Exercise 1.14

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If μ is a semifinite measure and $\mu(E) = \infty$, then for any $C > 0$ there is $F \subset E$ so that $C < \mu(F) < \infty$.

Solution. Basically, we need a sequence of sets $\{F_j\}$ which have finite measure diverging to ∞, for then we can choose one of arbitrarily large (finite) measure. So consider the collection $A = \{F \subset E \text{ such that } \mu(F) < \infty\}$. If the supremum of the measures of sets in this collection is ∞, then there must be a sequence contained therein whose measures converges to that supremum. So let $s = \sup_{F\in A} \mu(F)$. Then suppose for contradiction that $s < \infty$. Then there is a sequence $\{F_j\} \subset A$ so that $\lim_{j\to\infty} \mu(F_j) \to s$. Let $G_k = \bigcup_{j=1}^k F_j$. Then $\mu(G_k) \leq \sum_{j=1}^k \mu(F_j) < \infty$ since each $F_j \in A$. So $G_k \in A$ for all k, meaning $\mu(G_k) \leq s$ for all k. Thus $\lim_{k\to\infty} \mu(G_k) \leq s$, and by continuity from below, $\mu\left(\bigcup_{j=1}^\infty F_j\right) = \mu\left(\bigcup_{k=1}^\infty G_k\right) = \lim_{k\to\infty} \mu(G_k)$. Thus the union of the $G_k's$ (call it G) is in A. But since $\lim_{j\to\infty} \mu(F_j) = s$, we have $\mu(G) = s$. Then $E \setminus G$ has infinite measure, so there exists $H \subset (E \setminus G)$ so that $0 < \mu(H) < \infty$ and H is disjoint from G. Then $s = \mu(G) < \mu(H) + \mu(G) = \mu(H \cup G)$. But since H has finite measure and s is finite, $\mu(H \cup G) < \infty$, meaning $H \cup G \in A$, contradicting that s is the supremum. Thus $s = \infty$.