Exercise 3.8

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$\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Solution. Let $X = P \cup N$ be a Hahn decomposition of X for ν. Suppose $\nu \ll \mu$ and let $E \in \mathcal{M}$ be null for μ. Then E is also null for ν, so $\nu^+(E) = \nu(E \cap P) = 0$ since $E \cap P \subset E$ which is null. Similarly $\nu^-(E) = \nu(E \cap N) = 0$ since $E \cap N \subset E$. So $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0 + 0 = 0$, i.e. $|\nu| \ll \mu$. (Since $|\nu|$ is a positive measure, it is monotone, so all we need to show is that $|\nu|(E) = 0$.)

Now suppose $|\nu| \ll \mu$. Then if $E \in \mathcal{M}$ is null for μ, we have $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Since ν^+ and ν^- are positive measures, this means $\nu^+(E) = \nu^-(E) = 0$. So $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. (Again this is all that needs showing since ν^+ and ν^- are positive measures, and therefore monotone.)

Finally, suppose $\nu^+ \ll \mu$ and $\nu^- \ll \mu$. Then if $E \in \mathcal{M}$ is null for μ, we have $\nu^+(E) = \nu^-(E) = 0$. But this means that $\nu(E) = \nu^+(E) - \nu^-(E) = 0$. Any $F \subset E$ will also be null for μ and by the same argument can be shown to have zero ν-measure, so $\nu \ll \mu$.
