ESAM 446-1
Numerical Solution to PDEs
Finite Difference Methods

Lecture 6: 2D problems

Daniel Lecoanet

Outline

1. Explicit Timestepping
2. Advection Equation
3. Implicit Timestepping: Operator Splitting
4. Diffusion Equation

Explicit Timestepping

$$\partial_t c + u \partial_x c = 0 \qquad\qquad c(x,t)$$

$$\partial_t c + \boldsymbol{u} \cdot \boldsymbol{\nabla} c = \partial_t c + u \partial_x c + v \partial_y c = 0$$

$$\boldsymbol{u} = (u, v) \qquad\qquad c(x,y,t)$$

$$\{x_j\},\ \{y_k\}$$

$$c(x_j, y_k, t_n) = c_{j,k}^n$$

Explicit Timestepping

$$\partial_t c + u \partial_x c = 0$$

$$c_j^{n+1} = c_j^n - \frac{u \Delta t}{h} \left(c_{j+1}^n - c_j^n \right)$$

$$\partial_t c + u \partial_x c + v \partial_y c = 0$$

$$c_{j,k}^{n+1} = c_{j,k}^n - \frac{u \Delta t}{h} \left(c_{j+1,k}^n - c_{j,k}^n \right) - \frac{v \Delta t}{h} \left(c_{j,k+1}^n - c_{j,k}^n \right)$$

Explicit Timestepping

$$\partial_t c + u\partial_x c = 0$$

$$c_j^{n+1} = c_j^n - \frac{u\Delta t}{h}\left(c_{j+1}^n - c_j^n\right)$$

$$c_j^n = z^n \exp(ikhj)$$

$$z = 1 - \frac{u\Delta t}{h}\left(\exp(ikh) - 1\right)$$

$$\Delta t < \frac{h}{u}$$

Explicit Timestepping

$$\partial_t c + u\partial_x c + v\partial_y c = 0$$

$$c_{j,k}^{n+1} = c_{j,k}^n - \frac{u\Delta t}{h}\left(c_{j+1,k}^n - c_{j,k}^n\right) - \frac{v\Delta t}{h}\left(c_{j,k+1}^n - c_{j,k}^n\right)$$

$$c_{j,k}^n = z^n \exp(ik_x hj)\exp(ik_y hk)$$

$$z = 1 - \frac{u\Delta t}{h}\left(\exp(ik_x h) - 1\right) - \frac{v\Delta t}{h}\left(\exp(ik_y h) - 1\right)$$

$$u = v \qquad\qquad k_x h = k_y h$$

Explicit Timestepping

$$\partial_t c + u\partial_x c + v\partial_y c = 0 \qquad\qquad u = v$$

$$c_{j,k}^{n+1} = c_{j,k}^{n} - \frac{u\Delta t}{h}\left(c_{j+1,k}^{n} - c_{j,k}^{n}\right) - \frac{v\Delta t}{h}\left(c_{j,k+1}^{n} - c_{j,k}^{n}\right)$$

$$z = 1 - 2\frac{u\Delta t}{h}\left(\exp(ik_x h) - 1\right)$$

$$\Delta t < \frac{h}{2u}$$

Explicit Timestepping

$$\partial_t c + u\partial_x c + v\partial_y c = 0$$

$$c = \exp(-i\omega t)\exp(ik_x x)\exp(ik_y y)$$

$$\omega = |\boldsymbol{k}|\left(u\sin(\theta) + v\cos(\theta)\right) = \boldsymbol{u}\cdot\boldsymbol{k}$$

Explicit Timestepping

$$\partial_t c + u\partial_x c + v\partial_y c = 0$$

$$\partial_t c_{j,k} = -\frac{u}{2h}\left(c_{j+1,k} - c_{j-1,k}\right) - \frac{v}{2h}\left(c_{j,k+1} - c_{j,k-1}\right)$$

$$c = \exp(-i\omega' t)\exp(ik_x hj)\exp(ik_y hk)$$

$$\omega' = \frac{u\sin(k_x h)}{h} - \frac{v\sin(k_y h)}{h}$$

Explicit Timestepping

$$\partial_t c + u\partial_x c + v\partial_y c = 0$$

$$\partial_t c_{j,k} = -\frac{u}{2h}\left(c_{j+1,k} - c_{j-1,k}\right) - \frac{v}{2h}\left(c_{j,k+1} - c_{j,k-1}\right)$$

$$c = \exp(-i\omega' t)\exp(ik_x hj)\exp(ik_y hk)$$

$$\Delta\omega = \omega' - \omega = -\frac{uk_x(hk_x)^2}{6} - \frac{vk_y(hk_y)^2}{6}$$

Implicit Timestepping

$$\partial_t c + D\nabla^2 c = \partial_t c + D(\partial_x^2 + \partial_y^2)c = 0$$

$$\partial_t c + D\partial_x^2 c = 0 \qquad \partial_t c + D\partial_y^2 c = 0$$

$$\partial_t c + D(\partial_x^2 + \partial_y^2)c \approx (\partial_t + D\partial_x^2)(\partial_t + D\partial_y^2)c$$

Operator splitting

Operator Formulation

$$\partial_t c + D\partial_x^2 c = 0$$

$$\partial_t c + Ac = 0$$

$$c(t) = \exp(-At)c(0)$$

Operator Formulation

$$\partial_t c + D\partial_x^2 c + D\partial_y^2 c = 0$$

$$\partial_t c + Ac + Bc = 0$$

$$c(t) = \exp\left[-(A+B)t\right] c(0)$$

Operator Formulation

$$\partial_t c + D\partial_x^2 c + D\partial_y^2 c = 0$$

$$c(t) = \exp\left[-(A+B)t\right] c(0)$$

$$c(\Delta t) = \exp\left[-(A+B)\Delta t\right] c(0)$$

Operator Splitting

$$\partial_t c + D\partial_x^2 c + D\partial_y^2 c = 0$$

$$I - (A+B)\Delta t + \frac{1}{2}\left(A^2 + AB + BA + B^2\right)\Delta t^2 + \dots$$

$$\exp(-B\Delta t)\exp(-A\Delta t) = \left(I - B\Delta t + \frac{B^2\Delta t^2}{2} + \dots\right)\left(I - A\Delta t + \frac{A^2\Delta t^2}{2} + \dots\right)$$

$$\exp(-(A+B)\Delta t) = \exp(-B\Delta t)\exp(-A\Delta t) + \frac{1}{2}(AB - BA)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

Operator Splitting

$$\partial_t c + D\partial_x^2 c + D\partial_y^2 c = 0$$

$$\exp(-(A+B)\Delta t) \approx \exp(-B\Delta t)\exp(-A\Delta t)$$

$\exp(-A\Delta)c \iff$ timestepping $\partial_t c + D\partial_x^2 c = 0$

$$c^{n+1/2} + \Delta t D\partial_x^2 c^{n+1/2} = c^n$$

$$c^{n+1} + \Delta t D\partial_y^2 c^{n+1} = c^{n+1/2}$$

extends to >2 operators

Operator Splitting

$$\exp(-(A+B)\Delta t) = \exp(-B\Delta t)\exp(-A\Delta t) + \frac{1}{2}(AB - BA)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$\exp(-(A+B)\Delta t) = \exp(-B\Delta t/2)\exp(-A\Delta t/2)\exp(-A\Delta t/2)\exp(-B\Delta t/2) + \mathcal{O}(\Delta t^3)$$

$$\exp(-(A+B)\Delta t) = \exp(-B\Delta t/2)\exp(-A\Delta t)\exp(-B\Delta t/2) + \mathcal{O}(\Delta t^3)$$

Strang Splitting