Exercise 1.1

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A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is a called a **ring** if it is closed under finite unions and set differences. A ring that is closed under countable unions is called a σ-ring. Prove the following:

a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

b. If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) if and only if $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

d. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Solution.

a. Let $\mathcal{R}$ be a ring. Let $E_1, \ldots, E_n \in \mathcal{R}$ and let $E = \bigcup_{j=1}^n E_j$. Then $E \in \mathcal{R}$. Now,

$$E \setminus \left(\bigcap_{j=1}^n E_j \right) = \bigcup_{j=1}^n E \setminus E_j \in \mathcal{R} \tag{1}$$

since each $E \setminus E_j \in \mathcal{R}$. Then

$$\bigcap_{j=1}^n E_j = E \setminus \left(E \setminus \left(\bigcap_{j=1}^n E_j \right) \right) \in \mathcal{R}.$$

Now let $\mathcal{R}$ be a σ-ring, and let $\{E_j\}_1^\infty \subset \mathcal{R}$. The exact same argument as above works, since DeMorgan's laws work for arbitrary intersections and unions.

b. Let $\mathcal{R}$ be a ring and suppose it is also an algebra. Then $\mathcal{R}$ is closed under complements, so if $E \in \mathcal{R}$, then $E^c \in \mathcal{R}$. But then $\mathcal{R}$ is closed under finite unions, so $X = E \cup E^c \in \mathcal{R}$. Conversely, suppose $\mathcal{R}$ is a ring and $X \in \mathcal{R}$. Then if $E \in \mathcal{R}$, $E^c = X \setminus E \in \mathcal{R}$ as well since rings are closed under set differences. So $\mathcal{R}$ is an algebra.

Again, an exactly analogous argument works since rings are to algebras as σ-rings are to σ-algebras. The only difference from the former to the latter is closedness under complements.

c. Let $\mathcal{A} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. Then let $E \in \mathcal{A}$ and show $E^c \in \mathcal{A}$. There are two cases. If $E \in \mathcal{R}$, then $E^c \in \mathcal{A}$ since $(E^c)^c = E \in \mathcal{R}$. On the other hand, if $E^c \in \mathcal{R}$, then immediately $E^c \in \mathcal{A}$. First, let $E, F \in \mathcal{A}$. Then if both E and F are in $\mathcal{R}$, then $E \cup F \in \mathcal{A}$. If $E, F^c \in \mathcal{R}$, then by the above, $F = (F^c)^c \in \mathcal{R}$ and we are in the first case. A similar logic handles the other two possibilities. By induction, $\mathcal{A}$ is closed under finite unions. By part **a.**, $\mathcal{A}$ is also closed under finite intersections. So, given a collection $\{E_j\}_1^\infty \subset \mathcal{A}$, we can define a new collection $\{F_j\}_1^\infty$ by setting $F_k = E_k \setminus \bigcup_{j=1}^{k-1} E_j$ for each k. Then these F_j are disjoint and $\bigcup F_j = \bigcup E_j$. Also $F_k = E_k \cap \left(\bigcup_{j=1}^{k-1} E_j \right)$, so $F_k \in \mathcal{A}$ since each $E_k \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite unions and intersections. Thus, we have reduced our problem to showing that $\mathcal{A}$

is closed under countable disjoint unions. So let $\{E_j\}_1^\infty \subset \mathcal{A}$ be a collection of disjoint sets in $\mathcal{A}$. Let $F \subset \mathbb{N}$ be the collection of indices j such that $E_j \in \mathcal{R}$. Then $F^c = \mathbb{N} \setminus F$ is the collection of indices so that $E_j^c \in \mathcal{R}$. Then define

$$E = \bigcup_{j \in F} E_j$$

which is in $\mathcal{R}$ as a countable union. Then

$$E \setminus \left(\bigcup_{j=1}^{\infty} E_j \right)^c = E \cap \bigcup_{j=1}^{\infty} E_j = \bigcup_{j \in F} E_j \in \mathcal{R}$$

since the E_j are all disjoint. Then

$$E \setminus \left(E \setminus \left(\bigcup_{j=1}^{\infty} E_j \right)^c \right) = \left(\bigcup_{j=1}^{\infty} E_j \right)^c \in \mathcal{R},$$

meaning $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$.

d. Let $\mathcal{A} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Then $X \in \mathcal{A}$ since $X \cap F = F \in \mathcal{R}$ for all $F \in \mathcal{R}$. Let $E_1, E_2 \in \mathcal{A}$ and $F \in \mathcal{R}$. Then show

$$(E_1 \setminus E_2) \cap F = (E_1 \cap F) \setminus (E_2 \cap F). \tag{1}$$

If x is an element in the LHS, then $x \in E_1$ and not in E_2 and $x \in F$. This means $x \notin E_2 \cap F$, so we have shown $\subseteq$. Now suppose x is an element in the RHS. Then $x \in E_1$ and in F, but x is not in the intersection $E_2 \cap F$. If $x \in F$ and $x \notin E_2 \cap F$, then x cannot be in E_2. So we have shown $\supseteq$ (note that this equality has nothing to do with the rings or σ-algebras, and is general to set theory). Now since $F \in \mathcal{R}$ and $E_1, E_2 \in \mathcal{A}$, $E_j \cap F \in \mathcal{R}$ for $j = 1, 2$. Since $\mathcal{R}$ is closed under set differences, the RHS of (1) is in $\mathcal{R}$, meaning $E_1 \setminus E_2 \in \mathcal{A}$. Now suppose $\{E_j\}_1^\infty \subset \mathcal{A}$. Then for all $F \in \mathcal{R}$,

$$F \cap \bigcup E_j = \bigcup (F \cap E_j) \in \mathcal{R}$$

as a countable union. So $\mathcal{A}$ is a σ-ring which contains X, meaning it is a σ-algebra by part **b.**