Exercise 1.13

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Every σ-finite measure is semi-finite.

Solution. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then there are sets $\{X_j\}_1^\infty$ so that $X = \bigcup_{j=1}^\infty X_j$ and $\mu(X_j) < \infty$ for all j. Let $E \in \mathcal{M}$ have infinite measure. Then $E = E \cap X = \bigcup_{j=1}^\infty E \cap X_j$, so $\infty = \mu(E) \leq \sum_{j=1}^\infty \mu(E \cap X_j)$. But this means that there must be some $k \in \mathbb{N}$ so that $\mu(E \cap X_k) > 0$. But also $\mu(E \cap X_k) < \mu(X_k) < \infty$. So μ is semi-finite.