Real Analysis; Modern Techniques and their Applications Folland Solutions

Commands

Chapter 1

Section 2: σ-algebras

Exercise 1

A family of sets $\mathcal{R}\subset\mathcal{P}(X)$ is a called a **ring** if it is closed under finite unions and set differences. A ring that is closed under countable unions is called a σ-ring. Prove the following:
a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
b. If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) if and only if $X\in\mathcal{R}$.
c. If $\mathcal{R}$ is a σ-ring, then $\{E\subset X : E\in\mathcal{R}\text{ or }E^c\in\mathcal{R}\}$ is a σ-algebra.
d. If $\mathcal{R}$ is a σ-ring, then $\{E\subset X : E\cap F\in\mathcal{R}\text{ for all }F\in\mathcal{R}\}$ is a σ-algebra.

Solution.

a. Let $\mathcal{R}$ be a ring. Let $E_1,\ldots,E_n\in\mathcal{R}$ and let $E=\bigcup_{j=1}^n E_j$. Then $E\in\mathcal{R}$. Now,

$$E\setminus\left(\bigcap_{j=1}^n E_j\right)=\bigcup_{j=1}^n E\setminus E_j\in\mathcal{R}\$\$sinceeach\$E\setminus E_j\in\mathcal{R}.\$Then$$

\bigcap*{j=1}^nE_j=E\setminus\left(E\setminus\left(\bigcap*{j=1}^nE_j\right)\right)\in\mathcal R.

$$Nowlet\$\mathcal{R}\$bea\$\sigma\$-ring, andlet\$\{E_j\}_1^\infty\subset\mathcal{R}.\$Theexactsameargumentasaboveworks, sin$$

E=\bigcup_{j\in F}E_j

$$whichisin\$\mathcal{R}\$asacountableunion.Then$$

E\setminus\left(\bigcup{j=1}^\infty E_j\right)^c=E\cap\bigcup{j=1}^\infty Ej=\bigcup{j\in F}E_j\in\mathcal R

$$sincethe\$E_j\$arealldisjoint.Then$$

E\setminus\left(E\setminus\left(\bigcup{j=1}^\infty E_j\right)^c\right)=\left(\bigcup{j=1}^\infty E_j\right)^c\in\mathcal R,

$meaning\$\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}.\$**d.**Let\$\mathcal{A}=\{E\subset X: E\cap F\in\mathcal{R}\text{ for all }F\in\mathcal{R}\}.\$Then\$X\in$

(E_1\setminus E_2)\cap F=(E_1\cap F)\setminus(E_2\cap F).\tag{1}

$If\$x\$isanelementintheLHS, then\$x\in E_1\$andnotin\$E_2\$and\$x\in F.\$Thismeans\$x\notin E_2$

F\cap\bigcup E_j=\bigcup(F\cap E_j)\in\mathcal R

as a countable union. So $\mathcal A$ is a σ-ring which contains $X,$ meaning it is a $\sig

(a,b)=\bigcup_{n=1}^\infty\left[\frac{a+b}{2}-\left(1-\frac{1}{n+1}\right)\frac{b-a}{2},\frac{a+b}2+\left(1-\frac{1}{n+1}\right)\frac{b-a}2\right],

$since\$\frac{a+b}{2}\$istheexactmidpointbetween\$a\$and\$b\$and\$1-\frac{1}{n+1}\to 1\$as\$n\to\infty,\$butis$

[a,b]=\bigcap_{n=1}^\infty\left(a-\frac{1}{n},b+\frac{1}n\right),

$and\$\sigma\$-algebrasareclosedundercountableintersections, so\$\mathcal{E}_2\subset\sigma(\mathcal{E}_1)\$andwehaveshown$

(a,b)=\bigcup_{n=N}^\infty\left(a,b-\frac{1}n\right]

$$sothat\$\mathcal{E}_1\subset\sigma(\mathcal{E}_3).\$Conversely, foranyhalf-openinterval\$(a,b],\$$$

(a,b]=\bigcap_{n=1}^\infty\left(a,b+\frac{1}n\right)

$sothat\$\mathcal{E}_3\subset\sigma(\mathcal{E}_1).\$Thus\$\sigma(\mathcal{E}_1)=\sigma(\mathcal{E}_3).\$Anexactlyanalogousargumentcanbeusedtoshow$

(-\infty,a)=\bigcup_{n=1}^\infty\left[a-n,a\right)

$so\$\mathcal{E}_5\subset\sigma(\mathcal{E}_4).\$Foranyhalf-closedinterval\$[a,b)\in\mathcal{E}_4,\$\$(-\infty,b)\setminus(-\infty,a)=[a,b)\in\sigma($

(-\infty,a]=\bigcup_{n=1}^\infty(a-n,b]

so that $\cal E_7\subset\sigma(\cal E_3).$ So the desired equality is shown. The same argument ca

\cal B_\R=\sigma(\cal E_i)=\sigma(\cal E_j)

for any $i,j\in\{1,2,3,\ldots,8\}.$ Whether we have explicitly written the expression or not, any set

F*k=E_k\setminus\left(\bigcup*{n=1}^{k-1}E_n\right)=\emptyset,

then $E_k=\bigcup_{n=1}^{k-1}E_n.$ Then if $$F_{k+1}=E_{k+1}\setminus\left(\bigcup_{n=1

F*k=\bigcup*{j=1}^kE_j

$foreach\$k.\$Since\$\mathcal{A}\$isanalgebra, each\$F_k \in \mathcal{A}\$asafiniteunion. Also\$F_1 \subset F_2 \subset F_3 \subset \ldots$

\bigcup*{k=1}^\infty F_k=\bigcup*{k=1}^\infty\bigcup*{j=1}^kE_j=\bigcup*{j=1}^\infty E_j,

so $\bigcup_{j=1}^\infty E_j\in\cal A.$ ### Exercise 5 If $\cal M$ is a σ-algebra generated

\cal M:=\sigma(\cal E)=\bigcup_{\cal F\in K}\sigma(\cal F).\tag{1}

$LettheRHSof(1)be\$\mathcal{A}.\$Wewillshowthat\$\mathcal{A}\$isa\$\sigma\$-algebra. Let\$E \in \mathcal{A}.\$Then\$E \in \sigma($

{E*j}_1^\infty\subset\bigcup*{j=1}^\infty\sigma(\cal F_j).

$Then\$\bigcup_{j=1}^{\infty}\mathcal{F}_j\$isalsocountableasacountableunionofcountablesets, andisthusin\$K.\$Since\$$

\bigcup*{j=1}^\infty\sigma(\cal F_j)\subset\sigma\left(\bigcup*{j=1}^\infty\cal F_j\right).

$Thensincethelatterobjectis\$\sigma\$-algebrainwhichthecountablecollection\$\{E_j\}\$iscontained$

\bigcup*{j=1}^\infty E_j\in\sigma\left(\bigcup*{j=1}^\infty\cal F*j\right)\subset\bigcup*{\cal F\in K}\sigma(\cal F).

Thus $\cal A$ is closed under countable unions. Given that the $\cal A$ is a σ-algebra, we

(E\cup F)^c=E^c\cap F^c=(E^c\cap N^c)\cup(F^c\cap N)

$and\$E^c \cap N^c \in \mathcal{M}\$and\$F^c \cap N \subset N\$, so\$(E \cup F)^c \in \overline{\mathcal{M}}.\$Thus\$\overline{\mathcal{M}}\$isclosedundercomple$

\bigcup*1^\infty (E_j\cup F_j)=\bigcup*{j=1}^\infty E*j\cup\bigcup*{j=1}^\infty F_j

$$and\$\bigcup E_j \in \mathcal{M}\$and\$\bigcup F_j \subset \bigcup N_j\$and\$\mu(\bigcup N_j) \leq \sum \mu(N_j) = 0, \$so$$

\overline\mu\left(\bigcup*{j=1}^\infty (E_j\cup F_j)\right)=\mu\left(\bigcup*{j=1}^\infty E*j\right)\leq\sum*{j=1}^\infty\mu(E*j)=\sum*{j=1}^\infty \overline\mu(E_j\cup F_j).

$Thus\$\overline{\mu}\$isameasureon\$\overline{\mathcal{M}}.\$Nowshow\$\overline{\mu}\$iscomplete. Let\$E\cup F\in\overline{\mathcal{M}}\$besuchthat\$\overline{\mu}(E\cup$

\overline\mu(E\cup F)=\mu(E)=\nu(E)\leq\nu(E\cup F)
\$\$ since $E\subset E\cup F$ and ν is a measure on $\overline{\mathcal{M}}$. Conversely,

$$\nu(E\cup F)\leq\nu(E)+\nu(F)=\nu(E)=\overline{\mu}(E\cup F),$$

as $\nu(F)=0$ since ν is complete. So $\overline{\mu}$ is the unique complete extension of μ to $\overline{\mathcal{M}}$.

Exercise 7

If $\mu_1,\ldots,\mu_n$ are measures on $(X,\mathcal{M})$ and $a_1,\ldots,a_n\in[0,\infty)$, then $\mu=\sum_{j=1}^n a_j\mu_j$ is a measure on $(X,\mathcal{M})$.

Solution.

Since μ_j is a measure for all $j=1,\ldots,n$, $\mu_j(\emptyset)=0$ for all j. Thus

$$\mu(\emptyset)=\sum_{j=1}^n a_j\mu_j(\emptyset)=\sum_{j=1}^n a_j\cdot 0=0.$$

Now suppose $\{E_k\}_1^\infty\subset\mathcal{M}$. Then since all $a_j\geq 0$ and all μ_j are subadditive,

$$\begin{aligned}\mu\left(\bigcup_{k=1}^\infty E_k\right)&=\sum_{j=1}^n a_j\mu_j\left(\bigcup_{k=1}^\infty E_k\right)\leq\sum_{j=1}^n a_j\sum_{k=1}^\infty\mu_j(E_k)\\&=\sum_{j=1}^n\sum_{k=1}^\infty a_j\mu_j(E_k)=\sum_{k=1}^\infty\sum_{j=1}^n a_j\mu_j(E_k)=\sum_{k=1}^\infty\mu(E_k)\end{aligned}$$

so μ is subadditive. This interchanging of finite and infinite sums works since it is equivalent to the identity

$$\begin{aligned}&\sum_{k=1}^\infty a_1\mu_1(E_k)+\sum_{k=1}^\infty a_2\mu_2(E_k)+\cdots+\sum_{k=1}^\infty a_n\mu_n(E_k)\\&\qquad=\sum_{k=1}^\infty[a_1\mu_1(E_k)+a_2\mu_2(E_k)+\cdots+a_n\mu_n(E_k)],\end{aligned}$$

which is itself equivalent to rearranging the terms in the left hand side. The sum of an absolutely convergent series, as this one is, is independent of any rearrangement of the terms.

Exercise 8

Show that if $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$, provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

Solution.

First, note that since $\mathcal{M}$ is closed under countable unions and countable intersections, $\bigcup_{j=k}^\infty E_j \in \mathcal{M}$ for all $k \in \mathbb{N}$ and thus $\liminf E_j = \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty E_j \in \mathcal{M}$. Then by subadditivity,

$$\mu(\liminf E_j) \leq \sum_{k=1}^\infty \mu\left(\bigcap_{j=k}^\infty E_j\right).$$

Since $\mu(\bigcap_{j=k}^\infty E_j) \leq \mu(E_j)$ for all $j \geq k$ and $k \in \mathbb{N}$,

$$\mu\left(\bigcap_{j=k}^\infty E_j\right) \leq \inf_{j \geq k} \mu(E_j)$$

for all $k \in \mathbb{N}$. Then

$$\mu(\liminf E_j) \leq \sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j)$$

If $\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) = \infty$, then the sequence $\{\inf_{j \geq k} \mu(E_j)\}_{k=1}^\infty$ cannot converge to a finite number, so $\sup_k \inf_{j \geq k} \mu(E_j) = \infty$ also. On the other hand, if $\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) < \infty$, then the sequence $\inf_{j \geq k} \mu(E_j) \to 0$ as $k \to \infty$. But since this is an increasing sequence and μ is nonnegative, $\inf_{j \geq k} \mu(E_j) = 0$ for all $k \in \mathbb{N}$. In either case,

$$\sum_{k=1}^\infty \inf_{j \geq k} \mu(E_j) \leq \sup_k \inf_{j \geq k} \mu(E_j) = \liminf \mu(E_j).$$

For the other inequality, notice that for similar reasoning, $\limsup E_j \in \mathcal{M}$. Then notice that

$$
\begin{aligned}
\limsup E_j &= \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j \\
&= \bigcap_{k=1}^{\infty} \left(\bigcap_{j=k}^{\infty} E_j^c \right)^c \\
&= \left(\bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j^c \right)^c \\
&= (\liminf E_j^c)^c.
\end{aligned}
$$

Next, knowing that $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) < \infty$, we would like to have $(\liminf E_j^c)^c \subset \bigcup E_j$. But

$$
\liminf E_j^c = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j^c \supseteq \bigcap_{j=1}^{\infty} E_j^c \implies (\liminf E_j^c)^c \subseteq \left(\bigcap_{j=1}^{\infty} E_j^c \right)^c = \bigcup_{j=1}^{\infty} E_j
$$

So since the union has finite measure, and by the first proof,

$$
\begin{aligned}
\mu(\limsup E_j) &= \mu((\liminf E_j^c)^c) \\
&= \mu\left(\bigcup_{j=1}^{\infty} E_j\right) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j\right) \setminus (\liminf E_j^c)^c\right) \\
&= \mu\left(\bigcup_{j=1}^{\infty} E_j\right) - \mu\left(\left(\bigcup_{j=1}^{\infty} E_j\right) \cap (\liminf E_j^c)\right) \\
&\geq \mu\left(\bigcup_{j=1}^{\infty} E_j\right) - \mu(\liminf E_j^c) \\
&\geq \mu\left(\bigcup_{j=1}^{\infty} E_j\right) - \liminf \mu(E_j^c)
\end{aligned}
$$

Exercise 9

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then show
$\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Solution.

First notice that $E \cup F = E \cup (F \setminus (E \cap F))$. This choice of rewriting $E \cup F$ is motivated by the de
sire to have some expression which includes $E \cap F$. The sets E and $F \setminus (E \cap F)$ are disjoint so we have

$$\mu\big[E \cup (F \setminus (E \cap F))\big] = \mu(E) + \mu(F \setminus (E \cap F)).$$

If $\mu(F) = +\infty$, then $\mu(E) + \mu(F) = +\infty$ and $\mu(E \cup F) = +\infty$ since $F \subset E \cup F$. Thus the desired equality holds. If, on the other hand, $\mu(F) < +\infty$, then $\mu(E \cap F) < +\infty$ since $E \cap F \subset F$. Thus we can write

$$\mu\big[E \cup (F \setminus (E \cap F))\big] = \mu(E) + \mu(F) - \mu(E \cap F),$$

which implies the desired equality.

Exercise 11

Given a finitely additive measure μ on a measurable space $(X, \mathcal{M})$, show the following:
a) μ is a measure if and only if it is continuous from below.
b) If $\mu(X) < \infty$, then μ is a measure if and only if μ is continuous from above.

Solution.

The forwards direction on both of these follows from Theorem 1.8. For the backwards direction, all that must be shown is the countable additivity over disjoint unions.
(a) Suppose μ is a finitely additive measure which is continuous from below and let $\{E_j\} \subset \mathcal{M}$ be a collection of pairwise disjoint measurable subsets of X. Define F_k to be $\bigcup_{j=1}^k E_j$ for each $k \in \mathbb{N}$. Then $\bigcup_{k=1}^\infty F_k = \bigcup_{j=1}^\infty E_j$ and $F_1 \subset F_2 \subset F_3 \subset \ldots$ and all F_k are measurable. So by the continuity from below and finite additivity of μ,

$$\begin{aligned}
\mu\left(\bigcup_{j=1}^\infty E_j\right) &= \mu\left(\bigcup_{k=1}^\infty F_k\right) \\
&= \lim_{k\to\infty} \mu(F_k) \\
&= \lim_{k\to\infty} \mu\left(\bigcup_{j=1}^k E_j\right) \\
&= \lim_{k\to\infty} \sum_{j=1}^k \mu(E_j) \\
&= \sum_{j=1}^\infty \mu(E_j).
\end{aligned}$$

(b) Suppose μ is a finitely additive measure which is continuous from above and let $\{E_j\} \subset \mathcal{M}$ be a collection of pairwise disjoint measurable subsets of X. Then define

F_k to be $\bigcup_{j=1}^{k} E_j$ as in part (a) so that the unions of each collection are the same and $F_1 \subset F_2 \subset \ldots$ Then define G_k to be F_k^c for each k. Thus $\mu\left(\bigcap_{k=1}^{\infty} G_k\right) = \mu\left(X \setminus \left(\bigcup_{j=1}^{\infty} E_j\right)\right) = \mu(X) - \mu\left(\bigcup_{j=1}^{\infty} E_j\right)$. Then we have $G_1 \supset G_2 \supset G_3 \supset \ldots$ and $\mu(G_1) < \infty$ since $\mu(X) < \infty$, so the continuity from above and finite additivity of μ give

$$\begin{aligned}
\mu\left(\bigcap_{k=1}^{\infty} G_k\right) &= \lim_{k\to\infty} \mu(G_k) \\
&= \lim_{k\to\infty}\left[\mu(X) - \mu\left(\bigcup_{j=1}^{k} E_j\right)\right] \\
&= \mu(X) - \lim_{k\to\infty}\sum_{j=1}^{k}\mu(E_j) \\
&= \mu(X) - \sum_{j=1}^{\infty}\mu(E_j),
\end{aligned}$$

so $\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty}\mu(E_j)$.

Section 4: Outer Measures

Exercise 24

Let μ be a finite measure on $(X, \mathcal{M})$ and let μ^* be the outer measure induced by μ. Suppose that $E \subset X$ is such that $\mu^*(E) = \mu^*(X)$, but not necessarily that $E \in \mathcal{M}$. Show the following:
(a) If $A, B \in \mathcal{M}$ and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.
(b) Let $\mathcal{M}_E = \{A \cap E \colon A \in \mathcal{M}\}$, and define the function $\nu\colon \mathcal{M}_E \to [0, \infty)$ by $\nu(A \cap E) = \mu(A)$. This is well-defined by part (a). Then $\mathcal{M}_E$ is a σ-algebra on E and ν is a measure on $\mathcal{M}_E$.

Solution.

(a) First we show that $\mu^*(A \cap E) = \mu(A)$. This will also apply to B. Since $A \in \mathcal{M}$, the HKC Extension theorem says that A is μ_*-measurable, since $\mathcal{M} \subset \mathcal{M}^*$, the set of μ^*-measurable sets. Thus, $\mu^*(A) = \mu(A)$. Since $A \cap E \subset A$ and μ^* is monotone, $\mu^*(A \cap E) \leq \mu^*(A) = \mu(A)$.

Now, since μ is finite, $\mu(X) < \infty$, so we can say that $\mu(X) = \mu(A) + \mu(A^c)$. Note that $A^c \in \mathcal{M}$, and is thus also μ^*-measurable, so $\mu(A^c) = \mu^*(A^c)$. By definition of μ^*-measurability and since $\mu^*(X) = \mu^*(E)$, we have

$$\mu^*(A) + \mu^*(A^c) = \mu^*(X) = \mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E),$$

meaning

$$\mu^*(A \cap E) = \mu^*(A) + \mu^*(A^c) - \mu^*(A^c \cap E).$$

Then by the monotonicity of μ^*, $\mu^*(A^c \cap E) \leq \mu^*(A^c)$, so $\mu^*(A^c) - \mu^*(A \cap E) \geq 0$. Thus $\mu^*(A \cap E) \geq \mu^*(A)$. So the two are equal.
(b) First show $\mathcal{M}_E$ is a σ-algebra. $\emptyset \in \mathcal{M}_E$ since $\emptyset \in \mathcal{M}$ and $\emptyset = \emptyset \cap E$. Also $E \in \mathcal{M}_E$ since $E = X \cap E$ and $X \in \mathcal{M}$. Now let $F \in \mathcal{M}_E$. Then $F = A \cap E$ for some $A \in \mathcal{M}$. Then $E \setminus F = A^c \cap E \in \mathcal{M}_E$ since $A^c \in \mathcal{M}$. Now let $\{F_j\}_1^\infty \subset \mathcal{M}_E$. Then there is $\{A_j\}_1^\infty \subset \mathcal{M}$ so that $\{F_j\} = \{A_j \cap E\}$. Thus

$$\bigcup_{j=1}^\infty F_j = E \cap \bigcup_{j=1}^\infty A_j \in \mathcal{M}_E$$

since $\bigcup A_j \in \mathcal{M}$.
Now show ν is a measure on $\mathcal{M}_E$. First, $\nu(\emptyset) = \nu(\emptyset \cap E) = \mu(\emptyset) = 0$. Now we prove a lemma to show the countable additivity of ν.
Lemma. Given a measure space $(X, \mathcal{M}, \mu)$ and a collection of sets $\{A_j\}_1^\infty \subset \mathcal{M}$, so that $\mu(A_i) \cap \mu(A_j) = 0$ for all $i \neq j$, the following equality holds:

$$\mu\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty \mu(A_j).$$

Proof. Let $A = \bigcup_{j=1}^\infty A_j$ and let $N = \bigcup_{i \neq j} \mu(A_i \cap A_j)$. By subadditivity of μ, we have $\mu(N) \leq \sum_{i \neq j} \mu(A_i \cap A_j) = 0$, so $\mu(N) = 0$. Thus $\mu(A \setminus N) = \mu(A)$. But

$$\begin{aligned} A \setminus N &= \left(\bigcup_{j=1}^\infty A_j\right) \setminus \left(\bigcup_{i \neq j} A_i \cap A_j\right) \\ &= \bigcup_{j=1}^\infty \left[A_j \setminus \left(\bigcup_{\substack{i=1 \\ i \neq j}}^\infty A_i \cap A_j \right) \right]. \end{aligned}$$

This is a disjoint union, so

$$\mu(A\setminus N)=\sum_{j=1}^{\infty}\mu\left(A_j\setminus\left(\bigcup_{\substack{i=1\\ i\neq j}}^{\infty}A_i\cap A_j\right)\right).\$\$Theinnerunionsallhavezeromeasureassubsets$$

\begin{align}
\nu\left(\bigcup{j=1}^\infty(A_j\cap E)\right)=\mu(\bigcup{j=1}^\infty
Aj)=\sum{j=1}^\infty\mu(Aj)=\sum{j=1}^\infty\nu(A_j\cap E).
\end{align}

Section 5: Borel measures on the Real line ### Exercise 25 ### Exercise 26 Prove Proposition

\mu(U)=\sum_{j=1}^\infty\mu(I_j)<\infty,

$$andthetailofaconvergentseriesisarbitrarilysmall, thereexistsan\$N\in\mathbb{N}\$sothat\$\sum_{j=N+1}^{\infty}\mu(I$$

\begin{align}
\mu(A\triangle E)&=\mu(A\setminus E)+\mu(E\setminus A) \
&\leq\mu(U\setminus E)+\mu(\bigcup{j=N+1}^\infty I_j) \
&<\frac{\epsilon}{2}+\sum{j=N+1}^\infty\mu(I_j) \
&<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{align}

Exercise 27 # Chapter 2 ## Section 1: Measurable Functions ### Exercise 1 ## Section 2: Inte

\begin{align}
\mu({x:f(x)>\lambda})\leq\frac{1}{\lambda}\int_X f;d\mu.
\end{align}

$$**Proof.**Notethattheset\$\{x:f(x)>\lambda\}\in\mathcal{M}\$sinceitismerely\$f^{-1}((\lambda,\infty))\$and\$$$

\begin{align}
\mu({x:f(x)>1})=\sup_\phi\mu({x:\phi(x)>1}).
\end{align}

$$Foranysuch\$\phi,\$thereisastandardexpression\$\phi=\sum_{j=1}^{n}a_j\chi_{E_j}\$fordisjoint\$E_j\$anddistinct\$$$

\begin{align}
\mu({x:\phi(x)>1})=\sum{j\in J}\mu(E_j)\leq\sum{j\in

J}aj\mu(E_j)\leq\sum{j=1}^na_j\mu(E_j)=\int\phi.
\end{align}

$$Thus,$$

\begin{align}
\sup\phi\mu({x:\phi(x)>1})\leq\sup\phi\left{\int\phi\right}=\int f,
\end{align}

wherethesupremumsaretakenoverthecollectionofallsimplefunctions$ \phi $*with*$0 \leq \phi \leq f.$

\mu({x:\frac{1}{\lambda}f(x)>1})\leq\int\frac{1}{\lambda}f

whichreducestothedesiredinequality. $\square$ *Toprovethefirstsetisanullset, considerthatforal*

\mu({x:f(x)>n})\leq\frac{1}{n}\int f

bythelemma. Since$ \int f < \infty $*andthecontinuityfrombelowofmeasuresgivesthat*$\mu(\{x : f($

\mu({x:f(x)=\infty})\leq\lim_{n\to\infty}\frac{1}{n}\int f=0.

For the second set, the conclusion follows from noting that $$\{x:f(x)>0\}=\bigcup_{n=1}^\infty\{

\begin{align}
\lambda(E)&=\sup\left{\intE\phi; d\mu\mid\phi\text{ is simple and }0\leq\phi\leq f\right}
\
&=\sup\left{\sum{j=1}^\infty\int{E_j}\phi; d\mu\right} \
&=\sum{j=1}^\infty\sup\left{\int{E_j}\phi; d\mu\right} \
&=\sum{j=1}^\infty\int{E_j}f;d\mu=\sum{j=1}^\infty\lambda(E_j),
\end{align}

andλ*isameasure. Suppose*$g \in L^+$*issimple. Then*g*hasstandardexpression*$g = \sum_{j=1}^{n} a$

\begin{align}
\int g;d\lambda&=\sum{j=1}^na_j\lambda(E_j) \
&=\sum{j=1}^naj\int{Ej}f;d\mu \
&=\sum{j=1}^n\int{E_j}a_jf;d\mu \
&=\int_X\sum{j=1}^naj\chi{E_j}f;d\mu \

&=\int_Xgf;d\mu.
\end{align}

$Nowlet$g$beany$L^+$function. Thenbytheapproximationtheorem, thereisanincreasingseq$

\begin{align}
\int
g;d\lambda&=\lim{n\to\infty}\int\phi_n;d\lambda=\lim{n\to\infty}\int\phinf;d\mu=\int\lim{
n\to\infty}\phi_nf;d\mu=\int gf;d\mu.
\end{align}

Exercise 15 If $\{f_n\}\subset L^+,$ f_n decreases pointwise to f, and $\int f_1<\infty,$ t

\begin{align}
\int f_1=\int(f_1-f_n+f_n)=\int(f_1-f_n)+\int f_n \tag{1}
\end{align}

$i.e. \$ \int f_1 - \int f_n = \int (f_1 - f_n)\$ foreach\$ n. \$ Similarly, since\$ f_1 \geq f\$, \$ f_1 - f \in L^+\$ andt$

\begin{align}
\int f_1-\int f=\int(f_1-f).
\end{align}

$Since\$ f_1 - f_n\$ increasesto\$ f_1 - f\$ as\$ n \to \infty\$ andallaremeasurable, theMCTgives$

\begin{align}
\int f_1=\lim\left[\int(f_1-f_n)+\int f_n\right]=\int (f_1-f)+\lim\int f_n
\end{align}

$$sothat$$

\begin{align}
\int f_1=\int f_1-\int f+\lim\int f_n,
\end{align}

i.e. $\int f=\lim\int f_n.$ ### Exercise 16 Given $f\in L^+$ and $\int f<\infty,$ show that for every

\mu(E_n)\leq n\int f<\infty.

$So\$\mu(E_n)\$ isalwaysfinite. Alsoitisclearthat\$ E_1 \nearrow X\$, since\$ f(x) > \frac{1}{n} \implies f(x) > \frac{1}{n+}$

\int f=\lim\int f_n.

$Thus, forany$\epsilon > 0, $thereexists$N \in \mathbb{N}$sothat$ \int f - \int f_n < \epsilon$forall$n \geq N. But \int$

\int_{E_n}>\left(\int f\right)-\epsilon

for all $n\geq N.$ ## Section 3: Integration of Complex Functions ## Section 4: Modes of Converg