Exercise 3.1

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Prove Proposition 3.1, which states: Let ν be a signed measure on $(X, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, then $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$. If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu(\bigcap_{j=1}^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

Solution. Let $\{E_j\}$ be an increasing sequence of measurable functions. First we rewrite $\bigcup_{j=1}^\infty E_j$ as a disjoint union like so: $\bigcup_{j=1}^\infty (E_j \setminus E_{j-1})$ assuming that $E_0 = \emptyset$. These unions are the same and the latter is disjoint because $\{E_j\}$ is increasing. Then we can apply the additivity of ν to get

$$\begin{aligned} \nu\left(\bigcup_{j=1}^\infty E_j\right) &= \nu\left(\bigcup_{j=1}^\infty (E_j \setminus E_{j-1})\right) \\ &= \sum_{j=1}^\infty \nu(E_j \setminus E_{j-1}) \\ &= \lim_{j\to\infty} \sum_{k=1}^j \nu(E_k \setminus E_{k-1}) \\ &= \lim_{j\to\infty} \nu\left(\bigcup_{k=1}^j E_k\right) \\ &= \lim_{j\to\infty} \nu(E_j) \end{aligned} \tag{1}$$

Now let $\{E_j\}$ be a decreasing sequence and let $\nu(E_1) < \infty$. Consider the collection $\{E_1 \setminus E_j\}$. Since $\nu(E_1)$ is finite and ν is additive over disjoint sets, we have $\nu(E_1 \setminus E_j) = \nu(E_1) - \nu(E_j)$ for all j. Also $E_1 \setminus E_1 \subset E_1 \setminus E_2 \subset E_1 \setminus E_3 \subset \dots$ since $\{E_j\}$ is decreasing. Then we can apply the first part to get

$$\nu(\bigcup_{j=1}^\infty (E_1 \setminus E_j)) = \lim_{j\to\infty} \nu(E_1 \setminus E_j) = \nu(E_1) - \lim_{j\to\infty} \nu(E_j). \tag{2}$$

Then notice that $\bigcup_{j=1}^\infty (E_1 \setminus E_j) = E_1 \setminus (\bigcap_{j=1}^\infty E_j)$ so that

$$\nu(E_1) - \lim_{j\to\infty} \nu(E_j) = \nu(E_1 \setminus (\bigcap_{j=1}^\infty E_j)) = \nu(E_1) - \nu(\bigcap_{j=1}^\infty E_j) \tag{3}$$

and we are done.