Problem

Show that the series $\sum\limits_{k=1}^{\infty} \frac{1}{2^{\ln(k)}}$ diverges.

Solution

By the change-of-base formula,

$$\ln(k) = \frac{\log_2(k)}{\log_2(e)}.$$

Therefore, we can write

$$\begin{aligned} 2^{\ln(k)} &= 2^{\log_2(k)/\log_2(e)} \\ &= \left(2^{\log_2(k)}\right)^{1/\log_2(e)} \\ &= k^{1/\log_2(e)}. \end{aligned}$$

We can now see that

$$\sum_{k=1}^{\infty} \frac{1}{2^{\ln(k)}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/\log_2(e)}},$$

which is a p-series. Since $e > 2$, it follows that $\log_2(e) > 1$. Hence $p = 1/\log_2(e) < 1$, meaning the series diverges.

■

Generalization

Let's suppose $a, b > 1$. When do series of the form

$$\sum_{k=1}^{\infty} \frac{1}{b^{\log_a(k)}}$$

converge? Using the change-of-base formula, we have

$$\log_a(k) = \frac{\log_b(k)}{\log_b(a)}.$$

Guided by what's written in the previous section, we conclude that

$$\sum_{k=1}^{\infty} \frac{1}{b^{\log_a(k)}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/\log_b(a)}},$$

which will converge if $\frac{1}{\log_b(a)} > 1$, i.e., if $\log_b(a) < 1$. This occurs when $b > a$.