Exercise 1.18

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Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, and $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$ and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a. For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.

b. If $\mu^*(E) < \infty$, then E is μ^*-measurable if and only if there exists $B \in A_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.

c. If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in b. is superfluous.

Solution. a. This part is basically saying that the outer measure of any subset of X can be approximated from above by sets in the given algebra. Since μ^* is the outer measure induced by μ_0, we have for any $E \subset X$ that

$$\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j), E \subset \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}\}.$$

Now let $\epsilon > 0$. By definition of infimum, there exists a collection of sets $\{A_j\} \subset \mathcal{A}$ so that $E \subset \bigcup_{j=1}^{\infty} A_j$ and $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \mu^*(E) + \epsilon$. But $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}_\sigma$ by definition, so we are done.

b. Suppose $\mu^*(E) < \infty$ and E is μ^*-measurable. By part a., for any $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}_\sigma$ so that $\mu^*(A_n) \leq \mu^*(E) + 1/n$, or in other words $\mu^*(A_n) - \mu^*(E) \leq 1/n$. Since μ^* is the outer measure generated by μ_0 on the algebra $\mathcal{A}$, each element of $\mathcal{A}$ is μ^*-measurable. Since the set of μ^*-measurable sets forms a σ-algebra, the sets A_n (which are countable unions of sets in $\mathcal{A}$) are necessarily μ^*-measurable. Since E is also μ^* measurable and μ^* is a complete measure on the set of μ^*-measurable sets, we have

$$\mu^*(A_n) - \mu^*(E) = \mu^*(A_n \setminus E)$$

for each n. Then let $B = \bigcap_{k=1}^{\infty} A_k$. This B is in $A_{\sigma\delta}$ as a countable intersection. Then for each n, we have

$$\mu^*(B \setminus E) = \mu^*(\bigcap_{k=1}^{\infty}(A_k \setminus E)) \leq \mu^*(A_n \setminus E) = \mu^*(A_n) - \mu^*(E) \leq 1/n.$$

But then $\mu^*(B \setminus E)$ can only be zero.

Conversely suppose such a set B exists. Then let $C \subset X$. We must show that

$$\mu^*(C) = \mu^*(C \cap E) + \mu^*(C \cap E^c).$$

Since $E \subset B$, $B = (B \setminus E) \cup E$ and $\mu^*(B \setminus E) = 0$, we have

$$\begin{aligned} \mu^*(C \cap E) &\leq \mu^*(C \cap B) & (1)\\ &= \mu^*(C \cap ((B \setminus E) \cup E)) & (2)\\ &\leq \mu^*(C \cap (B \setminus E)) + \mu^*(C \cap E) & (3)\\ &= \mu^*(C \cap E) & (4) \end{aligned}$$

i.e. $\mu^*(C \cap E) = \mu^*(C \cap B)$. On the other hand, $E^c = (E^c \setminus B^c) \cup B^c$ since $B^c \subset E^c$. But

$$E^c \setminus B^c = E^c \cap B = B \cap E^c = B \setminus E$$

so $E^c = (B \setminus E) \cup B^c$. Then we have

$$\mu^*(C \cap B^c) \leq \mu^*(C \cap E^c) \leq \mu^*(C \cap (B \setminus E)) + \mu^*(C \cap B^c) = \mu^*(C \cap B^c)$$

i.e. $\mu^*(C \cap B^c) = \mu^*(C \cap E^c)$. This string of inequalities also works since $\mu^*(B \setminus E) = 0$. Finally,

$$\mu^*(C \cap E) + \mu^*(C \cap E^c) = \mu^*(C \cap B) + \mu^*(C \cap B^c) = \mu^*(C)$$

since B is μ^*-measurable as a countable intersection of μ^*-measurable sets (the μ^*-measurable sets form a σ-algebra). So E is μ^*-measurable.

c. Note that the reverse implication doesn't use the assumption that $\mu^*(E) < \infty$, so we only need to show the forwards implication is independent of this assumption, supposing that μ_0 is σ-finite. If it is, then there exists $\{X_j\} \subset \mathcal{P}(X)$ so that $X_j \in \mathcal{A}$ and $\mu_0(X_j) < \infty$ for each j and $X = \bigcup_{j=1}^{\infty} X_j$. It is possible to "disjointify" this collection and still have it satisfy all requirements, so we will simply assume it is disjoint. Then for any μ^*-measurable E (with possibly infinite outer measure) we can write $E = \bigcup_{j=1}^{\infty}(E \cap X_j)$. If $E_j = E \cap X_j$ for each j, then $E = \bigcup_{j=1}^{\infty} E_j$ and each E_j is μ^*-measurable as the intersection of two μ^*-measurable sets, and $\mu^*(E_j) \leq \mu^*(X_j) = \mu_0(X_j) < \infty$ since μ^* restricted to $\mathcal{A}$ is μ_0. Then by part b., there exists a $B_j \in \mathcal{A}_{\sigma\delta}$ so that $E_j \subset B_j$ and $\mu^*(B_j \setminus E_j) = 0$. Then

$$E = \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} B_j := B$$

Also since $E_j \subset B_j$ for each j and all the E_j are disjoint, we have

$$\mu^*(B \setminus E) = \mu^*\left(\bigcup_{j=1}^{\infty} B_j \setminus E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(B_j \setminus E_j) = 0.$$

Now, is $B \in \mathcal{A}_{\sigma\delta}$?