Exercise 1.4

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Given an algebra $\mathcal{A}$, show that $\mathcal{A}$ is a σ-algebra if and only if $\mathcal{A}$ is closed under countable increasing unions (i.e. if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset E_3 \subset \ldots$, then $\bigcup_1^\infty E_j \in \mathcal{A}$).

Solution. ($\Longrightarrow$) This direction is immediate, since σ-algebras are closed under any countable unions.

($\Longleftarrow$) Conversely, suppose that $\mathcal{A}$ is an algebra which is closed under increasing unions, as described in the problem. Since $\mathcal{A}$ is an algebra, all that remains to be shown is that $\mathcal{A}$ is closed under any countable unions. Let $\{E_j\}_1^\infty$ be a collection of sets in $\mathcal{A}$. Define a new collection $\{F_k\}_1^\infty$ via the definition

$$F_k = \bigcup_{j=1}^{k} E_j$$

for each k. Since $\mathcal{A}$ is an algebra, each $F_k \in \mathcal{A}$ as a finite union. Also $F_1 \subset F_2 \subset F_3 \subset \ldots$, so this is an increasing collection of sets in $\mathcal{A}$. Thus $\bigcup_{k=1}^\infty F_k \in \mathcal{A}$. But

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{k} E_j = \bigcup_{j=1}^{\infty} E_j,$$

so $\bigcup_{j=1}^\infty E_j \in \mathcal{A}$.