Exercise 1.29

Nolan Hauck

Last updated Thursday 16th May, 2024 at 10:29

If $E \in \mathcal{L}$ and $m(E) > 0$, then for any $\alpha < 1$, there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Solution. If $\alpha \leq 0$, then there is no problem to solve. So suppose $0 < \alpha < 1$. First suppose that the conclusion holds for $m(E) < \infty$. Then if $m(E) = \infty$, we can find $E_0 \subseteq E$ with $E \in \mathcal{L}$ and $0 < m(E_0) < \infty$ since m is semi-finite. Then there exists I so that $m(E \cap I) \geq m(E_0 \cap I) > \alpha m(I)$ so the conclusion holds. Thus it suffices to prove the conclusion for $m(E) < \infty$.

For contradiction, suppose that $m(E) \in \mathcal{L}$ with finite non-zero measure and there exists $0 < \alpha_0 < 1$ so that for all open intervals I, $m(E \cap I) \leq \alpha_0 m(I)$. Then for any $\epsilon > 0$, there exists U open in $\mathbb{R}$ so that $E \subseteq U$ and $m(U) < m(E) + \epsilon$, or $m(U) - m(E) = m(U \setminus E) < \epsilon$. Then since $\mathbb{R}$ is second countable, there exists a collection of disjoint open intervals $\{I_n\}_1^\infty$ so that $U = \bigcup_{n=1}^\infty I_n$. Then by assumption, $m(E \cap I_n) \leq \alpha_0 m(I_n)$ for all n. We would like somehow to use this inequality to get $m(U \setminus E)$ in terms of the $I_n \setminus E$, so we can see what ϵ we should choose to draw a contradiction. We have

$$I_n = (I_n \setminus E) \cup (E \cap I_n) \tag{1}$$

as a disjoint union, so that according to our inequality on the I_n,

$$m(I_n) = m(I_n \setminus E) + m(E \cap I_n) \leq m(I_n \setminus E) \leq m(I_n \setminus E) + \alpha_0 m(I_n). \tag{2}$$

Rewriting this, we see that $(1 - \alpha_0)m(I_n) \leq m(I_n \setminus E)$. Then

$$\begin{aligned} m(U \setminus E) &= m\Big(\big(\bigcup_{n=1}^\infty I_n\big) \setminus E\Big) \\ &= m\Big(\bigcup_{n=1}^\infty (I_n \setminus E)\Big) \\ &= \sum_{n=1}^\infty m(I_n \setminus E) \\ &\geq (1 - \alpha_0) \sum_{n=1}^\infty m(I_n) = (1 - \alpha_0)m(U). \end{aligned} \tag{3}$$

So we have

$$(1 - \alpha_0)m(U) \leq m(U \setminus E) < \epsilon. \tag{4}$$

If we choose $\epsilon = m(E)(1 - \alpha_0)$ and find the corresponding U and $\{I_n\}$, the inequality will still hold and we get

$$m(U) < m(E) \leq m(U) \tag{5}$$

since $E \subseteq U$. This is clearly a contradiction, so no such α_0 can exist. By *reductio ad absurdum*, the conclusion is proved.