Exercise 1.17

Nolan Hauck

If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\bigcup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

Solution. By the monotonicity of outer measures, we have

$$\mu^*(E \cap \bigcup_1^\infty A_j) = \mu^*(\bigcup_1^\infty E \cap A_j) \leq \sum_1^\infty \mu^*(E \cap A_j),$$

so we must show the reverse inequality. If we can show for each n that

$$\mu^*(E \cap \bigcup_{j=1}^\infty A_j) \geq \sum_{j=1}^n \mu^*(E \cap A_j)$$

then we are done since the left hand side does not depend on n. Let $B_n = \bigcup_{j=1}^n A_j$. By monotonicity, we know

$$\mu^*(E \cap \bigcup_1^\infty A_j) \geq \mu^*(E \cap B_n).$$

Then using the μ^*-measurability of each A_j we obtain the following for any $k \leq n$

$$\mu^*(E \cap B_n) = \mu^*((E \cap B_n) \cap A_k) + \mu^*((E \cap B_n) \cap A_k^c) \tag{1}$$

$$= \mu^*(E \cap A_k) + \mu^*(E \cap \bigcup_{j \neq k}^n A_j). \tag{2}$$

The second part of the RHS collapses in that way since the A_j's are disjoint, meaning $\bigcup_{j \neq k}^n A_j \subset A_k^c$ for all $k \leq n$. We can continue inductively, "removing" $\mu^*(E \cap A_{k_1})$ from $\mu^*(E \cap \bigcup_{j \neq k}^n A_j)$ for some $k_1 \in \{1, \ldots, k-1, k+1, \ldots, n\}$ until we are left with (after reordering the sum)

$$\mu^*(E \cap B_n) = \mu^*(E \cap A_1) + \cdots + \mu^*(E \cap A_n) = \sum_{j=1}^n \mu^*(E \cap A_j).$$

Thus

$$\mu^*(E \cap \bigcup_j^\infty A_j) \geq \sum_{j=1}^n \mu^*(E \cap A_j)$$

for all n, and we are done.