

Hamiltonians whose low-energy states require $\Omega(n)$ T gates

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Abstract

• The QMA-hardness of the Local Hamiltonian (LH) problem with $1/\text{poly}(n)$ gap implies that we cannot $1/\text{poly}(n)$ -approximate the ground-state energy of arbitrary local Hamiltonians in BQP, unless $\text{BQP}=\text{QMA}$.

• The **Quantum PCP Conjecture** (QPCP) asserts that the same is true even for a constant, $\Omega(1)$, relative promise gap. If true, QPCP implies the existence of Hamiltonians with interesting low-energy space properties.

• **We exhibit one such prerequisite to QPCP:** an explicit local Hamiltonian whose low-energy states all require $\omega(\log n)$ T gates, i.e. they are very non-stabilizer. In fact, we show a stronger result that the low-energy states require $\Omega(n)$ T gates, which is not necessarily implied by QPCP.

• Applying our procedure to the NLTS family from [ABN22] **yields an NLTS Hamiltonian whose low-energy states also require $\Omega(n)$ T gates.**

"QPCP Hamiltonians"

Local Hamiltonian: $H = \frac{1}{m} \sum_{i=1}^m h_i \otimes I_{2^{n-k}}$

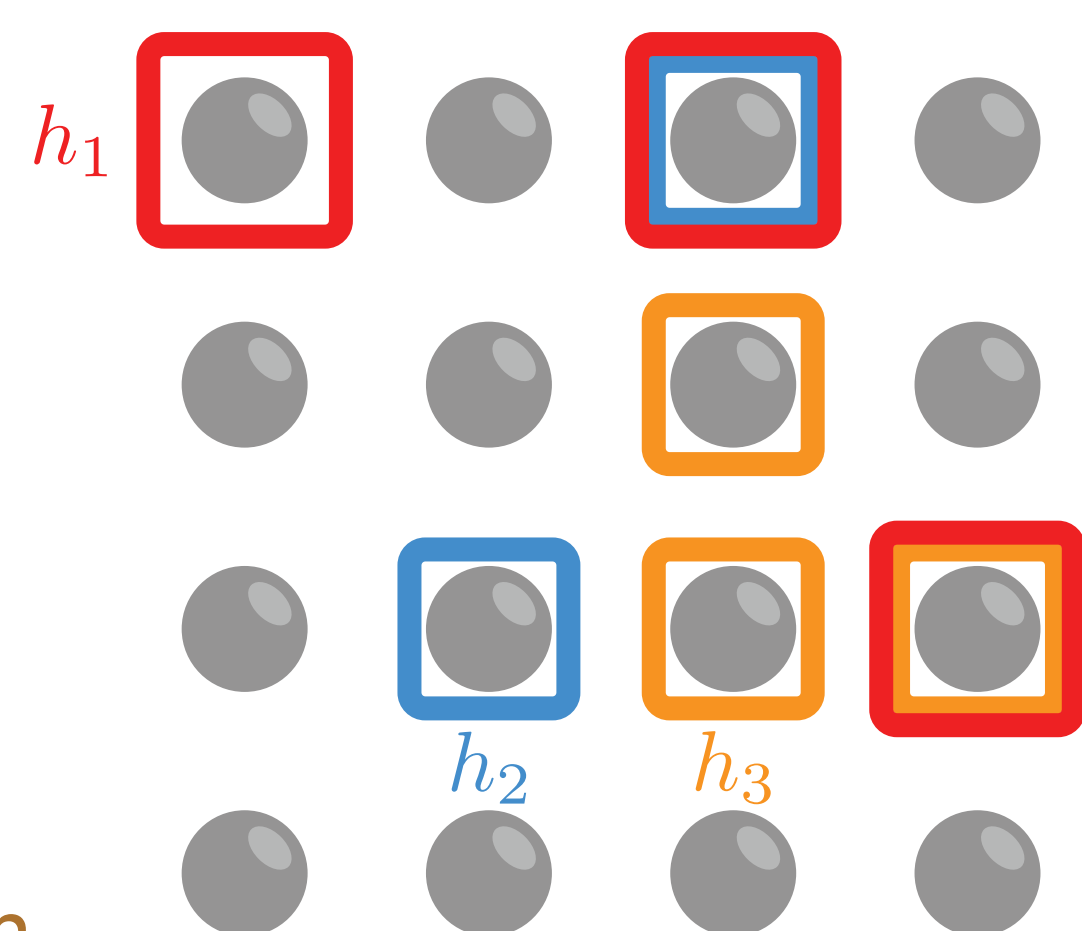
Ground-state energy: $E_{gs} \equiv \min_{|\psi\rangle} \langle \psi | H | \psi \rangle$

LH Problem: given $H, \epsilon > 0, \delta(n)$, decide between

$$(1) E_{gs} \leq \epsilon \quad (2) E_{gs} \geq \epsilon + \delta(n)$$

If $\mathcal{C} \subsetneq \text{QMA} = \text{QPCP}$ is conjectured, then for some constant $\delta > 0$ we cannot estimate $E_{gs} \pm \delta$ in \mathcal{C} .

\Rightarrow **No low-energy state of H can admit energy estimation in \mathcal{C} .**



Types of States

Stabilizer group: $S(|\psi\rangle) \equiv \{P \in \mathcal{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Stabilizer state: prepared by only Clifford gates $\Leftrightarrow |S(|\psi\rangle)| = 2^n$

Size of $S(|\psi\rangle)$ vs. T-count: prepared by $\leq t$ T gates $\Rightarrow |S(|\psi\rangle)| \geq 2^{n-t}$

Almost-Clifford state: prepared by $\leq \log n$ T gates

States and Hamiltonians

State	Energy Estimation Algorithm	Hamiltonian Implication
Trivial (low-depth circuit)	NP via a light-cone argument	NLTS [ABN22]
"Sampleable"	MA via dequantizing QSVT [GL22]	NLSS [GL22] (open)
Stabilizer	NP via stabilizer generators	NLCS [CCNN23]
Almost-Clifford	NP via combo of light-cone + stabilizer generators	NLACS [this work]
...

Rotating CSS Hamiltonians

CSS Hamiltonian: $H = \frac{1}{m} \sum_{i=1}^{m_x} \frac{1}{2} (I - X^{\otimes k})_i + \frac{1}{m} \sum_{i=1}^{m_z} \frac{1}{2} (I - Z^{\otimes k})_i$

Ground-states of a CSS Hamiltonian are highly-stabilizer.
Key idea: rotate to a basis which is far from stabilizer.

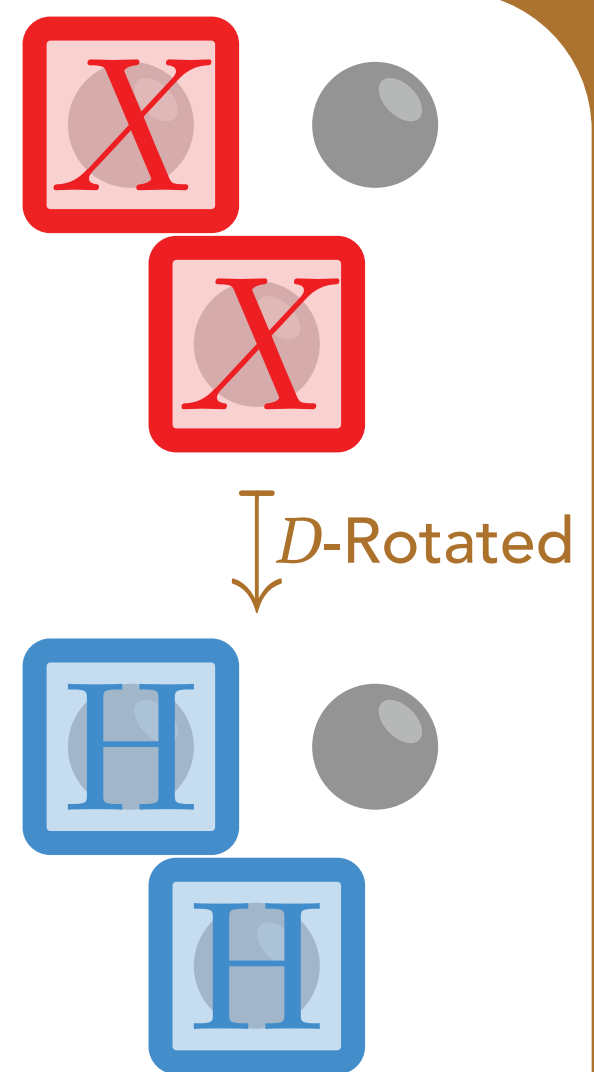
Single-qubit gates: H, Hadamard, and $D \equiv e^{i\frac{\pi}{8}}$

Rotated CSS Hamiltonian: $\tilde{H} \equiv D^{\otimes n} H D^{\dagger \otimes n} = \frac{1}{m} \sum_{i=1}^{m_x} \frac{1}{2} (I - H^{\otimes k})_i + \dots$

Theorem. There are $\epsilon > 0$ and $0 < c < 1$ s.t. if $|\psi\rangle$ can be prepared by $\leq cn$ T gates, then $\langle \psi | \tilde{H} | \psi \rangle \geq \epsilon$.

Corollary 1. For every $0 < \epsilon < \sin^2(\pi/8)$, ϵ -low-energy states of $\frac{1}{n} \sum_{i=1}^n D |-\rangle\langle -|_i D^\dagger$ require $\Omega(n)$ T gates to prepare.

Corollary 2. For the D -rotated NLTS family from [ABN22], all states of low-enough constant energy require $\Omega(n)$ T gates and $\Omega(\log n)$ depth to prepare.



Local Energy Bound

Consider a local term, $h = \frac{1}{2} (I - H^{\otimes k})$, and a state $|\psi\rangle$ with $S(|\psi\rangle) \equiv G$, prepared by $\leq cn$ T gates.

Step 1: If G "looks like" a full stabilizer group at h , then a local energy bound holds!

$$\langle \psi | h | \psi \rangle \geq \sin^2\left(\frac{\pi}{8}\right)$$

Locally-commuting at h : ignoring the Paulis outside of $\text{supp}(h)$, all terms commute.

$$\begin{pmatrix} I & I \\ I & X \\ X & I \\ X & X \end{pmatrix} \begin{pmatrix} X & I \\ Z & Y \\ Z & X \\ Y & I \end{pmatrix}$$

Consider these four Pauli operators. The "local views" of every operator on qubits 1 and 2 mutually commute, even though the four operators mutually anti-commute.

Pseudo-stabilizer state at h : G has a subgroup, G_h , which is locally-commuting at h and has size 2^k . (Largest possible size)

A Pauli group which locally-commutes on qubits 1 and 2, and has maximal possible size $2^2=4$.

$$\begin{pmatrix} I & I & I & I \\ I & X & I & Y \\ X & I & Z & I \\ X & X & Z & Y \end{pmatrix}$$

If G contains these four operators, then

$$\langle \psi | \frac{1}{2} (I^{\otimes 4} - H \otimes H \otimes I \otimes I) | \psi \rangle \geq \sin^2\left(\frac{\pi}{8}\right)$$

Global Energy Bound

Step 2: $|\psi\rangle$ is pseudo-stabilizer at $\Omega(n)$ local terms of $\frac{1}{m} \sum_{i=1}^{m_x} \frac{1}{2} (I - H^{\otimes k})_i$!

T-count gives a lower bound on $|G|$, we give an upper bound which depends on the sizes of "locally-commuting" subgroups of G .

Lemma. There are subgroups of G , $\{G_i\}$, which (1) locally-commute at h_i , (2) are the largest subgroups of G with this property, and (3) satisfy

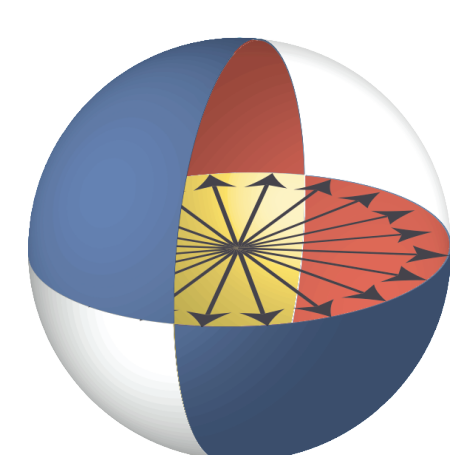
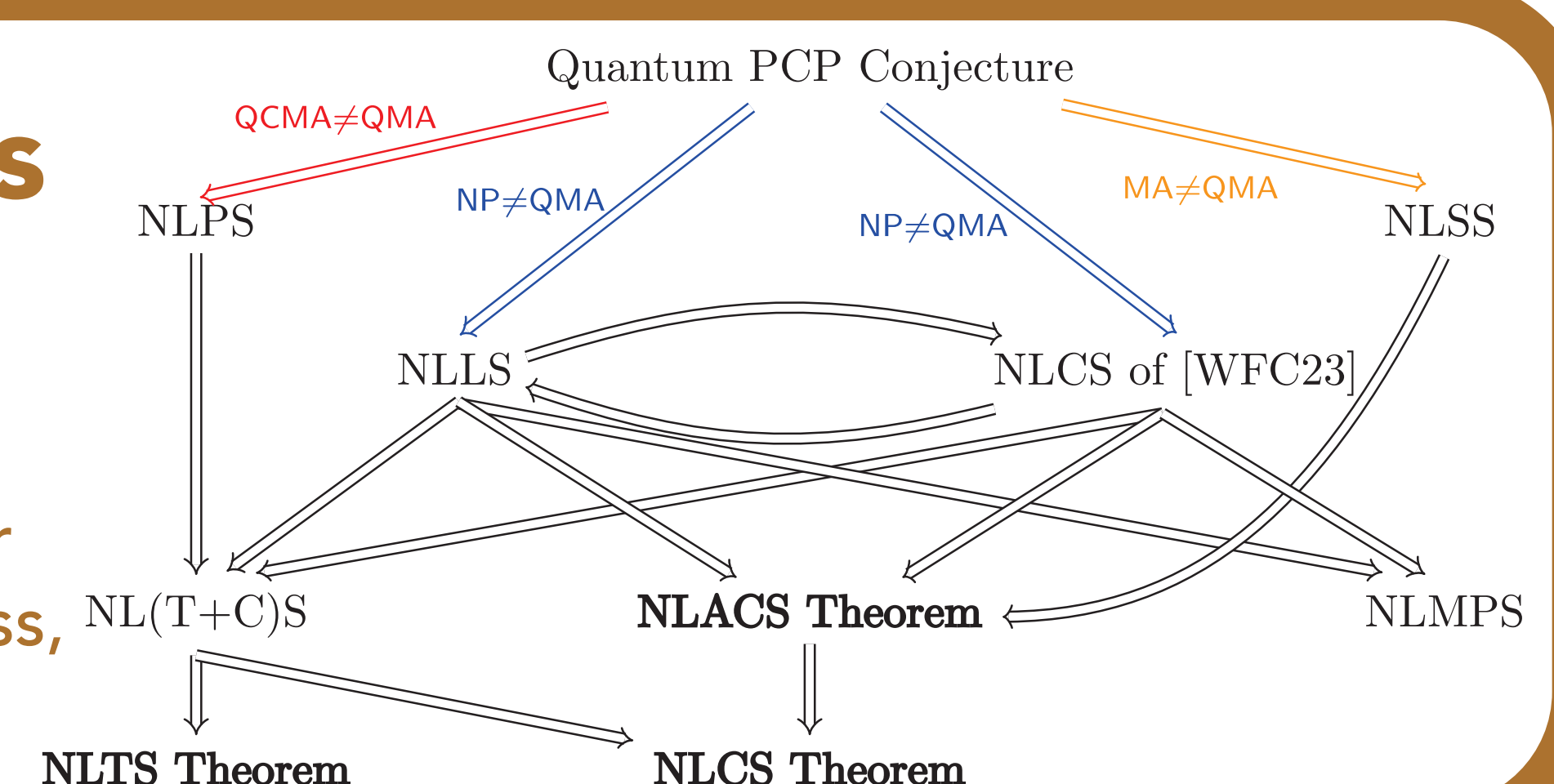
$$|G| \leq \prod_i |G_i|$$

Combined with the lower-bound, $|G| \geq 2^{(1-c)n}$, (3) implies that $\Omega(n)$ of the subgroups have size $|G_i| = 2^k$.

Future Directions

• Examine other Hamiltonian implications of QPCP:

• Improved hardness results for constant-gap LH: BQP-hardness, MA-hardness, ...



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