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MAT 524: Functions of a Real Variable II
Homework VIII

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[[Problem One]]:

Problem Statement

Let $f(x) = \frac{1}{2} - x$ on $[0, 1)$, viewed as an element of $L^2([0, 1])$. Show that $\hat{f}(0) = 0$ and $\hat{f}(n) = \frac{1}{2\pi i n}$ if $n \neq 0$.

Apply Parseval's identity to compute $\zeta(2)$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution

Note that the set $\mathcal{B} = \{e^{2\pi i n x} \mid n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0, 1])$. Using this set, we have

$$\begin{aligned}
\hat{f}(0) &= \left\langle f(x), e^0 \right\rangle && (1)\\
&= \int_0^1 \left(\frac{1}{2} - x\right) dx && (2)\\
&= \left[\frac{1}{2}x - \frac{x^2}{2}\right]_0^1 && (3)\\
&= \frac{1}{2} - \frac{1}{2} && (4)\\
&= 0. && (5)
\end{aligned}$$

For $n \neq 0$, we have

$$\begin{aligned}
\hat{f}(n) &= \left\langle f(x), e^{2\pi i n x} \right\rangle && (6)\\
&= \int_0^1 \left(\frac{1}{2} - x\right) \overline{e^{2\pi i n x}} dx && (7)\\
&= \int_0^1 \left(\frac{1}{2} - x\right) e^{-2\pi i n x} dx && (8)\\
&= \int_0^1 \frac{1}{2} e^{-2\pi i n x} dx - \int_0^1 x e^{-2\pi i n x} dx. && (9)
\end{aligned}$$

For the first integral in Equation (9), we have

$$\begin{aligned}\int_0^1 \frac{1}{2} e^{-2\pi inx} dx &= \frac{1}{-4\pi in}\left[e^{-2\pi in} - 1\right] &(10)\\ &= 0. &(11)\end{aligned}$$

To compute the second integral from Equation (9), we use integration by parts:

$$\begin{aligned}-\int_0^1 xe^{-2\pi inx} dx &= -\left[\frac{-x}{2\pi in}e^{-2\pi inx}\right]_0^1 - \int_0^1 \frac{1}{2\pi in}e^{-2\pi inx} dx &(12)\\ &= \frac{1}{2\pi in} - \left[\frac{1}{-4\pi^2 i^2 n^2}e^{-2\pi inx}\right]_0^1 &(13)\\ &= \frac{1}{2\pi in}. &(14)\end{aligned}$$

Thus, $\hat{f}(n) = \frac{1}{2\pi in}$.

Now, since $\mathcal{B}$ is an orthonormal basis for $L^2([0,1])$, Bessel's inequality becomes Parseval's identity. So

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 &= 2\sum_{n=1}^{\infty} \left|\frac{1}{2\pi in}\right|^2 &(15)\\ &= \sum_{n=1}^{\infty} \frac{1}{2\pi^2 n^2} &(16)\\ &= \|f(x)\|^2 &(17)\\ &= \int_0^1 \left|\frac{1}{2} - x\right|^2 dx &(18)\\ &= \int_0^{0.5} (0.5 - x)^2 \, dx + \int_{0.5}^1 (x - 0.5)^2 \, dx &(19)\\ &= \left[0.25x - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^{0.5} + \left[\frac{x^3}{3} - \frac{x^2}{2} + 0.25x\right]_{0.5}^1 &(20)\\ &= \frac{1}{12}. &(21)\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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