# Exercise 1.26

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Prove Proposition 1.20. (Use Theorem 1.18.) Proposition 1.20 states: If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\epsilon > 0$ there is a set $A$ that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

**Solution.** Let $E \in \mathcal{M}_\mu$ and let $\epsilon > 0$. By Theorem 1.18, there is $U$ open in $\mathbb{R}$ so that $E \subseteq U$ and $\mu(U) < \mu(E) + \epsilon/2$. Since $\mu(E) < \infty$, $\mu(U) < \infty$ and the $\epsilon$-inequality is equivalent to $\mu(U \setminus E) < \epsilon/2$. Since $\mathbb{R}$ is second countable, $U = \bigcup_1^\infty (a_j, b_j)$ for some disjoint intervals $(a_j, b_j)$. Then since $\mu(U) < \infty$ and the intervals are disjoint, there is $N \in \mathbb{N}$ so that $\mu(\bigcup_{j=N+1}^\infty) = \sum_{j=N+1}^\infty \mu(a_j, b_j) < \epsilon/2$. Let $A = \bigcup_{j=1}^N (a_j, b_j)$. Then

$$\begin{aligned} \mu(E \triangle A) &= \mu(E \setminus A) + \mu(A \setminus E) \\ &\leq \mu(\bigcup_{j=N+1}^\infty (a_j, b_j)) + \mu(U \setminus E) \qquad (1) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

The first inequality is because $A \setminus E \subset U \setminus E$ and since $E \subset U$, $E \setminus A \subset \bigcup_{j=N+1}^\infty (a_j, b_j)$. Essentially the rest of the intervals have to contain the rest of $A$.

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