Exercise 3.2

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If ν is a signed measure, E is ν-null if and only if $|\nu|(E) = 0$. Also, if ν and μ are signed measures, then $\nu \perp \mu$ if and only if $|\nu| \perp \mu$ if and only if $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. Denote this final condition by $\nu^\pm \perp \mu$.

Solution. Throughout the solution, let $X = P \cup N$ be a Hahn decomposition of X for ν. For the first equivalence, first suppose that E is ν-null. Then $\nu(E) = 0$ and $\nu(F) = 0$ for every measurable $F \subset E$. In particular, $P \cap E$ and $N \cap E$ are such measurable subsets and thus have zero measure. But $\nu(P \cap E) = \nu^+(E)$ and $\nu(P \cap N) = \nu^-(E)$, so $\nu^+(E) = \nu^-(E) = 0$. Thus $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Conversely, suppose $|\nu|(E) = 0$. Then $\nu^+(E) + \nu^-(E) = 0$, so that $\nu^-(E) = \nu^+(E) = 0$ since $\nu^\pm$ are both positive measures. Thus $\nu(E) = \nu^+(E) - \nu^-(E) = 0$.

Now, suppose ν and μ are signed measures and $\nu \perp \mu$ and show $|\nu| \perp \mu$. Then there exist sets $E, F \in \mathcal{M}$ so that $E \cup F = X$, $E \cap F = \emptyset$ and E is ν-null and F is μ-null. By the first equivalence, E is also $|\nu|$-null so the sets E and F show that $|\nu| \perp \mu$.

Now suppose $|\nu| \perp \mu$ and show $\nu^\pm \perp \mu$. Again, we have $X = E \cup F$ with $E \cap F = \emptyset$ and $|\nu|(E) = 0$ and F is μ-null. But since $|\nu|(E) = \nu^+(E) + \nu^-(E)$, we get that $\nu^+(E) = \nu^-(E) = 0$. Thus $\nu^\pm \perp \mu$ using the same two sets E and F.

Finally, suppose $\nu^\pm \perp \mu$ and show $\nu \perp \mu$. There exist sets E, F, E', F' so that $E \cup F = E' \cup F' = X$, $E \cap F = E' \cap F' = \emptyset$ and $\nu^+(E) = \nu^-(E') = 0$ and F and F' are both null for μ (note since ν^+ and ν^- are positive measures, it is enough to have $\nu^\pm(E) = 0$ to have E be $\nu^\pm$-null). Then $F \cup F'$ is null for μ and $X \setminus F \cup F' = E \cap E'$ is null for $|\nu|$ since $|\nu|(E \cap E') = \nu^+(E \cap E') + \nu^-(E \cap E') \leq \nu^+(E) + \nu^-(E') = 0$. So $|\nu| \perp \mu$.