Exercise 1.6

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Prove Theorem 1.9, which states: Suppose that $(X, \mathcal{M}, \mu)$ is a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\overline{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$. This the so-called "Completion Theorem."

Solution. Since $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\overline{\mathcal{M}}$ ($\mathcal{N}$ is due to the subadditivity of μ). Let $E \cup F \in \overline{\mathcal{M}}$ so that $E \in \mathcal{M}$ and $F \subset N$ for some $N \in \mathcal{N}$. Then we can assume E and F are disjoint, since $F \setminus E \subset N$ and $E \cup F = (F \setminus E) \cup E$. But then we can assume that E and N are disjoint as well, since if E and F are disjoint but $E \cap N \neq \emptyset$, we have $N \setminus E \in \mathcal{M}$ and $\mu(N \setminus E) \leq \mu(N) = 0$ and $F \subset N \setminus E$. So the collection with these restrictions is the same as the collection without them. So we have

$$(E \cup F)^c = E^c \cap F^c = (E^c \cap N^c) \cup (F^c \cap N)$$

and $E^c \cap N^c \in \mathcal{M}$ and $F^c \cap N \subset N$, so $(E \cup F)^c \in \overline{\mathcal{M}}$. Thus $\overline{\mathcal{M}}$ is closed under complements.

Now define $\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty]$ by $\overline{\mu}(E \cup F) = \mu(E)$. This is well-defined since if $E_1 \cup F_1 = E_2 \cup F_2$ are two different representations of the same set in $\overline{\mathcal{M}}$ so that $F_j \subset N_j \in \mathcal{N}$ for $j = 1, 2$, then $\overline{\mu}(E_1 \cup F_1) = \mu(E_1) \leq \mu(E_2 \cup N_2) = \mu(E_2) = \overline{\mu}(E_2 \cup F_2)$. The same can be done in the other direction, so $\overline{\mu}(E_1 \cup F_1) = \overline{\mu}(E_2 \cup F_2)$.

Now show $\overline{\mu}$ is a measure: $\overline{\mu}(\emptyset) = \overline{\mu}(\emptyset \cup \emptyset) = \mu(\emptyset) = 0$. Suppose $\{E_j \cup F_j\}_1^\infty \subset \overline{\mathcal{M}}$. Then there is $N_j \in \mathcal{N}$ so that $F_j \subset N_j$ for all j. Thus

$$\bigcup_1^\infty (E_j \cup F_j) = \bigcup_{j=1}^\infty E_j \cup \bigcup_{j=1}^\infty F_j$$

and $\bigcup E_j \in \mathcal{M}$ and $\bigcup F_j \subset \bigcup N_j$ and $\mu(\bigcup N_j) \leq \sum \mu(N_j) = 0$, so

$$\overline{\mu}\left(\bigcup_{j=1}^\infty (E_j \cup F_j)\right) = \mu\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu(E_j) = \sum_{j=1}^\infty \overline{\mu}(E_j \cup F_j).$$

Thus $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$.

Now show $\overline{\mu}$ is complete. Let $E \cup F \in \overline{\mathcal{M}}$ be such that $\overline{\mu}(E \cup F) = 0$ and let $A \subset E \cup F$. Since $\overline{\mu}(E \cup F) = 0$, we have $\mu(E) = 0$. If $F \subset N$ so that $N \in \mathcal{M}$ and $\mu(N) = 0$, then $E \cup F \subset E \cup N$ and $E \cup N \in \mathcal{M}$ and $\mu(E \cup N) \leq \mu(E) + \mu(N) = 0$. Then notice that $A = A \cap (E \cup F) = (A \cap E) \cup (A \cap F)$, and $A \cap E$ and $A \cap F$ are both subsets of null sets in $\mathcal{M}$, since $\mu(E) = 0$, meaning both are in $\overline{\mathcal{M}}$. Thus $A \in \overline{\mathcal{M}}$, so $\overline{\mu}$ is complete.

Finally, suppose $\nu : \overline{\mathcal{M}} \to [0, +\infty]$ is another complete measure on $\overline{\mathcal{M}}$ so that $\nu(E) = \mu(E)$ for all $E \in \mathcal{M}$. Then for $E \cup F \in \overline{\mathcal{M}}$,

$$\overline{\mu}(E \cup F) = \mu(E) = \nu(E) \leq \nu(E \cup F)$$

since $E \subset E \cup F$ and ν is a measure on $\overline{\mathcal{M}}$. Conversely,

$$\nu(E \cup F) \leq \nu(E) + \nu(F) = \nu(E) = \overline{\mu}(E \cup F),$$

as $\nu(F) = 0$ since ν is complete. So $\overline{\mu}$ is the unique complete extension of μ to $\overline{\mathcal{M}}$.