Exercise 1.27

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Prove Proposition 1.22a. (Show that if $x, y \in C$ and $x < y$, there exists $z \notin C$ such that $x < z < y$.) Proposition 1.22a states C is compact, nowhere dense, and totally disconnected, (i.e. the only connected subsets of C are single points). Moreover, C has no isolated points.

Solution. C is compact since it is an intersection of closed sets and is clearly bounded. Let $x < y \in C$. Then $x, y \in \{\sum_{j=1}^{\infty} a_j 3^{-j} \mid a_j \in \{0, 2\}\}$. So

$$x = \sum_{j=1}^{\infty} a_j 3^{-j} \quad \text{and} \quad y = \sum_{j=1}^{\infty} b_j 3^{-j}. \tag{1}$$

Since $x \neq y$ there and by the well-ordering of the integers, there exists as smallest j, call it j_0, so that $a_j \neq b_j$. Since a_{j_0}, b_{j_0} can only be either 0 or 2, it must be the case that $a_{j_0} = 0$ and $b_{j_0} = 2$ as otherwise we would have $x \geq y$. Then let $z = 1 \cdot 3^{-j_0} + \sum_{j \neq j_0}^{\infty} a_j 3^{-j}$. Then $z \notin C$ since z has a 1 in its ternary expansion. Also $x < y$ since z only differs from x at one ternary spot j_0, and its value there is larger. Also also $z < y$ since the j_0^{th} ternary spot of z is necessarily smaller than that of y.

Now suppose that $(\overline{C})^\circ \neq \emptyset$. Since C is closed this is just supposing that $C^\circ \neq \emptyset$. So there is $x \in C$ and $\epsilon > 0$ so that $(x - \epsilon, x + \epsilon) \subset C$. But then there is another point $y \in (x-\epsilon, x+\epsilon) \cap C$ with $y \neq x$. By the previous paragraph there is $z \notin C$ so that $x < z < y$ or $y < z < x$, contradicting that $(x - \epsilon, x + \epsilon) \subset C$. So $C^\circ = \emptyset$ i.e. C is nowhere dense.

Any connected subset of C which contains more than one point would necessarily contain the interval between those two points, contradicting that C is nowhere dense. So C is totally disconnected.

Suppose $x = \sum_{j=1}^{\infty} a_j 3^{-j}$ is such that there exists $N \in \mathbb{N}$ so that $a_j = 0$ for all $j \geq N$. Then for all $\epsilon > 0$, there is $K \in \mathbb{N}$ so that $y = 2 \cdot 3^{-K} \in (x - \epsilon, x + \epsilon)$. Thus x is not isolated.

On the other hand, if the ternary expansion of x does not terminate, we can construct a sequence $\{x_n\} \subset C$ so that $x_n \to x$. Let $x_1 = a_1 3^{-1}$, $x_2 = \sum_{j=1}^{2} a_j 3^{-j}$ and so on, letting $x_k = \sum_{j=1}^{k} a_j 3^{-j}$. Then it is clear that $x_n \to x$ and $x_n \in C$ for all C. Also $x_n \neq x$ for all n, so x cannot be isolated (there is a sequence converging to x which has no terms equal to x).
