Mathematical Statistics
Homework 3

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C & B 8.12

For samples of size $n = 1, 4, 16, 64, 100$ from a normal population with mean μ and known variance σ^2, plot the power function of the following LRTs. Take $\alpha = 0.05$.

(a) $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$

(b) $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$

Solution

(a) Following Example 8.3.3 (from Casella and Berger page 384), we see that the power function for this hypothesis test is

$$\beta(\mu) = P\left(Z > c + \frac{\mu_0 - \mu}{\sigma/\sqrt{n}}\right) \tag{1}$$

$$= P\left(Z > c - \frac{\mu}{\sigma/\sqrt{n}}\right). \tag{2}$$

Our task now is to find the value of c that forces the maximum probability of committing a Type I Error to be $\alpha = 0.05$. To this end, let $M_0 = (-\infty, 0]$ be the parameter space for the null hypothesis. If we force

$$\beta(\sup M_0) = \beta(0) = 0.05,$$

then, due to the fact that $\beta(\mu)$ is an increasing function, we will have $\beta(\mu) \leq 0.05$ for all $\mu \in M_0$. But then we have guaranteed that the maximum probability of committing a Type I Error is 0.05. Now, since we require $\beta(0) = P(Z > c) = 1 - P(Z \leq c) = 0.05$, we must choose $c = 1.645$ (obtained from the Standard Normal Probability Table). Therefore, the power function for this test is

$$\beta(\mu) = P\left(Z > 1.645 - \frac{\mu}{\sigma/\sqrt{n}}\right).$$

Plots of this function for $n = 1, 4, 16, 64, 100$ are provided on the next page. Note that for graphing purposes, we took $\sigma = 1$.

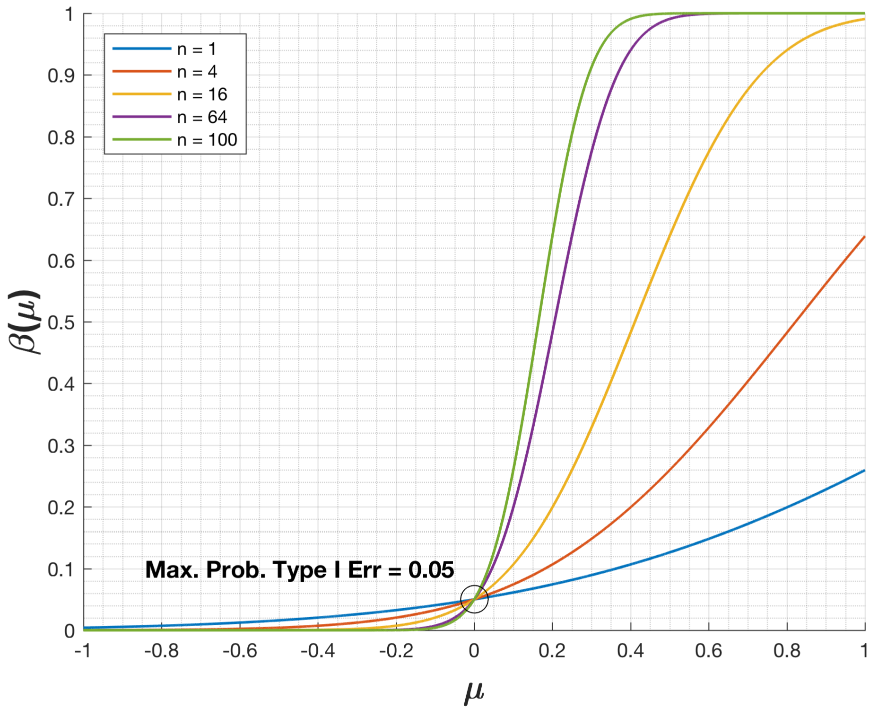

Figure 1: Power function for the test $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$

From the plot above, it is clear that, as n increases, the probability of committing a Type II Error decreases uniformly.

(b) The power function for this test is

$$\beta(\mu) = P\left(Z \geq c - \frac{\mu}{\sigma/\sqrt{n}} \text{ or } Z \leq -c - \frac{\mu}{\sigma/\sqrt{n}}\right) \tag{3}$$

$$= 1 - P\left(-c - \frac{\mu}{\sigma/\sqrt{n}} \leq Z \leq c - \frac{\mu}{\sigma/\sqrt{n}}\right) \tag{4}$$

$$= 1 - P\left(Z \leq c - \frac{\mu}{\sigma/\sqrt{n}}\right) + P\left(Z \leq -c - \frac{\mu}{\sigma/\sqrt{n}}\right). \tag{5}$$

We require that

$$\beta(0) = 1 - P(Z \leq c) + P(Z \leq -c) \tag{6}$$

$$= P(Z > c) + P(Z \leq -c) \tag{7}$$

$$= 2P(Z \leq -c) \tag{8}$$

$$= 0.05, \tag{9}$$

in order to control the maximum Type I Error probability. But this means $c = 1.96$ (from the Standard Normal Probability Table). Therefore, the power function of this hypothesis test is

$$\beta(\mu) = 1 - P\left(Z \leq 1.96 - \frac{\mu}{\sigma/\sqrt{n}}\right) + P\left(Z \leq -1.96 - \frac{\mu}{\sigma/\sqrt{n}}\right).$$

Plots of this function for $n = 1, 4, 16, 64, 100$ are provided below.

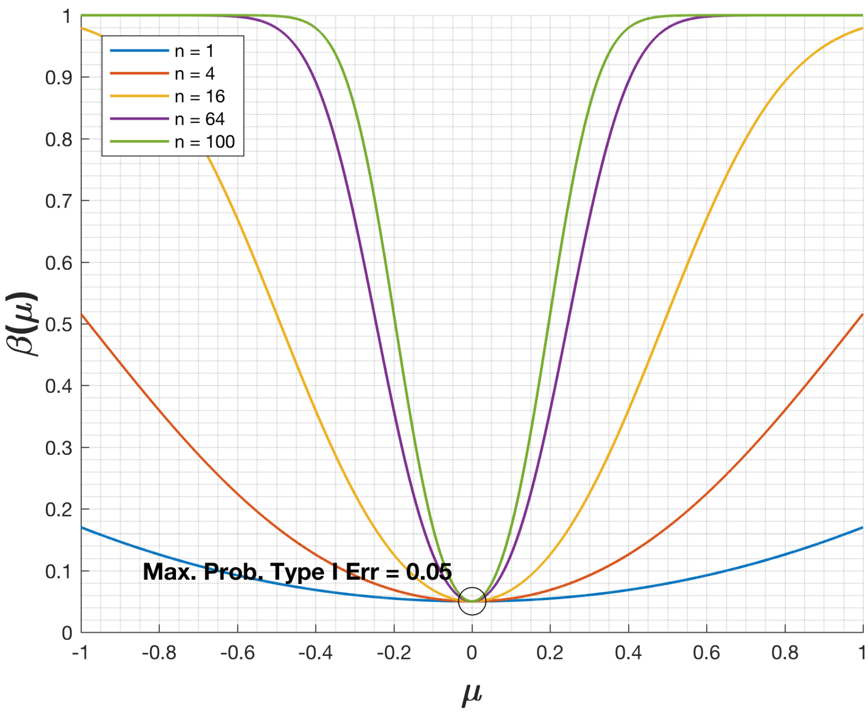

Figure 2: Power function for the test $H_0 : \mu = 0$ versus $H_1 : \mu \neq 0$

Again, we see that, as n increases, the probability of committing a Type II Error decreases uniformly.

$\blacksquare$

C & B 8.13

Let X_1, X_2 be iid uniform$(\theta, \theta+1)$. For testing $H_0 : \theta = 0$ versus $H_1 : \theta > 0$, we have two competing tests:

$$\phi_1(X_1): \text{ Reject } H_0 \text{ if } X_1 > 0.95,$$
$$\phi_2(X_1, X_2): \text{ Reject } H_0 \text{ if } X_1 + X_2 > C$$

(a) Find the value of C so that ϕ_2 has the same size as ϕ_1.

(b) Calculate the power function of each test. Draw a well-labeled graph of each power function.

(c) Prove or disprove: ϕ_2 is a more powerful test than ϕ_1.

(d) Show how to get a test that has the same size but is more powerful than ϕ_2.

Solution

(a) Let $\beta_1(\theta) = P_\theta(X > 0.95)$ be the power function for ϕ_1. The size of ϕ_1 is

$$\begin{aligned} \beta_1(0) &= P_0(X_1 > 0.95) && (10)\\ &= \int_{0.95}^{1} 1 \ dx_1 && (11)\\ &= 0.05. && (12) \end{aligned}$$

Now let $\beta_2(\theta) = P_\theta(X_1 + X_2 > C) = 1 - P_\theta(X_1 + X_2 \leq C)$ be the power function for ϕ_2. The size of ϕ_2 is $\beta_2(0) = 1 - P_0(X_1 + X_2 \leq C)$. Examine the figure below.

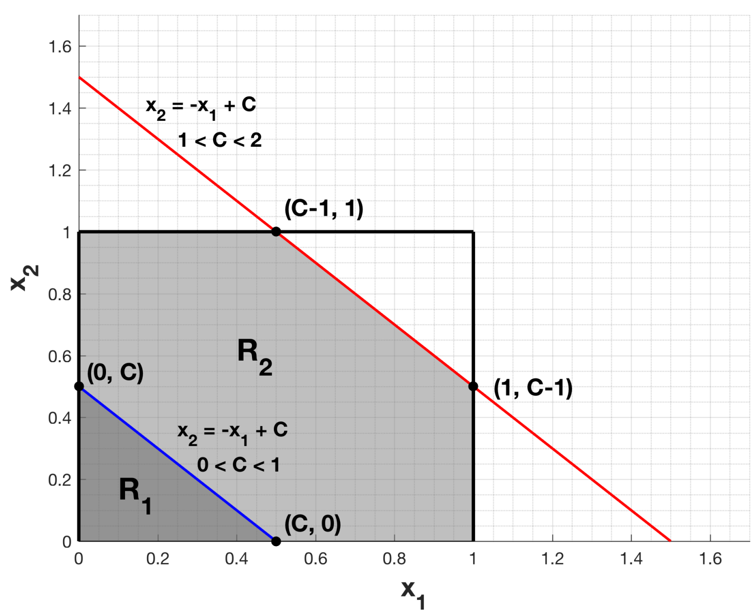

Figure 3: Regions of Integration for calculating $P_0(X_1 + X_2 \leq C)$

This figure shows the regions of integration for calculating $P_0(X_1 + X_2 \leq C)$. If we require $\beta_2(0) = 0.05$, it must be the case that $P_0(X_1 + X_2 \leq C) = 0.95$. Now, since the joint pdf of X_1 and X_2 is $f(x_1, x_2) = 1$

for $0 \leq x_1, x_2 \leq 1$ (0, otherwise), it follows that calculating $P_0(X_1 + X_2 \leq C)$ is the same as calculating the area of $\mathbf{R}_1$ or $\mathbf{R}_2$ (depending on the value of C). It is clear from the figure, however, that the area of $\mathbf{R}_1$ **cannot** be 0.95. The largest it can be is 0.5, i.e., half the area of the unit square drawn in the figure. Thus, our region of integration must be $\mathbf{R}_2$, which means C must be some number between 1 and 2. So we have

$$\begin{aligned}
\beta_2(0) &= P_0(X_1 + X_2 > C) && (13)\\
&= 1 - P_0(X_1 + X_2 \leq C) && (14)\\
&= 1 - \left(1 - \int_{C-1}^{1} \int_{C-x_2}^{1} 1\, dx_1 dx_2\right) && (15)\\
&= \int_{C-1}^{1} \left[(1-C) + x_2\right]\, dx_2 && (16)\\
&= (1-C) + 0.5 + (1-C)^2 - 0.5(1-C)^2. && (17)
\end{aligned}$$

Setting this equal to 0.05 and rearranging yields the quadratic equation

$$C^2 - 4C + 3.9 = 0,$$

which has solutions $C_1 \approx 1.68377$ and $C_2 \approx 2.31623$. Clearly C cannot be greater than 2, else $P_0(X_1+X_2 > C) = 0$. Thus, we choose $C = C_1 \approx 1.68377$.

(b) The power function of ϕ_1 is $\beta_1(\theta) = P_\theta(X_1 > 0.95)$. If $\theta > 0.95$, then X_1 is necessarily greater than 0.95, so $P_\theta(X_1 > 0.95) = 1$. If $\theta < -0.05$, then it is impossible for X_1 to be greater than 0.95, so $P_\theta(X_1 > 0.95) = 0$. Finally, if $-0.05 < \theta < 0.95$, then

$$\begin{aligned}
P_\theta(X_1 > 0.95) &= \int_{0.95}^{\theta+1} 1\, dx_1 && (18)\\
&= \theta + 1 - 0.95 && (19)\\
&= \theta + 0.05. && (20)
\end{aligned}$$

In summary,

$$\beta_1(\theta) = \begin{cases} 0 & \text{if } \theta < -0.05 \\ \theta + 0.05 & \text{if } -0.05 < \theta < 0.95 \\ 1 & \text{if } \theta > 0.95. \end{cases}$$

A plot of $\beta_1(\theta)$ is given on the next page.

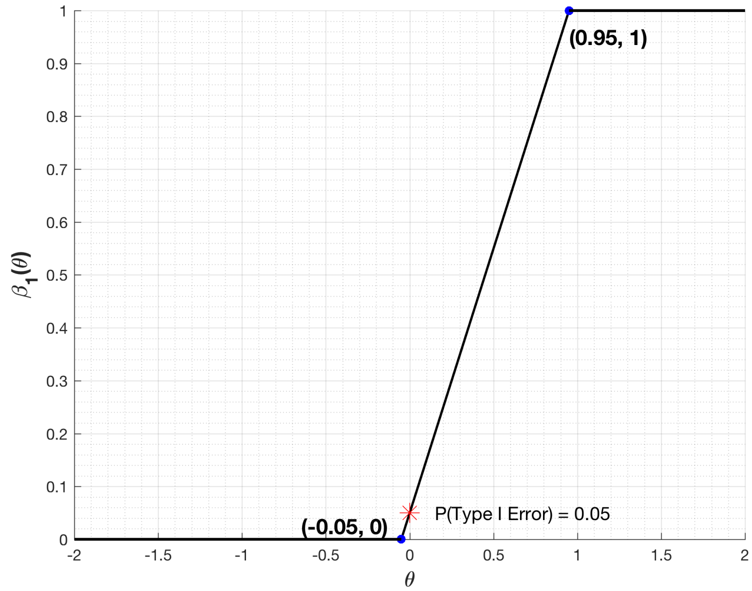

Figure 4: Plot of $\beta_1(\theta)$

Now $\beta_2(\theta) = 1 - P_\theta(X_1 + X_2 \leq 1.68377)$ will also be a piecewise function. Note that the smallest $X_1 + X_2$ can be is 2θ, and the largest it can be is $2\theta + 2$. If $\theta > 1.68377/2 = 0.8419$, then $P_\theta(X_1 + X_2 \leq 1.68377) = 0$ and $\beta_2(\theta) = 1$. If $\theta \leq \frac{1.68377-2}{2} = -0.1581$, then $P_\theta(X_1 + X_2 \leq 1.68377) = 1$, and $\beta_2(\theta) = 0$. The figure below will help us understand the behavior of $\beta_2(\theta)$ for $-0.1581 \leq \theta \leq 0.8419$.

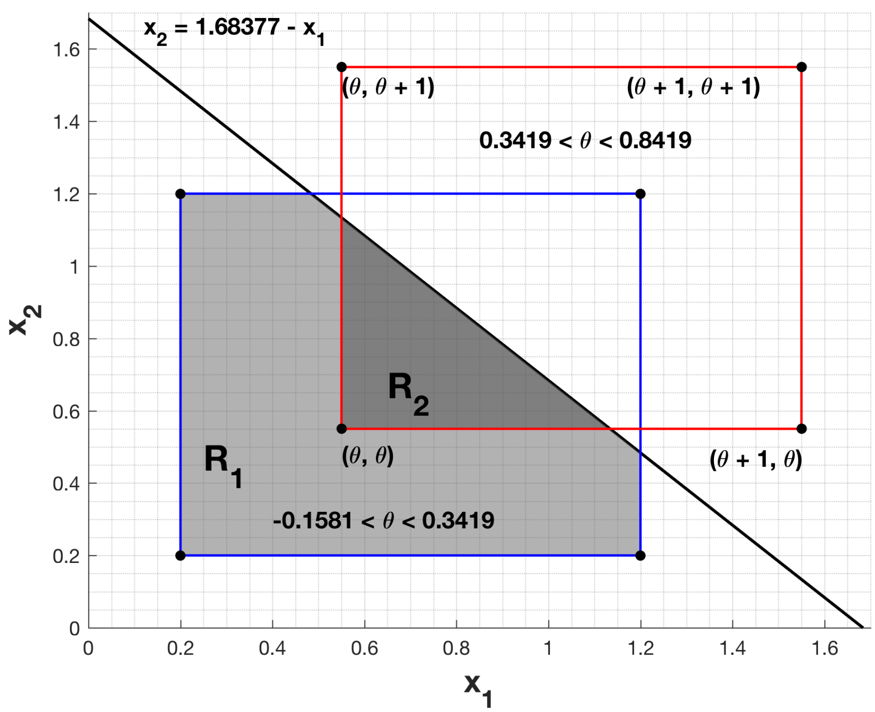

Figure 5: Regions of integration for $-0.1581 \leq \theta \leq 0.8419$

As can be seen in the figure, when $-0.1581 \leq \theta \leq 0.3419$,

$$\begin{aligned}
\beta_2(\theta) &= P_\theta(X_1 + X_2 > 1.68377) && (21)\\
&= \int\limits_{1.68377-(\theta+1)}^{\theta+1} \int\limits_{1.68377-x_2}^{\theta+1} 1 \, dx_1 dx_2 && (22)\\
&= \int\limits_{0.68377-\theta}^{\theta+1} (\theta - 0.68377 + x_2) \, dx_2 && (23)\\
&= \left[(\theta - 0.68377)x_2 + \frac{x_2^2}{2}\right]_{0.68377-\theta}^{\theta+1} && (24)\\
&= \frac{[1.68377 - 2(\theta+1)]^2}{2} && (25)\\
&= \frac{(0.31623 + 2\theta)^2}{2}. && (26)
\end{aligned}$$

For $0.3419 \leq \theta \leq 0.8419$,

$$\begin{aligned}
\beta_2(\theta) &= P_\theta(X_1 + X_2 > 1.68377) && (27)\\
&= 1 - P_\theta(X_1 + X_2 \leq 1.68377) && (28)\\
&= 1 - \int\limits_{\theta}^{1.68377-\theta} \int\limits_{\theta}^{1.68377-x_2} 1 \, dx_1 dx_2 && (29)\\
&= 1 - \int\limits_{\theta}^{1.68377-\theta} (1.68377 - x_2 - \theta) \, dx_2 && (30)\\
&= 1 - \left[(1.68377-\theta)^2 - \frac{(1.68377-\theta)^2}{2}\right] + \left[(1.68377-\theta)\theta - \frac{\theta^2}{2}\right] && (31)\\
&= \frac{-0.83508 + 6.735508\theta - 4\theta^2}{2}. && (32)
\end{aligned}$$

In summary,

$$\beta_2(\theta) = \begin{cases} 0 & \text{if } \theta \leq -0.1581 \\ \frac{(0.31623+2\theta)^2}{2} & \text{if } -0.1581 \leq \theta \leq 0.3419 \\ \frac{-0.83508+6.735508\theta-4\theta^2}{2} & \text{if } 0.3419 \leq \theta \leq 0.8419 \\ 1 & \text{if } \theta \geq 0.8419. \end{cases}$$

A plot of $\beta_2(\theta)$ is given on the next page.

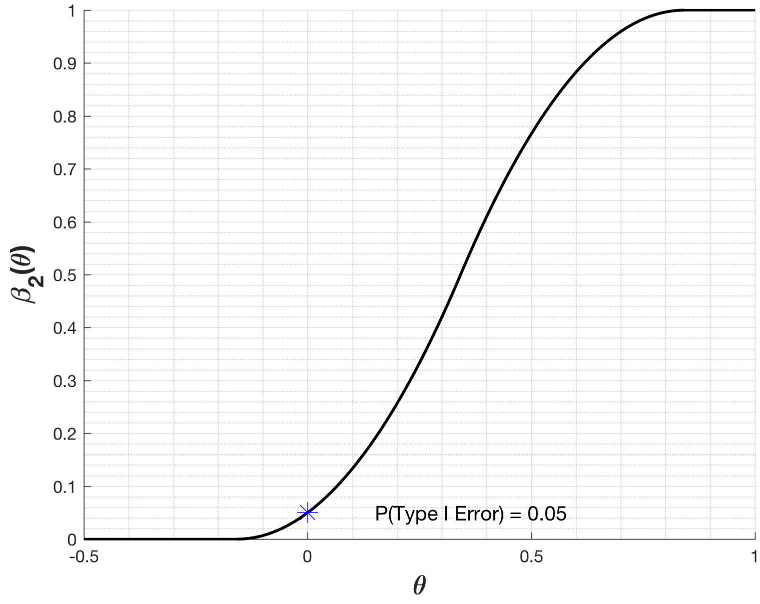

Figure 6: Plot of $\beta_2(\theta)$

(c) A plot comparing $\beta_1(\theta)$ and $\beta_2(\theta)$ is provided below.

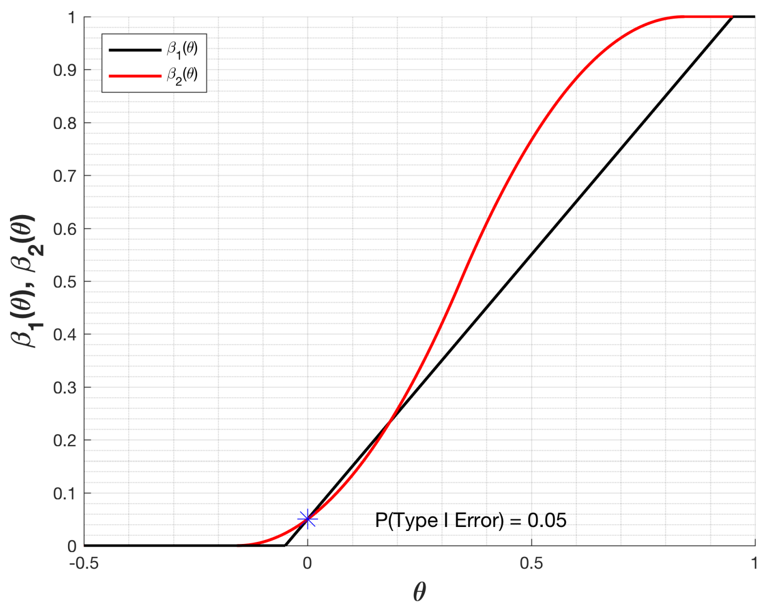

Figure 7: Comparison of $\beta_1(\theta)$ and $\beta_2(\theta)$

From the plot above, it is clear that ϕ_2 is not uniformly more powerful than ϕ_1.

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C & B 8.14

For a random sample $X_1, X_2, \ldots, X_n$ of Bernoulli(p) variables, it is desired to test

$$H_0 : p = 0.49 \text{ versus } H_1 : p = 0.51.$$

Use the Central Limit Theorem to determine, approximately, the sample size needed so that the two probabilities of error are both about 0.01. Use a test function that rejects H_0 if $\sum_{i=1}^{n} X_i$ is large.

Solution

By the Central Limit Theorem, the distribution of $\frac{\sqrt{n}(\bar{X}_n - p)}{\sqrt{p(1-p)}}$ is approximately standard normal for large n. But we can write

$$\frac{\sqrt{n}(\bar{X}_n - p)}{\sqrt{p(1-p)}} = \frac{\sqrt{n}}{\sqrt{p(1-p)}} \left(\frac{1}{n} \sum_{i=1}^{n} X_i - p \right) \tag{33}$$

$$= \frac{\sqrt{n}}{\sqrt{p(1-p)}} \left(\frac{\sum_{i=1}^{n} X_i - np}{n} \right) \tag{34}$$

$$= \frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}}. \tag{35}$$

Now, the power function for this hypothesis test is

$$\beta(p) = P_p\left(\sum_{i=1}^{n} X_i > c \right) \tag{36}$$

$$= P_p\left(\frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} > \frac{c - np}{\sqrt{np(1-p)}} \right) \tag{37}$$

$$\approx P\left(Z > \frac{c - np}{\sqrt{np(1-p)}} \right) \tag{38}$$

$$= 1 - P\left(Z \leq \frac{c - np}{\sqrt{np(1-p)}} \right), \tag{39}$$

where Z has the standard normal distribution. The probability of committing a Type I Error is

$$\beta(0.49) = 1 - P\left(Z \leq \frac{c - 0.49n}{\sqrt{n(0.49)(1-0.49)}} \right)$$

and the probability of committing a Type II Error is

$$1 - \beta(0.51) = P\left(Z \leq \frac{c - 0.51n}{\sqrt{n(0.51)(1-0.51)}} \right).$$

We want both of these probabilities to be approximately 0.01, which gives us the two equations

$$P\left(Z \leq \frac{c - 0.49n}{\sqrt{n(0.49)(1-0.49)}}\right) = 0.99$$

$$P\left(Z \leq \frac{c - 0.51n}{\sqrt{n(0.51)(1-0.51)}}\right) = 0.01.$$

Using the Standard Normal Probability Table, these equations become

$$\frac{c - 0.49n}{\sqrt{n(0.49)(1-0.49)}} = 2.33$$

$$\frac{c - 0.51n}{\sqrt{n(0.51)(1-0.51)}} = -2.33.$$

The solution to this system of equations is $(c, n) \approx (6783.5, 13567)$. Thus, a sample of size $n = 13567$ is required to achieve the specified error probabilities.

■

C & B 8.15

Show that for a random sample $X_1, X_2, \ldots, X_n$ from a $n(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$, is given by

$$\phi\left(\sum X_i^2\right) = \begin{cases} 1 & \text{if } \sum X_i^2 > c \\ 0 & \text{if } \sum X_i^2 \leq c. \end{cases}$$

For a given value of α, the size of the Type I Error, show how the value of c is explicitly determined.

Solution

Let f be the pdf of the X_i. Suppose

$$f(\mathbf{x}|\sigma_1) > kf(\mathbf{x}|\sigma_0),$$

where $k \geq 0$. Then

$$\begin{aligned} \frac{f(\mathbf{x}|\sigma_1)}{f(\mathbf{x}|\sigma_0)} &= \frac{\left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^n \exp\left\{\frac{-1}{2\sigma_1}\sum_{i=1}^n X_i^2\right\}}{\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\left\{\frac{-1}{2\sigma_0}\sum_{i=1}^n X_i^2\right\}} && (40) \\ &= \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left\{\left(\frac{1}{2\sigma_0} - \frac{1}{2\sigma_1}\right)\sum_{i=1}^n X_i^2\right\} && (41) \\ &> k. && (42) \end{aligned}$$

Multiplying by $(\sigma_0/\sigma_1)^n$ on both sides of the inequality and taking the natural logarithm yields

$$\left(\frac{1}{2\sigma_0} - \frac{1}{2\sigma_1}\right)\sum_{i=1}^n X_i^2 > \ln\left[\left(\frac{\sigma_1}{\sigma_0}\right)^n k\right]. \qquad (43)$$

Now, since $\sigma_0 < \sigma_1$, the quantity $\left(\frac{1}{2\sigma_0} - \frac{1}{2\sigma_1}\right)$ is positive, and dividing by it on both sides of the above inequality does not change the direction of the inequality symbol. So we have

$$\sum_{i=1}^n X_i^2 > \frac{\ln\left[\left(\frac{\sigma_1}{\sigma_0}\right)^n k\right]}{\left(\frac{1}{2\sigma_0} - \frac{1}{2\sigma_1}\right)} = c.$$

Therefore, if $f(\mathbf{x}|\sigma_1) > kf(\mathbf{x}|\sigma_0)$, then $\mathbf{x}$ is in the rejection region. By similar reasoning, we can show that $\mathbf{x}$ is not in the rejection region (i.e., $\sum_{i=1}^n X_i^2 \leq c$) if $f(\mathbf{x}|\sigma_1) < kf(\mathbf{x}|\sigma_0)$. Thus, by the Neyman-Pearson Lemma, ϕ is the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$.

Now, for a given value of α, we can calculate c using

$$P_{\sigma_0}\left(\sum_{i=1}^{n} X_i^2 > c\right) = P\left(\sigma_0^2 \sum_{i=1}^{n}\left(\frac{X_i}{\sigma_0}\right)^2 > c\right) \tag{44}$$

$$= P\left(\sigma_0^2 Y > c\right) \tag{45}$$

$$= P\left(Y > c/\sigma_0^2\right) \tag{46}$$

$$= \alpha, \tag{47}$$

where $Y \sim \chi^2(n)$. Thus $c/\sigma_0^2 = \chi^2_{n,\alpha}$, or

$$c = \chi^2_{n,\alpha}\sigma_0^2.$$

■

C & B 8.19

The random variable X has pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ needs to be constructed. Find the UMP level $\alpha = 0.10$ test and compute the Type II Error probability.

Solution

The pdf of X indicates that it has an exponential distribution with parameter $\beta = 1$. Therefore, Y must have a Weibull distribution with parameters $\gamma = \frac{1}{\theta}, \beta = 1$. The pdf of Y is then

$$g(y|\theta) = \begin{cases} \frac{1}{\theta} y^{1/\theta - 1} \exp\left\{-y^{1/\theta}\right\} & \text{if } 0 \leq y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

By the Neyman-Pearson Lemma, a test ϕ with rejection region R satisfying $\mathbf{y} \in R$ if $g(\mathbf{y}|2) > kg(\mathbf{y}|1)$ (with $k \geq 0$) and power function $\beta(\theta)$ satisfying $\beta(1) = 0.10$ is an UMP level $\alpha = 0.10$ test of the hypotheses given in the problem. Now, because only one observation is taken on Y, the inequality $g(\mathbf{y}|2) > kg(\mathbf{y}|1)$ becomes

$$\frac{1}{2} y^{-1/2} \exp\left\{-y^{1/2}\right\} > k \exp\{-y\}$$