Exercise 1.23

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Last updated Monday 11th March, 2024 at 21:17

Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a,b]\cap\mathbb{Q}$ where $-\infty\leq a<b\leq\infty$.

a. $\mathcal{A}$ is an algebra on $\mathbb{Q}$.

b. The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

c. Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset)=0$ and $\mu_0(A)=\infty$ for $A\neq\emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction to $\mathcal{A}$ is μ_0.

Solution. **a.** We would like to show the collection $\mathcal{E}=\{(a,b]\cap\mathbb{Q}|-\infty\leq a<b\leq\infty\}$ is an elementary family. This, however, is impossible since $\emptyset\notin\mathcal{E}$ (since $a<b$, there is always at least one rational number in $(a,b]$. Boohoo.). Instead, we will show that $\mathcal{E}_1=\{(a,b]\cap\mathbb{Q}|-\infty\leq a\leq b\leq\infty\}$ is an elementary family which generates the same algebra by finite unions. In fact, we could just allow a and b to line up at single point to get the empty set in our collection. Now $\emptyset\in\mathcal{E}_1$ clearly. Let $(a_1,b_1]\cap\mathbb{Q}$ and $(a_2,b_2]\cap\mathbb{Q}$ be in $\mathcal{E}_1$. Then without loss of generality, we can assume $(a_2,b_2]$ is "to the right" of $(a_1,b_1]$ i.e. $a_1\leq a_2$ and $a_1\leq b_2$. Note this includes the possiblility that $(a_2,b_2]\subseteq(a_1,b_1]$, and that

$$((a_1,b_1]\cap\mathbb{Q})\cap((a_2,b_2]\cap\mathbb{Q})=(a_1,b_1]\cap(a_2,b_2]\cap\mathbb{Q} \tag{1}$$

no matter what. Then there are several cases to handle:

1. $b_1<a_2$ means that $(a_1,b_1]\cap(a_2,b_2]\cap\mathbb{Q}=\emptyset$.
2. $b_1=a_2$ means that $(a_1,b_1]\cap(a_2,b_2]\cap\mathbb{Q}=(a_1,b_2]\cap\mathbb{Q}$.
3. $a_2<b_1\leq b_2$ means that $(a_1,b_1]\cap(a_2,b_2]\cap\mathbb{Q}=(a_2,b_1]\cap\mathbb{Q}$.
4. $a_1\leq a_2<b_2\leq b_1$ means that $(a_1,b_1]\cap(a_2,b_2]\cap\mathbb{Q}=(a_2,b_2]\cap\mathbb{Q}$.

Now suppose $-\infty\leq a\leq b\leq\infty$ and consider $(a,b]\cap\mathbb{Q}$. Then note that we are considering these as subsets of $\mathbb{Q}$, so the complements we are taking are relative to $\mathbb{Q}$. Then

$$((a,b]\cap\mathbb{Q})^c=(\mathbb{Q}\setminus(a,b])\cup(\mathbb{Q}\setminus\mathbb{Q})=((-\infty,a]\cap\mathbb{Q})\cup((b,\infty)\cap\mathbb{Q}) \tag{2}$$

So $\mathcal{E}_1$ is an elementary family on $\mathbb{Q}$, as a collection of subsets of $\mathbb{Q}$. By Proposition 1.7, the colleciton of finite disjoint unions of sets in $\mathcal{E}_1$ is an algebra on $\mathbb{Q}$, call it $\mathcal{A}_1$.

Suppose $A\in\mathcal{A}$. Then A is a finite union of sets from $\mathcal{E}$, say $A=\bigcup_{j=1}^N(a_j,b_j]$ where $-\infty\leq a_j<b_j\leq\infty$ for all $j=1,\dots,N$. Let $(a_{j_1},b_{j_1}],(a_{j_2},b_{j_2}]$ be two intervals in this finite union. Then we can assume that $a_{j_1}\leq a_{j_2}$ and $a_{j_1}\leq b_{j_2}$, as we did above (the first to "to the left" of the second). If the two intervals intersect, their intersection is either $(a_{j_1},b_{j_2}]$, $(a_{j_2},b_{j_1}]$, or $(a_{j_2},b_{j_1}]$, all still elements of $\mathcal{E}$. So we can rewrite the union A as a disjoint union of elements of $\mathcal{E}_1$, so $\mathcal{A}\subseteq\mathcal{A}_1$.

On the other hand, if $A\in\mathcal{A}_1$ with the same expression, we can throw out any intervals with $a_{j_0}=b_{j_0}$ since those will be the empty set. Then A is a union of sets in $\mathcal{E}$, so $\mathcal{A}_1\subseteq\mathcal{A}$. Note that the empty set is in $\mathcal{A}$ as an empty (finite) union.

b. Let $b \in \mathbb{Q}$. Then since $\sigma(\mathcal{A})$ is a σ-algebra, it is closed under countable unions and intersections. For every $n \in \mathbb{N}$, $b - \frac{1}{n} \in \mathbb{Q}$, so $(b - \frac{1}{n}, b] \cap \mathbb{Q} \in \mathcal{A}$. Then $\bigcap_{n=1}^{\infty} (b - \frac{1}{n}, b] \cap \mathbb{Q} = \{b\}$. Since $\mathbb{Q}$ is countable, every subset of $\mathbb{Q}$ can be written as a countable union of these singeltons, so $\sigma(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$.

c. We already know $\mu_0(\emptyset) = 0$ so all that remains is to show that μ_0 is additive where it is defined. Let $\{A_j\} \subset \mathcal{A}$ be disjoint so that $\bigcup_1^{\infty} A_j \in \mathcal{A}$. Then $\bigcup_1^{\infty} A_j = \emptyset \iff A_j = \emptyset$ for all j, so $\mu_0(\bigcup_1^{\infty} A_j) = 0 \iff \mu_0(A_j) = 0$ for all j. This proves that μ_0 is a premeasure.

Let μ_1 the measure generated by μ_0, which agrees with μ_0 on $\sigma(\mathcal{A}) = \mathcal{P}(\mathbb{Q})$ by the HKC theorem (1.11,1.13,1.14 in the book).

Let μ_c be the counting measure on $\mathcal{P}(\mathbb{Q})$. This is certainly a different measure, since finite sets will have finite measure, but any set in $\mathcal{A}$ (other than the empty set) contains infinitely many rationals, by density of rationals. Essentially if there is any space between a and b, then $(a, b] \cap \mathbb{Q}$ is countably infinite. So $\mu_c = \mu_0$ on $\mathcal{A}$.

Note these measures are not unique because μ_0 is not σ-finite.
