Real Analysis; Modern Techniques and their Applications Folland Solutions

Commands

Chapter 1

Section 2: σ-algebras

Exercise 1

A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is a called a **ring** if it is closed under finite unions and set differences. A ring that is closed under countable unions is called a σ-ring. Prove the following:

a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

b. If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) if and only if $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

d. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Solution.

a. Let $\mathcal{R}$ be a ring. Let $E_1, \ldots, E_n \in \mathcal{R}$ and let $E = \bigcup_{j=1}^n E_j$. Then $E \in \mathcal{R}$. Now,

$$E \setminus \left(\bigcap_{j=1}^n E_j\right) = \bigcup_{j=1}^n E \setminus E_j \in \mathcal{R}\$\$sinceeach\$E \setminus E_j \in \mathcal{R}.\$Then$$

\bigcap{j=1}^nE_j=E\setminus\left(E\setminus\left(\bigcap{j=1}^nE_j\right)\right)\in\mathcal R.

$Nowlet\$\mathcal{R}\$bea\$\sigma\$-ring, andlet\$\{E_j\}_1^\infty \subset \mathcal{R}.\$Theexactsameargumentasaboveworks, sin$

E=\bigcup_{j\in F}E_j

$$whichisin\$\mathcal{R}\$asacountableunion.Then$$

E\setminus\left(\bigcup{j=1}^\infty E_j\right)^c=E\cap\bigcup{j=1}^\infty Ej=\bigcup{j\in F}E_j\in\mathcal R

sincethe$E_j$arealldisjoint. Then

E\setminus\left(E\setminus\left(\bigcup{j=1}^\infty E_j\right)^c\right)=\left(\bigcup{j=1}^\infty E_j\right)^c\in\mathcal R,

meaning$\bigcup_{j=1}^\infty E_j \in \mathcal{A}$.$ * *d. * * Let$\mathcal{A} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$.$Then$X \in

(E_1\setminus E_2)\cap F=(E_1\cap F)\setminus(E_2\cap F).\tag{1}

If$x$isanelementintheLHS, then$x \in E_1$andnotin$E_2$and$x \in F$.$Thismeans$x \notin E_2

F\cap\bigcup E_j=\bigcup(F\cap E_j)\in\mathcal R

as a countable union. So $\mathcal A$ is a σ-ring which contains $X,$ meaning it is a $\sig

(a,b)=\bigcup_{n=1}^\infty\left[\frac{a+b}{2}-\left(1-\frac{1}{n+1}\right)\frac{b-a}{2},\frac{a+b}2+\left(1-\frac{1}{n+1}\right)\frac{b-a}2\right],

since$\frac{a+b}{2}$istheexactmidpointbetween$a$and$b$and$1 - \frac{1}{n+1} \to 1$as$n \to \infty$, $butis

[a,b]=\bigcap_{n=1}^\infty\left(a-\frac{1}{n},b+\frac{1}n\right),

andσ - algebrasareclosedundercountableintersections, so$\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$andwehaveshown

(a,b)=\bigcup_{n=N}^\infty\left(a,b-\frac{1}n\right]

sothat$\mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$.$Conversely, foranyhalf - openinterval$(a, b]$, $

(a,b]=\bigcap_{n=1}^\infty\left(a,b+\frac{1}n\right)

sothat$\mathcal{E}_3 \subset \sigma(\mathcal{E}_1)$.$Thus$\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_3)$.$Anexactlyanalogousargumentcanbeusedtoshow

(-\infty,a)=\bigcup_{n=1}^\infty\left[a-n,a\right)

so$\mathcal{E}_5 \subset \sigma(\mathcal{E}_4)$.$Foranyhalf - closedinterval$[a, b) \in \mathcal{E}_4$, $$(-\infty, b) \setminus (-\infty, a) = [a, b) \in \sigma(

(-\infty,a]=\bigcup_{n=1}^\infty(a-n,b]

so that $\cal E_7\subset\sigma(\cal E_3).$ So the desired equality is shown. The same argument ca

\cal B_\R=\sigma(\cal E_i)=\sigma(\cal E_j)

for any $i,j\in\{1,2,3,\ldots,8\}.$ Whether we have explicitly written the expression or not, any set

Fk=E_k\setminus\left(\bigcup{n=1}^{k-1}E_n\right)=\emptyset,

then $E_k=\bigcup_{n=1}^{k-1}E_n.$ Then if $$F_{k+1}=E_{k+1}\setminus\left(\bigcup_{n=1

Fk=\bigcup{j=1}^kE_j

foreach$k.$Since$𝒜$isanalgebra,each$F_k ∈ 𝒜$asafiniteunion.Also$F_1 ⊂ F_2 ⊂ F_3 ⊂ …

\bigcup{k=1}^\infty F_k=\bigcup{k=1}^\infty\bigcup{j=1}^kE_j=\bigcup{j=1}^\infty E_j,

so $\bigcup_{j=1}^\infty E_j\in\cal A.$ ### Exercise 5 If $\cal M$ is a σ-algebra generated

\cal M:=\sigma(\cal E)=\bigcup_{\cal F\in K}\sigma(\cal F).\tag{1}

LettheRHSof(1)be$𝒜.$Wewillshowthat$𝒜$isa$σ$ − algebra.Let$E ∈ 𝒜.$Then$E ∈ σ(

{Ej}_1^\infty\subset\bigcup{j=1}^\infty\sigma(\cal F_j).

Then$$\bigcup_{j=1}^{\infty}$$ℱ_j$isalsocountableasacountableunionofcountablesets,andisthusin$K.$Since$

\bigcup{j=1}^\infty\sigma(\cal F_j)\subset\sigma\left(\bigcup{j=1}^\infty\cal F_j\right).

Thensincethelatterobjectis$σ$ − algebrainwhichthecountablecollection${E_j}$iscontained

\bigcup{j=1}^\infty E_j\in\sigma\left(\bigcup{j=1}^\infty\cal Fj\right)\subset\bigcup{\cal F\in K}\sigma(\cal F).

Thus $\cal A$ is closed under countable unions. Given that the $\cal A$ is a σ-algebra, we

(E\cup F)^c=E^c\cap F^c=(E^c\cap N^c)\cup(F^c\cap N)

and$E^c ∩ N^c ∈ ℳ$and$F^c ∩ N ⊂ N$,so$(E ∪ F)^c ∈ $\overline{ℳ}$.$Thus$$\overline{ℳ}$$isclosedundercomple

\bigcup1^\infty (E_j\cup F_j)=\bigcup{j=1}^\infty Ej\cup\bigcup{j=1}^\infty F_j

and$⋃E_j ∈ ℳ$and$⋃F_j ⊂ ⋃N_j$and$μ(⋃N_j) ≤ ∑μ(N_j) = 0,$so

\overline\mu\left(\bigcup{j=1}^\infty (E_j\cup F_j)\right)=\mu\left(\bigcup{j=1}^\infty Ej\right)\leq\sum{j=1}^\infty\mu(Ej)=\sum{j=1}^\infty \overline\mu(E_j\cup F_j).

$Thus\$\overline{\mu}\$isameasureon\$\overline{\mathcal{M}}.\$Nowshow\$\overline{\mu}\$iscomplete.Let\$E\cup F\in\overline{\mathcal{M}}\$besuchthat\$\overline{\mu}(E\cup$

\overline\mu(E\cup F)=\mu(E)=\nu(E)\leq\nu(E\cup F)
\$\$ since $E\subset E\cup F$ and ν is a measure on $\overline{\mathcal{M}}$. Conversely,

$$\nu(E\cup F)\leq\nu(E)+\nu(F)=\nu(E)=\overline{\mu}(E\cup F),$$

as $\nu(F)=0$ since ν is complete. So $\overline{\mu}$ is the unique complete extension of μ to $\overline{\mathcal{M}}$.

Exercise 7

If $\mu_1,\dots,\mu_n$ are measures on $(X,\mathcal{M})$ and $a_1,\dots,a_n\in[0,\infty)$, then $\mu=\sum_{j=1}^n a_j\mu_j$ is a measure on $(X,\mathcal{M})$.

Solution.

Since μ_j is a measure for all $j=1,\dots,n$, $\mu_j(\emptyset)=0$ for all j. Thus

$$\mu(\emptyset)=\sum_{j=1}^n a_j\mu_j(\emptyset)=\sum_{j=1}^n a_j\cdot 0=0.$$

Now suppose $\{E_k\}_1^\infty\subset\mathcal{M}$. Then since all $a_j\geq 0$ and all μ_j are subadditive,

$$\begin{aligned}\mu\left(\bigcup_{k=1}^\infty E_k\right)&=\sum_{j=1}^n a_j\mu_j\left(\bigcup_{k=1}^\infty E_k\right)\leq\sum_{j=1}^n a_j\sum_{k=1}^\infty\mu_j(E_k)\\&=\sum_{j=1}^n\sum_{k=1}^\infty a_j\mu_j(E_k)=\sum_{k=1}^\infty\sum_{j=1}^n a_j\mu_j(E_k)=\sum_{k=1}^\infty\mu(E_k)\end{aligned}$$

so μ is subadditive. This interchanging of finite and infinite sums works since it is equivalent to the identity

$$\begin{aligned}&\sum_{k=1}^\infty a_1\mu_1(E_k)+\sum_{k=1}^\infty a_2\mu_2(E_k)+\cdots+\sum_{k=1}^\infty a_n\mu_n(E_k)\\&\qquad=\sum_{k=1}^\infty[a_1\mu_1(E_k)+a_2\mu_2(E_k)+\cdots+a_n\mu_n(E_k)],\end{aligned}$$

which is itself equivalent to rearranging the terms in the left hand side. The sum of an absolutely convergent series, as this one is, is independent of any rearrangement of the terms.

Exercise 8

Show that if $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$, provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

Solution.

Lemma. Given $E \in \mathcal{M}$ with $\mu(E) < \infty$ and $F \subset E$ measurable, $\mu(E \setminus F) = \mu(E) = \mu(F)$.
Proof. Notice $(E \setminus F) \cup F = E$, and $E \cap F$ and F are disjoint, so

$\mu(E) = \mu(E \setminus F) + \mu(F)\$\$ andtheequalityisshown.\$\square\$First,notethatsince\$\mathcal{M}\$isclosedu$

\mu(\liminf Ej)\leq \sum{k=1}^\infty\mu\left(\bigcap_{j=k}^\infty E_j\right).

$$Since\$\mu(\bigcap_{j=k}^\infty E_j) \leq \mu(E_j)\$forall\$j \geq k\$and\$k \in \mathbb{N},\$$$

\mu\left(\bigcap{j=k}^\infty E_j\right)\leq\inf{j\geq k}\mu(E_j)

$$forall\$k \in \mathbb{N}.\$Then$$

\mu(\liminf Ej)\leq\sum{k=1}^\infty\inf_{j\geq k}\mu(E_j)

$If\$\sum_{k=1}^\infty \inf_{j\geq k} \mu(E_j) = \infty,\$thenthesequence\$\{\inf_{j\geq k} \mu(E_j)\}_{k=1}^\infty\$cannotconvergetoafinitenumber$

\sum{k=1}^\infty\inf{j\geq k}\mu(Ej)\leq\sup_k\inf{j\geq k}\mu(E_j)=\liminf\mu(E_j).

$Fortheotherinequality,noticethatforsimilarreasoning,\$\limsup E_j \in \mathcal{M}.\$Thennoticetha$

\begin{align}
\limsup Ej&=\bigcap{k=1}^\infty\bigcup{j=k}^\infty E_j \
&=\bigcap{k=1}^\infty\left(\bigcap{j=k}^\infty E_j^c\right)^c \
&=\left(\bigcup{k=1}^\infty \bigcap_{j=k}^\infty E_j^c\right)^c \
&=(\liminf E_j^c)^c.
\end{align}

$Next, knowingthat\$\mu\left(\bigcup_{j=1}^{\infty}E_j\right)<\infty, \$wewouldliketohave\$(\liminf E_j^c)^c\subset\bigcup E_j. \$But$

\liminf Ej^c=\bigcup{k=1}^\infty\bigcap{j=k}^\infty E_j^c\supseteq\bigcap{j=1}^\infty Ej^c\implies (\liminf E_j^c)^c\subseteq\left(\bigcap{j=1}^\infty Ej^c\right)^c=\bigcup{j=1}^\infty E_j

$Thenwewanttobeabletodiscussthemeasureof\$\left(\bigcup_{j=1}^{\infty}E_j\right)\cap(\liminf E_j^c)\$toconvertthis\$ \liminf$

\begin{align}
E\cap(\liminf Ej^c)&=E\cap\left(\bigcup{k=1}^\infty\bigcap{j=k}^\infty E_j^c\right) \
&=\bigcup{k=1}^\infty\left(E\cap\left(\bigcap{j=k}^\infty E_j^c\right)\right) \
&=\bigcup{k=1}^\infty\bigcap_{j=k}^\infty E\cap E_j^c=\liminf(E\cap E_j^c).
\end{align}

$Sosincetheunionhasfinitemeasure, andsinceforasequenceofrealnumbers\$\{a_n\}, \$\$ \limsup$

\begin{align}
\mu(\limsup E_j)&=\mu((\liminf E_j^c)^c) \
&=\mu(E)-\mu\left(E\setminus(\liminf E_j^c)^c\right) \
&=\mu\left(E\right)-\mu\left(E\cap(\liminf E_j^c)\right) \
&=\mu\left(E\right)-\mu(\liminf(E\cap E_j^c)) \
&\geq\mu\left(E\right)-\liminf\mu(E\cap E_j^c) \
&=\mu\left(E\right)-\liminf\mu(E\setminus E_j) \
&=\mu(E)-\liminf(\mu(E)-\mu(E_j)) \
&=\mu(E)+\limsup(\mu(E_j)-\mu(E))=\limsup(E_j).
\end{align}

Exercise 9 If $(X,\mathcal{M},\mu)$ is a measure space and $E,F\in\mathcal{M},$ then show

\muE\left(\bigcup{j=1}^\infty Aj\right)=\mu\left(E\cap\bigcup{j=1}^\infty Aj\right)=\mu\left(\bigcup{j=1}^\infty E\cap Aj\right)\leq\sum{j=1}^\infty\mu(E\cap Aj)=\sum{j=1}^\infty\mu_E(A_j).

So μ is a measure. ### Exercise 11 Given a finitely additive measure μ on a measurable

\begin{align}
\mu\left(\bigcup{j=1}^\infty E_j\right)&=\mu\left(\bigcup{k=1}^\infty Fk\right) \
&=\lim{k\to\infty}\mu(Fk) \

&=\lim{k\to\infty}\mu\left(\bigcup{j=1}^kE_j\right) \
&=\lim{k\to\infty}\sum{j=1}^k\mu(E_j) \
&=\sum{j=1}^\infty\mu(E_j).
\end{align}

$(b) Suppose μ is a finitely additive measure which is continuous from above and let $\{E_j\} \subset \mathcal{M}$

\begin{align}
\mu\left(\bigcap{k=1}^\infty G_k\right)&=\lim{k\to\infty}\mu(Gk) \
&=\lim{k\to\infty}\left[\mu(X)-\mu\left(\bigcup_{j=1}^k E_j\right)\right] \
&=\mu(X)-\lim{k\to\infty}\sum{j=1}^k\mu(Ej) \
&=\mu(X)-\sum{j=1}^\infty\mu(E_j),
\end{align}

so $\mu\left(\bigcup_{j=1}^\infty E_j\right)=\sum_{j=1}^\infty\mu(E_j).$ ### Exercise 12 Let $(

0=\mu(E\triangle F)=\mu((E\cup F)\setminus(E\cap F))=\mu(E\cup F)-\mu(E\cap F),

$$meaning \$\mu(E \cup F) = \mu(E \cap F).\$ Then$$

\begin{align}
\mu(E)&\leq\mu(E\cup F)=\mu(E\cap F)\leq\mu(F) \
\text{and }\mu(F)&\leq\mu(E\cup F)=\mu(E\cap F)\leq\mu(E)
\end{align}

$so \$\mu(E) = \mu(F).\$ **b.** For \$E \in \mathcal{M}, \$\$E \sim E\$ since \$\mu(E \triangle E) = \mu(\emptyset) = 0.\$ Suppose \$E,$

\mu(E\setminus F)=\mu(F\setminus E)=\mu(F\setminus G)=\mu(G\setminus F)=0.

$$But$$

E\setminus G\subset(E\setminus F)\cup(F\setminus G)\quad\text{and}\quad
G\setminus E\subset(G\setminus F)\cup(F\setminus E),

$so \$\mu(E \setminus G) = \mu(G \setminus E) = 0.\$ **c.** To be pedantic here, the question is asking the reader to s$

\begin{align}
\rho^\ast([E],[G])=\rho(E,G)&=\mu(E\triangle G)=\mu(E\setminus G)+\mu(G\setminus E) \
&\leq\mu(E\setminus F)+\mu(F\setminus G)+\mu(G\setminus F)+\mu(F\setminus E) \
&=\mu(E\setminus F)+\mu(F\setminus E)+\mu(F\setminus G)+\mu(G\setminus F) \

&=\mu(E\triangle F)+\mu(F\triangle G) \
&=\rho(E,F)+\rho(F,G)=\rho^\ast([E],[F])+\rho^\ast([F],[G]).
\end{align}

The remaining conditions of being a metric are now shown. - $\rho^\ast([E],[E])=\rho(E,E)=\mu(E

\mu0\left(\bigcup{j=1}^\infty Ej\right)\leq\sum{j=1}^\infty\mu_0(E_j)

$$isthesameasshowing$$

\sup{F\in A}\mu(F)\leq\sum{j=1}^\infty\sup_{F\in A_j}\mu(F).

$$Further, thiswillbeequivalenttoshowingthatforevery\$\epsilon>0,\$$$

\sup{F\in A}\mu(F)\leq\sum{j=1}^\infty\left(\sup{F\in A_j}\mu(F)+\epsilon2^{-j}\right)=\sum{j=1}^\infty\sup_{F\in A_j}\mu(F)+\epsilon.

$Solet\$\epsilon>0.\$Bydefinitionofsupremum, forevery\$j\in\mathbb{N},\$thereexists\$F_j\subset E_j\$with\$\mu(F_j$

\mu(Fj)<\sup{F\in A_j}\mu(F)+\epsilon2^{-j}=\mu_0(E_j)+\epsilon2^{-j}.

$$Thensince\$\mu\$isameasureandallthe\$F_j\$aremeasurable,$$

\begin{align}
\mu\left(\bigcup{j=1}^\infty F_j\right)\leq\sum{j=1}^\infty\mu(Fj)
<\sum{j=1}^\infty\left(\mu0(E_j)+\epsilon2^{-j}\right)=\sum{j=1}^\infty\mu_0(E_j)+\epsilon.\tag{1}
\end{align}

$Nowdefine\$B\$tobethecollection\$\{\bigcup_{j=1}^{\infty}F_j:F_j\subset E_j\text{ for all }j\text{ and }\mu(F_j)<\infty\}.\$Theninequali$

\sup{F\in B}\mu(F)\leq\sum{j=1}^\infty\mu_0(E_j).

Exercise 16 ## Section 4: Outer Measures ### Exercise 24 Let μ be a finite measure on $

\mu^\ast(A)+\mu^\ast(A^c)=\mu^\ast(X)=\mu^\ast(E)=\mu^\ast(A\cap E)+\mu^\ast(A^c\cap E),

$$meaning$$

\mu^\ast(A\cap E)=\mu^\ast(A)+\mu^\ast(A^c)-\mu^\ast(A^c\cap E).

$Thenbythemonotonicityof\$\mu^*\$,\$\mu^*(A^c\cap E)\leq\mu^*(A^c),\$so\$\mu^*(A^c)-\mu^*(A\cap E)\geq 0.\$Thus$

\bigcup{j=1}^\infty F_j=E\cap\bigcup{j=1}^\infty A_j\in\mathcal{M}_E

$since\$\bigcup A_j \in \mathcal{M}. \$Nowshow\$\nu\$isameasureon\$\mathcal{M}_E. \$First, \$\nu(\emptyset) = \nu(\emptyset \cap E) = \mu(\emptyset) = 0.$

\mu\left(\bigcup{j=1}^\infty A_j\right)=\sum{j=1}^\infty\mu(A_j).

$**Proof.**Let\$A = \bigcup_{j=1}^{\infty} A_j\$andlet\$N = \bigcup_{i\neq j} \mu(A_i \cap A_j). \$Bysubadditivityof\$\mu\$, wehave\$,$

\begin{align}
A\setminus N&=\left(\bigcup{j=1}^\infty A_j\right)\setminus\left(\bigcup{i\neq j}Ai\cap A_j\right) \
&=\bigcup{j=1}^\infty\left[Aj\setminus\left(\bigcup{\substack{i=1 \ i\neq j}}^\infty A_i\cap A_j\right)\right].
\end{align}

$$Thisisadisjointunion, so$$

\begin{align}
\mu(A\setminus N)=\sum{j=1}^\infty\mu\left(A_j\setminus\left(\bigcup{\substack{i=1 \ i\neq j}}^\infty A_i\cap A_j\right)\right).
\end{align}
$$ The inner unions all have zero measure as subsets of N, so $\mu(A \setminus N) = \sum_{j=1}^{\infty} \mu(A_j)$. $\square$
So let $\{A_j \cap E\}_1^\infty \subset \mathcal{M}_E$ be a disjoint collection of sets. Since the sets are disjoint, we have that $\nu((A_i \cap E) \cap (A_j \cap E)) = \nu(\emptyset) = 0$ for all $i \neq j$. But $(A_i \cap E) \cap (A_j \cap E) = (A_i \cap A_j) \cap E$, so by definition $\mu(A_i \cap A_j) = 0$ for all $i \neq j$. Then by the lemma,

$$\nu\left(\bigcup_{j=1}^{\infty}(A_j \cap E)\right) = \mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j) = \sum_{j=1}^{\infty} \nu(A_j \cap E).$$

Section 5: Borel measures on the Real line

Exercise 25

Exercise 26

Prove Proposition 1.20, which says the following: If $E \in \mathcal{M}_\mu$, the collection of sets measurable by a Lebesgue-Stieltjes measure μ, then for every $\epsilon > 0$, there exists a set $A \subset \mathbb{R}$ which is a finite disjoint union of open intervals so that $\mu(A \triangle E) < \epsilon$ ($\triangle$ is the symmetric difference).

Solution.

Let $\epsilon > 0$. By Theorem 1.18 and the definition of infimum, there exists an open set U with $E \subset U$ so that $\mu(E) + \frac{\epsilon}{2} > \mu(U)$. Since $\mathbb{R}$ is a second-countable topological space, U can be written as a disjoint union of open intervals $I_j = (a_j, b_j)$. Some of these intervals could be unbounded; it makes no difference to our purpose. Since $\mu(E) < \infty$ and ϵ is finite, we also have $\mu(U) < \infty$. Thus $\mu(U \setminus E) = \mu(U) - \mu(E) < \frac{\epsilon}{2}$. Also, since

$$\mu(U) = \sum_{j=1}^{\infty} \mu(I_j) < \infty,$$

and the tail of a convergent series is arbitrarily small, there exists an $N \in \mathbb{N}$ so that $\sum_{j=N+1}^{\infty} \mu(I_j) < \frac{\epsilon}{2}$. Now let A be the part of the union corresponding to the head of this series: $\bigcup_{j=1}^{N} I_j$. Then since $A \setminus E \subset U \setminus E$ and $E \setminus A \subset \bigcup_{j=N+1}^{\infty} I_j$, we have

$$\begin{aligned}
\mu(A \triangle E) &= \mu(A \setminus E) + \mu(E \setminus A) \\
&\leq \mu(U \setminus E) + \mu(\bigcup_{j=N+1}^{\infty} I_j) \\
&< \frac{\epsilon}{2} + \sum_{j=N+1}^{\infty} \mu(I_j) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Exercise 27

Chapter 2

Section 1: Measurable Functions

Exercise 1

Section 2: Integration of Nonnegative Functions

Exercise 12

Prove Proposition 2.20, which states that if $f \in L^+$ and $\int f < \infty$, then $\{x : f(x) = \infty\}$ is a null set and $\{x : f(x) > 0\}$ is σ-finite. The reader is instructed to refer to Proposition 0.20, which proves a special case.

Solution.

First we state and prove, as a lemma, Chebyshev's inequality.
Lemma. Given $f \in L^+$ and $\lambda > 0$,

$$\mu(\{x : f(x) > \lambda\}) \leq \frac{1}{\lambda} \int_X f \, d\mu.$$

Proof. Note that the set $\{x : f(x) > \lambda\} \in \mathcal{M}$ since it is merely $f^{-1}((\lambda, \infty))$ and f is measurable. First we prove the case when $\lambda = 1$. This reduces the problem to showing that

$$\mu(\{x : f(x) > 1\}) \leq \int f \, d\mu. \tag{1}$$

Since $f \in L^+$, the set on the left hand side can be written in terms of the supremum of all simple functions ϕ such that $0 \leq \phi \leq f$:

$$\mu(\{x : f(x) > 1\}) = \sup_\phi \mu(\{x : \phi(x) > 1\}).$$

For any such ϕ, there is a standard expression $\phi = \sum_{j=1}^n a_j \chi_{E_j}$ for disjoint E_j and distinct a_j. Then, there is a subset of indices J so that $a_j > 1$ for all $j \in J$ (J could be empty). So, since measures are always positive and all a_j are positive,

$$\mu(\{x : \phi(x) > 1\}) = \sum_{j \in J} \mu(E_j) \leq \sum_{j \in J} a_j \mu(E_j) \leq \sum_{j=1}^n a_j \mu(E_j) = \int \phi.$$

Thus,

$$\sup_\phi \mu(\{x : \phi(x) > 1\}) \leq \sup_\phi \left\{ \int \phi \right\} = \int f,$$

where the supremums are taken over the collection of all simple functions ϕ with $0 \leq \phi \leq f$. It should be mentioned that these supremums exist by Theorem 2.10, the approximation theorem.

Since inequality (1) is true for any L^+ function, and $\frac{1}{\lambda} f \in L^+$ for any $f \in L^+$ and $\lambda > 0$, we have

$$\mu(\{x : \frac{1}{\lambda} f(x) > 1\}) \le \int \frac{1}{\lambda} f$$

which reduces to the desired inequality. □

To prove the first set is a null set, consider that for all $n \in \mathbb{N}$,

$$\mu(\{x : f(x) > n\}) \le \frac{1}{n} \int f$$

by the lemma. Since $\int f < \infty$ and the continuity from below of measures gives that $\mu(\{x : f(x) = \infty\}) = \lim_{n\to\infty} \mu(\{x : f(x) > n\})$, we have

$$\mu(\{x : f(x) = \infty\}) \le \lim_{n\to\infty} \frac{1}{n} \int f = 0.$$

For the second set, the conclusion follows from noting that

$$\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \frac{1}{n}\}$$

and $\mu(\{x : f(x) > \frac{1}{n}\}) \le n \int f < \infty$ for all n by the lemma.

Exercise 13

Exercise 14

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then show that λ is a measure on $\mathcal{M}$ and that for any $g \in L^+$, $\int g d\lambda = \int f g d\mu$.

Solution.

First, $\lambda(\emptyset) = \int_\emptyset f d\mu = 0$. Now let $\{E_j\}_1^\infty \subset \mathcal{M}$ be a collection of pairwise disjoint sets. Then if $E = \bigcup_{j=1}^\infty E_j$, since the E_j are all disjoint, the integral of any L^+ simple function over E (being a finite sum) can be split into a countable sum of integrals over each individual E_j. Also taking the supremum of a sum of values which are dependent on disjoint sets is the same as the taking the sum of the supremums over each individual set. Thus

$$\begin{aligned}
\lambda(E) &= \sup\left\{\int_E \phi\, d\mu \mid \phi \text{ is simple and } 0 \le \phi \le f\right\} \\
&= \sup\left\{\sum_{j=1}^{\infty}\int_{E_j} \phi\, d\mu\right\} \\
&= \sum_{j=1}^{\infty}\sup\left\{\int_{E_j} \phi\, d\mu\right\} \\
&= \sum_{j=1}^{\infty}\int_{E_j} f\, d\mu = \sum_{j=1}^{\infty}\lambda(E_j),
\end{aligned}$$

and λ is a measure.

Suppose $g \in L^+$ is simple. Then g has standard expression $g = \sum_{j=1}^n a_j\chi_{E_j}$ for disjoint sets E_j and distinct a_j. Then in the following string of equalities, we can switch the order of sum and integral since the sets over which we are integrating are all disjoint, by adding the relevant characteristic functions:

$$\begin{aligned}
\int g\, d\lambda &= \sum_{j=1}^{n} a_j\lambda(E_j) \\
&= \sum_{j=1}^{n} a_j\int_{E_j} f\, d\mu \\
&= \sum_{j=1}^{n}\int_{E_j} a_j f\, d\mu \\
&= \int_X \sum_{j=1}^{n} a_j\chi_{E_j} f\, d\mu \\
&= \int_X gf\, d\mu.
\end{aligned}$$

Now let g be any L^+ function. Then by the approximation theorem, there is an increasing sequence $\{\phi_n\}$ of simple functions with $0 \le \phi_n \le g$ for all n so that $\phi_n \to g$ pointwise on X. Then since $f \in L^+$, the sequence $\phi_n f \to gf$ as well. Then by the Monotone Convergence theorem (applied twice) and the previous identity,

$$\int g\, d\lambda = \lim_{n\to\infty}\int \phi_n\, d\lambda = \lim_{n\to\infty}\int \phi_n f\, d\mu = \int \lim_{n\to\infty}\phi_n f\, d\mu = \int gf\, d\mu.$$

Exercise 15

If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then show $\int f = \lim \int f_n$.

Solution.

Since f_n decreases to f, which is measurable as the limit of a sequence of measurable functions, we have $f_1 \geq f_n$ for all n and thus $\infty > \int f_1 \geq \int f_n$ for all n. Then $f_1 - f_n \geq 0$ so that $f_1 - f_n \in L^+$ for all n. So for each n,

$$\int f_1 = \int (f_1 - f_n + f_n) = \int (f_1 - f_n) + \int f_n \tag{1}$$

i.e. $\int f_1 - \int f_n = \int (f_1 - f_n)$ for each n. Similarly, since $f_1 \geq f$, $f_1 - f \in L^+$ and the following equality holds:

$$\int f_1 - \int f = \int (f_1 - f).$$

Since $f_1 - f_n$ increases to $f_1 - f$ as $n \to \infty$ and all are measurable, the MCT gives

$$\int f_1 = \lim \left[\int (f_1 - f_n) + \int f_n \right] = \int (f_1 - f) + \lim \int f_n$$

so that

$$\int f_1 = \int f_1 - \int f + \lim \int f_n,$$

i.e. $\int f = \lim \int f_n$.

Exercise 16

Given $f \in L^+$ and $\int f < \infty$, show that for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Solution.

Notice that the desired inequality is equivalent to $\int f - \int_E f < \epsilon$. This motivates one to try describing a sequence of sets $\{E_n\}$ so that $\int_{E_n} f \to \int f$, since $\int_E f = \int f \chi_E$. This necessitates that this sequence converges from below to X. Well, define $E_n = \{x : f(x) > \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then by the Lemma in 2.12, the following inequality holds for all n:

$$\mu(E_n) \leq n \int f < \infty.$$

So $\mu(E_n)$ is always finite. Also it is clear that $E_1 \nearrow X$, since $f(x) > \frac{1}{n} \implies f(x) > \frac{1}{n+1}$. Now define the sequence $\{f_n\}$ of functions by $f_n = \chi_{E_n} f$ for all n. Then $f_n \in L^+$ for all n and $f_1 \leq f_2 \leq f_3 \leq \ldots$ and $f_n \nearrow f$. By the Monotone Convergence Theorem,

$$\int f = \lim \int f_n.$$

Thus, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $\int f - \int f_n < \epsilon$ for all $n \geq N$. But $\int f_n = \int f\chi_{E_n} = \int_{E_n} f$. So

$$\int_{E_n} > \left(\int f\right) - \epsilon$$

for all $n \geq N$.

Section 3: Integration of Complex Functions

Section 4: Modes of Convergence

Section 5: Product Measures

Section 6: The n-dimensional Lebesgue Integral

Section 7: Integration in Polar Coordinates