**Proposition 2.17** *For any digraphs $F, G$, and $H$*

1. $H^{G+F} \simeq H^G \times H^F$
2. $H^{G\times F} \simeq (H^G)^F$.

**Proof** We prove 2 and leave 1 as an exercise. To each mapping $m$ of $V(F)$ to $V(H^G)$ we assign the mapping $\Phi(m)$ of $V(G\times F)$ to $V(H)$ which maps the vertex $(u,v)$ of $G \times F$ to the vertex $(m(v))(u)$ of $H$. (Recall that $m(v)$ is a mapping from $V(G)$ to $V(H)$.) It is clear that $\Phi$ is a bijective mapping between $V((G^H)^F)$ and $V(H^{G\times F})$. It is also easy to check that $\Phi$ is a homomorphism from $(G^H)^F$ to $H^{G\times F}$. Indeed, if $mm' \in E((G^H)^F)$, then $\Phi(m)\Phi(m') \in E(H^{G\times F})$, since $(\Phi(m))(u,v) = (m(v))(u)$ is adjacent to $(\Phi(m'))(u',v') = (m'(v'))(u')$ whenever $(u,v)$ is adjacent to $(u',v')$ in $G \times F$. □

**Corollary 2.18** *$G \times F \to H$ if and only if $F \to H^G$.*

**Proof** In fact, it is clear that $\hom(G \times F, H) = \hom(F, H^G)$, as both sides count the number of loops in $(H^G)^F$. □

We make two other observations.

**Proposition 2.19** *For any digraphs $G$ and $H$*

1. *$H$ is isomorphic to an induced subgraph of $H^G$*
2. $G \times H^G \to H$

**Proof** The constant mapping $f_h : V(G) \to V(H)$, which takes all vertices of $G$ to the vertex $h \in V(H)$, is a vertex of $H^G$. It is easy to check that the association $h \mapsto f_h$ is an isomorphism of $H$ onto an induced subgraph of $H^G$, proving 1. To prove 2, define $\eta$ to be the *evaluation* mapping on $V(G) \times V(H)^{V(G)}$, i.e., let $\eta(u,m) = m(u)$, where $m \in V(H)^{V(G)}, u \in V(G)$. It is again a simple exercise to show that $\eta$ is a homomorphism $G \times H^G \to H$. □

## 2.5 Shift graphs

Theorem 1.9 states that there exist graphs of arbitrarily high chromatic number and girth. The constructive proofs of this result are quite difficult. In contrast, there are simple constructions of graphs with high *odd* girth and chromatic number. Recall that the odd girth of a nonbipartite graph $G$ is the length of a shortest odd cycle in $G$, and that the odd girth is more relevant than girth from the perspective of graph homomorphisms (cf. Exercise 2 in Chapter 1).

General shift graphs have as vertices fixed-length strings over a fixed alphabet, and adjacency is defined as follows. If $\ell \geq 2$, the string $a_1a_2\cdots a_\ell$ is adjacent to $b_1b_2\cdots b_\ell$ just if $b_i = a_{i+1}$ for all $i = 1, 2, \cdots, \ell - 1$. The generic shift graph (known as the *de Bruijn graph*) $dB(n,\ell)$, has as vertices all strings of length $\ell$ over the alphabet $0, 1, 2, \cdots, n-1$. For our purposes we shall take an induced subgraph of the de Bruijn graph. The *shift graph* $R(n,\ell)$ is the subgraph of $dB(n,\ell)$ induced by all monotone strings, i.e., strings $a_1a_2\cdots a_\ell$ with $a_1 < a_2 < \cdots < a_\ell$.

We observe that shift graphs have an inherent orientation—reading the definition ('is adjacent *to*') in the context of digraphs results in a *shift digraph* $\vec{R}(n,\ell)$.

This digraph has arcs from a vertex $a_1a_2\cdots a_\ell$ to all vertices $a_2a_3\cdots a_\ell a$, for all characters $a, a > a_\ell$. In this way the arc from $a_1a_2\cdots a_\ell$ to $a_2a_3\cdots a_\ell a_{\ell+1}$ can be associated with a string of length $\ell+1$, namely $a_1a_2\cdots a_\ell, a_{\ell+1}$. Thus there is a natural connection to line digraphs. The *line digraph* $L(G)$ of a digraph $G$ has as its vertices the arcs of $G$, and has an arc from $ab \in E(G)$ to $cd \in E(G)$, just if $b = c$. It follows from the above observation that shift digraphs satisfy $\vec{R}(n, \ell+1) = L(\vec{R}(n, \ell))$. From the definition of $\vec{R}(n, 2)$ we see that $\vec{R}(n, 2) = L(\vec{T}_n)$, therefore, it is convenient to define $\vec{R}(n, 1)$ as $\vec{T}_n$, the transitive tournament, this time taken on the vertices $0, 1, \cdots, n-1$.

**Lemma 2.20** *Let $n$ be a positive integer. Then*

- $\vec{R}(n, 1) = \vec{T}_n$, *and*
- $\vec{R}(n, \ell+1) = L(\vec{R}(n, \ell))$, *for all* $\ell \geq 1$. □

Note that all shift digraphs are irreflexive.

In the following, we take the chromatic number of an irreflexive digraph to be the chromatic number of the underlying graph. The chromatic number of the line digraph of $G$ is bounded on both sides by a (logarithmic) function of the chromatic number of $G$.

**Lemma 2.21** *For any digraph $G$,*

$$\log_2 \chi(G) \leq \chi(L(G)) \leq \min\left\{k : \chi(G) \leq \binom{k}{\lfloor k/2 \rfloor}\right\}.$$

**Proof** In this section we only need the lower bound, which we prove first. A *proper colouring* of $L(G)$ is a colouring of the arcs of $G$ in which no two arcs $ab$ and $bc$ have the same colour. Thus consider such a proper colouring of $L(G)$ with $t$ colours, and form the spanning subgraphs $G_1, G_2, \cdots, G_t$ of $G$, where $E(G_i)$ consists of all the arcs that obtained colour $i$. Since each vertex of $G_i$ has either outdegree zero or indegree zero, the underlying graph of $G_i$ is bipartite, for every $i$. It is easy to check (cf. Exercise 11 in Chapter 1) that if each $c_i$ is a two-colouring of $G_i$, then $c$ defined by $c(v) = (c_1(v), c_2(v), \cdots, c_t(v))$ is a proper colouring of $G$, and so $\chi(G) \leq 2^t$.

To prove the upper bound, we show that $\chi(G) \leq \binom{k}{\lfloor k/2 \rfloor}$ implies that $\chi(L(G)) \leq k$. Indeed, any colouring $c$ of $G$ with $\binom{k}{\lfloor k/2 \rfloor}$ colours can be viewed as an assignment of $\lfloor k/2 \rfloor$-subsets of $\{1, 2, \cdots, k\}$ to the vertices of $G$, in which adjacent vertices obtain disjoint subsets. Define a mapping $f$ of $V(L(G))$ to $\{1, 2, \cdots, k\}$ in which the arc $xy$ of $G$ (being a vertex of $L(G)$) receives the value $f(xy) = u$ where $u$ is any element of $f(x) - f(y)$. It is easy to see that $f$ is indeed a $k$-colouring of $L(G)$, i.e., that arcs $xy$ and $yz$ cannot obtain the same colour. Note that the upper bound is roughly of the order $\log_4 \chi(G)$; it will be used in Section 2.6. □

**Lemma 2.22** *The graph $R(n, \ell)$ has odd girth at least $2\ell + 1$.*

**Proof** We proceed by induction on $\ell$. Indeed, the statement is trivial when $\ell = 1$. Hence consider an odd cycle $C$ in $R(n, \ell)$. It consists of vertices $r_1, r_2, \cdots, r_t$ ($t$ odd) of $R(n, \ell)$. Thus the strings $r_1, r_2, \cdots, r_t$ are arcs of $R(n, \ell - 1)$, and for each consecutive pair $r_i, r_{i+1}$ of arcs, the head of one equals the tail of the other. This defines a closed walk of the same length $t$ in $R(n, \ell - 1)$. Of course, a closed odd walk must contain a closed odd cycle, so we only have to observe that the closed walk $r_1, r_2, \cdots, r_t$ is itself not a cycle (hence the odd cycle is shorter and we can apply the induction hypothesis). This follows from the fact that the arc that is the lexicographically smallest string has both its neighbours in $C$ have tails equal to its head, and the arc that is the lexicographically largest string has both its neighbours in $C$ have heads equal to its tail (Fig. 2.6). □

The situation is illustrated in Fig. 2.6; for convenience we show parts of the *graph* $R(n, 2)$ and the *digraph* $\vec{R}(n, 1)$.

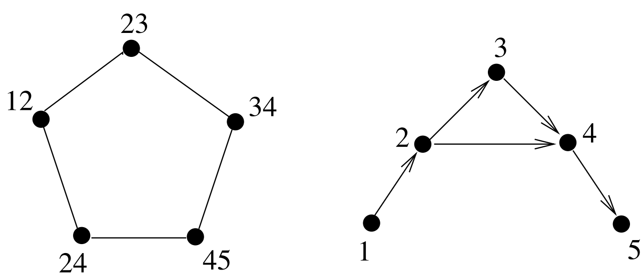


FIG. 2.6. A five-cycle in $R(n, 2)$ corresponds to a closed walk of length five in $\vec{R}(n, 1) = \vec{T}_n$.

Since the chromatic number of $R(n, \ell)$ is controlled by both $n$ and $\ell$, while the odd girth only by $\ell$, we can now construct graphs of arbitrarily high odd girth and chromatic number.

**Theorem 2.23** *Let $g, k$ be positive integers, $g \geq 3$ odd. Then there exists a graph $S(g, k)$ with odd girth at least $g$ and chromatic number at least $k$.*

**Proof** It is enough to set $S(g, k) = R(n, \ell)$ with $2\ell + 1 = g$ and $n = 2^{2^{2^{\cdot^{\cdot^{k}}}}}$, where the tower of powers has length $\ell$. (In other words, $n = n_\ell$ is defined recursively as $n_1 = 2^k$, and $n_{\ell+1} = 2^{n_\ell}$.) □

It may appear that it will be much harder to construct graphs of arbitrarily high odd girth $g$ and chromatic number $k$ which are *uniquely $k$-colourable*, but the product construction makes this easy.

**Proposition 2.24** *If $G$ is a connected graph which is not $k$-colourable, then $G \times K_k$ is uniquely $k$-colourable.*

**Proof** The second projection $\rho : G \times K_k \to K_k$ (taking each $(u, x)$ to $x$) is a $k$-colouring of $G \times K_k$. We shall show that every other $k$-colouring is obtained from

$\rho$ by renaming colours, i.e., is the projection $\rho$ followed by an automorphism of $K_k$. Thus let $c : G \times K_k \to K_k$ be a homomorphism, i.e., a $k$-colouring of $G \times K_k$. Suppose first that for some vertex $u \in V(G)$ the values $c(u, x) = x, x \in V(K_k)$, are all distinct. In this case, for any $v$ adjacent to $u$, and any $x \in V(K_k)$, the vertex $(v, x)$ is adjacent to all $(u, y), y \neq x$, and so $c(v, x) = c(u, x)$. In particular, all $c(v, x), x \in V(K_k)$, are again distinct. By the connectivity of $G$ we conclude that all values $c(u, x)$ with the same $x$ are equal, i.e., that the colouring $c$ differs from $\rho$ only in renaming the colours.

We prove the existence of such a vertex $u$ by contradiction. Suppose that for each $u \in V(G)$ there are two vertices $x_u, y_u$ of $K_k$ with $c(u, x_u) = c(u, y_u)$, and denote by $c(u)$ this common value $c(u, x_u) = c(u, y_u)$. We claim that $c$ is a $k$-colouring of $G$. Consider an edge $uv \in E(G)$. Note that for every $x \in V(K_k)$ at least one of $(u, x_u), (u, y_u)$ is adjacent to $(v, x)$, and so $c(v, x) \neq c(u)$; in particular, $c(v) \neq c(u)$. □

**Corollary 2.25** *Let $g, k$ be positive integers, $g$ odd. Then there exists a uniquely $k$-colourable graph of odd girth at least $g$.*

(Note that a uniquely $k$-colourable graph must have chromatic number $k$.)

**Proof** Let $G$ be a graph of chromatic number at least $k+1$ from the theorem (which clearly may be chosen to be connected). Then the proposition implies that $G \times K_k$ is uniquely $k$-colourable. It only remains to observe that $G \times K_k \to G$ and hence must not have odd girth smaller than $G$. □

The existence of uniquely $k$-colourable graphs with high *girth* is discussed in Corollary 3.17.

The proof of Proposition 2.24 has a nice interpretation in the power $K_k^{K_k}$. According to the isomorphism of $K_k^{K_k \times G}$ and $(K_k^{K_k})^G$ from Proposition 2.17, 2, each homomorphism $c : G \times K_k \to K_k$ corresponds to a set of vertices of $K_k^{K_k}$— one for each vertex $u \in V(G)$ (being the mapping that takes $x$ to $c(u, x)$). Since this mapping is a homomorphism $K_k \to K_k$ if and only if all values $c(u, x) = x$, for $x \in V(K_k)$, are distinct, the two parts of the proof of Proposition 2.24 correspond respectively to the following two assertions.

**Corollary 2.26** *Consider the power $K_k^{K_k}$.*

- *The subgraph induced by the vertices with loops has no other edges.*
- *The subgraph induced by the vertices without loops is $k$-colourable.* □

## 2.6 The Product Conjecture and graph multiplicativity

One of the most challenging open problems in this area is the following *Product Conjecture.*

**Conjecture 2.27** *If $G, H$ are graphs, then*

$$\chi(G \times H) = \min(\chi(G), \chi(H)).$$