Exercise 1.7

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If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\mu = \sum_{j=1}^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Solution. Since μ_j is a measure for all $j = 1, \ldots, n$, $\mu_j(\emptyset) = 0$ for all j. Thus

$$\mu(\emptyset) = \sum_{j=1}^n a_j \mu_j(\emptyset) = \sum_{j=1}^n a_j \cdot 0 = 0.$$

Now suppose $\{E_k\}_1^\infty \subset \mathcal{M}$. Then since all $a_j \geq 0$ and all μ_j are subadditive,

$$\mu\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{j=1}^n a_j \mu_j \left(\bigcup_{k=1}^\infty E_k\right) \leq \sum_{j=1}^n a_j \sum_{k=1}^\infty \mu_j(E_k) \tag{1}$$

$$= \sum_{j=1}^n \sum_{k=1}^\infty a_j \mu_j(E_k) = \sum_{k=1}^\infty \sum_{j=1}^n a_j \mu_j(E_k) = \sum_{k=1}^\infty \mu(E_k) \tag{2}$$

so μ is subadditive. This interchanging of finite and infinite sums works since it is equivalent to the identity

$$\sum_{k=1}^\infty a_1 \mu_1(E_k) + \sum_{k=1}^\infty a_2 \mu_2(E_k) + \cdots + \sum_{k=1}^\infty a_n \mu_n(E_k)$$

$$= \sum_{k=1}^\infty \left[a_1 \mu_1(E_k) + a_2 \mu_2(E_k) + \cdots + a_n \mu_n(E_k)\right], \tag{3}$$

which is itself equivalent to rearranging the terms in the left hand side. The sum of an absolutely convergent series, as this one is, is independent of any rearrangement of the terms.