Exercise 1.3

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Let $\mathcal{M}$ be an infinite σ-algebra. Then show the following:

a. $\mathcal{M}$ contains an infinite sequence of disjoint sets.

b. $\operatorname{card}(\mathcal{M}) \geq \mathfrak{c}$.

Solution. a. Since $\mathcal{M}$ is infinite, there exists a countable collection $\{E_n\}_1^\infty \subset \mathcal{M}$ comprised of distinct sets. If this collection is already pairwise disjoint we are done. If $F_k = E_k \setminus \left(\bigcup_{n=1}^{k-1} E_n\right) \neq \emptyset$ for all $k \underset{\neq}{>} 1$ (it's fine if $E_1 = F_1 = \emptyset$), then we are also done, as the collection $\{F_k\}$ will be a subset of $\mathcal{M}$ and pairwise disjoint. So suppose there is at least one such F_k with $k \underset{\neq}{>} 1$ which is empty, and that $E_1 \neq \emptyset$. Note that $F_2 = E_1 \setminus E_2$ is never empty, since the E_n are distinct. We prove that $F_k = \emptyset \implies F_{k+1} \neq \emptyset$ for all k. If

$$F_k = E_k \setminus \left(\bigcup_{n=1}^{k-1} E_n\right) = \emptyset,$$

then $E_k = \bigcup_{n=1}^{k-1} E_n$. Then if

$$F_{k+1} = E_{k+1} \setminus \left(\bigcup_{n=1}^{k} E_n\right) = E_{k+1} \setminus \left(E_k \cup \bigcup_{n=1}^{k-1} E_n\right) = E_{k+1} \setminus E_k = \emptyset,$$

then $E_{k+1} = E_k$, contradicting the distinctness of the collection. Now, if there are only finitely many non-empty F_k, then there exists $K \in \mathbb{N}$ so that $F_k = \emptyset$ for all $k \geq K$. But this is a contradiction since there cannot be consecutive empty F_k's. Thus there is a countably infinite sub-collection $\{F_{k_j}\}_{j=1}^\infty$ of disjoint sets. Finally, each $F_k \in \mathcal{M}$ since σ-algebras are closed under finite unions and set differences.

b. Suppose $\mathcal{M}$ were countable. Then suppose $\mathcal{M} = \{E_n\}_1^\infty$ is a distinct labelling of all the elements of $\mathcal{M}$. If all the E_n are disjoint, then $E_1 \cup E_2 \in \mathcal{M}$ and is not equal to any E_n for $n \geq 3$. This contradicts that we have labelled all elements of $\mathcal{M}$. If not all the elements of E_n are disjoint, then by part **a.,** we can construct a disjoint sequence $\{F_k\}_1^\infty$ from the E_n's. At least one of these F_k's must be distinct from all the E_n's as otherwise the original labelling would be disjoint and we are in the first case. Thus the original labelling cannot be complete. So $\mathcal{M}$ is uncountable.