Exercise 1.15

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Given a measure μ on $(X, \mathcal{M})$, define $\mu_0(E) = \sup\{\mu(F) : F \subset E \text{ and } \mu(F) < \infty\}$ for each $E \in \mathcal{M}$. Show the following:

a. μ_0 is a semifinite measure. It is called the **semifinite part** of μ.

b. If μ is semifinite, then $\mu = \mu_0$ (Use Exercise 14).

c. There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ and such that $\mu = \mu_0 + \nu$.

Solution. **a.** First, $\mu_0(\emptyset) = \mu(\emptyset) = 0$ since $\emptyset \subset \emptyset$ and $\mu(\emptyset) < \infty$. Actually for any set $E \in \mathcal{M}$ with $\mu(E) < \infty$, the measure $\mu_0(E) = \mu(E)$ since E will be in the collection $\{F \subset E \text{ such that } \mu(F) < \infty\}$. Now suppose $\{E_j\} \subset \mathcal{M}$ is a collection of disjoint measurable sets. Then let $E = \bigcup_{j=1}^{\infty} E_j$, let $A = \{F \in \mathcal{M} : \mathcal{F} \subset \mathcal{E} \text{ and } \mu(\mathcal{F}) < \infty\}$ and let $A_j = \{F \in \mathcal{M} : \mathcal{F} \subset \mathcal{E}_| \text{ and } \mu(\mathcal{F}) < \infty\}$. If $\mu_0(E)$ is finite, then $\mu_0(E) = \mu(E)$ and the countable additivity is inherited from μ. So suppose $\mu_0(E) = \infty$. Then showing the countable additivity amounts to showing that

$$\sum_{j=1}^{\infty} \mu_0(E_j) = \infty.$$

Suppose for contradiction that $\sum_{j=1}^{\infty} \mu_0(E_j) = s < \infty$. Note that since μ_0 is non-negative, the sequence of partial sums must approach s from below. Then for all $k \in \mathbb{N}$ (large enough), there exists $N \in \mathbb{N}$ so that $\sum_{j=1}^{n} \mu_0(E_j) > s - \frac{1}{k}$ for all $n \geq N$. Now consider a set E so that $\mu_0(E) = \infty$. Since $\mu_0(E)$ is a supremum of finite numbers, one of these finite numbers must be non-zero, i.e. there is a set $F \subset E$ so that $0 < \mu(F) < \infty$. Thus μ_0 is semifinite.

b. Suppose μ is semifinite. Then let $E \in \mathcal{M}$ so that $\mu(E) = \infty$. Then by Exercise 14, for each $n \in \mathbb{N}$, there is a set $F_n \subset E$ so that $n < \mu(F_n) < \infty$. Then $\{F_n\} \subset \{F : F \subset E \text{ and } \mu(F) < \infty\}$, so $\mu_0(E) \geq \mu(F_n)$ for all n. That is, $\mu_0(E) = \infty = \mu(E)$. If $\mu(E) < \infty$, then we have already shown that $\mu(E) = \mu_0(E)$ in part **a.**

c. Call $E \in \mathcal{M}$ μ-semifinite if $\mu(E) = \infty$ and there is $F \subset E$ so that $0 < \mu(F) < \infty$. Define $\nu : \mathcal{M} \to [\prime, \infty]$ in the following way, for $E \in \mathcal{M}$:

$$\nu(E) = \begin{cases} \infty & \text{if } E \text{ is } \mu\text{-semifinite} \\ \mu(E) - \mu_0(E) & \text{otherwise} \end{cases}$$

So ν only takes the values zero or ∞. If E is μ-semifinite, then we are done. If E is not μ-semifinite, but $\mu(E) = \infty$, then $\mu_0(E) = 0$, so $\nu(E) = \infty$. If $\mu(E) < \infty$, then E is certainly not μ-semifinite, and $\mu(E) = \mu_0(E)$, so $\nu(E) = 0$. Now show that ν is indeed a measure. Well, $\mu(\emptyset) = 0 < \infty$, so $\nu(\emptyset) = \mu(\emptyset) - \mu_0(\emptyset) = 0$. If $E \in \mathcal{M}$, then for all $F \subset E$ with $\mu(F) < \infty$, $\mu(F) \leq \mu(E)$ as well. This means that $\mu_0(E) \leq \mu(E)$ and that ν is indeed a non-negative function. Then let $\{E_j\}_1^{\infty} \subset \mathcal{M}$ be a disjoint collection of measurable sets. If there is $k \in \mathbb{N}$ so that E_k is μ-semifinite, then the union $E = \bigcup_{j=1}^{\infty} E_j$ is μ-semifinite, since such a set $F \subset E_k$ will also be contained in E. So $\nu(E) = \infty = \sum_{j=1}^{\infty} \nu(E_j)$ since $\nu(E_k) = \infty$.

On the other hand, if there is no μ-semifinite set in the collection $\{E_j\}$ and all are such that $\mu(E_j) < \infty$, then by the additivity of μ and μ_0, we have

$$\nu(E) = \sum_{j=1}^{\infty} \mu(E_j) - \sum_{j=1}^{\infty} \mu_0(E_j) = \sum_{j=1}^{\infty} (\mu(E_j) - \mu_0(E_j)) = \sum_{j=1}^{\infty} \nu(E_j).$$

Note that if the first sum is convergent, then the second sum definitely is since $\mu_0(E) \leq \mu(E)$ for all E, so in that case the rearrangement does not affect the value of the sum. If the first sum is divergent, then the intermediate steps aren't necessary, since $\nu(E) = \infty$ immediately.