Exercise 1.28

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Let F be increasing and right continuous, and let μ_F be the associated measure. Then $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a,b)) = F(b-) - F(a-)$, $\mu_F([a,b]) = F(b) - F(a-)$, and $\mu_F((a,b)) = F(b-) - F(a)$.

Solution. The singleton $\{a\}$ can be written as $\bigcap_{n=1}^{\infty}(a-1/n, a]$. Then since μ_F is a measure, it is continuous from above, and since $\mu_F((a-1,a]) = F(a) - F(a-1) < \infty$ and $(a-1/n, a] \supset (a-1/(n+1), a]$ for all n, we have

$$\begin{aligned}\mu_F(\{a\}) &= \mu_F\Big(\bigcap_{n=1}^{\infty}(a-1/n, a]\Big) = \lim_{n\to\infty}\mu_F((a-1,a]) \\ &= \lim_{n\to\infty}(F(a) - F(a-1/n)) = F(a) - F(a-).\end{aligned} \tag{1}$$

Next, $[a,b) = (\{a\}\cup(a,b])\setminus\{b\}$, so since all sets are of finite measure and $\{a\}\cap(a,b] = \emptyset$, we have

$$\begin{aligned}\mu_F([a,b)) &= \mu_F(\{a\}) + \mu_F((a,b]) - \mu_F(\{b\}) \\ &= F(a) - F(a-) + F(b) - F(a) - F(b) + F(b-) \\ &= F(b-) - F(a-).\end{aligned} \tag{2}$$

Next, $[a,b] = \{a\}\cup(a,b]$, so

$$\begin{aligned}\mu_F([a,b]) &= \mu_F(\{a\}) + \mu_F((a,b]) \\ &= F(a) - F(a-) + F(b) - F(a) = F(b) - F(a-).\end{aligned} \tag{3}$$

Finally, $(a,b) = (a,b]\setminus\{b\}$, so

$$\mu_F((a,b)) = F(b) - F(a) - F(b) + F(b-) = F(b-) - F(a). \tag{4}$$
