Maths-I (LAC) Linear algebra & calculus

UNIT-I

Matrices

1. Matrix def:-

An arrangement of elements (real & complex) or functions b/w a pair of brackets [ ] or ( ) is called a matrix

Ex:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$ is a matrix of order 2×3

3. Rectangular matrix :-

A matrix which is not square matrix is called a rectangular matrix.

Ex: $A = \begin{bmatrix} 1 & 4 & 3 & 5 \\ 2 & 3 & 4 & -6 \end{bmatrix}_{2\times4}$ is a 2×4 matrix.

2. Square matrix :-

A matrix with no. of rows is equal to no. of columns then matrix is called a square matrix.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$ is a 3×3 matrix.

4. Unit matrix :-

If $A = [a_{ij}]_{m\times n}$ such that $a_{ij} = 1$ for $i=j$ and $a_{ij} = 0$ for $i \neq j$ then A is called a unit matrix it is denoted by I_n

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero matrix or Null matrix:—

If $A = [a_{ij}]_{m\times n}$ such that $a_{ij} = 0 \ \forall$ i and j, then A is called a Zero matrix or Null matrix it is denoted by

$O_{m\times n}$ Eg: $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Diagonal matrix:—

A Square matrix ~~is said to be diagonal matrix~~ ~~if all the off diagonal elements are Zero~~ all of whose elements except those in leading diagonal are zero is called diagonal matrix

Ex:— $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$A = dia[9, 6, 3)$

Scalar matrix:—

A diagonal matrix is said to be a scalar matrix if all the diagonal elements are equal

Ex: $A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

Triangular matrix:—

Upper Triangular matrix:—

A Square matrix all of whose elements below the leading diagonal are Zero is called an upper Triangular matrix

Ex: $A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ Ex: $A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 \\ 4 & -2 & 3 & 0 \\ 6 & 7 & 8 & 9 \end{bmatrix}$

Lower Triangular matrix:—

A Square matrix all of whose elements above the leading diagonal are Zero is called a Lower Triangular matrix

Transpose of a matrix :—

The matrix obtain from A by interchanging rows & colomns is called the transpose of a matrix A and it is denoted by A' or A^T.

Ex: $A = \begin{bmatrix} 6 & 2 & 3 & 5 \\ -2 & 3 & 7 & 9 \end{bmatrix}_{2\times 4}$ then $A^T = \begin{bmatrix} 6 & -2 \\ 2 & 3 \\ 3 & 7 \\ 5 & 9 \end{bmatrix}_{4\times 2}$

Defi:—

1) If $A^T = A$, then A is Symmetric.
2) If $A^T = -A$, then A is Skew symmetric
3) If $A \cdot A^T = I$, then A is orthogonal matrix.

Note: If A^T and B^T be the transpose of A and B respectively then

① $(A^T)^T = A$
② $(A+B)^T = A^T + B^T$
③ $(KA)^T = K \cdot A^T$, K is a scalar
④ $(AB)^T = B^T A^T$

Minors and co-factors of a square matrix :—

A let $A = [a_{ij}]_{m\times n}$ be $m\times n$ matrix, the determinate of square sub matrix of A is called a minor of the matrix and it is denoted by $|M_{ij}|$. The signed minor $(-1)^{i+j}|M_{ij}|$ is called the co-factor of a_{ij} and is denoted by A_{ij}

Thus, If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Adjoint of a Square matrix :-

The Transpose of the co-factor matrix of A is called the adjoint matrix of A and it is denoted by "adj A"

Singular and Non-Singular matrix :-

A square matrix A is said to be singular if $|A|=0$ if $|A|\neq 0$, then A is said to be non-singular

Inverse of a matrix :-

Let A be any square matrix, then a matrix B if exists such that $AB=BA=I$, then B is called Inverse of A and is denoted by A^{-1}

Complex Matrices :-

A matrix with atleast one complex number is called a complex matrix.

Ex: $\begin{bmatrix} 3+9i & -4-2i \\ 4 & 6+4i \end{bmatrix}$

Complex Number :-

If $z=a+ib$, $i^2=-1$ & $a,b\in R$.

Conjugate of z is $\bar{z}=a-ib$

$\Rightarrow z\cdot\bar{z} = (a+ib)(a-ib)$

$= a^2-i^2b^2$

$z\cdot\bar{z} = a^2+b^2$

$|z| = \sqrt{a^2+b^2}$

Conjugate of a complex matrix :-

If $A = \begin{bmatrix} 3+9i & 5i-3 \\ 4 & 6-4i \end{bmatrix}$ then

Conjugate of A is $\bar{A} = \begin{bmatrix} 3-9i & -5i-3 \\ 4 & 6+4i \end{bmatrix}$

Complex Matrices :—

A matrix with atleast one complex number is called a complex matrix.

Ex: $\begin{bmatrix} 3+9i & -4-2i \\ 4 & 6+4i \end{bmatrix}$.

Conjugate of a matrix :—

The matrix obtained from A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and it is denoted by $\overline{A}$

Ex :— $A = \begin{bmatrix} 3+9i & -4-2i \\ -4 & 6-4i \end{bmatrix}$

then $\overline{A} = \begin{bmatrix} 3-9i & -4+2i \\ -4 & 6+4i \end{bmatrix}$.

The Transpose of the conjugate of a Square matrix:—

If A is a square matrix and its conjugate is $\overline{A}$. then the transpose of $\overline{A}$ is $(\overline{A})^T$ (or) $(\overline{A})^T = (\overline{A^T})$

i.e The Transpose of the conjugate of a square matrix is same as the conjugate of its Transpose.

and it is denoted by A^θ.

$$\therefore A^\theta = (\overline{A})^T = (\overline{A^T})$$

Hermitian matrix :—

A Square matrix A such that $\left((\overline{A^T}) = (\overline{A})^T\right) \Rightarrow A^\theta = A$

is called a hermitian matrix.

A Square matrix A is said to be a hermitian matrix if $A^\theta = A$ (or) $A^T = \overline{A}$

Eg:— $A = \begin{bmatrix} 4 & 1+3i \\ 1-3i & 7 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ and $\overline{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$

$\therefore A^T = \overline{A}$

$\therefore$ A is hermitian.

Skew hermitian matrix :—

A Square matrix A is said to be a Skew-hermitian matrix if $A^\theta = -A$ or $A^T = -\overline{A}$

Ex: Let $A = \begin{bmatrix} -3i & 2+i \\ -2+i & -i \end{bmatrix}$ Then

$A^T = \begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix}$ and $\overline{A} = \begin{bmatrix} 3i & 2-i \\ -2-i & i \end{bmatrix}$

$= -\overline{A} = \begin{bmatrix} -3i & -2+i \\ 2+i & -i \end{bmatrix}$

$\therefore A^T = -\overline{A}$

$\therefore$ A is Skew hermitian

Unitary matrix:—

A Square matrix A such that $(\bar{A})^T = A^{-1}$ (or) $A^{\theta}A = AA^{\theta} = I$ is called a unitary matrix.

Eg: $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$ is a unitary matrix.

S.M.

Ex: 1) Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

2) Show that $A = \begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if $a^2+b^2+c^2+d^2=1$

* IMP Rank of a matrix:—

The rank of a non-zero matrix $A_{m\times n}$ is a +ve Integer r ~~is the rank of A~~ If.

i) All minors of order > r are zero

ii) Atleast one minor of order 'r' is not zero.

Rank of A is denoted by $\rho(A)$.

Q. Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Sol $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}_{3\times 3}$

$|A| = 1[24-25] - 2[18-20] + 3[15-16]$

$|A| = -1-2(-2)+3(-1) = -1+4-3 = 0$

Minor of order 3 is zero $\therefore \rho(A) \neq 3$

$$|A| = \begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24-25 = -1 \neq 0$$

$\therefore$ there is a minor of order 2 which is not equal to zero.

$\therefore \rho(A) = 2.$

(2) find the rank of the matrix $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}_{3\times 3}$

Sol $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

we have det $A = \begin{vmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{vmatrix}$

$$= -1(18-1) - 0(9+5) + 6(3+30)$$

$$= -17 - 0 + 6(33)$$

$$= -17 + 198$$

$$= 181$$

$$\neq 0$$

$\therefore$ there is a minor of order 3 which is not equal to zero. $\therefore \rho(A) = 3$

(3) find the value of K such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & K & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

Sol let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & K & 7 \\ 3 & 6 & 10 \end{bmatrix}$

Given Rank of $A = \rho(A) = 2,$

$\therefore |A| = 0$

$$1[10K-42] - 2[20-21] + 3[12-3K] = 0$$

$$10K - 42 + 2 + 36 - 9K = 0$$

$$K - 4 = 0$$

$$\boxed{K = 4}$$

8) Prove that $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

Sol Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ Then $A^T = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix}$

$A^\theta = (\overline{A^T}) = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$

Now $AA^\theta = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+1+1+1 & 0 \\ 0 & 1+1+1+1 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$

Thus $A^\theta A = I$ Hence A is a unitary matrix.

9) find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Sol Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$\det A = 1[24-25] - 2[18-20] + 3[15-16]$

$= -1 - 2(-2) + 3(-1)$

$\det A = 0$

Rank(A) $\neq 3$. So it must be less than 3.

Consider the minor of order 2 $= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4-6 = -2 \neq 0$

Hence there is at least a minor of order 2 which is not zero $\therefore$ Rank of (A) $= 2$.

10) Symmetric matrix :—

A Square matrix A is said to be symmetric if $A^T = A$

Ex: $A = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 5 & 0 \\ 4 & 0 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 5 & 0 \\ 4 & 0 & 5 \end{bmatrix}$

5a) Find the rank of the value of K, If the rank of the matrix A is 2

where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

Sol Given matrix is $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & K & 0 \end{bmatrix}$

Applying $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & K-1 & -1 \end{bmatrix}$$

Given that $\text{rank}(A) = 2$.

We must have 3 rows identical.

$K-1 = -3 \Rightarrow K = -2$

Elementary operations :-

There are three elementary operations (row (or) columns) given by

E_1 :- Any row (or) column can be multiplied by non-zero real number.

KR_i where $K \neq 0$.

(or) KC_i where $K \neq 0$.

E_2 :- Any two rows (or) column can be interchanged.

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

E_3 :- A Row or column can be multiplied by a non-zero. Real number and the results can be added to any other row (or) column.

Ex: E_1 - $\begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ 10 & 15 \end{bmatrix}$

$2R_1, \frac{1}{5}R_2$ we get $\sim \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$

$\frac{1}{3}C_2$, we get $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

E_2 : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E_3 : $\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix}$

$R_2 = R_2 + 2R_1$

$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 10 \end{bmatrix}$

Rank of the matrix can be followed by following methods.

(i) Echelon form

(2) Normal form

Echelon form :-

A matrix is said to be in Echelon form if its satisfy the two conditions.

(i) Zero rows (if any) must follow non-zero rows.
(Zero rows should be bottom)

(ii) The no. of zero's before the first non-zero elements must be less than such no. of zero's in the next row
(It should be in assending order)

<u>Ex:</u>

(also $0<1<2$)

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$ it is in echelon form.

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}$ It is in Echelon form.

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ it is not in echelon form.

$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} 1 \\ 1 \\ \end{matrix}$ it is not in echelon form.

In this method we need to use only Row operations.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 6 & 7 & 9 \\ 3 & 8 & 4 & 5 \end{bmatrix}$$

Zero In Step1 Zero In Step2

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 6 & 7 & 9 \\ 5 & 3 & 4 & 2 \\ 8 & 3 & 1 & 7 \end{bmatrix}$$

Zero In first step. Zero In Step2 zero in Step3.

The rank of a matrix which is in Echelon form is it's no.of non-zero rows.

(9) Reduce the matrix $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ into Echelon form. Hence find its rank.

Sol: Given $A = \begin{bmatrix} 2 & -4 & 3 & -1 & 0 \\ 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

Applying $R_1 \leftrightarrow R_2$ $A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 2 & -4 & 3 & -1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_4 \to R_4 - 4R_1$, $A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 0 & 5 & 7 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 8 & 12 & -3 \end{bmatrix}$

$R_2 \leftrightarrow R_4$, $A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$

Applying $R_3 \to R_3 - R_2$ $A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 5 & 7 & -4 \end{bmatrix}$

Applying $R_4 \to 9R_4 + 5R_3$, we get

$$A = \begin{bmatrix} 1 & -2 & -1 & -4 & 2 \\ 0 & 1 & 8 & 12 & -3 \\ 0 & 0 & -9 & -9 & 4 \\ 0 & 0 & 0 & 18 & -16 \end{bmatrix}$$

This is in Echelon form

Number of non-zero rows is 4. Thus, Rank of $(A) = \rho(A) = 4$.

2a) Find the rank of the matrix $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by reducing to echelon form.

Sol: Let $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Then Applying $R_1 \leftrightarrow R_2$ $\quad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Applying $R_3 \to R_3 - 3R_1$
$\qquad\quad\; R_4 \to R_4 - R_1$ $\quad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$

Applying $R_3 = R_3 - R_2$
$\qquad\quad\; R_4 = R_4 - R_2$ $\quad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in Echelon form.

$\text{Rank}(A) = \rho(A) =$ Number of non-zero rows in the matrix $= 2$.

Normal form:—

Every matrix can be reduced into one of the following four forms by applying elementary operations. These four forms are called "Normal form"

The rank of a matrix which is in Normal form is the order of identity matrix.

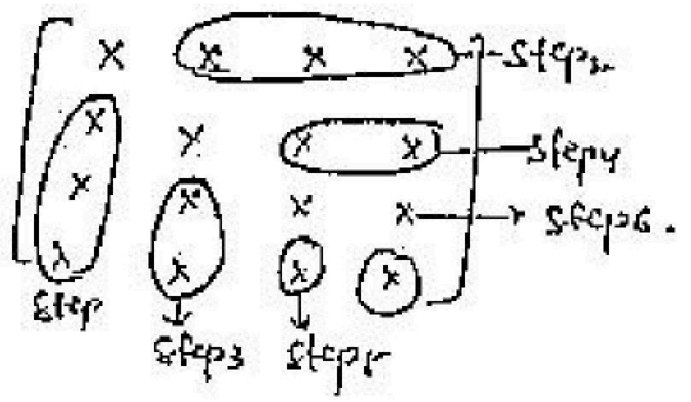
i.e, r.

$$[I_r],\ \begin{bmatrix} I_r \\ 0 \end{bmatrix}\ [I_r \mid 0]\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Ex: $[I_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \begin{bmatrix} I_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\ [I_2, 0] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In Normal form we can use both row and column operations.



① Find the Rank of the matrix $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$ by reducing in to the normal form.

Sol

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

Applying = $R_1 \to R_1 - 2R_2$
$R_4 \to R_4 - R_2$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 - R_3$

$$= \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Applying $C_4 \to C_4 - 2C_1$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is of the form

$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$ which is normal form.

$\therefore$ Rank of the matrix is 3.

Inverse by Gauss - Jordan method :-

A is a non-singular square matrix of order n. We write $A = I_n A$. Now we apply elementary row operations only to the matrix A and the prefactor I_n of the R.H.S. we will do this till we get an eqn of the form.

$I_n = BA$ Then obviously B is the inverse of A.

① Find the Inverse of the matrix $\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

we write $A = I_3 A$

Now we apply only row operations to the matrix A until it is reduced to the form I_3 same operations will be ~~prefact~~ performed on prefactor I_3 of R.H.S.

$$\begin{bmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 0 & -1 \\ 4 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -7 \\ 0 & 3 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & -4 \end{bmatrix} A$$

Applying $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & -7 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & -2 \end{bmatrix} A$$

Applying $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -4 \\ 0 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -2 \\ 0 & 1 & -2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_2$
$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -2 \\ -2 & 3 & 2 \end{bmatrix} A$$

Applying $R_1 \to R_1 + R_3$
$R_2 \to R_2 + 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix} A$$

This is of the form $I_3 = BA$.

Thus $B = A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ -7 & 11 & 6 \\ -2 & 3 & 2 \end{bmatrix} A$.

System of Linear Equations :-

consider a system of 3 linear eqn in 3 variables.

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \longrightarrow (1)$$

above System can be Represented in a matrix form $AX=B$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is called coefficient matrix.

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is called variable matrix.

$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is called constant matrix.

The System of linear eqns are into two types.

Homogeneous System.

The above system of (1) is said to be homogeneous if $b_1 = b_2 = b_3 = 0$

Ex:- $\left.\begin{aligned} 2x+3y+4z &= 0 \\ 2y+3z &= 0 \\ 2x+y+z &= 0 \end{aligned}\right\}$ Homogeneous.

Non-Homogeneous System :-

The above system of (1) is said to be Non-homogeneous if atleast one $b_i \neq 0$

Ex $\left.\begin{aligned} x+2y+4z &= 1 \\ x+3y-8z &= -3 \\ 2x+4y+3z &= 0 \end{aligned}\right\}$ Non-Homogeneous.

Solution of Non-homogeneous System:-

(a) The system of eqn $AX=B$ is consistent
ie it has a solution (unique or Infinite) if and only if.

The augmented matrix of above system of eqn is $[A|B]$

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Rank of A = Rank of $[A|B]$

(i) Rank of A = Rank of $[A|B] = n$ then the system will have unique solution.

$\therefore \boxed{\rho(A) = \rho[A|B] = n}$

(ii) Rank of A = Rank $[A|B] < n$ then the system will have no. of solution.

(b) The system of eqn $AX=B$ is inconsistent

If $\rho(A) \neq \rho[A|B]$ The system has no solution.

Working Rule:-

Step1: consider a non-sin homogeneous system of Linear eqns.
Write the augmented matrix of the system $[A|B]$

Step2: Reduce the augmented matrix into echelon form.

Step3: and decide below. n = no. of variables $\rho(A)$, $\rho(A|B)$

Ex: $\left[\begin{array}{ccc|c} 2 & 3 & 4 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{array}\right]$ no. of variables $=3$
$\rho(A)=3$ Ran of A
$\rho[A|B]=3 \Rightarrow$ Rank of A given B
$\rho(A|B) = \rho(A) = n$

Step4:

1) If $\rho(A) = \rho(A|B) = n$ then the system has unique solution
If $\rho(A) = \rho(A|B) < n$ then the system has infinite no. of solu
If $\rho(A) \neq \rho(A|B) > n$ then the system has no solution.

26)

Examine for what values of p and q, so that the eqs $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+pz=q$ have (i) No solution (ii) Unique solution (iii) infinitely many solutions.

3. **Sol** Given eqn are $2x+3y+5z=9$

$$7x+3y+2z=8$$
$$2x+3y+pz=q$$

The eqns can be written in the matrix form $AX=B$.

i.e $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ q \end{bmatrix}$

Applying $R_2 \to 2R_2 - 7R_1$, $R_3 = R_3 - R_1$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -15 & -31 \\ 0 & 0 & p-5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -47 \\ q-9 \end{bmatrix}$$

The Augmented matrix $[A,B] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -31 & -47 \\ 0 & 0 & p-5 & q-9 \end{bmatrix}$

we have $\det A = \begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & 2 \\ 2 & 3 & p \end{vmatrix} \Rightarrow 2[3p-6]-3[7p-4]+5[21-6] = -15p+75$

$\therefore \det A = 0$

$-15p = -75 \Rightarrow \boxed{p=5}$

Case 1 :- when $p=5$, $q \neq 9$

The rank$(A)=2$ and Rank$[A,B]=3$

The system will be inconsistent.

The system has no solution.

Case II :- when $p \neq 5$, $\det A \neq 0$, $q \neq 9$

The system will have unique solution.

Case (ii) when $p=5$, $q=17$

Rank(A) = 2, Rank[A|B] = 2

Number of variables = 3

$\therefore$ The system will be consistent and will have infinite no. of solution.

(2) Show that the eqns $x+2y-z=3$, $3x-y+2z=1$, $2x-2y+3z=2$, $x-y+z=-1$ are consistent and solve them.

Sol Given eqns are

$$x+2y-z=3$$
$$3x-y+2z=1$$
$$2x-2y+3z=2$$
$$x-y+z=-1$$

The eqns can be written in the matrix form $AX=B$.

i.e
$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

Applying $R_2 = R_2 - 3R_1$, $R_3 = R_3 - 2R_1$, $R_4 = R_4 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ -4 \\ -4 \end{bmatrix}$$

Applying $R_1 = 7R_1 + 2R_2$, $R_3 = 7R_3 - 6R_2$, $R_4 = 7R_4 - 3R_2$

$$\begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 20 \\ -4 \end{bmatrix}$$

Applying $\frac{R_3}{5}$, $\frac{R_4}{-1}$ $\Rightarrow$ $\begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 4 \\ 4 \end{bmatrix}$

Applying $R_4 \Rightarrow R_4 - R_3$

$$\begin{bmatrix} 7 & 0 & 3 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 4 \\ 0 \end{bmatrix}$$

we can observe that A is in Echelon form

Number of non-zero rows = 3.

Rank(A) = $\rho(A) = 3$ and Rank $[A/B] = 3$ = no. of Variables.

The system is consistent and solution is Unique.

$\boxed{z = 4}$, $-7y + 5z = -8$, $7x + 3z = 5$

$-7y + 20 = -8$ $\quad$ $7x + 12 = 5$

$-7y = -28$ $\quad$ $7x = -7$

$\boxed{y = 4}$ $\quad$ $\boxed{x = -1}$

$\therefore$ The solution is $x = -1, y = 4, z = 4$.

Solution of Homogeneous system:—

The system of eqn. $AX=0$ is always consistent i.e it has a solution.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

It is clear that $x=y=z=0$ is a solution of (1)

This is called trivial solution of $AX=0$ it is linearly dependent.

The trivial solution is called zero solution

(i) If Rank of $(A) = n$ (no. of variable) $\Rightarrow$ The system of eqn have only trivial solution.

(ii) If Rank of $(A) < n$ The system of eqn have an infinite no. of non-trivial solutions.

we shall have $n-r$ linearly independent solutions.

(1) Solve the system of eqns $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$

Sol let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Then the given system can be written as $AX=0$

Now $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus the matrix is in echelon form.

No. of Non-zero rows is 2.

The rank of the matrix is 2.

∴ Since no. of variable is 3.

This will have $3-2=1$ non-zero solution.

The corresponding eqn are $x+3y-2z=0$ and $-7y+8z=0$

Let $z=K$ then $y=\frac{-8K}{-7}$

$y=\frac{8}{7}K$ and $x=2K-3\left(\frac{8}{7}K\right)$

$x=\frac{14K-24K}{7}$

$x=-\frac{10}{7}K$

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -\frac{10}{7}K \\ \frac{8}{7}K \\ K \end{bmatrix}$$

$$X=\frac{K}{7}\begin{bmatrix} -10 \\ +8 \\ 7 \end{bmatrix}$$

which is the general solution of the given system.

Gauss elimination method :—

consider the system of eqn.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The augmented matrix is

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Applying, $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$; a_{11} is called first pivot—

$R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$ we get—, here a_{11} is called first pivot—

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_{22} & \alpha_{23} & \beta_{2} \\ 0 & \beta_{32} & \alpha_{33} & \beta_3 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{\beta_{32}}{\alpha_{22}} R_2$$

$$[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 0 & \beta_2 & \beta_3 \end{bmatrix}$$

α_{22} is called new pivot—

① The augmented matrix corresponds to an upper triangular system

③ This procedure is called partial pivoting.

② → which can be solved by backward substitution.

Gauss elimination method :-

① Solve the eqns $3x+y+2z=3$, $-3y+2x-z=-3$, $x+2y+z=4$ using Gauss elimination method.

Sol

$$3x+y+2z=3$$
$$2x-3y-z=-3$$
$$x+2y+z=4.$$

The given system of eqn can be written in the matrix form as $AX=B$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

The augmented matrix of the given system is

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

performing $R_2 \to R_2 - \frac{2}{3}R_1$; $R_3 \to R_3 - \frac{1}{3}R_1$, we get

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -\frac{11}{3} & -\frac{7}{3} & -5 \\ 0 & \frac{5}{3} & \frac{1}{3} & 3 \end{bmatrix}$$

performing $R_3 \to R_3 - \frac{5/3}{-\frac{11}{3}} R_2$

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -\frac{11}{3} & -\frac{7}{3} & -5 \\ 0 & 0 & -\frac{\cancel{24}\,8}{\cancel{33}\,11} & \frac{8}{11} \end{bmatrix}$$

$$[A|B] = \begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11/3 & -7/3 & -5 \\ 0 & 0 & -8/11 & 8/11 \end{bmatrix}$$

This augmented matrix corresponds to the following upper Triangular system.

$$3x + 1y + 2z = 3 \longrightarrow ①$$

$$-\frac{11}{3}y - \frac{7}{3}z = -5 \longrightarrow ②$$

$$-\frac{8}{11}z = \frac{8}{11}$$

$$\boxed{z = -1}$$

By Back substitution, we have.

eq ② $$-\frac{11}{3}y = -5 - \frac{7}{3}$$

$$-\frac{11}{3}y = \frac{-15-7}{3}$$

$$y = \frac{-22}{-11} \; 2$$

$$\boxed{y = 2}$$

eq ① $$\Rightarrow 3x + \cancel{2} + (\cancel{-2}) = 3$$

$$\boxed{x = 1}$$

The solution is $x = 1, y = 2, z = -1$

① Solve the equations $x+y+z=6$, $3x+3y+4z=20$, $2x+y+3z=13$ using Gauss elimination method.

Sol

$$x+y+z=6$$
$$3x+3y+4z=20$$
$$2x+y+3z=13$$

$$[A/B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right]$$

Applying $R_2 \to R_2 - \frac{3}{1}R_1$

$R_3 \to R_3 - \frac{2}{1}R_1$

$$[A|B] = \left[\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$[A|B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

By back substitution we have.

$$x+y+z=6 \longrightarrow ①$$
$$-y+z=1 \longrightarrow ②$$
$$\boxed{z=2}$$

eq ② $-y+2=1$
$$\boxed{y=1}$$

eq ① $x+1+2=6$
$$\boxed{x=3}$$

Gauss-Seidel Iteration method

The system of eq^n

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Digonal dominant property must be satisfied.

(i.e) $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$

Rewriting the eq^{n}s for x, y and z respectively.

$$x_0 = \frac{1}{a_{11}}[b_1 - a_{12}y - a_{13}z]$$

$$y_0 = \frac{1}{a_{22}}[b_2 - a_{21}x - a_{23}z] = 0$$

$$z_0 = \frac{1}{a_{33}}[b_3 - a_{31}x - a_{32}y] = 0$$

Iteration 1: $x_1 = \frac{1}{a_{11}}[b_1 - a_{12}y_0 - a_{13}z_0]$

$$y_1 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}z_0]$$

$$z_1 = \frac{1}{a_{33}}[b_3 - a_{31}x_1 - a_{32}y_1]$$

Iteration 2: $x_2 = \frac{1}{a_{11}}[b_1 - a_{12}y_1 - a_{13}z_1]$

$$y_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_2 - a_{23}z_1]$$

$$z_2 = \frac{1}{a_{33}}[b_3 - a_{31}x_2 - a_{32}y_2]$$

The above iteration process is continued until two successive approximations are equal.

① Solve the system of eqns by Gauss - Seidel method.

$20x+y-2z=17,\ 3x+20y-z=-18,\ 2x-3y+20z=25$

Sol: Given System of eqn is

$$\left.\begin{array}{l} 20x+y-2z=17 \\ 3x+20y-z=-18 \\ 2x-3y+20z=25 \end{array}\right\} \longrightarrow ①$$

Since the diagonal elements are dominant in the coefficient matrix of eq ①

$|20| > |1| + |-2|$

$|20| > |3| + |-1|$

$|20| > |2| + |-3|$

Hence the convergence condition is satisfied.

we apply Gauss-Seidel method for the system of eq ①

$$x = \frac{1}{20}[17-y+2z]$$

$$y = \frac{1}{20}[-18-3x+z]$$

$$z = \frac{1}{20}[25-2x+3y]$$

Gauss Seidel

Iteration 1: putting $y=z=0$ in R.H.S of eq ① we get

$$x_1 = \frac{1}{20}[17-0+0] = \frac{17}{20} = 0.8500$$

$$y_1 = \frac{1}{20}[-18-3(0.8500)+0] = -1.0275$$

$$z_1 = \frac{1}{20}[25-2(0.8500)+3(-1.0275)] = +1.0109$$

Iteration 2: $x_2 = \frac{1}{20}[17-y_1+2z_1] = \frac{1}{20}[17+1.0275+2(+1.0109)]$

$$x_2 = 1.0025$$

$$y_2 = \frac{1}{20}[-18-3x_2+z_1] = \frac{1}{20}[-18-3(1.0025)+(+1.0109)]$$

$$y_2 = \cancel{-1.0092}\ -0.99983$$

$$z_2 = \frac{1}{20}[25-2x_2+3y_2] = \frac{1}{20}[25-2(1.0025)+3(-0.9998)]$$

$$z_2 = 0.9998$$

Iteration 3 :-

$$x_3 = \frac{1}{20}\left[17 - y_2 + 2z_2\right]$$

$$= \frac{1}{20}\left[17 + 0.9998 + 2(0.9998)\right]$$

$$x_3 = 0.99997 = 1.00$$

$$y_3 = \frac{1}{20}\left[-18 - 3x_3 + z_2\right]$$

$$y_3 = \frac{1}{20}\left[-18 - 3(1) + 0.9998\right] = -1.00$$

$$z_3 = \frac{1}{20}\left[25 - 2x_3 + 3y_3\right]$$

$$= \frac{1}{20}\left[25 - 2(1) + 3(-1)\right]$$

$$z_3 = 1$$

The values in the 2nd and 3rd iteration are almost same. Hence the solution is $x = 1, y = -1, z = 1$.

LU Decomposition Method :—

Every Square matrix A can be expressed as the product of a lower triangular matrix & an upper triangular matrix.

Working Rule :—

Consider the system of eq^ns.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

In matrix form $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

let $A = L.U$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

then $L.UX = B \longrightarrow (1)$

let $UX = Y$

eq (1) $\Rightarrow$ $L.Y = B$.

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ solve to find y_1, y_2, y_3 then from

$$UX = Y.$$

we can find. $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \longrightarrow$ Required Solution.

① Solve the following eqn by L.U Decomposition method.

Sol $x+y+z=1$

$4x+3y-z=6$

$3x+5y+3z=4$

Sol Given system of eqn in matrix form.

$AX=B$

$$A=\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix}, \quad B=\begin{bmatrix}1\\6\\4\end{bmatrix}$$

$A=L\cdot U$

$$A=\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\4&3&-1\\3&5&3\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

$\Rightarrow u_{11}=1, \quad u_{12}=1, \quad u_{13}=1$

$l_{21}u_{11}=4$; $\boxed{l_{21}=4}$

$l_{21}u_{12}+u_{22}=3$; $4(1)+u_{22}=3$; $u_{22}=3-4$; $\boxed{u_{22}=-1}$

$l_{21}u_{13}+u_{23}=-1$; $4(1)+u_{23}=-1$; $\boxed{u_{23}=-5}$

$l_{31}u_{11}=3$; $\boxed{l_{31}=3}$

$l_{31}u_{12}+l_{32}u_{22}=5$; $3(1)+l_{32}(-1)=5$; $-l_{32}=5-3$; $\boxed{l_{32}=-2}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=3$; $(3)(1)+(-2)(-5)+u_{33}=3$; $\boxed{u_{33}=-10}$

$$A = L \cdot U$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

Now $AX = B$

$LUX = B$

$L(UX) = B \longrightarrow (1)$

$\Rightarrow$ let $UX = Y$

eq (1) $\Rightarrow LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$\Rightarrow \boxed{y_1 = 1}$, $4y_1 + y_2 = 6$, $3y_1 - 2y_2 + y_3 = 4$

$\boxed{y_2 = 2}$

$3(1) - 2(2) + y_3 = 4$

$y_3 = 8 - 3$

$\boxed{y_3 = 5}$

Now $UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$x + y + z = 1 \longrightarrow (2)$

$-y - 5z = 2 \longrightarrow (3)$

$-10z = 5$

$z = \frac{5}{-10} = \frac{-1}{2} \Rightarrow \boxed{z = -1/2}$

eq (3) $-y - 5\left(\frac{-1}{2}\right) = 2$

$-y = 2 - \frac{5}{2}$

$-y = -\frac{1}{2} \Rightarrow y = 1/2$

$x + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} = 1 \Rightarrow \boxed{x = 1}$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$