Exercise 3.9

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Last updated Wednesday 28th February, 2024 at 20:48

Suppose $\{\nu_j\}$ is a sequence of positive measures. If $\nu_j \perp \mu$ for all j, then $\sum_1^\infty \nu_j \perp \mu$ and if $\nu_j \ll \mu$, then $\sum_1^\infty \nu_j \ll \mu$.

Solution. Since $\nu_j \perp \mu$ for all j, there are sequences of measurable sets $\{E_j\}$ and $\{F_j\}$ so that for each j, $E_j \cup F_j = X$, $E_j \cap F_j = \emptyset$, E_j null for ν_j, and F_j null for μ. Then $F = \bigcup_1^\infty F_j \in \mathcal{M}$, so $\left(\bigcup_1^\infty F_j\right)^c = \bigcap_1^\infty E_j \in \mathcal{M}$ as well. Then $F \cup F^c = X$ and $F \cap F^c = \emptyset$. Then

$$\mu(F) \leq \sum_1^\infty \mu(F_j) = 0 \tag{1}$$

and

$$\left(\sum_1^\infty \nu_j\right)(F^c) = \sum_1^\infty \nu_j(F^c) \leq \sum_1^\infty \nu(F_j) = 0 \tag{2}$$

since $F^c \subset F_j$ and F_j is null for ν_j for all j. Since we are dealing exclusively with positive measures, the condition that $\mu(F) = 0$ and $\left(\sum_1^\infty \nu_j\right)(F^c) = 0$ is enough to conclude that F is null for μ and F^c is null for $\left(\sum_1^\infty \nu_j\right)$. So $\sum_1^\infty \nu_j \perp \mu$.

Now let $E \in \mathcal{M}$ be null for μ. Then E is null for ν_j for all j, so

$$\left(\sum_1^\infty \nu_j\right)(E) = \sum_1^\infty \nu_j(E) = 0. \tag{3}$$

Thus $\sum_1^\infty \nu_j \ll \mu$.
