GW calculations and

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Outline

I. Introduction: going beyond DFT

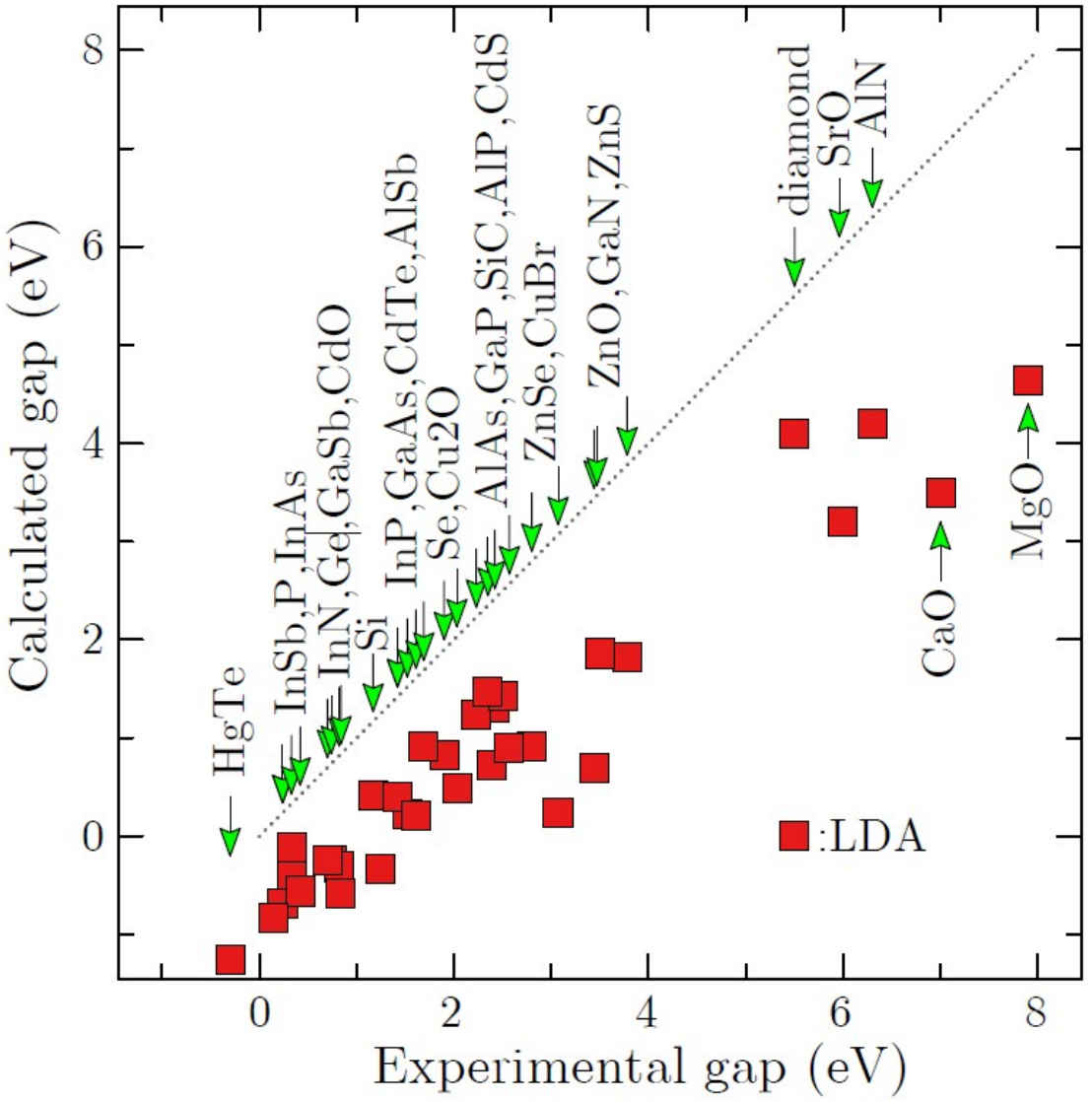
II. Introduction of the Green's function

III. Exact Hedin's equations and the *GW* approximation

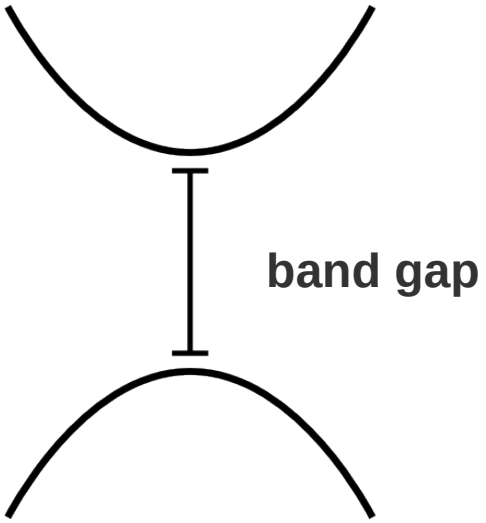
IV. Calculating the *GW* self-energy in practice

V. Applications

Standard DFT has unfortunately some shortcomings



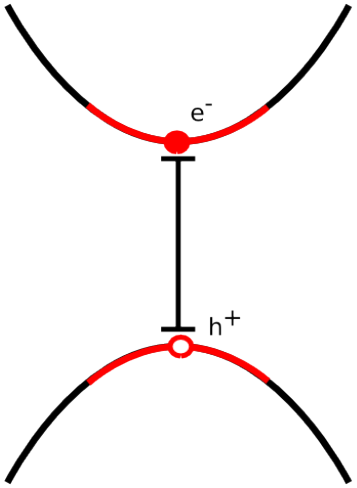
after van Schilfgaarde *et al* PRL **96** 226402 (2008)



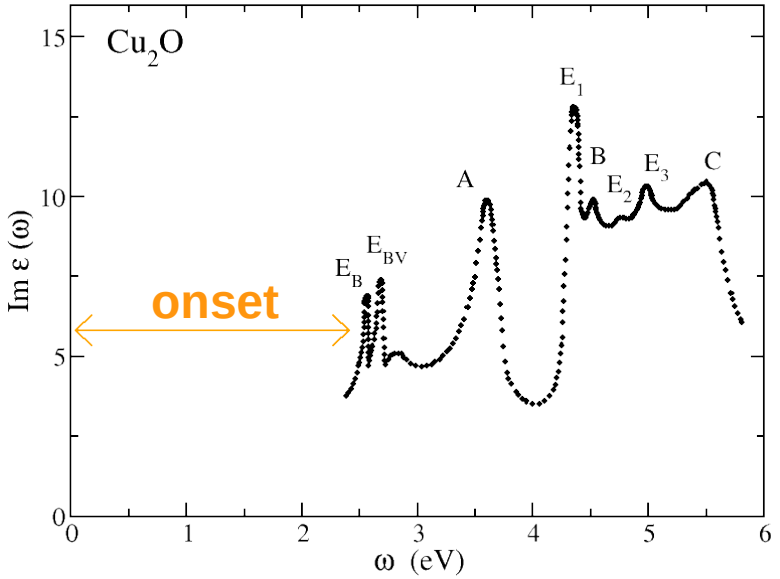
Band gap problem!

A pervasive problem

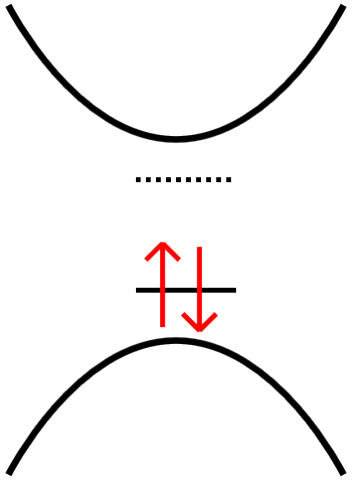
Effective masses
for transport in semiconductors



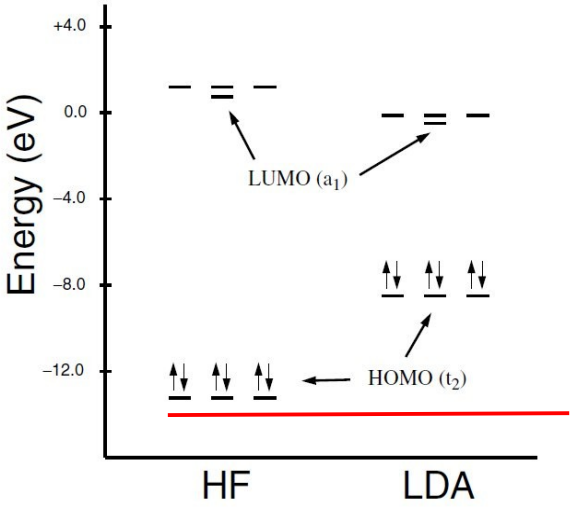
Optical absorption



Defect formation energy,
dopant solubility



Photoemission



Exp.

FIG. 1. Single-particle Hartree-Fock and local density approximation eigenvalue spectra (eV) for the SiH_4 molecule.

Gap re-normalization by a (metallic) substrate

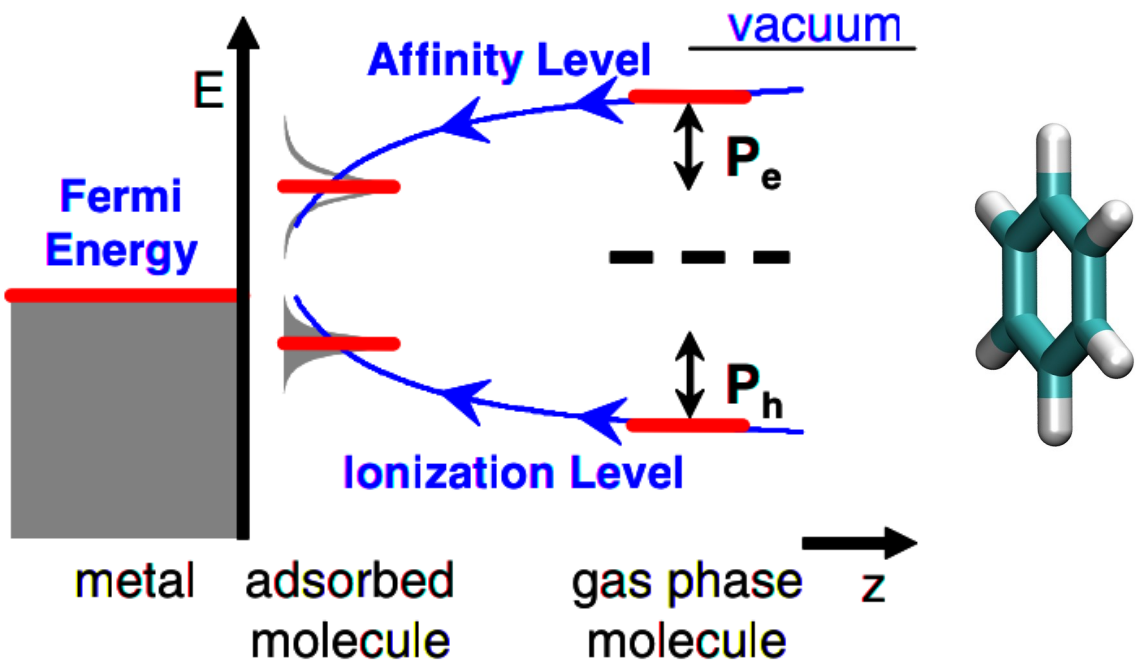


FIG. 1 (color online). Schematic energy level diagram indicating polarization shifts in the frontier energy levels (ionization and affinity) of a molecule upon adsorption on a metal surface.

Benzene deposited on copper, gold, graphite

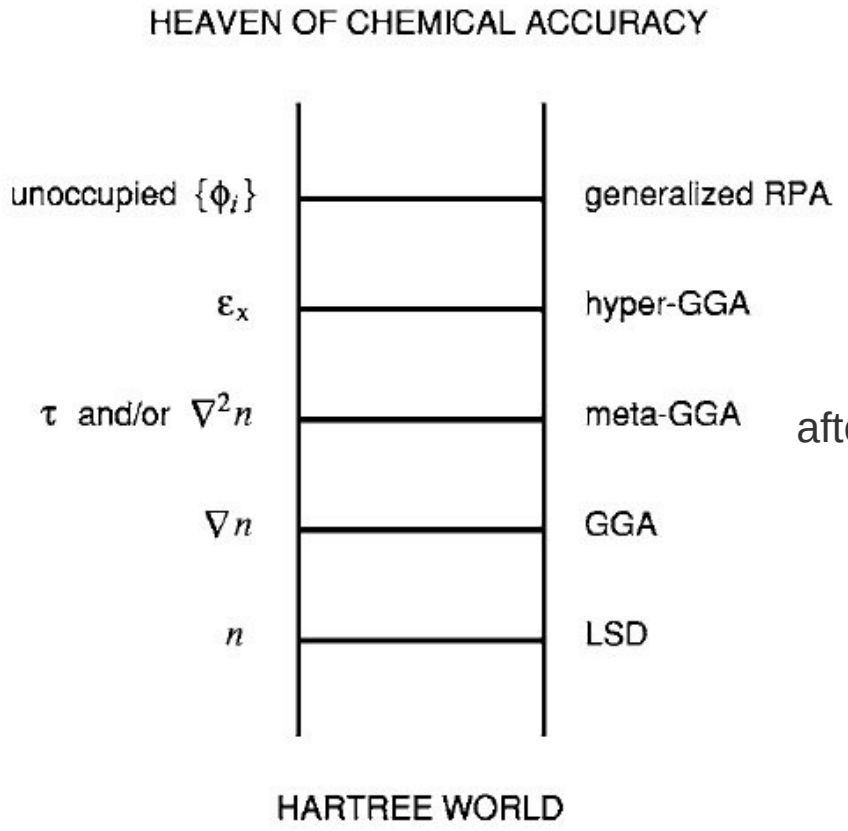
Neaton, Hybertsen, Louie PRL (2006)

How do go beyond within the DFT framework?

Not easy to find improvement within DFT framework

There is no such thing as a perturbative expansion

Perdew's Jacob's ladder does not help for the band gap



after J. Perdew JCP (2005).

FIG. 1. Jacob's ladder of density functional approximations to the exchange-correlation energy.

Need to change the overall framework!

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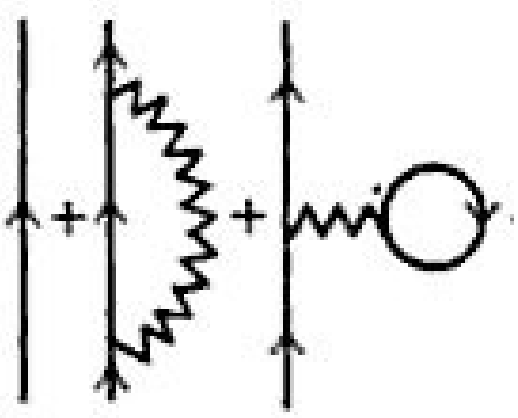
Many-body perturbation theory

Historically older than the DFT (from the 40-50's)!
Big names: Feynman, Schwinger, Hubbard, Hedin, Lundqvist

Green's functions
= propagator

$$G(\boldsymbol{r}t, \boldsymbol{r}'t') =$$



The Green's function

Exact ground state wavefunction: $|N,0\rangle$

Creation, annihilation operator: $\Psi^\dagger(\boldsymbol{r}\,t),\ \Psi(\boldsymbol{r}\,t)$

1. $\Psi^\dagger(\boldsymbol{r}\,t)|N,0\rangle$ is a (N+1) electron wavefunction not necessarily in the ground state

2. $\Psi^\dagger(\boldsymbol{r}'\,t')|N,0\rangle$ is another (N+1) electron wavefunction

Let's compare the two of them!

Green's function definition

$$\underbrace{\langle N,0|\Psi(\boldsymbol{r}\,t)}_{1}\underbrace{\Psi^{\dagger}(\boldsymbol{r}'\,t')|N,0\rangle}_{2}$$

$$=iG^{e}(\boldsymbol{r}\,t,\boldsymbol{r}'\,t') \quad \text{for} \quad t>t'$$

Mesures how an extra electron propagates from (*r't'*) to (*rt*).

Green's function definition

$$\underbrace{\langle N,0|\Psi^{\dagger}(\boldsymbol{r}'t')}_{2}\underbrace{\Psi(\boldsymbol{r}t)|N,0\rangle}_{1}$$

$$= i G^{h}(\boldsymbol{r}'t',\boldsymbol{r}t) \quad \text{for} \quad t' > t$$

Mesures how a missing electron (= a hole) propagates from (*rt*) to (*r't'*).

Final expression for the Green's function

$$iG(\boldsymbol{r}t,\boldsymbol{r}'t') = \langle N,0 | T\left[\Psi(\boldsymbol{r}t)\Psi^{\dagger}(\boldsymbol{r}'t')\right] | N,0 \rangle$$

time-ordering operator

$$G(\boldsymbol{r}t,\boldsymbol{r}'t') = G^{e}(\boldsymbol{r}t,\boldsymbol{r}'t') - G^{h}(\boldsymbol{r}'t',\boldsymbol{r}t)$$

Compact expression that describes both the propagation of an extra electron and an extra hole

Lehman representation

$$iG(\boldsymbol{r},\boldsymbol{r}',t-t') = \langle N,0 | T\left[\Psi(\boldsymbol{r}t)\Psi^{+}(\boldsymbol{r}'t')\right] | N,0\rangle$$

Closure relation

$$\sum_{M,i} |M,i\rangle\langle M,i|$$

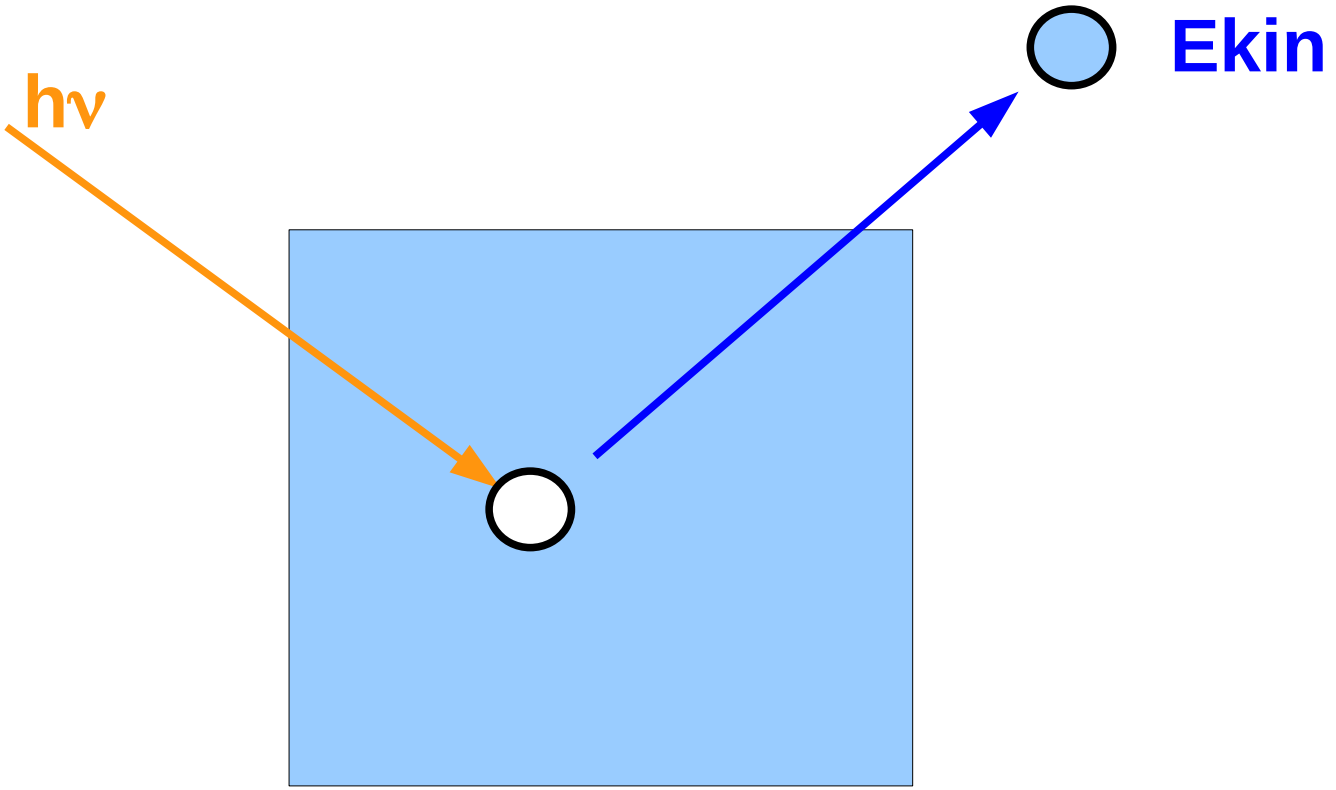
Lehman representation:

$$G(\boldsymbol{r},\boldsymbol{r}',\omega) = \sum_{i} \frac{f_i(\boldsymbol{r})\, f_i^{*}(\boldsymbol{r}')}{\omega - \epsilon_i \pm i\eta}$$

where $\epsilon_i = \begin{cases} E(N+1,i) - E(N,0) \\ E(N,0) - E(N-1,i) \end{cases}$

Exact excitation energies!

Related to photoemission spectroscopy

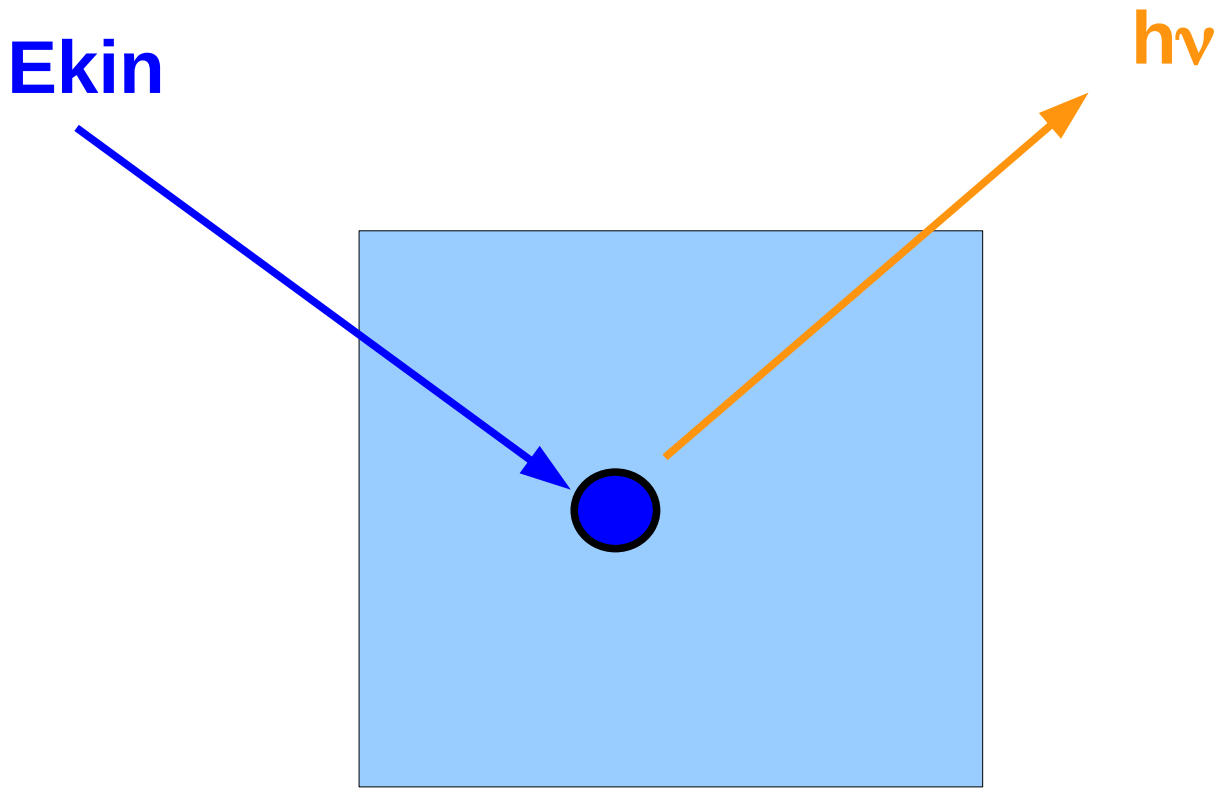


Energy conservation: before after

$$h\nu + E(N,0) = E_{kin} + E(N-1,i)$$

Quasiparticle energy: $$\epsilon_i = E(N,0) - E(N-1,i) = E_{kin} - h\nu$$

And inverse photoemission spectroscopy



Energy conservation: before after

$$E_{kin} + E(N,0) = h\nu + E(N+1,i)$$

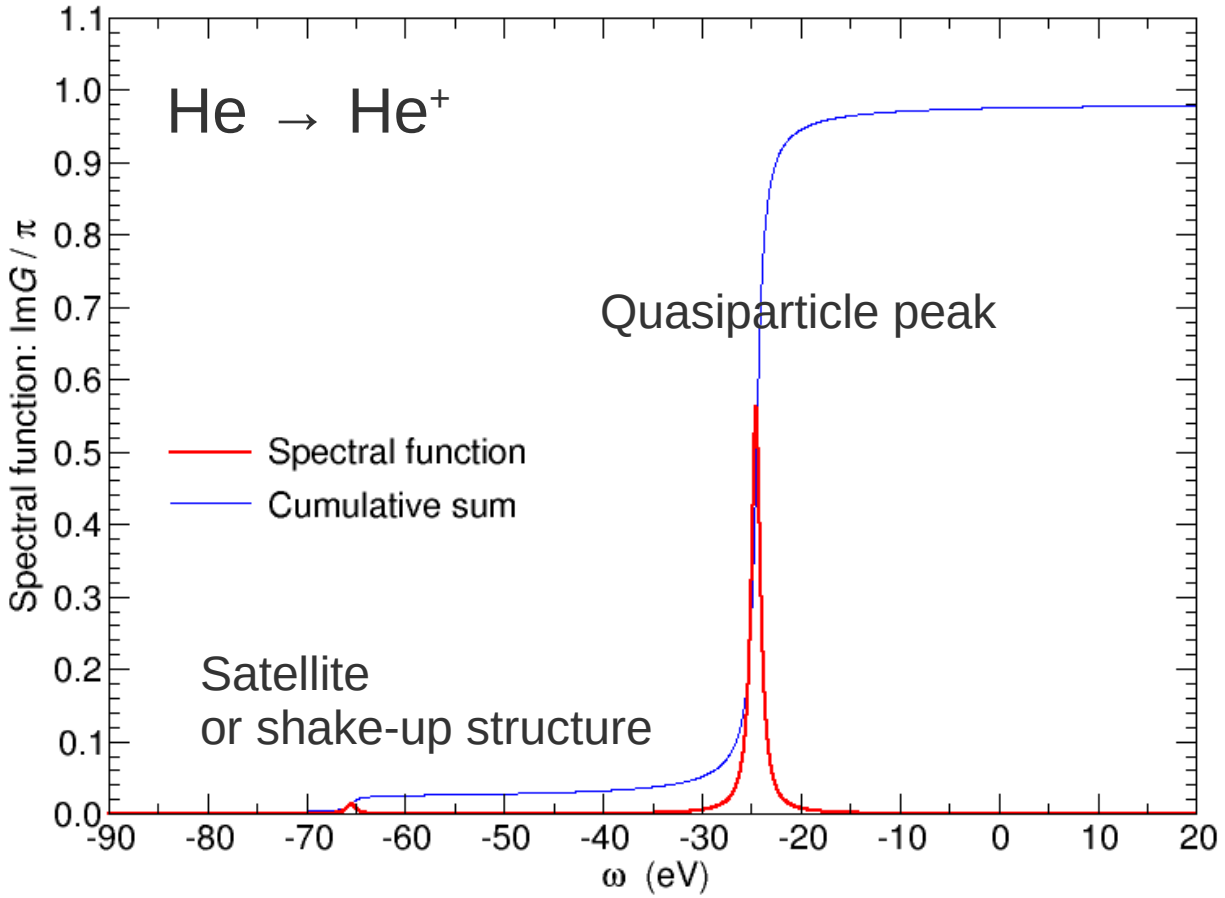
Quasiparticle energy: $$\epsilon_i = E(N+1,i) - E(N,0) = E_{kin} - h\nu$$

Exact realization of the Lehman decomposition

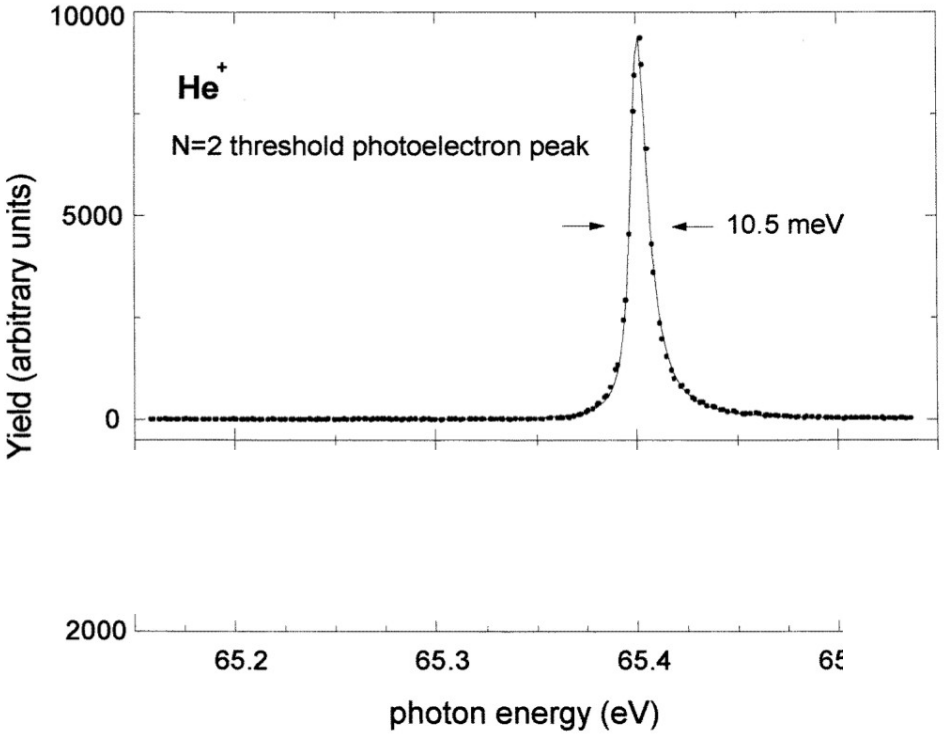
$$\langle m|G^h(\omega)|m\rangle = \sum_i \frac{\langle N\,0|\hat{c}_m^+|N-1\,i\rangle\langle N-1\,i|\hat{c}_m|N\,0\rangle}{\omega-\epsilon_i-i\eta}$$

$$N=2$$
$$N-1=1$$
$$m=1s$$

Obtained from FCI calculations



Satellites in reality?

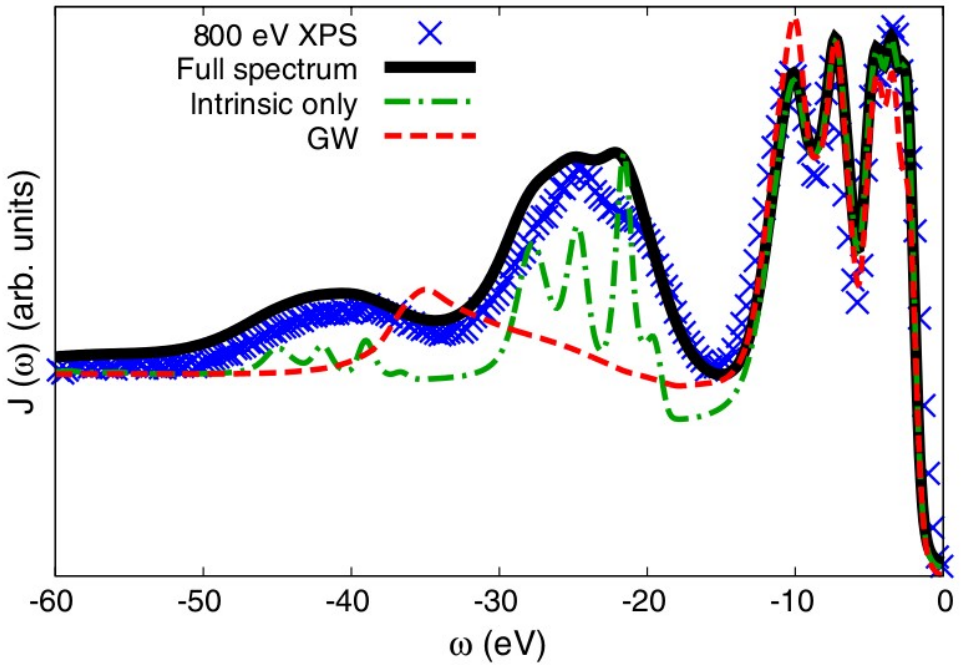


Helium gas

Thompson *et al.*
J. Phys. B: At. Mol. Opt. Phys. 1998

Silicon crystal

Guzzo *et al.* PRL 2011



Other properties of the Green's function

Get the electron density:

$$\rho(\boldsymbol{r}) = -i\,G(\boldsymbol{r}\,t^{-},\boldsymbol{r},t)$$

Galitskii-Migdal formula for the total energy:

$$E_{total} = \frac{1}{\pi}\int_{-\infty}^{\mu} d\,\omega\,\mathrm{Tr}\left[\left(\omega - h_0\right)\mathrm{Im}\,G\left(\omega\right)\right]$$

Expectation value of any 1 particle operator (local or non-local)

$$\langle O \rangle = \lim_{t \to t'} Tr\left[O\,G\right]$$

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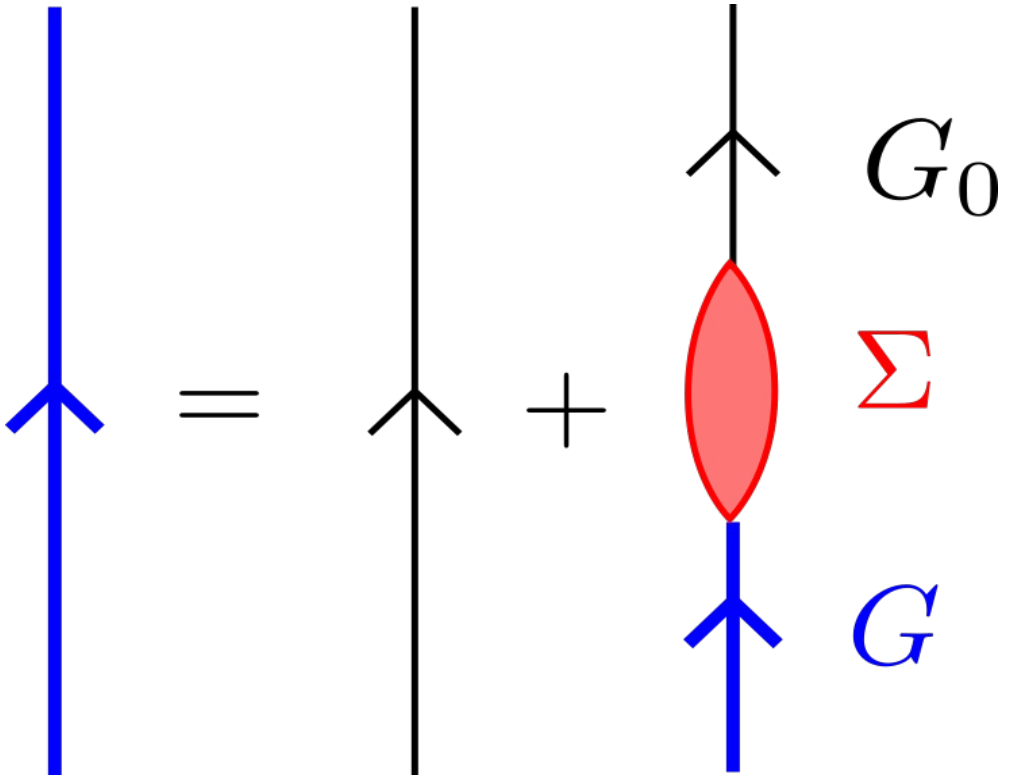
Dyson equation for the exact Green's function

Imagine there exists an operator that generates the exact *G*



$$G(1,2) = G_0(1,2) + \int d(34)\, G_0(1,3)\Sigma(3,4)G(4,2)$$

This operator is the famous “self-energy”:

- non-local in space
- time-dependent
- non-Hermitian

Everything else now deals with finding expressions for the self-energy!

A hierarchy of equations of motion

In fact there is an exact expression for the self-energy as a function of the **two-particle Green's function**

$$\left[G_0^{-1} - \Sigma\right] G = 1$$

$$\left[G_0^{-1} - G_2\right] G = 1$$

$$G_2(1,2;3,4) = \langle N,0 | T[\Psi(1)\Psi(2)\Psi^+(3)\Psi^+(4)] | N,0 \rangle$$

And try to guess the equation of motion for the two-particle Green's function?

$$G_2 \quad \text{needs} \quad G_3$$

$$G_3 \quad \text{needs} \quad G_4$$

$$G_4 \quad \text{needs} \quad G_5$$

.....

An expression for the self-energy

Trick due to Schwinger (1951):

- Introduce a small external potential *U* (that will be made equal to zero at the end)
- Calculate the variations of *G* with respect to *U*

$$G_2(1,3;2,3^+) = G(1,2)G(3,3^+) - \frac{\delta G(1,2)}{\delta U(3)}$$

Obtain a perturbation theory with basic ingredients *G* and *v*
1st order is Hartree-Fock
2nd order is MP2

However MP2 diverges for metals!

Trick due to Hubbard+Hedin (late 1950's – early 1960's):

- Introduce the electrostatic response *V* to *U*
- Calculate the variations of *G* with respect to *V*

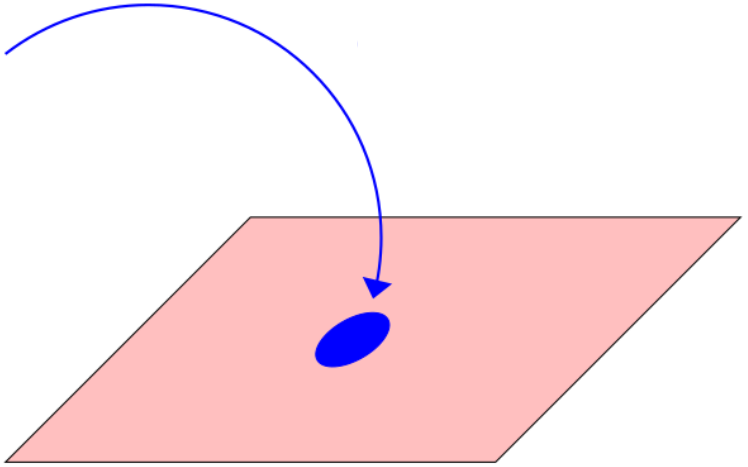
$$V(1) = U(1) - i\int d2\, v(1,2)\delta G(2,2)$$

Obtain **a new renormalized perturbation theory** with basic ingredients *G* and *W*

1st order is *GW*

Shifting from U to V

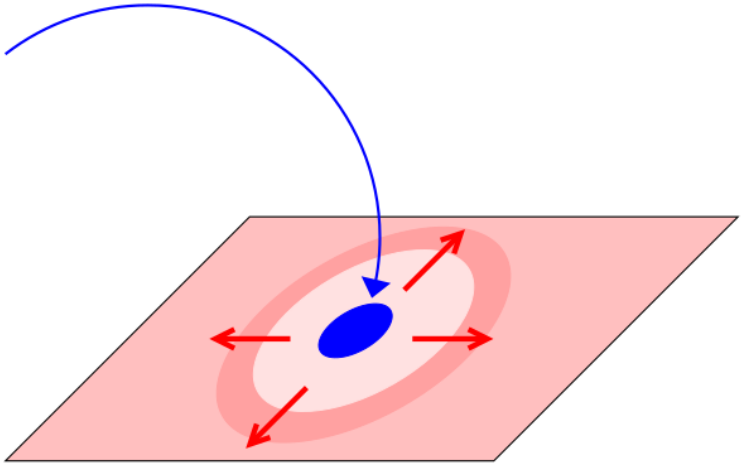
$$U(1)=\varepsilon\,\delta(\boldsymbol{r}-\boldsymbol{r_1})\delta(t-t_1)$$



Everything is functional of U

$$G[U]$$

$$U(1)=\varepsilon\,\delta(\boldsymbol{r}-\boldsymbol{r_1})\delta(t-t_1)$$



$$V(1)=U(1)+\int d\boldsymbol{r}\, v(r_1-r)\,\delta\rho(\boldsymbol{r})$$

V also includes the electrostatic response

Everything is functional of V

$$G[V]$$

Hedin's coupled equations

5 coupled equations: $1=(\boldsymbol{r}_1 t_1 \sigma_1)$ $2=(\boldsymbol{r}_2 t_2 \sigma_2)$

$$G(1,2)=G_0(1,2)+\int d34\, G_0(1,3)\Sigma(3,4)G(4,2)$$ **Dyson equation**

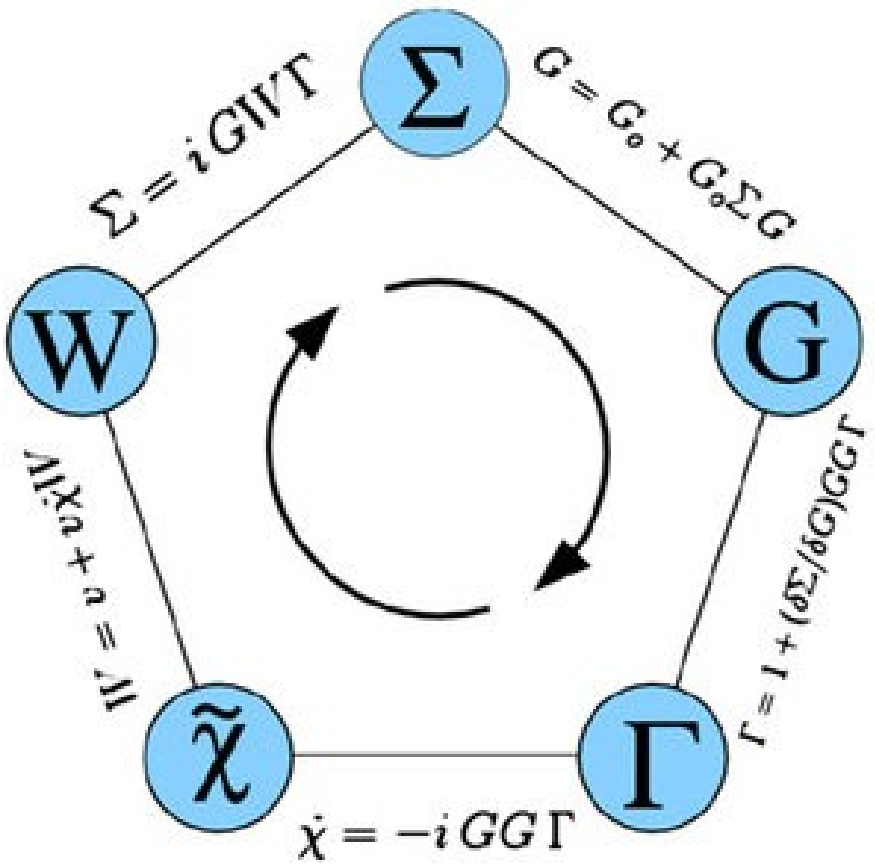
$$\Sigma(1,2)=i\int d34\, G(1,3)W(1,4)\Gamma(4,2,3)$$ **self-energy**

$$\Gamma(1,2,3)=\delta(1,2)\delta(1,3)+\int d4567\frac{\delta\Sigma(1,2)}{\delta G(4,5)}G(4,6)G(5,7)\Gamma(6,7,3)$$ **vertex**

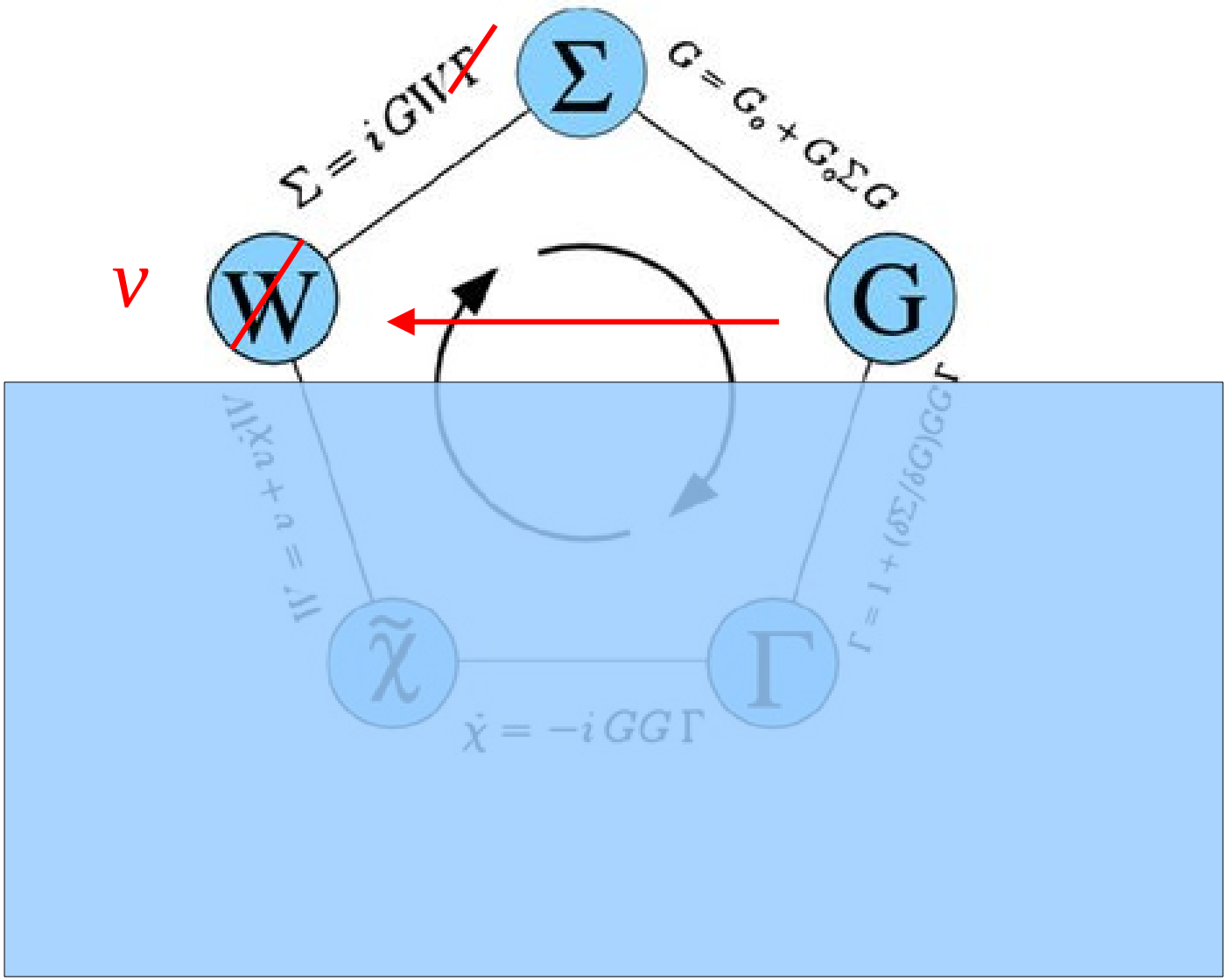
$$\chi_0(1,2)=-i\int d34\, G(1,3)G(4,1)\Gamma(3,4,2)$$ **polarizability**

$$W(1,2)=v(1,2)+\int d34\, v(1,3)\chi_0(3,4)W(4,2)$$ **screened Coulomb interaction**

Hedin's pentagram

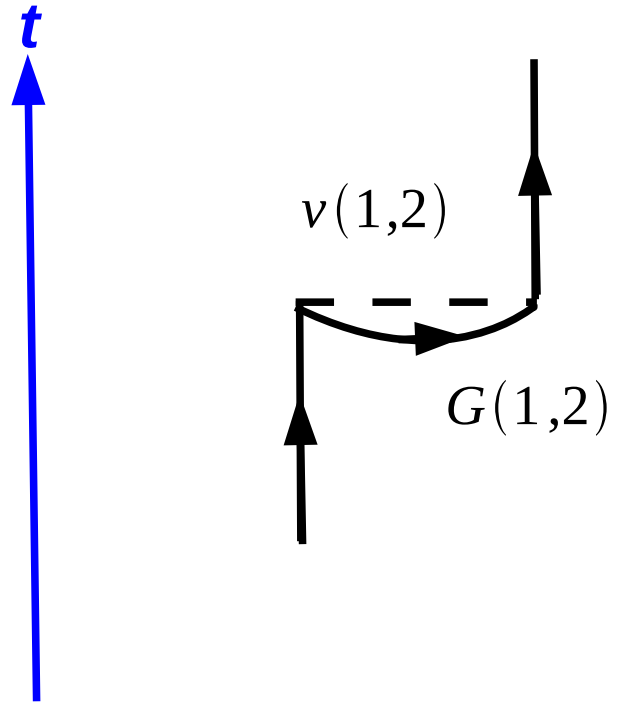


Hedin's pentagram approximated



Simplest approximation

$$\Sigma(1,2) = iG(1,2)v(1^+,2)$$ ⟹ Fock exchange



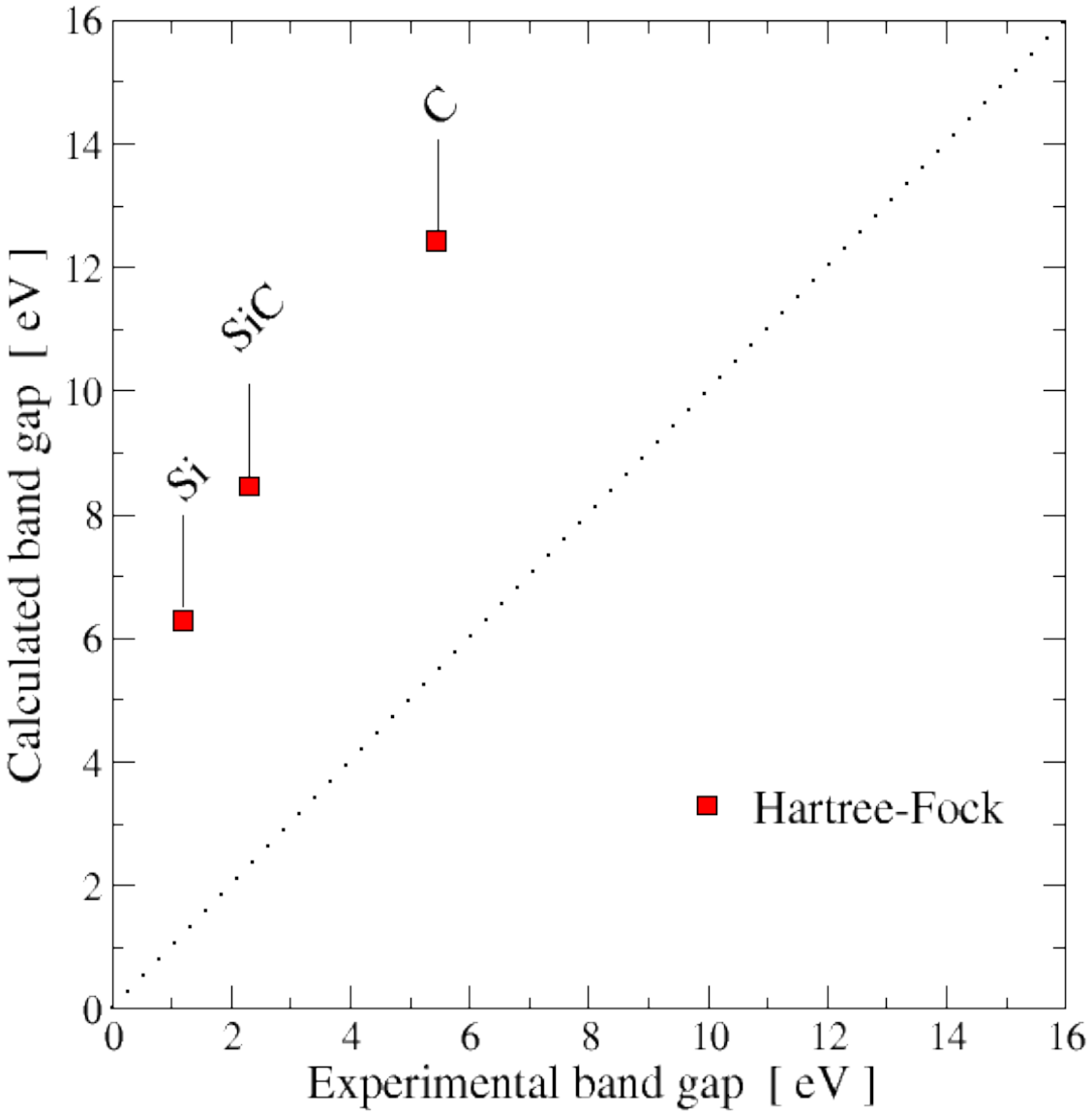
Dyson equation:

$$G = G_0 + G_0 \Sigma G$$

$$G = G_0 + G_0 \Sigma G_0 + \ldots$$

Not enough: Hartree-Fock is known to perform poorly for solids

Hartree-Fock approximation for band gaps



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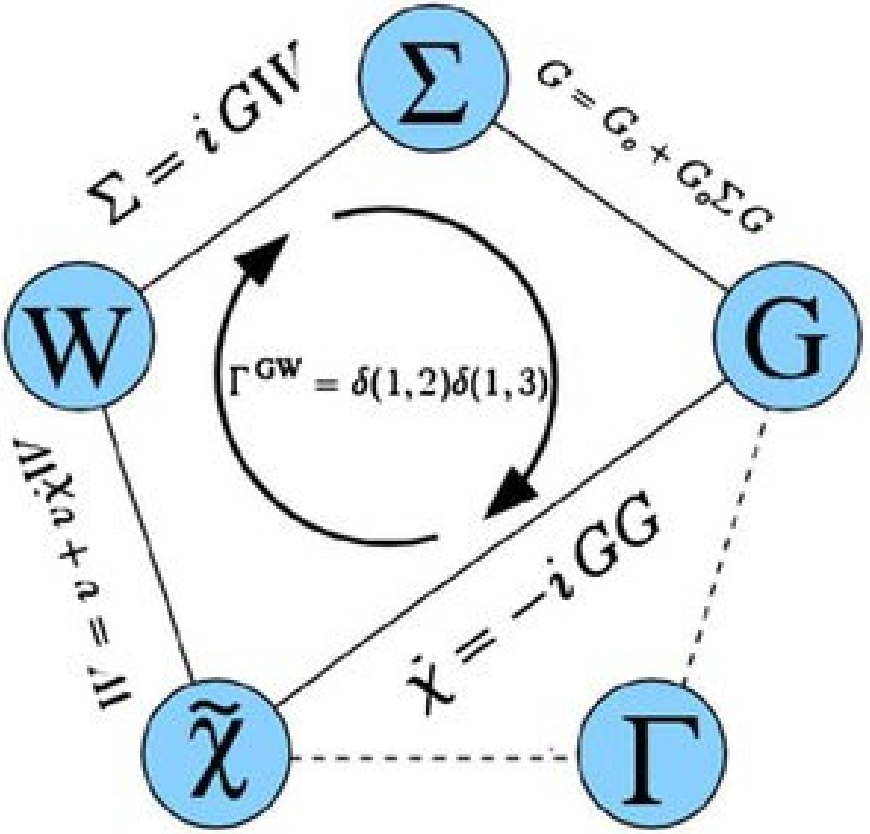
Truncated Hedin's pentagram



short-cutting the vertex

Here comes the *GW* approximation

$$\Sigma(1,2) = iG(1,2)W(1,2)$$ *GW* approximation

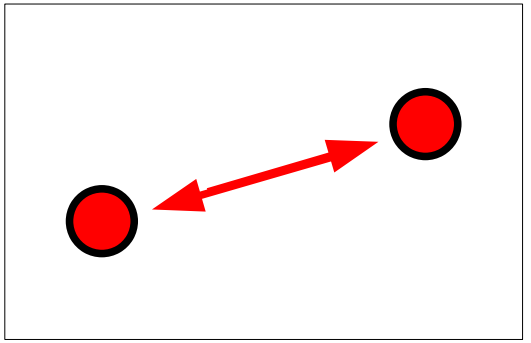
$$\chi_0(1,2) = -iG(1,2)G(2,1)$$ RPA approximation

$$W(1,2) = v(1,2) + \int d34\, v(1,3)\chi_0(3,4)W(4,2)$$ Dyson-like equation

What is W?

Interaction between electrons in vacuum:

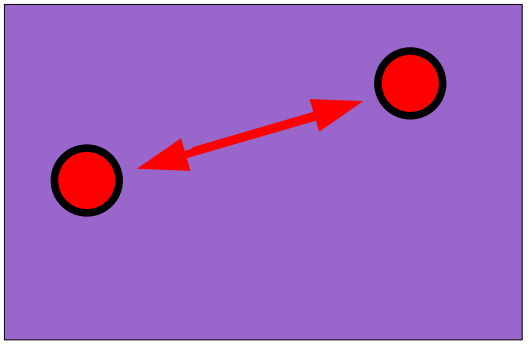
$$v(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\boldsymbol{r}-\boldsymbol{r}'|}$$



Interaction between electrons in a homogeneous polarizable medium:

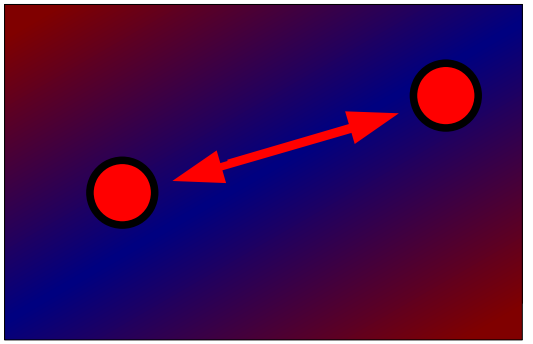
$$W(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{e^2}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Dielectric constant of the medium



Dynamically screened interaction between electrons in a general medium:

$$W(\boldsymbol{r},\boldsymbol{r}',\omega) = \frac{e^2}{4\pi\varepsilon_0}\int d\,r'' \frac{\varepsilon^{-1}(\boldsymbol{r},\boldsymbol{r}'',\omega)}{|\boldsymbol{r}''-\boldsymbol{r}'|}$$



Summary

GW viewed as a “super” Hartree-Fock

Hartree-Fock Approximation

$$\Sigma_x(\boldsymbol{r}_1,\boldsymbol{r}_2)=$$

$$\frac{i}{2\pi}\int_{-\infty}^{\mu} d\,\omega' G(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega')v(\boldsymbol{r}_1,\boldsymbol{r}_2)$$

= bare exchange

GW Approximation

$$\Sigma_{xc}(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)=$$

$$\frac{i}{2\pi}\int d\,\omega' G(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega+\omega')W(\boldsymbol{r}_2,\boldsymbol{r}_1,\omega')$$

Exercice

$$\Sigma_x(\boldsymbol{r}_1,\boldsymbol{r}_2)$$

Bare exchange

$$\Sigma_c(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega)$$

+ correlation

Non Hermitian dynamic

***GW* is nothing else but a “screened” version of Hartree-Fock.**

DFT vs *GW*

Electronic density

$$\rho(\boldsymbol{r})$$

Local and static

Green's function

$$G(\boldsymbol{r}t,\boldsymbol{r}'t')$$

Non-local, dynamic
Depends onto empty states

exchange-correlation potential

$$v_{xc}(\boldsymbol{r})$$

exchange-correlation operator = self-energy

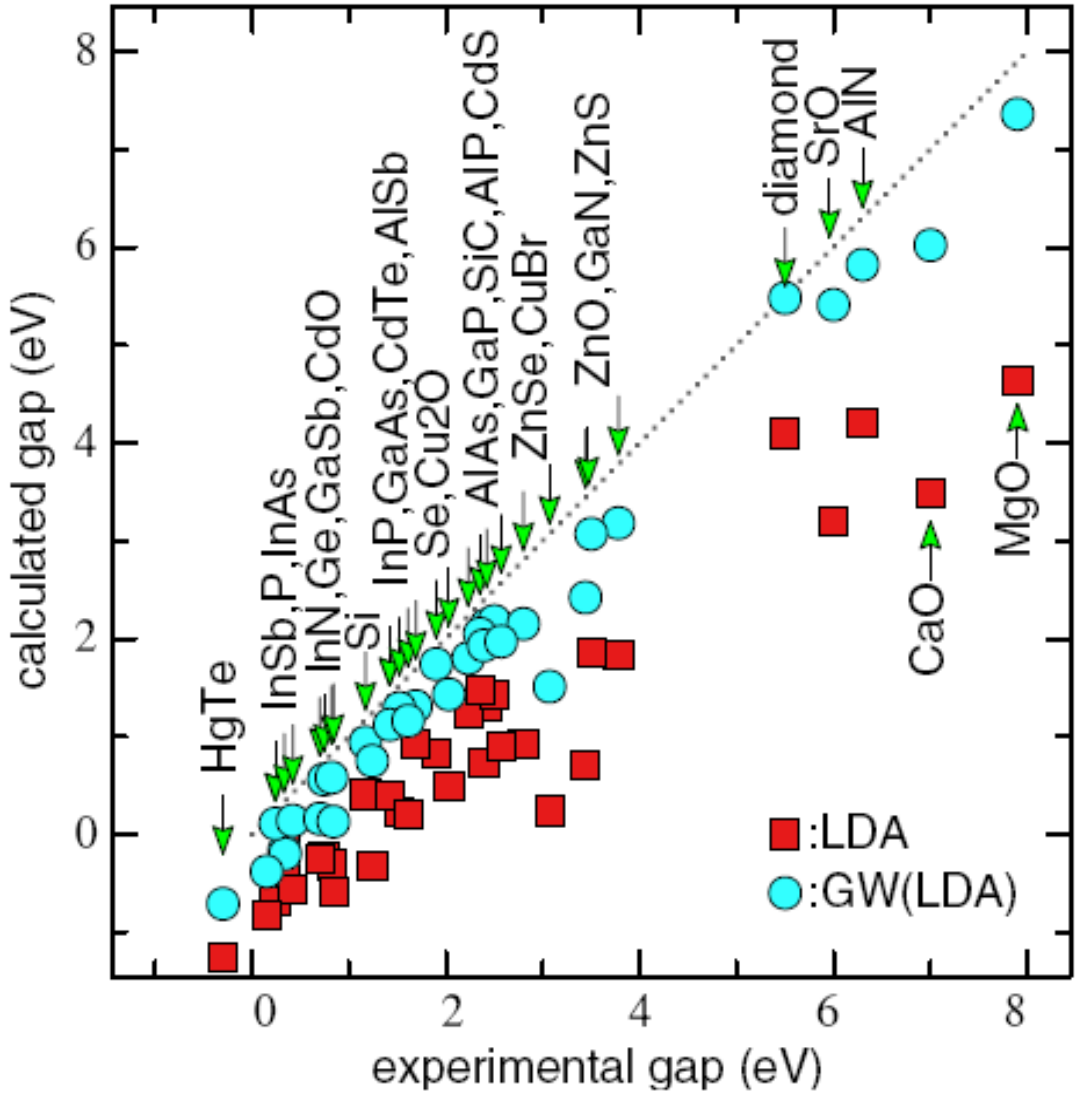
$$\Sigma_{xc}(\boldsymbol{r}t,\boldsymbol{r}'t')$$

Approximations: LDA, GGA, hybrids

GW approximation

$$\Sigma_{GW}(\boldsymbol{r}t,\boldsymbol{r}'t')=iG(\boldsymbol{r}t,\boldsymbol{r}',t')W(\boldsymbol{r}t,\boldsymbol{r}'t')$$

GW approximation gets good band gap



after van Schilfgaarde *et al* PRL **96** 226402 (2008)

No band gap problem anymore!

Outline

I. Introduction: going beyond DFT

II. Introduction of the Green's function

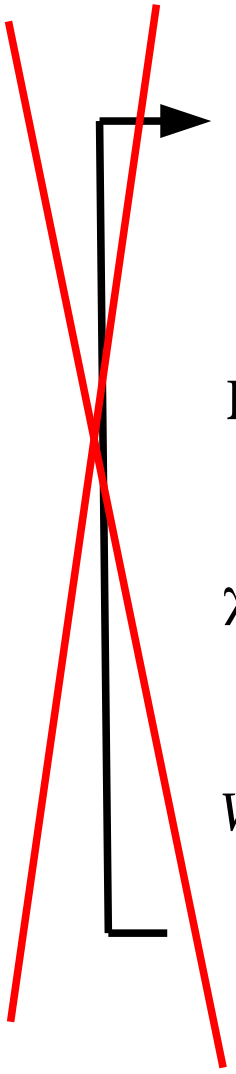
III. Exact Hedin's equations and the *GW* approximation

IV. Calculating the *GW* self-energy in practice

V. Applications

Hedin's coupled equations

5 coupled equations: $1=(\boldsymbol{r}_1 t_1 \sigma_1)$ $2=(\boldsymbol{r}_2 t_2 \sigma_2)$



$$G(1,2)=G_0(1,2)+\int d34\, G_0(1,3)\Sigma(3,4)G(4,2)$$ **Dyson equation**

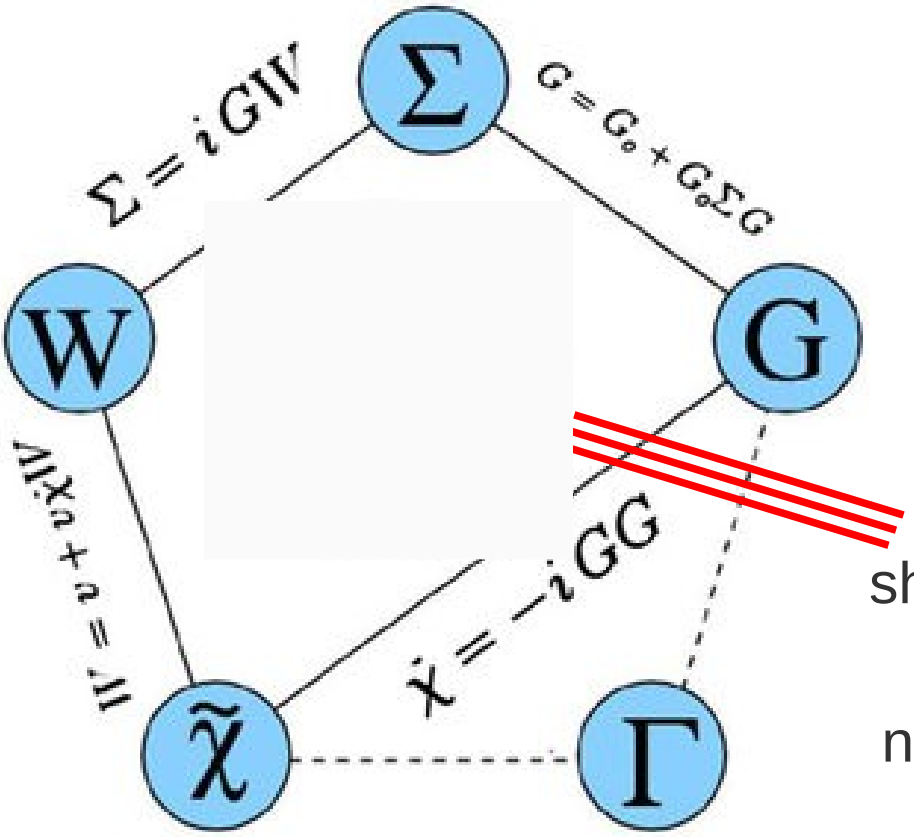
$$\Sigma(1,2)=i\int d34\, G(1,2)W(1,2)\Gamma(4,2,3)$$ **self-energy**

$$\Gamma(1,2,3)=\delta(1,2)\delta(1,3)+\int d4567\,\frac{\delta\Sigma(1,2)}{\delta G(4,5)}G(4,6)G(5,7)\Gamma(6,7,3)$$ **vertex**

$$\chi_0(1,2)=-i\int d34\, G(1,2)G(2,1)\Gamma(3,4,2)$$ **polarizability**

$$W(1,2)=v(1,2)+\int d34\, v(1,3)\chi_0(3,4)W(4,2)$$ **screened Coulomb interaction**

Super-truncated Hedin's pentagram



short-cutting the vertex

no self-consistency

Historical recap of *GW* calculations

- 1965: Hedin's calculations for the homogeneous electron gas
  Phys Rev **2201 citations**

- 1967: Lundqvist's calculations for the homogeneous electron gas
  Physik der Kondensierte Materie **299 citations**

- 1982: Strinati, Mattausch, Hanke for real semiconductors but within tight-binding
  PRB **154 citations**

- 1985: Hybertsen, Louie for real semiconductors with ab initio LDA
  PRL **711 citations** & PRB **1737 citations**

- 1986: Godby, Sham, Schlüter for real semiconductors to get accurate local potential
  PRL **544 citations** & PRB **803 citations**

- ~2001: First publicly available *GW* code in ABINIT

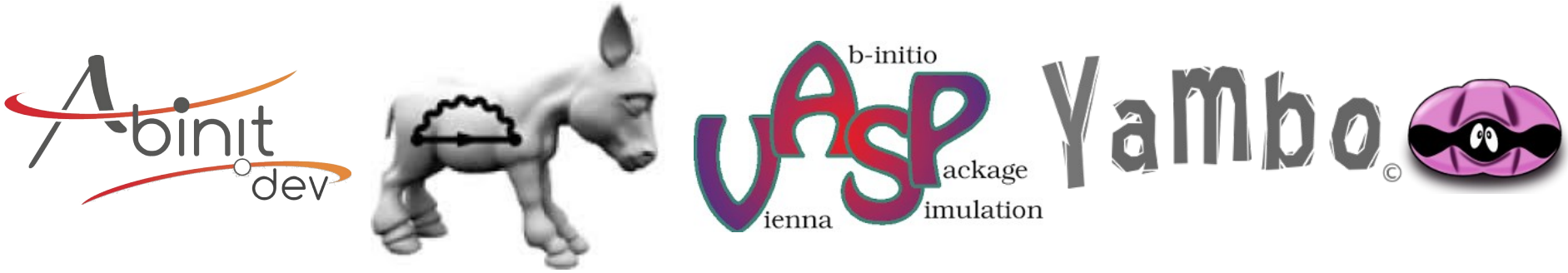
- 2003: Arnaud, Alouani for extension to Projector Augmented Wave
  PRB **102 citations**

- 2006: Shishkin, Kresse for extension to Projector Augmented Wave (again)
  PRB **256 citations**

GW approximation in practice

- For periodic solids: Abinit, BerkeleyGW, VASP, Yambo

based on plane-waves (with pseudo or PAW)



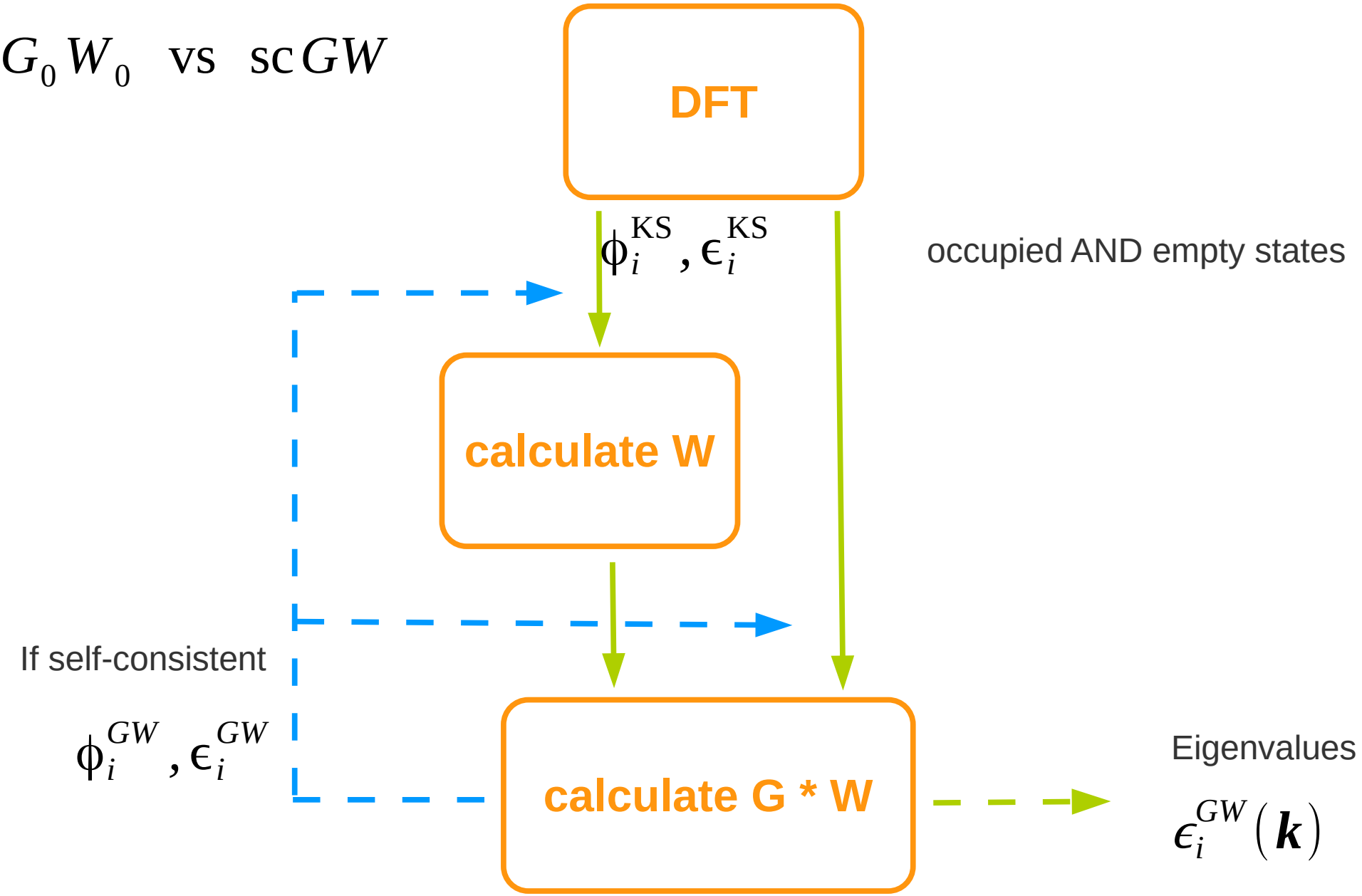
- For finite systems: MOLGW, Fiesta, FHI-AIMS

based on localized orbitals (Gaussians or Slater or other)

What is common to all implementations

Workflow of a typical *GW* calculation

G_0W_0 vs scGW



How to get *G*?

G from Kohn-Sham DFT hamiltonian

$$G^{\mathrm{KS}} = \left[\omega - h_{\mathrm{KS}}\right]^{-1}$$

which implies

$$G^{\mathrm{KS}\ -1}(\mathbf{r},\mathbf{r}',\omega) = \sum_{\mathrm{i}} \phi_{\mathrm{i}}^{\mathrm{KS}}(\mathbf{r}) \left[\omega - \epsilon_{\mathrm{i}}^{\mathrm{KS}}\right] \phi_{\mathrm{i}}^{\mathrm{KS}\,*}(\mathbf{r}')$$

$$G^{\mathrm{KS}}(\boldsymbol{r},\boldsymbol{r}',\omega) = \sum_{i} \frac{\phi_i^{\mathrm{KS}}(\boldsymbol{r})\phi_i^{\mathrm{KS}\,*}(\boldsymbol{r}')}{\omega - \epsilon_i^{\mathrm{KS}} \pm i\eta}$$

This expression will be used to get W and Σ

How to get *W*?

From the RPA equation

$$\chi_0(1,2) = -i\, G^{\mathrm{KS}}(1,2)\, G^{\mathrm{KS}}(2,1)$$

which translates into

$$\chi_0(\boldsymbol{r_1}, \boldsymbol{r_2}, \omega) = \sum_{\substack{i\,\mathrm{occ} \\ j\,\mathrm{virt}}} \phi_i(\boldsymbol{r_1})\phi_i^*(\boldsymbol{r_2})\phi_j(\boldsymbol{r_2})\phi_j^*(\boldsymbol{r_1}) \times \left[\frac{1}{\omega - (\epsilon_j - \epsilon_i) - i\eta} - \frac{1}{\omega - (\epsilon_i - \epsilon_j) + i\eta} \right]$$

This is the **Alder-Wiser formula** or the **SOS formula**

It involves empty states!

Then $\chi_0(1,2)$  $W(1,2)$

Diagonal self-energy correction approximation

Dyson equation: $$G^{-1} = G^{\mathrm{KS}\ -1} - \left(\Sigma - v_{xc}\right)$$

And remember: $$G^{\mathrm{KS}\ -1}\left(\mathbf{r},\mathbf{r}',\omega\right) = \sum_{i} \phi_i^{\mathrm{KS}}\left(\mathbf{r}\right)\left[\omega - \epsilon_i^{\mathrm{KS}}\right]\phi_i^{\mathrm{KS}\,*}\left(\mathbf{r}'\right)$$

G^{KS} is **diagonal** in KS basis

$$G_{ij}^{\mathrm{KS}\ -1}\left(\omega\right) = \delta_{ij}\left(\omega - \epsilon_i^{\mathrm{KS}}\right)$$

Approximation: $$\left\langle i\middle|\Sigma\left(\omega\right) - v_{xc}\middle|j\right\rangle \approx \delta_{ij}\left\langle i\middle|\Sigma\left(\omega\right) - v_{xc}\middle|i\right\rangle$$

Hence G is **diagonal** in KS basis

$$G_{ij}^{-1}\left(\omega\right) \approx \delta_{ij}\left(\omega - \epsilon_i^{\mathrm{KS}} - \left\langle i\middle|\Sigma\left(\omega\right) - v_{xc}\middle|i\right\rangle\right)$$

Diagonal self-energy correction approximation

G is **diagonal** in KS basis

$$G_{ij}^{-1}(\omega) \approx \delta_{ij}\left(\omega - \epsilon_i^{KS} - \left\langle i\left|\Sigma(\omega) - v_{xc}\right|i\right\rangle\right)$$

Excitation energies are the poles of *G* or the zeros of G^{-1}

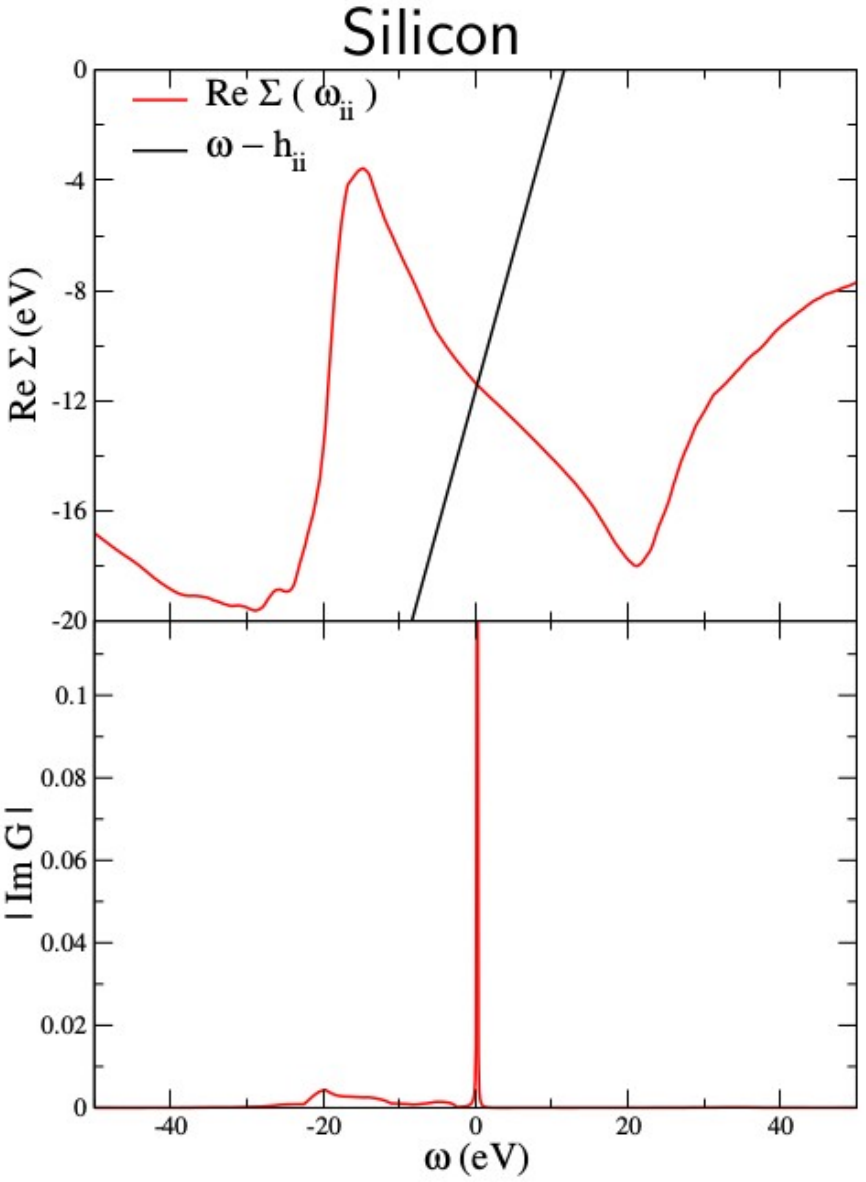
$$G^{-1}\left(\epsilon_i^{GW}\right) = 0$$

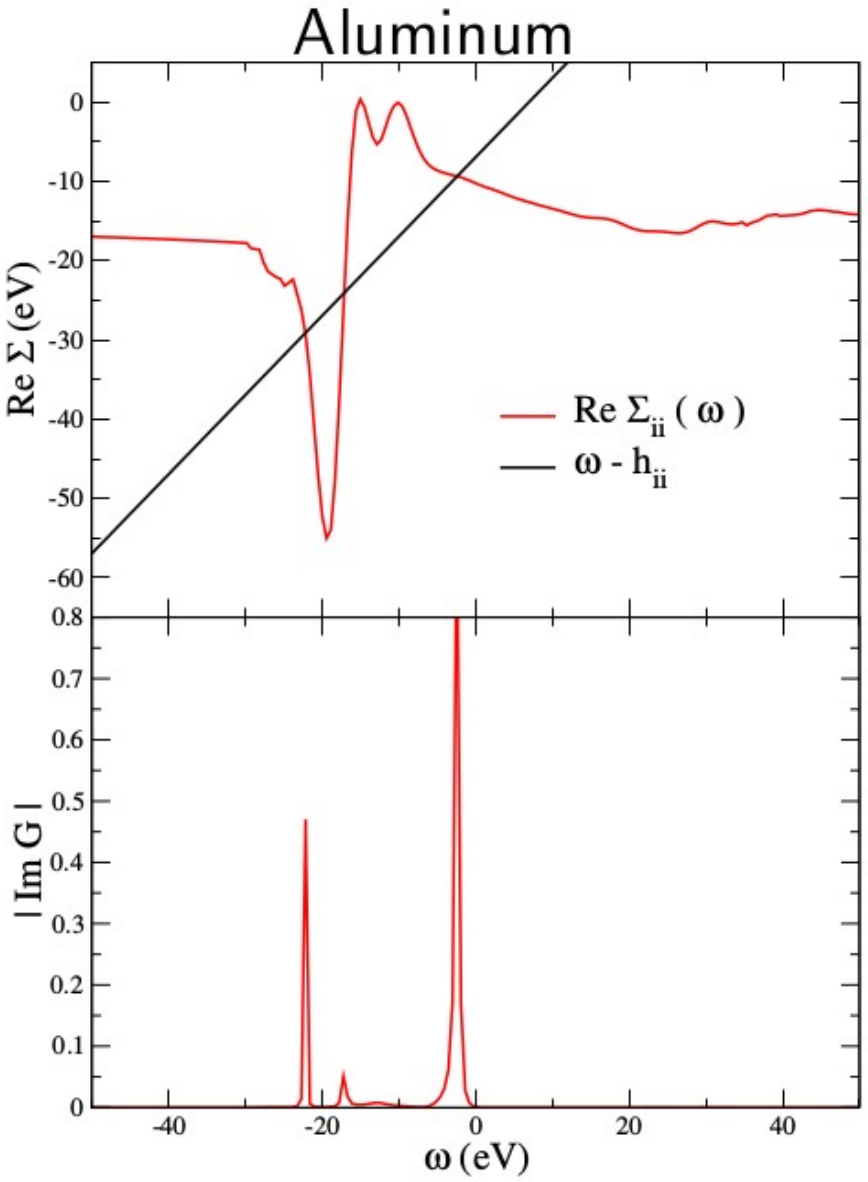
Quasiparticle energies:

$$\epsilon_i^{GW} = \epsilon_i^{KS} + \left\langle i\left|\Sigma\left(\epsilon_i^{GW}\right) - v_{xc}\right|i\right\rangle$$

Full quasiparticle solution

$$\epsilon_i^{GW} - \epsilon_i^{\mathrm{KS}} + \langle i | v_{xc} | i \rangle = \langle i | \Sigma(\epsilon_i^{GW}) | i \rangle$$





Linearization of the energy dependance

$$\epsilon_i^{GW} - \epsilon_i^{KS} = \left\langle \phi_i^{KS} \middle| \left[\Sigma\left(\epsilon_i^{GW}\right) - v_{xc} \right] \middle| \phi_i^{KS} \right\rangle$$

Not yet known

Taylor expansion:

$$\Sigma\left(\epsilon_i^{GW}\right) = \Sigma\left(\epsilon_i^{KS}\right) + \left(\epsilon_i^{GW} - \epsilon_i^{KS}\right) \frac{\partial \Sigma}{\partial \epsilon} + \ldots$$

Final result:

$$\epsilon_i^{GW} = \epsilon_i^{KS} + Z_i \left\langle \phi_i^{KS} \middle| \left[\Sigma\left(\epsilon_i^{KS}\right) - v_{xc} \right] \middle| \phi_i^{KS} \right\rangle$$

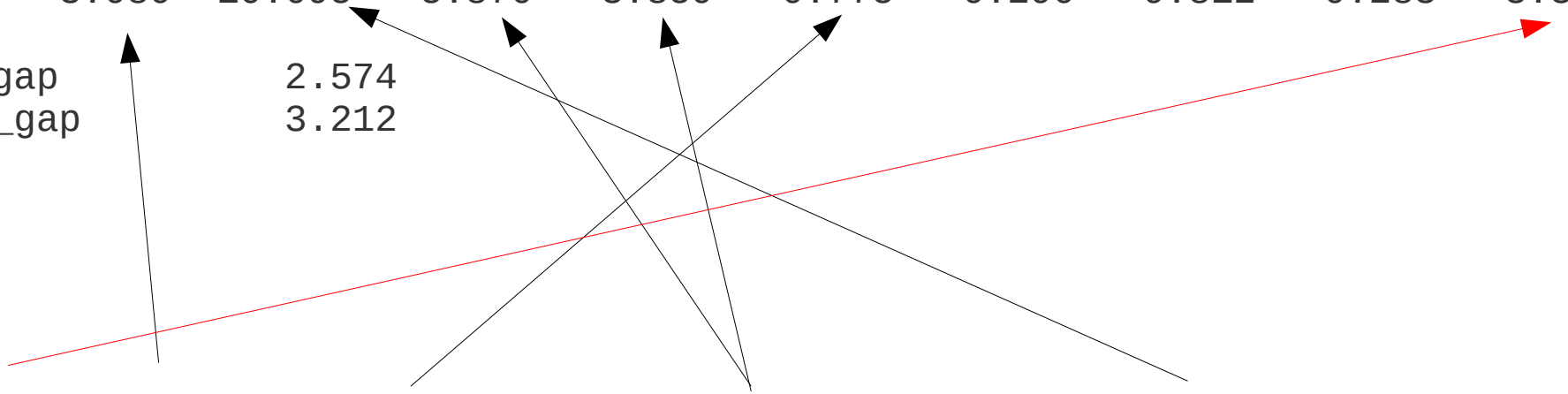
where $$Z_i = 1 \Big/ \left(1 - \left\langle i \middle| \frac{\partial \Sigma}{\partial \epsilon} \middle| i \right\rangle \right)$$

Quasiparticle equation

A typical ABINIT ouptput for Silicon at Gamma point

```
k =    0.000   0.000   0.000
Band      E0 <VxcLDA>   SigX SigC(E0)      Z dSigC/dE  Sig(E)    E-E0       E
   4   0.506 -11.291 -12.492   0.744   0.775  -0.291 -11.645  -0.354   0.152
   5   3.080 -10.095  -5.870  -3.859   0.775  -0.290  -9.812   0.283   3.363

E^0_gap         2.574
E^GW_gap        3.212
```



$$\epsilon_i^{GW} = \epsilon_i^{\mathrm{LDA}} + Z_i \left\langle \varphi_i^{\mathrm{LDA}} \middle| \left[\Sigma_{xc}\left(\epsilon_i^{\mathrm{LDA}}\right) - v_{xc}^{\mathrm{LDA}} \right] \middle| \varphi_i^{\mathrm{LDA}} \right\rangle$$

Self energy evaluation in *GW*

Correlation part of the GW self energy requires a convolution in frequency:

$$\Sigma_c(\omega) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\omega' G(\omega+\omega') W_p(\omega')$$

$$G(\omega) = \sum_i \frac{\phi_i(\boldsymbol{r})\phi_i^*(\boldsymbol{r}')}{\omega - \epsilon_i \pm i\eta}$$

?

How to deal with the frequency dependence in W?

How do we perform the convolution?
How do we treat the frequency dependence in *W*?

Dealing with two-point functions with plane-waves

Remember 1-point functions are

$$\phi_{\boldsymbol{k}}(\boldsymbol{r}) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{kG}} c_{\boldsymbol{k}}(\boldsymbol{G}) e^{i(\boldsymbol{k}+\boldsymbol{G}).\boldsymbol{r}}$$

1 vector of coefficients per k-point in the Brillouin zone

Then 2-point functions are

$$W(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\Omega} \sum_{\boldsymbol{qGG'}} e^{i(\boldsymbol{q}+\boldsymbol{G}).\boldsymbol{r}_1} W_{\boldsymbol{GG'}}(\boldsymbol{q}) e^{-i(\boldsymbol{q}+\boldsymbol{G'}).\boldsymbol{r}_2}$$

a matrix of coefficients per q-point in the BZ due to translational symmetry:

$$W(\boldsymbol{r}_1,\boldsymbol{r}_2) = W(\boldsymbol{r}_1+\boldsymbol{R},\boldsymbol{r}_2+\boldsymbol{R})$$

W in plane-waves and frequency space

(1) $$\chi_0(\boldsymbol{r_1},\boldsymbol{r_2},\omega)=\sum_{\substack{i\,\mathrm{occ}\\ j\,\mathrm{virt}}} \phi_i(\boldsymbol{r_1})\phi_i^*(\boldsymbol{r_2})\phi_j(\boldsymbol{r_2})\phi_j^*(\boldsymbol{r_1}) \times\left[\frac{1}{\omega-(\epsilon_j-\epsilon_i)-i\eta}-\frac{1}{\omega-(\epsilon_i-\epsilon_j)+i\eta}\right]$$

(2) $$\varepsilon(1,2)=\delta(1,2)-\int d3\, v(1,3)\chi_0(3,2)$$

(3) $$W(1,2)=\int d3\,\varepsilon^{-1}(1,3)v(3,2)$$

---

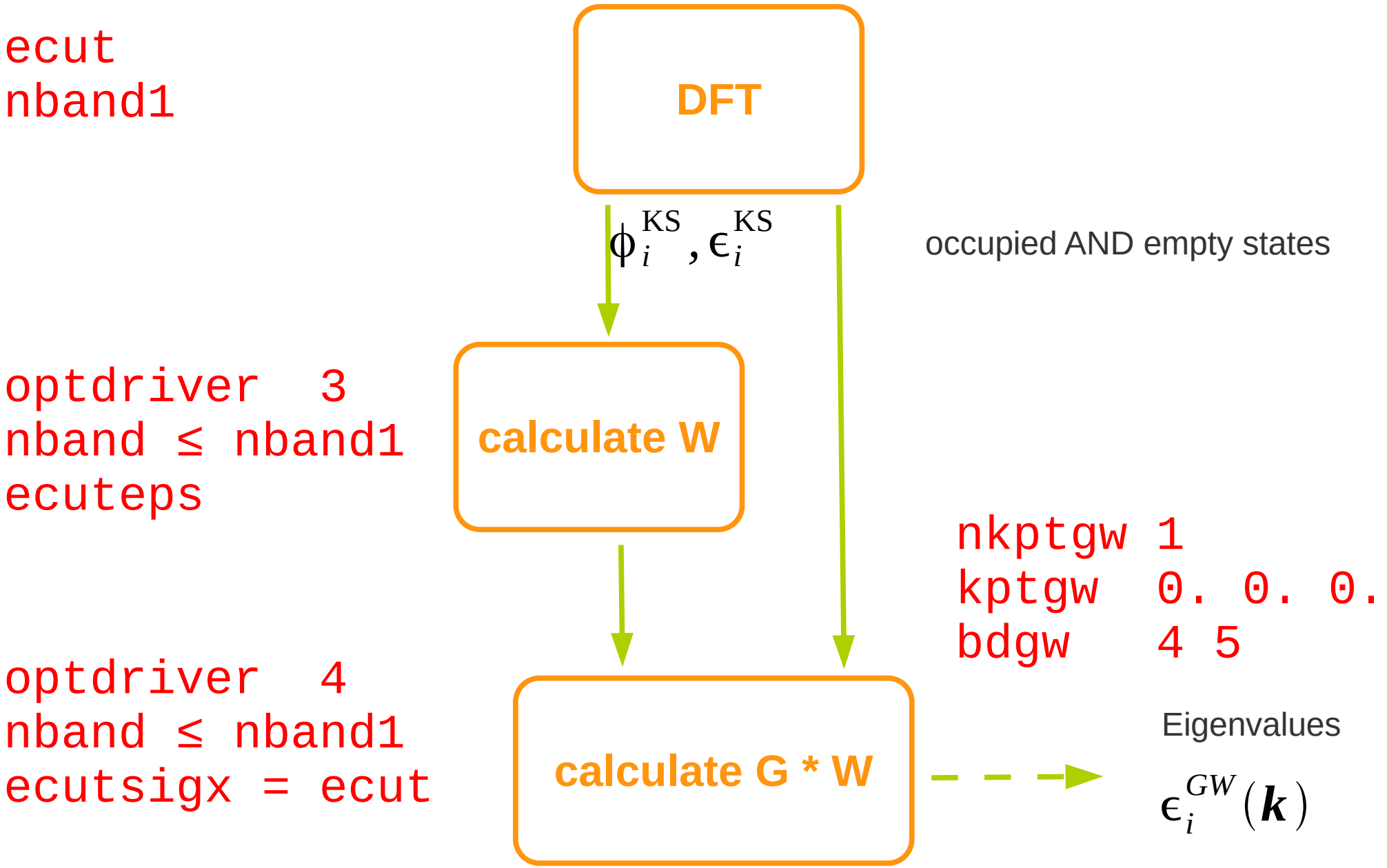
(1) $$\chi_{0\boldsymbol{GG'}}(\boldsymbol{q},\omega)=\sum_{\substack{\boldsymbol{k}\\ i\,\mathrm{occ}\\ j\,\mathrm{virt}}} \left\langle j\,\boldsymbol{k}-\boldsymbol{q}\middle|e^{-i(\boldsymbol{q}+\boldsymbol{G}).\boldsymbol{r_1}}\middle|i\,\boldsymbol{k}\right\rangle\left\langle i\,\boldsymbol{k}\middle|e^{i(\boldsymbol{q}+\boldsymbol{G'}).\boldsymbol{r_2}}\middle|j\,\boldsymbol{k}-\boldsymbol{q}\right\rangle \times\left[\frac{1}{\omega-(\epsilon_j-\epsilon_i)-i\eta}-\frac{1}{\omega-(\epsilon_i-\epsilon_j)+i\eta}\right]$$

ecuteps nband

(2) $$\varepsilon_{\boldsymbol{GG'}}(\boldsymbol{q},\omega)=\delta_{\boldsymbol{G},\boldsymbol{G'}}-\sum_{G''} v_{\boldsymbol{GG''}}(\boldsymbol{q})\chi_{0\boldsymbol{G''G'}}(\boldsymbol{q},\omega) \quad\Leftarrow\quad v_{\boldsymbol{GG''}}(\boldsymbol{q})=\frac{4\pi}{|\boldsymbol{q}+\boldsymbol{G}|^2}\delta_{\boldsymbol{G},\boldsymbol{G''}}$$

(3) $$W_{\boldsymbol{GG'}}(\boldsymbol{q},\omega)=\varepsilon^{-1}_{\boldsymbol{GG'}}(\boldsymbol{q},\boldsymbol{G'})\,v_{\boldsymbol{G'}}(\boldsymbol{q}) \quad\Leftarrow\quad \text{matrix inversion}$$

G_0W_0 calculation in ABINIT



Plasmon-Pole Models in *GW*

Correlation part of the GW self energy requires a convolution in frequency:

$$\Sigma_c(\omega) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} d\,\omega' G(\omega+\omega') W_p(\omega')$$

Generalized Plasmon-Pole Model:

$$\varepsilon^{-1}(\omega') - 1 = \frac{\Omega^2}{2\tilde{\omega}}\left[\frac{1}{\omega' - \tilde{\omega} + i\eta} - \frac{1}{\omega' + \tilde{\omega} - i\eta}\right]$$

Amplitude of the pole

Position of the pole

small real number

2 parameters need two constraints:

- Hybertsen-Louie (HL): $\varepsilon^{-1}(0)$ and f sum rule $\int_0^{+\infty} \omega\, \mathrm{Im}\,\varepsilon^{-1}(\omega) = -\frac{\pi}{2}\omega_p^2$
- Godby-Needs (GN): $\varepsilon^{-1}(0)$ and $\varepsilon^{-1}(i\omega)$

Silicon band gap with PPM

Silicon unit cell:
k-points: 5x5x5
bands: 190 empty states
cutoff energy for epsilon: 8 Ry

	HL	GN	Expt.
Γ_v	5.45	5.65	
Γ_c	8.71	8.87	
Direct Band gap	3.26	3.22	3.40 eV

Band gaps are almost the same
However, the absolute positioning of the bands is not (**0.2 eV difference!**)

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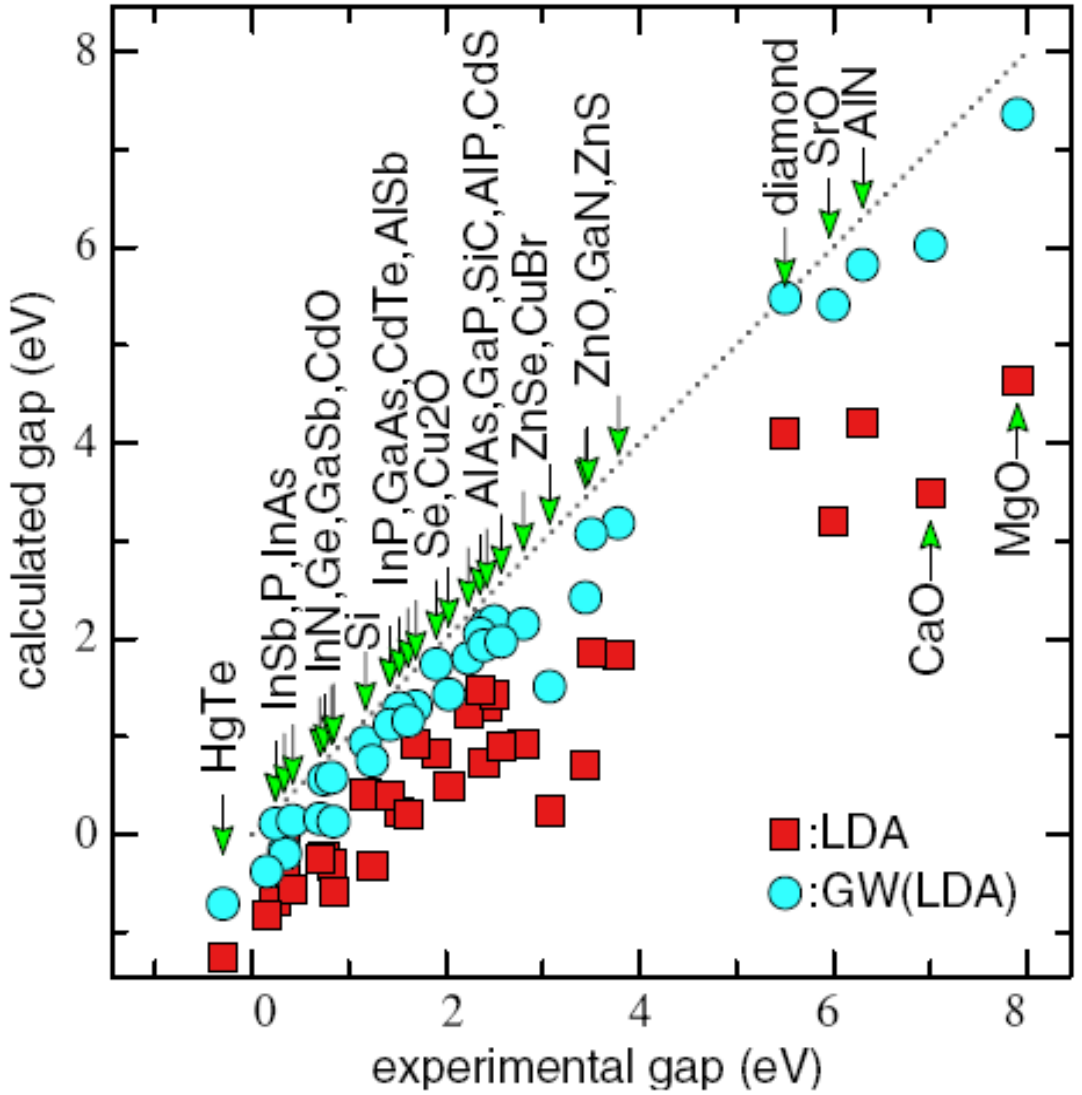
GW approximation gets good band gap



van Schilfgaarde *et al* PRL **96** 226402 (2008)

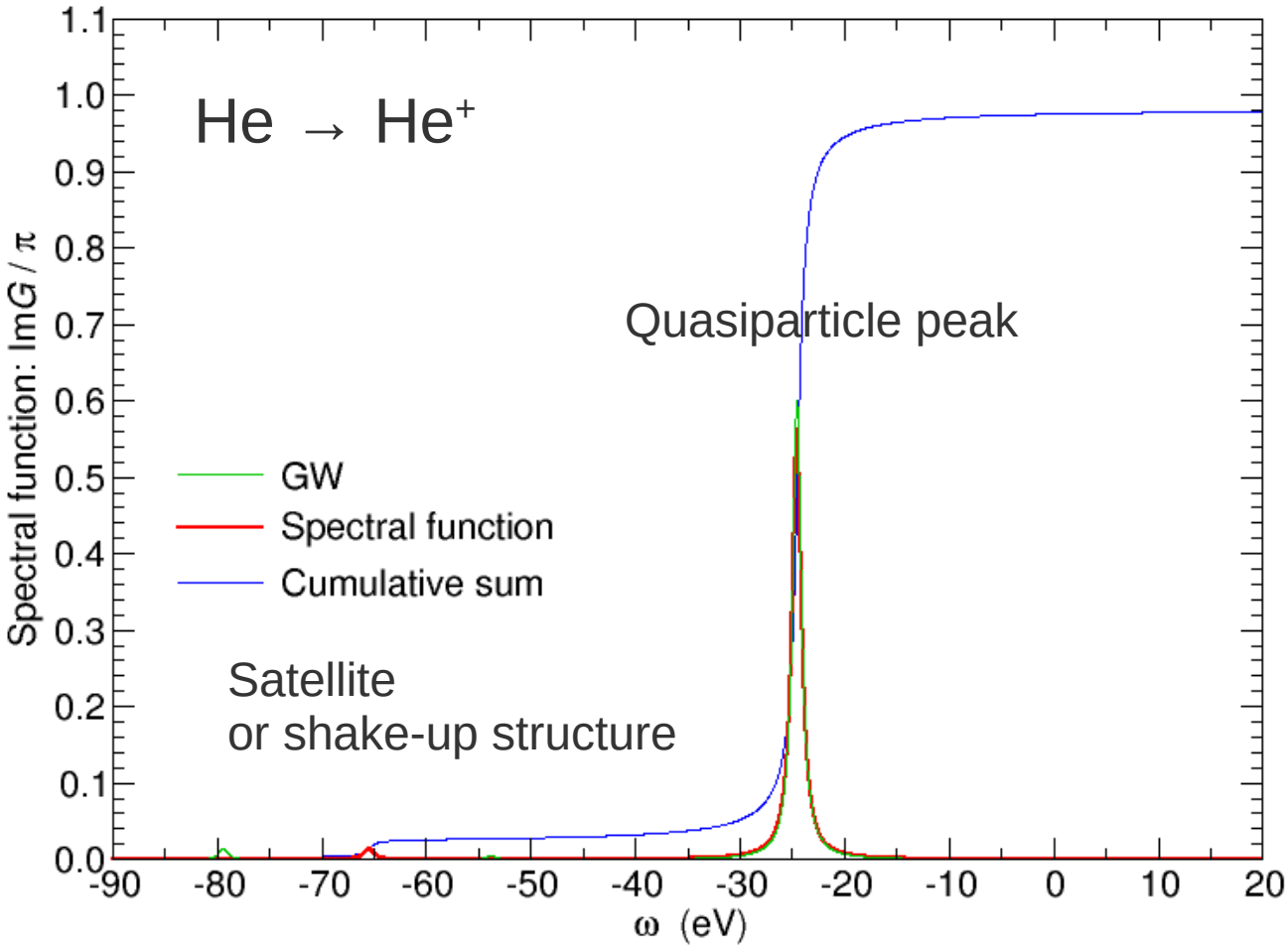
No more a band gap problem !

Exact realization of the Lehman decomposition

$$\langle m|G^h(\omega)|m\rangle = \sum_i \frac{\langle N\,0|\hat{c}_m^+|N-1\,i\rangle\langle N-1\,i|\hat{c}_m|N\,0\rangle}{\omega-\epsilon_i-i\eta}$$

$$N=2$$
$$N-1=1$$
$$m=1s$$

Obtained from FCI calculations



What is the best starting point for G_0W_0?

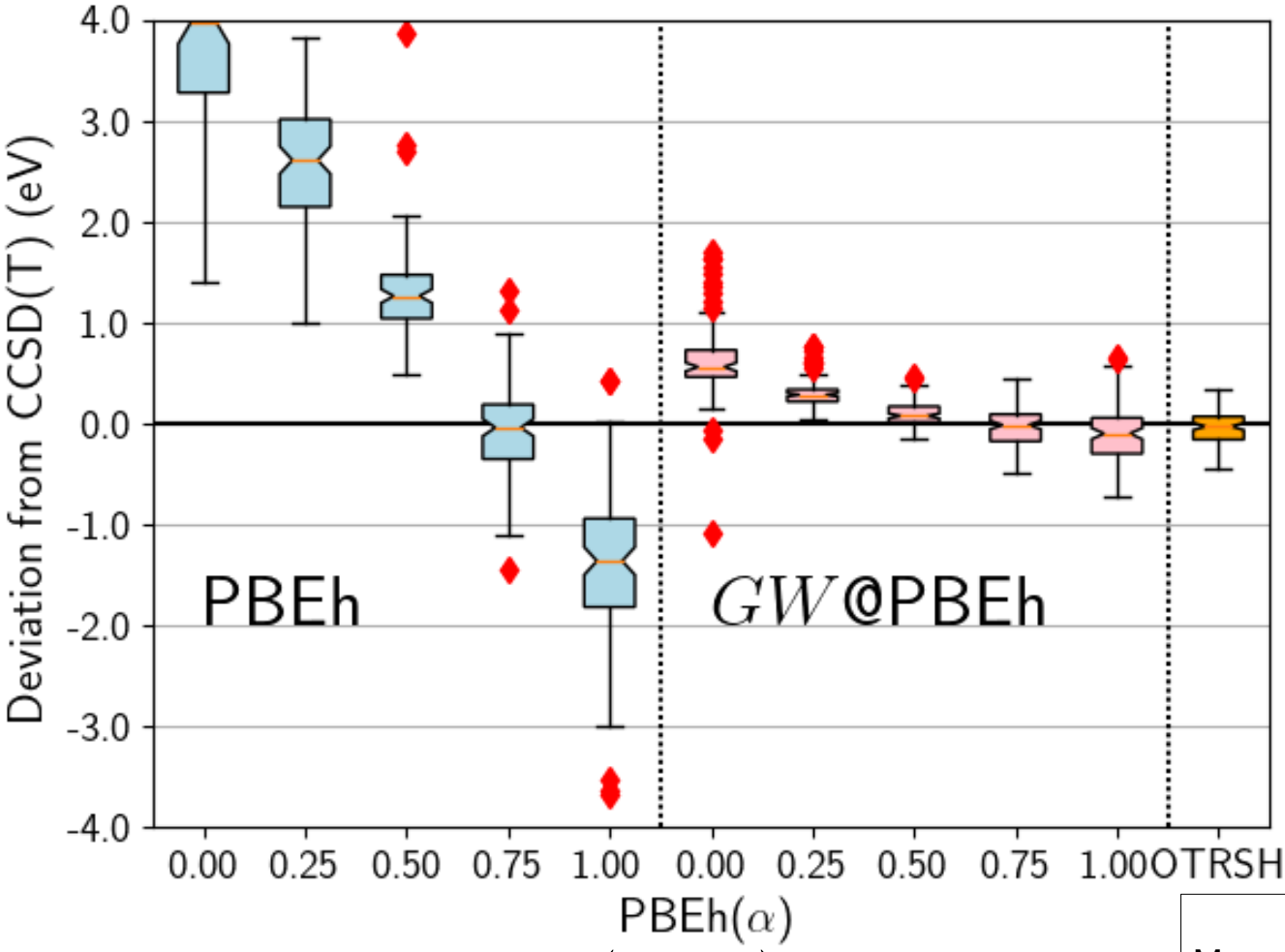
Ionization potential of 100 small molecules

van Setten *et al.* JCTC (2015)

https://gw100.wordpress.com/

but containing difficult elements: Rb, Cs, Br, As etc...

GW versus Coupled-cluster $E_{\mathrm{CCSD(T)}}\left(\mathrm{X}^0\right) - E_{\mathrm{CCSD(T)}}\left(\mathrm{X}^+\right)$

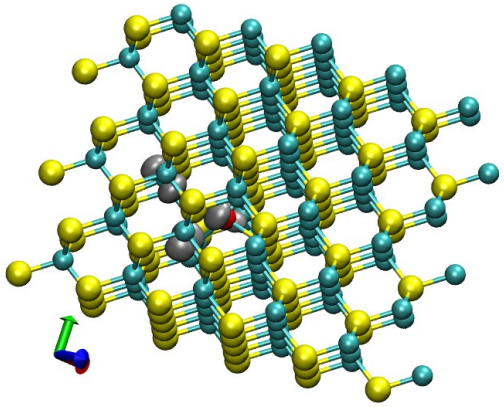


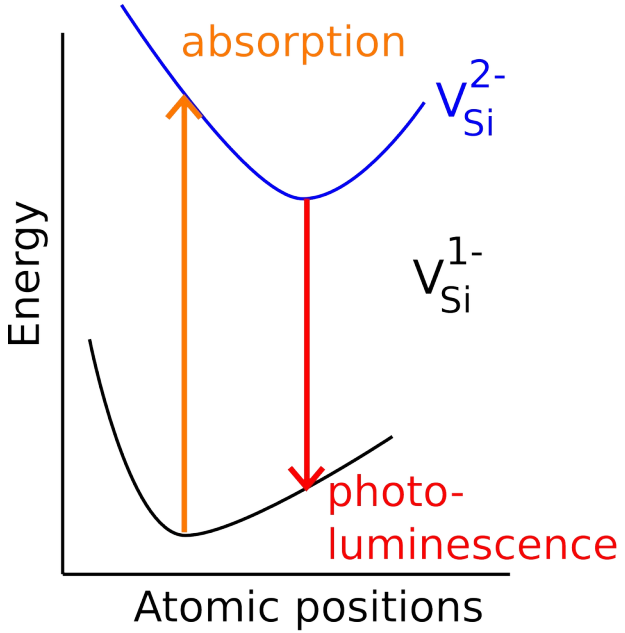
$$v_{xc} = \alpha \Sigma_x + (1-\alpha) v_x^{\mathrm{PBE}} + v_c^{\mathrm{PBE}}$$

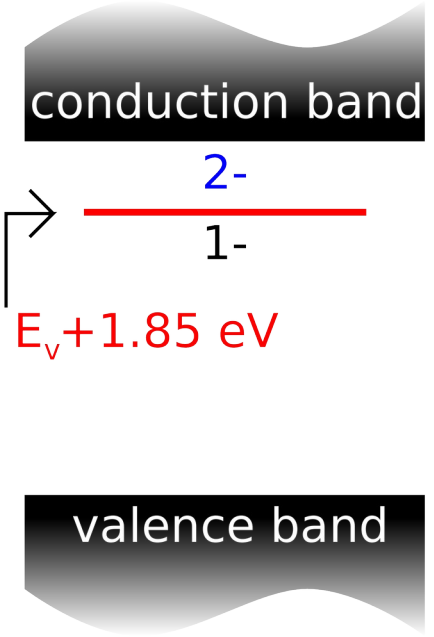
F. Bruneval, N. Dattani, M. van Setten, Frontiers in Chemistry (2021)

Defect calculation within GW approximation

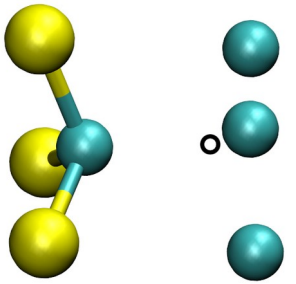
215 atoms

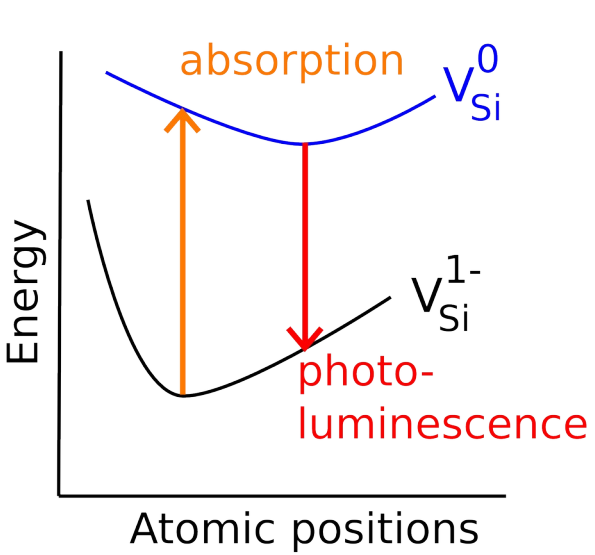


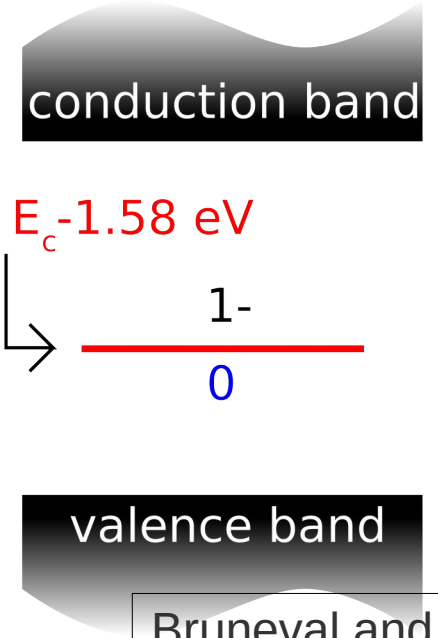




2.19 eV







2.19 eV

Bruneval and Roma PRB (2011)

Getting the density-matrix from GW

Electronic density:

$$\rho(\mathbf{r}) = \sum_i f_i \, \varphi_i^*(\mathbf{r}) \varphi_i(\mathbf{r})$$

Density-matrix:

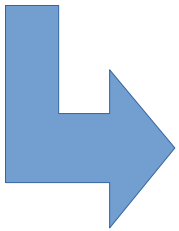
$$\gamma(\mathbf{r}, \mathbf{r}') = \sum_i f_i \, \varphi_i^*(\mathbf{r}) \varphi_i(\mathbf{r}')$$

An example: the kinetic energy is a explicit functional of $\gamma(\mathbf{r}, \mathbf{r}')$

$$T = -\frac{1}{2} \sum_i f_i \iint d\mathbf{r} d\mathbf{r}' \varphi_i^*(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') \nabla^2_{\mathbf{r}'} \varphi_i^*(\mathbf{r}')$$

GW density matrix in ABINIT

$$\langle \mathbf{k}i|\Sigma^{GW}(\mu_F + i\omega)|\mathbf{k}j\rangle$$



$$\langle \mathbf{k}i|\gamma^{GW}|\mathbf{k}j\rangle = -\frac{1}{2\pi}\int d\omega \frac{\langle \mathbf{k}i|\Sigma^{GW}(\mu_F + i\omega)|\mathbf{k}j\rangle}{(\mu_F + i\omega - \epsilon_{\mathbf{k}i})(\mu_F + i\omega - \epsilon_{\mathbf{k}j})}$$

already available in ABINIT v9.4

Denawi, Bruneval, Torrent, Rodriguez-Mayorga PRB (2023)

Surprising facts with density-matrices 1

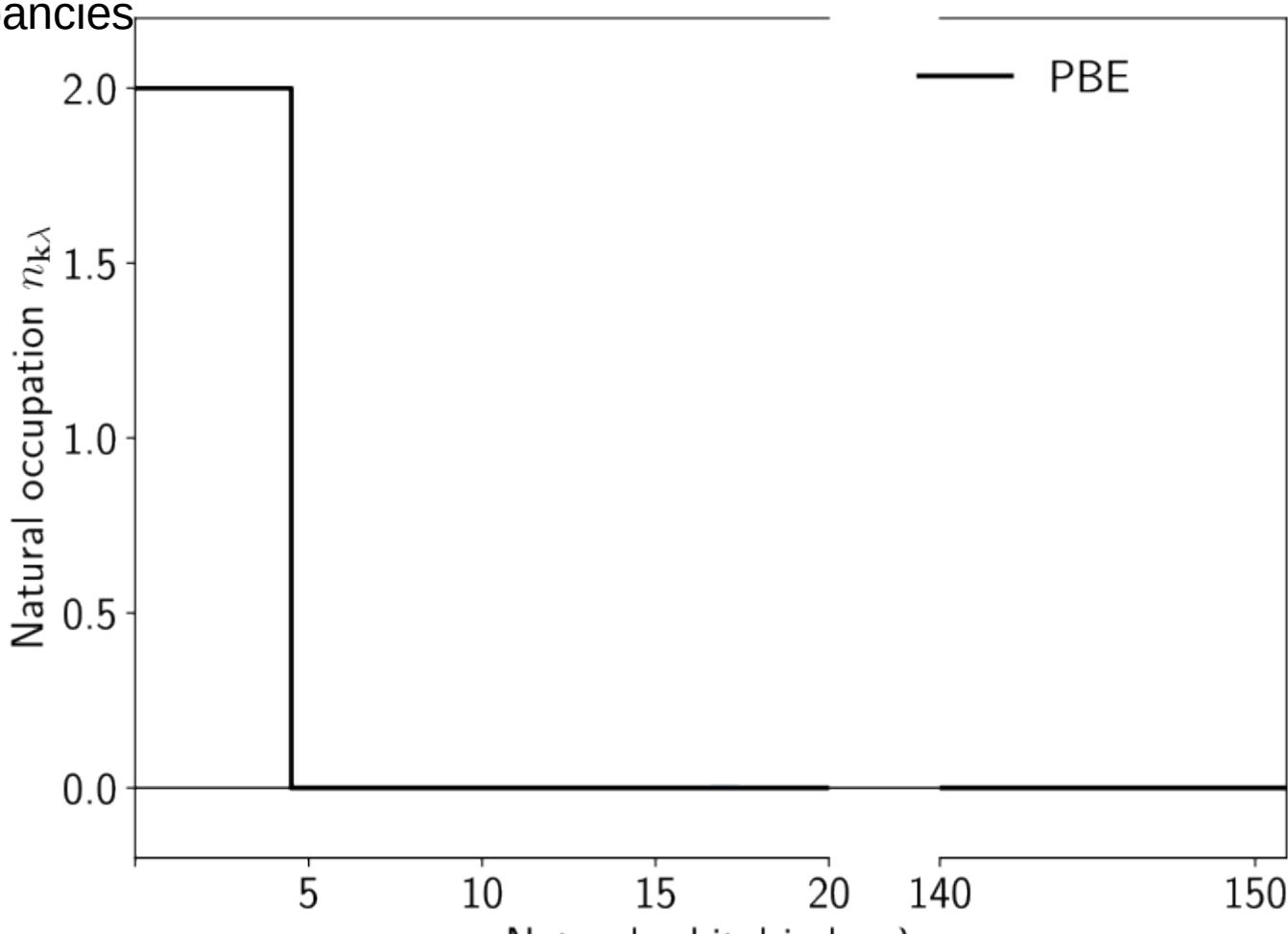
Diagonalization:

$$\sum_{j=1}^{N_b} \gamma_{\mathbf{k}ij} U_{\mathbf{k}j\lambda} = n_{\mathbf{k}\lambda} U_{\mathbf{k}i\lambda}$$

Natural orbitals, natural occupancies

$$\phi_{\mathbf{k}\lambda}(\mathbf{r}) = \sum_{i=1}^{N_b} U_{\mathbf{k}i\lambda} \varphi_{\mathbf{k}i}(\mathbf{r}).$$

Si



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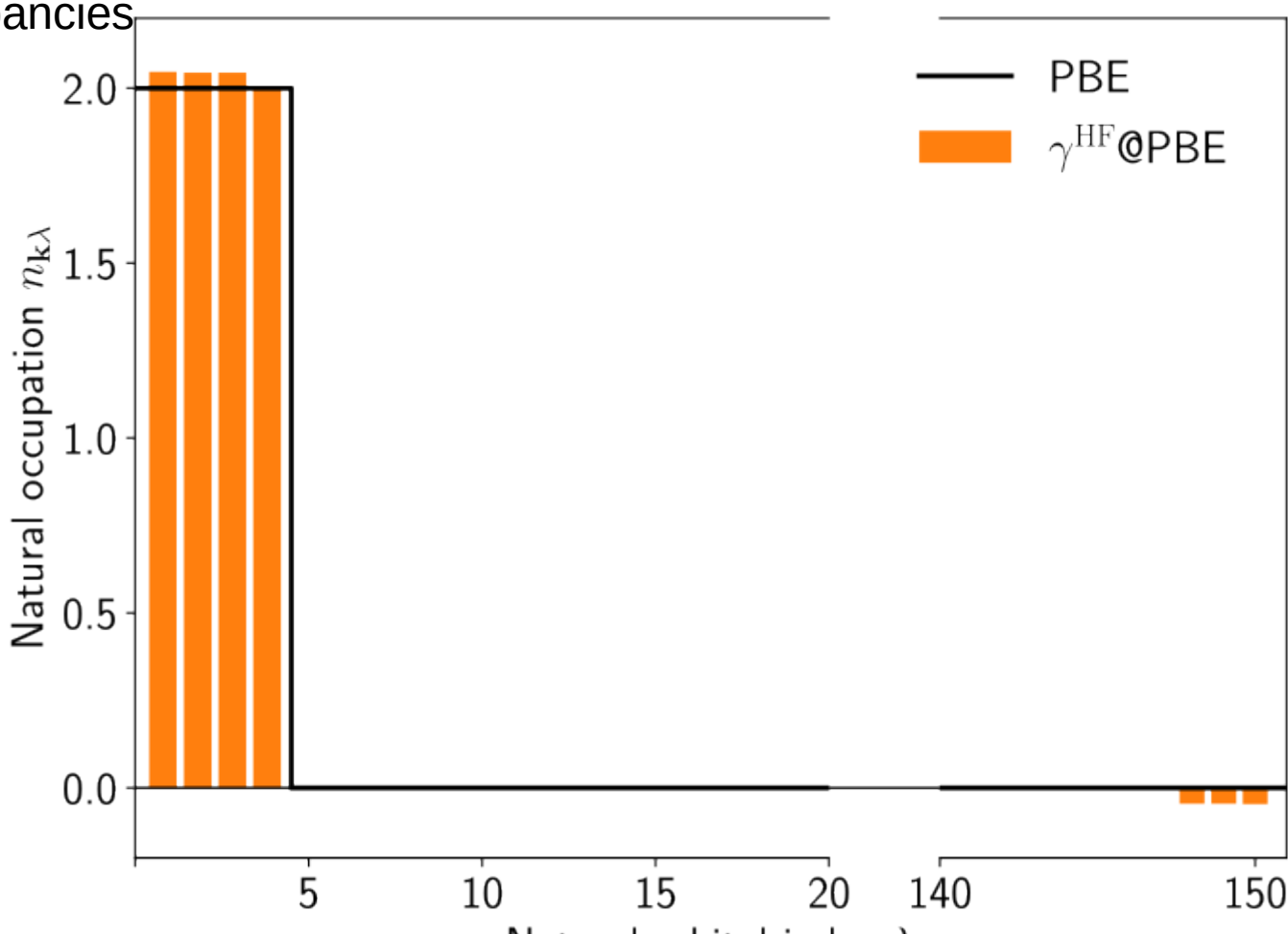

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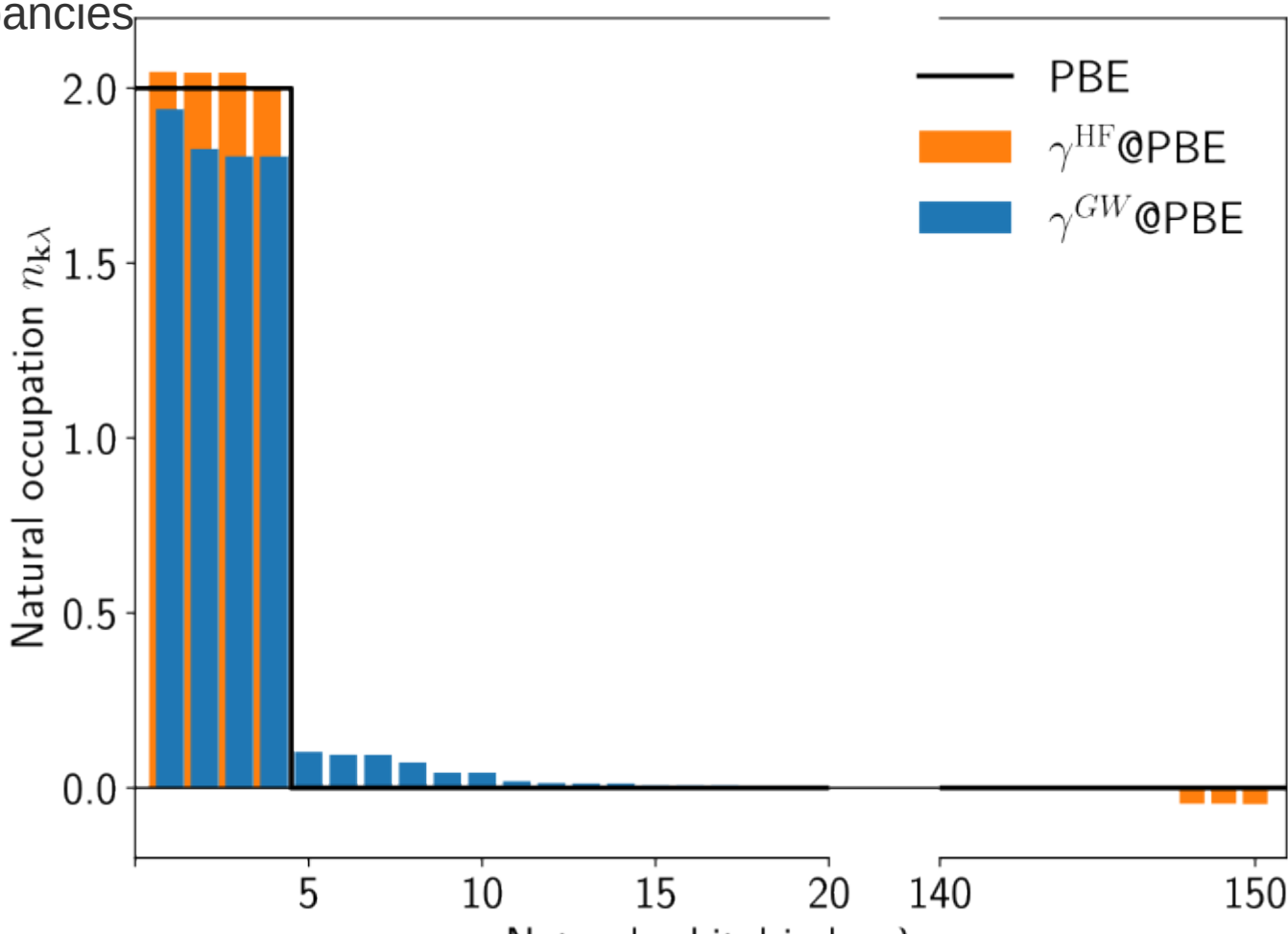
Si

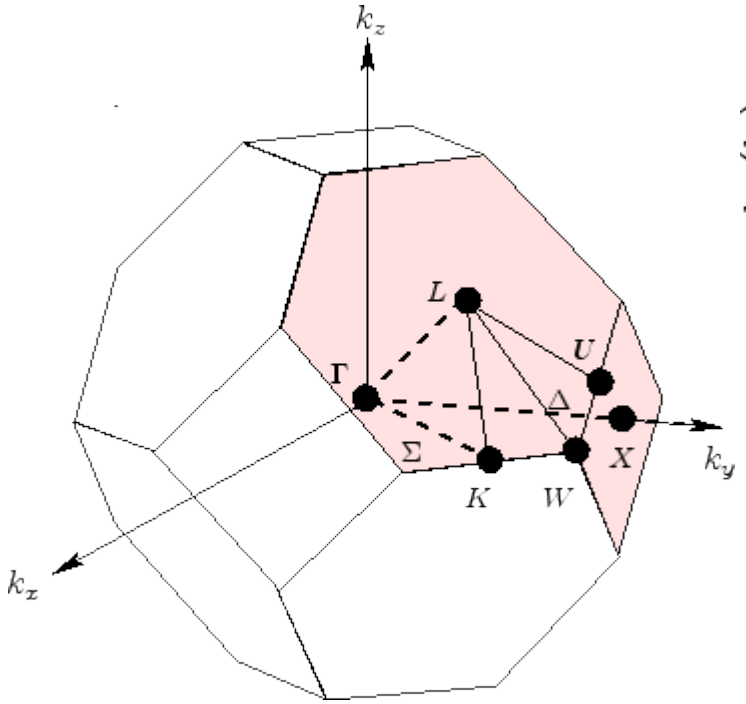


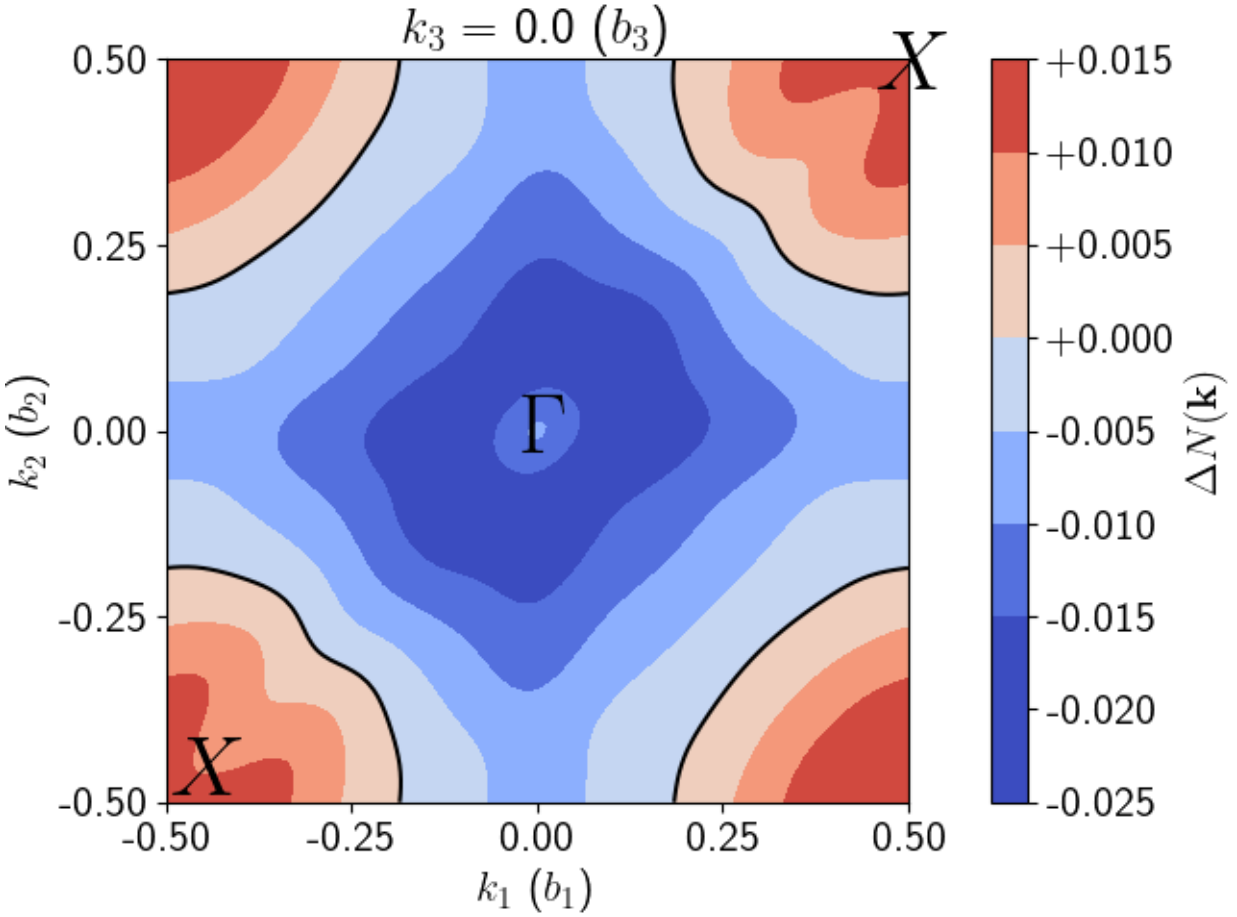
Surprising facts with density-matrices 2

Si

Electron count across the BZ

$$\sum_{\mathbf{k}} \sum_{\lambda} n_{\mathbf{k}\lambda} = N_e$$

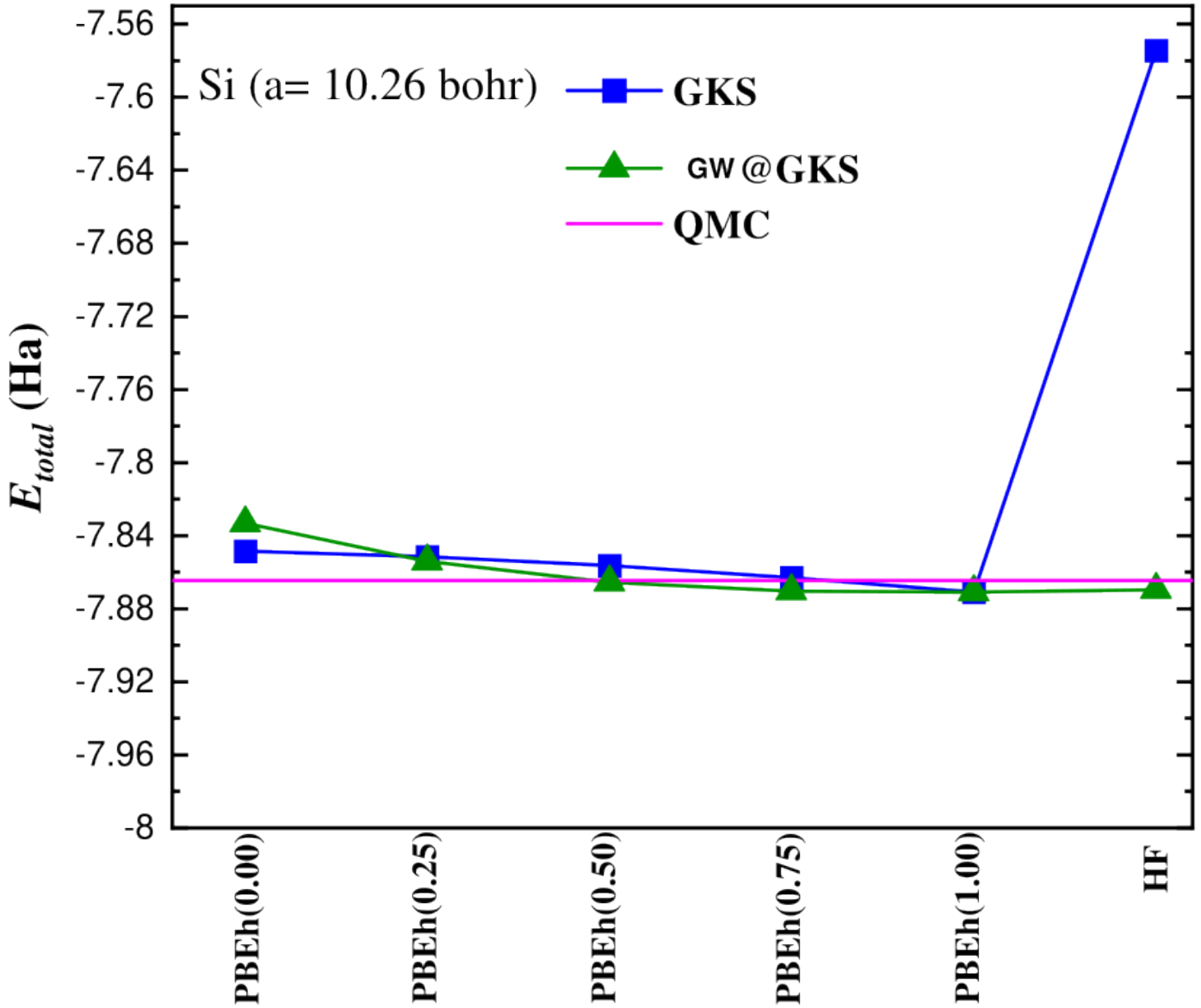




Varying number of electrons that sum up to the correct count!

Density matrix to estimate self-consistent GW energy

Denawi, Bruneval, Torrent, Rodriguez-Mayorga PRB (2023)



Summary

- The *GW* approximation **solves the band gap problem!**
- The calculations are extremely heavy, so that we resort to many additional technical approximations: **method named G_0W_0**
- The complexity comes from
  - Dependence upon empty states
  - Non-local operators
  - Dynamic operators that requires freq. convolutions

Reviews - Links

Reviews:

• L. Hedin, Phys. Rev. **139** A796 (1965).
• L. Hedin and S. Lunqdvist, in Solid State Physics, Vol. **23** (Academic, New York, 1969), p. 1.
• F. Aryasetiawan and O. Gunnarsson, Rep. Prog. Phys. **61** 237 (1998).
• W.G. Aulbur, L. Jonsson, and J.W. Wilkins, Sol. State Phys. **54** 1 (2000).
• G. Strinati, Riv. Nuovo Cimento **11** 1 (1988).

• F. Bruneval and M. Gatti, “Quasiparticle Self-Consistent GW Method for the Spectral Properties of Complex Materials”, Top. Curr. Chem (2014) 347: 99–136

Codes:

• http://www.abinit.org

• http://www.vasp.org
• http://www.berkeleygw.org/
• https://github.com/bruneval/molgw

Exercice: H_2 in minimal basis: GW@HF

Find the location of the poles of the self-energy

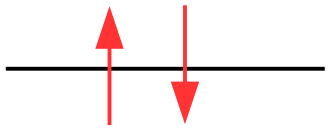
Szabo-Ostlung book chapter 3 teaches how to perform HF in this example:

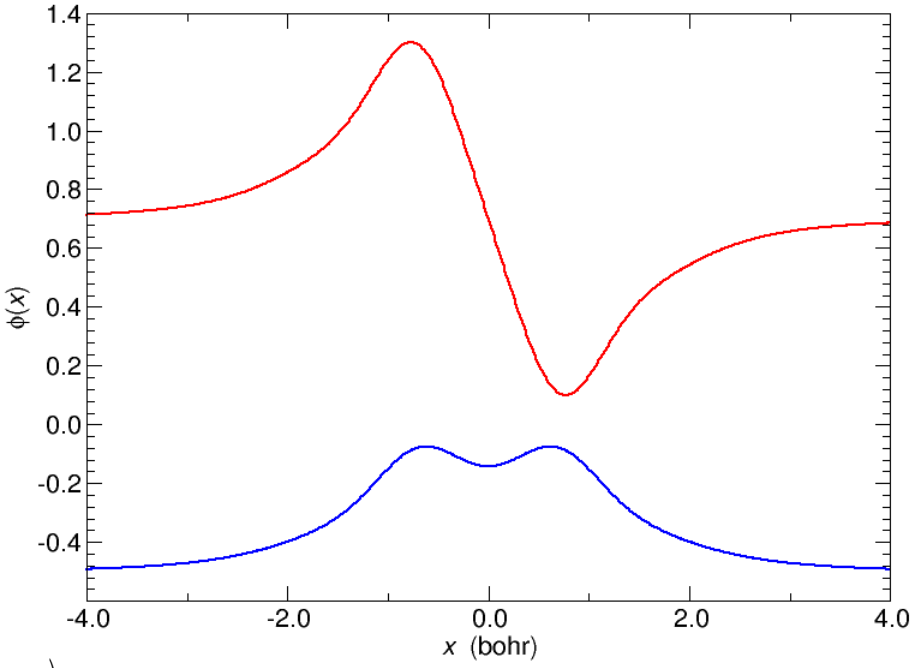
Basis: STO-3G r(H-H) = 1.4 bohr

2 basis functions → 2 eigenstates:

LUMO anti-bonding

HOMO bonding





In eigenvector basis:
Hamiltonian

$$C^T H C = \begin{pmatrix} -0.578 & 0 \\ 0 & 0.670 \end{pmatrix}$$

Coulomb interaction:

$$\left(11\left|\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right|11\right) = 0.675$$

$$\left(12\left|\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right|12\right) = 0.181$$

$$\left(22\left|\frac{1}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\right|22\right) = 0.697$$

Atomic units

Exercice: H_2 in minimal basis: GW@HF

Find the location of the poles of W

Diagonalize the RPA equation

$$\chi^{-1}(\omega) = \begin{matrix} \\ \langle i\,j| \end{matrix} \begin{matrix} |k\,l\rangle \\ \begin{pmatrix} \frac{\omega-(\epsilon_j-\epsilon_i)}{f_i-f_j} & & \\ & \ddots & \\ & & \end{pmatrix} \end{matrix} - \begin{pmatrix} & \\ (i\,j|\frac{1}{r}|k\,l) & \\ & \end{pmatrix}$$

$\Delta\epsilon = \epsilon_2 - \epsilon_1 = 1.248$

$v = (12|1/r|12) = 0.181$

$$\begin{matrix} & |1\,2\rangle & |2\,1\rangle \\ \langle 1\,2| & \frac{\omega-\Delta\epsilon}{2} & 0 \\ \langle 2\,1| & 0 & \frac{\omega+\Delta\epsilon}{-2} \end{matrix} - \begin{pmatrix} v & v \\ v & v \end{pmatrix}$$

$$\Omega = \pm\sqrt{\Delta\epsilon^2 + 4v\,\Delta\epsilon} = \pm 1.569$$

Exercice: H_2 in minimal basis: GW@HF

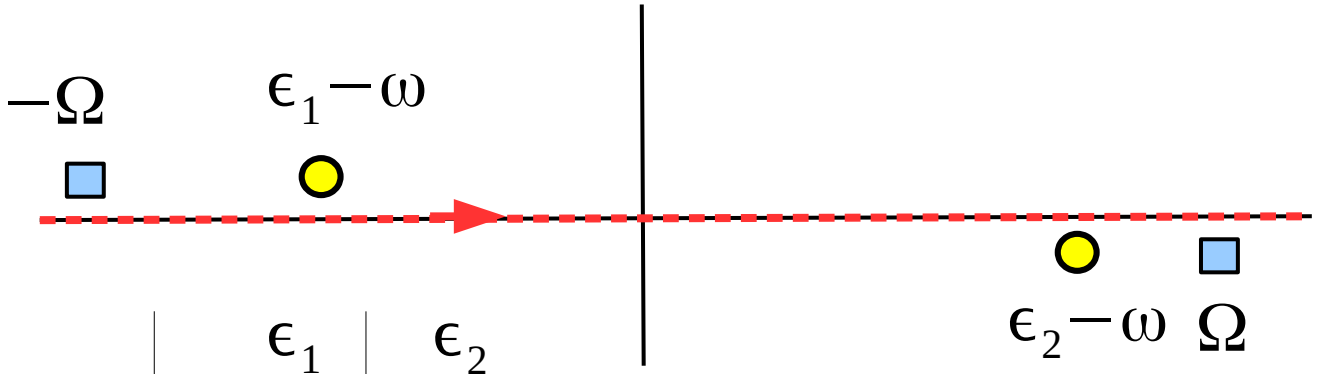
$$\Sigma_c(\omega) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega' G(\omega+\omega') W_p(\omega')$$

$$G(\omega) = \sum_i \frac{\phi_i(\boldsymbol{r})\phi_i(\boldsymbol{r}')}{\omega - \epsilon_i \pm i\eta}$$

$$W_p(\omega) = \sum_s \frac{R_s(\boldsymbol{r})R_s(\boldsymbol{r}')}{\omega - \Omega_s \pm i\eta}$$

$$\Sigma_c(\omega) = \frac{i}{2\pi} \sum_{i\in\{1,2\}} \sum_{s\in\{1\to2,2\to1\}} \int_{-\infty}^{+\infty} d\omega' \frac{\alpha}{\omega+\omega'-\epsilon_i \pm i\eta} \times \frac{\beta}{\omega'-\Omega \pm i\eta}$$

Integration in the complex plane:



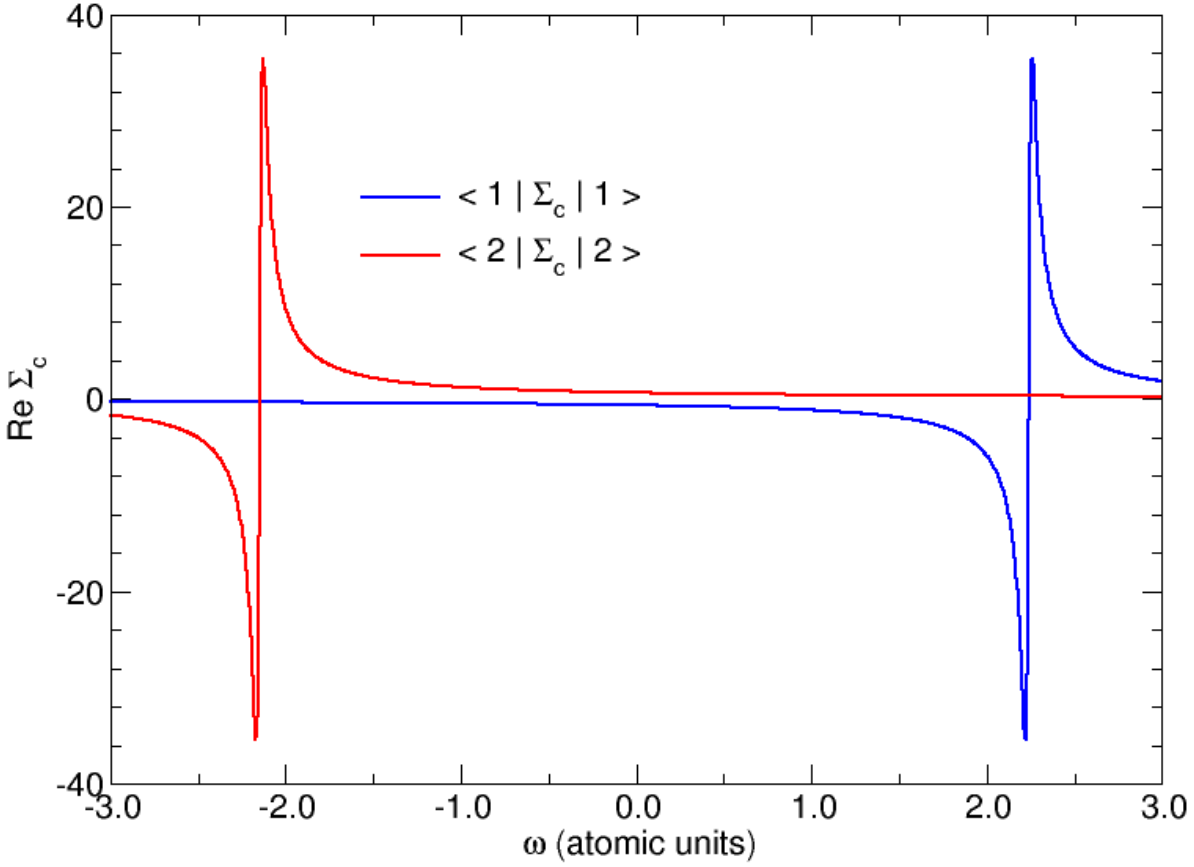
Pole table:

	ϵ_1	ϵ_2
$-\Omega$	✗	$\epsilon_2+\Omega$
Ω	$\epsilon_1-\Omega$	✗

Exercice: H_2 in minimal basis: GW@HF

$$\epsilon_2 + \Omega = 2.239$$
$$\epsilon_1 - \Omega = -2.147$$

Real part of the self-energy from MOLGW



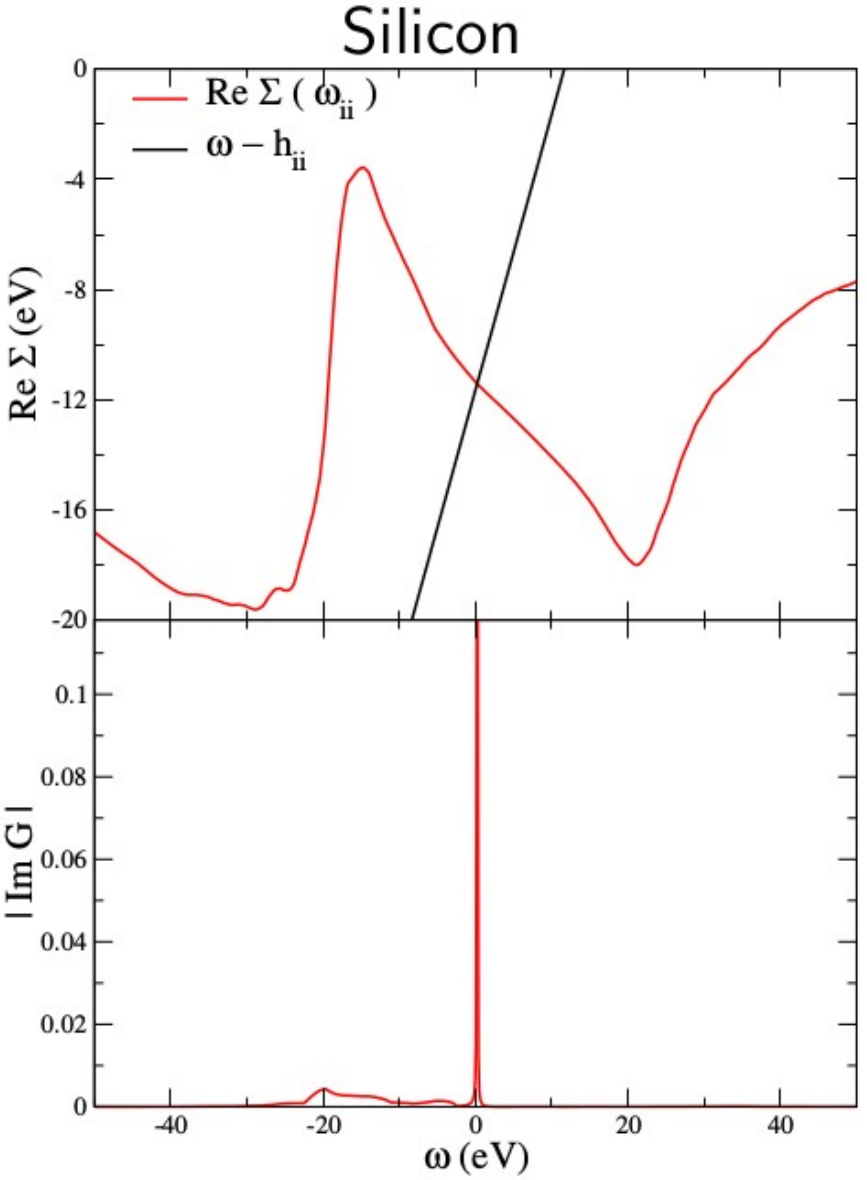
$$\epsilon^{GW}_{\mathrm{HOMO}} = -16.23 \ \mathrm{eV}$$
$$\epsilon^{GW}_{\mathrm{LUMO}} = 18.74 \ \mathrm{eV}$$

Exercice: H_2 in minimal basis: GW@HF

Same conclusions hold for
a many-state case:

Bulk silicon

Plasmon frequency ~ 17 eV
Occupied states ~ -5 - 0 eV
Empty states ~ +2 - … eV



Exercise 0: Where the spectral weight comes from?

Ex: A complex function made of single poles:

$$f(z) = \frac{A_1}{z - a_1} + \frac{A_2}{z - a_2} + \frac{A_3}{z - a_3} + \dots = \sum_i \frac{A_i}{z - a_i}$$

poles : a_i Residus : A_i

$$(z - a_1) f(z) = A_1 + A_2 \frac{z - a_1}{z - a_2} + A_3 \frac{z - a_1}{z - a_3} + \dots$$

$$\lim_{z \to a_1} (z - a_1) f(z) = A_1$$

Now with G $\lim_{z \to a} (z - a) G(z) = \lim_{z \to a} \frac{z - a}{z - \epsilon - \Sigma(z)}$ $\frac{0}{0}$ undetermined

$$G^{-1}(z) = G_0^{-1}(z) - \Sigma(z)$$
$$= z - \epsilon - \Sigma(z)$$

L'Hopital rule : $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

$$= \lim_{z \to a} \frac{1}{1 - \Sigma(z)} = \frac{1}{1 - \Sigma(a)}$$

$$= Z(a)$$

spectral weight

Exercise 1

Green's function in frequency domain

$$iG(\mathbf{r_1}t_1,\mathbf{r_2}t_2)=\theta(t_1-t_2)\sum_{i\,\mathrm{virt}}\phi_i(\mathbf{r_1})\phi_i^*(\mathbf{r_2})e^{-i\epsilon_i(t_1-t_2)} - \theta(t_2-t_1)\sum_{i\mathrm{occ}}\phi_i(\mathbf{r_2})\phi_i^*(\mathbf{r_1})e^{-i\epsilon_i(t_2-t_1)}$$

$$G(\mathbf{r_1},\mathbf{r_2},\omega)=\int d(t_1-t_2)e^{i\omega(t_1-t_2)}G(\mathbf{r_1}t_1,\mathbf{r_2}t_2)$$

$$G(\mathbf{r_1},\mathbf{r_2},\omega)=\sum_i \frac{\phi_i(\mathbf{r_1})\phi_i^*(\mathbf{r_2})}{\omega-\epsilon_i\pm i\eta}$$

Exercise 2:

Fock exchange from Green's functions

$$\Sigma_x(1,2) = iG(1,2)v(1^+,2) \quad \Longrightarrow \quad \Sigma_x(\boldsymbol{r_1},\boldsymbol{r_2},\omega) = -\sum_{i\text{occ}} \frac{\phi_i(\boldsymbol{r_1})\phi_i^*(\boldsymbol{r_2})}{|\boldsymbol{r_1}-\boldsymbol{r_2}|}$$

Exercise 3: let's play with Dyson equations

1) The multiple faces of the Dyson equation

$$\left[\omega - h_{\mathrm{KS}}\right] G_{\mathrm{KS}} = 1$$

$$\left[\omega - h_0 - v_{xc}\right] G_{\mathrm{KS}} = 1$$

$$\left[G_0^{-1} - v_{xc}\right] G_{\mathrm{KS}} = 1$$

$$G_{\mathrm{KS}} = G_0 + G_0 v_{xc} G_{\mathrm{KS}}$$

$$G_{\mathrm{KS}} = G_0 + G_0 v_{xc} G_0 + G_0 v_{xc} G_0 v_{xc} G_0 + \ldots$$

$$G_{\mathrm{KS}}^{-1} = G_0^{-1} - v_{xc}$$

2) Combining the Dyson equations

$$\left.\begin{aligned} G^{-1} &= G_0^{-1} - \Sigma \\ G_{\mathrm{KS}}^{-1} &= G_0^{-1} - v_{xc} \end{aligned}\right\} \quad G^{-1} = G_{\mathrm{KS}}^{-1} - \left(\Sigma - v_{xc}\right)$$

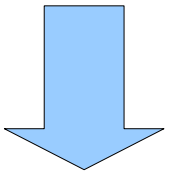
$$1 = \left[G_{\mathrm{KS}}^{-1} - \left(\Sigma - v_{xc}\right)\right] G$$

$$1 = \left[\omega - h_0 - \Sigma\right] G$$

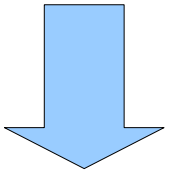
Exercise 4

Derive the standard Adler-Wiser formula (1963):

$$\chi_0(1,2) = -i\,G(1,2)G(2,1)$$



$$\chi_0(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = -\frac{i}{2\pi}\int d\omega' G(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega+\omega')G(\boldsymbol{r}_2,\boldsymbol{r}_1,\omega')$$



$$\chi_0(\boldsymbol{r}_1,\boldsymbol{r}_2,\omega) = \sum_{\substack{i\,\mathrm{occ}\\ j\,\mathrm{virt}}} \phi_i(\boldsymbol{r}_1)\phi_i^*(\boldsymbol{r}_2)\phi_j(\boldsymbol{r}_2)\phi_j^*(\boldsymbol{r}_1) \times\left[\frac{1}{\omega-(\epsilon_j-\epsilon_i)-i\eta}-\frac{1}{\omega-(\epsilon_i-\epsilon_j)+i\eta}\right]$$

Exercice 4: solution (1/3)

Definitions:

$$G(\omega) = \int d\tau \; G(\tau) \; e^{i\omega\tau} \qquad G(\tau) = \frac{1}{2\pi} \int d\omega \; G(\omega) \; e^{-i\omega\tau} \qquad \int d\omega \; e^{i\omega\tau} = 2\pi \, \delta(\tau)$$

$$\chi(\tau) = -i \; G(\tau) \; G(-\tau)$$

Exercice 4: solution (2/3)

$$\chi(\omega) = \int d\tau\, \chi(\tau)\, e^{i\omega\tau} = -\frac{i}{(2\pi)^2} \int d\tau \int d\omega_1\, G(\omega_1)\, e^{-i\omega_1\tau} \int d\omega_2\, G(\omega_2)\, e^{+i\omega_2\tau}\, e^{i\omega\tau}$$

$$= -\frac{i}{(2\pi)^2} \int d\omega_1 \int d\omega_2\, G(\omega_1)\, G(\omega_2) \underbrace{\int d\tau\, e^{+i\tau(\omega+\omega_2-\omega_1)}}_{2\pi\,\delta(\omega+\omega_2-\omega_1)}$$

$$= -\frac{i}{2\pi} \int d\omega_2\, G(\omega+\omega_2)\, G(\omega_2)$$

$$= -\frac{i}{2\pi} \int d\omega_2 \sum_p \frac{\phi_p(r)\,\phi_p^*(r')}{\omega_{+\omega_2} - \epsilon_p \pm i\eta} \times \sum_q \frac{\phi_q(r')\,\phi_q^*(r)}{\omega_2 - \epsilon_q \pm i\eta}$$

Exercice 4: solution (3/3)

$$\chi(\omega) = \int d\tau\, \chi(\tau)\, e^{i\omega\tau} = -\frac{i}{(2\pi)^2} \int d\tau \int d\omega_1\, G(\omega_1)\, e^{-i\omega_1\tau} \int d\omega_2\, G(\omega_2)\, e^{+i\omega_2\tau}\, e^{i\omega\tau}$$

$$= -\frac{i}{(2\pi)^2} \int d\omega_1 \int d\omega_2\, G(\omega_1)\, G(\omega_2) \int d\tau\, e^{+i\tau(\omega+\omega_2-\omega_1)}$$

$$\int d\tau\, e^{+i\tau(\omega+\omega_2-\omega_1)} = 2\pi\,\delta(\omega+\omega_2-\omega_1)$$

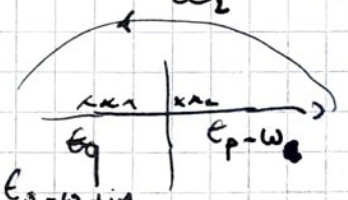
$$= -\frac{i}{2\pi} \int d\omega_2\, G(\omega+\omega_2)\, G(\omega_2)$$

$$= -\frac{i}{2\pi} \int d\omega_2 \sum_p \frac{\phi_p(r)\,\phi_p^*(r')}{\omega+\omega_2-\epsilon_p \pm i\eta} \times \sum_q \frac{\phi_q(r')\,\phi_q^*(r)}{\omega_2-\epsilon_q \pm i\eta}$$

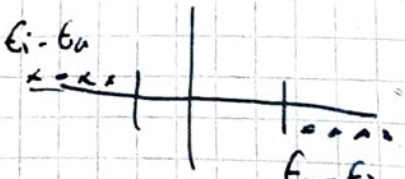
$$= \sum_{ia} \frac{\phi_i(r)\,\phi_i^*(r')\,\phi_a(r')\,\phi_a^*(r)}{\epsilon_i - \epsilon_a - \omega + 2i\eta} + \sum_{ia} \frac{\phi_a(r)\,\phi_a^*(r')\,\phi_i(r')\,\phi_i^*(r)}{\omega + \epsilon_i - \epsilon_a + 2i\eta}$$

pole $\omega_2 = \epsilon_p - \omega + i\eta$

pole : $\epsilon_q + i\eta$



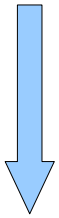
$$= \sum_{ia} \frac{\phi\phi\phi\phi}{\omega - (\epsilon_a - \epsilon_i) + 2i\eta} - \sum_{i0} \frac{\phi\phi\phi\phi}{\omega - (\epsilon_i - \epsilon_a) - 2i\eta}$$



Exercise 5

Derive that the product in time becomes a convolution in frequency:

$$\Sigma(\boldsymbol{r_1}, \boldsymbol{r_2}, t_1 - t_2) = i G(\boldsymbol{r_1}, \boldsymbol{r_2}, t_1 - t_2) W(\boldsymbol{r_2}, \boldsymbol{r_1}, t_2 - t_1)$$



$$G(\boldsymbol{r_1}, \boldsymbol{r_2}, \omega) = \int d(t_1 - t_2) e^{i\omega(t_1 - t_2)} G(\boldsymbol{r_1} t_1, \boldsymbol{r_2} t_2)$$

$$G(\boldsymbol{r_1}, \boldsymbol{r_2}, t_1 - t_2) = \frac{1}{2\pi} \int d\omega e^{-i\omega(t_1 - t_2)} G(\boldsymbol{r_1}, \boldsymbol{r_2}, \omega)$$

$$\Sigma(\boldsymbol{r_1}, \boldsymbol{r_2}, \omega) = \frac{i}{2\pi} \int d\omega' G(\boldsymbol{r_1}, \boldsymbol{r_2}, \omega + \omega') W(\boldsymbol{r_2}, \boldsymbol{r_1}, \omega')$$

Exercice 6: Feynman diagram drawing

a) Draw all the 1st order diagrams for the self-energy
b) Draw all the 2nd order diagrams for the self-energy
c) What is the difference between the proper and the improper self-energy
d) How self-consistency can simplify the expansion?

Self-energy diagram drawing rules:

1. Diagrams are combinations of arrows (Green's function) and horizontal lines (Coulomb interaction).

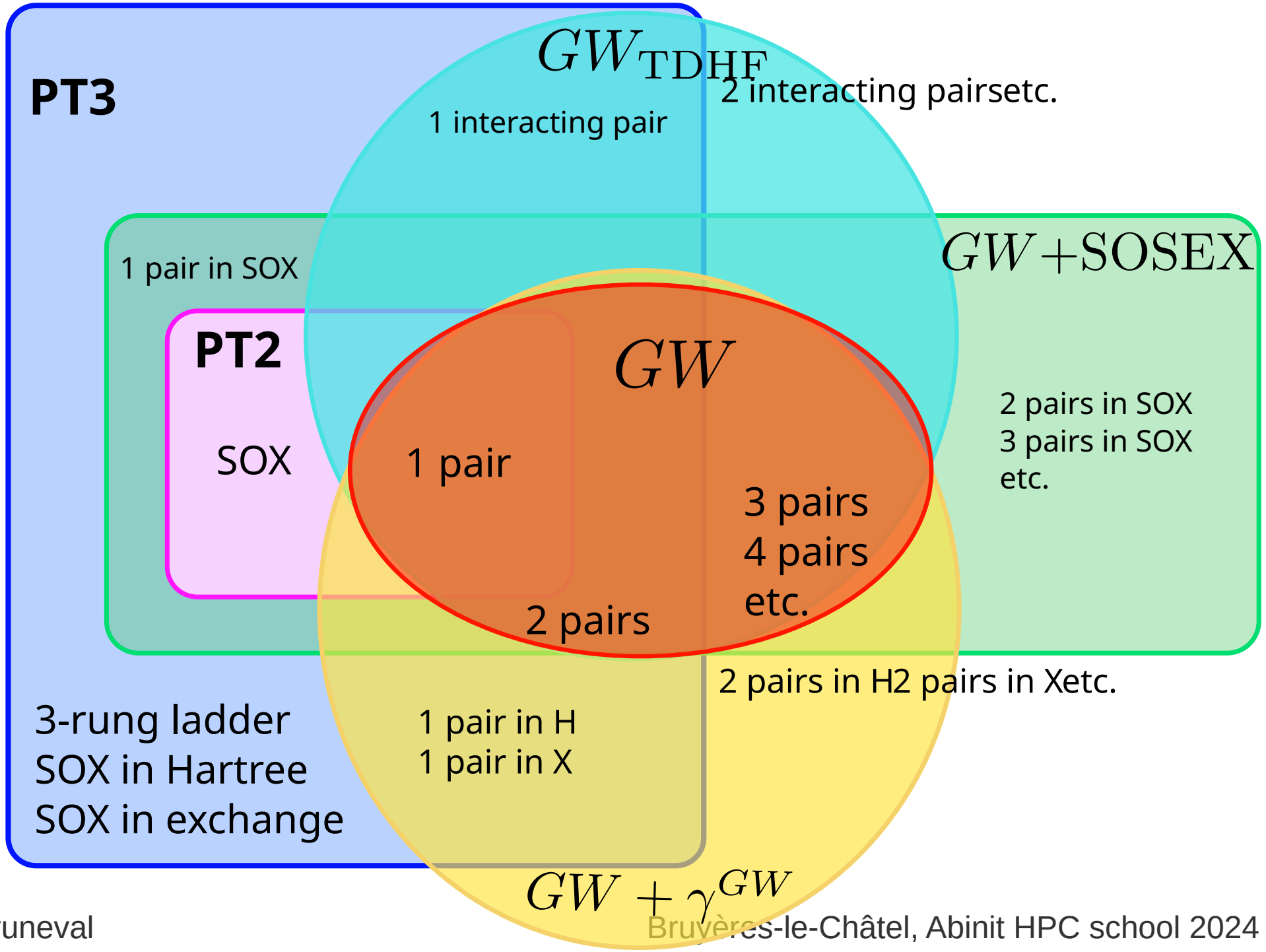
2. Diagrams should be connected.

3. Self-energy have an entry point and an exit point (possibly the same).

4. Each intersection (=vertex) should conserve the particle numbers.

5. A proper self-energy diagram cannot be cut (by removing an arrow) into another smaller self-energy.

Exercise 6:


PT3
GW_{TDHF}
1 interacting pair
2 interacting pairsetc.
1 pair in SOX
GW+SOSEX
PT2
GW
SOX
1 pair
2 pairs in SOX
3 pairs in SOX
etc.
3 pairs
4 pairs
etc.
2 pairs
2 pairs in H2 pairs in Xetc.
3-rung ladder
SOX in Hartree
SOX in exchange
1 pair in H
1 pair in X
$GW + \gamma^{GW}$

Perturbation theory up to third order

$$\Sigma^{(3)}_{pq}(\omega) = \sum_{I=1}^{6} (AI + CI + DI)$$

$$A1 = -\sum \frac{(2V_{pkqj} - V_{pkjq})(2V_{jiab} - V_{jiba})V_{abki}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_k + \varepsilon_i - \varepsilon_a - \varepsilon_b)}$$

$$A2 = \sum \frac{(2V_{pcqb} - V_{pcbq})(2V_{jiab} - V_{jiba})V_{jica}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_c)}$$

$$A3 = \sum \frac{(2V_{pcqj} - V_{pcjq})(2V_{jiab} - V_{jiba})V_{abci}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j - \varepsilon_c)}$$

$$A4 = \sum \frac{(2V_{pjqc} - V_{pjcq})(2V_{jiab} - V_{jiba})V_{abci}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j - \varepsilon_c)}$$

$$A5 = -\sum \frac{(2V_{pbqk} - V_{pbkq})(2V_{jiab} - V_{jiba})V_{ijka}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_k - \varepsilon_b)}$$

$$A6 = -\sum \frac{(2V_{pkqb} - V_{pkbq})(2V_{jiab} - V_{jiba})V_{ijka}}{(\varepsilon_j + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_k - \varepsilon_b)}$$

$$C1 = \sum \frac{(2V_{piab} - V_{piba})V_{abcd}V_{qicd}}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\omega + \varepsilon_i - \varepsilon_c - \varepsilon_d)}$$

$$C2 = \sum \frac{(2V_{piab} - V_{piba})V_{abjk}V_{qijk}}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j + \varepsilon_k - \varepsilon_a - \varepsilon_b)}$$

$$C3 = \sum \frac{(2V_{pijk} - V_{pikj})V_{abjk}V_{qiab}}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j + \varepsilon_k - \varepsilon_a - \varepsilon_b)}$$

$$C4 = \sum \frac{(2V_{paij} - V_{paji})V_{ijbc}V_{qabc}}{(\omega + \varepsilon_a - \varepsilon_i - \varepsilon_j)(\varepsilon_i + \varepsilon_j - \varepsilon_b - \varepsilon_c)}$$

$$C5 = \sum \frac{(2V_{pabc} - V_{pacb})V_{ijbc}V_{qaij}}{(\omega + \varepsilon_a - \varepsilon_i - \varepsilon_j)(\varepsilon_i + \varepsilon_j - \varepsilon_b - \varepsilon_c)}$$

$$C6 = -\sum \frac{(2V_{pakl} - V_{palk})V_{klij}V_{qaij}}{(\omega + \varepsilon_a - \varepsilon_i - \varepsilon_j)(\omega + \varepsilon_a - \varepsilon_k - \varepsilon_l)}$$

$$D1 = \sum \left\{ \frac{V_{piab}\left[V_{ajic}(V_{qjcb} - 2V_{qjbc}) + V_{ajci}(V_{qjbc} - 2V_{qjcb})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\omega + \varepsilon_j - \varepsilon_b - \varepsilon_c)} + \frac{V_{piba}\left[V_{ajic}(4V_{qjbc} - 2V_{qjcb}) + V_{ajci}(V_{qjcb} - 2V_{qjbc})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\omega + \varepsilon_j - \varepsilon_b - \varepsilon_c)} \right\}$$

$$D2 = \sum \left\{ \frac{V_{pica}\left[V_{abij}(4V_{qbcj} - 2V_{qbjc}) + V_{abji}(V_{qbjc} - 2V_{qbcj})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_c)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} + \frac{V_{piac}\left[V_{abij}(V_{qbjc} - 2V_{qbcj}) + V_{abji}(V_{qbcj} - 2V_{qbjc})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_c)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} \right\}$$

$$D3 = \sum \left\{ \frac{V_{pcja}\left[V_{jicb}(V_{qiba} - 2V_{qiab}) + V_{jibc}(V_{qiab} - 2V_{qiba})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j + \varepsilon_i - \varepsilon_b - \varepsilon_c)} + \frac{V_{pcaj}\left[V_{jicb}(4V_{qiab} - 2V_{qiba}) + V_{jibc}(V_{qiba} - 2V_{qiab})\right]}{(\omega + \varepsilon_i - \varepsilon_a - \varepsilon_b)(\varepsilon_j + \varepsilon_i - \varepsilon_b - \varepsilon_c)} \right\}$$

$$D4 = \sum \left\{ \frac{V_{pakj}\left[V_{jiab}(4V_{qikb} - 2V_{qibk}) + V_{jiba}(V_{qibk} - 2V_{qikb})\right]}{(\omega + \varepsilon_a - \varepsilon_j - \varepsilon_k)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} + \frac{V_{pajk}\left[V_{jiab}(V_{qibk} - 2V_{qikb}) + V_{jiba}(V_{qikb} - 2V_{qibk})\right]}{(\omega + \varepsilon_a - \varepsilon_j - \varepsilon_k)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} \right\}$$

$$D5 = \sum \left\{ \frac{V_{pibk}\left[V_{jiab}(V_{qajk} - 2V_{qakj}) + V_{jiba}(V_{qakj} - 2V_{qajk})\right]}{(\omega + \varepsilon_a - \varepsilon_j - \varepsilon_k)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} + \frac{V_{pikb}\left[V_{jiab}(4V_{qakj} - 2V_{qajk}) + V_{jiba}(V_{qajk} - 2V_{qakj})\right]}{(\omega + \varepsilon_a - \varepsilon_j - \varepsilon_k)(\varepsilon_i + \varepsilon_j - \varepsilon_a - \varepsilon_b)} \right\}$$

$$D6 = -\sum \left\{ \frac{V_{paki}\left[V_{ibaj}(4V_{qbkj} - 2V_{qbjk}) + V_{ibja}(V_{qbjk} - 2V_{qbkj})\right]}{(\omega + \varepsilon_a - \varepsilon_i - \varepsilon_k)(\omega + \varepsilon_b - \varepsilon_j - \varepsilon_k)} + \frac{V_{paik}\left[V_{ibaj}(V_{qbjk} - 2V_{qbkj}) + V_{ibja}(V_{qbkj} - 2V_{qbjk})\right]}{(\omega + \varepsilon_a - \varepsilon_i - \varepsilon_k)(\omega + \varepsilon_b - \varepsilon_j - \varepsilon_k)} \right\}$$

L.S. Cederbaum, W. Domcke (1977)

Exercice 6: effect of the other diagrams

Ionization potentials of the **GW100** benchmark (reference CCSD(T))