Exercise 1.20

Nolan Hauck

Last updated Saturday 9th March, 2024 at 13:36

Let μ^* be an outer measure on X, $\mathcal{M}^*$ the σ-algebra of μ^*-measurable sets, $\overline{\mu} = \mu^*\big|_{\mathcal{M}^*}$, and μ^+ the outer measure indced by $\overline{\mu}$ as in (1.12) (with $\overline{\mu}$ and $\mathcal{M}^*$ replacing μ_0 and $\mathcal{A}$).

a. If $E \subset X$, we have $\mu^*(E) \leq \mu^+(E)$ with equality if there exists $A \in \mathcal{M}^*$ with $A \supset E$ and $\mu^*(A) = \mu^*(E)$.

b. If μ^* is induced from a premeasure, then $\mu^* = \mu^+$. (Use Exercise 18a)

c. If $X = \{0, 1\}$, there exists an outer measure μ^* on X such that $\mu^* \neq \mu^+$.

Solution. **a.** Note that for $E \subset X$,

$$\mu^+(E) = \inf\{\sum_{j=1}^{\infty} \mu^*(A_j),\ E \subset \bigcup_{j=1}^{\infty} A_j,\ A_j \in \mathcal{M}^*\}. \tag{1}$$

(We could have written this in terms of $\overline{\mu}$, but $\overline{\mu}$ is μ^* on $\mathcal{M}^*$, so there is no difference. In fact, I believe introducing $\overline{\mu}$ is just cluttering notation. For any $A = \bigcup_{j=1}^{\infty} A_j$ which covers E by sets in $\mathcal{M}^*$, $\mu^*(E) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$ by the monotonicity of μ^*. Thus $\mu^*(E) \leq \mu^+(E)$, by eq. (1).

On the other hand, if there exsts $A \in \mathcal{M}^*$ with $E \subset A$ and $\mu^*(E) = \mu^*(A)$, then A is one of the sets over which the infimum in eq. (1) is taken, so $\mu^+(E) \leq \mu^*(A) = \mu^*(E)$ and by the first argument, the two functions agree.

b. By part a., we already know $\mu^* \leq \mu^+$. Suppose μ^* is induced by the premeasure μ_0 on the algebra $\mathcal{A}$. Then for any $E \subset X$,

$$\mu^*(E) = \inf\{\sum_{j=1}^{\infty} \mu_0(A_j),\ E \subset \bigcup_{j=1}^{\infty} A_j,\ A_j \in \mathcal{A}\}. \tag{2}$$

By Exercise 18a, for any $E \subset X$ and for any $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$. Since μ^* was generated by μ_0 and since each of the sets which are being unioned to create A are themselves in $\mathcal{A}$, their union A is in $\mathcal{M}^*$. Also if $A = \bigcup_1^\infty A_j$, then $\mu_0(A_j) = \mu^*(A_j)$ for all j. Thus the collection $\{A, \emptyset, \emptyset, \ldots\}$ is one of those over which the infimum in eq. (1) is begin taken. So for all $\epsilon > 0$,

$$\mu^+(E) \leq \mu^*(A) \leq \mu^*(E) + \epsilon \tag{3}$$

i.e. $\mu^+(E) \leq \mu^*(E)$. So we are done.

c. Something fun to notice here: the actual symbols that we use for the two elements in X are irrelevant. Since $\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, X\}$, there are only very few rules that μ^* needs to satisfy to be an outer measure. $\mu^*(\emptyset) = 0$ no matter what. Then we can arbitrarily assign values to $\mu^*(\{0\})$ and $\mu^*(\{1\})$. We must only ensure that $\mu^*(X) \leq \mu^*(\{0\}) + \mu^*(\{1\})$. In order to construct a working counterexample, however, we should choose values so that μ^* is not additive, i.e. the previous inequality is strict. Then $\{0\}$ and $\{1\}$ will not be μ^*-measurable, as we will see.

Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ be given by $\mu^*(\emptyset) = 0$, $\mu^*(\{0\}) = 2$, $\mu^*(\{1\}) = 3$ and $\mu^*(X) = 4$. Then μ^* is clearly monotone, and $\mu^*(\{0\} \cup \{1\}) = \mu^*(X) = 4 < 2 + 3 = \mu^*(\{0\}) + \mu^*(\{1\})$, so μ^* is sub-additive and therefore an outer measure on X.

Now we find $\mathcal{M}^*$. The empty set and X are always in $\mathcal{M}^*$. However, neither $\{0\}$ nor $\{1\}$ are, since $\{0\}^c = \{1\}$ and

$$4 = \mu^*(X) \neq \mu^*(\{0\} \cap X) + \mu^*(\{1\} \cap X) = 2 + 3 = 5. \tag{4}$$

Then by eq. (1), $\mu^+(\{0\}) = \mu^*(X) = 4$, but $\mu^*(\{0\}) = 2$, so $\mu^* \neq \mu^+$.
