• Galerkin approximation

$\mathcal{S}$, $\mathcal{V}$ are ∞-dim spaces. $\Rightarrow$ Select a finite-dimensional sub-space.

$$\mathcal{S}^h \subset \mathcal{S}$$

$$\mathcal{V}^h \subset \mathcal{V}$$

h here is not $\mathscr{h}$, it is the length scale of a mesh, to which $\mathcal{S}^h$ & $\mathcal{V}^h$ are associated with.

Remark: Strictly speaking, $\mathcal{S}^h$ and $\mathcal{S}$ are not <u>spaces</u>.
See the discussion on page 8.

Bubnov - Galerkin method:

Idea: $\mathcal{S}^h$ is constructed by $\mathcal{V}^h$ and a function that enforces the essential BC.

$$\underset{\mathcal{S}^h}{\underset{\cap}{u^h}} = \underset{\mathcal{V}^h}{\underset{\cap}{v^h}} + g^h$$

a function that satisfies $g^h(1) = g$.

$$
(G)\begin{cases}
\text{Given } f,\ g \ \& \ h,\ \text{find } u^h = v^h + g^h \text{ where} \\
v^h \in \mathcal{V}^h \ \& \ g^h(1) = g,\ \text{such that for all } w^h \in \mathcal{V}^h \\
\qquad a(w^h, v^h) = (w^h, f) + w^h h - a(w^h, g^h)
\end{cases}
$$

The Galerkin formulation of the model problem

$\longrightarrow$ It is nothing but a re-statement of (W) in terms of a finite dimensional collection of functions, $\mathcal{V}^h$.

$\longrightarrow$ $(W) \approx (G)$.

- <u>Matrix problem</u>

For $w^h \in \mathcal{V}^h$, there is a set of basis $N_A : \bar{\Omega} \to \mathbb{R}$ such that

dim of $\mathcal{V}^h$.

$$w^h = \sum_{A=1}^{n} c_A N_A$$

basis, shape, interpolation functions.

Apparently, $N_A(1) = 0$ for $A = 1, \cdots, n$.

We introduce N_{A+1} which satisfies $N_{A+1}(1) = 1$.

$$\Rightarrow \quad g^h(x) = g\, N_{A+1}(x)$$

$$\Rightarrow \quad u^h(x) = v^h(x) + g^h(x)$$
$$= \sum_{A=1}^{n} d_A N_A(x) + g\, N_{A+1}(x)$$

Now, the (G) problem can be writen further as

$$a\Big(\sum_{A=1}^{n} C_A N_A,\ \sum_{B=1}^{n} d_B N_B\Big) = \Big(\sum_{A=1}^{n} C_A N_A,\ f\Big)$$
$$+ \sum_{A=1}^{n} C_A N_A(0)\, h$$
$$- a\Big(\sum_{A=1}^{n} C_A N_A,\ g N_{A+1}\Big)$$

$$\Leftrightarrow \sum_{A=1}^{n} C_A \Big\{\sum_{B=1}^{n} a(N_A, N_B)\, d_B - (N_A, f) - N_A(0)\, h + a(N_A, N_{A+1})\, g\Big\}$$
$$= 0$$

$$\Leftrightarrow \underbrace{\sum_{B=1}^{n} a(N_A, N_B)}_{K_{AB}}\, d_B = \underbrace{(N_A, f) + N_A(0)\, h - a(N_A, N_{A+1})\, g}_{F_A}$$

$$\sum_{B=1}^{n} K_{AB}\, d_B = F_A. \quad \text{for } A = 1, \cdots, n.$$

(M) $\begin{cases} \text{Given the coefficients of } K \text{ \& } F, \text{ find } d \text{ such that} \\ \qquad Kd = F \end{cases}$

stiffness matrix — displacement vector — force vector

$$u^h(x) = \sum_{B=1}^{n} d_B N_B(x) + g N_{n+1}(x)$$

or we simply write

$$u^h(x) = \sum_{B=1}^{n+1} d_B N_B(x) \quad \text{with} \quad d_{n+1} = g.$$

If one wants to know the flux σ (e.g. heat flux, stress, etc.), one may calculate

$$u^h_{,x} = \sum_{B=1}^{n+1} d_B N_{B,x} \quad \& \quad \sigma^h = \kappa(x)\, u^h_{,x}.$$

Remark: $K = K^T$.

Remark: $(G) \Leftrightarrow (M)$

assuming we get all integrals calculated accurately.

Example: $n = 2$.

$$w^h = c_1 N_1 + c_2 N_2 \qquad u^h = d_1 N_1 + d_2 N_2 + g N_3.$$

We give the shape functions:

$$N_1 = \begin{cases} 1 - 2x & 0 \le x < 1/2 \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \qquad N_{1,x} = \begin{cases} -2 & \dots \\ 0 & \dots \end{cases}$$

$$N_2 = \begin{cases} 2x & 0 \le x < 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases} \qquad N_{2,x} = \begin{cases} 2 & \dots \\ -2 & \dots \end{cases}$$

$$N_3 = \begin{cases} 0 & 0 \le x < 1/2 \\ 2x - 1 & 1/2 \le x \le 1 \end{cases} \qquad N_{3,x} = \begin{cases} 0 & \dots \\ 2 & \dots \end{cases}$$

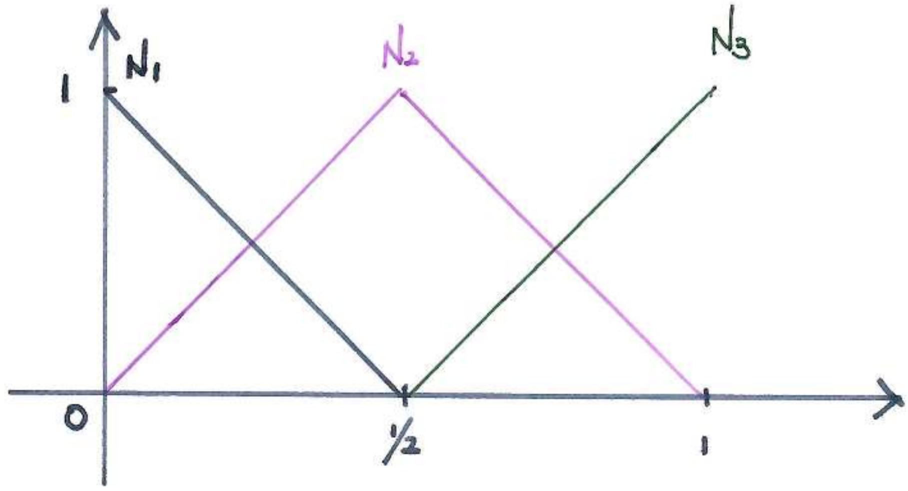
$$K = 2 \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$F_A = \int_0^1 N_A f \, dx + N_A(0)\, h - \int_{1/2}^1 N_{A,x} \overset{N_{3,x}}{2} \, dx \, g$$

$\Rightarrow$

$$F_1 = \int_0^{1/2} (1 - 2x) f \, dx + h$$

$$F_2 = \int_0^{1/2} 2x f \, dx + \int_{1/2}^1 2(1-x) f \, dx + 2g.$$



Consider f is linear : $f = ax$

exact solution : $u = g + (1-x)h + \frac{a}{6}(1-x^3)$

$$F_1 = \int_0^{1/2} (1-2x)\, ax\, dx + h = \frac{1}{24}a + h.$$

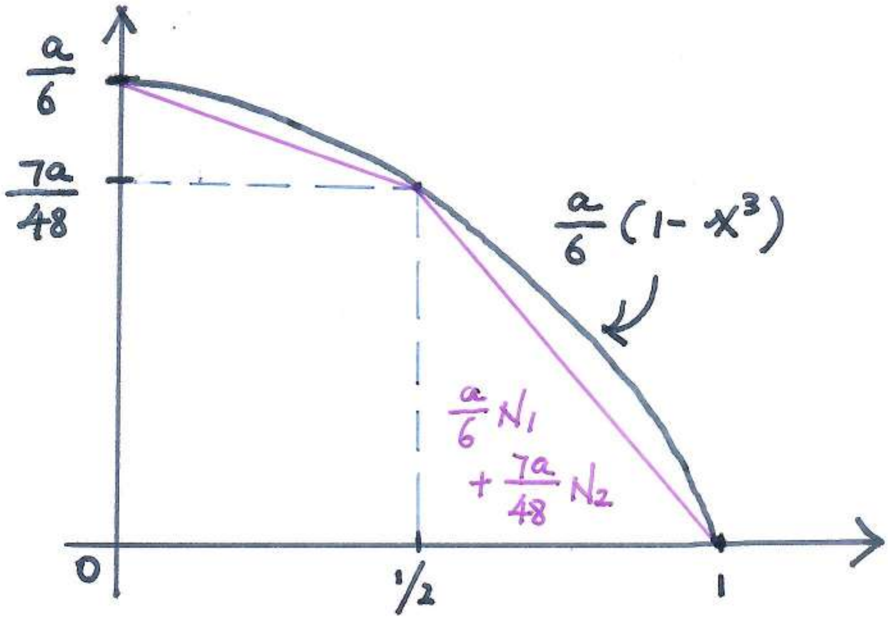
$$F_2 = \int_0^{1/2} 2x\, ax\, dx + \int_{1/2}^{1} 2(1-x)ax\, dx + 2g$$

$$= \frac{1}{4}a + 2g.$$

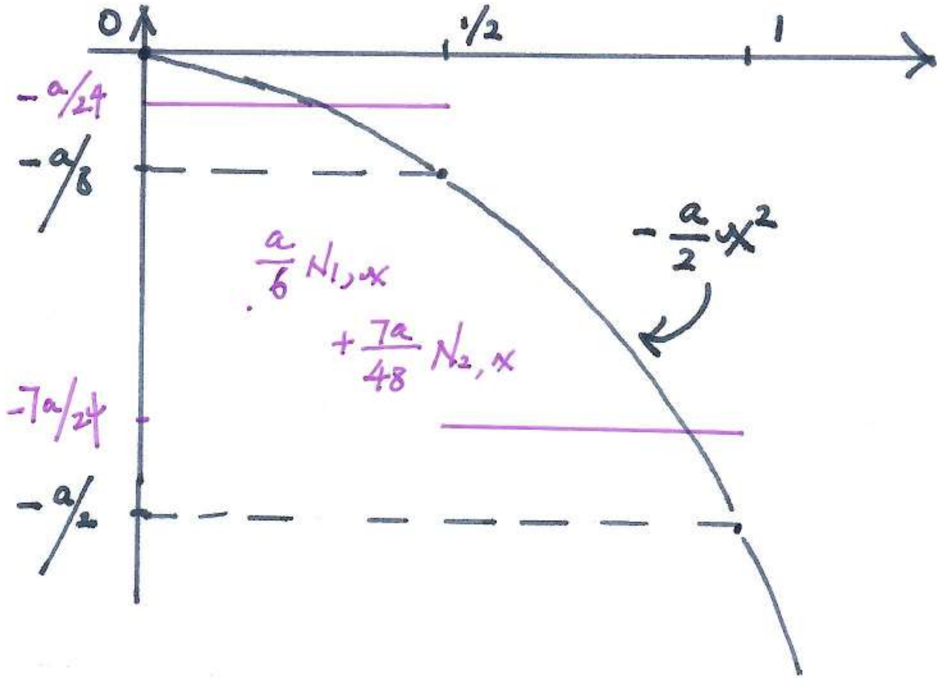
$$d = K^{-1} F = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \frac{a}{24} + h \\ \frac{a}{4} + 2g \end{bmatrix} = \begin{bmatrix} \frac{a}{6} + h + g \\ \frac{7a}{48} + \frac{h}{2} + g \end{bmatrix}$$

$$u^h = d_1 N_1 + d_2 N_2 + g N_3$$

$$= g + (1-x)h + \frac{a}{6} N_1 + \frac{7a}{48} N_2.$$



$\frac{a}{6}$
$\frac{7a}{48}$
$\frac{a}{6}(1-x^3)$
$\frac{a}{6}N_1$
$+\frac{7a}{48}N_2$
0
1/2
1



0
1/2
1
$-a/24$
$-a/8$
$-\frac{a}{2}x^2$
$\frac{a}{6}N_{1,x}$
$+\frac{7a}{48}N_{2,x}$
$-7a/24$
$-a/2$