# Exercise 1.25

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Complete the proof of Theorem 1.19, which states: If $\subset \mathbb{R}$, the following are equivalent:

1. $E \in \mathcal{M}_\mu$
2. $E = V \setminus N_1$ where $V$ is a $G_\delta$ set and $\mu(N_1) = 0$.
3. $E = H \cup N_2$ where $H$ is an $F_\sigma$ set and $\mu(N_2) = 0$.

**Solution.** The book states that (b) and (c) implying (a) are both "obvious", since $\mu$ is complete on $\mathcal{M}_\mu$. If $E = V \setminus N_1$ for some $G_\delta$ set $V$ and some $\mu(N_1) = 0$, then $E = V \cap N_1^c$. $V$ is a $G_\delta$, so is Borel, and $\mathcal{M}_\mu \supset \mathcal{B}_\mathbb{R}$. The way the theorem is stated seems to imply that $N_1 \in \mathcal{M}_\mu$, so $E$ is an intersection between two sets in a $\sigma$-algebra, and is thus in that $\sigma$-algebra.

The same can be said if $E = H \cup N_2$ for some $F_\sigma$ set $H$ and some $\mu(N_2) = 0$. Then $H$ is a Borel set and $N_2$ is assumed to be measurable, so $E$ is a union of two measurable sets and is therefore measurable.

My only problem with this is that there seems to be no reason to mention the completeness of $\mu$ on $\mathcal{M}_\mu$. If what the statement of the theorem means instead is that $\mu(N_1) = 0$ in the sense of the outer measure generated by some IRC function $F$, then $N_1$ and $N_2$ will be subsets of null sets in $\mathcal{M}_\mu$ and thus be measurable themselves by completeness.

The book proves that (a) $\implies$ (b) and (a) $\implies$ (b) in the case that $\mu(E) < \infty$. So suppose $\in \mathcal{M}_\mu$ and $\mu(E) = \infty$. Then consider $\mathbb{R} = \bigcup_{n=1}^\infty I_n$ where $I_n(-n, n)$. Then $E \cap I_n \in \mathcal{M}_\mu$ for all $n$ since $I_n \in \mathcal{B}_\mathbb{R}$, and $\mu(E \cap I_n) \le \mu(-n, n) = 2n < \infty$. By outer regularity, for each $n$ and for each $k \in \mathbb{N}$, there exists $V_{kn}$ open such that $E \cap I_n \subseteq V_{kn}$ and $\mu(V_{kn}) \le \mu(E \cap I_n) + 2^{-n}\frac{1}{k}$. Since we are dealing with finite measures here, we have $\mu(V_{kn} \setminus (E \cap I_n)) \le 2^{-n}\frac{1}{k}$. Let $V_k = \bigcup_{n=1}^\infty V_{kn}$. Then $V_k$ is open for all $k$, and $V = \bigcap_{k=1}^\infty$ is a $G_\delta$. Note that $E \cap I_n \nearrow E$ so that $E \subset V_k$ for all $k$, and thus $E \subset V$. Then for each $k$,

$$
\begin{aligned}
\mu(V \setminus E) &= \mu\left(\bigcap_{k=1}^\infty (V_k \setminus E)\right) \\
&\le \mu(V_k \setminus E) \\
&= \mu\left(\bigcup_{n=1}^\infty (V_{kn} \setminus E)\right) \\
&\le \sum_{n=1}^\infty \mu(V_{kn} \setminus E) \\
&\le \sum_{n=1}^\infty \mu(V_{kn} \setminus (E \cap I_n)) \le \sum_{n=1}^\infty 2^{-n}\frac{1}{k} = \frac{1}{k}.
\end{aligned} \tag{1}
$$

since $V_{kn} \setminus E \subseteq V_{kn} \setminus (E \cap I_n)$ for each $n$ and $k$. Thus $\mu(V \setminus E) = 0$. So let $N_1 = V \setminus E$. Then $E = V \setminus (V \setminus E)$ and (a) $\Longrightarrow$ (b).

Now we show (a) $\Longrightarrow$ (c). This argument is *almost* the same. By inner regularity, for each $n$ and for each $k$, there is a compact set $H_{kn}$ so that $H_{kn} \subset E \cap I_n$ and $\mu(H_{kn}) \geq \mu(E \cap I_n) - 2^{-n}\frac{1}{k}$. Since $\mu(E \cap I_n) < \infty$, this is the same as saying $\mu((E \cap I_n) \setminus H_{kn}) \leq 2^{-n}\frac{1}{k}$. For each $k$, let $H_k = \bigcap_{n=1}^{\infty} H_{kn}$. Then $H_k$ is closed (and compact) and $H_k \subseteq E$ for all $k$. Let $H = \bigcup_{k=1}^{\infty} H_k$. Then $H \subseteq E$. Then note that $\bigcup_{n=1}^{\infty} E \cap I_n = E$. So for all $k$,

$$\begin{aligned}
\mu(E \setminus H) &= \mu\left(E \setminus \bigcup_{k=1}^{\infty} H_k\right) \\
&= \mu\left(\bigcap_{k=1}^{\infty} E \setminus H_k\right) \\
&\leq \mu(E \setminus H_k) \\
&= \mu\left(E \setminus \bigcap_{n=1}^{\infty} H_{kn}\right) \\
&= \mu\left(\bigcup_{n=1}^{\infty} E \setminus H_{kn}\right) \\
&= \mu\left(\bigcup_{n=1}^{\infty} (E \cap I_n) \setminus H_{kn}\right) \\
&\leq \sum_{n=1}^{\infty} 2^{-n}\frac{1}{k} = \frac{1}{k}
\end{aligned} \tag{2}$$

So $\mu(E \setminus H) = 0$. Let $N_2 = E \setminus H$. Then $E = H \cup (E \setminus H)$ and (a) $\Longrightarrow$ (c).

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