Exercise 1.12

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Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Show the following:

a. If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

b. Say that $E \sim F$ if $\mu(E \triangle F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.

c. For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ for all $G \in \mathcal{M}$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

Solution. A note about the interpretation of this exercise. The statement "$(X, \mathcal{M}, \mu)$ is a finite measure space" could be ambiguous: does it mean that the space X is finite? The collection $\mathcal{M}$? One can try to complete this exercise using one of these assumptions, but will quickly find that the correct interpretation is that μ is in fact a finite measure on $(X, \mathcal{M})$, an arbitrary measurable space.

a. Let $E, F \in \mathcal{M}$ so that $\mu(E \triangle F) = 0$. Since $\mu(X) < \infty$, $\mu(E \cup F) < \infty$. Also $E \cap F \subset E \cup F$ and all are measurable. Then by the lemma in Exercise 8,

$$0 = \mu(E \triangle F) = \mu((E \cup F) \setminus (E \cap F)) = \mu(E \cup F) - \mu(E \cap F),$$

meaning $\mu(E \cup F) = \mu(E \cap F)$. Then

$$\begin{aligned} \mu(E) &\leq \mu(E \cup F) = \mu(E \cap F) \leq \mu(F) \\ \text{and } \mu(F) &\leq \mu(E \cup F) = \mu(E \cap F) \leq \mu(E) \end{aligned}$$

so $\mu(E) = \mu(F)$.

b. For $E \in \mathcal{M}$, $E \sim E$ since $\mu(E \triangle E) = \mu(\emptyset) = 0$. Suppose $E, F \in \mathcal{M}$ are such that $E \sim F$. Then $F \sim E$ since $E \triangle F = F \triangle E$. Now suppose $E \sim F$ and $F \sim G$ for $E, F, G \in \mathcal{M}$. Then showing $E \sim G$ amounts to showing that $\mu(E \triangle G) = \mu((E \setminus G) \cup (G \setminus E)) = \mu(E \setminus G) + \mu(G \setminus E) = 0$. But since μ is positive, this can only happen if $\mu(E \setminus G) = \mu(G \setminus E) = 0$. Since $E \sim F$ and $F \sim G$ we have

$$\mu(E \setminus F) = \mu(F \setminus E) = \mu(F \setminus G) = \mu(G \setminus F) = 0.$$

But

$$E \setminus G \subset (E \setminus F) \cup (F \setminus G) \quad \text{and} \quad G \setminus E \subset (G \setminus F) \cup (F \setminus E),$$

so $\mu(E \setminus G) = \mu(G \setminus E) = 0$.

c. To be pedantic here, the question is asking the reader to show that ρ^* defined as $\rho^*([E], [F]) = \rho(E, F)$ for $[E], [F] \in \mathcal{M}/\sim$ is a metric, so that's what we will do. The triangle inequality is proven to be satisfied using the same method as when showing transitivity in part **b:** for $[E], [F], [G] \in \mathcal{M}/\sim$,

$$\begin{aligned}
\rho^*([E],[G]) = \rho(E,G) &= \mu(E\triangle G) = \mu(E\setminus G) + \mu(G\setminus E) \\
&\leq \mu(E\setminus F) + \mu(F\setminus G) + \mu(G\setminus F) + \mu(F\setminus E) \\
&= \mu(E\setminus F) + \mu(F\setminus E) + \mu(F\setminus G) + \mu(G\setminus F) \\
&= \mu(E\triangle F) + \mu(F\triangle G) \\
&= \rho(E,F) + \rho(F,G) = \rho^*([E],[F]) + \rho^*([F],[G]).
\end{aligned}$$

The remaining conditions of being a metric are now shown. - $\rho^*([E],[E]) = \rho(E,E) = \mu(E\triangle E) = 0$. - $\rho^*([E],[F]) = \mu(E\triangle F) = \mu(F\triangle E) = \rho^*([F],[E])$. - For $[E] \neq [F]$, we have $\mu(E\triangle F) \neq 0$, so $\rho^*([E],[F]) \neq 0$. (This condition is really the whole point of modding out by $\sim$. The symmetric difference of two sets might be null set if we aren't dealing with equivalence classes.)