Exercise 1.11

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Given a finitely additive measure μ on a measurable space $(X, \mathcal{M})$, show the following:

a) μ is a measure if and only if it is continuous from below.

b) If $\mu(X) < \infty$, then μ is a measure if and only if μ is continuous from above.

Solution. The forwards direction on both of these follows from Theorem 1.8. For the backwards direction, all that must be shown is the countable additivity over disjoint unions.

(a) Suppose μ is a finitely additive measure which is continuous from below and let $\{E_j\} \subset \mathcal{M}$ be a collection of pairwise disjoint measurable subsets of X. Define F_k to be $\bigcup_{j=1}^k E_j$ for each $k \in \mathbb{N}$. Then $\bigcup_{k=1}^\infty F_k = \bigcup_{j=1}^\infty E_j$ and $F_1 \subset F_2 \subset F_3 \subset \ldots$ and all F_k are measurable. So by the continuity from below and finite additivity of μ,

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^\infty E_j\right) &= \mu\left(\bigcup_{k=1}^\infty F_k\right) \\
&= \lim_{k\to\infty} \mu(F_k) \\
&= \lim_{k\to\infty} \mu\left(\bigcup_{j=1}^k E_j\right) \\
&= \lim_{k\to\infty} \sum_{j=1}^k \mu(E_j) \\
&= \sum_{j=1}^\infty \mu(E_j).
\end{aligned}$$

(b) Suppose μ is a finitely additive measure which is continuous from above and let $\{E_j\} \subset \mathcal{M}$ be a collection of pairwise disjoint measurable subsets of X. Then define F_k to be $\bigcup_{j=1}^k E_j$ as in part (a) so that the unions of each collection are the same and $F_1 \subset F_2 \subset \ldots$ Then define G_k to be F_k^c for each k. Thus $\mu\left(\bigcap_{k=1}^\infty G_k\right) = \mu\left(X \setminus \left(\bigcup_{j=1}^\infty E_j\right)\right) = \mu(X) - \mu\left(\bigcup_{j=1}^\infty E_j\right)$. Then we have $G_1 \supset G_2 \supset G_3 \supset \ldots$ and $\mu(G_1) < \infty$ since $\mu(X) < \infty$, so the continuity from above and finite additivity of μ give

$$\begin{aligned}
\mu\left(\bigcap_{k=1}^\infty G_k\right) &= \lim_{k\to\infty} \mu(G_k) \\
&= \lim_{k\to\infty} \left[\mu(X) - \mu\left(\bigcup_{j=1}^k E_j\right)\right] \\
&= \mu(X) - \lim_{k\to\infty} \sum_{j=1}^k \mu(E_j) \\
&= \mu(X) - \sum_{j=1}^\infty \mu(E_j),
\end{aligned}$$

so $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$.