On the Calibration of Multiclass Classification with Rejection

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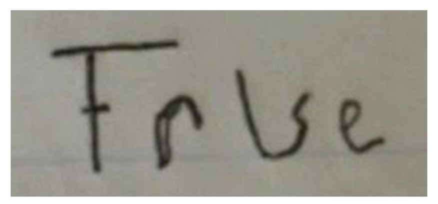
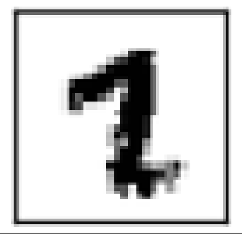
Introduction



Q: True or False!?



Q: 1,2 or 7 !?

Saying **"I don't know"** can **prevent misclassification**.

Most theoretical works in this problem focused on binary case.

Only Ramaswamy+ 2018 considered confidence-based approach in multiclass case.

Contributions:

- **An analysis of a recent classifier-rejector approach in multiclass case.**
- **Theoretical guarantee for well-known surrogate losses for confidence-based approach.**

Multiclass classification with rejection

(Chow 1970, Ramaswamy+ 2018)

Given: Labeled data $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n \overset{\text{i.i.d.}}{\sim} p(\boldsymbol{x}, y)$

Rejection cost $c \in (0, 0.5)$

Find: Classifier $f(\boldsymbol{x}) = \underset{y \in \mathcal{Y}}{\operatorname{argmax}}\, g_y(\boldsymbol{x})$

Rejector $r: \mathcal{X} \to \mathbb{R}$

$\boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^d$, $y \in \mathcal{Y} = \{1, \ldots, K\}$, $g_i(\boldsymbol{x}): \mathcal{X} \to \mathbb{R}$

that minimizes the following risk: $R_{\text{0-1-c}}(r, f) = \underset{p(\boldsymbol{x},y)}{\mathbb{E}}[\mathcal{L}_{\text{0-1-c}}(r, f; \boldsymbol{x}, y)]$

where $$\mathcal{L}_{\text{0-1-c}}(r, f; \boldsymbol{x}, y) = \underbrace{\mathbb{1}_{[f(\boldsymbol{x}) \neq y]}\mathbb{1}_{[r(\boldsymbol{x})>0]}}_{\text{misclassification loss}} + \underbrace{c\mathbb{1}_{[r(\boldsymbol{x}) \leq 0]}}_{\text{rejection loss}}$$

$\mathcal{L}_{\text{0-1-c}}(r, f; \boldsymbol{x}, y)$ **is hard to directly optimize.**

(Yuan+, 2010, Cortes+ (2015, 2016), Ramaswamy+ 2018)

A computationally-efficient and theoretically justified surrogate loss is needed.

Optimal solution of classification with rejection: (Chow 1970)

$$f^*(\boldsymbol{x}) = \arg\max_{y \in \mathcal{Y}} \eta_y(\boldsymbol{x})$$
$$r^*(\boldsymbol{x}) = \max_{y \in \mathcal{Y}} \eta_y(\boldsymbol{x}) - (1 - c)$$
$$\eta_y(\boldsymbol{x}) = p(y|\boldsymbol{x})$$

Calibration

Calibration ensures that minimizing a surrogate loss will lead to an optimal solution

- (r, f) is **calibrated** if $R_{\text{0-1-c}}(r, f) = R_{\text{0-1-c}}(r^*, f^*)$
- f is **classification-calibrated** if $f(\boldsymbol{x}) = f^*(\boldsymbol{x})$
- r is **rejection-calibrated** if $\operatorname{sign}[r(\boldsymbol{x})] = \operatorname{sign}[r^*(\boldsymbol{x})]$

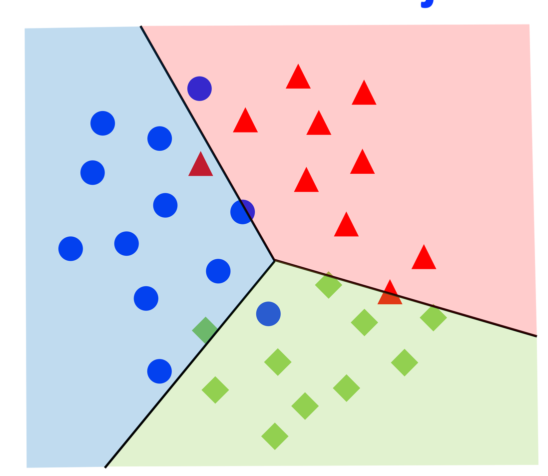
If (r, f) is calibrated, r must be rejection-calibrated.

A minimizer of a surrogate loss should give a calibrated (r, f)

Classifier-rejector approach

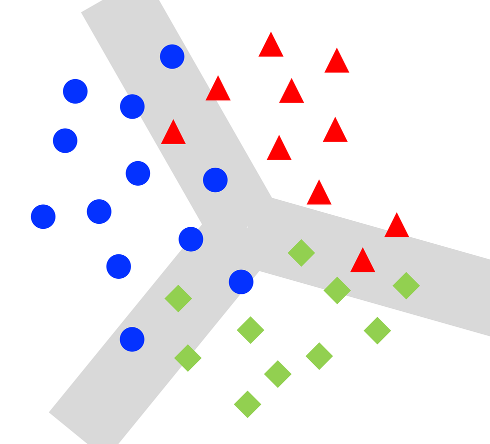
(Cortes+, 2015, 2016)

Classifier and **rejector** are trained simultaneously



Classifier

(colored area indicates classifier prediction)



Rejector

(gray area indicates rejection area)

$(r_{\boldsymbol{\eta}}^\dagger, f_{\boldsymbol{\eta}}^\dagger) = \underset{r \in \mathbb{R},\ \boldsymbol{g} \in \mathbb{R}^K}{\arg\min}\ W(r, f; \boldsymbol{\eta})$ $\quad \boldsymbol{\eta}(\boldsymbol{x}) = [\eta_1(\boldsymbol{x}), \ldots, \eta_K(\boldsymbol{x})]^\top$ $\quad W(r(\boldsymbol{x}), f(\boldsymbol{x}); \boldsymbol{\eta}(\boldsymbol{x})) = \sum_{y \in \mathcal{Y}} \eta_y(\boldsymbol{x})\mathcal{L}(r, f; \boldsymbol{x}, y)$

Corollary 5: (Necessary condition for rejection calibration)

For $\mathcal{L}(r, f; \boldsymbol{x}, y)$ **that is convex with respect to** r **and** $\left.\frac{\partial^2 W(r, f_{\boldsymbol{\eta}}^\dagger; \boldsymbol{\eta})}{\partial r^2}\right|_{r=0} > 0$

$r^\dagger$ **is rejection-calibrated only if both conditions hold:**

Condition (1)
$$\underbrace{\sup_{\boldsymbol{\eta}:\, \max_y \eta_y = 1-c} \left.\frac{\partial W(r, f_{\boldsymbol{\eta}}^\dagger; \boldsymbol{\eta})}{\partial r}\right|_{r=0} = 0}_{\text{Condition for false reject rate to be zero}}$$

Condition (2)
$$\underbrace{\inf_{\boldsymbol{\eta}:\, \max_y \eta_y = 1-c} \left.\frac{\partial W(r, f_{\boldsymbol{\eta}}^\dagger; \boldsymbol{\eta})}{\partial r}\right|_{r=0} = 0}_{\text{Condition for false accept rate to be zero}}$$

Necessary and sufficient condition is also provided in our paper (Theorem 4)

$$\sup_{\boldsymbol{\eta}:\, \max_y \eta_y = 1-c} \left.\frac{\partial W(r, f_{\boldsymbol{\eta}}^\dagger; \boldsymbol{\eta})}{\partial r}\right|_{r=0} = \inf_{\boldsymbol{\eta}:\, \max_y \eta_y = 1-c} \left.\frac{\partial W(r, f_{\boldsymbol{\eta}}^\dagger; \boldsymbol{\eta})}{\partial r}\right|_{r=0} = 0$$

Supremum and infimum values coincide under the same constraint.

When $\max_y \eta_y = 1 - c$

- Binary case: $\boldsymbol{\eta}$ can only be either $[1-c, c]^\top$ or $[c, 1-c]^\top$
- Multiclass case: $\boldsymbol{\eta}$ can be arbitrary. **Both conditions can be very different and do not hold simultaneously!**

Case study:

$\alpha \in \mathbb{R}$ $\beta \in \mathbb{R}$ Hyperparameters

$\phi: \mathbb{R} \to \mathbb{R}$ $\psi: \mathbb{R} \to \mathbb{R}$ Convex margin losses

- **Multiplicative pairwise comparison (MPC) loss:**
$$\mathcal{L}_{\text{MPC}}(r, f; \boldsymbol{x}, y) = \sum_{y' \neq y} \phi\Big(\alpha\big(g_y(\boldsymbol{x}) - g_{y'}(\boldsymbol{x})\big)\Big)\psi(-\alpha r(\boldsymbol{x})) + c\psi(\beta r(\boldsymbol{x}))$$
- **Additive pairwise comparison (APC) loss:**
$$\mathcal{L}_{\text{APC}}(r, f; \boldsymbol{x}, y) = \sum_{y' \neq y} \phi\Big(\alpha\big(g_y(\boldsymbol{x}) - g_{y'}(\boldsymbol{x}) - r(\boldsymbol{x})\big)\Big) + c\psi(\beta r(\boldsymbol{x}))$$

Consider $\phi(z) = \psi(z) = \exp(-z)$

Condition (1) gives $\frac{\beta}{\alpha} = (K-2) + 2\sqrt{(K-1)\frac{1-c}{c}}$

Condition (2) gives $\frac{\beta}{\alpha} = 2\sqrt{\frac{1-c}{c}}$

Equivalent to a condition proved by (Cortes+, 2016) when considering a binary case $(K = 2)$.

In multiclass case, (α, β) **that satisfies both conditions simultaneously does not exist.**

Similar results also hold when using the logistic loss $\phi(z) = \psi(z) = \log(1 + \exp(-z))$.

References

[1] C. K. Chow. On optimum recognition error and reject tradeoff. IEEE Transaction on Information Theory, 1970

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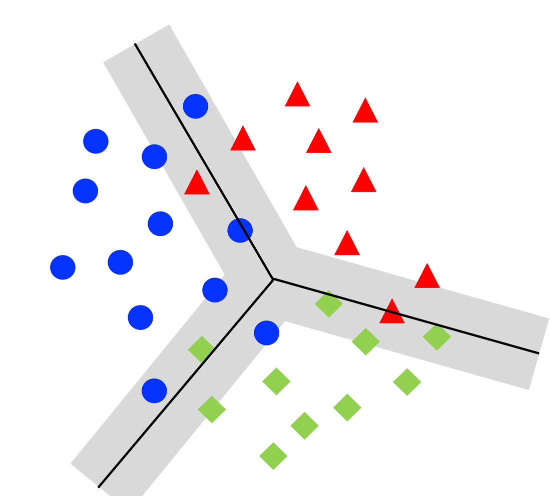
Confidence-based approach

(Bartlett+ 2008, Yuan+ 2010, Ramaswamy+ 2018)

Rejector depends solely on **classifier's** confidence



- **Cross-entropy (CE) loss:**
$$\mathcal{L}_{\text{CE}}(f; \boldsymbol{x}, y) = -g_y(\boldsymbol{x}) + \log \sum_{y' \in \mathcal{Y}} \exp\left(g_{y'}(\boldsymbol{x})\right)$$
- **One-versus-all (OVA) loss:**
$$\mathcal{L}_{\text{OVA}}(f; \boldsymbol{x}, y) = \phi\big(g_y(\boldsymbol{x})\big) + \sum_{y' \neq y} \phi\big(-g_{y'}(\boldsymbol{x})\big)$$
- **Rejector:**
$$r_f(\boldsymbol{x}) = \max_{y \in \mathcal{Y}} \Psi^{-1}(\boldsymbol{g}(\boldsymbol{x})) - (1 - c)$$

$\boldsymbol{g}(\boldsymbol{x}) = [g_1(\boldsymbol{x}), \ldots, g_K(\boldsymbol{x})]^\top$

$\Psi^{-1}: \mathbb{R}^K \to [0, 1]^K$ Inverse link function

$\Psi^{-1}_{y,\,\text{OVA}}(\boldsymbol{g}) = \frac{\phi'(-g_y)}{\phi'(-g_y) + \phi'(g_y)}$

See our paper for conditions of ϕ .

$\Psi^{-1}_{y,\,\text{CE}}(\boldsymbol{g}) = \frac{\exp(g_y)}{\sum_{y' \in \mathcal{Y}} \exp(g_{y'})}$

Softmax function

Excess risk:

$$\Delta R_{\text{0-1-c}}(r_f, f) = R_{\text{0-1-c}}(r_f, f) - \inf_{f':\text{measurable}} R_{\text{0-1-c}}(r_f, f)$$
$$\Delta R_\ell(f) = R_\ell(f) - \inf_{f':\text{measurable}} R_\ell(f')$$

If $\Delta R_{\text{0-1-}c}$ can be upper-bounded by ΔR_ℓ,

-> then the **minimizer of both risks are identical.**

Excess risk bound of OVA loss:

$$(2C)^{-s}\Delta R_{\text{0-1-c}}(r_f, f)^s \leq \Delta R_{\text{OVA}}(f)$$

Loss Name	$\phi(z)$	C	s
Logistic	$\log(1 + \exp(-z))$	$\frac{1}{2}$	2
Exponential	$\exp(-z)$	$\frac{1}{\sqrt{2}}$	2
Squared	$(1-z)^2$	$\frac{1}{2}$	2
Squared Hinge	$(1-z)_+^2$	$\frac{1}{2}$	2

Excess risk bound of CE loss:

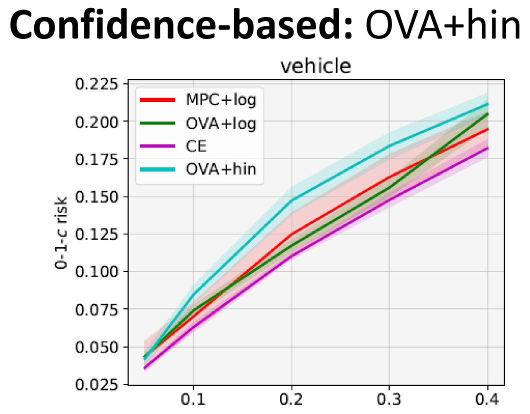
$$\frac{1}{2}\Delta R_{\text{0-1-c}}(r_f, f)^2 \leq \Delta R_{\text{CE}}(f)$$

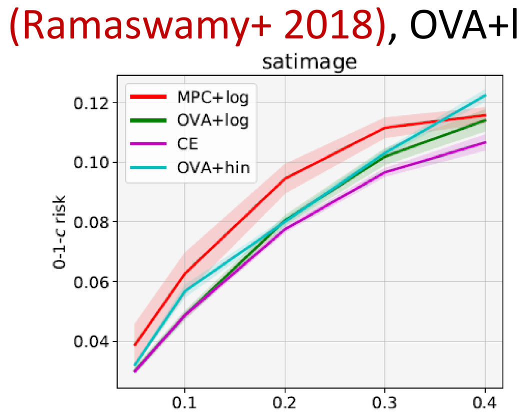
See our paper for more results, e.g., estimation error bound using Rademacher complexity.

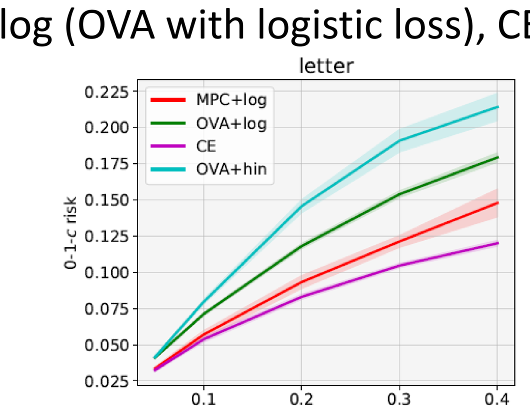
Experiment

Classifier-rejector: MPC+log (MPC with logistic loss), APC+log (APC with logistic loss)

Confidence-based: OVA+hin (Ramaswamy+ 2018), OVA+log (OVA with logistic loss), CE







Accuracy of non-rejected data: "- (-)" indicates all data were rejected.

dataset	c	APC+log	MPC+log	OVA+log	CE
vehicle	0.05	- (-)	96.6 (2.3)	100 (0.0)	**100 (0.0)**
	0.2	98.4 (1.9)	92.4 (3.0)	97.9 (0.7)	**97.4 (0.1)**
	0.4	**89.1 (2.9)**	85.3 (4.2)	90.2 (1.6)	**91.7 (0.9)**
satimage	0.05	**99.1 (0.2)**	97.2 (1.4)	**98.7 (0.1)**	**98.3 (0.1)**
	0.2	95.0 (1.0)	92.6 (1.2)	96.2 (0.2)	**95.7 (0.1)**
	0.4	91.5 (0.7)	89.0 (1.1)	92.2 (0.3)	**91.8 (0.2)**
yeast	0.05	- (-)	- (-)	- (-)	- (-)
	0.2	- (-)	- (-)	- (-)	**80.6 (6.2)**
	0.4	- (-)	- (-)	75.0 (3.9)	**76.6 (1.7)**

dataset	c	APC+log	MPC+log	OVA+log	CE
covtype	0.05	**79.5 (2.1)**	79.8 (1.7)	**82.1 (2.7)**	**82.0 (3.2)**
	0.2	74.0 (1.8)	73.8 (1.0)	74.9 (1.4)	**77.1 (0.3)**
	0.4	**69.8 (1.3)**	64.9 (3.4)	**68.7 (1.1)**	**69.4 (1.8)**
letter	0.05	**99.8 (0.1)**	98.6 (0.2)	**99.6 (0.2)**	**99 8 (0.0)**
	0.2	97.9 (0.3)	96.9 (0.5)	**98.3 (0.2)**	**98.4 (0.1)**
	0.4	**95.2 (0.5)**	94.6 (3.8)	94.6 (0.2)	**94.9 (0.3)**