Exercise 1.10

Nolan Hauck

Given a measure space $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then show that μ_E is a measure on $\mathcal{M}$.

Solution. We already know $(X, \mathcal{M})$ is a measurable space. Since $\mu \geq 0$, $\mu_E \geq 0$ as well. μ_E is well-defined since $E \cap A$ is entirely determined by A. First, we have $\mu_E(\emptyset) = \mu(\emptyset \cap E) = \mu(\emptyset) = 0$. Now let $\{A_j\}_1^\infty \subset \mathcal{M}$. Then since each $A_j \in \mathcal{M}$, each $E \cap A_j \in \mathcal{M}$. Then

$$\mu_E\left(\bigcup_{j=1}^\infty A_j\right) = \mu\left(E \cap \bigcup_{j=1}^\infty A_j\right) = \mu\left(\bigcup_{j=1}^\infty E \cap A_j\right) \leq \sum_{j=1}^\infty \mu(E \cap A_j) = \sum_{j=1}^\infty \mu_E(A_j).$$

So μ is a measure.