# Exercise 1.2

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Complete the proof of Proposition 1.2, which states that $\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

**a.** the open intervals $\mathcal{E}_1 = \{(a,b) : a < b\}$;

**b.** the closed intervals $\mathcal{E}_2 = \{[a,b] : a < b\}$;

**c.** the half-open intervals: $\mathcal{E}_3 = \{(a,b] : a < b\}$ or $\mathcal{E}_4 = \{[a,b) : a < b\}$;

**d.** The open rays in either direction; and

**e.** The closed rays in either direction.

**Solution.** First, note that $\mathcal{B}_{\mathbb{R}} = \sigma(\tau)$, where $\tau$ is the open sets on $\mathbb{R}$ in the usual topology, which has as countable basis the open intervals with rational endpoints. Thus, any open set in $\mathbb{R}$ is a countable union of open intervals.

**a.** Clearly, $\mathcal{E}_1 \subset \tau$ since open intervals are, well, open. So $\sigma(\mathcal{E}_1) \subset \sigma(\tau) = \mathcal{B}_{\mathbb{R}}$. Conversely, consider an open set $E \in \tau$. Then as in the introduction, $E = \bigcup_1^\infty (a_j, b_j)$, a countable union of open intervals (an infinite number of these could be empty, like if $E$ were itself an open interval). But this means $E \in \sigma(\mathcal{E}_1)$ since $\sigma$-algebras are closed under countable unions. So $\tau \subset \sigma(\mathcal{E}_1) \implies \sigma(\tau) \subset \sigma(\mathcal{E}_1)$.

**b.** We will show $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$ and apply **a.** For an open interval $(a,b) \in \mathcal{E}_1$,

$$(a,b) = \bigcup_{n=1}^\infty \left[\frac{a+b}{2} - \left(1 - \frac{1}{n+1}\right)\frac{b-a}{2}, \frac{a+b}{2} + \left(1 - \frac{1}{n+1}\right)\frac{b-a}{2}\right],$$

since $\frac{a+b}{2}$ is the exact midpoint between $a$ and $b$ and $1 - \frac{1}{n+1} \to 1$ as $n \to \infty$, but is never equal to one, and $\frac{b-a}{2}$ is the distance of the endpoints of an interval $(a,b)$ to their midpoint. We choose $n+1$ for the denominator of our proportion because the union starts from 1 and if the denominator were just $n$, the first "interval" would not be an element of $\mathcal{E}_2$. Anyway, this shows any element of $\mathcal{E}_1$ is a countable union of $\mathcal{E}_2$ and thus $\mathcal{E}_2 \subset \sigma(\mathcal{E}_2)$. Conversely, for any closed interval $[a,b] \in \mathcal{E}_2$,

$$[a,b] = \bigcap_{n=1}^\infty \left(a - \frac{1}{n}, b + \frac{1}{n}\right),$$

and $\sigma$-algebras are closed under countable intersections, so $\mathcal{E}_2 \subset \sigma(\mathcal{E}_1)$ and we have shown both inclusions.

**c.** Here we show that $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_3)$ and $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_4)$. For any open interval $(a,b)$, there exists $N \in \mathbb{N}$ so that for all $n \geq N$, we have $b - \frac{1}{n} > a$. Then

$$(a,b) = \bigcup_{n=N}^\infty (a, b - \frac{1}{n}]$$

so that $\mathcal{E}_1 \subset \sigma(\mathcal{E}_3)$. Conversely, for any half-open interval $(a, b]$,

$$(a, b] = \bigcap_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right)$$

so that $\mathcal{E}_3 \subset \sigma(\mathcal{E}_1)$. Thus $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_3)$. An exactly analogous argument can be used to show the second equality.

**d.** The left-open rays are $\mathcal{E}_5 = \{(-\infty, a) : a \in \mathbb{R}\}$. As before, it suffices to show $\sigma(\mathcal{E}_4) = \sigma(\mathcal{E}_5)$, for which it suffices to show that $\mathcal{E}_4 \subset \sigma(\mathcal{E}_5)$ and $\mathcal{E}_5 \subset \sigma(\mathcal{E}_4)$. For a left-open ray $(-\infty, a) \in \mathcal{E}_5$, we have

$$(-\infty, a) = \bigcup_{n=1}^{\infty} [a - n, a)$$

so $\mathcal{E}_5 \subset \sigma(\mathcal{E}_4)$. For any half-closed interval $[a, b) \in \mathcal{E}_4$, $(-\infty, b) \setminus (-\infty, a) = [a, b) \in \sigma(\mathcal{E}_5)$ since $\sigma$-algebras are closed under set differences. The same argument can be used to show that $\sigma(\mathcal{E}_6) = \sigma(\{(a, \infty) : a \in \mathbb{R}\}) = \sigma(\mathcal{E}_3)$.

**e.** The left-closed rays are $\mathcal{E}_7 = \{(-\infty, a] : a \in \mathbb{R}\}$. Here, it suffices to show that $\sigma(\mathcal{E}_7) = \sigma(\mathcal{E}_3)$. For a closed interval $(a, b] \in \mathcal{E}_3$, we have $(a, b] = (-\infty, b] \setminus (-\infty, a]$. So $\mathcal{E}_3 \subset \sigma(\mathcal{E}_7)$. Conversely, for any left-closed ray $(-\infty, a] \in \mathcal{E}_7$,

$$(-\infty, a] = \bigcup_{n=1}^{\infty} (a - n, b]$$

so that $\mathcal{E}_7 \subset \sigma(\mathcal{E}_3)$. So the desired equality is shown. The same argument can be used to show that $\sigma(\mathcal{E}_8) = \sigma(\{[a, \infty) : a \in \mathbb{R}\}) = \sigma(\mathcal{E}_4)$.

**Note.** This result is rather useful, since we have basically shown the following:

$$\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{E}_i) = \sigma(\mathcal{E}_j)$$

for any $i, j \in \{1, 2, 3, \ldots, 8\}$. Whether we have explicitly written the expression or not, any set in one family can be expressed using only sets from any other family.