Exercise 1.19

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Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$ define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^* measurable if and only if $\mu^*(E) = \mu_*(E)$. (Use Exercise 18.)

Solution. For the forwards direction, assuming E is μ^*-measurable, we get that for every $A \subseteq X$,

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A). \tag{1}$$

Then since μ^* is generated by μ_0, we have $\mu_0(X) = \mu^*(X)$ and letting $A = X$, we see that

$$\mu_0(X) = \mu^*(E) + \mu^*(E^c) \tag{2}$$

and the forwards direction follows.

Conversely, suppose $\mu^*(E) = \mu_*(E)$, or in other words that $\mu^*(E) + \mu^*(E^c) = \mu_0(X) = \mu^*(X)$. By Exercise 18a, for any $n \in \mathbb{N}$, there exists $A_n \in \mathcal{A}_\sigma$ with $E \subseteq A_n$ so that $\mu^*(A_n) - \mu^*(E) \leq 1/n$. Since $A_n \in \mathcal{A}_\sigma$, A_n is μ^*-measurable for all n. Then

$$\mu^*(E^c) = \mu^*(E^c \cap A_n) + \mu^*(E^c \cap A_n^c) \tag{3}$$

Since $E \subseteq A_n$ for all n, $A_n^c \subseteq E^c$ for all n. Also, $E^c \cap A_n = A_n \setminus E$. So eq. (3) gives us

$$\mu^*(E^c) = \mu^*(A_n \setminus E) + \mu^*(A_n^c). \tag{4}$$

Since A_n is μ^*-measurable, μ_0 generates μ^*, and μ^* is a measure on the σ-algebra of μ^*-measurable set, we have $\mu^*(X) = \mu^*(A_n) + \mu^*(A_n^c)$ for all n. Then using our assumption and eq. (4) and since μ^* is finite, we have for each n

$$\begin{aligned}
\mu^*(A_n \setminus E) &= \mu^*(E^c) - \mu^*(A_n^c) \\
&= \mu^*(X) - \mu^*(E) - \mu^*(A_n^c) \\
&= \mu^*(A_n) + \mu^*(A_n^c) - \mu^*(E) - \mu^*(A_n^c) \\
&= \mu^*(A_n) - \mu^*(E) \leq 1/n
\end{aligned} \tag{5}$$

Then let $A = \bigcap_{n=1}^\infty A_n$. Then $A \in \mathcal{A}_{\sigma\delta}$ and for all n,

$$\mu^*(A \setminus E) \leq \mu^*(A_n \setminus E) \leq 1/n, \tag{6}$$

so $\mu^*(A \setminus E) = 0$. By the equivalence in Exercise 18b, since μ_0 is finite, $\mu^*(E)$ is finite, so we have the existence of the required set in $A_{\sigma\delta}$, meaning E is μ^*-measurable.
