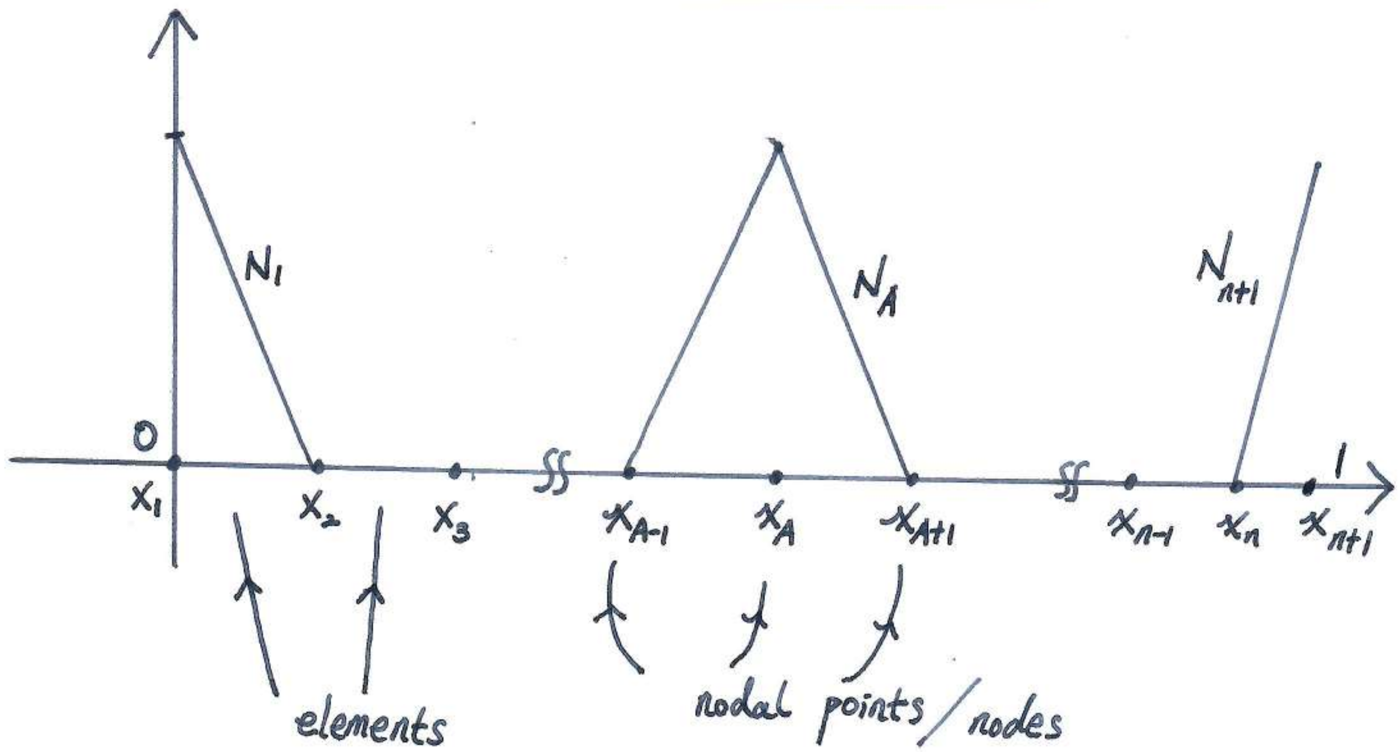
Piecewise Linear Finite Element Space.



$$h_A := x_{A+1} - x_A$$

$$h := \max h_A \quad \text{for } A = 1, \cdots, n.$$

For a typical interior node

$$N_A(x) = \begin{cases} \dfrac{x - x_{A-1}}{h_{A-1}} & x \in [x_{A-1}, x_A) \\ \dfrac{x_{A+1} - x}{h_A} & x \in [x_A, x_{A+1}) \\ 0 & \text{otherwise.} \end{cases}$$

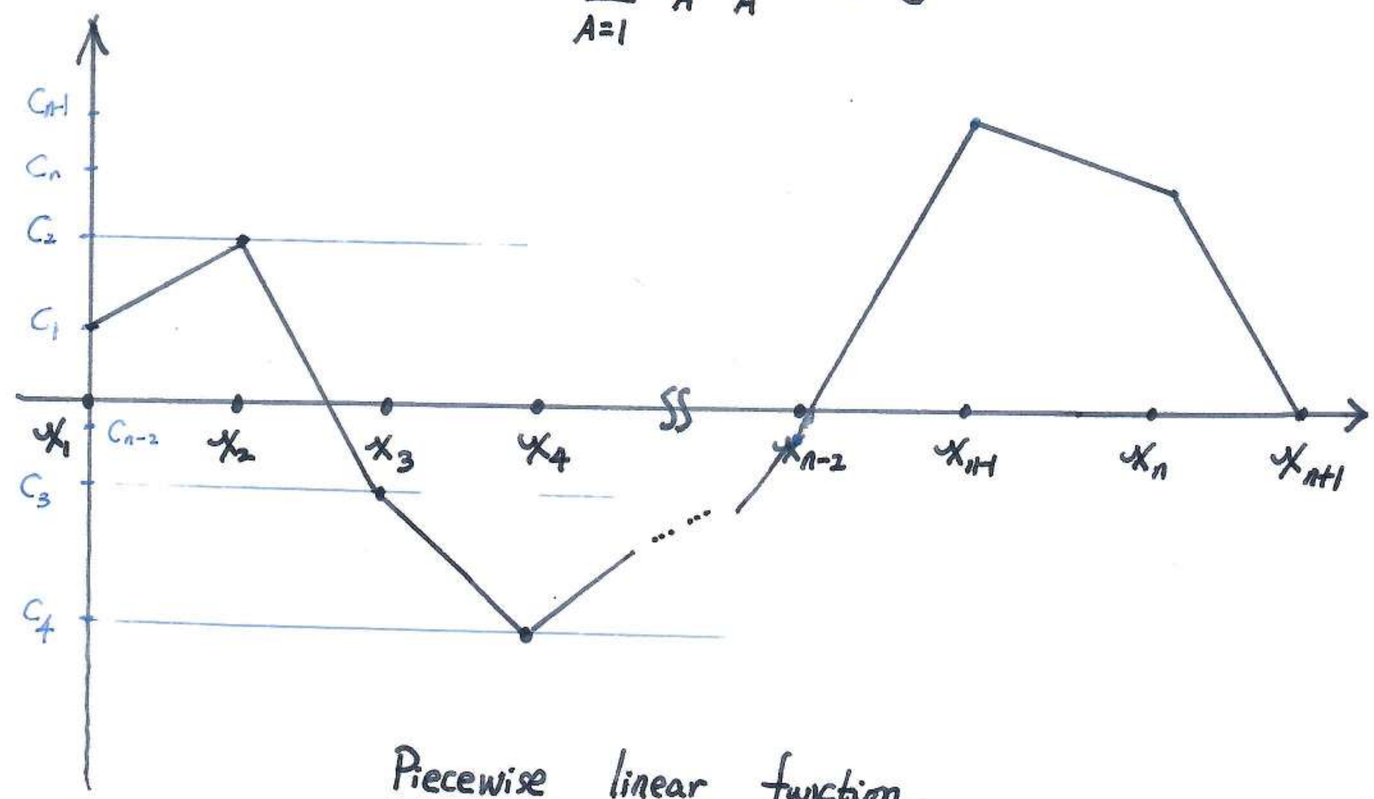
For the boundary two nodes

$$N_1(x) = \begin{cases} \dfrac{x_2 - x}{h_1} & x \in [x_1, x_2) \\ 0 & \text{otherwise} \end{cases}$$

$$N_{n+1}(x) = \begin{cases} \dfrac{x - x_n}{h_n} & x \in [x_n, \; x_{n+1}] \\ 0 & \text{otherwise.} \end{cases}$$

- $N_A(x)$ is referred to as 'hat' or 'roof' functions.
- $N_A(x_B) = \delta_{AB} = \begin{cases} 1 & A = B \\ 0 & A \neq B \end{cases}$ Kronecker delta.

Take a look at $w^h = \sum_{A=1}^{n} C_A N_A \in \mathcal{V}^h$



Piecewise linear function.

- $K_{AB} = \int_0^1 N_{A,x} N_{B,x} \, dx = 0$ if $B > A+1$ or $B < A-1$

$B = A-1$: $$K_{AB} = \int_0^1 N_{A,x} \, N_{A-1,x} \, dx$$

$$= \int_{x_{A-1}}^{x_A} \frac{1}{h_{A-1}} \, \frac{-1}{h_{A-1}} \, dx$$

$$= -\frac{1}{h_{A-1}}$$

$B = A$: $$K_{AB} = \int_0^1 \left(N_{A,x}\right)^2 dx$$

$$= \int_{x_{A-1}}^{x_A} \left(\frac{1}{h_{A-1}}\right)^2 dx + \int_{x_A}^{x_{A+1}} \left(\frac{1}{h_A}\right)^2 dx$$

$$= \frac{1}{h_{A-1}} + \frac{1}{h_A}$$

$B = A+1$: $$K_{AB} = \int_0^1 N_{A,x} \, N_{A+1,x} \, dx$$

$$= \int_{x_A}^{x_{A+1}} -\frac{1}{h_A} \cdot \frac{1}{h_A} \, dx$$

$$= -\frac{1}{h_A}.$$

$$
K = \begin{bmatrix}
\frac{1}{h_1} & \frac{-1}{h_1} & & & & \text{Zeros} & \\
\frac{-1}{h_1} & \frac{1}{h_1}+\frac{1}{h_2} & \frac{-1}{h_2} & & & & \\
& \ddots & \ddots & \ddots & & & \\
& & -\frac{1}{h_{A-1}} & \frac{1}{h_{A-1}}+\frac{1}{h_A} & -\frac{1}{h_A} & & \\
& & & \ddots & \ddots & \ddots & \\
\text{Zeros} & & & & -\frac{1}{h_{n-2}} & \frac{1}{h_{n-2}}+\frac{1}{h_{n-1}} & -\frac{1}{h_{n-1}} \\
& & & & & -\frac{1}{h_{n-1}} & \frac{1}{h_{n-1}}+\frac{1}{h_n}
\end{bmatrix}_{n \times n}
$$

- K is banded with bandwidth $= 3$ (tri-diagonal)

  This is due to our choice of the basis functions N_A, which is non-zero on a few elements (compactly supported).

- Again, we see that $K = K^T$.

- ~~Propo~~ Theorem: The matrix K is positive definite.

$c^T K c \geq 0$ for $\forall c \in R^n$

$c^T K c = 0$ implies $c = 0$

proof: (i) Pick any $\vec{c} \in \mathbb{R}^n$ $\quad \vec{c} = \{ c_A \}$

$$\vec{c}^T K \vec{c} = \sum_{A,B=1}^{n} c_A K_{AB} c_B = \sum_{A,B=1}^{n} c_A \, a(N_A, N_B) \, c_B$$

$$= a\left(\sum_{A=1}^{n} c_A N_A , \sum_{B=1}^{n} c_B N_B \right)$$

$$= \int_0^1 (w^h_{,x})^2 \, dx$$

$$\geq 0.$$

(ii) Assume $\vec{c}^T K \vec{c} = 0$, we have $\int_0^1 (w^h_{,x})^2 dx = 0$

which means $w^h_{,x} = 0$, or w^h is a constant function.

$w^h \in \mathcal{V}^h \Rightarrow w^h(1) = 0$. Then $w^h = 0$ for all x in $[0,1]$. Then. $c_A = 0$ for $A = 1, 2, \cdots, n$. Thus $\vec{c} = \vec{0}$. ∎

Remark: The above theorem guarantees that K is invertible, and its eigenvalues are ~~symmetric~~ real and positive.

In practice, one may solve the equation $Kd = F$ in an efficient manner.

Remark: Here we consider N_A as function defined over the whole domain $\Omega = [0,1]$, and derive the form of K. This is known as the "global point of view". We will switch to the "local/element point of view" subsequently.

- Load vector

$$F_A = (N_A, f) + N_A(0)h - a(N_A, N_{n+1})g$$

$$\approx \left(N_A, \sum_{B=1}^{n+1} f_B N_B\right) + N_A(0)h - a(N_A, N_{n+1})g$$

$$= \sum_{B=1}^{n+1} (N_A, N_B) f_B + \delta_{A1} h - a(N_n, N_{n+1}) g\, \delta_{An}$$

- $A = 1$

$$F_1 = (N_1, N_1) f_1 + (N_1, N_2) f_2 + h.$$

$$(N_1, N_1) = \int_{x_1}^{x_2} \frac{(x_2 - x)^2}{h_1^2} dx = \frac{1}{3h_1^2}(x - x_2)^3 \Big|_{x_1}^{x_2} = \frac{h_1}{3}$$

$$(N_1, N_2) = \int_{x_1}^{x_2} \frac{x_2 - x}{h_1} \frac{x - x_1}{h_1} dx = \frac{1}{h_1^2} \int_{x_1}^{x_2} (x_2 - x)\, d\frac{(x - x_1)^2}{2}$$

$$= \frac{1}{h_1^2}\left[(x_2 - x)\frac{(x - x_1)^2}{2}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{(x - x_1)^2}{2}\, d(x_2 - x)\right]$$

$$= \frac{1}{h_1^2}\left[0 + \int_{x_1}^{x_2} \frac{(x - x_1)^2}{2}\, d(x - x_1)\right]$$

$$= \frac{1}{h_1^2} \cdot \frac{1}{6}(x - x_1)^3 \Big|_{x_1}^{x_2}$$

$$= \frac{h_1}{6}.$$

$$\Rightarrow \ (N_A, N_{A+1}) = \frac{h_A}{6}$$

$$(N_A, N_{A-1}) = \frac{h_{A-1}}{6} \qquad \text{for } A = 2, \cdots, n-1.$$

$$(N_A, N_A) = \frac{1}{3}(h_{A-1} + h_A)$$

$$a(N_n, N_{n+1}) = \cancel{6}\int_{x_n}^{x_{n+1}} -\frac{1}{h_n} \cdot \frac{1}{h_n}\, dx = -\frac{1}{h_n}.$$

$$\Rightarrow \ F_1 = \frac{h_1}{3} f_1 + \frac{h_1}{6} f_2 + h.$$

$$F_A = \frac{h_{A-1}}{6} f_{A-1} + \frac{1}{3}(h_{A-1} + h_A) f_A + \frac{h_A}{6} f_{A+1} \qquad 2 \leq A \leq n-1$$

$$F_n \underbrace{=}_{\frac{1}{6}h_{n-1}f_{n-1}+} \frac{1}{3}(h_{n-1} + h_n) f_n + \frac{1}{6} h_n f_{n+1} + \frac{1}{h_n} g.$$

We got the matrix problem $\underset{n\times n}{K} d = F_{n\times 1}$

Consider performing a LU factorization.

$$K = \underbrace{L}_{\text{Lower triangular}}\ \underbrace{U}_{\text{Upper triangular.}}$$

$$L = \begin{bmatrix} 1 & & & & & \\ l_2 & 1 & & \text{Zeros} & & \\ & l_3 & 1 & & & \\ & & \ddots & \ddots & & \\ \text{Zeros} & & & l_{n-1} & 1 & \\ & & & & l_n & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_1 & c_1 & & & & \\ & u_2 & c_2 & & \text{Zeros} & \\ & & u_3 & c_3 & & \\ & & & \ddots & \ddots & \\ \text{Zeros} & & & & u_{n-1} & c_{n-1} \\ & & & & & u_n \end{bmatrix}$$

$$u_1 = \frac{1}{h_1}$$

$$l_i = -\frac{1}{h_{i-1}} \cdot \frac{1}{u_{i-1}}$$

$$2 \leq i \leq n$$

$$u_i = \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) + l_i \frac{1}{h_{i-1}}$$

$$2 \leq i \leq n$$

$$c_i = -\frac{1}{h_{i-1}}$$

$\Rightarrow$ $Kd = F$ is converted to $\begin{cases} L\hat{d} = F \\ Ud = \hat{d}. \end{cases}$

- Consider the 2-dof problem:

$$K = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} = LU$$

We solve $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

$$\Rightarrow \quad \hat{d}_1 = F_1 \ , \quad \hat{d}_2 = F_2 + \hat{d}_1 = F_2 + F_1$$

Then we solve $\begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_1 + F_2 \end{bmatrix}$

$$\Rightarrow \quad d_2 = \frac{1}{2}(F_1 + F_2)$$

$$d_1 = \frac{1}{2}F_1 + d_2 = F_1 + \frac{1}{2}F_2 .$$

- We consider a 3-dof problem with non-uniform mesh.

$$\underbrace{x_1 = 0 \quad x_2}_{h_1 = 1/2} = \underbrace{\frac{1}{2} \quad x_3}_{h_2 = 1/4} = \underbrace{\frac{3}{4} \quad x_4}_{h_3 = 1/4} = 1$$

$$K = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

To solve $K d = F$, we first solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$\Rightarrow \quad \hat{d}_1 = F_1 \quad \hat{d}_2 = F_1 + F_2, \quad \hat{d}_3 = F_1 + F_2 + F_3$

Then we solve

$$\begin{bmatrix} 2 & -2 & 0 \\ 0 & 4 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \\ \hat{d}_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_1 + F_2 \\ F_1 + F_2 + F_3 \end{bmatrix}$$

$$\Rightarrow \quad d_1 = F_1 + \frac{1}{2} F_2 + \frac{1}{4} F_3$$

$$= \frac{5}{24} f_1 + \frac{7}{32} f_2 + \frac{1}{16} f_3 + \frac{1}{96} f_4 + g + h$$

$$d_2 = \frac{1}{2} F_1 + \frac{1}{2} F_2 + \frac{1}{4} F_3$$

$$= \frac{1}{8} f_1 + \frac{17}{96} f_2 + \frac{1}{16} f_3 + \frac{1}{96} f_4 + g + h/2$$

$$d_3 = \frac{1}{4} F_1 + \frac{1}{4} F_2 + \frac{1}{4} F_3$$

$$= \frac{1}{16} f_1 + \frac{3}{32} f_2 + \frac{5}{96} f_3 + \frac{1}{96} f_4 + g + \frac{h}{4}$$

$$d_4 = g.$$

Now, let us go back to the case of $f = ax$.

Then $f_1 = 0$, $f_2 = \frac{a}{2}$, $f_3 = \frac{3a}{4}$, $f_4 = a$

$$\Rightarrow \quad d_1 = \frac{1}{6}a + g + h$$

$$d_2 = \frac{7}{48}a + g + h/2$$

$$d_3 = \frac{37}{384}a + g + h/4$$

$$d_4 = g$$