Exercise 1.9

Nolan Hauck

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then show $\mu(E)+\mu(F) = \mu(E\cup F)+\mu(E\cap F)$.

Solution. First notice that $E \cup F = E \cup (F \setminus (E \cap F))$. This choice of rewriting $E \cup F$ is motivated by the desire to have some expression which includes $E \cap F$. The sets E and $F \setminus (E \cap F)$ are disjoint so we have

$$\mu\Big[E \cup (F \setminus (E \cap F))\Big] = \mu(E) + \mu(F \setminus (E \cap F)).$$

If $\mu(F) = +\infty$, then $\mu(E) + \mu(F) = +\infty$ and $\mu(E \cup F) = +\infty$ since $F \subset E \cup F$. Thus the desired equality holds. If, on the other hand, $\mu(F) < +\infty$, then $\mu(E \cap F) < +\infty$ since $E \cap F \subset F$. Thus we can write

$$\mu\Big[E \cup (F \setminus (E \cap F))\Big] = \mu(E) + \mu(F) - \mu(E \cap F),$$

which implies the desired equality.