Exercise 1.16

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Let $(X, \mathcal{M}, \mu)$ be a measure space. A set $E \subset X$ is called *locally measurable* if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. Let $\widetilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \widetilde{\mathcal{M}}$. If $\mathcal{M} = \widetilde{\mathcal{M}}$ then μ is called *saturated.*

a. If μ is σ-finite, then μ is saturated.

b. $\widetilde{\mathcal{M}}$ is a σ-algebra.

c. Define $\widetilde{\mu}$ on $\widetilde{\mathcal{M}}$ by $\widetilde{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\widetilde{\mu}(E) = \infty$ otherwise. Then $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ called the *saturation* of μ.

d. If μ is complete, so is $\widetilde{\mu}$.

e. Suppose that μ is semifinite. For $E \in \widetilde{\mathcal{M}}$, define $\underline{\mu}(E) = \sup\{\mu(A) : A \in \mathcal{M} \text{ and } \mathcal{A} \subset \mathcal{E}\}$. Then $\underline{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$ that extends μ.

f. Let X_1, X_2 be disjoint uncountable sets so that $X = X_1 \cup X_2$ and $\mathcal{M}$ the σ-algebra of countable or co-countable sets in X. Let μ_0 be the counting measure on $\mathcal{P}(\mathcal{X}_\infty)$, and define μ on $\mathcal{M}$ by $\mu(E) = \mu_0(E \cap X_1)$. Then μ is a measure on $\mathcal{M}$, $\widetilde{\mathcal{M}} = \mathcal{P}(\mathcal{X})$, and in the notation of parts **c.** and **e.**, $\widetilde{\mu} \neq \underline{\mu}$.

Solution. a. Suppose μ is σ-finite. Then there exist $\{X_j\}_1^\infty \subset \mathcal{M}$ so that $\mu(X_j) < \infty$ for all j and $\bigcup_{j=1}^\infty X_j = X$. Let $E \in \widetilde{\mathcal{M}}$. Then $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ with $\mu(A) < \infty$. But this means $E \cap X_j \in \mathcal{M}$ for all j. Thus $E = E \cap X = \bigcup_{j=1}^\infty E \cap X_j \in \mathcal{M}$. So $\widetilde{\mathcal{M}} \subset \mathcal{M}$, meaning the two are equal.

b. Supposing $E \in \widetilde{\mathcal{M}}$ and $A \in \mathcal{M}$ with $\mu(A) < \infty$, we have $A \setminus (E \cap A) = E^c \cap A \in \mathcal{M}$ since A and $E \cap A$ are in $\mathcal{M}$ and $\mathcal{M}$ is a σ-algebra. This is true for any such A, so $E^c \in \widetilde{\mathcal{M}}$. If $\{E_j\}_1^\infty \subset \widetilde{\mathcal{M}}$, then for any $A \in \mathcal{M}$ with $\mu(A) < \infty$, $A \cap E_j \in \mathcal{M}$, so $A \cap \bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty A \cap E_j \in \mathcal{M}$, meaning $\bigcup_{j=1}^\infty \in \widetilde{\mathcal{M}}$. Thus $\widetilde{\mathcal{M}}$ is a σ-algebra.

c. Note $\widetilde{\mu}(\emptyset) = \mu(\emptyset) = 0$ since $\emptyset \in \mathcal{M}$. Now let $\{E_j\}_1^\infty$ be a disjoint collection of locally measurable sets. Let $E = \bigcup_{j=1}^\infty E_j$. If $E \in \mathcal{M}$, then we distinguish two cases: $\mu(E) < \infty$ and $\mu(E) = \infty$. If $\mu(E) < \infty$, then since $E \in \mathcal{M}$ and each $E_j \in \widetilde{\mathcal{M}}$, we have $E_j \cap E \in \mathcal{M}$ for all j. But $E_j \cap E = E_j$ for all j, so $E_j \in \mathcal{M}$ for all j. Thus

$$\widetilde{\mu}(E) = \mu(E) = \sum_{j=1}^\infty \mu(E_j) = \sum_{j=1}^\infty \widetilde{\mu}(E_j).$$

Now suppose $E \in \mathcal{M}$ and $\mu(E) = \infty$. Then if all $E_j \in \mathcal{M}$, we are done immediately by the previous line. If there is at least one $E_k \in \widetilde{\mathcal{M}} \setminus \mathcal{M}$, then $\widetilde{\mu}(E_k) = \infty$, so $\sum_{j=1}^\infty \widetilde{\mu}(E_j) =$

$\widetilde{\mu}(E_k) + \sum_{j \neq k} \widetilde{\mu}(E_j) = \infty$. Now suppose $E \in \widetilde{\mathcal{M}} \setminus \mathcal{M}$. Then $\widetilde{\mu}(E) = \infty$. It cannot be that all $E_j \in \mathcal{M}$ as otherwise E would be in $\mathcal{M}$. So there must be at least one $E_k \in \widetilde{\mathcal{M}} \setminus \mathcal{M}$, meaning $\sum_{j=1}^{\infty} \widetilde{\mu}(E_j) = \infty$, as in the previous case. So $\widetilde{\mu}$ is countably additive and therefore a measure on $\widetilde{\mathcal{M}}$. To show that $\widetilde{\mu}$ is a saturated measure on $\widetilde{\mathcal{M}}$, we are showing that the space $(X, \widetilde{\mathcal{M}}, \widetilde{\mu})$ is saturated. This means we need to show that $\widetilde{\widetilde{\mathcal{M}}} \subset \widetilde{\mathcal{M}}$, where

$$\widetilde{\widetilde{\mathcal{M}}} = \{E \subset X : B \cap E \in \widetilde{\mathcal{M}} \text{ for all } B \in \widetilde{\mathcal{M}} \text{ with } \widetilde{\mu}(B) < \infty\}.$$

So let $E \in \widetilde{\widetilde{\mathcal{M}}}$ and let $B \in \mathcal{M}$ so that $\mu(B) < \infty$. Then $B \in \widetilde{\mathcal{M}}$ as well and has $\widetilde{\mu}(B) = \mu(B) < \infty$, so $B \cap E \in \widetilde{\mathcal{M}}$ by definition of $\widetilde{\widetilde{\mathcal{M}}}$. But then $B \cap E = B \cap (B \cap E) \in \mathcal{M}$ by definition of $\widetilde{\mathcal{M}}$. So $E \in \widetilde{\mathcal{M}}$, since B was arbitrary, and we are done.

d. Suppose μ is complete. Let $E \in \widetilde{\mathcal{M}}$ so that $\widetilde{\mu}(E) = 0$. Let $F \subset E$. Then E must be in $\mathcal{M}$, so by the completeness of μ, $F \in \mathcal{M}$ as well. Thus $F \in \widetilde{\mathcal{M}}$ and $\widetilde{\mu}$ is complete.

e. First of all, $\underline{\mu}(\emptyset) = \sup\{\mu(A) : A \subset \emptyset \text{ and } A \subset \emptyset\} = \mu(\emptyset) = 0$. Now let $\{E_j\}_1^\infty \subset \widetilde{\mathcal{M}}$ be disjoint locally measurable sets. Let $E = \bigcup_{j=1}^\infty E_j$, $\mathcal{B} = \{\mathcal{A} \subset \mathcal{E} \text{ and } \mathcal{A} \in \mathcal{M}\}$ and let $\mathcal{B}_| = \{\mathcal{A} \subset \mathcal{E}_| \text{ and } \mathcal{A} \in \mathcal{M}\}$ for each j. Then $\underline{\mu}(E) = \sup_{A \in \mathcal{B}} \mu(A)$ and $\underline{\mu}(E_j) = \sup_{A \in \mathcal{B}_|} \mu(A)$ for each j. So we must show

$$\underline{\mu}(E) = \sum_{j=1}^\infty \underline{\mu}(E_j).$$

Let $\epsilon > 0$. Then by definition of supremum, for each j there exists $A_j \in \mathcal{B}_|$ so that $\sup_{A \in \mathcal{B}_|} \mu(A) < \mu(A_j) + \epsilon 2^{-j}$. Then

$$\sum_{j=1}^\infty \underline{\mu}(E_j) = \sum_{j=1}^\infty \sup_{A \in \mathcal{B}_|} \mu(A) < \sum_{j=1}^\infty (\mu(A_j) + \epsilon 2^{-j}) = \sum_{j=1}^\infty (\mu(A_j)) + \epsilon.$$

But since $A_j \in \mathcal{M}$ for each j and the disjoint-ness of the E_j implies disjoint-ness of the A_j, we have $\mu(\bigcup_{j=1}^\infty A_j) = \sum_{j=1}^\infty \mu(A_j)$. Since each $A_j \subset E_j$, we have $\bigcup A_j \subset \bigcup E_j$, so $\bigcup A_j \in \mathcal{B}$. These two facts taken together mean

$$\sum_{j=1}^\infty \underline{\mu}(E_j) < \mu\left(\bigcup_{j=1}^\infty A_j\right) + \epsilon \leq \sup_{A \in \mathcal{B}} \mu(A) + \epsilon = \underline{\mu}(E) + \epsilon$$

for all $\epsilon > 0$. That is,

$$\sum_{j=1}^\infty \underline{\mu}(E_j) \leq \underline{\mu}(E).$$

For the other inequality, first suppose that each $A \in \mathcal{B}$ is such that $\mu(A) < \infty$. Then since each E_j is locally measurable, the intersection $A \cap E_j \in \mathcal{M}$ for each j. Thus $A \cap E_j \in \mathcal{B}_|$ for each j. Each $A \cap E_j$ is disjoint since the E_j are disjoint, so

$$\mu(A) = \mu\left(\bigcup_{j=1}^\infty A \cap E_j\right) = \sum_{j=1}^\infty \mu(A \cap E_j) \leq \sum_{j=1}^\infty \underline{\mu}(E_j).$$

This is true for all $A \in \mathcal{B}$, so

$$\underline{\mu}(E) = \sup_{A \in \mathcal{B}} \mu(A) \leq \sum_{j=1}^{\infty} \underline{\mu}(E_j).$$

Now suppose that there is one $A \in \mathcal{B}$ so that $\mu(A) = \infty$. Then certainly $\underline{\mu}(E) = \infty$, so we must show $\sum \underline{\mu}(E_j) = \infty$. By problem 14, since μ is semifinite, for any $k \in \mathbb{N}$ there exists $C \subset A$ with $C \in \mathcal{M}$ and and $k < \mu(C) < \infty$. Such a C will always be in $\mathcal{B}$. Since $\mu(C) < \infty$, the intersection $E_j \cap C$ is in $\mathcal{M}$ and $\mu(C) = \sum \mu(E_j \cap C)$ since the E_j are disjoint. Thus for any $k \in \mathbb{N}$, we have the existence of a set $C \in \mathcal{B}$ so that

$$k < \sum_{j=1}^{\infty} \mu(E_j \cap C) \leq \sum_{j=1}^{\infty} \sup_{A \in \mathcal{B}_|} \mu(A) = \sum_{j=1}^{\infty} \underline{\mu}(E_j),$$

or in other words, $\sum \underline{\mu}(E_j) = \infty$. So $\underline{\mu}$ is a countably additive measure on $\widetilde{\mathcal{M}}$.

Similar to part c., showing $\underline{\mu}$ is saturated on $\widetilde{\mathcal{M}}$ amounts to showing that for any $E \in \widetilde{\widetilde{\mathcal{M}}}$ and any $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $B \cap E \in \mathcal{M}$, where $\widetilde{\widetilde{\mathcal{M}}}$ refers to the collection of sets locally measurable by $\underline{\mu}$. If $B \in \mathcal{M}$ and $\mu(B) < \infty$, then $E \in \widetilde{\mathcal{M}}$ and $\underline{\mu}(E) = \mu(E)$, since $E \subseteq E$. So $\underline{\mu}(E) < \infty$, meaning $B \cap E \in \widetilde{\mathcal{M}}$. Then $B \cap E = B \cap (B \cap E) \in \mathcal{M}$ since $B \in \mathcal{M}$ with $\mu(B) < \infty$. Thus $E \in \widetilde{\mathcal{M}}$, so $\underline{\mu}$ is saturated.

Clearly $\mu(E) = \underline{\mu}(E)$ for any $E \in \mathcal{M}$, so we are done.

f. First, $\mu(\emptyset) = \mu_0(\emptyset \cap X_1) = \mu_0(\emptyset) = 0$. If $\{E_j\}_1^\infty \subset \mathcal{M}$ is a collection of disjoint measurable sets, then since μ_0 is a measure on $\mathcal{P}(\mathcal{X}_\infty)$, $E_j \cap X_1$ is measurable by μ_0 for each E_j, and these intersections are all disjoint. Then by the additivity of μ_0, we have

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \mu_0(\bigcup_{j=1}^{\infty} E_j \cap X_1) = \sum_{j=1}^{\infty} \mu_0(E_j \cap X_1) = \sum_{j=1}^{\infty} \mu(E_j).$$

So μ is a measure on $\mathcal{M}$.

Since $\widetilde{\mathcal{M}}$ is clearly contained in $\mathcal{P}(\mathcal{X})$, all that needs showing is that any $E \subset X$ is locally measurable by μ. So let $E \subset X$ and let $A \in \mathcal{M}$ so that $\mu(A) < \infty$. We must show that $E \cap A \in \mathcal{M}$. The following table will be helpful.

However, since $\mu(A) = \mu_0(A \cap X_1) < \infty$, $A \cap X_1$ must be finite. Then since X_1 is uncountable, $X_1 \setminus A$ is uncountable. But $X_1 \setminus A \subset A^c$, which is supposed to be countable. So actually the only possible case is when A is countable, in which case $E \cap A$ is always countable, and this shows $\widetilde{\mathcal{M}} = \mathcal{P}(\mathcal{X})$ since $E \subseteq X$ was arbitrary.