[\bar{X}^3(\bar{X} - \theta)] + \theta E[\bar{X}^3] && (53)\\
&= \frac{3}{n}E[\bar{X}^2] + \theta E[\bar{X}^2(\bar{X} - \theta + \theta)] && (54)\\
&= \frac{3}{n}\left(\frac{1}{n} + \theta^2\right) + \theta E[\bar{X}^2(\bar{X} - \theta)] + \theta^2 E[\bar{X}^2] && (55)\\
&= \frac{3}{n^2} + \frac{3\theta^2}{n} + \frac{2\theta}{n}E[\bar{X}] + \theta^2\left(\frac{1}{n} + \theta^2\right) && (56)\\
&= \frac{3}{n^2} + \frac{3\theta^2}{n} + \frac{2\theta^2}{n} + \frac{\theta^2}{n} + \theta^4 && (57)
\end{aligned}$$

Thus, the variance of $\phi(T(\mathbf{X}))$ is

$$\begin{aligned}
\mathrm{Var}[\phi(T(\mathbf{X}))] &= \frac{3}{n^2} + \frac{3\theta^2}{n} + \frac{2\theta^2}{n} + \frac{\theta^2}{n} + \theta^4 - \frac{2}{n}\left(\frac{1}{n} + \theta^2\right) + \frac{1}{n^2} - \theta^4 && (58)\\
&= \frac{2}{n^2} + \frac{4\theta^2}{n}. && (59)
\end{aligned}$$

To compute the Crámer-Rao Lower Bound for variances of estimators of $\tau(\theta) = \theta^2$, we require the second partial derivative (with respect to θ) of the log-likelihood function of $X_1, X_2, \ldots, X_n$. We have

$$\begin{aligned}
\mathscr{L}(\theta|\mathbf{x}) &= \log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left\{-(x_i - \theta)/2\right\}\right) && (60)\\
&= n\log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{i=1}^{n}(x_i - \theta)^2. && (61)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^2 \mathscr{L}(\theta|\mathbf{x})}{\partial\theta^2} &= \frac{\partial}{\partial\theta}\sum_{i=1}^{n}(x_i - \theta) && (62)\\
&= -\sum_{i=1}^{n} 1 && (63)\\
&= -n. && (64)
\end{aligned}$$

With this, the Crámer-Rao Lower Bound is

$$\begin{aligned}
\text{CRLB} &= \frac{\left(\frac{\partial}{\partial\theta}\tau(\theta)\right)^2}{-E\left[\frac{\partial^2}{\partial\theta^2}\mathscr{L}(\theta|\mathbf{x})\right]} && (65)\\
&= \frac{4\theta^2}{-E[-n]} && (66)\\
&= \frac{4\theta^2}{n}. && (67)
\end{aligned}$$

Clearly $\text{Var}[\phi(T(\mathbf{X}))] > \text{CRLB}$.

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