Exercise 1.8

Nolan Hauck

Show that if $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$, provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

Solution.

Lemma. Given $E \in \mathcal{M}$ with $\mu(E) < \infty$ and $F \subset E$ measurable, $\mu(E \setminus F) = \mu(E) - \mu(F)$.

Proof. Notice $(E \setminus F) \cup F = E$, and $E \cap F$ and F are disjoint, so

$$\mu(E) = \mu(E \setminus F) + \mu(F) \tag{1}$$

and the equality is shown. $\square$

First, note that since $\mathcal{M}$ is closed under countable unions and countable intersections, $\bigcup_{j=k}^\infty E_j \in \mathcal{M}$ for all $k \in \mathbb{N}$ and thus $\liminf E_j = \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty E_j \in \mathcal{M}$. Then by subadditivity,

$$\mu(\liminf E_j) \leq \sum_{k=1}^\infty \mu\left(\bigcap_{j=k}^\infty E_j\right).$$

Since $\mu(\bigcap_{j=k}^\infty E_j) \leq \mu(E_j)$ for all $j \geq k$ and $k \in \mathbb{N}$,

$$\mu\left(\bigcap_{j=k}^\infty E_j\right) \leq \inf_{j \geq k} \mu(E_j)$$

for all $k \in \mathbb{N}$. Then

$$\mu(\liminf E_j) \leq \sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j)$$

If $\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) = \infty$, then the sequence $\{\inf_{j \geq k} \mu(E_j)\}_{k=1}^\infty$ cannot converge to a finite number, so $\sup_k \inf_{j \geq k} \mu(E_j) = \infty$ also. On the other hand, if $\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) < \infty$, then the sequence $\inf_{j \geq k} \mu(E_j) \to 0$ as $k \to \infty$. But since this is an increasing sequence and μ is nonnegative, $\inf_{j \geq k} \mu(E_j) = 0$ for all $k \in \mathbb{N}$. In either case,

$$\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) \leq \sup_k \inf_{j \geq k} \mu(E_j) = \liminf \mu(E_j).$$

For the other inequality, notice that for similar reasoning, $\limsup E_j \in \mathcal{M}$. Then notice that

$$\limsup E_j = \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty E_j \tag{2}$$

$$= \bigcap_{k=1}^\infty \left(\bigcap_{j=k}^\infty E_j^c\right)^c \tag{3}$$

$$= \left(\bigcup_{k=1}^\infty \bigcap_{j=k}^\infty E_j^c\right)^c \tag{4}$$

$$= (\liminf E_j^c)^c. \tag{5}$$

Next, knowing that $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$, we would like to have $(\liminf E_j^c)^c \subset \bigcup E_j$. But

$$\liminf E_j^c = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j^c \supseteq \bigcap_{j=1}^{\infty} E_j^c \implies (\liminf E_j^c)^c \subseteq \left(\bigcap_{j=1}^{\infty} E_j^c\right)^c = \bigcup_{j=1}^{\infty} E_j$$

Then we want to be able to discuss the measure of $\left(\bigcup_{j=1}^{\infty} E_j\right) \cap (\liminf E_j^c)$ to convert this $\liminf$ back into a $\limsup$. If $E = \bigcup_{j=1}^{\infty} E_j$, we have

$$\begin{aligned} E \cap (\liminf E_j^c) &= E \cap \left(\bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j^c\right) \\ &= \bigcup_{k=1}^{\infty} \left(E \cap \left(\bigcap_{j=k}^{\infty} E_j^c\right)\right) \\ &= \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E \cap E_j^c = \liminf(E \cap E_j^c). \end{aligned}$$

So since the union has finite measure, and since for a sequence of real numbers $\{a_n\}$, $\limsup a_n = -\liminf(-a_n)$, and by the first proof, and by the lemma,

$$\begin{aligned} \mu(\limsup E_j) &= \mu((\liminf E_j^c)^c) && (6) \\ &= \mu(E) - \mu\left(E \setminus (\liminf E_j^c)^c\right) && (7) \\ &= \mu(E) - \mu\left(E \cap (\liminf E_j^c)\right) && (8) \\ &= \mu(E) - \mu(\liminf(E \cap E_j^c)) && (9) \\ &\geq \mu(E) - \liminf \mu(E \cap E_j^c) && (10) \\ &= \mu(E) - \liminf \mu(E \setminus E_j) && (11) \\ &= \mu(E) - \liminf(\mu(E) - \mu(E_j)) && (12) \\ &= \mu(E) + \limsup(\mu(E_j) - \mu(E)) = \limsup(E_j). && (13) \end{aligned}$$