Exercise 1.5

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If $\mathcal{M}$ is a σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all the countable subsets of $\mathcal{E}$ (Hint: show the latter object is itself a σ-algebra).

Solution. Let $K = \{\mathcal{F} : \mathcal{F} \text{ is a countable subset of } \mathcal{E}\}$ Then we must show that

$$\mathcal{M} := \sigma(\mathcal{E}) = \bigcup_{\mathcal{F} \in K} \sigma(\mathcal{F}). \tag{1}$$

Let the RHS of (1) be $\mathcal{A}$. We will show that $\mathcal{A}$ is a σ-algebra. Let $E \in \mathcal{A}$. Then $E \in \sigma(\mathcal{F})$ for some $\mathcal{F} \in K$, so $E^c \in \sigma(\mathcal{F})$, meaning $E^c \in \mathcal{A}$. Now let $\{E_j\}_1^\infty$ be a collection of sets in $\mathcal{A}$. For each $j \in \mathbb{N}$, there is a countable $\mathcal{F}_j \subset \mathcal{E}$ so that $E_j \in \sigma(\mathcal{F}_j)$ meaning

$$\{E_j\}_1^\infty \subset \bigcup_{j=1}^\infty \sigma(\mathcal{F}_j).$$

Then $\bigcup_{j=1}^\infty \mathcal{F}_j$ is also countable as a countable union of countable sets, and is thus in K. Since $\mathcal{F}_\ell \subset \bigcup_{j=1}^\infty \mathcal{F}_j$ for each ℓ, $\sigma(\mathcal{F}_\ell) \subset \sigma(\bigcup_{j=1}^\infty \mathcal{F}_j)$ for each ℓ. But this means that

$$\bigcup_{j=1}^\infty \sigma(\mathcal{F}_j) \subset \sigma\left(\bigcup_{j=1}^\infty \mathcal{F}_j\right).$$

Then since the latter object is σ-algebra in which the countable collection $\{E_j\}$ is contained, and since the latter object is a subset of $\mathcal{A}$, we have

$$\bigcup_{j=1}^\infty E_j \in \sigma\left(\bigcup_{j=1}^\infty \mathcal{F}_j\right) \subset \bigcup_{\mathcal{F} \in K} \sigma(\mathcal{F}).$$

Thus $\mathcal{A}$ is closed under countable unions. Given that the $\mathcal{A}$ is a σ-algebra, we now show the identity in (1). Suppose $E \in \mathcal{E}$. Then $\{E\}$ is a countable subset of $\mathcal{E}$, so that $\sigma(\{E\}) \subset \mathcal{A}$. But $E \in \sigma(\{E\})$, so $E \in \mathcal{A}$. Thus $\mathcal{E} \subset \mathcal{A}$, and since $\mathcal{A}$ is a σ-algebra, $\sigma(\mathcal{E}) \subset \mathcal{A}$. Conversely, suppose $A \in \mathcal{A}$. Then $A \in \sigma(\mathcal{F})$ for some $\mathcal{F} \in K$. But $\mathcal{F} \subset \mathcal{E}$, so $\sigma(\mathcal{F}) \subset \sigma(\mathcal{E})$, meaning $A \in \sigma(\mathcal{E})$. Thus $\mathcal{A} \subset \sigma(\mathcal{E})$. We have shown inclusion in both directions, so the identity in (1) holds.