Exercise 2.16

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Given $f \in L^+$ and $\int f < \infty$, show that for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Solution. Notice that the desired inequality is equivalent to $\int f - \int_E f < \epsilon$. This motivates one to try describing a sequence of sets $\{E_n\}$ so that $\int_{E_n} f \to \int f$, since $\int_E f = \int f\chi_E$. This necessitates that this sequence converges from below to X. Well, define $E_n = \{x : f(x) > \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then by the Lemma in 2.12, the following inequality holds for all n:

$$\mu(E_n) \leq n \int f < \infty.$$

So $\mu(E_n)$ is always finite. Also it is clear that $E_1 \nearrow X$, since $f(x) > \frac{1}{n} \implies f(x) > \frac{1}{n+1}$. Now define the sequence $\{f_n\}$ of functions by $f_n = \chi_{E_n} f$ for all n. Then $f_n \in L^+$ for all n and $f_1 \leq f_2 \leq f_3 \leq \dots$ and $f_n \nearrow f$. By the Monotone Convergence Theorem,

$$\int f = \lim \int f_n.$$

Thus, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ so that $\int f - \int f_n < \epsilon$ for all $n \geq N$. But $\int f_n = \int f\chi_{E_n} = \int_{E_n} f$. So

$$\int_{E_n} > \left(\int f\right) - \epsilon$$

for all $n \geq N$.