Higher-order element

Lagrange polynomials:

order of poly.

$$l_a^{n_{en}-1}(\xi) := \frac{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi - \xi_b)}{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi_a - \xi_b)} .$$

node

- $l_a^{n_{en}-1}(\xi_b) = \delta_{ab}$. $\longleftarrow$ interpolation property.
- Shape functions of n_{en}-noded element: $N_a = l_a^{n_{en}-1}$.

Example: $n_{en} = 2$ $\quad \xi_1 = -1, \ \xi_2 = +1$

$$N_1 = l_1^1(\xi) = \frac{\xi - 1}{-1-1} = \frac{1}{2}(1-\xi),$$

$$N_2 = l_2^1(\xi) = \frac{\xi - (-1)}{1-(-1)} = \frac{1}{2}(1+\xi).$$

Example: $n_{en} = 3$. $\quad \xi_1 = -1, \ \xi_2 = 0, \ \xi_3 = +1$

$$N_1 = l_1^2(\xi) = \frac{(\xi-0)(\xi-1)}{(-1-0)(-1-1)} = \frac{1}{2}\xi(\xi-1)$$

$$N_2 = l_2^2(\xi) = \frac{(\xi-(-1))(\xi-1)}{(0-(-1))(0-1)} = 1-\xi^2$$

$$N_3 = l_3^2(\xi) = \frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)} = \frac{1}{2}\xi(\xi+1)$$

To use the higher-order element, all we need is a map that transform $N_a(\xi)$ to $N_A(x)$.

We assume that there are n_{en} nodes in each physical element.

and $\quad x^e : [\xi_1, \xi_{n_{en}}] \to [x_1^e, x_{n_{en}}^e]$

$$x^e(\xi) = \sum_{a=1}^{n_{en}} N_a(\xi) x_a^e .$$

$$\to \quad x^e_{,\xi} = \sum_{a=1}^{n_{en}} N_{a,\xi}(\xi) x_a^e$$

$$\xi_{,x} = (x_{,\xi})^{-1}$$

Quadrature

$$\int_a^b f(\xi)\,d\xi \approx \sum_{\ell=0}^{n_{int}} \underset{W}{\cancel{A}}_{\ell}\, f(x_\ell)$$

→ Idea: use $\hat{f}$, the interpolation function of f, and integrate $\hat{f}$.

linear interpolation

$$\hat{f}(\xi) = f(\xi_1)\, l_1^1(\xi) + f(\xi_2)\, l_2^1(\xi)$$

$$\Rightarrow \int_{\xi_1}^{\xi_2} \hat{f}(\xi)\,d\xi = f(-1)\int_{-1}^{1} \tfrac{1}{2}(1-\xi)\,d\xi + f(1)\int_{-1}^{1}\tfrac{1}{2}(1+\xi)\,d\xi$$

$$= f(-1) + f(1)$$

$$\left(= \frac{h}{2}\left[f(\xi_1) + f(\xi_2)\right]\right)$$ → Trapezoidal rule.

quadratic interpolation

$$\hat{f}(\xi) = f(\xi_1)\, l_1^2(\xi) + f(\xi_2)\, l_2^2(\xi) + f(\xi_3)\, l_3^2(\xi)$$

$$\Rightarrow \int_{\xi_1}^{\xi_2} \hat{f}(\xi)\,d\xi = f(\xi_1)\int_{\xi_1}^{\xi_3} \tfrac{1}{2}(\xi^2-\xi)\,d\xi$$

$$+ f(\xi_2)\int_{\xi_1}^{\xi_3} 1-\xi^2\,d\xi$$

$$+ f(\xi_3)\int_{\xi_1}^{\xi_3} \tfrac{1}{2}(\xi^2+\xi)\,d\xi$$

$$= f(\xi_1)\,\frac{1}{6}\left(\xi_3^3 - \xi_1^3\right) + f(\xi_2)\left[\xi_3 - \xi_1 - \frac{1}{3}\xi_3^3 + \frac{1}{3}\xi_1^3\right]$$

$$\frac{1}{3}$$

$$+ f(\xi_3)\,\frac{1}{6}\left(\xi_3^3 - \xi_1^3\right)$$

$$= \frac{1}{3}f(-1) + \frac{4}{3}f(0) + \frac{1}{3}f(1).$$

Simpson 's rule.

cubic interpolation :

$$\int_{\xi_1}^{\xi_4} \hat{f}(\xi)\,d\xi = \frac{1}{4}f(-1) + \frac{3}{4}f(-\tfrac{1}{3}) + \frac{3}{4}f(\tfrac{1}{3}) + \frac{1}{4}f(1).$$

Newton's rule.

quartic interpolation

$$\int_{\xi_1}^{\xi_5} \hat{f}(\xi)\,d\xi = \frac{1}{45}\left[7f(-1) + 32f(-\tfrac{1}{2}) + 12f(0) + 32f(\tfrac{1}{2}) + 7f(1)\right]$$

Cotes' rule.

Remark : Taylor expansion may reveal that

For the four rules, $R[f] = \int_{-1}^{1} f\,d\xi - \int_{-1}^{1}\hat{f}\,d\xi$

$$= \begin{cases} -\frac{2}{3}f''(\hat{\xi}_0), \\ -\frac{1}{90}f^{(4)}(\hat{\xi}), \\ -\frac{3}{80}\frac{2^5}{3^5}f^{(4)}(\hat{\xi}). \qquad \frac{32}{6480} \\ -\frac{8}{945}\frac{1}{27}f^{(6)}(\hat{\xi}). \qquad \frac{1}{15120} \end{cases}$$

Gaussian quadrature

- $n_{int} = 1$ $\qquad \xi_\ell = 0 \qquad W_\ell = 2 \qquad R = \frac{1}{3} g^{(2)}(\hat{\xi})$

- $n_{int} = 2$ $\qquad \xi_\ell = \pm \frac{1}{\sqrt{3}} \qquad W_\ell = 1 \qquad R = \frac{1}{135} g^{(4)}(\hat{\xi})$

- $n_{int} = 3$ $\qquad \xi_\ell = -\sqrt{\frac{3}{5}},\ 0,\ \sqrt{\frac{3}{5}} \qquad R = \frac{1}{15750} g^{(6)}(\hat{\xi}).$

  $W_\ell = 5/9 \quad 8/9 \quad 5/9$

... ...

Gaussian quadrature are optimal in the following sense: n_{int}-point Gaussian quadrature can integrate polynomials with degree up to $2n_{int} - 1$, and out textbook call this as ~~$2n_{int}$-th accurate~~ the quadrature accuracy of order $2n_{int}$.

Remark: The above gives the Gaussian quadrature in the 1D reference element.

Remark: We may use

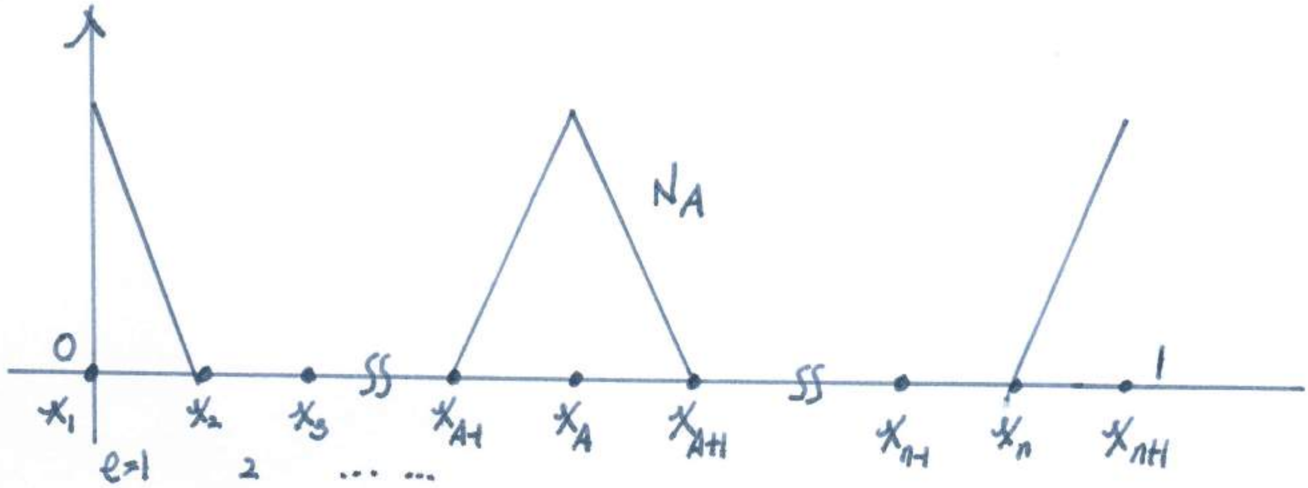
$$\int_a^b f(\eta)\, d\eta = \int_{-1}^{1} f\left(\frac{b-a}{2}\xi + \frac{a+b}{2}\right) \frac{b-a}{2}\, d\xi$$

$$\approx \frac{b-a}{2} \sum_{\ell=1}^{n_{int}} w_\ell f\left(\frac{b-a}{2}\xi_\ell + \frac{a+b}{2}\right)$$

to obtain 1D Gaussian quadrature in $[a, b]$.

Remark: In multi-D, we will have to determine the location of ξ_ℓ and w_ℓ based on the ref. element.

Two additional data structures



local node number

$$A = IEN(a, e)$$

global node number

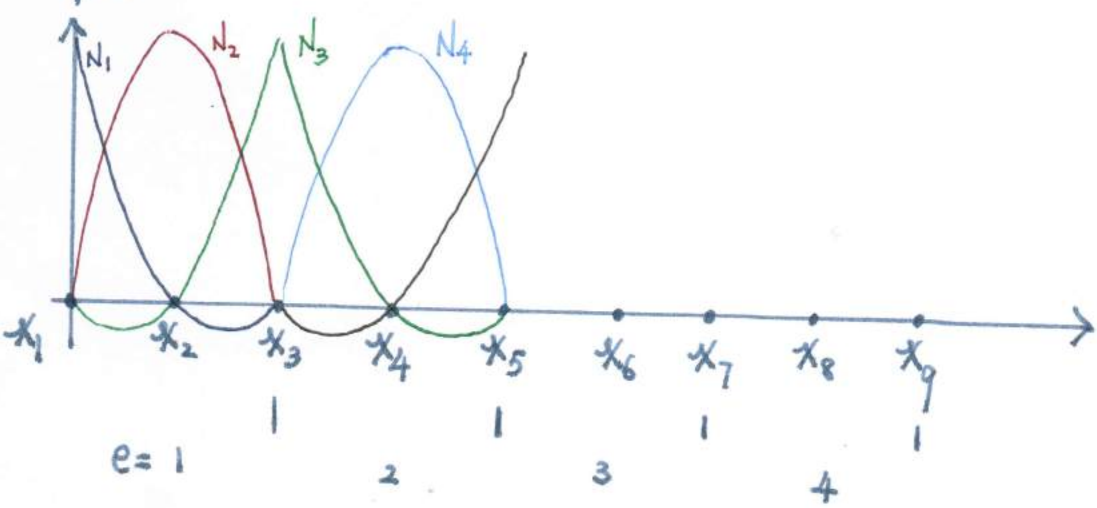
element-node array

element number

$$ID(A) = \begin{cases} P & \text{if } A \text{ is not on the Dirichlet BC.} \\ 0 & \text{otherwise.} \end{cases}$$

Thus. $LM(a, e) = ID(IEN(a, e))$.

Example:



IEN :

	e=1	2	3	4
a=1	1	3	5	7
2	2	4	6	8
3	3	5	7	9

ID :

A =	1	2	3	4	5	6	7	8	9
P =	1	2	3	4	5	6	7	8	0

LM :

	e=1	2	3	4
a=1	1	3	5	7
2	2	4	6	8
3	3	5	7	0

Remark: IEN only depends on the mesh.
ID depends on the mesh & BC.