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MAT 524: Functions of a Real Variable II
Homework VI

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[[Problem One]]:

Problem Statement

Let H be a Hilbert space. For $y \in H$, define $T_y \in H^*$ by

$$T_y(x) = \langle x, y \rangle .$$

Prove that $\|T_y\| = \|y\|$. Explain why it follows that the map $\phi : H \longrightarrow H^*$ defined by $y \mapsto T_y$ is one-to-one.

Solution

Let $y \in H$ and let $S = \{M : |T_y(x)| \leq M \|x\| \text{ for all } x \in H\}$. Note that $\|y\| \in S$, by the Cauchy-Schwarz inequality. Thus, to show that $\|y\| = \inf S = \|T_y\|$, we simply need to show that $\|y\|$ is a lower bound for S. Suppose otherwise. Then there exists $M \in S$ such that $M < \|y\|$ and $|T_y(x)| = |\langle x, y \rangle| \leq M \|x\|$ for all $x \in H$. But $|\langle y, y \rangle| = \|y\|^2 > M \|y\|$. This contradicts the fact that $M \in S$. Thus, $\|y\| = \inf S = \|T_y\|$.

To show that ϕ is one-to-one, let $y, z \in H$ and suppose $\phi(y) = \phi(z)$. Then for all $x \in H$, we have $\langle x, y \rangle = \langle x, z \rangle$. In particular, $\langle y - z, y \rangle = \langle y - z, z \rangle$, or

$$\begin{aligned}
0 &= \langle y - z, y \rangle - \langle y - z, z \rangle && (1) \\
&= \langle y - z, y \rangle + \overline{(-1)} \langle y - z, z \rangle && (2) \\
&= \langle y - z, y \rangle + \langle y - z, -z \rangle && (3) \\
&= \langle y - z, y - z \rangle . && (4)
\end{aligned}$$

But this implies that $y - z = 0$, i.e., $y = z$. Therefore, ϕ is one-to-one. ■