Exercise 3.7

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Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

a. $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

b. $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$.

Solution. Let $X = P \cup N$ be a Hahn decomposition of X with respect to ν.

a. Using the Hahn decomposition, we have $\nu^+(E) = \nu(E \cap P)$ and $E \cap P \subset E$ which is measurable. The collection over which the relevant supremum is taken is $\mathcal{P}(E) \cap \mathcal{M}$, so $\nu^+(E) = \nu(P \cap E) \leq \sup\{\nu(F) : F \subset E, F \in \mathcal{M}\}$. On the other hand, for any $F \subset E$ with $F \in \mathcal{M}$, $\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E)$ since $F \subset E$ and ν^+ is a positive measure. So $\sup\{\nu(F)\} \leq \nu^+(E)$.

For the second part note that $-\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\} = \sup\{-\nu(F) : F \in \mathcal{M}, F \subset E\}$. Then one can use the same proof as before to get that $\nu^-(E)$ is an upper bound for the collection of measures and that $\nu^-(E) \leq \sup\{-\nu(F) : F \in \mathcal{M}, F \subset E\}$ (since $\nu^-(E) = \nu(E \cap N)$ and $E \cap N \subset E$ which is measurable).

Note that by the uniqueness of the Jordan decomposition, the functions defined by $\mu_1(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\mu_2(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ for $E \in \mathcal{M}$ are actually positive and well-defined measures which are mutually singular.

b. For any disjoint collection $\{E_j\}_1^n$ of ν-measurable sets such that $\bigcup_1^n E_j = E$, we have

$$\sum_1^n |\nu(E_j)| \leq \sum_1^n |\nu|(E_j) = |\nu|(\bigcup_1^n E_j) = |\nu|(E) \tag{1}$$

so

$$\sup\{\sum_1^n |\nu(E_j)|\} \leq |\nu|(E). \tag{2}$$

Conversely, we can use the Hahn decomposition to write $E = (E \cap P) \cup (E \cap N)$ as a disjoint union of measurable sets, so that $|\nu|(E) = \nu^+(E) + \nu^-(E) = \nu(E \cap P) - \nu(E \cap N) \leq |\nu(E \cap P)| + |\nu(E \cap N)|$. However, since $(E \cap P) \cup (E \cap N)$ is a disjoint union, $|\nu(E \cap P)| + |\nu(E \cap N)|$ is in the set over which we are taking our supremum, so $|\nu|(E) \leq \sup\{\sum_1^n |\nu(E_j)|\}$. Thus the two quantities are equal.