Exercise 1.29

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Let E be a Lebesgue measurable set.

a. If $E \subset N$ where N is the nonmeasurable set described in §1.1, then $m(E) = 0$.

b. If $m(E) > 0$ then E contains a non-measurable set. (It suffices to assume $E \subset [0, 1]$. In the notation of §1.1, $E = \bigcup_{r \in R} E \cap N_r$.)

Solution. **a.** Let $E_r = E \cap N_r$ for each $r \in R = \mathbb{Q} \cap [0, 1)$. Then since $N_r = \big((N \cap [0, 1 - r)) + r\big) \cup ((N \cap [1 - r, 1)) + (r - 1))$ and Theorem 1.21 says m is translation invariant,

$$m(E_r) = m(E \cap N \cap [0, 1 - r)) + m(E \cap N \cap [1 - r, 1)) = m(E \cap N) = m(E) \tag{1}$$

for each $r \in R$. Note also by Theorem 1.21 that E_r is measurable since $E \cap N = E$ is measurable, $[0, 1 - r)$ and $[1 - r, 1)$ are measurable, and translating measurable sets yields measurable sets, so this computation is valid. Then since $[0, 1)$ is the disjoint union of the N_r, E is the disjoint union of the E_r ($E = E \cap [0, 1)$). So we have

$$m(E) = m\Big(\bigcup_{r \in R} E_r\Big) = \sum_{r \in R} m(E_r) = \sum_{r \in R} m(E). \tag{2}$$

The only way this makes sense is if $m(E) = \infty$ or $m(E) = 0$. However, $E \subseteq [0, 1)$, so $m(E) \leq 1$, i.e. $m(E) = 0$.

b. First suppose the conclusion is true for $E \subseteq [0, 1]$. Now let F be a measurable set in $\mathbb{R}$. Then $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n + 1)$ as a disjoint union, so $F = F \cap \mathbb{R} = \bigcup_{n \in \mathbb{Z}} F \cap [n, n + 1)$ is also a disjoint union. If $m(F) = \sum_{n \in \mathbb{Z}} m(F \cap [n, n + 1)) > 0$, then there exists $n \in \mathbb{Z}$ so that $F \cap [n, n + 1) > 0$. Then if $G = (F \cap [n, n + 1)) - n$, then $G \subseteq [0, 1]$ with $m(G) = m(F)$ (by Theorem 1.21), so G contains a non-measurable set. So it is sufficient to consider $E \subset [0, 1]$.

Suppose $m(E) > 0$ and suppose for contradiction that F is measurable for all $F \subseteq E$. Then in the notation of §1.1, the N_r and disjoint and $E \cap N_r \subseteq E \implies N_r$ is measurable for all r. Also recall that if μ is a translation invariant measure on $\mathcal{M}$, then $\mu(N_r) = \mu(N)$ for all $r \in R$. But m satisfies these requirements, so

$$m(E) = m(E \cap [0, 1]) = m\big(\bigcup_{r \in R} E \cap N_r\big) = \sum_{r \in R} m(E \cap N_r) = \sum_{r \in R} m(E \cap N). \tag{3}$$

As above, this only makes sense if $m(E) = \infty$ or if $m(E) = 0$, so we have drawn a contradiction, and there must be a non-measurable set contained in E.
