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The nonlinear Schrödinger equation is a well-known partial differential equation that provides a successful model in nonlinear optic theory, as well as other applications. In this dissertation, following a survey of mathematical literature, the geometric theory of differential equations is applied to the nonlinear Schrödinger equation.

The main result of this dissertation is that the known list of local conservation laws for the nonlinear Schrödinger equation is complete. A theorem is proved and used to produce a sequence of local conservation law characteristics of the nonlinear Schrödinger equation. The list of local conservation laws as given by Faddeev and Takhtajan and a theorem of Olver, which provides a one-to-one correspondence between equivalence classes of conservation laws and equivalence classes of their characteristics, are then used to prove the main result.

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Johnner Barrett, Author

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Chapter 1

Introduction

Differential equations form an interface between mathematical theory and concrete application. This introduction provides examples of differential equations and their applications. The nonlinear Schrödinger equation is introduced, along with background information and a review of the literature.

1.1 PDEs and Applications

Partial differential equations (PDEs, henceforth) are systems of equations relating partial derivatives of functions. A solution of a PDE is a function satisfying the system of equations and any other specified requirements such as boundary conditions, differentiability and so on. Given a system of differential equations, two main questions are the existence and uniqueness of solutions. For ordinary differential equations, where only one independent variable is involved, many techniques are available, both analytical and numerical, and the theory is quite satisfactory. For general situations involving partial derivatives, the results are much less comprehensive.

Differential equations arising as models are of primary importance, as these embody an element of truth. Such equations provide a means of quantifying a phenomenon, and give insight into the underlying situation. It can be argued that a model's merit hinges on its usefulness. Both ordinary and partial differential equations model a wide range of applications. Ordinary differential equations have their place in vibrations and electricity. For PDEs, the one-dimensional heat equation models heat flow [17] and is useful for many diffusion problems, including stock-option pricing, via the Black-Scholes model [7]. The wave equation models such phenomena as vibrations of strings, membranes, and water waves in a variety of contexts and dimensions [16]. One of the most well-known systems of differential equations is the system of Navier-Stokes equations [16], which are regularly used in aircraft design and nautical engineering, among other applications. Many special cases of the Navier-Stokes system have been resolved but finding a general solution remains one of the most significant open problems in mathematics, with a million dollar prize attached.

1.2 The Geometric Viewpoint

Geometry unlocks the secrets of PDEs. Solutions to a differential equation may be viewed as an abstract surface and the invariants of this surface will indicate properties of the PDEs.

Symmetries of PDEs are transformations which take solutions to solutions; a precise definition is given in Chapter 3. Symmetries may be continuous, such as translations and rotations, or discrete, such as a reflection. The use of symmetry methods to examine and solve ordinary differential equations (ODEs) dates back to Sophus Lie, who developed a theory concerning the local group of infinitesimal symmetries of solutions to a given ODE [25]. This theory parallels Galois theory for polynomials. It is no exaggeration that Lie theory unifies all basic techniques for solving ODEs. The Lie theory of ODEs and the Galois theory of polynomials represent a profound approach to problem solving by considering the symmetries of the solutions as a way to find them. While no appropriate analogue has been found for partial differential equations, Lie symmetry methods are powerful in characterizing the nature of certain systems, often leading to explicit solutions. Symmetries can often suggest a change of variable. Given a known solution, symmetries will generate more solutions, possibly infinitely many. In the case of the heat equation, a Lie symmetry produces the heat kernel, an integral function used to construct solutions to boundary value problems [25].

Conservation laws are fundamental to understanding any physical model. Conservation laws provide constraints on solutions, giving information on long-term behavior and forming the building blocks of further models; a formal definition appears in Chapter 4. The importance of knowing the conservation laws of a given system cannot be overstated. Well-known examples are the conservation of energy and the conservation of angular momentum. Determining conservation laws possessed by solutions to given models is an important step in the analysis of the model. Given a system whose solutions are preserved by rotations, it is not difficult to believe that this property is related to a statement about angular momentum.

Noether's theorem exhibits the relationship between symmetries and conservation laws for certain systems of PDEs [25], giving a means of identifying conservation laws for such systems. Many physically meaningful systems, such as the heat equation, do not satisfy the hypotheses of Noether's Theorem, making a connection between symmetries and conserved quantities difficult to establish in general. A full understanding of conservation laws and the symmetry structure of a given system of differential equations is a big step toward a full understanding of the phenomenon it models.

There is a class of PDEs that stand out from the rest. A nonlinear system of PDEs is *integrable* if it is solvable by the so-called inverse scattering method (see [3]). Many integrable systems share several remarkable traits, such as the existence of soliton solutions, possession of a Bäcklund transformation, a recursion operator, an infinite number of conservation laws and a Hamiltonian formulation [3, 15, 37]. Soliton solutions are solitary wave packets that are largely resistant to distortion from collision degradation and age [2]. Bäcklund transformations are symmetries of a PDE which are not continuous Lie symmetries [18]. Such transformations are often found by inspection. A recursion operator is a differential operator which takes symmetries to other symmetries and can be used to generate a sequence of conservation laws [25]. The Painlevé property for PDEs, which hypothesizes that solutions can be expressed as Laurent series, has been proposed as a test of integrability [37]. Integrable systems are the focus of much research and application.

Many integrable systems are physically significant. The Korteweg-deVries equation describes long waves in shallow water and is one of the most studied integrable equations [4]. It arises in connection with the Maurer-Cartan form on the Lie group $\mathrm{SL}_2(\mathbb{R})$, the special linear group of two-by-two matrices [11] and as the Euler equation for the Virasoro-Bott group of S^1–diffeomorphisms [29]. The Kadomtsev-Petviashvili equation models shallow water waves in two dimensions and is a higher-dimensional analogue of the Korteweg-deVries equation [3]. The self-dual Yang-Mills equation describes the weak interaction of particles [4]. There is speculation that the self-dual Yang-Mills equations act as a source system, from which all other integrable systems arise via symmetry reduction [4], a claim with intriguing physical ramifications.

1.2 The NLS

The nonlinear Schrödinger equation (NLS) is a well-known partial differential equation with widespread applications. The NLS appears in many forms, such as a coupled system [28], a vector version [30] and a nonlocal version involving integral terms [26]. The form used by Faddeev and Takhtajan [15], which will be used here, is

$$iq_t = -q_{xx} + 2\chi \left|q\right|^2 q, \tag{1.1}$$

where $\chi \in \mathbb{R}$ and q is a complex-valued probability density function. Equation (1.1) is known as the quantum NLS or cubic (1+1)-dimensional NLS.

The NLS is a cornerstone of quantum theory, modeling the interaction of an indeterminate number of particles. Originally constructed to give probability density functions for electron energy levels [6], its influence is now felt in many areas. The NLS models the general slow modulation of gravity waves on water layers of uniform depth [19] and leads to a successful three-dimensional model [13]. A two-dimensional form of the NLS describes the complex amplitude of wave trains in elastic solids [11]. It is enormously successful in nonlinear optic theory where it describes the propagation of a pulse envelope in a monomode optical fiber [9] and it acts as a model of self-focusing waves [38].

The NLS is known to be integrable and to possess soliton solutions [9, 20], several of which are computed in [11, 14, 21, 38]. An explicit example is the function $\psi(t,x)$ where

$$\psi(t,x) = \operatorname{sech}(x + y + 2k\eta t)e^{ikx+i(k^2-\eta^2)t} \tag{1.2}$$

where y, k and η are each constants. The soliton solution $\psi(t,x)$ is a solution of the NLS in the form $iq_t = q_{xx} + 2\alpha qq^*q$ for $\alpha = \pm 1$ where $q = \psi(t,x)$[21]. A Bäcklund transformation for the NLS is given by

$$\psi = \varphi + \frac{2(\lambda - \lambda^*)q^*}{2 - kqq^*} \tag{1.3}$$

where φ is solution of the NLS in the form $i\varphi_t = -\varphi_{xx} + k\varphi^*\varphi^2$ and q is obtained from the equations

$$q_x = i(\frac{1}{2}k\varphi q^2 + \varphi^* - \lambda q) \tag{1.4}$$

$$q_t = -\frac{1}{2}k\varphi_x q^2 + \varphi_x^* + ik\varphi\varphi^* q - \frac{i}{2}\lambda k\varphi q^2 - i\lambda\varphi^* + i\lambda^2 q \tag{1.5}$$

where λ is a complex constant and k is a real constant [12, 14]. The NLS has a nonlocal recursion operator (5.31), derived in [4] and used in [21, 22, 23]. The Hamiltonian, another integrability trait, is an observable corresponding to the energy integral for solutions of a system and is discussed for the NLS in [8, 14, 15, 26, 32]. The NLS is related to a loop algebra of the Kac-Moody type [29], and is a member of the AKNS hierarchy, which is dealt with in [2, 32], and is solvable by the inverse scattering method [8, 15, 38].

Ablowitz, Nixon, Horikis and Frantzeskakis make use of the conservation of energy and momentum of the NLS to investigate perturbations of dark solitons [1]. The conservation of charge, the number of particles and momentum for the NLS are discussed by Watanabe, Yajima and Miyakawa [36], among others. A conservation law corresponding to a boost into a moving coordinate frame is given in [24, 30, 36]. Both [14] and [30] look at generalized conservation laws of the NLS. Sequences of conservation laws for the NLS are given in [14, 15, 21, 27, 38]. Specifically, Faddeev and Takhtajan derive the conserved densities

$$P_n = q^* w_n \tag{1.6}$$

where

$$w_n = -i\partial_x w_{n-1} + \chi q^* \sum_{k=1}^{n-2} w_k w_{n-l-k} \tag{1.7}$$

and $w_1 = q$, giving an infinite sequence of conservation laws for (1.1) [15]. Ablowitz and Segur [31] show that this sequence may be written in terms of an asymptotic solution. Pritula and Vekslerchik [27] exhibit a generating function for conserved densities.

1.3 Main Result

The question of completeness of the known conservation laws for the NLS was posed by Terng, after working on the NLS [32], and by Pohjanpelto, after working on classifying conservation laws [5]. The main result of this dissertation proves the completeness of the known list of local conservation laws of the NLS by proving the following theorem.

Theorem 1 *The sequence (1.6) together with $\frac{x}{2}|q|^2 - tiq_x^*q$ forms a complete list of conserved densities of local conservation laws of the NLS.*

Chapter 2 provides the framework for the geometry of PDEs, including definitions, examples and a technical lemma. Chapter 3 defines symmetries of PDEs, states the method of computation and discusses examples. A proposition giving the geometric symmetries of the NLS is also proven in Chapter 3. Conservation laws are defined and examined through examples in Chapter 4. A proposition stating the low-order conservation laws of the NLS is proven in Chapter 4. Chapter 5 contains the main result of this dissertation. Theorem 6 and Corollary 7 in Chapter 5 derive the form of characteristics that depend only on spatial derivatives of order three and higher. The recursion operator of the NLS and the known geometric symmetries are used to generate a sequence of characteristics of conservation laws of the NLS, which is compared to the conservation laws given by (1.6). A theorem of Olver, stating that nontrivial conservation laws and nontrivial characteristics are in bijective correspondence for systems of PDEs satisfying mild regularity conditions, is invoked [25]. Appendix B proves that the NLS satisfy the hypotheses of Olver's theorem. This establishes Theorem 1. Sequences appearing in the literature are examined in Appendix A. Chapter 6 summarizes the results of this dissertation, outlines further steps in the analysis of the NLS, and proposes similar questions for a (2+1)-dimensional system derived from the NLS.

Chapter 2

Visualizing PDEs

The principle behind symmetry analysis of PDEs is to consider the set of solutions to a PDE as a geometric object, temporarily setting aside boundary and initial conditions.

2.1 Geometric Structure

The following material is based on Olver [25]. Consider a system of PDEs involving two independent variables, t and x, and p dependent variables, $u = (u^1, ..., u^p)$. Let $X = \mathbb{R}^2$ be the domain of the independent variables and $U = K^p$ where $K = \mathbb{R}$ or $\mathbb{C}$ be the domain of dependent variables. A *solution* of a system of PDEs is a smooth function $f : X \to U$, that satisfies the system of PDEs. Given a solution $f(t, x)$ of a system of PDEs, the *graph* of $f(t, x)$ is defined to be

$$\Gamma_f = \{(t, x, u) : u = f(t, x)\} \subset X \times U. \tag{2.1}$$

For a K-valued function $f(t, x)$, there are $\binom{1+k}{k}$ different possible k^{th}-order partial derivatives of f. If $f : X \to K^p$, then there are $p\binom{1+k}{k}$ different values needed to represent all k^{th}-order partial derivatives of f at the point (t,x). A *multi-index of order* n is an n-tuple $I = (i_1, ..., i_n)$ where each i_l is either zero or one, indicating a time or space derivatives, as in $\partial_I f(t,x) = \partial_{(1,0,0)} f(t,x) = \frac{\partial^3 f(t,x)}{\partial x \partial t \partial t}$, where $I = (1,0,0)$. Subscripts of dependent variables denote partial derivatives and the notation $u_n = \partial_x^n f$ will be used for spatial derivatives when $n > 1$. Let $d = p\binom{1+k}{k}$ and let $U_k = K^d$ with coordinates u_I^α, where I is of order k, represent the possible values of the k^{th}-order partial derivatives of $f(t,x)$. Set $U^{(n)} = U \times U_1 \times ... \times U_n$ to be the Cartesian product space whose coordinates represent the values of all partial derivatives of order up to, and including, n of f at (t,x). Observe that $U^{(n)} = K^m$ where $m = p\binom{2+n}{n}$. A point in $U^{(n)}$ is denoted $u^{(n)}$ and has m different components u_I^α where $\alpha = 1, ..., p$ and I runs over all multi-indices of order k where $0 \leq k \leq n$. Given a smooth function $f : X \to U$, there is an induced function $\mathrm{pr}^n f : X \to U^{(n)}$ which is the vector of all partial derivatives of f, up to and including n^{th}-order. The space $X \times U^{(n)}$ is the n^{th}*-order jet space* of $X \times U$ and is denoted $\mathcal{J}_{t,x}^n$ [25, 34]. The subscripts indicate the partial derivatives appearing in $U^{(n)}$.

Let $\mathcal{F}(\mathcal{J}_{t,x}^n)$ be the space of smooth functions on $\mathcal{J}_{t,x}^n$, called *differential functions.* Subscripts preceded by a comma will denote partial derivatives of differential functions. Recall that partial derivatives such as u_2 and u_t are, themselves, treated as variables and no comma will be used. Numeric subscripts such as u_n refer exclusively to x-derivatives.

An n^{th}-order system of PDEs consisting of l equations is a smooth map $\mathbf{\Delta} : \mathcal{J}_{t,x}^n \to K^l$. Often, the maps $\mathbf{\Delta}$ will allow for all t-derivatives to be rewritten in terms involving only x-derivatives and $\mathcal{J}_x^n$ will denote the restriction of $\mathcal{J}_{t,x}^n$ to the subspace containing only x-derivatives. The restriction of $\mathcal{J}_{t,x}^n$ to $\mathcal{J}_x^n$ induces a map $\widetilde{\mathbf{\Delta}}:\mathcal{J}_x^n \to K^l$.

The Jacobian matrix of a system of PDEs $\mathbf{\Delta}$ is the matrix $\mathrm{J}(\mathbf{\Delta})$ whose k^{th}-row is $\left(\frac{\partial \Delta^k}{\partial t} \frac{\partial \Delta^k}{\partial x} \frac{\partial \Delta^k}{\partial u_I}\right)$ for $0 \leq k \leq l$ and where $I = (i_1, ..., i_n)$ is a multi-index. If the matrix $\mathrm{J}(\mathbf{\Delta})$ is of maximal rank on some open subset of $\mathcal{J}^n_{t,x}$, then the system $\mathbf{\Delta}$ is said to be of *maximal rank.* The Implicit Function Theorem shows that if $\mathrm{J}(\mathbf{\Delta})$ is of maximal rank, then locally, $\mathbf{\Delta}$ defines a surface in the space $\mathcal{J}^n_{t,x}$. Define $\mathcal{S}_{\Delta} = \{(t, x, u^{(n)})$ such that $\mathbf{\Delta}(t, x, u^{(n)}) = 0\}$. The elements of this set are graphs in $\mathcal{J}^n_{t,x}$. If $f(t,x)$ is a solution of $\mathbf{\Delta}$, then $\mathbf{\Delta}(t, x, \mathrm{pr}^n f) \subset \mathcal{S}_{\Delta}$. The object $\mathcal{S}_{\Delta}$ is an example of a differential variety [34].

The heat equation is an example of a PDE whose solution is a real-valued function $f(t,x)$, representing the value of temperature at a point (t,x), specifying time and position, that satisfies the equation $u_t = u_{xx}$ where $u = f(t,x)$. To interpret the solution set of the heat equation as a surface, define a differential function

$$\Delta_{heat}(t, x, u^{(2)}) = u_t - u_{xx}. \tag{2.2}$$

Let $\mathcal{S}_{\Delta_{heat}} = \{(t, x, u^{(2)})$ such that $\Delta_{heat}(t, x, u^{(2)}) = 0\}$. The graph Γ_f is contained in $\mathcal{S}_{\Delta_{heat}}$ if $f(t,x)$ is a solution of the heat equation. The function $e^{-t} \sin x$ solves $\Delta_{heat}(t, x, u^{(2)}) = 0$ and thus, $(t, x, (e^{-t} \sin x)^{(2)}) \in \mathcal{S}_{\Delta_{heat}}$.

The Jacobian matrix of the heat equation is

$$\mathrm{J}(\Delta_{heat}) = \begin{pmatrix} \frac{\partial \Delta_{heat}}{\partial t} & \frac{\partial \Delta_{heat}}{\partial x} & \frac{\partial \Delta_{heat}}{\partial u} & \frac{\partial \Delta_{heat}}{\partial u_t} & \frac{\partial \Delta_{heat}}{\partial u_x} & \frac{\partial \Delta_{heat}}{\partial u_{tt}} & \frac{\partial \Delta_{heat}}{\partial u_{tx}} & \frac{\partial \Delta_{heat}}{\partial u_{xx}} \end{pmatrix} \tag{2.3}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.4}$$

which is rank one everywhere. By the Implicit Function Theorem, the function Δ_{heat} defines a surface in $\mathcal{J}^2_{t,x}$.

The wave equation is a PDE whose solution is a real-valued function $f(t,x)$ representing, for instance, the displacement of a vibrating string that satisfies the equation

$u_{tt} = u_{xx}$ where $u = f(t, x)$. A surface may be defined by

$$\Delta_{wave}(t, x, u^{(2)}) = u_{tt} - u_{xx} \tag{2.5}$$

as with the heat equation. In addition, the wave equation Δ_{wave} can also be expressed as a pair of equations by introducing a second dependent variable v and rewriting the wave equation as follows

$$\Delta_{wave}(t, x, u^{(1)}, v^{(1)}) = \begin{cases} u_t - v_x \\ v_t - u_x \end{cases} \tag{2.6}$$

where the second-order equation (2.5) is obtained by requiring partial derivatives to commute. Equation (2.6) is more general than (2.5), as solutions need only be once differentiable. A solution to (2.5) is $\sin(t-x)$ which means $\Gamma_{\sin(t-x)} \subset \mathcal{S}_{\Delta_{wave}}$. Also, $\binom{u}{v} = \binom{\sin(t-x)}{\sin(x-t)}$ solves (2.6).

The Jacobian matrix of (2.6) is

$$\mathrm{J}(\Delta_{wave}) = \begin{pmatrix} \frac{\partial\Delta^1}{\partial t} & \frac{\partial\Delta^1}{\partial x} & \frac{\partial\Delta^1}{\partial u} & \frac{\partial\Delta^1}{\partial v} & \frac{\partial\Delta^1}{\partial u_t} & \frac{\partial\Delta^1}{\partial v_t} & \frac{\partial\Delta^1}{\partial u_x} & \frac{\partial\Delta^1}{\partial v_x} \\ \frac{\partial\Delta^2}{\partial t} & \frac{\partial\Delta^2}{\partial x} & \frac{\partial\Delta^2}{\partial u} & \frac{\partial\Delta^2}{\partial v} & \frac{\partial\Delta^2}{\partial u_t} & \frac{\partial\Delta^2}{\partial v_t} & \frac{\partial\Delta^2}{\partial u_x} & \frac{\partial\Delta^2}{\partial v_x} \end{pmatrix} \tag{2.7}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}. \tag{2.8}$$

This matrix is of rank 2 everywhere. Hence, the function Δ_{wave} defines a surface in $\mathcal{J}^2_{t,x}$.

The nonlinear Schrödinger equation was given in (1.1). Let $U = \mathbb{C}^2$ with coordinates q and q^*. For reasons of analyticity, the NLS is viewed as

$$\begin{pmatrix} \Delta^*_{NLS} \\ \Delta_{NLS} \end{pmatrix} = \begin{pmatrix} 2\chi(q^*)^2 q - q^*_{xx} + iq^*_t \\ 2\chi q^2 q^* - q_{xx} - iq_t \end{pmatrix} \tag{2.9}$$

where the conjugate q^* is treated as a second dependent variable. The nonlinearity of the NLS makes finding solutions more challenging. However, solutions with only one independent variable can be sought. For example, the function $e^{-2\chi it}$ is easily seen to be a solution of equation (1.1). Hence, $(t, x, \mathrm{pr}^2 e^{-2\chi it}, \mathrm{pr}^2 e^{2\chi it}) \in \mathcal{S}_{\Delta_{NLS}}$. The Jacobian matrix for the NLS is

$$\mathrm{J}\begin{pmatrix}\Delta^*_{NLS}\\ \Delta_{NLS}\end{pmatrix} = \begin{pmatrix} \frac{\partial\Delta^*_{NLS}}{\partial t} & \frac{\partial\Delta^*_{NLS}}{\partial x} & \frac{\partial\Delta^*_{NLS}}{\partial q_t} & \frac{\partial\Delta^*_{NLS}}{\partial q_x} & \frac{\partial\Delta^*_{NLS}}{\partial q^*_t} & \frac{\partial\Delta^*_{NLS}}{\partial q^*_x} & \frac{\partial\Delta^*_{NLS}}{\partial q_{xx}} & \frac{\partial\Delta^*_{NLS}}{\partial q^*_{xx}} \\ \frac{\partial\Delta_{NLS}}{\partial t} & \frac{\partial\Delta_{NLS}}{\partial x} & \frac{\partial\Delta_{NLS}}{\partial q_t} & \frac{\partial\Delta_{NLS}}{\partial q_x} & \frac{\partial\Delta_{NLS}}{\partial q^*_t} & \frac{\partial\Delta_{NLS}}{\partial q^*_x} & \frac{\partial\Delta_{NLS}}{\partial q_{xx}} & \frac{\partial\Delta_{NLS}}{\partial q^*_{xx}} \end{pmatrix} \tag{2.10}$$

$$= \begin{pmatrix} 0 & 0 & 2\chi(q^*)^2 & 4\chi q^* q & 0 & i & 0 & -1 \\ 0 & 0 & 4\chi q^* q & 2\chi q^2 & -i & 0 & -1 & 0 \end{pmatrix} \tag{2.11}$$

where only the nonzero columns for the second-order partial derivative terms are given. The matrix $\mathrm{J}\binom{\Delta^*_{NLS}}{\Delta_{NLS}}$ is of rank 2 everywhere. Hence, $\binom{\Delta^*_{NLS}}{\Delta_{NLS}}$ defines a surface in $\mathcal{J}^2_{t,x}$.

2.2 Operators

Total derivatives on $\mathcal{F}(\mathcal{J}^n_{t,x})$ are differential operators that generalize the chain rule. Let $U = \mathbb{C}^2$. Given $P \in \mathcal{F}(\mathcal{J}^n_{t,x})$, the total derivative of P with respect to time is

$$D_t P = P_{,t} + P_{,q} q_t + P_{,q^*} q^*_t + \ldots + P_{,q_n} q_{nt} + P_{,q^*_n} q^*_{nt}. \tag{2.12}$$

The total derivative of P with respect to x is similar:

$$D_x P = P_{,x} + P_{,q} q_x + P_{,q^*} q^*_x + \ldots + P_{,q_n} q_{n+1} + P_{,q^*_n} q^*_{n+1}. \tag{2.13}$$

The following lemma is an application of the total derivative.

Lemma 2 *Let $\mathcal{J}^n_{t,x} = \mathbb{R}^2 \times (\mathbb{C}^2)^{(n)}$. Suppose $P \in \mathcal{F}(\mathcal{J}^n_x)$. If $D_x P = 0$, then $P = f(t)$, that is, P is only a function of t. Further, if $D_x P = g(t)F(q, q^*)$ for some non-constant function $F(q, q^*) \in \mathcal{F}(\mathcal{J}^0_x)$, then $D_x P = 0$ and $g(t) = 0$.*

Proof. The idea is to begin with the highest-order terms and work downward. If $P \in \mathcal{F}(\mathcal{J}^n_x)$, then P depends on $t, x, q^{(n)}, q^{*(n)}$. Since the q_n and q^*_n are all independent in $\mathcal{J}^n_x$, if $D_x P = 0$, then equation (2.13) gives $P_{,q^*_n} = 0$ and $P_{,q_n} = 0$ as the $n+1$ order terms must vanish, that is the coefficients of those terms must equal zero. As a result, P depends only on $t, x, q^{(n-1)}$ and $q^{*(n-1)}$. Hence, $D_x P = 0 \Rightarrow P_{,q^*_{n-1}} = 0$ and $P_{,q_{n-1}} = 0$. Repeat this process until $P_{,x} + P_{,q} q_x + P_{,q^*} q^*_x = 0 \Rightarrow P_{,x} = 0$, $P_{,q^*} = 0$ and $P_{,q} = 0$ which proves that P only depends on t. This proves the first part of the lemma. The second part is similar. Continue the steps until $P_{,x} + P_{,q} q_x + P_{,q^*} q^*_x = g(t) F(q, q^*)$ where $F(q, q^*) \in \mathcal{F}(\mathcal{J}^0_{t,x})$. The coefficient of q_x and q^*_x must equal zero and so P does not depend on q or q^*. Thus, $P_{,x} = g(t) F(q, q^*)$ only holds if both sides of the equality are zero and the second statement of the lemma follows. ■

A total divergence operator can also be defined in terms of the total derivative. For smooth functions $F_1, F_2 \in \mathcal{F}(\mathcal{J}^n_{t,x})$ define

$$\mathrm{Div}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = D_t F_1 + D_x F_2. \tag{2.14}$$

This operator is used in the definition of a conservation law in Chapter 4.

Associated to each system of PDEs is a differential operator called the Fréchet derivative which linearizes the differential equation by perturbing the dependent variables and differentiating with respect to the perturbation. Symbolically, the Fréchet derivative of $\mathbf{\Delta}(t, x, u^{(n)})$ is the operator $\mathrm{D}_{\mathbf{\Delta}}$ obtained from the equation

$$\mathrm{D}_{\mathbf{\Delta}} Q = \frac{d}{d\epsilon} \mathbf{\Delta}(t, x, (u + \epsilon Q)^{(n)})\bigg|_{\epsilon=0} \tag{2.15}$$

for every $Q \in \mathcal{F}(\mathcal{J}^n_{t,x})$. For PDEs that involve two dependent variables, $\mathrm{D}_{\boldsymbol{\Delta}}$ is obtained from

$$\mathrm{D}_{\boldsymbol{\Delta}}\begin{pmatrix} Q \\ R \end{pmatrix} = \frac{d}{d\epsilon}\boldsymbol{\Delta}(t, x, (u+\epsilon Q)^{(n)}, (v+\epsilon R)^{(n)})\Big|_{\epsilon=0} \tag{2.16}$$

for every $Q, R \in \mathcal{F}(\mathcal{J}^n_{t,x})$. The operator $\mathrm{D}_{\boldsymbol{\Delta}}$ plays an important role in the symmetry analysis of PDEs.

Consider the heat equation $\Delta_{heat}(t, x, u^{(2)}) = u_t - u_{xx}$. The Fréchet derivative is obtained by computing

$$\begin{aligned} \mathrm{D}_{\Delta_{heat}}Q &= \frac{d}{d\epsilon}\Delta_{heat}(t, x, (u+\epsilon Q)^{(n)})\Big|_{\epsilon=0} && (2.17)\\ &= \frac{d}{d\epsilon}(D_t(u+\epsilon Q) - D_{xx}(u+\epsilon Q))\Big|_{\epsilon=0} && (2.18)\\ &= (D_t - D_{xx})Q && (2.19) \end{aligned}$$

for every $Q \in \mathcal{F}(\mathcal{J}^n_{t,x})$ and thus, $\mathrm{D}_{\Delta_{heat}} = D_t - D_{xx}$. In this case, the "linearization" is the equation itself.

The Fréchet derivative of the wave equation (2.6) is

$$\begin{aligned} \mathrm{D}_{\Delta_{wave}}\begin{pmatrix} Q \\ R \end{pmatrix} &= \frac{d}{d\epsilon}\Delta_{wave}(t, x, (u+\epsilon Q)^{(n)}, (v+\epsilon R)^{(n)})\Big|_{\epsilon=0} && (2.20)\\ &= \begin{pmatrix} \frac{d}{d\epsilon}(D_t(u+\epsilon Q) - D_x(v+\epsilon R)\big|_{\epsilon=0} \\ \frac{d}{d\epsilon}D_t(v+\epsilon R) - D_x(u+\epsilon Q)\big|_{\epsilon=0} \end{pmatrix} && (2.21)\\ &= \begin{pmatrix} D_t & -D_x \\ -D_x & D_t \end{pmatrix}\begin{pmatrix} Q \\ R \end{pmatrix} && (2.22) \end{aligned}$$

for every $Q, R \in \mathcal{F}(\mathcal{J}^n_{t,x})$. One can also check that the Fréchet derivative of the wave equation in the form (2.5) is $\mathrm{D}_{\Delta_{wave}} = D_{tt} - D_{xx}$.

The Fréchet derivative of the NLS in the form (2.9) is

$$\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix}P\\P^*\end{pmatrix}=\frac{d}{d\epsilon}\begin{pmatrix}\Delta^*_{NLS}(t,x,(q+\epsilon P)^{(2)},(q^*+\epsilon P^*)^{(2)})\\\Delta_{NLS}(t,x,(q+\epsilon P)^{(2)},(q^*+\epsilon P^*)^{(2)})\end{pmatrix}\Bigg|_{\epsilon=0} \tag{2.23}$$

$$=\begin{pmatrix}\frac{d}{d\epsilon}\left(2\chi((q+\epsilon P)^*)^2(q+\epsilon P)-D_{xx}(q+\epsilon P)^*+iD_t(q+\epsilon P)^*\right)\big|_{\epsilon=0}\\\frac{d}{d\epsilon}(2\chi(q+\epsilon P)^{2(q+\epsilon P)*}-D_{xx}(q+\epsilon P)-iD_t(q+\epsilon P)\big|_{\epsilon=0}\end{pmatrix} \tag{2.24}$$

$$=\begin{pmatrix}2\chi(q^*)^2 & iD_t-D_{xx}+4\chi\left|q\right|^2\\-iD_t-D_{xx}+4\chi\left|q\right|^2 & 2\chi q^2\end{pmatrix}\begin{pmatrix}P\\P^*\end{pmatrix} \tag{2.25}$$

for $P,P^*\in\mathcal{F}(\mathcal{J}^n_{t,x})$. Note that just as q^* is treated as a second variable, despite its relation to q, P^* is treated as a second separate differential function, despite its relation to P. Thus,

$$\mathrm{D}_{\Delta_{NLS}}=\begin{pmatrix}2\chi(q^*)^2 & iD_t-D_{xx}+4\chi\left|q\right|^2\\-iD_t-D_{xx}+4\chi\left|q\right|^2 & 2\chi q^2\end{pmatrix}. \tag{2.26}$$

The nonlinearity of the NLS is evident by the presence of terms in $\mathrm{D}_{\Delta_{NLS}}$ not involving total derivative operators.

Chapter 3

Computation of Symmetries

The investigation of a system of PDEs begins with finding the set of transformations under which its solutions are invariant. This chapter presents a method to determine the symmetries of a given system of PDEs, using the NLS as an explicit example.

3.1 Symmetries of PDEs

Several definitions and basic results will be given here [25]. Given a manifold G with a group structure and an operation $*$, G is a *Lie group* if $*$ and its inverse are both smooth functions $G \times G \to G$. An action of G on a manifold $M \subset X \times U$ is a transformation

$$\Psi : G \times M \to M \tag{3.1}$$

whose image in M is denoted $\Psi(\epsilon, (t, x, u)) = (\widetilde{t}, \widetilde{x}, \widetilde{u})$. The action of G on M satisfies the properties

$$\Psi(0, (t, x, u)) = (t, x, u) \tag{3.2}$$

and

$$\Psi(\lambda, \Psi(\epsilon, (t, x, u))) = \Psi(\lambda * \epsilon, (t, x, u)) \tag{3.3}$$

for all $\epsilon, \lambda \in G$ and $(t, x, u) \in M$ where $0 \in G$ denotes the identity.

For differential equations, Lie groups act on the graphs of solutions. Specifically, suppose that $f(t, x)$ is a solution of a system of PDEs $\boldsymbol{\Delta}$. The *orbit* of the action Ψ on the graph Γ_f of f is

$$\Psi(\Gamma_f) = \{\Psi(\epsilon, (t, x, u)) : \forall \epsilon \in G, \forall\ (t, x, u) \in \Gamma_f\}. \tag{3.4}$$

A transformation Ψ is a *symmetry* of $\boldsymbol{\Delta}$ if $\Psi(\Gamma_f) \subset \mathcal{S}_\Delta$ whenever $\Gamma_f \subset \mathcal{S}_\Delta$.

For example, let $G = (\mathbb{R}, +)$ and $M = X \times U$. Define an action of G on M by $\Psi(\epsilon, (t, x, u)) = (t+\epsilon, x, u)$. The action Ψ shifts Γ_u along the t-axis; Ψ is a time-translation. This is a symmetry of solutions to the heat equation, the wave equation and the NLS, as will be shown in sections 3.2 and 3.3.

The *infinitesimal generator* of the action Ψ at $(t, x, u) \in M$ is the vector field $\mathbf{v}$ obtained by

$$\begin{aligned} \mathbf{v}|_{(t,x,u)} &= \frac{d}{d\epsilon}\Psi(\epsilon, (t, x, u))\Big|_{\epsilon=0} = \frac{d}{d\epsilon}(\widetilde{t}, \widetilde{x}, \widetilde{u})\Big|_{\epsilon=0} \\ &= \frac{d\widetilde{t}}{d\epsilon}\partial_{\widetilde{t}}(\widetilde{t}, \widetilde{x}, \widetilde{u})\Big|_{\epsilon=0} + \frac{d\widetilde{x}}{d\epsilon}\partial_{\widetilde{x}}(\widetilde{t}, \widetilde{x}, \widetilde{u})\Big|_{\epsilon=0} + \frac{d\widetilde{u}}{d\epsilon}\partial_{\widetilde{u}}(\widetilde{t}, \widetilde{x}, \widetilde{u})\Big|_{\epsilon=0}. \end{aligned} \tag{3.5}$$

From (3.5), the vector field v can be identified with the differential operator $\frac{d\widetilde{t}}{d\epsilon}\partial_{\widetilde{t}} + \frac{d\widetilde{x}}{d\epsilon}\partial_{\widetilde{x}} + \frac{d\widetilde{u}}{d\epsilon}\partial_{\widetilde{u}}$. The relation between a group action and its infinitesimal generator is the linchpin of the symmetry approach. Given a vector field, one can compute the action it generates by solving ODEs.

For instance, the vector field $\mathbf{v} = t\partial_t + x\partial_x + u\partial_u$ generates an action of the Lie group $(\mathbb{R}, +)$ on $X \times U$. Using a standard technique from differential geometry, one exponentiates the vector field to recover the transformation (see [25]). Comparing the coefficients of $\mathbf{v}$ with (3.5) leads to the ODEs

$$\frac{dt}{d\epsilon} = t, \ \frac{dx}{d\epsilon} = x, \ \frac{du}{d\epsilon} = u \tag{3.6}$$

subject to the initial conditions $t(0) = t_0$, $x(0) = x_0$ and $u(0) = u_0$. Solving (3.6) gives, as functions of ϵ,

$$t(\epsilon) = e^{\epsilon}t_0, \ x(\epsilon) = e^{\epsilon}x_0, \ \ u(\epsilon) = e^{\epsilon}u_0. \tag{3.7}$$

The transformation generated by $\mathbf{v}$ is therefore $\Psi(\epsilon, (t_0, x_0, u_0)) = (e^{\epsilon}t_0, e^{\epsilon}x_0, e^{\epsilon}u_0)$, which is a scaling transformation.

A function $f : X \times U \to U$ is *invariant* under the transformation Ψ generated by $\mathbf{v}$, meaning

$$f(\Psi(\epsilon, (t, x, u)) = f(t, x, u) \tag{3.8}$$

for all $\epsilon \in G$, if $\mathbf{v}(f)$=0 [25]. For example, observe that if $f(t, x, u) = e^{-\frac{t}{x}}\sin(\frac{x}{u})$, then

$$\mathbf{v}(f) = (t\partial_t + x\partial_x + u\partial_u)(e^{-\frac{t}{x}}\sin(\frac{x}{u})) \tag{3.9}$$

$$= t\left(\frac{-e^{-\frac{t}{x}}\sin(\frac{x}{u})}{x}\right) + x\left(\frac{te^{-\frac{t}{x}}\sin(\frac{x}{u})}{x^2} + \frac{e^{-\frac{t}{x}}\cos(\frac{x}{u})}{u}\right) + u\left(\frac{-xe^{-\frac{t}{x}}\cos(\frac{x}{u})}{u^2}\right) = 0. \tag{3.10}$$

This indicates that $f(t, x, u) = e^{-\frac{t}{x}}\sin(\frac{x}{u})$ is invariant under the scaling transformation.

3.2 Methodology

To determine the collection of symmetries of the solutions of a given system of PDEs, one can determine the generating vector fields by using the fact that

$$\mathbf{v}(f) = 0 \Rightarrow f(\Psi(\epsilon, (t, x, u)) = f(t, x, u) \tag{3.11}$$

and extending this result into the jet space $X \times U^{(n)}$. The goal is to find the Lie algebra of the vector fields which vanish on $\mathcal{S}_\Delta \subset \mathcal{J}^n_{t,x}$.

A *generalized vector field* on a space $X \times U$ with two dependent variables u and v is a linear combination of the form

$$\mathbf{v} = \xi\partial_x + \tau\partial_t + \varphi\partial_u + \psi\partial_v \tag{3.12}$$

where ξ, τ, φ and ψ are differential functions of $(t, x, u^{(n)}, v^{(n)})$ and n is a positive integer. Let

$$\mathbf{v}_{ev} = (\varphi - \xi u_x - \tau u_t)\partial_u + (\psi - \xi v_x - \tau v_t)\partial_v \tag{3.13}$$

be the *evolutionary* part of $\mathbf{v}$. Recall from (2.12) and (2.13) that $D_x = \partial_x + u_x\partial_u + v_x\partial_v$ and $D_t = \partial_t + u_t\partial_u + v_t\partial_v$. Let

$$\mathrm{tot}\mathbf{v} = \xi D_x + \tau D_t \tag{3.14}$$

be the *divergence* part of $\mathbf{v}$. Then it follows that

$$\mathbf{v} = \mathrm{tot}\mathbf{v} + \mathbf{v}_{ev}. \tag{3.15}$$

To determine the effect of the transformation generated by $\mathbf{v}$ in $\mathcal{J}^n_{t,x}$, the vector field must be extended there. This is done with an operation called *prolongation* which is defined as

$$\mathrm{pr}^n\mathbf{v} = \mathrm{tot}\mathbf{v} + \mathrm{pr}^n_{ev}\mathbf{v} \tag{3.16}$$

where

$$\mathrm{pr}^n_{ev}\mathbf{v} = \sum_{n\geq p\geq 0} D_{i_0}\cdots D_{i_p}((\varphi - \xi u_x - \tau u_t)\partial_{u_I} + (\psi - \xi v_x - \tau v_t)\partial_{v_I}) \tag{3.17}$$

and where $I = (i_0, ..., i_p)$ is a multi-index. Olver [25] proves the following result.

Lemma 3 *Suppose a system of PDEs* $\boldsymbol{\Delta}$ *is of maximal rank on* $\mathcal{S}_{\boldsymbol{\Delta}}$. *If* G *is a local Lie group of transformations of a system of PDEs* $\boldsymbol{\Delta}$ *and*

$$\mathrm{pr}^n\mathbf{v}(\boldsymbol{\Delta}) = 0 \tag{3.18}$$

on $\mathcal{S}_{\boldsymbol{\Delta}}$ *for every infinitesimal generator* $\mathbf{v}$ *of* G, *then* G *is a symmetry group of* $\boldsymbol{\Delta}$.

As noted earlier, the heat and wave equations and the NLS each admit time translation as a symmetry. The infinitesimal generator of this symmetry is the vector field ∂_t. Observe that $\mathrm{tot}\partial_t = \boldsymbol{D}_t$, $\partial_{t_{ev}} = -u_t\partial_u - v_t\partial_v$ and $\mathrm{pr}^n\partial_t = \partial_t + \sum_{I,|I|\geq 0} D_I(-u_t\partial_{u_I} - v_t\partial_{v_I})$ is a vector field on $\mathcal{J}^n_{t,x}$.

To illustrate, consider the heat equation. Applying the prolongation formula gives

$$\mathrm{pr}^n\partial_t\Delta_{heat} = (\partial_t + \sum_{I,|I|\geq 0} \boldsymbol{D}_I(-u_t)\partial_{u_I})(u_t - u_{xx}) \tag{3.19}$$

$$= \boldsymbol{D}_t(-u_t) - \boldsymbol{D}_{xx}(-u_t) = -u_{tt} + u_{txx} \tag{3.20}$$

which does not equal zero in general. However, on $\mathcal{S}_{\Delta_{heat}}$,

$$\mathrm{pr}^n\partial_t\Delta_{heat} = -u_{tt} + u_{txx} = -(u_{xx})_t + u_{txx}, \tag{3.21}$$

which vanishes if partial derivatives commute.

With the prolongation formula (3.16) in mind, it is not hard to see that a vector field generates a symmetry if and only if its evolutionary part does. Indeed, on $\mathcal{S}_{\boldsymbol{\Delta}}$, $\boldsymbol{\Delta}$ is identically zero, and hence the total derivatives with respect to the independent variables are zero. That is,

$$\mathrm{pr}^n\mathbf{v}(\boldsymbol{\Delta}) = (\mathrm{tot}\mathbf{v} + \mathrm{pr}^n\mathbf{v}_{ev})(\boldsymbol{\Delta}) = \mathrm{pr}^n\mathbf{v}_{ev}(\boldsymbol{\Delta}) \tag{3.22}$$

since $\text{tot}\mathbf{v}(\boldsymbol{\Delta}) = (\xi D_x + \tau D_t)(\boldsymbol{\Delta}) = 0$ on $\mathcal{S}_\Delta$. Thus, it is enough to set $\text{pr}^n\mathbf{v}_{ev}(\boldsymbol{\Delta})$ to zero to determine which vector fields $\mathbf{v}_{ev}$ generate transformations that do, in fact, leave the solutions $\mathcal{S}_\Delta$ intact. That is, it is enough to consider vector fields of the form

$$\mathbf{v} = Q(t, x, u^{(n)}, v^{(n)})\partial_u + R(t, x, u^{(n)}, v^{(n)})\partial_v \tag{3.23}$$

called *evolutionary vector fields* and determine the functions Q and R such that (3.22) vanishes. The pair of differential functions $\binom{Q}{R}$ is referred to as the *characteristic* of $\mathbf{v}$.

Finding all local symmetries for solutions of a given system of PDEs is therefore accomplished by finding characteristics of the generating vector fields. For an evolutionary vector field $\mathbf{v}$, it turns out that

$$\text{pr}^n\mathbf{v}(\Delta) = \text{D}_\Delta\mathbf{Q} \tag{3.24}$$

where $\mathbf{Q} = \binom{Q}{R}$ is the characteristic of $\mathbf{v}$ and D_Δ is the Fréchet derivative (see [25] for details). This is easily verified for the heat equation. If $\mathbf{v} = Q\partial_u$, then

$$\text{pr}^n\mathbf{v}(\Delta_{heat}) = \sum_{n \geq p \geq 0} D_I Q \partial_{u_I} \Delta_{heat} = (D_t Q \partial_{u_t} - D_{xx} Q \partial_{u_{xx}})\Delta_{heat} = \text{D}_{\Delta_{heat}} Q. \tag{3.25}$$

Observe that $\partial_{t_{ev}} = -u_t\partial_u$ and the characteristic of $\mathbf{v}$ is the differential function $-u_t$. It is easy to check that $\text{D}_{\Delta_{heat}}(-u_t) = 0$ on $\mathcal{S}_{\Delta_{heat}}$. Therefore, Δ_{heat} is invariant under a time translation as established using the Fréchet derivative. A similar argument holds for the wave equation.

3.3 Geometric Symmetries of the NLS

The *geometric symmetries* of a system of PDEs is the set of symmetries generated by vector fields of the form (3.12) where $n = 1$. The following result exhibits the method used to compute the infinitesimal generators of the local symmetry group. It also highlights crucial arguments that are important to the proof of Theorem 6 in Chapter 5. In the case of the NLS, the characteristic in (3.23) will be $\binom{Q}{Q^*}$ rather than $\binom{Q}{R}$, as in the transition from (2.22) to (2.23).

Proposition 4 *The group of geometric symmetries of $\mathcal{S}_{\Delta_{NLS}}$ is generated by the Lie algebra spanned by the five vector fields with characteristics given by*

$$\begin{pmatrix} Q_0 \\ Q_0^* \end{pmatrix} = \begin{pmatrix} iq \\ -iq^* \end{pmatrix}, \tag{3.26}$$

$$\begin{pmatrix} Q_1 \\ Q_1^* \end{pmatrix} = \begin{pmatrix} q_x \\ q_x^* \end{pmatrix}, \tag{3.27}$$

$$\begin{pmatrix} Q_2 \\ Q_2^* \end{pmatrix} = \begin{pmatrix} q_t \\ q_t^* \end{pmatrix} = \begin{pmatrix} i(q_2 - 2\chi q\,|q|^2) \\ -i(q_2^* - 2\chi q^*\,|q|^2) \end{pmatrix}, \tag{3.28}$$

$$\begin{pmatrix} Q_g \\ Q_g^* \end{pmatrix} = t\begin{pmatrix} q_x \\ q_x^* \end{pmatrix} - \frac{x}{2}\begin{pmatrix} iq \\ -iq^* \end{pmatrix}, \tag{3.29}$$

$$\begin{pmatrix} Q_{sc} \\ Q_{sc}^* \end{pmatrix} = \begin{pmatrix} -ti(q_2 - 2\chi q\,|q|^2) - \frac{x}{2}q_x - \frac{1}{2}q \\ ti(q_2^* - 2\chi q^*\,|q|^2) - \frac{x}{2}q_x^* - \frac{1}{2}q^* \end{pmatrix}. \tag{3.30}$$

Proof. This is proven by direct computation. On $\mathcal{S}_{\Delta_{NLS}}$, for a given $Q \in \mathcal{F}(\mathcal{J}^1_{t,x})$, all t-derivatives can be replaced using the NLS (2.9). Thus, it is enough to consider $\widetilde{Q} \in \mathcal{F}(\mathcal{J}^2_x)$. Unless there is room for confusion, $\widetilde{Q}$ and Q will be used interchangeably.

Every characteristic Q satisfies

$$\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix} = \begin{pmatrix} 2\chi(q^*)^2 & iD_t - D_{xx} + 4\chi\,|q|^2 \\ -iD_t - D_{xx} + 4\chi\,|q|^2 & 2\chi q^2 \end{pmatrix}\begin{pmatrix} Q \\ Q^* \end{pmatrix} = 0 \tag{3.31}$$

on $\mathcal{S}_{\Delta_{NLS}}$. Note that the components are simply conjugates of each other. Hence, it is enough to consider the second component of (3.31), namely

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = (-iD_t - D_{xx} + 4\chi\,|q|^2)Q + 2\chi q^2 Q^* = 0. \tag{3.32}$$

The Fréchet derivative (3.31) involves second-order x-derivatives. The characteristic Q depends upon $t, x, q^{(2)}$ and $q^{*(2)}$. It follows that $\left[\mathrm{D}_{\Delta_{NLS}}\binom{Q}{Q^*}\right]_2$ is a polynomial in $q^{(4)}$ and $q^{*(4)}$. As in the proof of Lemma 2, beginning with the highest-order terms, each coefficient can be set to zero, giving information about the form of the characteristic function. Here, extracting the highest-order term gives

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_4^*(-2Q_{,q_2^*}) + R_1(t, x, q^{(3)}, q^{*(3)}) \tag{3.33}$$

for some function R_1 containing all terms of order 3 and less. If this polynomial is to vanish, the coefficient of q_4^* must equal zero. This shows that Q cannot depend on q_2^*. Thus, $Q(t, x, q^{(2)}, q^{*(2)}) = Q'(t, x, q^{(2)}, q^{*(1)})$. The defining equation (3.31) now becomes

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q' \\ Q'^* \end{pmatrix}\right]_2 = q_3^{*2}(-2Q'_{,q_2q_2}) + R_2(t, x, q^{(3)}, q^{*(3)}) \tag{3.34}$$

for some function R_2 not depending on quadratic terms involving q_3 and q_3^*. Setting the coefficient of q_3^{*2} to zero shows that $Q'_{,q_2q_2} = 0$, which brings Q to the form

$$Q = Q_1(t, x, q^{(1)}, q^{*(1)})q_2 + Q_2(t, x, q^{(1)}, q^{*(1)}) \tag{3.35}$$

for some functions Q_1 and Q_2. Substituting (3.35) into (3.31) gives

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_3^*(-2(q_2Q_{1,q_1^*} + Q_{2,q_x^*})) - q_3D_xQ_1 + R_3(t, x, q^{(2)}, q^{*(2)}) \tag{3.36}$$

for some function R_3. From this, $q_2Q_{1,q_1^*} + Q_{2,q_x^*} = 0$ which implies that $Q_{2,q_x^*} = 0$ and $Q_{1,q_1^*} = 0$. Further, an application of Lemma 2 shows that Q_1 is a function of t alone. Thus,

$$Q = Q_1(t)q_2 + Q_2(t, x, q^{(1)}, q^*). \tag{3.37}$$

Now, substituting (3.37) into (3.31) gives

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_2^2(-2Q_{2,q_xq_x}) + R_4(t,x,q^{(2)},q^{*(2)}) \tag{3.38}$$

for some function R_4 not depending on quadratic terms involving q_2 and q_2^*. Setting the quadratic term coefficient to zero shows that Q_2 is linear in q_x. Then

$$Q_2 = Q_3(t,x,q,q^*)q_x + Q_5(t,x,q,q^*) \tag{3.39}$$

for some functions Q_3 and Q_5. Rewriting (3.31) using (3.39) results in

$$\begin{aligned}\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 &= q_2(-iQ_{1,t} - 2Q_{3,q}q_x - 2Q_{3,x} - 2Q_{3,q^*}q_x^*) + \\ &\quad q_2^*(-2Q_{3,q^*}q_x^* - 2Q_{5,q^*} - 2Q_1\chi q^2 + 2Q_1^*\chi q^2) + R_5(t,x,q^{(1)},q^{*(1)})\end{aligned} \tag{3.40}$$

for some function R_5. This leads in turn to

$$Q_{3,q} = 0,\; Q_{3,q^*} = 0,\; -iQ_{1,t} - 2Q_{3,x} = 0 \tag{3.41}$$

$$-2Q_{3,q^*}q_x^* - 2Q_{5,q^*} - 2Q_1\chi q^2 + 2Q_1^*\chi q^2 = 0 \tag{3.42}$$

which imply

$$Q_{3,x} = \frac{-i}{2}Q_{1,t} \tag{3.43}$$

$$Q_{5,q^*} = \chi q^2(Q_1^* - Q_1). \tag{3.44}$$

Integrating (3.43) and (3.44) yields

$$Q_3 = \frac{-ix}{2}Q_{1,t} + Q_6(t) \tag{3.45}$$

$$Q_5 = \chi q^2(Q_1^* - Q_1)q^* + Q_7(t,x,q). \tag{3.46}$$

for some functions Q_6 and Q_7. The characteristic therefore has the form

$$Q = Q_1(t)q_2 + (Q_6(t) - \frac{ix}{2}Q_{1,t})q_x + \chi q^2(Q_1^* - Q_1)q^* + Q_7(t,x,q). \tag{3.47}$$

Returning to (3.32),

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_x^2(-4\chi q^*(Q_1^* + Q_1) - 2Q_{7,qq}) + R_6(t, x, q^{(1)}, q^{*(1)}) \tag{3.48}$$

for some function R_6 not depending on quadratic terms involving q_x and q_x^*. Because Q_7 does not depend on q^* and $Q_1^* + Q_1 = 2\,\mathrm{Re}\, Q_1$, then Q_1 must be purely imaginary and Q_7 is linear in q. Let $Q_7 = Q_8(t,x)q + Q_9(t,x)$ for some functions Q_8 and Q_9. This information gives

$$\left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_x(-iQ_{6,t} - \frac{x}{2}Q_{1,tt} - 2Q_{8,x}) + q_x^*(2\chi q^2(Q_6^* - Q_6)) + R_7(t, x, q, q^*) \tag{3.49}$$

for some function R_7. As $Q_6^* - Q_6 = 2i\,\mathrm{Im}\, Q_6$, the coefficient of q_x^* requires Q_6 to be real. Also,

$$-iQ_{6,t} - \frac{x}{2}Q_{1,tt} - 2Q_{8,x} = 0 \tag{3.50}$$

which implies

$$Q_8 = \frac{-x^2}{8}Q_{1,tt} - \frac{ix}{2}Q_{6,t} + Q_{10}(t) \tag{3.51}$$

for some function Q_{10}. The form of the characteristic currently is

$$\begin{aligned} Q = {} & Q_1(t)q_2 + (\frac{-ix}{2}Q_{1,t} + Q_6(t))q_x \\ & + (\frac{-x^2}{8}Q_{1,tt} - \frac{ix}{2}Q_{6,t} + Q_{10}(t))q + Q_9(t,x) + \chi q q^*(Q_1^* - Q_1). \end{aligned} \tag{3.52}$$

Observe that no unknown functions involving q or q^* remain. The defining equations (3.31) become

$$\begin{aligned} \left[\mathrm{D}_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = {} & q^2 q^*(-i\chi D_t(Q_1^* - Q_1) + 2\chi(Q_{10}^* + Q_{10})) \\ & + q^2 Q_9 + q^{*2} Q_9 + q(-iD_t(\frac{-x^2}{8}Q_{1,tt} - \frac{ix}{2}Q_{6,t} + Q_{10}(t)) + \frac{1}{4}Q_{1,tt}). \end{aligned} \tag{3.53}$$

Setting (3.53) to zero immediately shows that Q_9 vanishes identically. The coefficient of q gives

$$\frac{ix^2}{8}D_t^3(Q_1) - \frac{ix}{2}D_t^2 Q_6 + D_t Q_{10}(t) + \frac{1}{4}Q_{1,tt} = 0. \tag{3.54}$$

Here, the coefficients of x must vanish as the unknown function depend only on t. Thus,

$$D_t^3 Q_1 = 0,\ D_t^2 Q_6 = 0 \tag{3.55}$$

which imply

$$Q_1 = i(c_1 t^2 + c_2 t + c_3) \tag{3.56}$$

$$Q_6 = c_7 t + c_8 \tag{3.57}$$

where c_1, c_2, c_3, c_7 and c_8 are all real constants. Moreover,

$$-iD_t Q_{10}(t) + \frac{1}{4}Q_{1,tt} = 0 \tag{3.58}$$

$$-i\chi D_t(Q_1^* - Q_1) + 2\chi(Q_{10}^* + Q_{10}) = 0. \tag{3.59}$$

First,

$$D_t Q_{10}(t) = \frac{-i}{4}Q_{1,tt} = \frac{-i}{4}(2ic_1) = \frac{c_1}{2} \Rightarrow Q_{10} = \frac{c_1}{2}t + c_4 \tag{3.60}$$

where $c_4 \in \mathbb{C}$. From equation (3.58)

$$-i\chi(D_t(2i\operatorname{Im} Q_1) + 4\chi \operatorname{Re} Q_{10} = -i\chi(D_t(2i(c_1t^2 + c_2t + c_3)) + 4\chi \operatorname{Re}(\frac{c_1}{2}t + c_4) \tag{3.61}$$

$$= 2\chi(2c_1 t + c_2) + 4\chi \operatorname{Re}(\frac{c_1}{2}t + c_4) = 0 \tag{3.62}$$

which implies

$$6\chi c_1 t + 2\chi c_2 + 4\chi \operatorname{Re} c_4 = 0 \Rightarrow c_1 = 0 \tag{3.63}$$

$$\operatorname{Re} c_4 = -\frac{c_2}{2}. \tag{3.64}$$

The imaginary part of c_4 is arbitrary; call it c_5.

Combining the results of the calculation gives the general form of a characteristic satisfying (3.31). Hence,

$$Q = i(c_2t + c_3)q_2 + (c_7t + c_8 + \frac{x}{2}c_2)q_x + (-\frac{c_2}{2} + ic_5 - \frac{ix}{2}c_7)q - 2i\chi q^2q^*(c_2t + c_3). \quad (3.65)$$

Each coefficient of a constant is a characteristic function for an infinitesimal generator of a local symmetry of the NLS. For example, let $c_2 = 1$ and set the rest of the constants to zero to obtain the characteristic $itq_2 + \frac{x}{2}q_x - \frac{1}{2}q - 2i\chi q^2q^*t = tq_t + \frac{x}{2}q_x - \frac{1}{2}q$ which is the characteristic for the scaling vector field for solutions of the NLS. Similarly, the coefficient of the constant c_8 gives q_x which is the $x-$translation. ∎

The local group of transformations generated by the vector fields obtained in Proposition 4 may be computed by exponentiating each vector field. For instance, the characteristic $\binom{iq}{-iq^*}$ represents $\mathbf{v} = i(q\partial_q - q^*\partial_{q^*})$ which generates the transformation

$$\Psi(\epsilon, (t, x, q, q^*)) = (t, x, e^{i\epsilon}q, e^{-i\epsilon}q^*), \quad (3.66)$$

and which is a symmetry of solutions of the NLS.

The symmetry generated by the vector field with characteristic (3.29) is a *Galilean boost* Ψ_g which represents a boost into a moving coordinate frame. Explicitly,

$$\Psi_g(\epsilon, (t, x, q, q^*)) = (t, x - \epsilon t, \frac{\epsilon^2t - 2\epsilon x}{4} + q(t, x + \epsilon t), \frac{\epsilon^2t - 2\epsilon x}{4} + q^*(t, x + \epsilon t)) \quad (3.67)$$

for every real number ϵ in some neighborhood containing zero.

Chapter 4

Finding Conservation Laws

This chapter defines conservation laws, gives examples and outlines a method of determining conservation laws for a given system of PDEs

Quantities such as mass and energy that remain unchanged for solutions of a system of PDEs are said to be conserved. Such quantities can often be observed. For instance, the wave equation in one dimension models the motion of an infinitesimal piece of a vibrating string. As the string vibrates, the mass of the piece of string is unchanged. Mass can then be represented as the definite integral of a density function. The rate of change of the integral's value over time is zero, which expresses conservation of mass in mathematical terms. For general PDEs, conservation laws are mathematical statements showing that conserved quantities are divergence free. Finding the conserved quantities that are not easily observable is an important goal.

A related notion to that of symmetries is that of a *differential invariant*, a function that is constant under a given transformation. Given the transformation group from the last chapter, one could piece together all differential invariants for each transformation. A good question is how close the set of differential invariants is to the set of conservation laws.

4.1 Definitions

A *conservation law* of a system of PDEs $\mathbf{\Delta}$ with m independent variables is a smooth m-component differential function $\mathbf{P}$, whose components each belong to $\mathcal{F}(\mathcal{J}^n_{t,x})$, such that $\mathrm{Div}\mathbf{P} = 0$ on $\mathcal{S}_\Delta$. If a system of PDEs $\mathbf{\Delta}$ consists of l equations, then a *characteristic* of the conservation law $\mathbf{P}$ of $\mathbf{\Delta}$ is an l-component differential function $\mathbf{Q}$ such that

$$\mathbf{Q} \cdot \mathbf{\Delta} = \mathrm{Div}\mathbf{P} \tag{4.1}$$

where the product is defined to be $\mathbf{Q} \cdot \mathbf{\Delta} = \sum_{k=1}^{l} Q_k \Delta_k$ and equation (4.1) must hold identically regardless of being on the solution set $\mathcal{S}_\Delta$. In the case of two independent variables, $\mathbf{P} = \binom{T}{X}$, T is called the conserved density and X is the corresponding flux.

Observe that the *order* of $\mathbf{P}$, which is the order of the highest-order derivative appearing in $\mathbf{P}$, and the order of $\mathbf{Q}$ are related. Moreover, $\mathbf{P}$ and $\mathbf{Q}$ are not unique. Determining all such $\mathbf{P}$ and $\mathbf{Q}$ for a given system is a challenging problem.

4.2 Criterion for Conservation Law Characteristics

Let $\mathcal{F}_c(\mathcal{J}^n_{t,x})$ be the subset of $\mathcal{F}(\mathcal{J}^n_{t,x})$ consisting of differential functions with compact support, meaning ones which vanish outside some compact set. The adjoint operator of the Fréchet derivative is the operator D^*_Δ satisfying the equation

$$\int_\Omega \langle \mathrm{D}_\Delta M, N\rangle\, dtdx = \int_\Omega \langle M, \mathrm{D}^*_\Delta N\rangle\, dtdx \tag{4.2}$$

for every $M, N \in \mathcal{F}_c(\mathcal{J}^n_{t,x})$ over a large enough open domain $\Omega \subset \mathbb{R}^2$, where $\langle M, N\rangle = M\cdot N$.

To determine the adjoint of the Fréchet derivative of the heat equation, let $M, N \in \mathcal{F}_c(\mathcal{J}^n_{t,x})$ and compute

$$\int_\Omega \langle \mathrm{D}_{\Delta_{heat}} M, N\rangle\, dtdx = \int_\Omega \langle (D_t - D_{xx})M, N\rangle\, dtdx = \int_\Omega (ND_tM - ND_{xx}M)dtdx \tag{4.3}$$

$$= NM|_\Omega - \int_\Omega QD_tNdtdx - ND_xM|_\Omega + \int_\Omega D_xMD_xNdtdx \tag{4.4}$$

$$= -\int_\Omega MD_tNdtdx + D_xND_xM|_\Omega - \int_\Omega MD_x^2Rdtdx \tag{4.5}$$

$$= \int_\Omega \langle M, (-D_t - D_{xx})N\rangle\, dtdx = \int_\Omega \left\langle M, \mathrm{D}^*_{\Delta_{heat}} N\right\rangle dtdx. \tag{4.6}$$

Notice that for each integration by parts, the boundary terms go to zero for Ω large enough, as M, N and their derivatives are of compact support. Observe that $\mathrm{D}_{\Delta_{heat}} \neq \mathrm{D}^*_{\Delta_{heat}}$.

For the wave equation, a similar computation shows

$$\int_\Omega \langle \mathrm{D}_{\Delta_{wave}} M, N\rangle\, dtdx = \int_\Omega N(D_{tt}M - D_{xx}M)dtdx \tag{4.7a}$$

$$= \int_\Omega M(D_{tt}N - D_{xx}N)dtdx \tag{4.7b}$$

for Ω large enough to ensure that the boundary terms go to zero. Hence, $\mathrm{D}^*_{\Delta_{wave}} = D_{tt} - D_{xx}$. This shows that $\mathrm{D}^*_{\Delta_{wave}} = \mathrm{D}_{\Delta_{wave}}$.

To determine the adjoint of the Fréchet derivative of the NLS, consider differential functions $P, Q \in \mathcal{F}_c(\mathcal{J}^n_{t,x})$ and compute

$$\int_\Omega \left\langle \mathrm{D}_{\Delta_{NLS}} \begin{pmatrix} P \\ P^* \end{pmatrix}, \begin{pmatrix} Q \\ Q^* \end{pmatrix} \right\rangle dtdx \tag{4.8}$$

$$= \int_\Omega \left\langle \begin{pmatrix} 2\chi(q^*)^2 P + (iD_t - D_{xx} + 4\chi\,|q|^2)P^* \\ (-iD_t - D_{xx} + 4\chi\,|q|^2)P + 2\chi q^2 P^* \end{pmatrix}, \begin{pmatrix} Q \\ Q^* \end{pmatrix} \right\rangle dtdx \tag{4.9}$$

$$= \int_\Omega \left\langle \begin{pmatrix} P \\ P^* \end{pmatrix}, \begin{pmatrix} 2\chi(q^*)^2 Q + (iD_t - D_{xx} + 4\chi\,|q|^2)Q^* \\ (-iD_t - D_{xx} + 4\chi\,|q|^2)Q + 2\chi q^2 Q^* \end{pmatrix} \right\rangle dtdx \tag{4.10}$$

$$= \int_\Omega \left\langle \begin{pmatrix} P \\ P^* \end{pmatrix}, \begin{pmatrix} 2\chi(q^*)^2 & iD_t - D_{xx} + 4\chi\,|q|^2 \\ -iD_t - D_{xx} + 4\chi\,|q|^2 & 2\chi q^2 \end{pmatrix} \begin{pmatrix} Q \\ Q^* \end{pmatrix} \right\rangle dtdx \tag{4.11}$$

where Ω is large enough to ensure each boundary term goes to zero. The adjoint of the operator $\mathrm{D}_{\Delta_{NLS}}$ is

$$\mathrm{D}^*_{\Delta_{NLS}} \begin{pmatrix} Q \\ Q^* \end{pmatrix} = \begin{pmatrix} 2\chi(q^*)^2 & iD_t - D_{xx} + 4\chi\,|q|^2 \\ -iD_t - D_{xx} + 4\chi\,|q|^2 & 2\chi q^2 \end{pmatrix} \begin{pmatrix} Q \\ Q^* \end{pmatrix}. \tag{4.12}$$

Observe that $\mathrm{D}_{\Delta_{NLS}} = \mathrm{D}^*_{\Delta_{NLS}}$, which shows that the Fréchet derivative of $\boldsymbol{\Delta}_{NLS}$ is self-adjoint. Systems of differential equations whose Fréchet derivative is self-adjoint are said to satisfy the Helmholtz conditions [25].

The calculus of variations examines extrema of integral functions. A differential operator arises in this context that is very useful in determining conservation laws. Let

$$\mathrm{E} = \begin{pmatrix} E_u \\ E_v \end{pmatrix} = \begin{pmatrix} \sum_{|I|\geq 0}(-D)_I \partial_{u_I} \\ \sum_{|I|\geq 0}(-D)_I \partial_{v_I} \end{pmatrix} \tag{4.13}$$

be the Euler operator E: $\mathcal{F}(\mathcal{J}^n_{t,x}) \to \mathcal{F}^p(\mathcal{J}^n_{t,x})$, where $(-D)_I \partial_{u_I} = (-D_{i_1})(-D_{i_2}) \cdots (-D_{i_k})\partial_{u_{i_1 \ldots i_k}}$, $\mathcal{F}^p(\mathcal{J}^n_{t,x})$ is the Cartesian product of p copies of $\mathcal{F}(\mathcal{J}^n_{t,x})$, and p is the number of dependent variables. Notice that the Euler operator is linear. For instance, if

$L = uv_t - u_{xxx}v^2$, then

$$\mathrm{E}(L) = \begin{pmatrix} E_u(L) \\ E_v(L) \end{pmatrix} = \begin{pmatrix} \partial_u(uv_t - u_{xxx}v^2) + (-D_x)^3(uv_t - u_{xxx}v^2)) \\ \partial_v(uv_t - u_{xxx}v^2) - D_t(uv_t - u_{xxx}v^2) \end{pmatrix} = \begin{pmatrix} v_t + (-D_x)^3(-v^2) \\ -2vu_{xxx} - D_t u \end{pmatrix}. \tag{4.14}$$

Another example will be helpful in Chapter 5. Let n be a natural number and compute

$$\mathrm{E}(q_n q_t^*) = \begin{pmatrix} E_q(q_n q_t^*) \\ E_{q^*}(q_n q_t^*) \end{pmatrix} = \begin{pmatrix} \sum_J (-D_J \partial_{q_J}) q_n q_t^* \\ \sum_J (-D_J \partial_{q_J^*}) q_n q_t^* \end{pmatrix} = \begin{pmatrix} (-1)^n D_x^n q_t^* \\ -D_t q_n \end{pmatrix} = \begin{pmatrix} (-1)^n q_{tn}^* \\ -q_{nt} \end{pmatrix}. \tag{4.15}$$

One property of the Euler operator is that its kernel is composed entirely of divergences [25]. That is, over a simply connected domain,

$$\mathrm{E}(G) = 0 \Leftrightarrow G = \mathrm{Div}\mathbf{P} \tag{4.16}$$

for some smooth differential function $\mathbf{P}$. For example,

$$\mathrm{E}(u_{xx}u + u_x^2 - v_{xx}v - v_x^2) = \begin{pmatrix} E_u(u_{xx}u + u_x^2 - v_{xx}v - v_x^2) \\ E_v(u_{xx}u + u_x^2 - v_{xx}v - v_x^2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.17}$$

indicating that

$$u_{xx}u + u_x^2 - v_{xx}v - v_x^2 = \mathrm{Div}G \tag{4.18}$$

for some differential function G. Notice that with only x-derivatives involved, the operator Div becomes D_x and G has only one component. It is easy to see that

$$\mathrm{Div}(u_x u - v_x v) = D_x(u_x u - v_x v) = u_{xx}u + u_x^2 - v_{xx}v - v_x^2. \tag{4.19}$$

For the product of F,$G \in \mathcal{F}^l(\mathcal{J}^n_{t,x})$, the Euler operator behaves according to the rule

$$\mathrm{E}(F \cdot G) = \mathrm{D}_F^* G + \mathrm{D}_G^* F \tag{4.20}$$

where D^*_F and D^*_G are the adjoints of D_F and D_G, respectively [25]. Applying the Euler operator to (4.1) and using (4.16) lead to

$$0 = \mathrm{E}(\mathrm{Div}\mathbf{P}) = \mathrm{E}(\mathbf{Q}\cdot\Delta) = \mathrm{D}^*_{\Delta}\mathbf{Q} + \mathrm{D}^*_{\mathbf{Q}}\Delta. \tag{4.21}$$

If (4.21) is restricted to $\mathcal{S}_\Delta$, then $\mathrm{D}^*_{\mathbf{Q}}\Delta = 0$ as Δ vanishes identically on $\mathcal{S}_\Delta$. Thus, $\mathrm{D}^*_{\Delta}\mathbf{Q} = 0$ if $\mathbf{Q}$ is a characteristic of a conservation law $\mathbf{P}$. This gives a necessary, but not sufficient, condition that a characteristic must satisfy to give rise to a conservation law. In other words, candidates for characteristics of conservation laws are obtained by finding $\mathbf{Q}$ such that $\mathrm{D}^*_{\Delta}\mathbf{Q} = 0$ on $\mathcal{S}_\Delta$.

The characteristic Q of any conservation law of the heat equation will satisfy

$$\mathrm{D}^*_{\Delta_{heat}}Q = -D_tQ - D_{xx}Q = 0 \tag{4.22}$$

on $\mathcal{S}_{\Delta_{heat}}$. It can be shown that only constant functions satisfy (4.22). Indeed,

$$\mathrm{Div}\begin{pmatrix} u \\ -u_x \end{pmatrix} = 1\cdot(u_t - u_{xx}). \tag{4.23}$$

Then

$$\int_I \mathrm{Div}\begin{pmatrix} u \\ -u_x \end{pmatrix} dx = \int_I (D_tu + D_x(-u_x))dx = D_t\int_I udx + \int_{\partial I}(-u_x)dx = 0 \tag{4.24}$$

by the divergence theorem, over any interval I for x. The integral $\int_I udx$ represents thermal energy and the integral $\int_{\partial I}(-u_x)dx$ represents the flux of heat across the boundary of I, by Fourier's law [16]. Equation (4.24) is the conservation of thermal energy, the foundation of deriving the heat equation.

For the wave equation, the differential function u_t is a characteristic of a conservation law. Computing

$$\mathrm{D}_{\Delta_{wave}}u_t = (D_{tt} - D_{xx})u_t = \frac{d}{dt}(u_{tt} - u_{xx}) = 0 \tag{4.25}$$

on $\mathcal{S}_{\Delta_{wave}}$. Observe that

$$u_t(u_{tt} - u_{xx}) = \mathrm{Div}\begin{pmatrix} \frac{1}{2}(u_t^2 + u_x^2) \\ -u_x u_t \end{pmatrix}. \tag{4.26}$$

Integrating (4.26) gives

$$D_t \int_I \frac{1}{2}(u_t^2 + u_x^2)dx + D_x \int_I -u_x u_t dx = 0. \tag{4.27}$$

which represents the change of the total energy of the vibrating string with time [16].

4.3 The First-order Conservation Laws of the NLS

The characteristic Q of any local conservation law of the NLS must satisfy

$$\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix} = 0 \tag{4.28}$$

on $\mathcal{S}_{\Delta_{NLS}}$. Because $\mathrm{D}^*_{\Delta_{NLS}} = \mathrm{D}_{\Delta_{NLS}}$, the issue of finding conservation law characteristics reduces to checking the characteristics obtained in Proposition 4.

Proposition 5 *The characteristics of the local conservation laws of the NLS corresponding to the geometric symmetries are linear combinations of*

$$\begin{pmatrix} Q_0 \\ Q_0^* \end{pmatrix} = \begin{pmatrix} iq \\ -iq^* \end{pmatrix}, \tag{4.29}$$

$$\begin{pmatrix} Q_1 \\ Q_1^* \end{pmatrix} = \begin{pmatrix} q_x \\ q_x^* \end{pmatrix}, \tag{4.30}$$

$$\begin{pmatrix} Q_2 \\ Q_2^* \end{pmatrix} = \begin{pmatrix} q_t \\ q_t^* \end{pmatrix} = \begin{pmatrix} i(q_{xx} - 2\chi q\,|q|^2) \\ -i(q_{xx}^* - 2\chi q^*\,|q|^2) \end{pmatrix}, \tag{4.31}$$

$$\begin{pmatrix} Q_g \\ Q_g^* \end{pmatrix} = t\begin{pmatrix} q_x \\ q_x^* \end{pmatrix} - \frac{x}{2}\begin{pmatrix} iq \\ -iq^* \end{pmatrix}. \tag{4.32}$$

Proof. This is proven by displaying the associated conservation laws. One can verify the following:

$$\operatorname{Div}\begin{pmatrix} qq^* \\ i(qq_x^* - q_xq^*) \end{pmatrix} = \begin{pmatrix} -iq \\ iq^* \end{pmatrix} \cdot \mathbf{\Delta}_{NLS}, \tag{4.33}$$

$$\operatorname{Div}\begin{pmatrix} -iq_xq^* \\ q_{xx}q^* - q_x^*q_x - \frac{1}{2}\chi\,|q|^2 \end{pmatrix} = \begin{pmatrix} q_x \\ q_x^* \end{pmatrix} \cdot \mathbf{\Delta}_{NLS}, \tag{4.34}$$

$$\operatorname{Div}\begin{pmatrix} q_xq_x^* + \chi\,|q|^4 \\ -q_tq_x^* - q_t^*q_x \end{pmatrix} = \begin{pmatrix} q_t \\ q_t^* \end{pmatrix} \cdot \mathbf{\Delta}_{NLS} \tag{4.35}$$

thus verifying that equations (4.29)-(4.31) are characteristics of conservation laws of the NLS. In addition, (4.32) is a characteristic of the infinitesimal generator of the Galilean boost and

$$\operatorname{Div}\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} Q_g \\ Q_g^* \end{pmatrix} \cdot \begin{pmatrix} \Delta_{NLS}^* \\ \Delta_{NLS} \end{pmatrix} \tag{4.36}$$

where $G = \frac{x}{2}\,|q|^2 - tiq_x^*q$ and $H = t(-\,|q_x|^2 + \chi\,|q|^4 + iq_t^*q) + \frac{ix}{2}(qq_x^* - q_xq^*) + \frac{1}{2}qq^*$. ∎

The scaling symmetry from Proposition 4 is a different matter. The scaling transformation has a characteristic function given by

$$\begin{pmatrix} Q_{sc} \\ Q_{sc}^* \end{pmatrix} = \begin{pmatrix} -ti(q_{xx} - 2\chi q\,|q|^2) - \frac{x}{2}q_x - \frac{1}{2}q \\ ti(q_{xx}^* - 2\chi q^*\,|q|^2) - \frac{x}{2}q_x^* - \frac{1}{2}q^* \end{pmatrix}. \tag{4.37}$$

Calculation shows that

$$\begin{pmatrix} Q_{sc} \\ Q_{sc}^* \end{pmatrix} \cdot \begin{pmatrix} \Delta_{NLS}^* \\ \Delta_{NLS} \end{pmatrix} = D_tR + D_xS - \frac{1}{2}L \tag{4.38}$$

where

$$R = tq^*q_{xx} - \frac{i}{4}\,|q|^2 - \frac{ix}{2}q_xq_t^* - \chi t\,|q|^4, \tag{4.39}$$

$$S = |q_x|^2\,\frac{x+1}{2} + \frac{1}{2}(q_x^*q - q^*q_x) - \chi\frac{x}{2}\,|q|^4 + \frac{ix}{2}q^*q_t + t(q_tq_x^* - q_{xt}q^*) \tag{4.40}$$

and

$$L = \chi\,|q|^4 + |q_x|^2 + \frac{i}{2}(q_t^*q - q^*q_t). \tag{4.41}$$

The function L is called the *Lagrangian* function of the NLS. Applying the Euler operator to equation (4.38) leads to

$$\mathrm{E}(D_t R + D_x S - \frac{1}{2}L) = -\frac{1}{2}\mathrm{E}(L) \tag{4.42}$$

which is nonzero in general. Direct computation shows

$$\mathrm{E}(L) = \begin{pmatrix} E_q(L) \\ E_{q^*}(L) \end{pmatrix} = \begin{pmatrix} \Delta^*_{NLS} \\ \Delta_{NLS} \end{pmatrix}. \tag{4.43}$$

Hence, equation (4.42) only vanishes on solutions of the NLS (see [15] and [22]). Thus, equation (4.38) shows that not every symmetry characteristic corresponds to a conservation law with the scaling symmetry being a well-known example.

Chapter 5

Main Result

The aim of this chapter is to prove that the known list of conservation laws of the NLS is complete.

First, it will be shown that every characteristic function satisfying (4.28) must be linear in its highest-order term. Second, to be a conservation law characteristic, the leading coefficient must be real or imaginary, depending on the parity of the characteristic's order. Third, a recursion operator is used to produce a sequence of characteristics, one of each order. Fourth, an argument based on the order is then used to conclude that the sequence (1.6), together with the density corresponding to the Galilean boost, forms the complete list of densities of local conservation laws of the NLS.

5.1 Conservation Law Characteristics

To attack the problem of finding conservation laws of the NLS, the characteristics are sought. Following the methods used in Chapter 3 and Chapter 4, the form of any conservation law characteristic is determined.

The first task is to prove that any characteristic of a conservation law for the NLS not dependent upon t-derivatives is *quasi-linear*, meaning linear in its highest-order term with constant coefficient. Every characteristic of a conservation law for the NLS necessarily satisfies equation (4.28). The following theorem is a statement regarding the form of such a function.

Theorem 6 *If $Q \in \mathcal{F}(\mathcal{J}_x^n)$ is a differential function, not dependent upon t-derivatives, satisfying equation (4.28) and $n \geq 6$, then Q is quasi-linear.*

Proof. Every characteristic must satisfy

$$\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix} = \begin{pmatrix} 2\chi(q^*)^2 & iD_t - D_{xx} + 4\chi\left|q\right|^2 \\ -iD_t - D_{xx} + 4\chi\left|q\right|^2 & 2\chi q^2 \end{pmatrix}\begin{pmatrix} Q \\ Q^* \end{pmatrix}$$

$$= \begin{pmatrix} 2\chi(q^*)^2 Q + (iD_t - D_{xx} + 4\chi\left|q\right|^2)Q^* \\ (-iD_t - D_{xx} + 4\chi\left|q\right|^2)Q + 2\chi q^2 Q^* \end{pmatrix} = \begin{pmatrix} [\mathrm{D}^*_{\Delta_{NLS}}\binom{Q}{Q^*}]_1 \\ [\mathrm{D}^*_{\Delta_{NLS}}\binom{Q}{Q^*}]_2 \end{pmatrix} = 0 \qquad (5.1)$$

on $\mathcal{S}_{\Delta_{NLS}}$. This leads to a conjugate pair of equations involving complex polynomials. Focusing on the second component

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = -i(Q_{,q_n}(iq_{n+2} + \ldots) + Q_{,q_n^*}(-iq_{n+2} + ..) + ..)$$

$$- (q_{n+2}Q_{,q_n} + q^*_{n+2}Q_{,q_n^*} + \ldots) +$$

$$= q^*_{n+2}(-2Q_{,q_n^*}) + R_1(t, x, q^{(n+1)}, q^{*(n+1)}) \qquad (5.2)$$

for some function R_1. Equation (5.2) must vanish on solutions. Thus, the coefficient q^*_{n+2} must be equal to zero, showing that Q does not depend on q^*_n. Now, equation (5.1) becomes

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q^2_{n+1}(-2Q_{q_n q_n}) + \widetilde{R_1}(t, x, q^{(n+1)}, q^{*(n+1)}) \qquad (5.3)$$

for some function $\widetilde{R_1}$ not dependent on q_{n+1}^2. The coefficient of the quadratic term of order $n+1$ is therefore $Q_{,q_n q_n}$. Setting this coefficient to zero shows that Q is linear in q_n. Thus,

$$Q = Q_1(t, x, q^{(n-1)}, q^{*(n-1)})q_n + Q_2(t, x, q^{(n-1)}, q^{*(n-1)}) \tag{5.4}$$

for some functions Q_1 and Q_2. Applying $\mathrm{D}^*_{\Delta_{NLS}}$ to (5.4) leads to

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_{n+1}(-2D_xQ_1) + q^*_{n+1}(-2Q_{1,q^*_{n-1}} - 2Q_{2,q^*_{n-1}}) + R_2(t, x, q^{(n)}, q^{*(n)}) \tag{5.5}$$

for some function R_2. Setting the coefficient of q_{n+1} to zero implies that $D_xQ_1 = 0$. By Lemma 2, Q_1 is a function of t alone.

Observe that (5.5) forces $2Q_{2,q^*_{n-1}}$ to be zero, also. The Fréchet derivative applied to Q results in

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = -Q_{2,q_{n-1}q_{n-1}}q_n^2 + \widetilde{R_2}(t, x, q^{(n)}, q^{*(n)}) \tag{5.6}$$

for some function $\widetilde{R_2}$ not dependent on quadratic terms involving q_n. The coefficient of q_n^2 must be zero and so $Q_{2,q_{n-1}q_{n-1}}$ must vanish. Thus, Q_2 is linear in q_{n-2} meaning that there are functions $\widetilde{Q_2}$ and $Q_3(t, x, q^{(n-2)}, q^{*(n-2)})$ such that

$$Q = Q_1(t)q_n + \widetilde{Q_2}q_{n-1} + Q_3(t, x, q^{(n-2)}, q^{*(n-2)}). \tag{5.7}$$

Recomputing the Fréchet derivative applied to (5.7) gives

$$\begin{aligned}\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 &= 2q_n^*(\chi q^2(Q_1^* - Q_1) - Q_{3,q^*_{n-2}} - q_{n-1}Q_{2,q^*_{n-2}}) \\ &\quad + q_n(-iQ_{1,t} - 2D_xQ_2) + R_3(t, x, q^{(n-1)}, q^{*(n-1)}),\end{aligned} \tag{5.8}$$

for some function $R_3(t, x, q^{(n-1)}, q^{*(n-1)})$, providing

$$D_xQ_2 = -\frac{i}{2}Q_{1,t} \tag{5.9}$$

$$Q_{3,q^*_{n-2}} = \chi q^2(Q_1^* - Q_1). \tag{5.10}$$

Integrating (5.9) gives

$$Q_2 = Q_4(t) - \frac{i}{2}Q_{1,t}x \tag{5.11}$$

for some function $Q_4(t)$. The general characteristic is now

$$Q = Q_1(t)q_n + (Q_4(t) - \frac{i}{2}Q_{1,t}x)q_{n-1} + Q_3(t, x, q^{(n-2)}, q^{*(n-2)}). \tag{5.12}$$

Applying $\mathrm{D}_{\Delta_{NLS}}$ to (5.12) produces

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = Q_{3,q_{n-2}q_{n-2}}q^2_{n-1} + R_4(t, x, q^{(n-1)}, q^{*(n-1)}) \tag{5.13}$$

for some $R_4(t, x, q^{(n-1)}, q^{*(n-1)})$ not depending on q^2_{n-1}. Thus, $Q_{3,q_{n-2}q_{n-2}} = 0$ which gives, using equation (5.10)

$$Q_3 = \chi q^2(Q_1^* - Q_1)q^*_{n-2} + Q_6 q_{n-2} + Q_7 \tag{5.14}$$

for some $Q_6(t, x, q^{(n-3)}, q^{*(n-3)})$ and $Q_7(t, x, q^{(n-3)}, q^{*(n-3)})$. The characteristic is now

$$Q = Q_1(t)q_n + (Q_4 - \frac{i}{2}Q_{1,t}x)q_{n-1} + \chi q^2(Q_1^* - Q_1)q^*_{n-2} + Q_6 q_{n-2} + Q_7. \tag{5.15}$$

Applying $\mathrm{D}_{\Delta_{NLS}}$ to (5.15) gives

$$\begin{aligned}\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 &= q^*_{n-1}(2\chi q^2(Q_4^* - Q_4 + \frac{ix}{2}(Q^*_{1,t} + Q_{1,t})) \\ &\quad - 4\chi q q_x((n-1)Q_1 + Q_1^*) - 2q_{n-2}Q_{6,q_{n-3}} - 2Q_{7,q^*_{n-3}}) \\ &\quad + q_{n-1}(-4n\chi Q_1 D_x \left|q\right|^2 - i(Q_{4,t} - \frac{ix}{2}Q_{1,tt}) - 2D_x Q_6) \\ &\quad + R_4(t, x, q^{(n-2)}, q^{*(n-2)})\end{aligned} \tag{5.16}$$

for some function $R_4(t, x, q^{(n-2)}, q^{*(n-2)})$. The coefficients of q^*_{n-1} and q_{n-1} must both be zero which leads to

$$Q_6 = -2\chi n Q_1 \left|q\right|^2 - \frac{i}{2}xQ_{4,t} - \frac{x^2}{8}Q_{1,tt} + Q_8(t) \tag{5.17}$$

and

$$2\chi q^2(Q_4^* - Q_4 + \frac{ix}{2}(Q_{1,t}^* + Q_{1,t})) - 4\chi q q_x((n-1)Q_1 + Q_1^*) - 2q_{n-2}Q_{6,q_{n-3}} - 2Q_{7,q_{n-3}^*} \quad (5.18)$$

for some function $Q_8(t)$. The general characteristic is now

$$\begin{aligned} Q &= Q_1(t)q_n + (Q_4 - \frac{i}{2}Q_{1,t}x)q_{n-1} + \chi q^2(Q_1^* - Q_1)q_{n-2}^* \\ &+ (-2\chi n Q_1 \left|q\right|^2 - \frac{i}{2}xQ_{4,t} - \frac{x^2}{8}Q_{1,tt} + Q_8(t))q_{n-2} + Q_7. \end{aligned} \quad (5.19)$$

Applying $\mathrm{D}^*_{\Delta_{NLS}}$ to (5.19) yields

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_{n-2}^2 Q_{7,q_{n-3}q_{n-3}} + q_{n-2}^{*2} Q_{7,q_{n-3}^* q_{n-3}^*} + \widetilde{R_4}(t, x, q^{(n-2)}, q^{*(n-2)}) \quad (5.20)$$

for some function $\widetilde{R_4}$ not depending on q_{n-2}^2 or q_{n-2}^{*2}. The coefficients of q_{n-2}^2 and q_{n-2}^{*2} must each be zero and hence,

$$Q_{7,q_{n-3}q_{n-3}} = 0 = Q_{7,q_{n-3}^* q_{n-3}^*}. \quad (5.21)$$

Combining the information in (5.18) and (5.21) results in

$$\begin{aligned} Q_7 &= (-2\chi((n-1)Q_1 + Q_1^*)qq_x + \chi q^2(Q_4^* - Q_4 + \frac{ix}{2}(Q_{1,t}^* + Q_{1,t})))q_{n-3}^* \\ &+ Q_9(t, x, q^{(n-4)}, q^{*(n-4)})q_{n-3} + Q_{10}(t, x, q^{(n-4)}, q^{*(n-4)}) \end{aligned} \quad (5.22)$$

for some functions $Q_9(t, x, q^{(n-4)}, q^{*(n-4)})$ and $Q_{10}(t, x, q^{(n-4)}, q^{*(n-4)})$. The characteristic is now of the form

$$\begin{aligned} Q &= Q_1(t)q_n + (Q_4 - \frac{i}{2}Q_{1,t}x)q_{n-1} + \chi q^2(Q_1^* - Q_1)q_{n-2}^* \\ &+ (-2\chi n Q_1 \left|q\right|^2 - \frac{i}{2}xQ_{4,t} - \frac{x^2}{8}Q_{1,tt} + Q_8(t))q_{n-2} \\ &+ (-2\chi((n-1)Q_1 + Q_1^*)qq_x + \chi q^2(Q_4^* - Q_4 + \frac{ix}{2}(Q_{1,t}^* + Q_{1,t})))q_{n-3}^* \\ &+ Q_9(t, x, q^{(n-4)}, q^{*(n-4)})q_{n-3} + Q_{10}(t, x, q^{(n-4)}, q^{*(n-4)}). \end{aligned} \quad (5.23)$$

Application of $\mathrm{D}^*_{\Delta_{NLS}}$ to (5.23) yields

$$\left[\mathrm{D}^*_{\Delta_{NLS}}\begin{pmatrix} Q \\ Q^* \end{pmatrix}\right]_2 = q_{n-2}(D_x(2\chi ix(n-1)Q_{1,t}\,|q|^2$$
$$-\chi n(n-1)Q_1 D_x\,|q|^2 - 2\chi(n-1)Q_4 D_x\,|q|^2)$$
$$-D_x Q_9 + 2i\chi Q_{1,t}\,|q|^2) + R_5(t,x,q^{(n-3)},q^{*(n-2)}) \tag{5.24}$$

for some function $R_5(t,x,q^{(n-3)},q^{*(n-2)})$. Thus,

$$D_x(2\chi ix(n-1)Q_{1,t}\,|q|^2 - \chi n(n-1)Q_1 D_x\,|q|^2 \tag{5.25}$$

$$-2\chi(n-1)Q_4 D_x\,|q|^2 - Q_9) + 2i\chi Q_{1,t}\,|q|^2 = 0. \tag{5.26}$$

By the second part of Lemma 2, equation (5.25) implies

$$2i\chi Q_{1,t}\,|q|^2 = 0 \Longrightarrow Q_{1,t} = 0 \tag{5.27}$$

which proves the theorem for $n > 5$. ∎

For $n \leq 5$, the resulting characteristics are known (see, for instance, [24] and [36]).

Corollary 7 *Every conservation law characteristic of order greater than or equal to 6 is equivalent to a characteristic of the form*

$$Q(t,x,q^{(n)},q^{*(n)}) = c_n q_n + R(t,x,q^{(n-1)},q^{*(n-1)}). \tag{5.28}$$

for some constant $c_n \in \mathbb{C}$ and some function $R \in \mathcal{F}(\mathcal{J}_x^{n-1})$. If n is even, then c_n is imaginary. If n is odd, then c_n is real.

Proof. Suppose that $Q \in \mathcal{F}(\mathcal{J}_x^n)$ is an n^{th}-order conservation law characteristic of the NLS, not dependent upon t-derivatives. By Theorem 6, the characteristic can be written as in (5.28). The product of the characteristic and the NLS is

$$\begin{pmatrix} Q \\ Q^* \end{pmatrix} \cdot \Delta_{NLS} = c_n q_n (i q_t^* - q_{xx}^*) - c_n^* q_n^* (i q_t + q_{xx}) + S(t, x, q^{(n-1)}, q^{*(n-1)}) \tag{5.29}$$

where $S(t, x, q^{(n-1)}, q^{*(n-1)})$ involves terms containing lower-order derivatives of q and q^*. In particular, no t-derivatives appear in S. Application of the Euler operator to (5.29) gives

$$\begin{aligned}
&\mathrm{E}(c_n q_n (i q_t^* - q_{xx}^*) + c_n^* q_n^* (-i q_t - q_{xx}) + S(t, x, q^{(n-1)}, q^{*(n-1)})) \\
&= \begin{pmatrix} E_q(c_n q_n (i q_t^* - q_{xx}^*) + c_n^* q_n^* (-i q_t - q_{xx}) + S(t, x, q^{(n-1)}, q^{*(n-1)})) \\ E_{q^*}(c_n q_n (i q_t^* - q_{xx}^*) + c_n^* q_n^* (-i q_t - q_{xx}) + S(t, x, q^{(n-1)}, q^{*(n-1)})) \end{pmatrix} \\
&= \begin{pmatrix} c_n (-1)^n (D_n i q_t^* - D_n q_{xx}^*) - c_n^* (D_x^2 q_n^* - i D_t q_n^*) + E_q(S(t, x, q^{(n-1)}, q^{*(n-1)})) \\ -c_n (-i D_t q_n - D_x^2 q_n) - c_n^* (-1)^n (D_n i q_t - D_n q_{xx}) + E_{q^*}(S(t, x, q^{(n-1)}, q^{*(n-1)})) \end{pmatrix} \\
&= \begin{pmatrix} (c_n^* + c_n (-1)^n) i q_{tn}^* + q_{xx}^* (-c_n^* - c_n (-1)^n) + E_q(S(t, x, q^{(n-1)}, q^{*(n-1)})) \\ (-c_n - c_n^* (-1)^n) i q_{tn} + q_{xx}^* (-c_n - c_n^* (-1)^n) + E_{q^*}(S(t, x, q^{(n-1)}, q^{*(n-1)})) \end{pmatrix}.
\end{aligned} \tag{5.30}$$

Equation (5.30) must vanish if the product $\binom{Q}{Q^*} \cdot \Delta_{NLS}$ is a divergence, in other words, a conservation law. In particular, the coefficients of the t-derivatives must vanish. Thus, if n is even, then $c_n^* + c_n = 2 \operatorname{Re} c_n = 0$, whence c_n is imaginary. If n is odd, then $c_n^* - c_n = 2i \operatorname{Im} c_n = 0$ showing that c_n is real. ∎

A recursion operator for the NLS (see [5]) is

$$y_{n+1} = \mathcal{R}\{y_n\} = i \partial_x y_n - 2 i \chi q \int (q y_n^* + q^* y_n) dx. \tag{5.31}$$

Here, q solves the NLS (2.9) and $\binom{y_n}{y_n^*}$ satisfies (5.1) and the integral is treated as an operator. Let $y_0 = -iq$. Then

$$y_1 = \mathcal{R}\{y_0\} = q_x \tag{5.32}$$

$$y_2 = \mathcal{R}\{y_1\} = i q_{xx} - 2 \chi i q \, |q|^2 \tag{5.33}$$

which agree with characteristics Q_1 and Q_2 in Proposition 4. Applying $\mathcal{R}$ to y_2 produces

$$y_3 = \mathcal{R}\{y_2\} = -q_3 + 6\chi q_x |q|^2 \tag{5.34}$$

$$y_4 = \mathcal{R}\{y_3\} = -iq_4 + 8\chi i q_{xx} |q|^2 + 2i\chi q^2 q_{xx}^* + 6i\chi q_x^2 q^* + 4i\chi q |q_x|^2 - 6i\chi^2 q |q|^4 . \tag{5.35}$$

By induction, a formula for the characteristics may be given along these lines:

$$y_{n+1} = \mathcal{R}^n\{y_1\} = (i\partial_x)^n q_x + \sum_{m=1}^{n} (i\partial_x)^{n-m}(-2i\chi q \int S(m)dx), \tag{5.36}$$

where $S(m) = q^* y_m + q y_m^*$. Hence, the recursion operator produces the sequence of characteristics $y_0, ..., y_n, ..,$ one for each order.

For $Q_1, Q_2 \in \mathcal{F}(\mathcal{J}_{t,x}^n)$, define an equivalence relation via

$$Q_1 \sim Q_2 \iff (Q_1 - Q_2)|_{\mathcal{S}_{\Delta_{NLS}}} = 0. \tag{5.37}$$

Now, form the set of equivalence classes containing y_n, namely

$$[y_n] = \{Q \in \mathcal{F}(\mathcal{J}_{t,x}^n) \colon \; Q \sim y_n\} \tag{5.38}$$

producing the set $\{[y_0], [y_1], ..., [y_n]...\}$. There is exactly one equivalence class for each $n \geq 3$ because the leading term of each such y_n is of order n. This is false for the cases $n < 2$, in which Q_g and Q_{sc} result in additional characteristics for orders 1 and 2, respectively.

Claim 8 *Every n^{th}-order characteristic function of a conservation law satisfying (4.28) is equivalent to a linear combination of $Q_g, y_0, y_1, ..., y_n ...$*

Proof. To show that any characteristic can be expressed as a linear combination of $\{Q_g, Q_{sc}, y_0, y_1, ..., y_n\}$, a descent argument will be used. Let $R \in \mathcal{F}(\mathcal{J}_{t,x}^n)$. Note that on $\mathcal{S}_{\Delta_{NLS}}$, there is an equivalent $\widetilde{R} \in \mathcal{F}(\mathcal{J}_x^n)$ obtained by replacing time derivatives using

(2.9). By Theorem 6, there is a constant α_n such that $\widetilde{R} - \alpha_n y_n$ is of order $n-1$. A constant α_{n-1} is then chosen such that $\widetilde{R} - \alpha_n y_n - \alpha_{n-1} y_{n-1}$ is of order $n-2$ and so on until $\widetilde{R} - \sum_{i=3}^{n} \alpha_i y_i$ is of order at most two. The low-order characteristics are given in Proposition 5. Hence, there exist real constants $\alpha_g, \alpha_0, \alpha_1$ and α_2 such that

$$\widetilde{R} - \sum_{i=3}^{n} \alpha_i y_i = \alpha_g Q_g + \alpha_0 Q_0 + \alpha_1 Q_1 + \alpha_2 Q_2. \tag{5.39}$$

Using these, one writes

$$R \sim \widetilde{R} = \sum_{i=3}^{n} \alpha_i y_i + \alpha_g Q_g + \alpha_0 Q_0 + \alpha_1 Q_1 + \alpha_2 Q_2 \tag{5.40}$$

where $\alpha_i \in \mathbb{C}$ for $i = 3, ..., n$. ∎

5.2 Conservation Laws of the NLS

Recall from (1.6) that the sequence of known conservation laws have conserved densities $P_n = q^* w_n$ where

$$w_n = -i\partial_x w_{n-1} + \chi q^* \sum_{k=1}^{n-2} w_k w_{n-l-k} \tag{5.41}$$

and $w_1 = q$. Induction shows that

$$P_{n+1} = q^*(-i\partial_x)^n q + q^* \sum_{l=0}^{n-1} (-i\partial_x)^l \chi q^* \sum_{k=1}^{n-1-l} w_k w_{n-l-k}, \tag{5.42}$$

for $n \geq 2$. Note that P_{n+1} only depends upon spatial derivatives and has order n. Each P_{n+1} is a conserved density and has a related flux X_{n+1} of order $n+1$ that is determined by the equation

$$\operatorname{Div}\begin{pmatrix} P_{n+1} \\ X_{n+1} \end{pmatrix} = D_t P_{n+1} + D_x X_{n+1} = 0 \tag{5.43}$$

on $\mathcal{S}_{\Delta_{NLS}}$. Hence, for each order n, there is one conservation law $\binom{P_n}{X_n}$ for each member of sequence (1.6).

A conservation law is trivial if there exist functions $M_1, M_2 \in \mathcal{F}(\mathcal{J}^n_{t,x})$ such that

$$M_1 = -M_2 \tag{5.44}$$

and

$$\begin{pmatrix} T \\ X \end{pmatrix} = \begin{pmatrix} D_x M_1 \\ D_t M_2 \end{pmatrix} \tag{5.45}$$

on $\mathcal{S}_{\Delta_{NLS}}$. Define two conservation laws to be equivalent if their difference is trivial. Form the set of equivalence classes by letting

$$\left[\begin{pmatrix} P_n \\ X_n \end{pmatrix}\right] = \left\{\begin{pmatrix} M \\ N \end{pmatrix} : \begin{pmatrix} M \\ N \end{pmatrix} \sim \begin{pmatrix} P_n \\ X_n \end{pmatrix} \; M, N \in \mathcal{F}(\mathcal{J}^n_{t,x})\right\}. \tag{5.46}$$

There is exactly one equivalence class for each n.

Olver proves (with some regularity conditions) that there is a one-to-one correspondence between the set of equivalence classes of conservation laws of a system of PDEs and the set of equivalence classes of characteristic functions of conservation laws (see Appendix B) [25]. For each order $n \geq 3$, there is therefore exactly one equivalence class of the form (5.38) and one equivalence class of the form (5.46). This proves Theorem 1 which states:

The sequence (1.6) together with $\frac{x}{2}|q|^2 - tiq_x^* q$ forms a complete list of the densities of the local conservation laws of the NLS (2.9).

Chapter 6

Conclusion

6.1 Summary

The main result of this dissertation is Theorem 1 which states that the known list of local conservation laws of the NLS is complete. The results of this dissertation are summarized by the following remark.

Remark 9 *If $\{D_n\}$ is any sequence of conserved densities of local conservation laws of the NLS, then there exist constants $\lambda_g, \lambda_1, .., \lambda_n, ..$ such that*

$$D_n = \lambda_g G + \sum_i \lambda_i P_i \tag{6.1}$$

where G is the density corresponding to the characteristic (4.32) and P_i is a member of the sequence (1.6).

6.2 Further Questions

It is desirable to have a general formula for the characteristics, not involving the integral operator as in (5.36). Direction calculation does not produce a formula and the use of Olver's equivalence theorem is necessary. This result will be available soon.

The Davey-Stewartson equation is accepted as a two-dimensional analogue of the NLS [13]. The Davey-Stewartson equation possesses a Bäcklund transformation (see [18]) and hence, is integrable. Determining a new conserved quantity or proving that no new conservation laws of the Davey-Stewartson equation exist would shed light on the nature of the phenomena it models. The methods of symmetry analysis employed in this paper will be used to answer these questions in a future publication.

Appendix A

Known Sequences

Many methods have been used to generate the conserved densities of the NLS. Notable examples are briefly shown to be equivalent to the sequence (1.6).

i) Ablowitz, Segur, Newell and Kaup [2] and Konno Sanuki and Ichikawa [23] derive (1.6) via the inverse scattering transform. This expresses the evolution equation as an eigenvalue problem written as

$$\partial_x \mathbf{v} = \begin{pmatrix} -i\zeta & q \\ r & i\zeta \end{pmatrix} \mathbf{v} \tag{A.1}$$

$$\partial_t \mathbf{v} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \mathbf{v} \tag{A.2}$$

for some A, B and C functions of t, x, ζ. Requiring integrability (that is, commutivity of partial derivatives) gives

$$A_x - rC + qB = 0$$

$$C_x - 2\eta C - r_t - 2rA = 0$$

$$B_x - 2\eta B + q_t + 2qA = 0. \tag{A.3}$$

The remarkable fact is that these equations are exactly those obtained as the pull-back of the Maurer-Cartan form on SL(2,$\mathbb{R}$) where $\eta = i\zeta(x)$ [10]. Further connections between the NLS and Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ are examined by Ratiu, Flaschka and Newell [29] and Terng and Ulenbeck [32]. An asymptotic expansion in $\frac{1}{\zeta}$ for the sequence of densities follows from

$$\varphi = \sum_i \frac{f_n}{(2i\zeta)^n} \tag{A.4}$$

where

$$f_{m+1} = r\partial_x \frac{1}{r} f_m + \sum_{k+j=m-1} f_k f_j,\ f_1 = |r|^2 \tag{A.5}$$

which leads to

$$(2i)^n C_n = \int_{-\infty}^{\infty} f_n dx. \tag{A.6}$$

One can follow [23], which gives a general format. Clearly, $P_1 = f_1$. Further,

$$c_n P_{n+1} - f_{n+1} = c_n q^* w_{n+1} - r\partial_x \frac{1}{r} f_m + \sum_{k+j=m-1} f_k f_j. \tag{A.7}$$

Let $r = q^*$ and the formulas match when $\chi = 1$ and $c_n = i$.

ii) Kodoma and Mikhailov [21] use a recursion operator to generate a sequence of generalized symmetries. Explicitly,

$$\begin{pmatrix} K_n \\ K_n^* \end{pmatrix} = \Lambda \begin{pmatrix} K_{n-1} \\ K_{n-1}^* \end{pmatrix} = -i\partial_x \begin{pmatrix} K_{n-1} \\ K_{n-1}^* \end{pmatrix} - i \begin{pmatrix} 2q\int q^* K_{n-1} + qK_{n-1}^* & 2q\int q^* K_{n-1} + qK_{n-1}^* \\ 2q\int q^* K_{n-1} + qK_{n-1}^* & 2q^* \int q^* K_{n-1} + qK_{n-1}^* \end{pmatrix}. \tag{A.8}$$

Two sequences are given there; only one is local. Notice that $\Lambda|_{\mathcal{S}_{\Delta_{NLS}}} = \mathcal{R}$ (see (5.31)) and choosing $\binom{K_0}{K_0^*} = \binom{iq}{-iq^*}$ gives

$$\begin{pmatrix} K_n \\ K_n^* \end{pmatrix} = \Lambda^n \begin{pmatrix} K_0 \\ K_0^* \end{pmatrix} = c_n \begin{pmatrix} y_n \\ y_n^* \end{pmatrix}. \tag{A.9}$$

iii) Pritula and Vekslerchik [27] use a functional representation of the AKNS hierarchy (a generalization of the NLS) involving q, r and ζ to produce $J(t,\zeta) = q(t + i[\zeta])r(t)$, a generating function for the conserved densities of

$$\begin{aligned} iq_t + q_{xx} + 2rq^2 &= 0 \\ -ir_t + r_{xx} + 2qr^2 &= 0. \end{aligned} \tag{A.10}$$

These are obtained from (A.3) via $A = -2i\zeta - irq$, $B = -iq_x$ and $C = ir_x$. Taking $J = \sum_{n=0}^{\infty} J_n \zeta^n$, the resulting sequence is

$$J_0 = qr,\ J_1 = q_x r \text{ and } J_{n+1} = r\partial_x \frac{J_n}{r} - \sum_{l=0}^{n-1} J_l J_{n-1-l}. \tag{A.11}$$

Let $J_k = q^* \widetilde{J}_k$ and $r = -q^*$ to see that the recurrence relation becomes

$$q^* \widetilde{J}_{n+1} = -q^* \partial_x \widetilde{J}_n - q^{*2} \sum_{l=0}^{n-1} \widetilde{J}_l \widetilde{J}_{n-1-l} = q^*(-\partial_x \widetilde{J}_n - q^* \sum_{l=0}^{n-1} \widetilde{J}_l \widetilde{J}_{n-1-l}). \tag{A.12}$$

Letting $\widetilde{J}_n = iw_n$ gives sequence (1.6).

Appendix B

Olver's Theorem

The following definitions are given by Olver [25]. A system of PDEs is in *Kovaleskaya form* if it can be written as

$$q^{\alpha}_{mt} = \Gamma_{\alpha}(t, x, q^{(n)}, q^{*(n)}),\ \alpha = 1, 2 \tag{B.1}$$

where $\Gamma_{\alpha}(t, x, q^{(n)}, q^{*(n)})$ are analytic functions. A system is said to be *locally solvable* if for prescribed initial conditions a smooth solution satisfying equation (B.1) in a neighborhood of every point exists. A noncharacteristic direction to Δ_{NLS} is a vector ω such that the matrix $\sum_{|J|=n} \frac{\partial \Delta}{\partial u^{\alpha}_{J}} \cdot \omega_J$ is nonsingular at $(x_0, u^{(n)}_0)$. A system Δ is said to be *normal* if it has at least one noncharacteristic direction. A system of PDEs is *totally nondegenerate* if it and all its prolongations are of maximal rank and locally solvable.

Olver proves the following theorem [25].

Theorem 10 *For a totally nondegenerate, normal system of PDEs, two conservation laws are equivalent if and only if their characteristics are.*

By definition, the NLS is in Kovaleskaya form. A theorem given by Olver states that a system of DEs is normal if and only if it is transformable into a system in Kovaleskaya form. Hence, the NLS are normal. Olver gives another theorem: An analytic system with the same number of equations as dependent variables is totally nondegenerate if and only if it is normal. The equation (2.9) has two dependent variables q and q^* and therefore the NLS are totally nondegenerate. This establishes the following result.

Lemma 11 *The NLS is a totally nondegenerate, normal system of PDEs.*

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