Problem

Derive the moment-generating function (MGF) of the Negative Binomial Distribution.

Solution

Let X be a negative binomial random variable with parameters p and r. The MGF of X is

$$E\left[e^{tX}\right] = \sum_{x=r}^{\infty} e^{tx}\binom{x-1}{r-1}p^r(1-p)^{x-r} \tag{1}$$

$$= \sum_{x=r}^{\infty}\binom{x-1}{r-1}\left(\frac{p}{1-p}\right)^r\left[e^t(1-p)\right]^x \tag{2}$$

$$= \left(\frac{p}{1-p}\right)^r\sum_{x=r}^{\infty}\binom{x-1}{r-1}\left[e^t(1-p)\right]^x. \tag{3}$$

This can be vastly simplified. Let $k = x - r$. Then

$$\left(\frac{p}{1-p}\right)^r\sum_{x=r}^{\infty}\binom{x-1}{r-1}\left[e^t(1-p)\right]^x = \left(\frac{p}{1-p}\right)^r\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}[e^t(1-p)]^{k+r} \tag{4}$$

$$= \left(pe^t\right)^r\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}[e^t(1-p)]^k. \tag{5}$$

For simplicity's sake, let $y = e^t(1-p)$, and let $a = r - 1$. The summation in (4) can now be written as

$$\sum_{k=0}^{\infty}\binom{k+a}{a}y^k.$$

This summation is computed as follows: First multiply and divide by $a!$. This yields

$$\sum_{k=0}^{\infty}\binom{k+a}{a}y^k = \frac{1}{a!}\sum_{k=0}^{\infty}a!\binom{k+a}{a}y^k \tag{6}$$

$$= \frac{1}{a!}\sum_{k=0}^{\infty}\left[\frac{(k+a)!}{k!}\right]y^k \tag{7}$$

$$= \frac{1}{a!}\sum_{k=0}^{\infty}\left(\prod_{j=0}^{a-1}[(a-j)+k]\right)y^k \tag{8}$$

$$= \frac{1}{a!}\sum_{k=0}^{\infty}\frac{d^a}{dy^a}\left[y^{a+k}\right] \tag{9}$$

$$= \frac{1}{a!}\frac{d^a}{dy^a}\left[\sum_{k=0}^{\infty}y^{a+k}\right] \tag{10}$$

$$= \frac{1}{a!}\frac{d^a}{dy^a}\left[\frac{1}{1-y} - \sum_{j=0}^{a-1}y^j\right] \tag{11}$$

$$= \frac{1}{a!}\left[\frac{d^a}{dy^a}\left(\frac{1}{1-y}\right) - \frac{d^a}{dy^a}\sum_{j=0}^{a-1}y^j\right] \tag{12}$$

$$= \frac{1}{a!}\left(\frac{a!}{(1-y)^{a+1}}\right) \tag{13}$$

$$= \frac{1}{(1-y)^{a+1}} \tag{14}$$

Note that it must be assumed that $|y| < 1$ for the above equations to be valid. Substituting (13) into (4) and putting everything in terms of r, p, and t yields

$$E[e^{tX}] = \frac{(pe^t)^r}{[1-(1-p)e^t]^r},$$

with the requirement that $(1-p)e^t < 1$, i.e., $t < \ln\left(\frac{1}{1-p}\right)$.

$\blacksquare$

Comment

This is the most interesting of the basic MGF derivations, in my opinion. Others that I tried didn't really involve any interesting techniques. Maybe this derivation can be done more simply, but I am not aware of how at the moment.