Exercise 3.5

Nolan Hauck

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If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$. (Use Exercise 4.)

Solution. Using the definition of sums of functions, we get

$$\nu_1 + \nu_2 = \nu_1^+ - \nu_1^- + \nu_2^+ - \nu_2^- = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-) \tag{1}$$

Since $\nu_j^\pm$ is a positive measure for $j = 1, 2$, Exercise 4 gives

$$(\nu_1 + \nu_2)^+ \leq \nu_1^+ + \nu_2^+ \text{ and } (\nu_1 + \nu_2)^- \leq \nu_1^- + \nu_2^- \tag{2}$$

so that

$$\begin{aligned} |\nu_1 + \nu_2| &= (\nu_1 + \nu_2)^+ + (\nu_1 + \nu_2)^- \\ &\leq \nu_1^+ + \nu_2^+ + \nu_1^- + \nu_2^- \\ &= (\nu_1^+ + \nu_1^-) + (\nu_2^+ + \nu_2^-) = |\nu_1| + |\nu_2| \, . \end{aligned} \tag{3}$$