Exercise 2.12

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Prove Proposition 2.20, which states that if $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite. The reader is instructed to refer to Proposition 0.20, which proves a special case.

Solution. First we state and prove, as a lemma, Chebyshev's inequality.

Lemma. Given $f \in L^+$ and $\lambda > 0$,

$$\mu(\{x : f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X f \, d\mu. \tag{1}$$

Proof. Note that the set $\{x : f(x) > \lambda\} \in \mathcal{M}$ since it is merely $f^{-1}((\lambda, \infty))$ and f is measurable. First we prove the case when $\lambda = 1$. This reduces the problem to showing that

$$\mu(\{x : f(x) > 1\}) \leq \int f \, d\mu. \tag{1}$$

Since $f \in L^+$, the set on the left hand side can be written in terms of the supremum of all simple functions ϕ such that $0 \leq \phi \leq f$:

$$\mu(\{x : f(x) > 1\}) = \sup_\phi \mu(\{x : \phi(x) > 1\}). \tag{2}$$

For any such ϕ, there is a standard expression $\phi = \sum_{j=1}^n a_j \chi_{E_j}$ for disjoint E_j and distinct a_j. Then, there is a subset of indices J so that $a_j > 1$ for all $j \in J$ (J could be empty). So, since measures are always positive and all a_j are positive,

$$\mu(\{x : \phi(x) > 1\}) = \sum_{j \in J} \mu(E_j) \leq \sum_{j \in J} a_j \mu(E_j) \leq \sum_{j=1}^n a_j \mu(E_j) = \int \phi. \tag{3}$$

Thus,

$$\sup_\phi \mu(\{x : \phi(x) > 1\}) \leq \sup_\phi \left\{ \int \phi \right\} = \int f, \tag{4}$$

where the supremums are taken over the collection of all simple functions ϕ with $0 \leq \phi \leq f$. It should be mentioned that these supremums exist by Theorem 2.10, the approximation theorem. Since inequality (1) is true for any L^+ function, and $\frac{1}{\lambda} f \in L^+$ for any $f \in L^+$ and $\lambda > 0$, we have

$$\mu(\{x : \frac{1}{\lambda} f(x) > 1\}) \leq \int \frac{1}{\lambda} f$$

which reduces to the desired inequality. $\square$

To prove the first set is a null set, consider that for all $n \in \mathbb{N}$,

$$\mu(\{x : f(x) > n\}) \leq \frac{1}{n} \int f$$

by the lemma. Since $\int f < \infty$ and the continuity from below of measures gives that $\mu(\{x : f(x) = \infty\}) = \lim_{n\to\infty} \mu(\{x : f(x) > n\})$, we have

$$\mu(\{x : f(x) = \infty\}) \leq \lim_{n\to\infty} \frac{1}{n} \int f = 0.$$

For the second set, the conclusion follows from noting that

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \frac{1}{n}\}$$

and $\mu(\{x : f(x) > \frac{1}{n}\}) \leq n \int f < \infty$ for all n by the lemma.