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MAT 524: Functions of a Real Variable II
Homework I

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[[R & F: Sec. 4.4, Pr. 28]]:

Statement

Let f be integrable over E and let C be a measurable subset of E. Then

$$\int_C f = \int_E f\chi_C.$$

Proof

Since f is integrable over E so is $|f|$, by definition. Clearly, $|f\chi_C| \leq |f|$ on E, so by the Integral Comparison Test (Proposition 16), it follows that $f\chi_C$ is integrable over E. Thus, we can write

$$\begin{aligned}
\int_E f\chi_C &= \int_E [f\chi_C]^+ - \int_E [f\chi_C]^- &(1)\\
&= \int_E \max\{f\chi_C, 0\} - \int_E \max\{-f\chi_C, 0\} &(2)\\
&= \int_E \max\{f, 0\}\chi_C - \int_E \max\{-f, 0\}\chi_C &(3)\\
&= \int_E f^+\chi_C - \int_E f^-\chi_C. &(4)
\end{aligned}$$

The functions f^+ and f^- are both non-negative, so using the theory from Section 4.3 of Royden and Fitzpatrick, we see that

$$\begin{aligned}
\int_E f^+\chi_C - \int_E f^-\chi_C &= \int_C f^+ - \int_C f^- &(5)\\
&= \int_C f. &(6)
\end{aligned}$$

Therefore, $\int_C f = \int_E f\chi_C$, as required.

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[[R & F: Sec. 4.4, Pr. 29]]:

Problem

For a measurable function f on $[1,\infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. **(a)** Is it true that f is integrable over $[1,\infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? **(b)** Is it true that f is integrable over $[1,\infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

Solution

(a) If f is integrable over $[1,\infty)$ then, by definition, $\int_{[1,\infty)} |f|$ converges. But

$$\begin{aligned} \int_{[1,\infty)} |f| &= \int_1^2 |f| + \int_2^3 |f| + \int_3^4 |f| + \dots && (7) \\ &\geq \left|\int_1^2 f\right| + \left|\int_2^3 f\right| + \left|\int_3^4 f\right| + \dots && (8) \\ &= \sum_{n=1}^{\infty} |a_n|, && (9) \end{aligned}$$

implying that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Thus, $\sum_{n=1}^{\infty} a_n$ converges.

As for the converse statement, it is not necessarily true. Consider the following counter-example. Let $f(x) = \pi\cos(\pi x)$. Then $a_n = \int_n^{n+1} \pi\cos(\pi x)\,dx = \sin((n+1)\pi) - \sin(n\pi) = 0$ for all n. Clearly $\sum_{n=1}^{\infty} a_n$ converges, but $\int_{[1,\infty)} \pi\cos(\pi x)\,dx = \lim_{x\to\infty} \sin(\pi x) - \sin(\pi)$ diverges.

(b) We already showed in the previous part that if f is integrable over $[1,\infty)$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

The converse statement also doesn't hold in this case. The same counter-example can be used here, since $\sum_{n=1}^{\infty} a_n$ (with a_n defined as in part (a)) also converges absolutely.

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