Exercise 1.24

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Let μ be a finite measure on $(X, \mathcal{M})$ and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ satisfies $\mu^*(E) = \mu^*(X)$ (but not that $E \in \mathcal{M}$).

a. If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

b. Let $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$, and define the function ν on $\mathcal{M}_E$ defined by $\nu(A \cap E) = \mu(A)$ (which makes sense by (a)). Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Solution. **a.** Since $A \in \mathcal{M}$ and μ^* is induced by μ, $\mu(A) = \mu^*(A)$. Then $\mu(A) = \mu(X \cap A) = \mu^*(X \cap A) \geq \mu^*(E \cap A)$. For the same reason, $\mu(B) \geq \mu^*(E \cap B)$. Then

$$\mu(A) - \mu(B) \geq \mu^*(E \cap A) - \mu^*(E \cap B) = \mu^*(E \cap A) - \mu^*(E \cap A) = 0 \tag{1}$$

since μ is finite. The same argument with A and B switched shows $\mu(B) - \mu(A) \geq 0$, so $\mu(A) = \mu(B)$.

b. Notice that we want $\mathcal{M}_E$ to be a σ-algebra on E, so when we take complements, we will be taking them relative to E. Let $A \cap E \in \mathcal{M}_E$. Then $E \setminus (A \cap E) = A^c \cap E \in \mathcal{M}_E$ since $A^c \in \mathcal{M}$.

Now suppose $\{A_j \cap E\}_1^\infty \subset \mathcal{M}_E$. Then $\bigcup_1^\infty (A_j \cap E) = E \cap \bigcup_1^\infty A_j \in \mathcal{M}_E$ since $\bigcup_1^\infty A_j \in \mathcal{M}$. So $\mathcal{M}_E$ is a σ-algebra on E.

First, $\nu(\emptyset) = \nu(\emptyset \cap E) = \mu(\emptyset) = 0$. Note that if $A \cap E$ and $B \cap E$ are disjoint, then $A \cap E = (A \setminus B) \cap E$. This is basically saying one can remove the "B part" either before or after intersecting with E. Now suppose $\{A_j \cap E\} \subset \mathcal{M}_E$ is disjoint. Then for each $j \in \mathbb{N}$, $A_j \cap E$ and $\bigcup_1^{j-1} A_k \cap E = (\bigcup_1^{j-1} A_k) \cap E$ are disjoint, so

$$A_j \cap E = (A_j \setminus \bigcup_1^{j-1} A_k) \cap E. \tag{2}$$

Then

$$\begin{aligned}
\nu\left(\bigcup_{j=1}^\infty A_j \cap E\right) &= \nu\left(\bigcup_{j=1}^\infty \left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right) \cap E\right) \\
&= \mu\left(\bigcup_{j=1}^\infty A_j \setminus \bigcup_{k=1}^{j-1} A_k\right) \\
&= \sum_{j=1}^\infty \mu\left(A_j \setminus \bigcup_{k=1}^{j-1} A_k\right) \\
&= \sum_{j=1}^\infty \nu\left(E \cap A_j \setminus \bigcup_{k=1}^{j-1} A_k\right) \\
&= \sum_{j=1}^\infty \nu(A_j \cap E)
\end{aligned} \tag{3}$$

since $A_{j_1} \setminus \bigcup_1^{j_1-1} A_k$ and $A_{j_1} \setminus \bigcup_1^{j_2-1} A_k$ are disjoint for $j_1 \neq j_2$.
