Exercise 1.22

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Let $(X, \mathcal{M}, \mu)$ be a measure space, μ^* the outer measure induced by μ according to (1.12) (the infimum definition), $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, and $\overline{\mu} = \mu^*\big|_{\mathcal{M}^*}$.

a. If μ is σ-finite, then $\overline{\mu}$ is the completion of μ. (Use Exercise 18.)

b. In general, $\overline{\mu}$ is the saturation of the completion of μ.

Solution. **a.** Let $\mathcal{N} = \{\text{null sets of } \mu\}$. Recall that the completion of $\mathcal{M}$ is

$$\underline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\} \tag{1}$$

and the completion of μ is the unique complete measure λ on $\underline{\mathcal{M}}$ such that $\lambda(E \cup F) = \mu(E)$ for $E \cup F \in \underline{\mathcal{M}}$. So λ extends μ to $\underline{\mathcal{M}}$. By the completion theorem, all we must show is that $\mathcal{M}^* = \underline{\mathcal{M}}$, since $\overline{\mu}$ will be a complete measure on μ^* (since it agrees with μ on $\mathcal{M}$) and thus can only be the completion of μ.

Let $E \in \mathcal{M}^*$. Since μ is σ-finite and μ^* was generated by μ, we can use 18b freely without supposing that $\mu^*(E) < \infty$. This means $E \in \mathcal{M}^*$ is equivalent to the statement "there exists $B \in \mathcal{M}_{\sigma\delta}$ (using the notation of Exercise 18) with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$." However, since $\mathcal{M}$ is a σ-algebra $\mathcal{M}_{\sigma\delta} \subseteq \mathcal{M}$, and the reverse inclusion is obvious so the two our equal. So the statement in question becomes "there exists $B \in \mathcal{M}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$." We show this implies $E \in \underline{\mathcal{M}}$ and the reverse.

Since μ^* is generated by μ,

$$\mu^*(B \setminus E) = \inf\{\sum_{j=1}^{\infty} \mu(A_j),\ B \setminus E \subseteq \bigcup_{j=1}^{\infty} A_j,\ A_j \in \mathcal{M}\}. \tag{2}$$

Since the A_j are in $\mathcal{A}$, this is equivalent to if the infimum were taken over disjoint collections $\{A_j\}$ and their disjoint unions. Then each of those unions would be in $\mathcal{M}$ themselves and have measure exactly equal to the sum of the measures of their parts. So we may say there exists a sequence $\{A_n\} \subset \mathcal{M}$ so that $A_n \searrow B \setminus E$ i.e. $\mu(A_n) \searrow \mu(B \setminus E)$ i.e. $\mu(\bigcap_{j=1}^{\infty} A_n) = 0$ and $B \setminus E \subset A_n$ for all n means that $B \setminus E \subset \bigcap_{j=1}^{\infty} A_n$. So $B \setminus E \in \underline{\mathcal{M}}$ since $\underline{\mathcal{M}}$ is complete.

But since $B \in \mathcal{M}$, $B^c \in \mathcal{M}$ and since $E \subset B$, $E^c = B^c \cup (B \setminus E)$, so $E^c \in \underline{\mathcal{M}}$. Thus $E \in \underline{\mathcal{M}}$ and $\mathcal{M}^* \subset \underline{\mathcal{M}}$.

Conversely, suppose $E \cup F \in \underline{\mathcal{M}}$. Then $E \in \mathcal{M}$ and there is $N \in \mathcal{N}$ so that $F \subset N$ and $\mu(N) = 0$. So $E \cup F \subset E \cup N$ and $(E \cup N) \setminus (E \cup F) = N \setminus F$. Then since $N \in \mathcal{M}$,

$$\mu^*(N \setminus F) \leq \mu^*(N) = \mu(N) = 0. \tag{3}$$

So there is $B \in \mathcal{M}$ with $E \cup F \subseteq B$ and $\mu^*(B \setminus (E \cup F)) = 0$. By the Exercise 18b equivalence discussed above, $E \cup F$ is μ^*-measurable. So $\underline{\mathcal{M}} \subset \mathcal{M}^*$, and we are done.

b. If $(X, \underline{\mathcal{M}}, \underline{\mu})$ is the completion of μ then according to exercise 16, we must show that $\overline{\mu}(E) = \underline{\mu}(E)$ if $E \in \underline{\mathcal{M}}$ and $\overline{\mu}(E) = \infty$ otherwise.

Suppose $E \cup F \in \underline{\mathcal{M}}$. Then there is $N \in \mathcal{N}$ so that $\mu(N) = 0$ and $F \subseteq N$. Then since $\overline{\mu}$ agrees with μ on $\mathcal{M}$ and $E \cup N \in \mathcal{M}$,

$$\overline{\mu}(E \cup F) \leq \overline{\mu}(E \cup N) = \mu(E \cup N) = \mu(E) + \mu(N \setminus E) = \mu(E). \tag{4}$$

Also $\mu(E) = \overline{\mu}(E) \leq \overline{\mu}(E \cup F)$ by subadditivity of outer measures, so the two are equal.

Now suppose $E \notin \underline{\mathcal{M}}$. If $\overline{\mu}(E) = \mu^*(E) < \infty$, the argument above using Exercise 18 applies to E and we can say that $E \in \underline{\mathcal{M}}$, a contradiction. So $\mu^*(E) = \overline{\mu}(E) = \infty$.

Since $\overline{\mu}$ is saturated by Exercise 21, $\overline{\mu}$ must be the saturation of $\underline{\mu}$.
