Exercise 3.3

Nolan Hauck

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Let ν be a signed measure on $(X, \mathcal{M})$.

a. $L^1(\nu) = L^1(|\nu|)$.

b. If $f \in L^1(\nu)$ then $\left|\int f d\nu\right| \leq \int |f|\, d\,|\nu|$.

c. If $E \in \mathcal{M}$ then $|\nu|(E) = \sup\{\left|\int_E f d\nu\right| : |f| \leq 1\}$.

Solution (INCOMPLETE). a. Note for a positive measurable function F and positive measures μ_1 and μ_2 on the same space we have $\int F d(\mu_1 + \mu_2) = \int F d\mu_1 + \int F d\mu_2$. Thus for the positive measures ν^+ and ν^- and a measurable function f, we have the following:

$$\int |f|\, d\nu^+ + \int |f|\, d\nu^- = \int |f|\, d(\nu^+ + \nu^-) = \int |f|\, d\,|\nu|\,. \tag{1}$$

If $f \in L^1(\nu)$ then $f \in L^1(\nu^+)$ and $f \in L^1(\nu^-)$ by definition, so that $\int |f|\, d\,|\nu| < \infty$ and thus $f \in L^1(|\nu|)$. Conversely, if $f \in L^1(|\nu|)$, then the right hand side of equation (1) is finite, meaning the two parts of the sum on the left hand side are finite, and that $f \in L^1(\nu)$.

b. Note that for a positive measure μ and a measurable function F, $\left|\int F d\mu\right| \leq \int |F|\, d\mu$. If $f \in L^1(\nu)$, then again by the note in part **a.** and since $\nu^\pm \geq 0$, we have

$$\begin{aligned}
\left|\int f d\nu\right| &= \left|\int f d\nu^+ - \int f d\nu^-\right| \\
&\leq \left|\int f d\nu^+\right| + \left|\int f d\nu^-\right| \\
&\leq \int |f|\, d\nu^+ + \int |f|\, d\nu^- \\
&= \int |f|\, d(\nu^+ + \nu^-) \\
&= \int |f|\, d\,|\nu|\,.
\end{aligned} \tag{2}$$

c. It should be mentioned that although it is not stated in the problem, this assumes that the f in the collection over which we are taking the supremum must be in $L^1(\nu)$ as otherwise their integrals with respect to ν would not be defined. Using the bound in part **b.**, we find that for any $f \in L^1(\nu)$ so that $|f| \leq 1$

$$\left|\int_E f d\nu\right| \leq \int_E |f|\, d\,|\nu| \leq \int_E d\,|\nu| = |\nu|(E). \tag{3}$$

Thus $|\nu|(E)$ is an upper bound for the set $\{\left|\int_E f d\nu\right| : |f| \leq 1\}$. If $|\nu|(E) = 0$, then $\left|\int_E f d\nu\right| = \left|\int_E f d\nu^+ - \int_E f d\nu^-\right| = 0$ since $\nu^+(E) = \nu^-(E) = 0$. Now suppose $|\nu|(E) = \infty$. Then either $\nu^+(E)$ or $\nu^-(E)$ is ∞. Assume that $\nu^+(E) = \infty$.

So let $0 < |\nu|(E) < \infty$. Then $\int_E 1 d|\nu| < \infty$, so the function $f(x) = 1$ for all $x \in E$ is in $L^1(|\nu|)$. By part **a.**, f is also in $L^1(\nu)$ and $|f| \leq 1$ obviously, so f is in the collection of functions over which we take the supremum. But this means $\left|\int_E d\nu\right|$ is in the set over which we take the supremum

Then for any $\epsilon > 0$,

$$|\nu|(E) - \epsilon = \int_E \left(1 - \frac{\epsilon}{|\nu|(E)}\right) d|\nu|. \tag{4}$$

Since $|\nu|(E)$ is positive and finite, we can choose ϵ small enough so that $\frac{\epsilon}{|\nu|(E)} < 1$. Then $\left|1 - \frac{\epsilon}{|\nu|(E)}\right| = 1 - \frac{\epsilon}{|\nu|(E)} < 1$. Thus the function $g(x) = 1 - \frac{\epsilon}{|\nu|(E)}$ is measurable as a constant function and sits in $L^1(E, |\nu|)$ since it has a finite integral over E. By part **a.**, $g \in L^1(E, \nu)$ as well. So we have

$$\int_E \left(1 - \frac{\epsilon}{|\nu|(E)}\right) d|\nu| \leq \int_E d|\nu| = |\nu|(E) \tag{5}$$

and g is in the collection of functions over which we take the supremum.