DEPTH MAP GENERATOR EQUATIONS

Shadow

Reflected Light:

The direction of reflected light is approximately parallel to the plane of the image and approximately the opposite direction as the direct light on that plane. The highest value $c_0 + c_1$ is observed in the image when the normal vector is parallel to the reflected light vector, and the lowest c_0 is observed when the normal vector is orthogonal to the reflected light vector.

$$p = c_0 + c_1 \vec{n} \cdot \vec{h}$$

In addition, the *xy* projection of the surface normal vector is parallel to the gradient of the surface projected onto the *xy* plane, whose direction is roughly estimated by the gradient of the image intensity function, assuming contour lines of the image intensity roughly correspond to contour lines in the surface. This means the normalized projection of the unit surface normal vector is equal to the unit gradient vector.

$$\hat{\nabla} p = \begin{bmatrix} \hat{\nabla} p_x \\ \hat{\nabla} p_y \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{\partial p}{\partial x}}{\sqrt{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2}} \\ \dfrac{\frac{\partial p}{\partial y}}{\sqrt{\left(\frac{\partial p}{\partial x}\right)^2 + \left(\frac{\partial p}{\partial y}\right)^2}} \end{bmatrix} = \begin{bmatrix} \dfrac{n_x}{\sqrt{n_x^2 + n_y^2}} \\ \dfrac{n_y}{\sqrt{n_x^2 + n_y^2}} \end{bmatrix}$$

Or equivalently

$$\frac{n_x}{n_y} = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}$$

Where
p = pixel value
c_0 = constant, set to lowest pixel value in image
c_1 = constant, set to highest pixel value in shadow mask minus c_0
$\vec{n}$ = surface normal unit vector
$\vec{h}$ = horizontal unit vector (normalized projection of light direction on *xy* image plane)
$\hat{\nabla} p$ = normalized gradient vector of image intensity function
θ = angle of reflected light, measured from the y-axis

$$|\vec{n}|^2 = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = n_x^2 + n_y^2 + n_z^2 = 1$$

$$\vec{h} = \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

$$n_x = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} n_y$$

$$p = c_0 + c_1 \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} \sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix} = c_0 + c_1\left(n_x \sin(\theta) + n_y \cos(\theta)\right) = c_0 + c_1 \sin(\theta)\, n_x + c_1 \cos(\theta)\, n_y$$

$$p = c_0 + c_1 \sin(\theta) \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} n_y + c_1 \cos(\theta)\, n_y = c_0 + \left(c_1 \sin(\theta) \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + c_1 \cos(\theta) \right) n_y$$

$$n_y = \frac{p - c_0}{c_1 \left(\sin(\theta) \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + \cos(\theta) \right)}$$

$$n_x = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} \frac{p - c_0}{c_1 \left(\sin(\theta) \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + \cos(\theta) \right)} = \frac{p - c_0}{c_1 \left(\sin(\theta) + \cos(\theta) \frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial x}} \right)}$$

$$n_z = \sqrt{1 - n_x^2 - n_y^2} = \sqrt{1 - \frac{(p - c_0)^2}{c_1^2} \left(\frac{1}{\left(\sin(\theta) \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + \cos(\theta) \right)^2} + \frac{1}{\left(\sin(\theta) + \cos(\theta) \frac{\frac{\partial p}{\partial y}}{\frac{\partial p}{\partial x}} \right)^2} \right)}$$

Note that n_z will be defined as always positive, which is appropriate because relevant normals of the surface should be pointing out of the image plane.

For computational purposes, the case where the gradient is zero must be dealt with.
If $\frac{\partial p}{\partial x} = 0,\ n_x = 0$ and if $\frac{\partial p}{\partial y} = 0,\ n_y = 0$

Based on the nature of the equations constraints arise on the angle ψ between the projections of $\vec{n}$ and $\vec{h}$ on the *xy* plane.

$$\cos(\psi) = \frac{\vec{n}_{xy} \cdot \vec{h}}{|\vec{n}_{xy}|} = \frac{\vec{n} \cdot \vec{h}}{|\vec{n}_{xy}|} = \frac{p - c_0}{c_1 |\vec{n}_{xy}|}$$

This angle is maximized (cosine minimized) when $n_z = 0$ and $|\vec{n}_{xy}| = 1$.

$$\psi_{max} = \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)$$

The resulting 2 angles are

$$\theta \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)$$

And the vectors are

$$\begin{bmatrix} \sin\left(\theta \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)\right) \\ \cos\left(\theta \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)\right) \end{bmatrix}$$

Note that angles are measured from y axis.

It may be that the angle between $\hat{\nabla} p$ and projection of $\vec{h}$ on the *xy* plane exceeds φ.

$$\frac{p - c_0}{c_1} > \hat{\nabla} p \cdot \vec{h}$$

In such cases $\hat{\nabla} p$ will be replaced with $\hat{\nabla} p^*$ such that

$$\frac{p - c_0}{c_1} = \hat{\nabla} p^* \cdot \vec{h}$$

$\hat{\nabla} p^*$ is set to

$$\begin{bmatrix} \sin\left(\theta \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)\right) \\ \cos\left(\theta \pm \cos^{-1}\left(\frac{p - c_0}{c_1}\right)\right) \end{bmatrix}$$

With signs chosen such that $\hat{\nabla} p \cdot \hat{\nabla} p^*$ is maximized. The "convexity" assumption means the sign of n_y and n_x must be the same as the sign of $\frac{\partial p}{\partial y}$ and $\frac{\partial p}{\partial x}$. By maximizing $\hat{\nabla} p \cdot \hat{\nabla} p^*$ the convexity assumption is upheld. If "concavity" is assumed this must be minimized instead.

This replacement represents the rotation of the plane of the surface contour away from the *xy* plane, such that the projection of the contour on the *xy* plane is unchanged.

Ambient Light:

The direction of light is approximately normal to the image plane, and is denoted $\hat{\boldsymbol{z}}$. The highest value $c_0 + c_1$ is observed in the image when the normal vector is normal to the image plane, and the lowest c_0 is observed when the normal vector is parallel to the image plane. This results in almost the same equation as for reflected light.

$p = c_0 + c_1 \vec{n} \cdot \hat{\boldsymbol{z}}$

$$|\vec{n}|^2 = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = n_x^2 + n_y^2 + n_z^2 = 1$$

$$\vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$n_x = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} n_y$$

$$p = c_0 + c_1 \vec{n} \cdot \hat{\mathbf{z}} = c_0 + c_1 \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = c_0 + c_1 n_z$$

$$\frac{p - c_0}{c_1} = n_z$$

$$n_x^2 + n_y^2 + n_z^2 = \left(\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} \right)^2 + 1 \right) n_y^2 + \left(\frac{p - c_0}{c_1} \right)^2 = 1$$

$$\left(\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} \right)^2 + 1 \right) n_y^2 = 1 - \left(\frac{p - c_0}{c_1} \right)^2$$

$$n_y = \sqrt{\frac{1 - \left(\frac{p - c_0}{c_1} \right)^2}{\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} \right)^2 + 1}}$$

The "convexity" assumption means the sign of n_y must be the opposite of the sign of $\frac{\partial p}{\partial y}$.

If $\frac{\partial p}{\partial y} = 0$ then $n_y = 0$ and $n_x = \sqrt{1 - n_z^2}$

If $\frac{\partial p}{\partial x}$ is also 0, then $\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Unlike with reflected light, based on the nature of the equations there are no constraints on the valid directions of the gradient.

Light

In the light, the brightest reflection is given by a surface perpendicular to the light direction.

$$p = c_0 + c_1 \vec{n} \cdot \vec{l}$$

The normal vector projection is parallel to the gradient once again.

$$\frac{n_x}{n_y} = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}$$

In addition, the estimated intensity of flat areas of the surface can be used as follows

$$p_{flat} = c_0 + c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = c_0 + c_1 l_z$$

Where
p = pixel value
p_{flat} = estimate pixel value for flat areas of the image
c_0 = constant, set to the shadow threshold
c_1 = constant, set to highest pixel value in the image minus the shadow threshold
$\vec{n}$ = surface normal unit vector
$\vec{l}$ = unit vector representing the direction of light
$\hat{\nabla} p$ = normalized gradient vector of image intensity function

$$\boldsymbol{l_z = \frac{p_{flat} - c_0}{c_1}}$$

$$l_x^2 + l_y^2 + l_z^2 = 1$$

$$\frac{l_x}{l_y} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$l_x = l_y \tan(\theta)$$

$$l_y^2 \tan^2(\theta) + l_y^2 + \left(\frac{p_{flat} - c_0}{c_1}\right)^2 = 1$$

$$l_y^2(1 + \tan^2(\theta)) = 1 - \left(\frac{p_{flat} - c_0}{c_1}\right)^2$$

$$\frac{l_y^2}{\cos^2(\theta)} = 1 - \left(\frac{p_{flat} - c_0}{c_1}\right)^2$$

$$l_y^2 = \cos^2(\theta) \left(1 - \left(\frac{p_{flat} - c_0}{c_1}\right)^2\right)$$

$$l_y = \cos(\theta)\sqrt{1 - \left(\frac{p_{flat} - c_0}{c_1}\right)^2}$$

$$l_x = \sin(\theta)\sqrt{1 - \left(\frac{p_{flat} - c_0}{c_1}\right)^2}$$

$$\frac{n_x}{n_y} = \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}$$

$$n_x = n_y \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$n_y^2\left(1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right) + n_z^2 = 1$$

$$n_z = \sqrt{1 - n_y^2\left(1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}$$

Once again n_z must be positive.

$$p = c_0 + c_1\vec{n}\cdot\vec{l}$$

$$p = c_0 + c_1\left(l_x n_x + l_y n_y + l_z n_z\right)$$

$$p = c_0 + c_1\left(l_x n_y \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + l_y n_y + l_z\sqrt{1 - n_y^2\left(1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}\right)$$

$$\frac{p - c_0}{c_1} = \left(l_x n_y \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}} + l_y n_y + l_z\sqrt{1 - n_y^2\left(1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}\right)$$

$$\frac{p-c_0}{l_z c_1}-\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)n_y=\sqrt{1-n_y^2\left(1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}$$

$$1-\left(\frac{p-c_0}{l_z c_1}-\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)n_y\right)^2=n_y^2\left(1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)$$

$$1-\left(\frac{p-c_0}{l_z c_1}\right)^2-\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)^2 n_y^2+2\left(\frac{p-c_0}{l_z c_1}\right)\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)n_y=n_y^2\left(1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)$$

$$1-\left(\frac{p-c_0}{l_z c_1}\right)^2-\left(\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)^2+\left(1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)\right)n_y^2+2\left(\frac{p-c_0}{l_z c_1}\right)\left(\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}\right)n_y=0$$

For simplicity let

$$a=\frac{p-c_0}{l_z c_1}$$

$$b=\frac{l_x\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}+l_y}{l_z}$$

Then

$$1-a^2-\left(b^2+1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)n_y^2+2abn_y=0$$

$$\left(b^2+1+\left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)n_y^2-2abn_y+a^2-1=0$$

$$n_y = \frac{2ab \pm \sqrt{4a^2b^2 - 4\left(b^2 + 1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)(a^2 - 1)}}{2\left(b^2 + 1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}$$

Then substitute into

$$n_x = n_y \frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}$$

and

$$n_z = \sqrt{1 - n_y^2\left(1 + \left(\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial y}}\right)^2\right)}$$

Note that there are 2 possible vectors for $\vec{n}$ that can satisfy all the equations. The cosine of the angle between each vector and the gradient of the image intensity function should be compared. The vector with the largest cosine is selected. (This indirectly enforces the convexity assumption, which means the sign of n_y and n_x must be the same as the sign of $\frac{\partial p}{\partial y}$ and $\frac{\partial p}{\partial x}$).

In the case that $\frac{\partial p}{\partial y} = 0$

$$n_z = \sqrt{1 - n_x^2}$$

$$p = c_0 + c_1(l_x n_x + l_z n_z)$$

$$\frac{p - c_0}{c_1} = l_x n_x + l_z\sqrt{1 - n_x^2}$$

$$\frac{p - c_0}{c_1} - l_x n_x = l_z\sqrt{1 - n_x^2}$$

$$\left(\frac{p - c_0}{c_1} - l_x n_x\right)^2 = l_z^2 - l_z^2 n_x^2$$

$$\left(\frac{p - c_0}{c_1}\right)^2 - 2\left(\frac{p - c_0}{c_1}\right) l_x n_x + l_x^2 n_x^2 = l_z^2 - l_z^2 n_x^2$$

$$(l_z^2 + l_x^2) n_x^2 - 2\left(\frac{p - c_0}{c_1}\right) l_x n_x + \left(\frac{p - c_0}{c_1}\right)^2 - l_z^2 = 0$$

$$n_x = \frac{2\left(\frac{p - c_0}{c_1}\right) l_x \pm \sqrt{4\left(\frac{p - c_0}{c_1}\right)^2 l_x^2 - 4(l_z^2 + l_x^2)\left(\left(\frac{p - c_0}{c_1}\right)^2 - l_z^2\right)}}{2(l_z^2 + l_x^2)}$$

The constraints in this case are

$$\cos(\psi) = \frac{\vec{n}_{xy} \cdot \vec{l}_{xy}}{|\vec{n}_{xy}||\vec{l}_{xy}|} = \frac{n_x l_x + n_y l_y}{|\vec{n}_{xy}||\vec{l}_{xy}|}$$

Combining with the following

$$\frac{p - c_0}{c_1} = n_x l_x + n_y l_y + n_z l_z$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

First

$$|\vec{n}_{xy}| = \sqrt{n_x^2 + n_y^2} = \sqrt{1 - n_z^2}$$

and

$$\frac{p - c_0}{c_1} - n_z l_z = n_x l_x + n_y l_y$$

Then

$$\cos(\psi) = \frac{\frac{p - c_0}{c_1} - n_z l_z}{|\vec{l}_{xy}|\sqrt{1 - n_z^2}}$$

The angle is maximized when $\cos(\psi)$ is minimized.

$$0 = \frac{d}{dn_z}\left(\frac{\frac{p - c_0}{c_1} - n_z l_z}{|\vec{l}_{xy}|\sqrt{1 - n_z^2}}\right) = \frac{-l_z|\vec{l}_{xy}|\sqrt{1 - n_z^2} + \left(\frac{n_z|\vec{l}_{xy}|}{\sqrt{1 - n_z^2}}\right)\left(\frac{p - c_0}{c_1} - n_z l_z\right)}{|\vec{l}_{xy}|^2(1 - n_z^2)}$$

$$0 = \frac{-l_z|\vec{l}_{xy}| + \left(\frac{n_z|\vec{l}_{xy}|}{1 - n_z^2}\right)\left(\frac{p - c_0}{c_1} - n_z l_z\right)}{|\vec{l}_{xy}|^2\sqrt{1 - n_z^2}}$$

$$l_z|\vec{l}_{xy}| = \left(\frac{n_z|\vec{l}_{xy}|}{1 - n_z^2}\right)\left(\frac{p - c_0}{c_1} - n_z l_z\right)$$

$$l_z|\vec{l}_{xy}|(1 - n_z^2) = n_z|\vec{l}_{xy}|\left(\frac{p - c_0}{c_1} - n_z l_z\right)$$

$$l_z|\vec{l}_{xy}| - l_z|\vec{l}_{xy}|n_z^2 = \frac{p - c_0}{c_1}|\vec{l}_{xy}|n_z - l_z|\vec{l}_{xy}|n_z^2$$

$$l_z|\vec{l}_{xy}| = \frac{p - c_0}{c_1}|\vec{l}_{xy}|n_z$$

$$l_z = \frac{p - c_0}{c_1}n_z$$

$$\frac{l_z c_1}{p - c_0} = n_z$$

By substitution

$$\frac{p - c_0}{c_1} - n_z l_z = n_x l_x + n_y l_y$$

$$\frac{p - c_0 - c_1(n_z l_z + n_x l_x)}{c_1 l_y} = n_y$$

$$n_x^2 + \left(\frac{p - c_0 - c_1(n_z l_z + n_x l_x)}{c_1 l_y}\right)^2 + n_z^2 = 1$$

$$c_1^2 l_y^2 n_x^2 + \big(p - c_0 - c_1(n_z l_z + n_x l_x)\big)^2 = c_1^2 l_y^2 (1 - n_z^2)$$

$$c_1^2 l_y^2 n_x^2 + (p - c_0)^2 - 2(p - c_0)c_1(n_z l_z + n_x l_x) + c_1^2(n_z l_z + n_x l_x)^2 = c_1^2 l_y^2 (1 - n_z^2)$$

$$c_1^2 l_y^2 n_x^2 + (p - c_0)^2 - 2(p - c_0)c_1(n_z l_z + n_x l_x) + n_z^2 l_z^2 c_1^2 + 2 n_z l_z l_x c_1^2 n_x + n_x^2 l_x^2 c_1^2 = c_1^2 l_y^2 (1 - n_z^2)$$

$$c_1^2\big(l_y^2 + l_x^2\big) n_x^2 + 2 c_1 l_x \big(n_z l_z c_1 - (p - c_0)\big) n_x + (p - c_0)^2 - 2(p - c_0) c_1 n_z l_z + n_z^2 l_z^2 c_1^2 - c_1^2 l_y^2 (1 - n_z^2) = 0$$

$$c_1^2\big(l_y^2 + l_x^2\big) n_x^2 + 2 c_1 l_x \big(n_z l_z c_1 - (p - c_0)\big) n_x + \big((p - c_0) - c_1 n_z l_z\big)^2 - c_1^2 l_y^2 (1 - n_z^2) = 0$$

$$n_x = \frac{-2 c_1 l_x \big(n_z l_z c_1 - (p - c_0)\big) \pm \sqrt{\Big(2 c_1 l_x \big(n_z l_z c_1 - (p - c_0)\big)\Big)^2 - 4 c_1^2 \big(l_y^2 + l_x^2\big)\Big(\big((p - c_0) - c_1 n_z l_z\big)^2 - c_1^2 l_y^2 (1 - n_z^2)\Big)}}{2 c_1^2 \big(l_y^2 + l_x^2\big)}$$

It may be that the angle between $\widehat{\nabla} p$ and projection of $\vec{l}$ on the *xy* plane exceeds φ.

$$\frac{\frac{p - c_0}{c_1} - n_z l_z}{\left|\vec{l}_{xy}\right| \sqrt{1 - n_z^2}} > \frac{\widehat{\nabla} p \cdot \vec{l}_{xy}}{\left|\vec{l}_{xy}\right|}$$

In such cases $\widehat{\nabla} p$ will be replaced with $\widehat{\nabla} p^*$ as defined above.

$\widehat{\nabla} p^*$ is set to the vector with the maximum possible φ as calculated above.

With signs chosen such that $\widehat{\nabla} p \cdot \widehat{\nabla} p^*$ is maximized.

This replacement represents the rotation of the plane of the surface contour away from the *xy* plane, such that the projection of the contour on the *xy* plane is unchanged.