Exercise 2.14

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Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^+$, let $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Then show that λ is a measure on $\mathcal{M}$ and that for any $g \in L^+$, $\int g d\lambda = \int fg d\mu$.

Solution. First, $\lambda(\emptyset) = \int_\emptyset f d\mu = 0$. Now let $\{E_j\}_1^\infty \subset \mathcal{M}$ be a collection of pairwise disjoint sets. Then if $E = \bigcup_{j=1}^\infty E_j$, since the E_j are all disjoint, the integral of any L^+ simple function over E (being a finite sum) can be split into a countable sum of integrals over each individual E_j. Also taking the supremum of a sum of values which are dependent on disjoint sets is the same as the taking the sum of the supremums over each individual set. Thus

$$\begin{aligned}
\lambda(E) &= \sup\left\{\int_E \phi \, d\mu \mid \phi \text{ is simple and } 0 \le \phi \le f\right\} \\
&= \sup\left\{\sum_{j=1}^\infty \int_{E_j} \phi \, d\mu\right\} \\
&= \sum_{j=1}^\infty \sup\left\{\int_{E_j} \phi \, d\mu\right\} \\
&= \sum_{j=1}^\infty \int_{E_j} f \, d\mu = \sum_{j=1}^\infty \lambda(E_j),
\end{aligned}$$

and λ is a measure.

Suppose $g \in L^+$ is simple. Then g has standard expression $g = \sum_{j=1}^n a_j \chi_{E_j}$ for disjoint sets E_j and distinct a_j. Then in the following string of equalities, we can switch the order of sum and integral since the sets over which we are integrating are all disjoint, by adding the relevant characteristic functions:

$$\begin{aligned}
\int g \, d\lambda &= \sum_{j=1}^n a_j \lambda(E_j) \\
&= \sum_{j=1}^n a_j \int_{E_j} f \, d\mu \\
&= \sum_{j=1}^n \int_{E_j} a_j f \, d\mu \\
&= \int_X \sum_{j=1}^n a_j \chi_{E_j} f \, d\mu \\
&= \int_X gf \, d\mu.
\end{aligned}$$

Now let g be any L^+ function. Then by the approximation theorem, there is an increasing sequence $\{\phi_n\}$ of simple functions with $0 \le \phi_n \le g$ for all n so that $\phi_n \to g$ pointwise on X. Then since $f \in L^+$, the sequence $\phi_n f \to gf$ as well. Then by the Monotone Convergence theorem (applied twice) and the previous identity,

$$\int g \, d\lambda = \lim_{n\to\infty} \int \phi_n \, d\lambda = \lim_{n\to\infty} \int \phi_n f \, d\mu = \int \lim_{n\to\infty} \phi_n f \, d\mu = \int gf \, d\mu.$$