Problem

Show that the series $\sum_{k=1}^{\infty} \frac{1}{2^{\ln(k)}}$ diverges.

Solution

By the change-of-base formula,

$$\ln(k) = \frac{\log_2(k)}{\log_2(e)}.$$

Therefore, we can write

$$\begin{aligned} 2^{\ln(k)} &= 2^{\log_2(k)/\log_2(e)} \\ &= \left(2^{\log_2(k)}\right)^{1/\log_2(e)} \\ &= k^{1/\log_2(e)}. \end{aligned}$$

We can now see that

$$\sum_{k=1}^{\infty} \frac{1}{2^{\ln(k)}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/\log_2(e)}},$$

which is a p-series. Since $e > 2$, it follows that $\log_2(e) > 1$. Hence $p = 1/\log_2(e) < 1$, meaning the series diverges.

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