Exercise 1.21

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Let μ^* be an outer measure induced from a premeasure and $\overline{\mu}$ the restriction of μ^* to μ^*-measurable sets. Then $\overline{\mu}$ is saturated. (Use Exercise 18.)

Solution. Let $\mathcal{M}^*$ be the σ-algebra of μ^*-measurable sets. Recall from Exercise 16 that $\overline{\mu}$ is said to be saturated if every locally measurable set is also in $\mathcal{M}^*$. That is, if $E \subset X$ is such that $E \cap A$ is μ^* measurable for all $A \in \mathcal{M}^*$ with $\overline{\mu}(A) < \infty$, then $E \in \mathcal{M}^*$.

With that being said, let E be locally measurable, and let $B \subset X$. Since μ^* is subadditive, all we must show is that

$$\mu^*(B) \geq \mu^*(E \cap B) + \mu^*(E^c \cap B). \tag{1}$$

If $\mu^*(B) = \infty$, the inequality is obvious. So suppose $\mu^*(B) < \infty$. Let μ_0 be the premeasure on the algebra $\mathcal{A}$ which generates μ^*. Then by Exercise 18a, for any $\epsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ so that $B \subseteq A$ and $\mu^*(A) < \mu^*(B) + \epsilon$. Since $\mu^*(B) < \infty$, $\mu^*(A) < \infty$ as well. A is a union of sets in $\mathcal{A}$, so $A \in \mathcal{M}^*$. Then since E is locally measurable, $E \cap A \in \mathcal{M}^*$. In exercise 16, we showed the set of locally measurable sets relative to any measure is a σ-algebra, so E^c is also locally measurable, meaning $E^c \cap A \in \mathcal{M}^*$ as well. Then since μ^* is a measure on $\mathcal{M}^*$,

$$\mu^*(A) = \mu^*((E \cap A) \cup (E^c \cap A)) = \mu^*(E \cap A) + \mu^*(E^c \cap A). \tag{2}$$

Thus by the monotonicity of μ^*,

$$\mu^*(E \cap B) + \mu^*(E^c \cap B) \leq \mu^*(E \cap A) + \mu^*(E^c \cap A) = \mu^*(A) \leq \mu^*(B) + \epsilon. \tag{3}$$

Then ϵ was arbitrary, so the inequality (1) is shown.
