Induction Motor - I_{dr} Full Derivation & Stamp

Friday, March 5, 2021 8:20 AM

The voltage equation for the d phase of the rotor in the stationary reference frame:

$$v_{dr} = R_r I_{dr} + p\psi_{dr} - \psi_{qr}\omega_r$$
$$\psi_{dr} = (L_{lr} + L_m)I_{dr} + L_m I_{ds}$$
$$\psi_{qr} = (L_{lr} + L_m)I_{qr} + L_m I_{qs}$$
$$L_r = L_{lr} + L_m$$

To solve any power grid or circuit element with nonlinearities through time, remember **EDLS**

- **E**xpand
- **D**iscretize
- **L**inearize
- **S**tamp

1. **Step 1 - Expand**: Expand the differential algebraic voltage equation.

   In our induction motor, the rotor circuit is shorted, so $V_{dr} = 0$

$$\rightarrow 0 = R_r i_{dr}(t) + p\left(L_m i_{ds}(t) + L_r i_{dr}(t)\right) - \omega_r(t)\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right)$$

$$\rightarrow 0 = L_r \frac{\partial i_{dr}(t)}{\partial t} + L_m \frac{\partial i_{ds}(t)}{\partial t} - \omega_r(t)\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right) + R_r i_{dr}(t)$$

$$\rightarrow -R_r i_{dr}(t) = L_r \frac{\partial i_{dr}(t)}{\partial t} + L_m \frac{\partial i_{ds}(t)}{\partial t} - \omega_r\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right)$$

$$\rightarrow -R_r i_{dr}(t) + \omega_r\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right) = \overbrace{L_r \frac{\partial i_{dr}(t)}{\partial t} + L_m \frac{\partial i_{ds}(t)}{\partial t}}^{V_{lrdr}(t) + V_{lmds}}$$

2. **Step 2 - Discretize**: Apply the trapezoidal rule to eliminate the differential terms and obtain an algebraic equation.

   First, multiply both sides of the equation by δt:

$$\rightarrow \left(-R_r i_{dr}(t) + \omega_r\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right)\right)\partial t = L_r \partial i_{dr}(t) + L_m \partial i_{ds}(t)$$

$$\rightarrow \int_t^{t+\Delta t} \left(-R_r i_{dr}(\tau) + \omega_r(\tau)\left(L_r i_{qr}(\tau) + L_m i_{qs}(\tau)\right)\right) d\tau = L_r \partial i_{dr}(t) + L_m \partial i_{ds}(t)$$

Apply the Trapezoidal Integration approximation:

$$\rightarrow -R_r\left(i_{dr}(t+\Delta t) + i_{dr}(t)\right) + \left[\left(\omega_r(t+\Delta t)\left(L_r i_{qr}(t+\Delta t) + L_m i_{qs}(t+\Delta t)\right)\right) + \left(\omega_r(t)\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right)\right)\right] = \frac{2L_r}{\Delta t}\left(i_{dr}(t+\Delta t) - i_{dr}(t)\right) + \frac{2L_m}{\Delta t}\left(i_{ds}(t+\Delta t) - i_{ds}(t)\right)$$

$$\rightarrow \underbrace{-R_r i_{dr}(t) + \omega_r(t)\left(L_r i_{qr}(t) + L_m i_{qs}(t)\right)}_{V_{lrdr}(t) + V_{lmds}(t)} = \left(R_r + \frac{2L_r}{\Delta t}\right) i_{dr}(t+\Delta t) + \frac{2L_m}{\Delta t} i_{ds}(t+\Delta t) - \omega_r(t+\Delta t)\left(L_r i_{qr}(t+\Delta t) + L_m i_{qs}(t+\Delta t)\right) - \frac{2L_r}{\Delta t} i_{dr}(t) - 2L_m i_{ds}(t)$$

$$\rightarrow V_{lrdr}(t) + V_{lmds}(t) = \left(R_r + \frac{2L_r}{\Delta t}\right) i_{dr}(t+\Delta t) + \frac{2L_m}{\Delta t} i_{ds}(t+\Delta t) - \omega_r(t+\Delta t)\left(L_r i_{qr}(t+\Delta t) + L_m i_{qs}(t+\Delta t)\right) - \frac{2L_r}{\Delta t} i_{dr}(t) - \frac{2L_m}{\Delta t} i_{ds}(t)$$

$$\rightarrow 0 = \left(R_r + \frac{2L_r}{\Delta t}\right) i_{dr}(t+\Delta t) + \frac{2L_m}{\Delta t} i_{ds}(t+\Delta t) - \omega_r(t+\Delta t)\left(L_r i_{qr}(t+\Delta t) + L_m i_{qs}(t+\Delta t)\right) + hist_{dr}(t)$$

Where $hist_{dr}(t) = -V_{lrdr}(t) - V_{lmds}(t) - \frac{2L_r}{\Delta t} i_{dr}(t) - \frac{2L_m}{\Delta t} i_{ds}(t)$

3. **Step 3 - Linearize**: Perform a 1st order Taylor Expansion to linearize your entire nonlinear algebraic equation.
$f(x^{k+1}) = f(x^k) + f'(x^k)\cdot(x^{k+1}-x^k)$

$$f_{dr}\left(I_{dr}^{K}(t+\Delta t), I_{ds}^{K}(t+\Delta t), \omega_r^{K}(t+\Delta t), I_{qr}^{K}(t+\Delta t), I_{qs}^{K}(t+\Delta t)\right) = \left(R_r + \frac{2L_r}{\Delta t}\right) i_{dr}(t+\Delta t) + \frac{2L_m}{\Delta t} i_{ds}(t+\Delta t) - \omega_r(t+\Delta t)\left(L_r i_{qr}(t+\Delta t) + L_m i_{qs}(t+\Delta t)\right) + hist_{dr}(t)$$

$$\begin{aligned}
\rightarrow f_{dr}\left(I_{dr}^{K+1}(t+\Delta t), I_{ds}^{K+1}(t+\Delta t), \omega_r^{K+1}(t+\Delta t), I_{qr}^{K+1}(t+\Delta t), I_{qs}^{K+1}(t+\Delta t)\right) = {} & \overbrace{f_{dr}\left(I_{dr}^{K}(t+\Delta t), I_{ds}^{K}(t+\Delta t), \omega_r^{K}(t+\Delta t), I_{qr}^{K}(t+\Delta t), I_{qs}^{K}(t+\Delta t)\right)}^{f_{dr}^{K}} \\
& + \frac{\partial f_{dr}}{\partial I_{dr}}\left(I_{dr}^{K+1}(t+\Delta t) - I_{dr}^{K}(t+\Delta t)\right) \\
& + \frac{\partial f_{dr}}{\partial I_{ds}}\left(I_{ds}^{K+1}(t+\Delta t) - I_{ds}^{K}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial I_{qr}}\right|_{\omega_r^K}\left(I_{qr}^{K+1}(t+\Delta t) - I_{qr}^{K}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial I_{qs}}\right|_{\omega_r^K}\left(I_{qs}^{K+1}(t+\Delta t) - I_{qs}^{K}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial \omega_r}\right|_{I_{qs}^K, I_{qr}^K}\left(\omega_r^{K+1}(t+\Delta t) - \omega_r^{K}(t)\right)
\end{aligned}$$

4. **Step 4 - Stamp**: Relate linearized equations to circuit elements. Add elements systematically to the Y matrix.

We now have a linearized equation. We must separate our final equation into our knowns (values at iteration k and time $t+\Delta t$ or values just at time t) and unknowns (values at iteration k and time $t+\Delta t$). Rearranging our previous equation and relating our partial derivatives to their corresponding circuit elements we get:

$$\begin{aligned}
\rightarrow f_{dr}\left(I_{dr}^{K+1}(t+\Delta t), I_{ds}^{K+1}(t+\Delta t), \omega_r^{K+1}(t+\Delta t), I_{qr}^{K+1}(t+\Delta t), I_{qs}^{K+1}(t+\Delta t)\right) = {} & \frac{\partial f_{dr}}{\partial I_{dr}}\left(I_{dr}^{K+1}(t+\Delta t)\right) \\
& + \frac{\partial f_{dr}}{\partial I_{ds}}\left(I_{ds}^{K+1}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial I_{qr}}\right|_{\omega_r^K}\left(I_{qr}^{K+1}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial I_{qs}}\right|_{\omega_r^K}\left(I_{qs}^{K+1}(t+\Delta t)\right) \\
& + \left.\frac{\partial f_{dr}}{\partial \omega_r}\right|_{I_{qs}^K, I_{qr}^K}\left(\omega_r^{K+1}(t+\Delta t)\right) \\
& + f_{dr}^{K} - \frac{\partial f_{dr}}{\partial I_{dr}}\left(I_{dr}^{K}(t+\Delta t)\right) \\
& - \frac{\partial f_{dr}}{\partial I_{ds}}\left(I_{ds}^{K}(t+\Delta t)\right) \\
& - \left.\frac{\partial f_{dr}}{\partial I_{qr}}\right|_{\omega_r^K}\left(I_{qr}^{K}(t+\Delta t)\right) \\
& - \left.\frac{\partial f_{dr}}{\partial I_{qs}}\right|_{\omega_r^K}\left(I_{qs}^{K}(t+\Delta t)\right) \\
& - \left.\frac{\partial f_{dr}}{\partial \omega_r}\right|_{I_{qs}^K, I_{qr}^K}\left(\omega_r^{K}(t+\Delta t)\right)
\end{aligned}$$

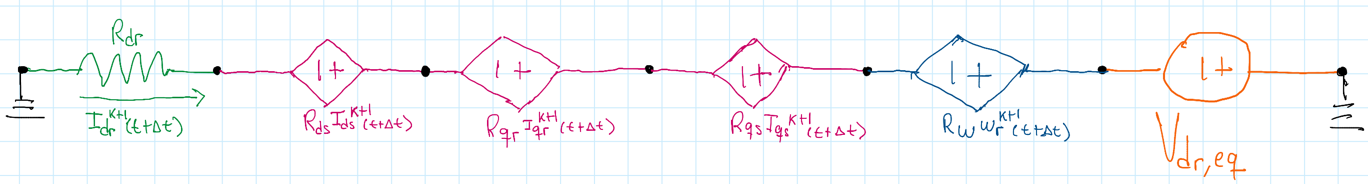
Resistance
R_{dr}

Current Controlled Voltage Sources
$R_{ds}\, I_{ds}^{K+1}(t+\Delta t)$
$R_{qr}\, I_{qr}^{K+1}(t+\Delta t)$
$R_{qs}\, I_{qs}^{K+1}(t+\Delta t)$

Voltage Controlled Voltage Source
R_{ω}

Independent Voltage Source
$V_{dr,eq}$

From our equations above, we know our final linearized equivalent circuit at k+1 that we stamp looks like this:



We can use KVL to stamp the circuit in the matrix. We will stamp each circuit element with respect to its controlling node in the Y matrix which houses all our partial derivatives at iteration $k+1$ and time $t+\Delta t$. We will stamp our independent voltage source (our known values evaluated at iteration k and time $t+\Delta t$) in our J vector. If we look at just the row corresponding to I_{dr}, our equivalent circuit stamps should look like this:

In row $I_{dr(t+\Delta t)}^{k+1}$ of our Y matrix

$$\begin{array}{c c} & \begin{array}{ccccc} I_{dr} & I_{ds} & I_{qr} & I_{qs} & \omega_r \end{array} \\ I_{dr} & \begin{bmatrix} R_{dr} & R_{ds} & R_{qr} & R_{qs} & R_{\omega} \end{bmatrix} \end{array}$$

In row I_{dr} of our J Vector

$$\begin{array}{cc} I_{dr} & \begin{bmatrix} -V_{dr,eq} \end{bmatrix} \end{array}$$