Exercise 2.15

Nolan Hauck

If $\{f_n\} \subset L^+$, f_n decreases pointwise to f, and $\int f_1 < \infty$, then show $\int f = \lim \int f_n$.

Solution. Since f_n decreases to f, which is measurable as the limit of a sequence of measurable functions, we have $f_1 \geq f_n$ for all n and thus $\infty > \int f_1 \geq \int f_n$ for all n. Then $f_1 - f_n \geq 0$ so that $f_1 - f_n \in L^+$ for all n. So for each n,

$$\int f_1 = \int (f_1 - f_n + f_n) = \int (f_1 - f_n) + \int f_n \tag{1}$$

i.e. $\int f_1 - \int f_n = \int (f_1 - f_n)$ for each n. Similarly, since $f_1 \geq f$, $f_1 - f \in L^+$ and the following equality holds:

$$\int f_1 - \int f = \int (f_1 - f). \tag{1}$$

Since $f_1 - f_n$ increases to $f_1 - f$ as $n \to \infty$ and all are measurable, the MCT gives

$$\int f_1 = \lim \left[\int (f_1 - f_n) + \int f_n \right] = \int (f_1 - f) + \lim \int f_n$$

so that

$$\int f_1 = \int f_1 - \int f + \lim \int f_n, \tag{2}$$

i.e. $\int f = \lim \int f_n$.