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Spectra of quaternion unit gain graphs

Francesco Belardo [a], Maurizio Brunetti [a], Nolan J. Coble [b], Nathan Reff [c,*], Howard Skogman [c]

[a] Department of Mathematics and Applications, University of Naples Federico II, Naples, Italy
[b] Applied Mathematics & Statistics, and Scientific Computation Program, University of Maryland, College Park, MD 20742, USA
[c] Department of Mathematics, SUNY Brockport, Brockport, NY 14420, USA

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ABSTRACT

A quaternion unit gain graph is a graph where each orientation of an edge is given a quaternion unit, which is the inverse of the quaternion unit assigned to the opposite orientation. In this paper we define the adjacency, Laplacian and incidence matrices for a quaternion unit gain graph and study their properties. These properties generalize several fundamental results from spectral graph theory of ordinary graphs, signed graphs and complex unit gain graphs. Bounds for both the left and right eigenvalues of the adjacency and Laplacian matrix are developed, and the right eigenvalues for the cycle and path graphs are explicitly calculated.

* Corresponding author.
E-mail addresses: fbelardo@unina.it (F. Belardo), mbrunett@unina.it (M. Brunetti), ncoble@terpmail.umd.edu (N.J. Coble), nreff@brockport.edu (N. Reff), hskogman@brockport.edu (H. Skogman).

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1. Introduction

While matrices of quaternions and their eigenvalues have been studied for almost a century [14,37,56], in the past few decades they have garnered even more attention [56]. Due to the noncommutativity of the quaternions, there are two separate eigenvalue problems to consider for quaternion matrices. The right eigenvalue problem is a natural extension of complex matrices, while the left eigenvalues have much more unexpected behavior [57].

Some spectral properties of quaternionic matrices have been codeveloped by physicists. In particular, the development of quaternion quantum mechanics and gauge theories by Birkhoff and von Neumann [13], and Finkelstein, Jauch and Speiser [29–32] inspired much research [1]. More recently, quaternion differential operators [22] and the right quaternionic eigenvalue problem [23,24] have been studied.

A signed graph is a graph where each edge is given a label of either $+1$ or -1. The spectral properties of signed graphs initially considered in [35] have been the topic of intense study [2,7,11,12,26,27,34,55]. In particular, signed graphs naturally generalize ordinary graphs and their associated matrices. Even the signless Laplacian of an ordinary graph can be considered as the Laplacian of a signed graph with all edges signed -1.

A *gain graph* (or *voltage graph*) is a graph with the additional structure that each orientation of an edge is given a group element, called a *gain*, which is the inverse of the group element assigned to the opposite orientation. *Complex unit gain graphs* are gain graphs where the gains are complex units. This type of gain graph and its associated matrices and eigenvalues were initially studied in [43] and further developed in [3,9,33,40,44,50,52]. This particular gain graph has the advantage that it generalizes both ordinary graphs and signed graphs, since these can be considered as the specialized cases of having all gains $+1$ and ± 1, respectively.

The main goal of this paper is to extend the foundations of spectral graph theory to gain graphs where the gains can be any quaternion unit, which we call *quaternion unit gain graphs*. This type of gain graph will generalize ordinary graphs, signed graphs and complex unit gain graphs since all of these types will be specialized cases. Furthermore, both the Laplacian of a graph, as well as its signless Laplacian can be viewed as specialized cases for the Laplacian of a quaternion unit gain graph.

The paper is organized as follows. Section 2 provides a background on the properties of quaternions, matrices of quaternions and their properties, and general gain graphs with some specialization to quaternion unit gain graphs. In Section 3, a suitably defined incidence matrix leads to an extension of the classical connection with the Laplacian matrix. Switching and switching matrices are also developed. In Sections 4 and 6, the eigenvalues of the adjacency and Laplacian matrices are examined, respectively. Section 7 is devoted to the explicit calculation of the right eigenvalues of the adjacency and Laplacian matrices of both the cycle and path quaternion unit gain graphs. Finally, Section 5 investigates the line graph of a quaternion unit gain graph and associated matrices.

2. Background

2.1. Quaternions

The algebra of real quaternions $\mathbb{H}$ is a unital associative $\mathbb{R}$-algebra with generators i, j and k such that

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.1}$$

An element $q \in \mathbb{H}$ can be written as $q = a + bi + cj + dk$, where a, b, c and $d \in \mathbb{R}$ are uniquely determined. Relations in (2.1) are called *Hamilton's relations*. For simplicity we will say the set of quaternions instead of real quaternions for the remainder of this paper.

Let $q = a + bi + cj + dk \in \mathbb{H}$. The *conjugate* of q is $\overline{q} = a - bi - cj - dk$ (in the literature this is also sometimes denoted $q^\star$). The *norm* (or *modulus*) of q is $|q| = \sqrt{q\overline{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. If $q \neq 0$, then the inverse of q is $q^{-1} = \overline{q}/|q|^2$. If $q, h \in \mathbb{H}$, then $\overline{qh} = \overline{h}\overline{q}$. The *real part* of q is $\mathrm{Re}(q) = a$. The *imaginary part* of q is $\mathrm{Im}(q) = bi + cj + dk$.

The quaternion exponential function can be defined, and a polar form for a quaternion can be derived [4, p. 214]. Consider a nonzero $q = a + bi + cj + dk = a + \mathbf{v} \in \mathbb{H}^*$, and $\mathbf{n} = \mathbf{v}/|\mathbf{v}|$. Then there exists $\theta \in [0, 2\pi)$, such that

$$q = |q|(\cos\theta + \mathbf{n}\sin\theta) = |q|e^{\mathbf{n}\theta},$$

where $\tan\theta = |\mathbf{v}|/a$.

Two quaternions q and h are *similar*, written $q \sim h$, if there exists a nonzero quaternion u, such that $q = u^{-1}hu$. This similarity relation $\sim$ is also an equivalence relation on $\mathbb{H}$. The equivalence class containing $q \in \mathbb{H}$ under this equivalence relation $\sim$ will be denoted by $[q]$. This equivalence relation has several nice properties that have been studied by Au-Yeung [6], Brenner [14] and Zhang [56]. If $x \in \mathbb{R}$ then the equivalence classes are trivial, namely $[x] = \{x\}$. However, for nonreal quaternions $q \in \mathbb{H}\backslash\mathbb{R}$, these equivalence classes have infinitely many quaternions, but also intersect the complex plane precisely at two conjugate pairs. Furthermore, $q \sim \overline{q}$ for every quaternion q.

The subsequent lemmas synthesize the above discussion.

Lemma 2.1 *([56], Lemma 2.1). Let $q = a+bi+cj+dk \in \mathbb{H}$. Then q and $a+\sqrt{b^2+c^2+d^2}\,i$ are similar. That is, $q \in \left[a + \sqrt{b^2+c^2+d^2}\,i\right]$.*

Lemma 2.2 *([14,6], [56], Theorem 2.2). Let $q_1 = a_1 + b_1i + c_1j + d_1k \in \mathbb{H}$ and $q_2 = a_2+b_2i+c_2j+d_2k \in \mathbb{H}$. Then $q_1 \sim q_2$ if and only if $a_1 = a_2$ and $b_1^2+c_1^2+d_1^2 = b_2^2+c_2^2+d_2^2$, that is, $Re(q_1) = Re(q_2)$ and $|Im(q_1)| = |Im(q_2)|$.*

The proof of the following lemma is left to the readers.

Lemma 2.3. *Let* (q_1, q_2) *be any pair of quaternions. Then,*

(i) $\mathrm{Re}(q_1) \leq |q_1|$*, and the equality holds if and only if* q_1 *is a nonnegative real number.*
(ii) $\mathrm{Re}(q_1) = \mathrm{Re}(\overline{q}_1)$ *and* $\mathrm{Re}(q_1q_2) = \mathrm{Re}(q_2q_1)$*.*
(iii) $|q_1q_2| = |q_1||q_2|$*.*

A *unit quaternion* (sometimes called a *versor*) is a quaternion of norm 1. We will denote the multiplicative group of unit quaternions as

$$U(\mathbb{H}) = \{q \in \mathbb{H} : |q| = 1\}.$$

Geometrically this is the 3-sphere, S^3, and one could also consider this as a real Lie group, denoted $Sp(1)$ or $U(1, \mathbb{H})$ in the literature.

Finally, we conclude this short introduction to quanternions by mentioning their representation by pairs of complex numbers. In fact,

$$q = a + bi + cj + dk = (a + bi) + (c + di)j = \gamma_1 + \gamma_2 j. \tag{2.2}$$

Hence, each quaternion q is uniquely represented by a pair of complex numbers (γ_1, γ_2).

2.2. Matrices of quaternions

Let $\mathbb{H}^n$ denote the collection of column vectors of length n with entries in $\mathbb{H}$. Let $\mathbb{H}^{m\times n}$ denote the collection of $m \times n$ matrices with quaternion entries in $\mathbb{H}$. For a column vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$, the conjugate transpose of $\mathbf{x}$ is $\mathbf{x}^* = \overline{\mathbf{x}}^{\mathrm{T}} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)^{\mathrm{T}}$. Suppose $A = (a_{ij}) \in \mathbb{H}^{m\times n}$, then the conjugate transpose of A is $A^* = \overline{A}^{\mathrm{T}} = (\overline{a}_{ji}) \in \mathbb{H}^{n\times m}$.

We next restrict to square matrices in $\mathbb{H}^{n\times n}$; due to the skew-field nature of $\mathbb{H}$, there are three different ways to define eigenvalues.

For a square matrix $A \in \mathbb{H}^{n\times n}$, we say $\lambda \in \mathbb{H}$ is a *left eigenvalue* of A if there exists a nonzero $\mathbf{x} \in \mathbb{H}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$. In this case, $(\lambda, \mathbf{x})$ is said to be a *left eigenpair* for A. The *left spectrum* of A, denoted by $\sigma_l(A)$, is the set of all left eigenvalues of A.

Similarly, λ is said to be a *right eigenvalue* of A if there exists a nonzero $\mathbf{x} \in \mathbb{H}^n$ such that $A\mathbf{x} = \mathbf{x}\lambda$. In this case, $(\lambda, \mathbf{x})$ is said to be a *right eigenpair* for A. The *right spectrum* of A, denoted by $\sigma_r(A)$, is the set of all right eigenvalues of A.

Finally, a third way to define eigenvalues comes from polynomials (cf. [28]). Let $\mathbb{R}[x]$ denote the ring of polynomials with real coefficients and indeterminate x. Since $\mathbb{H}^{n\times n}$ is a real algebra of finite dimension, for a matrix $A \in \mathbb{H}^{n\times n}$ there exists a polynomial $g(x) \in \mathbb{R}[x]$ such that $g(A) = 0$. There is a unique monic polynomial $m_A \in \mathbb{R}[x]$ of minimal degree for which $m_A(A) = 0$. The polynomial m_A is called the *minimal annihilating polynomial* of A. Hence, we define the *spectral eigenvalues* as the roots (in $\mathbb{C}$) of m_A, and we denote the set of spectral eigenvalues by $\sigma(A)$.

For many situations, the properties of right eigenvalues of a quaternion matrix parallel known results from classic spectral theory of complex matrices. However, there are many unique characteristics that arise. For example, if λ is a right eigenvalue of A, then $q^{-1}\lambda q$ is also a right eigenvalue of A for any nonzero quaternion q. In other words, all elements of the equivalence class $[\lambda]$ are right eigenvalues of A. This property does not hold for left eigenvalues, as can be illustrated in the following example from [56,57]. Consider the matrix $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$. Notice that, i, j and k are similar. However, j and k are left eigenvalues of A, but i is not. In fact, $(j, (k, 1))$ and $(k, (1, j))$ are two left eigenpairs. This example is particularly relevant to this paper because we will soon see that A is also an example of the adjacency matrix of a $U(\mathbb{H})$-gain graph.

Lemma 2.4 *([14] Theorem 10, [56] p.36, [28], Proposition 2.1). Let $A \in \mathbb{H}^{n\times n}$. If $(\lambda, \mathbf{x})$ be a right eigenpair of A then for any nonzero $q \in \mathbb{H}$, $(q^{-1}\lambda q, \mathbf{x}q)$ is an eigenpair of A.*

Consider a square quaternion matrix $A \in \mathbb{H}^{n\times n}$. Following (2.2), the matrix A can be uniquely written as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{n\times n}$. The *complex adjoint matrix* (or simply *adjoint*) of A is the matrix

$$f(A) = f(A_1 + A_2 j) = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \in \mathbb{C}^{2n\times 2n}.$$

The complex adjoint matrix verifies the following useful properties (cf. also Theorem 4.2 in [56]):

Proposition 2.5 *([37]). Let $A, B \in \mathbb{H}^{n\times n}$, and let $f(A) \in \mathbb{C}^{2n\times 2n}$ denote the complex adjoint of A. Then, the following hold:*

- *$f(I_n) = I_{2n}$;*
- *$f(AB) = f(A)f(B)$;*
- *$f(A + B) = f(A) + f(B)$;*
- *$f(A^*) = (f(A))^*$;*
- *$f(A^{-1}) = (f(A))^{-1}$, if A^{-1} exists;*
- *$f(A)$ is unitary, Hermitian, or normal if and only if A is unitary, Hermitian, or normal, respectively.*

The complex adjoint is an effective tool for studying the right eigenvalues of A.

Lemma 2.6 *([14,37], [56] Theorem 5.4). Let $A \in \mathbb{H}^{n\times n}$. Then A has exactly n right eigenvalues which are complex numbers with nonnegative imaginary parts.*

Lemma 2.7 *([37], [56] Corollary 5.1). Let $A \in \mathbb{H}^{n\times n}$. Then every real eigenvalue (if any) of the matrix $f(A)$ appears an even number of times, and the complex eigenvalues of that matrix appear in conjugate pairs.*

Lemma 2.8 *([56] Corollary 5.2). Let $A \in \mathbb{H}^{n\times n}$. Then A has exactly n right eigenvalues up to equivalence classes.*

Furthermore, the adjoint can be used explicitly calculate all of these right eigenvalue classes.

Lemma 2.9 *([28] Lemma 2.5). Let $A \in \mathbb{H}^{n\times n}$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $f(A)$, then every $q \in [\lambda]$ is a right eigenvalue of A.*

More can be said in the cases of specialized quaternion matrices, as Lee [37] discovered. For example, just as with Hermitian complex matrices, a Hermitian quaternion matrix will have real right eigenvalues.

Lemma 2.10 *([37] Theorem 9). Let $A \in \mathbb{H}^{n\times n}$. If $A = A^*$, then the right eigenvalues of A are real.*

Lemma 2.11 *([28] Proposition 3.8). If $A \in \mathbb{H}^{n\times n}$ is Hermitian, then every right eigenvalue of A is real. Furthermore, $\lambda \in \sigma(A)$ if and only if $\lambda \in \mathbb{R}$ and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero $\mathbf{x} \in \mathbb{H}^n$.*

Hence, in case of Hermitian quaternion matrix A, we have that $\sigma(A) = \sigma_r(A) \subseteq \sigma_l(A)$, and we have the following spectral decomposition:

Lemma 2.12 *([25] Theorem $\mathbf{1}_{\mathbb{H}}$). Let $A \in \mathbb{H}^{n\times n}$. Then A can be expanded as*

$$A = \sum_{m=1}^{n} \lambda_m \mathbf{x}_m \mathbf{x}_m^*,$$

where $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ an orthonormal basis of eigenvectors of A, with $\mathbf{x}_i$ being an eigenvector corresponding to the real eigenvalue λ_i ($i = 1, 2, \dots, m$).

Since the spectral (or equivalently, right) eigenvalues of any Hermitian matrix $A \in \mathbb{H}^{n\times n}$ are real, we will assume they are labeled and ordered according to the following convention:

$$\lambda_n^r(A) \leq \lambda_{n-1}^r(A) \leq \cdots \leq \lambda_2^r(A) \leq \lambda_1^r(A).$$

Here the superscript r is just to indicate these are right eigenvalues of a Hermitian matrix A. Since we will mostly consider the right/spectral eigenvalues, the superscript r will be omitted whenever there is no ambiguity. Note that when using the adjoint to calculate the right spectra of a Hermitian $A \in \mathbb{H}^{n\times n}$, the eigenvalues of $f(A)$ will match the right one of A, each with a double multiplicity.

Additionally, we can compute the right eigenvalues as spectral eigenvalues, that are, roots of the polynomial m_A defined as the determinant of $\lambda I - A$ (the characteristic polynomial). Attempts to define a determinant for matrices with noncommutative entries started more than 150 years ago, we refer the readers to [5] for a nice and brief introduction to determinants. The existing definitions are not equivalent on the entire set $\mathbb{H}^{n\times n}$. Here we only give the definition of the *Moore determinant* because of its more stringent relationship with the (right) eigenvalues of a Hermitian quaternion matrix.

Let $A = (a_{ij})$ be a matrix in $\mathbb{H}^{n\times n}$, and let σ be a permutation of $S_n = \{1,\dots,n\}$. We say that σ is expressed in the *Moore form* if it is written as a product of disjoint cycles (possibly of length 1)

$$\sigma = (n_{11}\cdots n_{1h_1})(n_{21}\cdots n_{2h_2})\cdots(n_{r1}\cdots n_{rh_r}),$$

with

$$n_{i1} < n_{ij} \quad \forall j > 1, \qquad n_{11} > n_{21} > \cdots > n_{r1} \qquad \text{and} \qquad h_1 + \cdots + h_r = n.$$

It is easily seen that the Moore form of each permutation is unique. Then, we define

$$\mathrm{MdetA} = \sum_{\sigma\in S_n} |\sigma|\, a_\sigma, \tag{2.3}$$

where $|\sigma|$ denotes the parity of σ, and

$$a_\sigma := a_{n_{11},n_{12}} a_{n_{12},n_{13}} \cdots a_{n_{1h_1},n_{11}} a_{n_{21},n_{22}} \cdots a_{n_{rh_r},n_{r1}}.$$

Clearly, $\mathrm{MdetA} = \det A$ whenever $A \in \mathbb{C}^{n\times n} \subset \mathbb{H}^{n\times n}$. The two parts of the following proposition can be deduced by [5, Theorems 9 and 10] and [36, Theorem 7.1], respectively.

Proposition 2.13. *Let $A \in \mathbb{H}^{n\times n}$ be a Hermitian matrix.*

(i) *The number* $\mathrm{Mdet}A$ *is real and it is equal to the product of the n (right) eigenvalues of A.*
(ii) $\mathrm{Mdet}A \neq 0$ *if and only if A is invertible.*

Let $A \in \mathbb{H}^{n\times n}$ be a Hermitian matrix. For $x \in \mathbb{R}$, the matrix $xI - A$ is also Hermitian, and the polynomial

$$p_A(x) = \mathrm{Mdet}(xI - A) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$$

is said to be the *characteristic polynomial* of A. The roots of $p_A(x)$ are the real left eigenvalues of A (this comes, for instance, from Proposition 2.13(ii)), which are its right eigenvalues as well.

The Moore determinant is not multiplicative, especially because the product of two Hermitian matrices is not necessarily Hermitian. However, we have the following result.

Lemma 2.14. *Let $A \in \mathbb{H}^{n\times n}$. Then AA^* and A^*A share the same nonzero (right) eigenvalues.*

Proof. Note that both AA^* and A^*A are Hermitian. Let us consider their complex adjoint $f(AA^*)$ and $f(A^*A)$. In view of Proposition 2.5, their complex adjoints are Hermitian and

$$f(AA^*) = f(A)f(A^*) = f(A)(f(A))^* \quad \text{and} \quad f(A^*A) = f(A^*)f(A) = (f(A))^*f(A).$$

Since $f(A)$ is a matrix with entries in $\mathbb{C}$, then the usual theory on complex Hermitian matrices guarantees that $f(A)(f(A))^*$ and $(f(A))^*f(A)$ share the same nonzero eigenvalues, and the same applies for the matrices AA^* and A^*A. $\square$

Two matrices $A, B \in \mathbb{H}^{n\times n}$ are said to be *similar* if there is an invertible $S \in \mathbb{H}^{n\times n}$ such that $A = S^{-1}BS$. Similar matrices have the same right eigenvalues, however, this is not the case for the left eigenvalues (for example, see [56], Example 7.1).

Lemma 2.15 *([14] Theorem 13). Suppose $A, B \in \mathbb{H}^{n\times n}$ are similar matrices. Then A and B have the same right eigenvalues.*

If $A \in \mathbb{H}^{n\times n}$ is Hermitian, then the quadratic form $\mathbf{x}^*A\mathbf{x}$, for some $\mathbf{x} \in \mathbb{H}^n\backslash\{\mathbf{0}\}$, can be used to calculate the right eigenvalues of A. In particular, we can calculate the smallest and largest eigenvalues using the following, which is a generalized version of Rayleigh-Ritz Theorem.

Lemma 2.16 *([39,49,51]). Let $A \in \mathbb{H}^{n\times n}$ be Hermitian. Then*

$$\lambda_1^r(A) = \max_{\mathbf{x}\in\mathbb{H}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}, \quad \text{and}$$

$$\lambda_n^r(A) = \min_{\mathbf{x}\in\mathbb{H}^n\backslash\{\mathbf{0}\}} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}},$$

where the maximum and minimum are achieved when $\mathbf{x}$ is a right eigenvector of the corresponding right eigenvalues.

A Hermitian matrix $A \in \mathbb{H}^{n\times n}$ is *positive semidefinite* if $\mathbf{x}^*A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{H}^n\backslash\{\mathbf{0}\}$.

Lemma 2.17 *([56], Remark 6.1). Let $A \in \mathbb{H}^{n\times n}$ be Hermitian. Then A is positive semidefinite if and only if A has only nonnegative right eigenvalues, and if and only if $f(A)$ is positive semidefinite.*

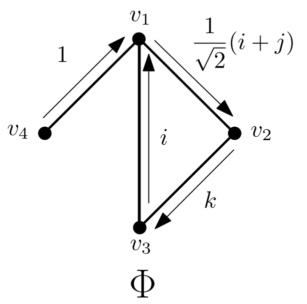

Fig. 1. A $U(\mathbb{H})$-gain graph Φ.

Zhang's work on quaternionic matrices also extends some classical bounds from complex matrix theory [57], including the following bound for both the right and left spectra.

Lemma 2.18 *([57], Theorem 2). Let $A \in \mathbb{H}^{n\times n}$ and let $\lambda \in \mathbb{H}$ be a left or right eigenvalue of A. Then*

$$|\lambda| \leq \max_i \sum_{j=1}^{n} |a_{ij}|.$$

Let $A \in \mathbb{H}^{n\times n}$. The left spectral radius of A is $\rho_l(A) = \max\{|\lambda|\colon \lambda \in \sigma_l(A)\}$. Similarly, the right spectral radius of A is $\rho_r(A) = \max\{|\lambda|\colon \lambda \in \sigma_r(A)\}$.

2.3. Gain graphs

Suppose $\Gamma = (V, E)$ is a graph. The set of oriented edges, denoted by $\vec{E}(\Gamma)$, contains two copies of each edge with opposite directions. An oriented edge from v_i to v_j is denoted by e_{ij}. Formally, a *gain graph* is a triple $\Phi = (\Gamma, \mathfrak{G}, \varphi)$ consisting of an *underlying graph* $\Gamma = (V, E)$, the *gain group* $\mathfrak{G}$ and a function $\varphi : \vec{E}(\Gamma) \to \mathfrak{G}$ (called the *gain function*), such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1}$. For brevity, we write $\Phi = (\Gamma, \varphi)$ for a gain graph if the gain group is clear from the context, and call Φ a $\mathfrak{G}$-gain graph. In this paper the focus will be on $U(\mathbb{H})$-*gain graphs* (or *quaternion unit gain graphs*). See Fig. 1 for an example. To make the picture less cluttered each edge will only have one oriented edge labeled with a gain (since the gain in the opposite direction is immediately determined as the inverse).

The set of complex units is $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and we will focus on this as a multiplicative group (sometimes called the *circle group* in the literature). Throughout this paper, a $\mathbb{T}$-*gain graph* may also be referred to as a *complex unit gain graph*.

We will always assume that Γ is simple. The set of vertices is $V := \{v_1, v_2, \ldots, v_n\}$. Edges in E are denoted by $e_{ij} = v_iv_j$. Even though this is the same notation for an oriented edge from v_i to v_j it will always be clear whether an edge or oriented edge is being used. We define $n := |V|$ and $m := |E|$. The degree of a vertex v_j is denoted by $d_j = \deg(v_j)$. The maximum degree is denoted by Δ.

A *switching function* is any function $\zeta : V \to \mathfrak{G}$. Switching the $\mathfrak{G}$-gain graph $\Phi = (\Gamma, \varphi)$ means replacing φ by φ^ζ, defined by: $\varphi^\zeta(e_{ij}) = \zeta(v_i)^{-1}\varphi(e_{ij})\zeta(v_j)$; producing the $\mathfrak{G}$-gain graph $\Phi^\zeta = (\Gamma, \varphi^\zeta)$. We say Φ_1 and Φ_2 are *switching equivalent*, written $\Phi_1 \sim \Phi_2$, when there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. Switching equivalence forms an equivalence relation on gain functions for a fixed underlying graph. An equivalence class under this equivalence relation is called a *switching class* of φ, and is denoted by $[\Phi]$.

The gain of a walk $W = v_1e_{12}v_2e_{23}v_3 \cdots v_{k-1}e_{k-1,k}v_k$ is $\varphi(W) = \varphi(e_{12})\varphi(e_{23}) \cdots \varphi(e_{k-1,k})$. A walk W is *neutral* if $\varphi(W) = 1_\mathfrak{G}$, where $1_\mathfrak{G}$ denotes the identity of $\mathfrak{G}$. An edge set $S \subseteq E$ is *balanced* if every cycle $C \subseteq S$ is neutral. A subgraph is *balanced* if its edge set is balanced. We write $b(\Phi)$ for the number of connected components of Φ that are balanced.

Switching conjugates the gain of a closed walk in a gain graph.

Lemma 2.19 *([44], Proposition 2.1). Let $W = v_1e_{12}v_2e_{23}v_3 \cdots v_ke_{k1}v_1$ be a closed walk in a $\mathfrak{G}$-gain graph Φ. Let $\zeta : V \to \mathfrak{G}$. Then $\varphi^\zeta(W) = \zeta(v_1)^{-1}\varphi(W)\zeta(v_1)$.*

If C is a cycle in the underlying graph Γ, then the gain of C in some fixed direction starting from a base vertex v is denoted by $\varphi(\vec{C}_v)$. Thus, $\vec{C}_v$ is called a *directed cycle with base vertex v*.

A *potential function* for φ is a function $\theta : V \to \mathfrak{G}$, such that for every $e_{ij} \in \vec{E}(\Gamma)$, $\theta(v_i)^{-1}\theta(v_j) = \varphi(e_{ij})$ [54]. We write $(\Gamma, 1_\mathfrak{G})$ for the $\mathfrak{G}$-gain graph with all neutral edges. The following result can be deduced from [53].

Lemma 2.20. *Let $\Phi = (\Gamma, \varphi)$ be a $\mathfrak{G}$-gain graph. Then the following are equivalent:*

(1) Φ is balanced.
(2) $\Phi \sim (\Gamma, 1_\mathfrak{G})$.
(3) φ has a potential function.

For a quaternion unit gain graph we can associate several different types of matrices. Suppose Φ is a $U(\mathbb{H})$-gain graph. The *adjacency matrix* $A(\Phi) = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined by

$$a_{ij} = \begin{cases} \varphi(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then $a_{ij} = \varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \overline{\varphi(e_{ji})}/|\varphi(e_{ji})|^2 = \overline{\varphi(e_{ji})} = \overline{a}_{ji}$. Therefore, $A(\Phi)$ is Hermitian.

The *Laplacian matrix* (*Kirchhoff matrix* or *admittance matrix*) is defined as $L(\Phi) = D(\Gamma) - A(\Phi)$, where $D(\Gamma)$ is the diagonal matrix of the degrees of vertices of Γ. Hence, $L(\Phi)$ is also Hermitian.

$$\begin{bmatrix} -1 & \frac{-1}{\sqrt{2}}(i+j) & i & 0 \\ 0 & 1 & 0 & -k \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -\alpha & \frac{-1}{\sqrt{2}}(i+j)\beta & i\delta & 0 \\ 0 & \beta & 0 & -k\epsilon \\ 0 & 0 & \delta & \epsilon \\ \alpha & 0 & 0 & 0 \end{bmatrix}$$

Fig. 2. Two examples of incidence matrices for the $U(\mathbb{H})$-gain graph Φ in Fig. 1. The incidence matrix on the left has $\alpha_e = 1$ for each column, and for the incidence matrix on the right α, β, δ and ϵ can be any element of $U(\mathbb{H})$.

Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. We denote by $-\Phi$ the $U(\mathbb{H})$-gain graph $(\Gamma, -\varphi)$ obtained by replacing each edge-gain with its opposite. Clearly $A(-\Phi) = -A(\Phi)$. Moreover, we say that Φ is *antibalanced* if and only if $-\Phi$ is balanced. We say Φ_1 and Φ_2 are *isomorphic* if there exists a graph isomorphism that preserves edge gains. When Φ_1 is isomorphic to a switching of Φ_2, then we say they are *switching isomorphic*. If Φ and $-\Phi$ are switching isomorphic, then Φ is said to be *gain-symmetric*. When Φ is gain-symmetric, since $A(\Phi)$ and $A(-\Phi) = -A(\Phi)$ are similar matrices, then the adjacency spectrum of A is symmetric w.r.t. the origin. Note, the latter is not a unique feature of gain-symmetric graphs, as counterexamples arise for signed graphs [10].

3. Incidence and switching matrices

Suppose $\Phi = (\Gamma, \varphi)$ is a $U(\mathbb{H})$-gain graph. An *incidence matrix* $\mathrm{H}(\Phi) = (\eta_{ve})$ is any $n \times m$ matrix, with entries in $U(\mathbb{H}) \cup \{0\}$, where each column corresponds to an edge $e = e_{ij} \in E$, and has all zero entries except two nonzero entries $\eta_{v_j e} = \alpha_e \in U(\mathbb{H})$ and $\eta_{v_i e} = -\varphi(e_{ij})\alpha_e$. This choice for $\alpha_e \in U(\mathbb{H})$ is arbitrary for each column, which is why we say "an" incidence matrix, because with this definition $\mathrm{H}(\Phi)$ is not unique, but instead a family of incidence matrices. For example, if we choose $\alpha_e = 1$ for all columns we will have $\eta_{v_j e} = 1$ so $\eta_{v_i e} = -\varphi(e_{ij})$ for each $e_{ij} \in E$. This particular incidence matrix can be viewed as a generalization of an oriented incidence matrix of an unsigned graph. The more general definition above is a generalization of the incidence matrix for signed graphs [55] as well as complex unit gain graphs [43]. Henceforth, we will write $\mathrm{H}(\Phi)$ to indicate that some fixed incidence matrix has been chosen. See Fig. 2 for an example.

The incidence matrix can also be used to calculate the Laplacian matrix, and thereby showing a connection to the adjacency matrix as well. This result is a natural generalization of the same result known for graphs, signed graphs [55], $\mathbb{T}$-gain graphs [44], and a particular case of Lemma 4.9 in [16] (when $G = U(\mathbb{H})$ and $s_1 = -1$). The following result is completely independent on the choice for α_e as mentioned above, so any legitimate incidence matrix from the family above may be used.

Lemma 3.1. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. Then $L(\Phi) = \mathrm{H}(\Phi)\mathrm{H}(\Phi)^*$.*

Proof. Notice that both $L(\Phi)$ and $\mathrm{H}(\Phi)\mathrm{H}(\Phi)^*$ are $n \times n$ matrices. Denote the i^{th} row of $\mathrm{H}(\Phi)$, indexed by $v_i \in V$ as $\mathbf{r}_i$. Denote the j^{th} column of $\mathrm{H}(\Phi)^*$, indexed by $v_j \in V$ as $\mathbf{c}_j$. Now the (i,j)-entry of $\mathrm{H}(\Phi)\mathrm{H}(\Phi)^*$ corresponds to the dot product $\mathbf{r}_i \cdot \mathbf{c}_j$.

Case 1: $i = j$ (same vertices). Then $\mathbf{c}_j = \mathbf{r}_i^*$ and the dot product can be simplified as follows,

$$\mathbf{r}_i \cdot \mathbf{c}_j = \mathbf{r}_i \cdot \mathbf{r}_i^* = \sum_{e \in E} \eta_{v_i e} \overline{\eta}_{v_i e} = \sum_{e \in E} |\eta_{v_i e}|^2 = d_i,$$

since $|\eta_{v_i e}| = 1$ if v_i is incident to e.

Case 2: $i \neq j$ (distinct vertices). Here the dot product simplifies as follows,

$$\mathbf{r}_i \cdot \mathbf{c}_j = \sum_{e \in E} \eta_{v_i e} \overline{\eta}_{v_j e}.$$

If there is no edge $e \in E$ that both v_i and v_j are incident to, then this sum is simply 0. However, if there is an edge which joins v_i and v_j as adjacent vertices, then there can only be one such edge since Γ is simple. This means that the sum can be further simplified to a single term $\eta_{v_i e} \overline{\eta}_{v_j e}$, where $e = e_{ij}$ is this single adjoining edge. Furthermore,

$$\mathbf{r}_i \cdot \mathbf{c}_j = \eta_{v_i e} \overline{\eta}_{v_j e} = -\varphi(e_{ij}) \alpha_e \overline{\alpha}_e = -\varphi(e_{ij}) |\alpha_e|^2 = -\varphi(e_{ij}) = -a_{ij},$$

where $\alpha_e \in U(\mathbb{H})$ can be arbitrarily chosen for each edge, as mentioned in the definition of $\mathrm{H}(\Phi)$.

Therefore, $\mathrm{H}(\Phi)\mathrm{H}(\Phi)^* = D(\Gamma) - A(\Phi) = L(\Phi)$. □

With this alternative form of the Laplacian calculation using an incidence matrix, the quadratic form $\mathbf{x}^* L(\Phi) \mathbf{x}$ can be simplified. This is a more general form than the specialized cases known for graphs, signed graphs [35], and complex unit gain graphs [43].

Corollary 3.2. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{H}^n$. Then*

$$\mathbf{x}^* L(\Phi) \mathbf{x} = \sum_{e_{ij} \in E(\Phi)} |x_i - \varphi(e_{ij}) x_j|^2.$$

Moreover, $L(\Phi)$ is positive semidefinite, and the right eigenvalues of $L(\Phi)$ are all non-negative.

Proof. Let $\mathbf{x} \in \mathbb{H}^n$. By Lemma 3.1 we can expand the quadratic form as follows,

$$\mathbf{x}^* L(\Phi) \mathbf{x} = \mathbf{x}^* \mathrm{H}(\Phi) \mathrm{H}^*(\Phi) \mathbf{x} = \left(\mathrm{H}^*(\Phi)\mathbf{x}\right)^* \left(\mathrm{H}^*(\Phi)\mathbf{x}\right) = \sum_{e \in E(\Gamma)} \overline{\left(\sum_{k=1}^{n} \overline{\eta_{v_k e}} x_k\right)} \left(\sum_{\ell=1}^{n} \overline{\eta_{v_\ell e}} x_\ell\right).$$

Since Γ is a simple graph, each edge e is incident to only two adjacency vertices, and we can simplify further. Now summing over edges $e = e_{ij}$ joining v_i and v_j.

$$\begin{aligned}
\mathbf{x}^*L(\Phi)\mathbf{x} &= \sum_{e_{ij}\in E(\Gamma)} \overline{(\overline{\eta_{v_ie_{ij}}}x_i + \overline{\eta_{v_je_{ij}}}x_j)}(\overline{\eta_{v_ie_{ij}}}x_i + \overline{\eta_{v_je_{ij}}}x_j) \\
&= \sum_{e_{ij}\in E(\Gamma)} \overline{x_i}\eta_{v_ie_{ij}}\overline{\eta_{v_ie_{ij}}}x_i + \overline{x_i}\eta_{v_ie_{ij}}\overline{\eta_{v_je_{ij}}}x_j + \overline{x_j}\eta_{v_je_{ij}}\overline{\eta_{v_ie_{ij}}}x_i + \overline{x_j}\eta_{v_je_{ij}}\overline{\eta_{v_je_{ij}}}x_j \\
&= \sum_{e_{ij}\in E(\Gamma)} \overline{x_i}|\eta_{v_ie_{ij}}|^2x_i - \overline{x_i}\varphi(e_{ij})|\alpha_{e_{ij}}|^2x_j - \overline{x_j}|\alpha_{e_{ij}}|^2\overline{\varphi(e_{ij})}x_i + \overline{x_j}|\eta_{v_je_{ij}}|^2x_j.
\end{aligned}$$

Furthermore, since $\eta_{v_ie_{ij}}, \eta_{v_je_{ij}}, \alpha_{e_{ij}}, \varphi(e_{ij}) \in U(\mathbb{H})$ this expansion can be further simplified to the desired form,

$$\mathbf{x}^*L(\Phi)\mathbf{x} = \sum_{e_{ij}\in E(\Gamma)} \overline{(x_i - \varphi(e_{ij})x_j)}(x_i - \varphi(e_{ij})x_j) = \sum_{e_{ij}\in E(\Gamma)} |x_i - \varphi(e_{ij})x_j|^2.$$

In this simplified form it is clear that $\mathbf{x}^*L(\Phi)\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{H}^n\backslash\{\mathbf{0}\}$. Hence, by definition, $L(\Phi)$ is positive semidefinite. Moreover, by Lemma 2.17 the right eigenvalues of $L(\Phi)$ are all nonnegative. □

Suppose $\zeta\colon V \to U(\mathbb{H})$ is a switching function for a $U(\mathbb{H})$-gain graph Φ. The *switching matrix* associated to ζ is the diagonal matrix $D(\zeta) := \mathrm{diag}(\zeta(v_i)\colon v_i \in V)$. Matrices associated to the switched gain graph Φ^ζ can be calculated directly from the matrices associated to Φ.

Lemma 3.3. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. Let $\zeta\colon V \to U(\mathbb{H})$ be a switching function for Φ. Then*

- $\mathrm{H}(\Phi^\zeta) = D(\zeta)^{-1}\mathrm{H}(\Phi) = D(\zeta)^*\mathrm{H}(\Phi)$,
- $A(\Phi^\zeta) = D(\zeta)^{-1}A(\Phi)D(\zeta) = D(\zeta)^*A(\Phi)D(\zeta)$,
- $L(\Phi^\zeta) = D(\zeta)^{-1}L(\Phi)D(\zeta) = D(\zeta)^*L(\Phi)D(\zeta)$.

Remark 3.4. Let Φ be a $U(\mathbb{H})$-gain graph and let $M = M(\Phi)$ an associated Hermitian matrix. The M-spectral theory of $U(\mathbb{H})$-gain graphs is based on the right/spectral eigenvalues of M. In the remainder of the paper, whenever there is no possibility of ambiguity, we refer to the right/spectral eigenvalues as *the (M-)eigenvalues of the $U(\mathbb{H})$-gain graph*. Similarly, the eigenvalues of the $U(\mathbb{H})$-gain graph Φ will form the *M-spectrum* $\sigma(M(\Phi)) = \sigma_M(\Phi)$.

4. Eigenvalues of the adjacency matrix

The A-spectral theory of $U(\mathbb{H})$-gain graph nicely extends the A-spectral theory of $\mathbb{T}$-gain graphs. In this section we will give the basic results for this spectral theory. The next lemma implies that a switching class has a (unique) adjacency spectrum. This is immediate from Lemma 3.3 and Lemma 2.15.

Lemma 4.1. *Let $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ both be $U(\mathbb{H})$-gain graphs. If $\Phi_1 \sim \Phi_2$, then $A(\Phi_1)$ and $A(\Phi_2)$ have the same spectrum. That is, $\sigma_A(\Phi_1) = \sigma_A(\Phi_2)$.*

Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph, and let X be a subset of $V(\Gamma)$. We write $\Phi[X]$ to denote the induced subgraph of Φ with vertex set X, and write $\Phi - X$ to denote $\Phi[V(\Gamma) \setminus X]$. If $X = \{v\}$, then we will write $\Phi - v$ in place of $\Phi - \{v\}$. It is natural to ask whether the (right) eigenvalues of $A(\Phi)$ and $A(\Phi[X])$ interlace. The answer is positive as the following result shows.

Theorem 4.2 *(Interlacing Theorem). Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph with n vertices, and let X be a subset of $V(\Gamma)$ containing k vertices. Denoted by*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \qquad \text{and} \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$$

the eigenvalues of $A(\Phi)$ an $A(\Phi[X])$ respectively, the following inequalities hold:

$$\lambda_i \geq \mu_i \geq \lambda_{n+i-k} \qquad \text{for } 1 \leq i \leq k.$$

Proof. By Lemma 2.10 it is not restrictive to assume that the vertices $A(\Phi[X])$ correspond to the principal submatrix of $A(\Phi)$ obtained by deleting the last $n-k$ rows and the last $n-k$ columns. The statement now comes from [47, Theorem 2.2]. □

Corollary 4.3. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph with n vertices. For every $v \in V(\Gamma)$, the right eigenvalues of $A(\Phi)$ and of $A(\Phi - v)$ labeled in decreasing order interlace as follows.*

$$\lambda_1(\Phi) \geqslant \lambda_1(\Phi - v) \geq \lambda_2(\Phi) \geq \lambda_2(\Phi - v) \geq \cdots \geq \lambda_{n-1}(\Phi - v) \geq \lambda_n(\Phi).$$

We are going to give a $U(\mathbb{H})$-version of the *Coefficient Theorem*, also known as *Sachs Formula*. This formula describes the connection between the (right) eigenvalues of the adjacency matrix of a $U(\mathbb{H})$-gain graph $\Phi = (\Gamma, \varphi)$ and the combinatorial structure of Φ. It was given for simple graphs in the 1960s independently by several researchers (with different notation), but possibly first stated by Sachs (see [20, Theorem 1.2] and the subsequent remark). In order to state Theorems 4.5 and 4.6, we need the following lemma.

Lemma 4.4. *Let $\Phi = (\Gamma, \varphi)$ a $U(\mathbb{H})$-gain graph and let $\mathcal{C}(\Gamma)$ be the set of cycles of Γ. It is well-defined the function*

$$\mathfrak{R} : C \in \mathcal{C}(\Gamma) \longmapsto \operatorname{Re}(\varphi(\vec{C}_v)) \in \mathbb{R}, \tag{4.1}$$

where v is any vertex of C, and $\vec{C}_v$ is any of the two directed cycles with base vertex v determined by C.

Proof. Let $\vec{C}_v = v_1 e_{12} v_2 \cdots v_l e_{l1} v_1$ and $\vec{C}_v^{-1}$ be the two different directed cycles with base vertex v. By definition, $\varphi(\vec{C}_v)\varphi(\vec{C}_v^{-1}) = 1$. Hence the two quaternions $\varphi(\vec{C}_v)$ and $\varphi(\vec{C}_v^{-1})$ share their real part, since they are conjugate.

Let now $w = v_h$ with $h > 1$. We want to compare the gains of $\vec{C}_v$ and of $\vec{C}_w = v_h e_{h,h+1} v_{h+1} \cdots v_l e_{l1} v_1 \cdots e_{h-1,h} v_h$. Once we set

$$q_1 = \varphi(e_{12})\varphi(e_{23}) \cdots \varphi(e_{h-1,h}) \qquad \text{and} \qquad q_2 = \varphi(e_{h,h+1})\varphi(e_{h+1,h+2}) \cdots \varphi(e_{l1}),$$

we observe that, by Lemma 2.3(ii),

$$\mathrm{Re}(\varphi(\vec{C}_v)) = \mathrm{Re}(q_1 q_2) = \mathrm{Re}(q_2 q_1) = \mathrm{Re}(\varphi(\vec{C}_w)).$$

This completes the proof. $\square$

Let C_r denote the cycle of order r, and let K_2 be the complete graph with two vertices. An elementary figure is any graph in the set $\{K_2, C_r \mid r \geq 3\}$. A *basic figure* is the disjoint union of elementary figures. If B is a basic figure, we denote by $\mathcal{C}(B)$ the class of cycles in B, by $p(B)$ the number of components of B, and, for $1 \leq i \leq |V(\Gamma)|$, by $\mathcal{B}_i(\Gamma)$ the set of subgraphs of Γ which are basic figures with i vertices. We also define

$$c(B) := |\mathcal{C}(B)| \qquad \text{and} \quad \mathfrak{R}(B) := \prod_{C \in \mathcal{C}(B)} \mathfrak{R}(C).$$

The latter makes sense by Lemma 4.4.

Theorem 4.5. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph with n vertices. Then,*

$$\mathrm{Mdet} A(\Phi) = \sum_{B \in \mathcal{B}_n(\Gamma)} (-1)^{n+p(B)} 2^{c(B)} \mathfrak{R}(B).$$

Proof. A cycle

$$\sigma = (n_{11} \cdots n_{1h_1})(n_{21} \cdots n_{2h_2}) \cdots (n_{r1} \cdots n_{rh_r}).$$

expressed in its Moore form gives a non-zero contribution in the sum (2.3) if and only if the sequences of vertices

$$E_1 = v_{n_{11}} v_{n_{12}} \cdots v_{n_{1h_1}}, \quad E_2 = v_{n_{21}} v_{n_{22}} \cdots v_{n_{2h_2}}, \quad \ldots \quad E_r = v_{n_{r1}} v_{n_{r2}} \cdots v_{n_{rh_r}}$$

determine a basic figure B in $\mathcal{B}_n(\Gamma)$, and each E_j can be regarded as a fixed elementary figure. For the rest of the proof we assume that this is the case. The argument to show that $|\sigma| = (-1)^{n+p(B)}$ is given along the proof of [20, Theorem 1.3]. We now consider the following closed walks

$$\vec{E}_j = v_{n_{j1}} v_{n_{j2}} \cdots v_{n_{jh_j}} v_{n_{j1}} \qquad \text{and} \qquad \vec{E}_j^{-1} = v_{n_{j1}} v_{n_{jh_j}} \cdots v_{n_{j2}} v_{n_{j1}}$$

for $1 \leq j \leq r$ (the connecting edges have been dropped for sake of clarity). Note that $\vec{E}_j = \vec{E}_j^{-1}$ if and only if $E_j = K_2$. By definition, we see that

$$a_\sigma = \prod_{i=1}^{r} \varphi(\vec{E}_j),$$

moreover, $\varphi(\vec{E}_j) + \varphi(\vec{E}_j^{-1}) = 2\Re(E_j)$ in particular for all j's such that $h_j \geq 3$, and $\varphi(\vec{E}_j) = 1$ whenever $h_j = 2$, since $a_{j1,j2} a_{j2,j1} = a_{j1,j2} a^*_{j1,j2} = |a_{j1,j2}| = 1$.

It is now routine to show that the contribution in the sum (2.3) of all σ's determining the same basic figures B is $(-1)^{n+p(B)} 2^{c(B)} \Re(B)$. $\square$

We are now ready to state and prove the Coefficient Theorem.

Theorem 4.6 *(Coefficient Theorem). Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph with n vertices, and let $p_{A(\Phi)}(x) = x^n + \sum_{i=1}^n a_i x^{n-i}$ be the characteristic polynomial of $A(\Phi)$. Then,*

$$a_i = \sum_{B \in \mathcal{B}_i(\Gamma)} (-1)^{p(B)} 2^{c(B)} \Re(B).$$

Proof. By [36, Theorem 16.1], for $1 \leq i \leq n$, the number $(-1)^i a_i$ computes the sum of the principal (Moore)-minors in $A(\Phi)$ of order i. The result now follows from Theorem 4.5. $\square$

It is worthy to notice that Theorem 4.6 reduces to the Coefficient Theorem for $\mathbb{T}$-gain graphs [41, Corollary 3.1] if the gains are all in $\mathbb{C}$, and to the Coefficient Theorem for signed graphs [10, Theorem 2.1] if the gains are in $\{-1, 1\}$.

Let $\rho_A = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$ be the spectral radius of the matrix A. Now, we consider some basic inequalities related to the largest eigenvalue λ_1 and the spectral radius ρ_A of the adjacency matrix of a $U(\mathbb{H})$-gain graph. In general, for a $U(\mathbb{H})$-gain graph, it is $\rho_A(\Phi) = \max\{\lambda_1(\Phi), -\lambda_n(\Phi)\}$. Notably, when Φ is a balanced graph it is $\rho_A(\Phi) = \lambda_1(\Phi)$, while $\rho_A(\Phi) = -\lambda_n(\Phi)$ holds for Φ antibalanced.

The famous Perron-Frobenius Theorem [20] is not valid for the generic graph Φ, however it does not completely disappear. As for signed graphs and for $\mathbb{T}$-gain graphs, the balanced $U(\mathbb{H})$-gain graph will enjoy the validity of the Perron-Frobenius Theory (the nonnegative eigenvector gets components switched in sign). The following lemma will be used later.

Lemma 4.7. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph with adjacency matrix $A = A(\Phi)$, and $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a unit eigenvector related to the eigenvalue $\lambda_k(\Phi)$. Then,*

$$\lambda_k(\Phi) = \sum_{ij \in E(\Phi)} 2\mathrm{Re}(\overline{x_i}\, \varphi(e_{ij})\, x_j). \tag{4.2}$$

Proof. Clearly,

$$
\begin{aligned}
\lambda_k(\Phi) &= \mathbf{x}^* A(\Phi)\mathbf{x} \\
&= \sum_{ij \in E(\Phi)} (\overline{x_i}\, \varphi(e_{ij})\, x_j) + (\overline{x_j}\, \varphi(e_{ji})\, x_i) \\
&= \sum_{ij \in E(\Phi)} (\overline{x_i}\, \varphi(e_{ij})\, x_j) + (\overline{x_j}\, \overline{\varphi(e_{ij})}\, x_i) \\
&= \sum_{ij \in E(\Phi)} 2\mathrm{Re}(\overline{x_i}\, \varphi(e_{ij})\, x_j),
\end{aligned}
$$

where the last equality is obtained as the sum of a quaternion with its conjugate. □

Let $\Phi = (\Gamma, \varphi)$ a $U(\mathbb{H})$-gain graph and let $\rho_A(\Gamma)$ be the adjacency spectral radius of the underlying graph Γ. The following result is consistent with the main result in [17] and it generalizes Lemma 2.1 of [46]. Note that $U(\mathbb{H})$ is an infinite group, and the proof of the characterization of balance in terms of adjacency spectral radius in [17] is for finite groups.

Theorem 4.8. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph with underlying graph Γ. Then*

$$\rho_A(\Phi) \le \rho_A(\Gamma).$$

Moreover, if Φ is connected, $\lambda_1(\Phi) = \rho_A(\Gamma)$ (resp. $-\lambda_n(\Phi) = \rho_A(\Gamma)$) if and only if Φ is balanced (resp. antibalanced).

Proof. To prove the main inequality, let $\mathbf{x}$ be a unit eigenvector of $\lambda_k(\Phi)$, where $\mathbf{x}^\top = (x_1, \dots, x_n)$, and let $\mathbf{y}^\top = (|x_1|, \dots, |x_n|)$. The following sequence of inequalities holds.

$$
\begin{aligned}
|\lambda_k(\Phi)| &= |\mathbf{x}^* A(\Phi)\mathbf{x}| && \text{(by definition)} \\
&= |\sum_{ij \in E} 2\mathrm{Re}(\overline{x_i}\, \varphi(e_{ij})\, x_j)| && \text{(by Lemma 4.7)} \\
&\le 2 \sum_{ij \in E} |\overline{x_i}\, \varphi(e_{ij})\, x_j| && \text{(by Lemma 2.3(i))} \\
&= 2 \sum_{ij \in E} |\overline{x_i}|\, |\varphi(e_{ij})|\, |x_j| && \text{(by Lemma 2.3(iii))} \\
&= 2 \sum_{ij \in E} |\overline{x_i}|\, |x_j| && \text{(since } \varphi(e_{ij}) \in U(\mathbb{H})) \\
&= \mathbf{y}^* A(\Gamma)\mathbf{y} && \text{(by definition of } A(\Gamma) \text{ and } \mathbf{y}) \\
&\le \rho_A(\Gamma) && \text{(by Lemma 2.16 applied to } A(\Gamma, 1_{U(\mathbb{H})}) = A(\Gamma)).
\end{aligned}
$$

We next prove the additional statements. Let Φ be connected and $\lambda_1(\Phi) = \rho_A(\Gamma)$, if so the inequalities above for $k = 1$ become equalities. In particular, we get

$$\overline{x_i}\varphi(e_{ij})x_j = |\overline{x_i}|\,|x_j|\,, \tag{4.3}$$

and $\mathbf{y}$ is an $A(\Gamma)$-eigenvalue with respect to $\rho_A(\Gamma)$. As a consequence of Perron-Frobenius Theorem applied to $A(\Gamma)$, the real number $|\overline{x_i}|$ and, *a fortiori*, the quaternion x_i are non-zero for $1 \leq i \leq n$. Therefore, the function

$$\theta : v_i \in V(\Gamma) \longmapsto \frac{\overline{x}_i}{|x_i|} \in U(\mathbb{H})$$

is well-defined. We now show that θ is a potential function for φ. Multiplying both sides of Equation (4.3) by x_i (resp. $\overline{x}_j$) on the left (resp., on the right), we obtain

$$x_i\overline{x}_i\varphi(e_{ij})x_j\overline{x}_j = x_i\,|\overline{x_i}|\,|x_j|\,\overline{x_j}.$$

Since $x_i\overline{x}_i = |x_i|^2 = |\overline{x}_i|^2 \neq 0$ for $1 \leq i \leq n$, we deduce

$$\varphi(e_{ij}) = \frac{x_i}{|x_i|}\frac{\overline{x}_j}{|\overline{x}_j|} = \theta(v_i)^{-1}\theta(v_j).$$

The balance of Φ is now ensured by Lemma 2.20.

Conversely, just note that if Φ is balanced, then $\lambda_1(\Phi) = \rho_A(\Gamma)$ by Lemma 4.1. This completes the proof.

Finally, let Φ be connected and $-\lambda_n(\Phi) = \rho_A(\Gamma)$. Evidently, $-\lambda_n(\Phi) = \lambda_1(-\Phi) = \rho(A(\Gamma))$. Hence, $-\Phi$ is balanced which implies that Φ is antibalanced.

This completes the proof. $\square$

A bound which involves structural properties of a $U(\mathbb{H})$-gain graph Φ and the left eigenvalues of its associated matrices is interesting because not much is known about the left eigenvalues of quaternion matrices. The following is a bound for both the left and right spectral radius of the adjacency matrix of a $U(\mathbb{H})$-gain graph Φ. This result can also be seen as a generalization of the same result known for $\mathbb{T}$-gain graphs [43] and thus, signed and ordinary graphs as well.

Lemma 4.9. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. Then*

$$\rho_l(A(\Phi)) \leq \Delta \quad \text{and} \quad \rho_r(A(\Phi)) \leq \Delta.$$

Proof. If $\lambda \in \mathbb{H}$ is a left or right eigenvalue of $A(\Phi) = (a_{ij})$, then by Lemma 2.18,

$$|\lambda| \leq \max_i \sum_{j=1}^{n} |a_{ij}| = \max_i \sum_{\substack{j=1\\ j\neq i}}^{n} |\varphi(e_{ij})| = \max_i d_i = \Delta. \quad \square$$

In view of Theorem 4.8 and Lemma 4.9, we immediately get the following corollary

Corollary 4.10. *Let $\Phi = (\Gamma, \varphi)$ be a connected $U(\mathbb{H})$-gain graph with spectral radius $\rho(\Phi)$ and largest vertex degree Δ. Then $\rho_A(\Phi) \leq \Delta$, with equality if and only if either $\Phi \sim (\Gamma', 1_{U(\mathbb{H})})$ or $\Phi \sim (\Gamma', -1_{U(\mathbb{H})})$, where Γ' is a Δ-regular graph.*

5. Line graphs

In [44] oriented gain graphs were defined and studied, which led to results involving the line graph and matrix properties for complex unit gain graphs. Here we specialize to the case of $U(\mathbb{H})$-gain graphs and study line graphs, but also generalize several matrix results to this setting of quaternion unit gain graphs. Our results are also consistent with those in [3,16].

A $\mathfrak{G}$*-phased graph* is a pair (Γ, ω), consisting of a graph Γ, and a $\mathfrak{G}$*-incidence phase function* $\omega : V \times E \to \mathfrak{G} \cup \{0\}$ that satisfies

$$\begin{aligned} &\omega(v, e) \neq 0 \text{ if } v \text{ is incident to } e, \\ &\omega(v, e) = 0 \text{ otherwise.} \end{aligned}$$

A (weak) *involution* of a group $\mathfrak{G}$ is any element $\mathfrak{s} \in \mathfrak{G}$ such that $\mathfrak{s}^2 = 1_{\mathfrak{G}}$. Henceforth, all involutions are weak involutions, thus the group identity will also be called an involution.

Let $\mathfrak{G}^{\mathfrak{s}}$ be a group with a distinguished central involution $\mathfrak{s}$. Let ω be a $\mathfrak{G}^{\mathfrak{s}}$-incidence phase function. The $\mathfrak{G}^{\mathfrak{s}}$*-gain graph associated to a* $\mathfrak{G}^{\mathfrak{s}}$*-phased graph* (Γ, ω), denoted by $\Phi(\omega)$, has its gains defined by

$$\varphi(e_{ij}) = \omega(v_i, e_{ij}) \cdot \mathfrak{s} \cdot \omega(v_j, e_{ij})^{-1}. \tag{5.1}$$

For a $\mathfrak{G}^{\mathfrak{s}}$-gain graph Φ, an *orientation of* Φ is any $\mathfrak{G}^{\mathfrak{s}}$-incidence phase function ω that satisfies Equation (5.1). An *oriented* $\mathfrak{G}^{\mathfrak{s}}$*-gain graph* is a pair (Φ, ω), consisting of a $\mathfrak{G}^{\mathfrak{s}}$-gain graph Φ and ω, an orientation of Φ.

See Fig. 3 for an example of each of these graph structures.

Let Λ_Γ denote the line graph of the unsigned graph Γ. Let ω_Λ be a $\mathfrak{G}^{\mathfrak{s}}$-incidence phase function on Λ_Γ defined by

$$\omega_\Lambda(e_{ij}, e_{ij}e_{jk}) = \omega(v_j, e_{ij})^{-1}. \tag{5.2}$$

Let (Φ, ω) be an oriented $\mathfrak{G}^{\mathfrak{s}}$-gain graph. The *line graph of* (Φ, ω) is the oriented $\mathfrak{G}^{\mathfrak{s}}$-gain graph $(\Phi(\omega_\Lambda), \omega_\Lambda)$. See Fig. 4 for an example.

Let (Φ, ω) be an oriented $\mathfrak{G}^{\mathfrak{s}}$-gain graph. The *incidence matrix* $\mathrm{H}(\Phi, \omega) = (\eta_{ve})$ is an $n \times m$ matrix with entries in $\mathfrak{G}^{\mathfrak{s}} \cup \{0\}$, defined by

$$\eta_{ve} = \omega(v, e) \text{ for every } (v, e) \in V \times E. \tag{5.3}$$

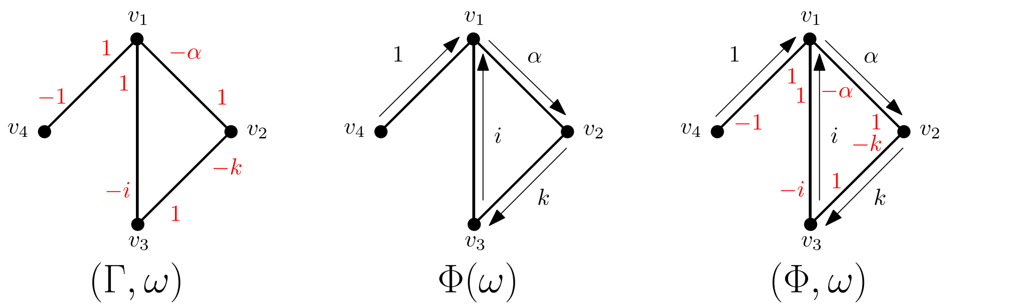

Fig. 3. Here $\alpha = \frac{1}{\sqrt{2}}(i + j)$. On the left, (Γ, ω) is a $U(\mathbb{H})$-phased graph. In the middle, $\Phi(\omega)$ is the associated $U(\mathbb{H})^{-1}$-gain graph to (Γ, ω) on the left. Note that $\Phi(\omega)$ is the same gain graph as Φ from Fig. 1. On the right, this is an oriented $U(\mathbb{H})^{-1}$-gain graph. The incidence phase function values of ω are labeled in red to distinguish them from the gain function values of φ. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

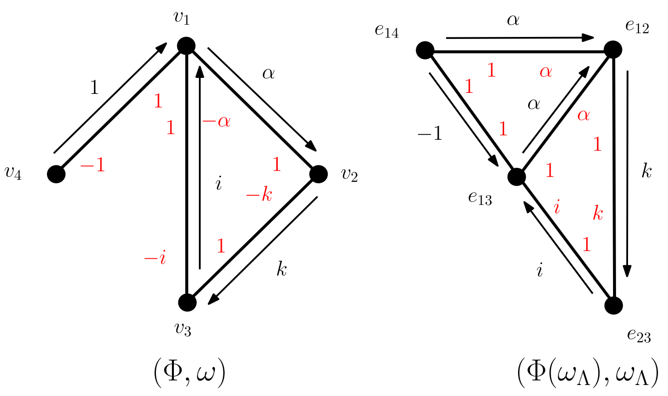

Fig. 4. An oriented $U(\mathbb{H})^{-1}$-gain graph (Φ, ω) and its line graph $(\Phi(\omega_\Lambda), \omega_\Lambda)$.

Notice that, in the case of $U(\mathbb{H})$-gain graphs, there is a slight difference between this definition and the incidence matrix introduced in Section 3. In the general version from Section 3 there is no orientation, and so an arbitrary choice $\alpha_e \in U(\mathbb{H})$ can be made for the column associated to e. Here, this choice is made specifically by choosing an orientation, that is a phase function ω. In other words giving and oriented gain graph (Φ, ω) is tantamount to fix an incidence matrix $\mathrm{H}(\Phi, \omega)$, among those possibly associated to Φ.

For the adjacency and Laplacian matrices associated to an oriented $U(\mathbb{H})^{\mathfrak{s}}$-gain graph (Φ, ω), one can use the definitions above and ignore the extra structure ω provides. That is $A(\Phi, \omega) = A(\Phi)$ and $L(\Phi, \omega) = L(\Phi)$.

For ordinary graphs, there is a well-known relationship between the oriented incidence matrix of a graph and the adjacency matrix of the line graph. This was generalized to signed graphs by Zaslavsky [55], further generalized to complex unit gain graphs in [44], and more generally in Lemma 4.17 of [16]. Here we show that this also works for quaternion unit gain graphs.

Theorem 5.1. *Let (Φ, ω) be an oriented $U(\mathbb{H})^{\mathfrak{s}}$-gain graph, where $\mathfrak{s}$ is fixed as either $+1$ or -1. Then*

$$\mathrm{H}(\Phi, \omega)^* \mathrm{H}(\Phi, \omega) = 2I + \mathfrak{s} A(\Phi(\omega_\Lambda), \omega_\Lambda). \tag{5.4}$$

Proof. The proof here is identical to that of Theorem 5.1 in [44]. Notice that $\mathrm{H}(\Phi, \omega)^* \mathrm{H}(\Phi, \omega)$ is an $|E| \times |E|$ matrix. Consider the dot product of row $\mathbf{r}_i$ of $\mathrm{H}(\Phi, \omega)^*$ with column $\mathbf{c}_j$ of $\mathrm{H}(\Phi, \omega)$. Suppose column $\mathbf{c}_i$ corresponds to edge e_{qr} and $\mathbf{c}_j$ corresponds to edge e_{lk}.

Case 1: $i = j$ (same edge). Then $\mathbf{r}_i = \mathbf{c}_j^*$ and thus,

$$\begin{aligned} \mathbf{r}_i \cdot \mathbf{c}_j = \mathbf{c}_j^* \cdot \mathbf{c}_j &= \overline{\eta}_{v_k e_{lk}} \eta_{v_k e_{lk}} + \overline{\eta}_{v_l e_{lk}} \eta_{v_l e_{lk}} \\ &= \overline{\omega(v_k, e_{lk})} \omega(v_k, e_{lk}) + \overline{\omega(v_l, e_{lk})} \omega(v_l, e_{lk}) \\ &= |\omega(v_k, e_{lk})|^2 + |\omega(v_l, e_{lk})|^2 \\ &= 2. \end{aligned}$$

Case 2: $i \neq j$ and $r = l$ (distinct adjacent edges).

$$\begin{aligned} \mathbf{r}_i \cdot \mathbf{c}_j = \mathbf{c}_i^* \cdot \mathbf{c}_j &= \overline{\eta}_{v_l e_{ql}} \eta_{v_l e_{lk}} \\ &= \omega(v_l, e_{ql})^{-1} \omega(v_l, e_{lk}) \\ &= \omega_\Lambda(e_{ql}, e_{ql} e_{lk}) \omega_\Lambda(e_{lk}, e_{ql} e_{lk})^{-1} \\ &= \mathfrak{s} \varphi_\Lambda(e_{ql} e_{lk}). \end{aligned}$$

Therefore, $\mathrm{H}(\Phi, \omega)^* \mathrm{H}(\Phi, \omega) = 2I + \mathfrak{s} A(\Phi(\omega_\Lambda), \omega_\Lambda)$. $\square$

Corollary 5.2. *Let (Φ, ω) be an oriented $U(\mathbb{H})^{-1}$-gain graph. If λ is a right eigenvalue of $A(\Phi(\omega_\Lambda), \omega_\Lambda)$, then $\lambda \leq 2$.*

Corollary 5.3. *Let (Φ, ω) be an oriented $U(\mathbb{H})^{+1}$-gain graph. If λ is a right eigenvalue of $A(\Phi(\omega_\Lambda), \omega_\Lambda)$, then $-2 \leq \lambda$.*

Remark 5.4. The Line graphs have a special role in conjoining the spectral theories of graphs. In fact, in view of Lemma 2.14, $L(\Phi) = \mathrm{H}(\Phi, \omega)\mathrm{H}(\Phi, \omega)^*$ and $\mathrm{H}(\Phi, \omega)^* \mathrm{H}(\Phi, \omega) = 2I + \mathfrak{s} A(\Phi(\omega_\Lambda), \omega_\Lambda)$ do share the same nonzero eigenvalues. Hence, the results obtained in one spectral theory can be ported to the other one theory. An example of this fact will be given in the next section, where the Laplacian variant of Interlacing Theorem and the Matrix-Tree theorem can be indirectly obtained by means of $(\Phi(\omega_\Lambda), \omega_\Lambda)$.

6. Eigenvalues of the Laplacian matrix

The next lemma implies that a switching class has a Laplacian (right) spectrum. The following is immediate from Lemma 3.3 and Lemma 2.15.

Lemma 6.1. *Let $\Phi_1 = (\Gamma, \varphi_1)$ and $\Phi_2 = (\Gamma, \varphi_2)$ both be $U(\mathbb{H})$-gain graphs. If $\Phi_1 \sim \Phi_2$, then $L(\Phi_1)$ and $L(\Phi_2)$ have the same (right) spectrum. In particular, $\sigma_L(\Phi_1) = \sigma_L(\Phi_2)$.*

Similarly to what done in Section 4, let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph, and let e be an edge in $E(\Gamma)$. Denote by $\Phi - e$ the $U(\mathbb{H})$-graph obtained from Φ by deleting the edge e of Γ. Hence the eigenvalues of $L(\Phi)$ and $L(\Phi - e)$ interlace.

Theorem 6.2 *(Interlacing Theorem – edge variant). Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph with n vertices, and let $\Phi - e$ the $U(\mathbb{H})$-graph obtained from Φ by deleting the edge e. Denoted by*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \qquad \text{and} \qquad \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$$

the eigenvalues of $L(\Phi)$ and $L(\Phi - e)$ respectively, the following inequalities hold:

$$\lambda_1 \geqslant \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n$$

Proof. Since $L(\Phi) = \mathrm{H}(\Phi, \omega)\mathrm{H}(\Phi, \omega)^*$ and $\mathrm{H}(\Phi, \omega)^*\mathrm{H}(\Phi, \omega) = 2I + \mathfrak{s}A(\Phi(\omega_\Lambda), \omega_\Lambda)$ do share the same nonzero eigenvalues, deleting an edge in Φ means deleting a vertex in the line graph $(\Phi(\omega_\Lambda), \omega_\Lambda)$. Now the assertion comes from Corollary 4.3. □

The next main result is the variant of the Matrix-Tree theorem for the Laplacian matrix of $U(\mathbb{H})$-graphs. Before stating it, we need a few preparatory lemmas. The proof is an adaption of Cvetković et al. method [18], whose procedure has been used for the analogous result for signed graphs [12]. In fact, we will strictly follow it. As we need to use the concept of line graph, let us denote by $\mathcal{L}(\Phi)$ the $U(\mathbb{H})$-graph whose adjacency matrix is $A(\Phi(\omega_\Lambda), \omega_\Lambda) = \mathrm{H}(\Phi, \omega)^*\mathrm{H}(\Phi, \omega) - 2I$, that is, by setting $\mathfrak{s} = 1$ in Equation (5.4). Using the latter setting, the least A-eigenvalue of $\mathcal{L}(\Phi)$ is greater than or equal to -2. Of course, by setting $\mathfrak{s} = -1$, we obtain analogous results for 2 as the largest A-eigenvalue of the corresponding line graph.

This first lemma is a result stated for the adjacency matrix of simple graphs. Since it depends on the characteristic polynomials of vertex deleted subgraphs, it is still valid in the context of the adjacency matrix of $U(\mathbb{H})$-graphs. Hence, the proof is omitted. For a $U(\mathbb{H})$-graph Φ, let $p_A(\Phi, x)$ be the adjacency characteristic polynomial.

Lemma 6.3. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-graph with n vertices, and $p_A^{(k)}(\Phi, x)$ be the k-th derivative of the adjacency characteristic polynomial of Φ. Then*

$$p_A^{(k)}(\Phi, x) = k! \sum_{|U|=k} p_A(\Phi - U, x),$$

where the summation is taken over all k-vertex subsets U of Φ.

The *rank* of $A \in \mathbb{H}^{m\times n}$ is the maximum number of columns of A that are right linearly independent, that is the number of rows that are left linearly independent [56].

Lemma 6.4. *Let* $\mathrm{H}(\Phi) = \mathrm{H}(\Phi, \omega)$ *be an incident matrix of a connected* $U(\mathbb{H})$*-graph* $\Phi = (\Gamma, \varphi)$ *with* n *vertices, then*

$$\operatorname{rank}(\mathrm{H}(\Phi)) = \begin{cases} n-1 & \text{if } \Phi \text{ is balanced,} \\ n & \text{if } \Phi \text{ is unbalanced.} \end{cases}$$

Proof. According to Lemma 3.3, any two incidence matrices of Φ differ by a left multiplication with a diagonal matrix. Hence, without loss of generality, we can take the incidence matrix $\mathrm{H}(\Phi, \omega) = \mathrm{H}(\Phi) = (\mathrm{H}_{ve})$ be column-wise defined as $\mathrm{H}_{ie} = \varphi(e_{ij})$, $\mathrm{H}_{je} = -1$ and $\mathrm{H}_{ve} = 0$ for $v \notin \{i, j\}$. Each column corresponds to an edge, say $e = e_{ij}$, joining two vertices v_i and v_j of Φ. Consider the rows H_1, H_2, …, H_n, and assume that the rows are (left) linearly dependent, say

$$c_1H_1 + c_2H_2 + \cdots + c_nH_n = (0, 0, \ldots, 0) \quad \text{and} \quad (c_1, c_2, \ldots, c_n) \neq (0, 0, \ldots, 0). \tag{6.1}$$

Each column of $\mathrm{H}(\Phi)$ contains exactly two non zero entries. Consequently, from the latter consideration and from (6.1), for each edge e_{ij} we obtain $c_i\varphi(e_{ij}) - c_j = 0$. Note that, due to connectivity, $c_i \neq 0$ for each $i = 1, 2, \ldots, n$ and the vector $(c_1, c_2, \ldots, c_n)$ is unique up to a scalar factor. Hence, from $c_i\varphi(e_{ij}) - c_j = 0$, obtain $c_i\varphi(e_{ij})c_j^{-1} = 1$. Now, consider the vertex valued function

$$\theta(v_i) = c_i.$$

It is not difficult to see that θ is a potential function on the vertices, and therefore Φ is balanced. Conversely, if Φ is balanced, then we can consider the trivial signature assigning $+1$ to each edge, and the result comes from the theory of simple graphs. □

Lemma 6.5. *Let* $\Phi = (\Gamma, \varphi)$ *be a* $U(\mathbb{H})$*-graph of order* n *and size* m*, and let* $\mathcal{L}(\Phi)$ *be the line graph of* Φ*. If* $\mu_\Phi(\lambda)$ *denotes the multiplicity of the scalar* λ *in* $\sigma_A(\Phi)$*, then*

$$\mu_{\mathcal{L}(\Phi)}(-2) = \begin{cases} m-n+1 & \text{if } \Gamma \text{ is balanced,} \\ m-n & \text{if } \Gamma \text{ is unbalanced.} \end{cases}$$

Proof. According to (5.4), $2I_m + A(\mathcal{L}(\Phi))$ is positive semidefinite. Hence, $\lambda_m(\mathcal{L}(\Phi)) \geq -2$. Note, the multiplicity of -2 is equal to the nullity of the matrix $\mathrm{H}(\Phi)^*\mathrm{H}(\Phi)$, which is given by $m - \operatorname{rank}(\mathrm{H}(\Phi))$. The rest of the proof follows by Lemma 6.4. □

Corollary 6.6. *Let* $\Phi = (\Gamma, \varphi)$ *be a connected* $U(\mathbb{H})$*-graph and* $\mathcal{L}(\Phi)$ *be its line graph. We have that* -2 *is not an eigenvalue of* $A(\mathcal{L}(\Phi))$ *if and only if* Φ *is either a tree or an unbalanced unicyclic graph.*

The subsequent lemma generalizes the analogous result given in [8] for $\mathbb{T}$-gain graphs.

Lemma 6.7. *Let $\Phi = (\Gamma, \varphi)$ be a unicyclic $U(\mathbb{H})$-graph, and let C be the unique cycle of Γ. Then,*

$$\mathrm{Mdet}(L(\Phi)) = 2 - 2\mathfrak{R}(C). \tag{6.2}$$

Proof. Let v and w be two adjacent vertices on the cycle C. We set $\varphi(\vec{C}_v) = q \in U(\mathbb{H})$, where $\vec{C}_v$ is one of the two different directed cycles with base vertex v.

By Lemma 6.1, we may assume that $\varphi(vw) = q$, $\varphi(wv) = \bar{q}$, and all other directed edges are neutral. Now, the entries of the square matrices $\mathrm{H}(\Phi)$, $\mathrm{H}(\Phi)^*$ and $L(\Phi)$ are in the commutative ring $R = \mathbb{R}[q, \bar{q}]/(q\bar{q} - 1)$. Hence, their Moore determinants enjoy the properties of the complex Hermitian matrices determinant. In particular, the determinant remains the same if we add to a row A_i a different row A_j multiplied by some non-zero number in R; moreover, $(\mathrm{Mdet}(\mathrm{H}(\Phi))(\mathrm{Mdet}\mathrm{H}(\Phi)^*) = \mathrm{Mdet}(L(\Phi))$. That is why the proofs of Lemmas 2.2 and 2.4 in [50] work in our context, showing that

$$\mathrm{Mdet}(L(\Phi)) = \mathrm{Mdet} L(\vec{C}_v) = |1 - \varphi(\vec{C}_v)|^2 = 2 - 2\mathfrak{R}(C). \quad \square$$

The following result extends to $U(\mathbb{H})$-gain graphs similar statements proved for signed graphs (see, for example, [12]) and for $\mathbb{T}$-gain graphs [8].

Lemma 6.8. *Let $\Phi = (\Gamma, \gamma)$ be a connected $U(\mathbb{H})$-gain graph of order n and size m, and $\mathcal{C}(\Gamma)$ be the set of cycles in Γ. Then,*

$$(-1)^m p_A(\mathcal{L}(\Phi), -2) = \begin{cases} m+1 & \text{if } \Gamma \text{ is a tree;} \\ 2 - 2\mathfrak{R}(C) & \text{if } \Gamma \text{ is unicyclic and } \mathcal{C}(\Gamma) = \{C\}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If Γ is a tree, then Φ is balanced; therefore, $\Phi \sim (\Gamma, 1)$, and the equality $(-1)^m \varphi(\mathcal{L}(\Phi), -2) = m + 1$ is well-known in the literature (e.g., [19, Lemma 7.5.2(i)]).

Let now Γ be unicyclic. By Lemma 6.7, Equation (5.4) specializes to

$$p_A(\mathcal{L}(\Phi), -2) = p_L(\Phi, 0) = \mathrm{Mdet}(L(\Phi)) = 2 - 2\mathfrak{R}(C).$$

Finally, if Γ is neither a tree nor a unicyclic graph, then $m > n$, and $(-1)^m \varphi(\mathcal{L}(\Phi), -2) = 0$ by Lemma 6.5. $\square$

A *gain-TU-subgraph* of a $U(\mathbb{H})$-graph Φ is a subgraph whose components are trees or unbalanced unicyclic graphs, namely the unique cycle is unbalanced. If Θ is a $U(\mathbb{H})$-TU-subgraph, then $\Theta = T_1 \cup T_2 \cup \cdots \cup T_r \cup U_1 \cup U_2 \cup \cdots \cup U_s$, where the T_i's are trees and the U_i's are unbalanced unicyclic graphs. For each U_i, let C_i be its unique cycle. The weight of the gain-TU-subgraph Θ is defined as $\gamma(\Theta) = \prod_{i=1}^r |T_i| \prod_{j=1}^s (2 - 2\mathfrak{R}(C_i))$.

We are now in position to give the Matrix-Tree theorem.

Theorem 6.9 *(Matrix-Tree Theorem). Let Φ be a $U(\mathbb{H})$-graph and let $p_L(\Phi, x) = x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ be the Laplacian characteristic polynomial of Φ. Then*

$$b_i = (-1)^i \sum_{\Theta \in \mathcal{TU}_i} \gamma(\Theta) \quad (i = 1, 2, \ldots, n),$$

where $\mathcal{TU}_i$ denotes the set of gain-TU-subgraph of Φ containing i edges.

Proof. From (5.4) and by the MacLaurin expansion we obtain

$$\begin{aligned} p_L(\Phi, x) &= x^{n-m} p_A(\mathcal{L}(\Phi), x-2) \\ &= x^{n-m} \sum_{k=0}^{m} p_A^{(k)}(\mathcal{L}(\Phi), -2) \frac{x^k}{k!} \\ &= x^{n-m} \sum_{k=m-n}^{m} x^k \frac{1}{k!} p_A^{(k)}(\mathcal{L}(\Phi), -2), \end{aligned}$$

due to the fact that -2 is an eigenvalue of $A(\mathcal{L}(\Gamma))$ with multiplicity at least $m-n$. In view of Lemma 6.3 we arrive at

$$p_L(\Phi, x) = x^{m-n} \sum_{k=m-n}^{m} x^k \sum_{|U|=k} p_A(\mathcal{L}(\Gamma) - U, -2). \tag{6.3}$$

Clearly, $\mathcal{L}(\Phi) - U$ is still a line graph. From Corollary 6.6, line graphs have -2 as an A-eigenvalue unless all components are line graphs of trees or unbalanced unicyclic graphs. Hence, nonzero contributions in (6.3) come from line graphs whose roots are gain-TU-subgraph. Therefore, by Lemma 6.8, we have

$$\sum_{|S|=k} p_A(\mathcal{L}(\Phi) - S, -2) = (-1)^{m-k} \sum_{H \in \mathcal{H}_{m-k}} \gamma(\Theta).$$

Next, from (6.3) we have

$$p_L(\Phi, x) = x^{m-n} \sum_{k=m-n}^{m} x^k (-1)^{m-k} \sum_{\Theta \in \mathcal{TU}_{m-k}} \gamma(\Theta),$$

and by putting $i = m-k$ we obtain

$$p_L(\Phi, x) = \sum_{i=0}^{n} x^{n-i} (-1)^i \sum_{\Theta \in \mathcal{TU}_i} \gamma(\Theta).$$

This completes the proof. $\square$

For graphs Γ, the signless Laplacian $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ has received a growing amount of attention. When finding upper bounds for the Laplacian right spectral radius, it turns out that signless Laplacian can be used since $\lambda_1^r(L(\Gamma)) \leq \lambda_1^r(Q(\Gamma))$. This universal upper bound extends to the more general settings of signed graphs [35] and $\mathbb{T}$-gain graphs [43]. This further generalizes to $U(\mathbb{H})$-gain graphs.

We write $(\Gamma, -1)$ for the $U(\mathbb{H})$-gain graph with all gains -1. Let $\rho_L(\Phi) = \lambda_1^r(L(\Phi))$.

Theorem 6.10. *Let $\Phi = (\Gamma, \varphi)$ be a connected $U(\mathbb{H})$-gain graph, and let $\rho_Q(\Gamma)$ be the signless Laplacian spectral radius of Γ. Then*

$$\rho_L(\Phi) \leq \rho_L(\Gamma, -1) = \rho_Q(\Gamma).$$

Furthermore, equality holds if and only if $\Phi \sim (\Gamma, -1)$.

Proof. The first part of this proof is similar to the signed graphic proof in [35, Lemma 3.1]. The second part of this proof is a natural generalization of the complex unit case [43, Theorem 5.4].

Suppose $\left(\lambda_1^r(L(\Phi)), \mathbf{x}\right)$ is a right eigenpair of $L(\Phi)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{H}^n$ is a unit vector. By Lemmas 2.16 and 3.2:

$$\rho_L(\Phi) = \mathbf{x}^* L(\Phi)\mathbf{x} = \sum_{e_{ij} \in E(\Phi)} |x_i - \varphi(e_{ij})x_j|^2 \leq \sum_{e_{ij} \in E(\Phi)} (|x_i| + |x_j|)^2 \leq \rho_L(\Gamma, -1) = \rho_Q(\Gamma).$$

If $\Phi \sim (\Gamma, -1)$, then $\rho_L(\Phi) = \rho_L(\Gamma, -1) = \rho_Q(\Gamma)$ by Lemma 6.1.

Now suppose that $\rho_L(\Phi) = \rho_L(\Gamma, -1) = \rho_Q(\Gamma)$. Then,

$$|x_i|^2 + 2|x_i||x_j| + |x_j|^2 = |x_i - \varphi(e_{ij})x_j|^2 = |x_i|^2 - 2 \cdot \mathrm{Re}(\varphi(e_{ij})x_j\bar{x}_i) + |x_j|^2.$$

Therefore,

$$-\mathrm{Re}(\varphi(e_{ij})x_j\bar{x}_i) = |x_i||x_j|.$$

Also, $(x_1, \ldots, x_n)$ is a right eigenvector of $Q(\Gamma)$ with corresponding right eigenvalue $\rho_Q(\Gamma)$. Applying the Perron-Frobenius Theorem to $Q(\Gamma)$, one can conclude $x_l \neq 0$ for every $l \in \{1, \ldots, n\}$.

Let

$$x_i = |x_i|e^{\mathbf{n}_i\theta_i} \in \mathbb{H}, \text{ and}$$
$$x_j = |x_j|e^{\mathbf{n}_j\theta_j} \in \mathbb{H}.$$

Now by substitution,

$$-\mathrm{Re}\big(|x_i||x_j|\varphi(e_{ij})e^{\mathbf{n}_j\theta_j}e^{-\mathbf{n}_i\theta_i}\big) = |x_i||x_j|.$$

That is,

$$\mathrm{Re}\big(\varphi(e_{ij})e^{\mathbf{n}_j\theta_j}e^{-\mathbf{n}_i\theta_i}\big) = -1.$$

Since $\varphi(e_{ij})e^{\mathbf{n}_j\theta_j}e^{-\mathbf{n}_i\theta_i} \in U(\mathbb{H})$ it must be that

$$\varphi(e_{ij})e^{\mathbf{n}_j\theta_j}e^{-\mathbf{n}_i\theta_i} = -1.$$

Rewriting this equation one can obtain

$$\frac{x_i}{|x_i|}\frac{\bar{x}_j}{|x_j|} = -\varphi(e_{ij}).$$

Let $\theta\colon V \to \mathbb{T}$ be defined as $\theta(v_i) = \bar{x}_i/|x_i|$, for all $v_i \in V$. Since $\bar{x}_i/|x_i| \in U(\mathbb{H})$ for every $i \in \{1, \dots, n\}$, θ is a potential function for $-\varphi$. Therefore, $(\Gamma, -\varphi)$ is balanced by Corollary 2.20. □

Using Theorem 6.10, one can obtain a multitude of upper bounds for the Laplacian right spectral $\lambda_1^r(L(\Phi))$ because many upper bounds for the right spectral radius of the signless Laplacian of the underlying graph $\lambda_1^r(Q(\Gamma))$ have already been established. For example, it is well-known that $\lambda_1^r(Q(\Gamma)) \leq 2\Delta$ [18]. Therefore, for a connected $U(\mathbb{H})$-gain graph Φ, it is immediate by Theorem 6.10 that $\lambda_1^r(L(\Phi)) \leq 2\Delta$. Many other similar bounds can be stated from known upper bounds for $\lambda_1^r(Q(\Gamma))$ [21,38,42,48,58]. However, for the left spectral radius, much less is known.

The following is a bound for the left spectral radius of the Laplacian of a $U(\mathbb{H})$-gain graph Φ. It is also a bound for the right spectral radius $\lambda_1^r(L(\Phi))$ as mentioned in the previous paragraph.

Lemma 6.11. *Let $\Phi = (\Gamma, \varphi)$ be a $U(\mathbb{H})$-gain graph. Then*

$$\rho_l(L(\Phi)) \leq 2\Delta.$$

Proof. If $\lambda \in \mathbb{H}$ is a left eigenvalue of $L(\Phi) = (l_{ij})$, then by Lemma 2.18,

$$|\lambda| \leq \max_i \sum_{j=1}^{n} |l_{ij}| = \max_i \Big(d_i + \sum_{\substack{j=1\\ j\neq i}}^{n} |\varphi(e_{ij})|\Big) = 2\Delta. \quad \square$$

We conclude this section by giving a result which preludes to a generalization of the well-known Fiedler theory for the Laplacian of simple graphs.

Theorem 6.12. *Let $\Phi = (\Gamma, \varphi)$ be a connected $U(\mathbb{H})$-gain graph, and let $\mu(\Phi) = \lambda_n(L(\Phi))$. Then $\mu(\Phi) \geq 0$, with equality if and only if $\Phi \sim (\Gamma, \mathbf{1}_{\mathbb{H}})$.*

Proof. In view of Corollary 3.2, $\mu(\Phi) \geq 0$ and $\mu(\Phi) = 0$ if and only if there exists a unit eigenvector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ such that $\sum_{e_{ij} \in E(\Phi)} |x_i - \varphi(e_{ij})x_j|^2 = 0$. Hence, for each edge $e_{ij} \in E(\Phi)$ we have $x_i - \varphi(e_{ij})x_j = 0$. Due to the connectivity assumption, $x_i \neq 0$ for each $i = 1, 2, \ldots, n$, otherwise $\mathbf{x}$ becomes a zero vector. Now, $\theta(i) = x_i^* \varphi(e_{ij})x_j$ defines a potential function for Φ. Hence, by Corollary 2.20, Φ is balanced.

The converse is immediate, as the all-ones vector is an eigenvector for $(\Gamma, \mathbf{1}_{\mathbb{H}})$ corresponding to the eigenvalue 0. $\square$

7. Examples - the cycle and path $U(\mathbb{H})$-gain graphs

The adjacency and Laplacian eigenvalues of a cycle and path $\mathbb{T}$-gain graph have been calculated explicitly in [43] (see also [45] for the signed variant). A goal of this section is to do the same for cycle and path $U(\mathbb{H})$-gain graphs.

Lemma 7.1 *([43], Theorem 5.1). Suppose $\Phi = (C_n, \mathbb{T}, \varphi)$ with $\varphi(C_n) = e^{i\theta} \in \mathbb{T}$. Then*

$$\sigma_A(\Phi) = \left\{ 2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \ldots, n-1\} \right\}, \tag{7.1}$$

and

$$\sigma_L(\Phi) = \left\{ 2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \ldots, n-1\} \right\}. \tag{7.2}$$

Note, this calculation is independent of how you calculate the cycle gain because $\mathbb{T}$ is an abelian group, and also if traversing the cycle clockwise or counterclockwise the cycle gain will be either $e^{i\theta}$ or $e^{-i\theta}$ but the above calculation is the same for either θ or $-\theta$.

Now we can prove a more general version for cycle $U(\mathbb{H})$-graphs, and explicitly calculate the right eigenvalues of both the adjacency and Laplacian matrices. This calculation will also be independent of starting vertex choice and directed orientation as in the complex unit gain graph version above.

Theorem 7.2. *Let C_n be the cycle graph on n vertices. Suppose $\Phi = (C_n, U(\mathbb{H}), \varphi)$.*

(1) Then there exists a switching function $\zeta \colon V(\Phi) \to U(\mathbb{H})$ such that Φ^ζ has gain function values all within $\mathbb{T}$. Specifically, consider the closed walk

$$\mathcal{C}_1 = v_1 e_{12} v_2 e_{23} v_3 \cdots v_{n-1} e_{n-1,n} v_n e_{n,1} v_1$$

in Φ with gain $\varphi(\mathcal{C}_1) = \varphi(e_{12})\varphi(e_{23}) \cdots \varphi(e_{n-1,n})\varphi(e_{n,1}) = q_1 \in U(\mathbb{H})$. Then there exists a switching function ζ such that

$$\varphi^\zeta(e_{i,i+1}) = 1 \text{ for } i \in \{1, 2, \ldots, n-1\}, \text{ and}$$
$$\varphi^\zeta(e_{n,1}) = Re(q_1) + |Im(q_1)|i = e^{i\theta} \in \mathbb{T}.$$

(2) *Considering the calculation in (1) above where* $Re(q_1) + |Im(q_1)|i = e^{i\theta}$. *The eigenvalues of* $A(\Phi)$ *and* $L(\Phi)$ *can be calculated as*

$$\sigma_A(\Phi) = \left\{ 2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \dots, n-1\} \right\}, \tag{7.3}$$

and

$$\sigma_L(\Phi) = \left\{ 2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \dots, n-1\} \right\}. \tag{7.4}$$

Proof. (1): We first define a switching function $\xi\colon V(\Phi) \to U(\mathbb{H})$ as follows:

$$\xi(v_i) = \begin{cases} 1 & i = 1, \text{ and} \\ \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} & i \neq 1. \end{cases}$$

In other words, the switch for the starting vertex is 1, and the switch for each vertex $i > 1$ is the inverse of the gain of the path from vertices 1 to i. By definition the switched edges can be calculated as:

$$\varphi^{\xi}(e_{i,i+1}) = \xi(v_i)^{-1}\varphi(e_{i,i+1})\xi(v_{i+1}).$$

For $i = 1$,

$$\varphi^{\xi}(e_{12}) = \varphi(e_{12})\xi(v_2) = \varphi(e_{12})\varphi(e_{12})^{-1} = 1,$$

for $1 < i < n$,

$$\begin{aligned} \varphi^{\xi}(e_{i,i+1}) &= \xi(v_i)^{-1}\varphi(e_{i,i+1})\xi(v_{i+1}) \\ &= \left[\left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1}\right]^{-1} \varphi(e_{i,i+1}) \left(\prod_{j=1}^{i} \varphi(e_{j,j+1})\right)^{-1} \\ &= \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right) \varphi(e_{i,i+1})\varphi(e_{i,i+1})^{-1} \left(\prod_{j=1}^{i-1} \varphi(e_{j,j+1})\right)^{-1} = 1, \end{aligned}$$

and lastly, for $i = n$,

$$\begin{aligned} \varphi^{\xi}(e_{n,1}) = \xi(n)^{-1}\varphi(e_{n,1}) = \left[\left(\prod_{j=1}^{n-1} \varphi(e_{j,j+1})\right)^{-1}\right]^{-1} \varphi(e_{n,1}) &= \left(\prod_{j=1}^{n-1} \varphi(e_{j,j+1})\right)\varphi(e_{n,1}) \\ &= \varphi(\mathcal{C}_1) = q_1. \end{aligned}$$

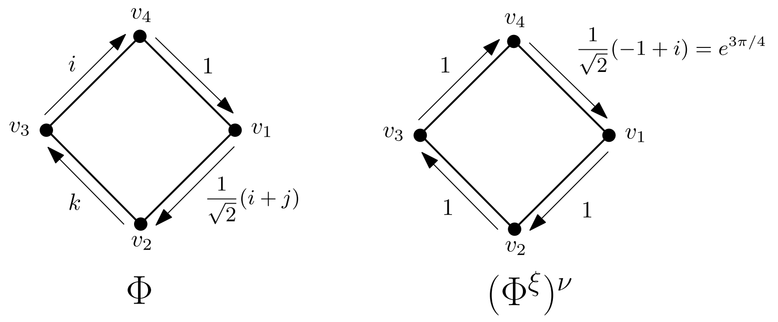

Fig. 5. A $U(\mathbb{H})$-gain graph $\Phi = (C_4, U(\mathbb{H}), \varphi)$ and a switching equivalent $\mathbb{T}$-gain graph $(\Phi^{\xi})^{\nu}$.

Suppose we write $q_1 = a + bi + cj + dk$. By Lemma 2.1, the complex number $z = a + \sqrt{b^2 + c^2 + d^2}i = \mathrm{Re}(q_1) + |\mathrm{Im}(q_1)|i$ is similar to q_1. That is, there exists a nonzero quaternion $h \in \mathbb{H}$ such that $z = h^{-1}q_1h$. Moreover, we can normalize h to get a nonzero $u = h/|h| \in U(\mathbb{H})$, and clearly, $z = u^{-1}q_1u$. Furthermore, $|z| = |u^{-1}q_1u| = |u^{-1}||q_1||u| = 1$, so $z \in \mathbb{T}$.

Now define a switching function $\nu\colon V(\Phi^{\xi}) \to U(\mathbb{H})$, by

$$\nu(v_i) = u \text{ for each } v_i \in V(\Phi^{\xi}).$$

The result of this switching function produces the gain graph $(\Phi^{\xi})^{\nu}$ with gain function

$$\varphi^{\xi\nu}(e_{i,i+1}) = u^{-1}u = 1 \text{ for } 1 \leq i < n, \text{ and}$$
$$\varphi^{\xi\nu}(e_{n,1}) = u^{-1}q_1u = z \in \mathbb{T}.$$

Therefore, the original Φ can be switched to $(\Phi^{\xi})^{\nu}$ for the desired results of the theorem.
(2): Notice that the edge gains of $(\Phi^{\xi})^{\nu}$ are all elements of $\mathbb{T}$, and $\varphi^{\xi\nu}(\mathcal{C}_1) = z = \mathrm{Re}(q_1) + |\mathrm{Im}(q_1)|i = e^{i\theta}$. Thus, $(\Phi^{\xi})^{\nu}$ can be thought of as a $\mathbb{T}$-gain graph, and by Lemma 7.1, the right eigenvalues the adjacency and Laplacian matrices for $(\Phi^{\xi})^{\nu}$ can be computed as

$$\sigma_A(\Phi^{\xi})^{\nu} = \left\{2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \dots, n-1\}\right\},$$

and

$$\sigma_L(\Phi^{\xi})^{\nu} = \left\{2 - 2\cos\left(\frac{\theta + 2\pi j}{n}\right) : j \in \{0, \dots, n-1\}\right\}.$$

By Lemmas 4.1 and 6.1, switching preserves the right spectrum for both the adjacency and Laplacian matrices of a $U(\mathbb{H})$-gain graph. That is, $\sigma_r(A(\Phi)) = \sigma_A((\Phi^{\xi})^{\nu})$ and $\sigma_L(\Phi) = \sigma_L((\Phi^{\xi})^{\nu})$. $\square$

Example. Consider the $U(\mathbb{H})$-gain graph $\Phi = (C_4, U(\mathbb{H}), \varphi)$ in Fig. 5. Let $\mathcal{C}_1 = v_1e_{12}v_2e_{23}v_3e_{34}v_4e_{41}v_1$. Then $\varphi(\mathcal{C}_1) = -\frac{1}{\sqrt{2}}(1-k)$. By the switching procedure of part (1) in the proof of Theorem 7.2, Φ can be switched to $(\Phi^{\xi})^{\nu}$ as seen in Fig. 5, where $\varphi^{\xi\nu}(e_{12}) = 1$, $\varphi^{\xi\nu}(e_{23}) = 1$, $\varphi^{\xi\nu}(e_{34}) = 1$, and

$$\varphi^{\xi\nu}(e_{41}) = \mathrm{Re}\left(-\frac{1}{\sqrt{2}}(1-k)\right) + \left|\mathrm{Im}\left(-\frac{1}{\sqrt{2}}(1-k)\right)\right| i = -\frac{1}{\sqrt{2}}(1-i) = e^{i\theta}.$$

Using $\theta = 3\pi/4$ yields the following right spectra by Theorem 7.2:

$$\sigma_A(\Phi) = \left\{\pm\sqrt{2+\sqrt{2-\sqrt{2}}}, \pm\sqrt{2-\sqrt{2-\sqrt{2}}}\right\}, \text{ and}$$
$$\sigma_L(\Phi) = \left\{2\pm\sqrt{2+\sqrt{2-\sqrt{2}}}, 2\pm\sqrt{2-\sqrt{2-\sqrt{2}}}\right\}.$$

Alternatively, the complex adjoint can be used to calculate these right eigenvalues, and indeed these match the above calculation:

$$\sigma(f(A(\Phi))) = \sigma_A(\Phi) = \{\pm 1.111, \pm 1.663\}, \text{ and}$$
$$\sigma(f(L(\Phi))) = \sigma_L(\Phi) = \{0.337, 0.889, 3.111, 3.663\}.$$

Theorem 7.3. *Let P_n be the path graph on n vertices. Suppose $\Phi = (P_n, U(\mathbb{H}), \varphi)$. The right eigenvalues of $A(\Phi)$ and $L(\Phi)$ can be calculated as*

$$\sigma_A(\Phi) = \left\{2\cos\left(\frac{\pi j}{n+1}\right) : j \in \{1, \dots, n\}\right\}, \tag{7.5}$$

and

$$\sigma_L(\Phi) = \left\{2 - 2\cos\left(\frac{\pi j}{n}\right) : j \in \{0, \dots, n-1\}\right\}. \tag{7.6}$$

Proof. Let P_n be a path graph with n vertices. Since any $U(\mathbb{H})$-gain graph $\Phi = (P_n, \varphi)$ is balanced, Φ and P_n have the same adjacency and Laplacian right spectra by Lemmas 4.1 and 6.1. The adjacency and Laplacian eigenvalues of P_n are above, and can be found in [15, p. 9]. $\square$

8. Concluding remarks

In this paper we have considered the basic results for a foundation of a spectral theory of quaternion unit gain graphs. The results obtained here are a generalization of well-known results for the spectral theories of graphs, signed graphs and complex unit gain graphs.

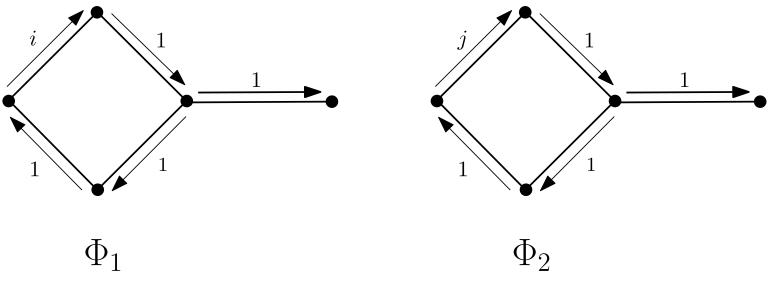

Fig. 6. Two cospectral non-isomorphic $U(\mathbb{H})$-gain graphs.

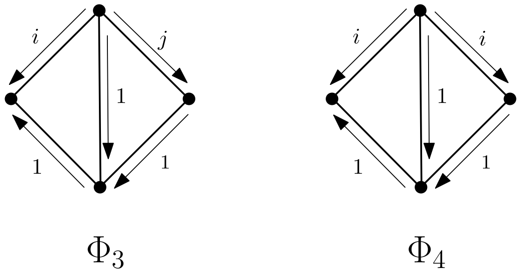

Fig. 7. Two non-cospectral $U(\mathbb{H})$-gain graphs.

The careful reader has noticed that Theorem 4.6 (the Coefficient Theorem of the adjacency matrix) says that the eigenvalues are affected from the real part of the weight of the cycles in the $U(\mathbb{H})$-gain graphs. The latter means that $U(\mathbb{H})$-gain graphs whose cycles have gains differing only on the complex part will get the same spectrum. For example, let Γ be a lollipop graph as a cycle C_4 with a hanging edge. Let Φ_1 (resp. Φ_2) be the $U(\mathbb{H})$-gain graph (Γ, φ_1) (resp. (Γ, φ_2)) whose unique cycle has weight i (resp. j), as depicted in Fig. 6. The gain graphs Φ_1 and Φ_2 are switching nonisomorphic, but $p_A(\Phi_1) = p_A(\Phi_2) = \lambda(\lambda^4 - 5\lambda^2 + 4)$.

Hence, let $\Phi = (\Gamma, \varphi)$ be a unicyclic $U(\mathbb{H})$-gain graphs. If the gain of the unique cycle is, say, $q = a + bi + cj + dk \in \mathbb{H}$, then we can consider its similar number in $\mathbb{C}$, namely $q' = a + \sqrt{b^2 + c^2 + d^2}\, i$. Let $\Phi' = (\Gamma, \varphi')$ be the unicyclic $\mathbb{T}$-gain graph, whose gain of the unique cycle is q', then Φ and Φ' will be cospectral graphs.

The latter fact might be surprising, but we have a similar phenomenon when we restrict to balanced graphs: the spectral theory of signed graphs restricted to balanced graphs just "reduces" to the usual spectral theory of unsigned graphs. So, we can similarly say that the spectral theory of $U(\mathbb{H})$-gain graphs restricted to unicyclic graphs reduces to the spectral theory of $\mathbb{T}$-gain graphs.

However, the here developed spectral theory of $U(\mathbb{H})$-gain graphs is in general different from that of $\mathbb{T}$-gain graphs. To show that, we consider Γ as the bicyclic graph known as diamond with two different gains as in Fig. 7.

It is easy to check that $p_A(\Phi_3) = \lambda^4 - 5\lambda^2 + 2$, and $p_A(\Phi_4) = \lambda^2(\lambda^2 - 5)$. Notably, if Γ gets two different and non-opposite complex quaternion units, then the resulting polynomial is the same of $p_A(\Phi_3)$, and the motivation is that the real parts of all cycles are always equal to 0. Evidently, with quaternions we get many more switching nonisomorphic cospectral graphs.

We expect that many other results from the spectral theories of signed and complex unit gain graphs can be generalized to the quaternions settings.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] S.L. Adler, Quaternionic Quantum Mechanics and Quantum Fields, International Series of Monographs on Physics, Oxford University Press, 1995.

[2] Saieed Akbari, Francesco Belardo, Farideh Heydari, Mohammad Maghasedi, Mona Souri, On the largest eigenvalue of signed unicyclic graphs, Linear Algebra Appl. 581 (2019) 145–162, MR 3982012.

[3] Abdullah Alazemi, Milica Anđelić, Francesco Belardo, Maurizio Brunetti, Carlos M. da Fonseca, Line and subdivision graphs determined by $\mathbb{T}_4$-gain graphs, Mathematics 7 (10) (2019) 926.

[4] Simon L. Altmann, Rotations, Quaternions, and Double Groups, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1986, MR 868858.

[5] Helmer Aslaksen, Quaternionic determinants, Math. Intell. 18 (1996) 57–65, MR 1412993 (97j:16028).

[6] Yik Hoi Au-Yeung, On the convexity of numerical range in quaternionic Hilbert spaces, Linear Multilinear Algebra 16 (1–4) (1984) 93–100, MR 768998.

[7] Francesco Belardo, Balancedness and the least eigenvalue of Laplacian of signed graphs, Linear Algebra Appl. 446 (2014) 133–147, MR 3163133.

[8] Francesco Belardo, Maurizio Brunetti, Line graphs of complex unit gain graphs with least eigenvalue −2, Electron. J. Linear Algebra 37 (2021) 14–30, MR4246513.

[9] Francesco Belardo, Maurizio Brunetti, Nathan Reff, Balancedness and the least Laplacian eigenvalue of some complex unit gain graphs, Discuss. Math., Graph Theory 40 (2) (2020) 417–433, MR 4060993.

[10] Francesco Belardo, Sebastian M. Cioabă, Jack Koolen, Jianfeng Wang, Open Problems in the Spectral Theory of Signed Graphs, Art Discrete Appl. Math., vol. 1, 2018, 2, #P2.10, 23. MR 3997096.

[11] Francesco Belardo, Paweł Petecki, Spectral characterizations of signed lollipop graphs, Linear Algebra Appl. 480 (2015) 144–167, MR 3348518.

[12] Francesco Belardo, Slobodan K. Simić, On the Laplacian coefficients of signed graphs, Linear Algebra Appl. 475 (2015) 94–113, MR 3325220.

[13] Garrett Birkhoff, John von Neumann, The logic of quantum mechanics, Ann. Math. 37 (4) (1936) 823–843, MR 1503312.

[14] J.L. Brenner, Matrices of quaternions, Pac. J. Math. 1 (1951) 329–335, MR 43761.

[15] Andries E. Brouwer, Willem H. Haemers, Spectra of Graphs, Universitext, Springer, New York, 2012, MR 2882891.

[16] Matteo Cavaleri, Daniele D'Angeli, Alfredo Donno, Gain-line graphs via G-phases and group representations, Linear Algebra Appl. 613 (2021) 241–270, MR 4199816.

[17] Matteo Cavaleri, Daniele D'Angeli, Alfredo Donno, A group representation approach to balance of gain graphs, J. Algebraic Comb. 54 (1) (2021) 265–293, MR 4290425.

[18] Dragoš Cvetković, Peter Rowlinson, Slobodan K. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (1) (2007) 155–171, MR 2312332 (2008c:05105).

[19] Dragoš Cvetković, Peter Rowlinson, Slobodan Simić, On graphs with least eigenvalue −2, in: Spectral Generalizations of Line Graphs, in: London Mathematical Society Lecture Note Series, vol. 314, Cambridge University Press, Cambridge, 2004, MR 2120511.

[20] Dragoš M. Cvetković, Michael Doob, Horst Sachs, Spectra of Graphs. Theory and Applications, third ed., Johann Ambrosius Barth, Heidelberg, 1995, MR 1324340.

[21] Kinkar Ch. Das, A characterization on graphs which achieve the upper bound for the largest Laplacian eigenvalue of graphs, Linear Algebra Appl. 376 (2004) 173–186, MR 2015532 (2004g:05102).

[22] Stefano De Leo, Gisele Ducati, Quaternionic differential operators, J. Math. Phys. 42 (5) (2001) 2236–2265.

[23] Stefano De Leo, Giuseppe Scolarici, Right eigenvalue equation in quaternionic quantum mechanics, J. Phys. A 33 (15) (2000) 2971–2995, MR 1766502.

[24] Stefano De Leo, Giuseppe Scolarici, Luigi Solombrino, Quaternionic eigenvalue problem, J. Math. Phys. 43 (11) (2002) 5815–5829.

[25] T. Dray, C.A. Manogue, The octonionic eigenvalue problem, Adv. Appl. Clifford Algebras 8 (1998) 341–364, MR 1698296 (2000i:17003).

[26] Yi-Zheng Fan, On the least eigenvalue of a unicyclic mixed graph, Linear Multilinear Algebra 53 (2) (2005) 97–113, MR 2133313.

[27] Yi-Zheng Fan, Bit-Shun Tam, Jun Zhou, Maximizing spectral radius of unoriented Laplacian matrix over bicyclic graphs of a given order, Linear Multilinear Algebra 56 (4) (2008) 381–397, MR 2434109.

[28] Douglas R. Farenick, Barbara A.F. Pidkowich, The spectral theorem in quaternions, Linear Algebra Appl. 371 (2003) 75–102, MR 1997364.

[29] D. Finkelstein, J.M. Jauch, S. Schiminovich, D. Speiser, Principle of general Q covariance, J. Math. Phys. 4 (1963) 788–796, MR 151124.

[30] D. Finkelstein, J.M. Jauch, D. Speiser, Quaternionic representations of compact groups, J. Math. Phys. 4 (1963) 136–140, MR 142690.

[31] D. Finkelstein, J.M. Jauch, D. Speiser, Notes on Quaternion Quantum Mechanics. i, the Logico-Algebraic Approach to Quantum Mechanics, Vol. II, Univ. Western Ontario Ser. Philos. Sci., vol. 5, Reidel, Dordrecht-Boston, Mass., 1979, pp. 367–421, MR 591061.

[32] David Finkelstein, Josef M. Jauch, Samuel Schiminovich, David Speiser, Foundations of quaternion quantum mechanics, J. Math. Phys. 3 (1962) 207–220, MR 137500.

[33] Shengjie He, Rong-Xia Hao, Fengming Dong, The rank of a complex unit gain graph in terms of the matching number, Linear Algebra Appl. 589 (2020) 158–185, MR 4046874.

[34] Yao Ping Hou, Bounds for the least Laplacian eigenvalue of a signed graph, Acta Math. Sin. Engl. Ser. 21 (4) (2005) 955–960, MR 2156977.

[35] Yaoping Hou, Jiongsheng Li, Yongliang Pan, On the Laplacian eigenvalues of signed graphs, Linear Multilinear Algebra 51 (1) (2003) 21–30, MR 1950410.

[36] I.I. Kyrchei, The Theory of the Column and Row Determinants in a Quaternion Linear Algebra, vol. 15, Nova Science Publishers, 2012, cited by [26].

[37] H.C. Lee, Eigenvalues and canonical forms of matrices with quaternion coefficients, Proc. R. Ir. Acad., Sect. A 52 (1949) 253–260, MR 0036738.

[38] Jiong-Sheng Li, Xiao-Dong Zhang, On the Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1–3) (1998) 305–307, MR 1653547 (99g:05133).

[39] Ying Li, Jian-li Zhao, Proof of Rayleigh-Ritz theorem of quaternion matrices, Acta Sci. Nat. Univ. Neimongol. 2006 (01) (2006) 5–8.

[40] Lu Lu, Jianfeng Wang, Qiongxiang Huang, Complex unit gain graphs with exactly one positive eigenvalue, Linear Algebra Appl. 608 (2021) 270–281, MR 4152791.

[41] Ranjit Mehatari, M. Rajesh Kannan, Aniruddha Samanta, On the adjacency matrix of a complex unit gain graph, Linear Multilinear Algebra (2020) 1–16.

[42] Carla Silva Oliveira, Leonardo Silva de Lima, Nair Maria Maia de Abreu, Pierre Hansen, Bounds on the index of the signless Laplacian of a graph, Discrete Appl. Math. 158 (4) (2010) 355–360, MR 2588119 (2011d:05228).

[43] Nathan Reff, Spectral properties of complex unit gain graphs, Linear Algebra Appl. 436 (9) (2012) 3165–3176, MR 2900705.

[44] Nathan Reff, Oriented gain graphs, line graphs and eigenvalues, Linear Algebra Appl. 506 (2016) 316–328, MR 3530682.

[45] Slobodan K. Simić, Zoran Stanić, Polynomial reconstruction of signed graphs, Linear Algebra Appl. 501 (2016) 390–408, MR 3485074.

[46] Zoran Stanić, Integral regular net-balanced signed graphs with vertex degree at most four, Ars Math. Contemp. 17 (1) (2019) 103–114, MR 3998150.

[47] T.-Y. Tam, Interlacing inequalities and Cartan subspaces of classical real simple Lie algebras, SIAM J. Matrix Anal. Appl. 21 (2) (1999) 581–592, MR 1742811 (2001a:15022).

[48] Shang-wang Tan, Ji-ming Guo, Jian Qi, The spectral radius of Laplacian matrices and quasi-Laplacian matrices of graphs, Gongcheng Shuxue Xuebao 20 (6) (2003) 69–74, MR 2031534 (2004k:05137).

[49] Bo Xun Tu, Weak direct products and weak circular products of matrices over the division ring of real quaternions, J. Fudan Univ. Nat. Sci. 30 (3) (1991) 331–340, MR 1148481.
[50] Yi Wang, Shi-Cai Gong, Yi-Zheng Fan, On the determinant of the Laplacian matrix of a complex unit gain graph, Discrete Math. 341 (1) (2018) 81–86, MR 3713385.
[51] J.L. Wu, Distribution and estimation for eigenvalues of real quaternion matrices, Comput. Math. Appl. 55 (9) (2008) 1998–2004, MR 2401108.
[52] Feng Xu, Qi Zhou, Dein Wong, Fenglei Tian, Complex unit gain graphs of rank 2, Linear Algebra Appl. 597 (2020) 155–169, MR 4080071.
[53] Thomas Zaslavsky, Biased graphs. I. Bias, balance, and gains, J. Combin. Theory Ser. B 47 (1989) 32–52, MR 1007712.
[54] Thomas Zaslavsky, Totally frustrated states in the chromatic theory of gain graphs, European J. Combin. 30 (1) (2009) 133–156, MR 2460223.
[55] Thomas Zaslavsky, Matrices in the Theory of Signed Simple Graphs, Advances in Discrete Mathematics and Applications: Mysore, 2008, Ramanujan Math. Soc. Lect. Notes Ser., vol. 13, Ramanujan Math. Soc., Mysore, 2010, pp. 207–229, MR 2766941.
[56] Fuzhen Zhang, Quaternions and matrices of quaternions, Linear Algebra Appl. 251 (1997) 21–57, MR 1421264.
[57] Fuzhen Zhang, Geršgorin type theorems for quaternionic matrices, Linear Algebra Appl. 424 (1) (2007) 139–153, MR 2324381.
[58] Xiao-Dong Zhang, Two sharp upper bounds for the Laplacian eigenvalues, Linear Algebra Appl. 376 (2004) 207–213, MR 2015534 (2004m:05173).