Problem

Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{c}$ be $n \times 1$ column vectors, and let $\mathbf{M}_1$ and $\mathbf{M}_2$ be $n \times n$ matrices. Suppose

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} \mathbf{c}.$$

Show that if $\det(\mathbf{M}_1) \neq 0$, then

$$\mathbf{v}_2 = \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{v}_1.$$

Solution

Left-multiply both sides of the given equation by the $n \times 2n$ block matrix $\begin{bmatrix} \mathbf{M}_2 \mathbf{M}_1^{-1} & -\mathbf{I}_{n\times n} \end{bmatrix}$, where $\mathbf{I}_{n\times n}$ is the $n \times n$ identity matrix. On the left-hand side of the equation, we obtain

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_2 \mathbf{M}_1^{-1} & -\mathbf{I}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} &= \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{v}_1 - \mathbf{I}_{n\times n} \mathbf{v}_2 \\ &= \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{v}_1 - \mathbf{v}_2. \end{aligned}$$

On the right-hand side of the equation, we obtain

$$\begin{aligned} \begin{bmatrix} \mathbf{M}_2 \mathbf{M}_1^{-1} & -\mathbf{I}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} \mathbf{c} &= \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{M}_1 \mathbf{c} - \mathbf{I}_{n\times n} \mathbf{M}_2 \mathbf{c} \\ &= \mathbf{M}_2 \mathbf{c} - \mathbf{M}_2 \mathbf{c} \\ &= \mathbf{0}. \end{aligned}$$

Therefore,

$$\mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0};$$

i.e.,

$$\mathbf{v}_2 = \mathbf{M}_2 \mathbf{M}_1^{-1} \mathbf{v}_1,$$

as asserted.

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Remark

The interesting thing about this problem is that the equation

$$\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \end{bmatrix} \mathbf{c}$$

looks like the equation

$$\frac{a}{b} = \frac{c}{d}.$$

The solution for b in the latter equation is

$$b = \frac{d}{c}a = dc^{-1}a,$$

while the solution for $\mathbf{v}_2$ in the former equation is

$$\mathbf{v}_2 = \mathbf{M}_2\mathbf{M}_1^{-1}\mathbf{v}_1.$$

Strangely enough, you can solve for $\mathbf{v}_2$ almost by pretending that you have fractions instead of block matrices.