M-II

Advanced calculus

1. UNIT-I

   first order ordinary Differential Equations.

2. UNIT-II

   Ordinary differential Equations of Higher order.

3. UNIT-III

   Multivariable calculus

4. UNIT-IV

   Vector differentiation

5. UNIT-V

   Vector Integration.

Differential Equations of first order and first degree

An eqn of the form $\frac{dy}{dx} = f(x,y)$ is called a differential eqn of first order and first degree.

Exact differential Equation:-

Let $M(x,y)dx + N(x,y)\,dy = 0$ be a first order and first degree differential Equations. where M and N are real valued func for some x,y. Then the differential eqn $Mdx + Ndy = 0$ is said to be an exact differential equation if it satisfies the condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

The general solution of the exact differential eqn. $M\,dx + N\,dy = 0$ is

$$\int_{(y\ \text{constant})} M\,dx + \int (\text{terms not containing } x \text{ in } N)\,dy = c$$

where c is a constant.

(1) $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$

Sol Given eqn is $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0 \longrightarrow (1)$

This is of the form $M\,dx + N\,dy = 0$

Here $M = x^3 + 3xy^2$ and $N = y^3 + 3x^2y$.

$$\therefore \frac{\partial M}{\partial y} = 3x(2y) = 6xy$$

$$\frac{\partial N}{\partial x} = 3y(2x) = 6xy.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ (The given differential eqn is exact)

Hence the general solution is

$$\int_{(y\ \text{constant})} M\,dx + \int (\text{terms not containing } x \text{ in } N)\,dy = c$$

$$\int x^3+3xy^2\,dx+\int y^3\,dy=c$$

$$\frac{x^4}{4}+3y^2\frac{x^2}{2}+\frac{y^4}{4}=c$$

$$x^4+6x^2y^2+y^4=4c$$

$$\boxed{\therefore x^4+6x^2y^2+y^4=c}$$

This is the general solution of eq ①

② $(e^y+1)\cos x\,dx+(e^y)\sin x\,dy=0$

Non Exact differential equation :-

Let $M(x,y)\,dx+N(x,y)\,dy=0$ be not an exact differential eqn. if $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ then make that eqn exact by finding Integrating factor of $Mdx+Ndy=0$

Integrating factor :-

In general, for differential eqn $Mdx+Ndy=0$ is not an exact.

In such situation, we find a func λ such that by multiplying λ to the eqn, it becomes an exact eqn

so $\lambda M(x,y)\,dx+\lambda N(x,y)\,dy=0$ becomes exact diff eq

Here the the func $\lambda=\lambda(x,y)$ is then called an Integrating factor.

Methods to find an integrating factor (I.F) :-

for given non exact diff eqn
$$M dx + N dy = 0$$

Cases **Case I**

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous differential equation and $M dx + N dy = 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor of $M dx + N dy = 0$

① find the integrating factor of
$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

Sol Given eqn is $(x^2+y^2)dx - 2xy\,dy = 0$

This is of the form $M dx + N dy = 0$

Here $M = x^2+y^2$ and $N = -2xy$

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ The given eqn is not exact

But it is homogeneous eqn.

Thus $I.F = \frac{1}{M dx + N dy}$

Now $M dx + N dy = M\cdot x + N\cdot y$

$= (x^2+y^2)x + (-2xy)y$

$= x^3 + xy^2 - 2y^2x$

$Mx + Ny = x^3 - y^2x$

$$I.F = \frac{1}{x^3 - y^2x} = \frac{1}{x(x^2-y^2)}$$

① Solve the following differential eqn:

① $y(xy+e^x)dx - e^x dy = 0$

Sol Given eqn is $y(xy+e^x)dx - e^x dy = 0$ ⟶ ①

This is of the form $Mdx + Ndy = 0$

Here $M = y(xy+e^x)$ and $N = -e^x$

$$\frac{\partial M}{\partial y} = 2xy + e^x \text{ and } \frac{\partial N}{\partial x} = -e^x$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given diff eqn is not exact different eqn.

Dividing with y^2 in eq ①, we get

$$\frac{y(xy+e^x)}{y^2}dx - \frac{e^x}{y^2}dy = 0$$

$$\left[\frac{xy^2}{y^2} + \frac{ye^x}{y^2}\right]dx - \left[\frac{e^x}{y^2}\right]dy = 0$$

$$\left[x + \frac{e^x}{y}\right]dx - \left[\frac{e^x}{y^2}\right]dy = 0 \longrightarrow ②$$

This is of the form $M_1dx + N_1dy = 0$

$$M_1 = x + \frac{e^x}{y} \text{ and } N_1 = \frac{-e^x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{e^x}{y^2} \text{ and } \frac{\partial N_1}{\partial x} = -\frac{e^x}{y^2}$$

$$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

eq ② is an exact differential eqn.

The general solution is

$\int M_1 dx + \int (\text{terms not containing } x \text{ in } N_1) dy = c$

(y constant)

$$\int \left(x + \frac{e^x}{y}\right) dx + \int 0 \, dy = c$$

$$\boxed{\frac{x^2}{2} + \frac{e^x}{y} = c}$$

Which is the required general solution.

Exact differential eqn :—

** (1) $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$

Sol Given eqn is $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$

This is of the form $Mdx+Ndy=0$

Here $M=x^3+3xy^2$ and $N=y^3+3x^2y$

$\therefore \frac{\partial M}{\partial y}=6xy$ and $\frac{\partial N}{\partial x}=6xy$

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

The given differential eqn is exact

The general solution is

$\int M dx + \int (\text{terms independent of } x \text{ in } N)dy = c.$
(y constant)

$\int [x^3+3y^2x]dx+\int y^3 dy=c$

$\frac{x^4}{4}+3y^2\frac{x^2}{2}+\frac{y^4}{4}=c$

$x^4+6x^2y^2+y^4=4C$

$\boxed{\therefore x^4+6x^2y^2+y^4=c}$

Imp ** Method-I

(2) $y(xy+e^x)dx-e^x dy=0$

Sol Given eqn is $y(xy+e^x)dx-e^x dy=0 \longrightarrow (1)$

This is of the form $(xy^2+ye^x)dx-e^x dy=0$

$Mdx+Ndy$

Here $M=xy^2+ye^x$, $N=-e^x$

$$\frac{\partial M}{\partial y} = 2xy + e^x \quad , \quad \frac{\partial N}{\partial x} = -e^x$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ is not}$$

The given eqn is not exact differential eqn.

Dividing with y^2 in ① , we get

$$\frac{1}{y^2}[xy^2 + ye^x]dx - \frac{1}{y^2}e^x dy = 0$$

$$\frac{\cancel{y}}{y^{\cancel{2}}}[xy + e^x]dx - \frac{1}{y^2}e^x dy = 0$$

$$\frac{1}{y}[xy + e^x]dx - \frac{1}{y^2}e^x dy = 0$$

$$\left[\frac{xy}{y} + \frac{e^x}{y}\right]dx - \frac{1}{y^2}e^x dy = 0$$

$$\left[x + \frac{e^x}{y}\right]dx - \frac{1}{y^2}e^x dy = 0 \longrightarrow ②$$

This is of the form $M_1 dx + N_1 dy = 0$

Here $M_1 = x + \frac{e^x}{y}$, $N_1 = -\frac{1}{y^2}e^x$

$$\frac{\partial M_1}{\partial y} = \frac{-e^x}{y^2} \quad , \quad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2}e^x$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ eq ② is an exact diff eqn.

The general solution is

$\int M_1 dx + \int (\text{terms of } N \text{ not containing } x) dy = c$
(y constant)

$$\int \left[x + \frac{e^x}{y}\right]dx + \int 0\, dy = c$$

$$\frac{x^2}{2} + \frac{1}{y}e^x = c$$

$$\boxed{\frac{x^2}{2} + \frac{1}{y}e^x = c}$$

Solve: $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$.

Solution:- The given equation is $(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$ → (1)

This is of the form $M\,dx + N\,dy = 0$

where $M = (e^y+1)\cos x$ and $N = e^y \sin x$

$$\therefore \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, The given differential eqn is exact

Terms in N which do not contain x are 0

$\therefore$ The general solution is given by

$$\underset{(y\,\text{constant})}{\int M\,dx} + \int (\text{terms independent of } x \text{ in } N)\,dy = c$$

$$\Rightarrow \underset{(y\,\text{constant})}{\int (e^y+1)\cos x\,dx} + \int 0.dy = c$$

$$\Rightarrow (e^y+1)\int \cos x\,dx = c \qquad \left\{\because \int \cos x\,dx = \sin x + c\right\}$$

$$\boxed{(e^y+1)\sin x = c}$$

This is the general solution of eq (1)

Method-I

Solve: $(1+xy)x\,dy + (1-yx)y\,dx = 0$

Solution:-

The given equation is $(1+xy)x\,dy + (1-yx)y\,dx = 0$

$\Rightarrow \quad (1-yx)y\,dx + (1+xy)x\,dy = 0$

$\Rightarrow \quad (y-y^2x)\,dx + (x+x^2y)\,dy = 0$

This is of the form $M\,dx + N\,dy = 0$

where $M = y - y^2x$, $N = x + x^2y$.

$\therefore \quad \frac{\partial M}{\partial y} = 1 - 2yx \;, \quad \frac{\partial N}{\partial x} = 1 + 2xy$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that $M\,dx + N\,dy = 0$ is not an exact differential equation.

Dividing the given equation with x^2y^2, we get

$$\frac{(y-y^2x)}{x^2y^2}\,dx + \frac{(x+x^2y)}{x^2y^2}\,dy = 0$$

$$\left[\frac{y}{x^2y^2} - \frac{y^2x}{x^2y^2}\right]dx + \left[\frac{x}{x^2y^2} + \frac{x^2y}{x^2y^2}\right]dy = 0$$

$$\frac{x\,dy + y\,dx}{x^2y^2} + \frac{1}{y}\,dy - \frac{1}{x}\,dx = 0$$

$$\frac{d(xy)}{(xy)^2} - \frac{1}{x}\,dx + \frac{1}{y}\,dy = 0$$

Integrating, we get

$$\int \frac{d(xy)}{(xy)^2} - \int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy = 0$$

$$\int \frac{1}{(xy)^2}\,d(xy) - \int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy = 0 \qquad \left\{ \because \int \frac{1}{x^2}\,dx = -\frac{1}{(x)} + c \right\}$$

$$-\frac{1}{xy} - \log x + \log y = \log c$$

Which is the required general solution.

* **Solve:** Find the integrating factor of

$$(x^2+y^2)\,dx - 2xy\,dy = 0$$

Solution: The given eqn is $(x^2+y^2)dx - 2xy\,dy = 0 \longrightarrow (1)$

This is of the form $M\,dx + N\,dy = 0$

where $M = x^2+y^2$, $N = -2xy$

then $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = -2y$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So that $M\,dx + N\,dy = 0$ is not an exact differential eqn

But eq (1) is of the ~~form~~ a homogeneous differential eqn

and ~~and~~ $Mx + Ny = (x^2+y^2)x + (-2xy)y$

$$Mx + Ny = x^3 + xy^2 - 2xy^2 \neq 0$$

Thus $I.F = \frac{1}{Mx+Ny} = \frac{1}{x^3+xy^2-2xy^2} = \frac{1}{x^3 - xy^2}$

$$\boxed{I.F = \frac{1}{x(x^2-y^2)}}$$

Solve: $(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$

<u>Sol:</u> Given equation is

$$[xy\sin xy + \cos xy]\,y\,dx + [xy\sin xy - \cos xy]\,x\,dy = 0$$

This is of the form $M\,dx + N\,dy = 0$

where $M = xy^2\sin xy + y\cos xy$

$N = x^2y\sin xy - x\cos xy$

$$\frac{\partial M}{\partial y} = x[2y\sin xy + y^2(-\cos xy)\,x] + [\cos xy\cdot 1 + y\sin xy]$$

$$(y - y^2x)\,dx + (x + x^2y)\,dy = 0$$

$M = y - y^2x \qquad N = x + x^2y$

$\frac{\partial M}{\partial y} = 1 - 2xy \qquad \frac{\partial N}{\partial x} = 1 + 2xy$

dividing with x^2y^2, we

$$\frac{1}{x^2y^2}(y - y^2x)\,dx + \frac{1}{x^2y^2}(x + x^2y)\,dy = 0$$

$$\left[\frac{1}{x^2y} - \frac{1}{x}\right]dx + \left[\frac{1}{xy^2} + \frac{1}{y}\right]dy = 0$$

$\frac{\partial M_1}{\partial y} = \frac{-1}{x^2y^2} \qquad \frac{\partial N'}{\partial x} = \frac{-1}{x^2y^2}$

$\vdots$

$$\frac{1}{y}\int \frac{1}{x^2}\,dx - \int \frac{1}{x}\,dx + \int \frac{1}{y}\,dy = 0$$

$\frac{1}{y}\int x^{-2}\,dx$

$\frac{\int x^3}{3} \qquad \frac{1}{y}\,\frac{x^{-2+1}}{-2+1}$

$\frac{-1}{xy}$

Method 2

Solve: $x^2y\,dx - (x^3+y^3)\,dy = 0$

Solution: Given eqn is $x^2y\,dx-(x^3+y^3)dy=0 \longrightarrow (1)$

This is of the form $M\,dx+N\,dy=0$

where $M = x^2y$, $N = -(x^3+y^3)$

we have $\frac{\partial M}{\partial y} = x^2$, $\frac{\partial N}{\partial x} = -3x^2$

$\therefore$ We have $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ so that the given differential eqn is not exact

But eq (1) is a homogeneous differential eqn

Now $Mx+Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$

$\therefore$ Integrating factor $= \frac{1}{M\,dx+N\,dy} = -\frac{1}{y^4}$

Multiplying eq (1) with $-1/y^4$, we get

$$-\frac{1}{y^{4}_{3}}(x^2y)dx - \frac{1}{y^4}\left[-(x^3+y^3)\right]dy = 0$$

$$-\frac{x^2}{y^3}dx + \frac{x^3+y^3}{y^4}dy = 0 \longrightarrow (2)$$

This is of the form $M_1dx+N_1dy=0$

where $M_1 = -\frac{x^2}{y^3}$ and $N_1 = \frac{x^3+y^3}{y^4}$

we have $\frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4}$ and $\frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$

$\therefore$ Eq (2) is an Exact differential equation.

General Solution of eq (2) is given by

$$\int M_1\,dx + \int (\text{terms independent of } x \text{ in } N_1)\,dy = c$$
(y constant)

$$\int -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{1}{y^3}\int x^2\,dx + \log y = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

The general solution of (2) is same as the general solution of (1)

Solve:

Method 3:

If the equation $Mdx + Ndy = 0$ is of the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor of $Mdx + Ndy = 0$

Imp
**

(1) Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$

Sol: Given eqn is

$$(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0 \longrightarrow (1)$$

This is of the form $Mdx + Ndy = 0$

where $M = xy^2\sin xy + y\cos xy$

$N = x^2y\sin xy - x\cos xy$

we have $\frac{\partial M}{\partial y} = x[2y\sin xy + y^2\cos xy \cdot x] + [1\cdot\cos xy + y(-\sin xy)x]$

$$\frac{\partial M}{\partial y} = 2xy\sin xy + x^2y^2\cos xy + \cos xy - xy\sin xy$$

$$\frac{\partial M}{\partial y} = xy\sin xy + x^2y^2\cos xy + \cos xy.$$

and $\frac{\partial N}{\partial x} = y[2x\sin xy + x^2\cos xy \cdot y] - [1\cdot\cos xy + x(-\sin xy)y]$

$$= 2xy\sin xy + x^2y^2\cos xy - \cos xy + xy\sin xy$$

$$\frac{\partial N}{\partial x} = 3xy\sin xy + x^2y^2\cos xy - \cos xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ The given differential eqn is ~~exact~~ not an exact diff eqn

Since eq ① is of the form $y \cdot f(xy)dx + x \cdot g(xy)dy = 0$ and

$$Mx - Ny = (xy^2 \sin xy + y\cos xy)x - (x^2y\sin xy - x\cos xy)y$$

$$= \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy$$

$$Mx - Ny = 2xy\cos xy \neq 0$$

Thus I.F $= \dfrac{1}{Mx-Ny} = \dfrac{1}{2xy\cos xy}$

Multiplying eq ① with I.F i.e $\dfrac{1}{2xy\cos xy}$, we get

$$= \frac{1}{2xy\cos xy}\left[xy^2\sin xy + y\cos xy\right]dx + \frac{1}{2xy\cos xy}\left[x^2y\sin xy - x\cos xy\right]dy = 0$$

$$= \left[\frac{\cancel{x}y^2\sin xy}{2\cancel{x}\cancel{y}\cos xy} + \frac{\cancel{y}\cos \cancel{xy}}{2x\cancel{y}\cos \cancel{xy}}\right]dx + \left[\frac{x^2y\sin xy}{2x y\cos xy} - \frac{\cancel{x}\cos \cancel{xy}}{2\cancel{x}y\cos \cancel{xy}}\right]dy = 0$$

$$\left[\frac{y}{2}\tan xy + \frac{1}{2x}\right]dx + \left[\frac{x}{2}\tan xy - \frac{1}{2y}\right]dy = 0 \longrightarrow ②$$

This is of the form $M_1dx + N_1dy = 0$ ~~and it is exa~~

where $M_1 = \dfrac{y}{2}\tan xy + \dfrac{1}{2x}$ and $N_1 = \dfrac{x}{2}\tan xy - \dfrac{1}{2y}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2}\left[y\sec^2 xy \cdot x + \tan xy(1)\right] + 0$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{xy}{2}\sec^2 xy + \frac{1}{2}\tan xy}$$

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$$\frac{\partial N_1}{\partial x} = \frac{1}{2}[x\sec^2 xy \cdot y + \tan xy(1)] - 0$$

$$\boxed{\frac{\partial N_1}{\partial x} = \frac{xy}{2}\sec^2 xy + \frac{1}{2}\tan xy}$$

$\therefore$ eq ② is an exact differential eqn.

The general Solution is given by.

$$\int M_1 dx + \int (\text{terms independent of } x \text{ in } N_1) dy = C$$

(y constant)

$$\int \frac{1}{2}\left[y\tan xy + \frac{1}{x}\right]dx + \int -\frac{1}{2y}dy = c$$

$$\frac{1}{2}\int y\tan xy\, dx + \frac{1}{2}\int \frac{1}{x}dx - \frac{1}{2}\int \frac{1}{y}dy = c$$

$$\frac{y}{2}\int \tan xy\, dx + \frac{1}{2}\log(x) - \frac{1}{2}\log y = \log c_1 \qquad \left\{\int \tan ax\, dx = \frac{\log|\sec ax|}{a} + c\right.$$

$$\frac{\cancel{y}}{2}\frac{\log|\sec xy|}{\cancel{y}} + \frac{1}{2}\log x - \frac{1}{2}\log y = \log c_1 \qquad \left\{\log\frac{m}{n} = \log m - \log n\right]$$

$$\frac{1}{2}[\log(\sec xy) + \log(x) - \log y] = \log c_1 \qquad \left\{\log m^n = n\log m\right\}$$

$$\log\left(\frac{x\sec(xy)}{y}\right) = 2\log c_1$$

$$\log\left[\frac{x\sec(xy)}{y}\right] = \log c_1^2$$

$$\frac{x\sec xy}{y} = c_1^2$$

$$\frac{x\sec xy}{y} = c$$

This is the required general solution.

④ $y(1+xy)dx + x(1-xy)dy = 0$

② Solve the

① $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0$

② $(y-xy^2)dx - (x+x^2y)dy = 0.$

① Solve the following differential eqns

① $(y-xy^2)dx - (x+x^2y)dy = 0$

② $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0$

Method-3

Solve: $y(1+xy)dx + x(1-xy)dy = 0$ (10)

Sol: Given D.E is $y(1+xy)dx + x(1-xy)dy = 0 \longrightarrow (1)$

This is of the form $Mdx + Ndy = 0$

where $M = y + xy^2$, $N = x - x^2y$

then $\frac{\partial M}{\partial y} = 1 + 2xy$ and $\frac{\partial N}{\partial x} = 1 - 2xy$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, The given eqn is not exact D.E

The given D.E is of the form $y \cdot f(xy) + x \cdot g(xy)dy = 0$

Now $Mx - Ny = (y + xy^2)x - (x - x^2y)y$

$Mx - Ny = \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$

$Mx - Ny = 2x^2y^2 \neq 0$

$\therefore$ I.F $= \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$

Multiplying eq(1) with $\frac{1}{2x^2y^2}$, we get

$$\frac{1}{2x^2y^{\cancel{2}}} \cancel{y}(1+xy)dx + \frac{1}{2x^{\cancel{2}}y^2} \cancel{x}(1-xy)dy = 0$$

$$\left[\frac{1}{2x^2y} + \frac{\cancel{x}y}{2x^{\cancel{2}}y}\right]dx + \left[\frac{1}{2y^2x} - \frac{xy}{2xy^{\cancel{2}}}\right]dy = 0$$

$$\left[\frac{1}{2x^2y} + \frac{1}{2x}\right]dx + \left[\frac{1}{2y^2x} - \frac{1}{2y}\right]dy = 0 \longrightarrow (2)$$

This is of the form $M_1dx + N_1dy = 0$

Here $M_1 = \frac{1}{2x^2y} + \frac{1}{2x}$, $N_1 = \frac{1}{2xy^2} - \frac{1}{2y}$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2} \text{ and } \frac{\partial N_1}{\partial x} = \frac{-1}{2x^2y^2} \quad \left\{ \frac{d}{dx}(1/x) = -1/x^2 \right\}$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, the eqn (2) is an exact differential eqn

$\therefore$ The general Solution of (1) is

$$\underset{(y\,\text{constant})}{\int M_1\,dx} + \underset{(\text{terms not containing } x)}{\int N_1\,dy} = c$$

$$\int \left[\frac{1}{2x^2y} + \frac{1}{2x}\right]dx + \int \frac{-1}{2y}\,dy = c$$

$$\frac{1}{2y}\left[\frac{x^{-2+1}}{-2+1}\right] + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\frac{1}{2y}\left(\frac{x^{-1}}{-1}\right] + \frac{1}{2}[\log x - \log y] = c \qquad \left\{ \int x^n dx = \frac{x^{n+1}}{n+1} + c \right.$$

$$-\frac{1}{2xy} + \frac{1}{2}\left[\log\left(\frac{x}{y}\right)\right] = c \qquad \left\{ \int \frac{1}{x}dx = \log x + c \right.$$

$$\frac{1}{2}\left[\log\left(\frac{x}{y}\right) - \frac{1}{xy}\right] = c.$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = 2c$$

$$\log\left(\frac{x}{y}\right) - \frac{1}{xy} = c_1$$

which is the required solution.

Method : 4

If there exists a continuous single variable function $f(x)$ such that $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=f(x)$ then $e^{\int f(x)dx}$ is an integrating factor of $Mdx+Ndy=0$.

① Solve $2xy\,dy-(x^2+y^2+1)dx=0$

Sol: Given eqn is $-(x^2+y^2+1)dx+2xy\,dy=0 \longrightarrow ①$

This is of the form $Mdx+Ndy=0$ where

$$M=-(x^2+y^2+1) \text{ and } N=2xy.$$

we have $\frac{\partial M}{\partial y}=-2y$ and $\frac{\partial N}{\partial x}=2y$

So that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

$\therefore$ The given eqn is not exact

we have $\frac{1}{N}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=\frac{1}{2xy}[-2y-2y]=\frac{-\cancel{4y}^{\,2}}{\cancel{2xy}}=\frac{-2}{x}=f(x)$

$$\therefore I.F=e^{\int f(x)dx}=e^{-2\int\frac{1}{x}dx}=e^{-2\log x}$$

$$I.F=e^{\log_e x^{-2}}=x^{-2}=\frac{1}{x^2}$$

$$I.F=\frac{1}{x^2}$$

Multiplying eq ① with $\frac{1}{x^2}$, we get

$$\frac{-[x^2+y^2+1]}{x^2}dx+\frac{2xy}{x^2}dy=0$$

$$-\left[\frac{\cancel{x^2}}{\cancel{x^2}}+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx+\frac{2y}{x}dy=0$$

$$-\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx+\frac{2y}{x}dy=0 \longrightarrow (2)$$

This is of the form $M_1 dx + N_1 dy = 0$ where

$$M_1 = -\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right] \text{ and } N_1 = \frac{2y}{x}$$

we have $\frac{\partial M_1}{\partial y} = -\left[0+\frac{2y}{x^2}+0\right]$ and $\frac{\partial N_1}{\partial x} = 2y\left(\frac{-1}{x^2}\right)$

$$\frac{\partial M_1}{\partial y} = \frac{-2y}{x^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ eq (2) is exact D.E.

Hence the general solution is given by.

$$\underset{(y\text{ constant})}{\int M_1 dx} + \int (\text{terms not containing } x \text{ in } N)\, dy = c$$

$$\int -\left[1+\frac{y^2}{x^2}+\frac{1}{x^2}\right]dx + \int 0 \cdot dy = c.$$

$$-\int 1 \cdot dx - y^2 \int \frac{1}{x^2}dx - \int \frac{1}{x^2}dx = c$$

$$-x - y^2\left[\frac{x^{-2+1}}{-2+1}\right] - \left[\frac{x^{-2+1}}{-2+1}\right] = c \qquad \left\{\int x^n dx = \frac{x^{n+1}}{n+1}+c\right\}$$

$$-x - y^2\left[\frac{x^{-1}}{-1}\right] - \left[\frac{x^{-1}}{-1}\right] = c$$

$$\boxed{-x + \frac{y^2}{x} + \frac{1}{x} = c}$$

This is the general solution of eq (2) and hence of eq (1)

(12)

Solve: $2xy\,dy - (x^2+y^2+1)dx = 0$

Solution: Given eqn is $2xy\,dy - (x^2+y^2+1)dx = 0 \longrightarrow (1)$

This is of the form $M\,dx + N\,dy = 0$, where

$$M = -(x^2+y^2+1) \quad , \quad N = 2xy$$

we have $\frac{\partial M}{\partial y} = -2y \quad , \quad \frac{\partial N}{\partial x} = 2y$

So that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ The given eqn is not exact

we have $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2xy}[-2y-2y] = \frac{-4y}{2xy} = \frac{-2}{x}$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{-2}{x} = f(x)$$

$$\therefore I.F = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$$

$$I.F = \frac{1}{x^2}$$

Multiplying eq (1) with $1/x^2$, we get

$$\frac{1}{x^2}\left[2xy\,dy - (x^2+y^2+1)dx\right] = 0$$

$$\frac{2y}{x}dy - \left[1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right]dx = 0 \longrightarrow (2)$$

This is of the form $M_1 dx + N_1 dy = 0$, where

$$M_1 = -\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right) \quad , \quad N_1 = \frac{2y}{x}$$

we have $\frac{\partial M_1}{\partial y} = \frac{-2y}{x^2} \quad , \quad \frac{\partial N_1}{\partial x} = \frac{-2y}{x^2}$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ eq ② is exact D.E

Hence the general solution is given by

$$= \int M_1 dx + \int (\text{terms independent of } x \text{ in } N_1) dy = c.$$

(y constant)

$$\Rightarrow \int \left[-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right] dx + \int 0 . dy = c$$

$$-x + \frac{y^2}{x} + \frac{1}{x} = c$$

This is the general solution of eq ② and hence of eq ①

② Solve $(3y^2+4xy-x)dx + x(x+2y)dy = 0$ ⑬ ⑩

Sol: The given D.E is $(3y^2+4xy-x)dx + (x^2+2xy)dy = 0 \longrightarrow$ ①

This is of the form $Mdx + Ndy = 0$

where $M = 3y^2+4xy-x$ and $N = x^2+2xy$

we have $\frac{\partial M}{\partial y} = 3\frac{\partial}{\partial y}y^2 + 4x\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(x)$ and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x^2) + 2y\frac{\partial}{\partial x}(x)$

$$\frac{\partial M}{\partial y} = 3(2y) + 4x(1) - 0 \qquad \frac{\partial N}{\partial x} = 2x+2y$$

$$\frac{\partial M}{\partial y} = 6y+4x$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given D.E is not exact

consider $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{x^2+2xy}\left[6y+4x-(2x+2y)\right]$

$$= \frac{1}{x^2+2xy}\left[6y+4x-2x-2y\right]$$

$$= \frac{1}{x(x+2y)}\left[4y+2x\right] \Rightarrow \frac{2\cancel{(x+2y)}}{x\cancel{(x+2y)}} = \frac{2}{x} = f(x)$$

$$\therefore \text{I.F} = e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\log x}$$

$$\text{I.F} = e^{\log_e x^2} = x^2.$$

Multiplying eq ① with x^2, we get

$$x^2[3y^2+4xy-x]dx + x^2[x^2+2xy]dy = 0$$

$$[3x^2y^2+4x^3y-x^3]dx + [x^4+2x^3y]dy = 0 \longrightarrow ②$$

This is of the form $M_1dx + N_1dy = 0$

$M_1 = 3x^2y^2 + 4x^3y - x^3$ and $N_1 = x^4 + 2x^3y$

we have $\frac{\partial M_1}{\partial y} = 6x^2y + 4x^3 - 0$ and $\frac{\partial N_1}{\partial x} = 4x^3 + 6x^2y$

$$\frac{\partial M_1}{\partial y} = 6x^2y + 4x^3 \text{ and } \frac{\partial N_1}{\partial x} = 4x^3 + 6x^2y$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ eq ② is exact diff eqn.

The general solution is given by

$\int M_1 \, dx$ + $\int$ (terms not containing x in N) $dy = c$
(y constant)

$$\int (3x^2y^2 + 4x^3y - x^3)\,dx + \int 0 \, dy = c$$

$$3y^2 \int x^2 dx + 4y \int x^3 dx - \int x^3 dx = c$$

$$\cancel{3}y^2\left[\frac{x^3}{\cancel{3}}\right] + \cancel{4}y\left[\frac{x^4}{\cancel{4}}\right] - \left[\frac{x^4}{4}\right] = c$$

$$x^3y^2 + yx^4 - \frac{x^4}{4} = c$$

~~Which is~~ the this is the general solution of eq ②
and hence of eq ①

Asst ① Solve the following differential eqns.

① $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

② $(x^2 + y^2)\,dx - 2xy\,dy = 0$.

Method 5:

If there exists a continuous and differentiable single variable function $g(y)$ such that $\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=g(y)$ then $e^{\int g(y)dy}$ is an integrating factor of $Mdx+Ndy=0$

① Solve the differential equation $y(xy+e^x)dx-e^x dy=0$

Sol: Given eqn is $y(xy+e^x)dx-e^x dy=0 \longrightarrow ①$

This is of the form ~~y(xy~~ $Mdx+Ndy=0$

where $M=xy^2+ye^x \qquad N=-e^x$

we have $\frac{\partial M}{\partial y}=2xy+e^x \qquad \frac{\partial N}{\partial x}=-e^x$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ The given D.E is not exact

Now $\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=\frac{1}{xy^2+ye^x}\left[-e^x-2xy-e^x\right]$

$=\frac{1}{xy^2+ye^x}\left[-2e^x-2xy\right]$

$=\frac{1}{y[\cancel{xy+e^x}]}(-2(\cancel{e^x+xy})]$

$\frac{1}{M}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=\frac{-2}{y}=f(y)$

$\therefore$ I.F $=e^{\int f(y)dy}=e^{-2\int\frac{1}{y}dy}=e^{-2\log y}=e^{\log y^{-2}}=y^{-2}$

I.F $=\frac{1}{y^2}$

Multiplying the given D.E with $\frac{1}{y^2}$, we get

$\frac{1}{y^2}[xy^2+e^x y]dx-\frac{e^x}{y^2}dy=0$

$$\left[\frac{xy^2}{y^2}+\frac{e^{x}y}{y^2}\right]dx-\frac{e^x}{y^2}dy=0$$

$$\left[x+\frac{e^x}{y}\right]dx-\frac{e^x}{y^2}dy=0 \longrightarrow (2)$$

This is of the form $M_1dx+N_1dy=0$

$$M_1=x+\frac{e^x}{y} \qquad N_1=-\frac{e^x}{y^2}$$

we have $\frac{\partial M_1}{\partial y}=0+e^x\left(-\frac{1}{y^2}\right)$ and $\frac{\partial N_1}{\partial x}=-\frac{1}{y^2}(e^x)$

$$\frac{\partial M_1}{\partial y}=-\frac{e^x}{y^2} \text{ and } \frac{\partial N_1}{\partial x}=-\frac{e^x}{y^2}$$

$$\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$ eq (2) is exact diff eqn.

The general solution is given by.

$$\int\limits_{(y\text{ constant})} M_1dx+\int(\text{terms not containing } x \text{ in } N)dy=c$$

$$\int\left(x+\frac{e^x}{y}\right)dx+\int 0\,dy=c$$

$$\int x\,dx+\frac{1}{y}\int e^x\,dx=c$$

$$\frac{x^2}{2}+\frac{e^x}{y}=c$$

which is the general solution.

Asst (1) Solve: (1) $y(2xy+e^x)dx-e^xdy=0$

(2) $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0.$

(3) $(xy^2-x^2)dx+(3x^2y^2+x^2y-2x^3)dy=0.$

(15)

② Solve $y(2xy+e^x)dx - e^x dy = 0$

Sol Given eqn is $y(2xy+e^x)dx - e^x dy = 0$

This is of the form $Mdx + Ndy = 0$

where $M = y(2xy+e^x)$ and $N = -e^x$

We have $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy^2 + ye^x)$ and $\frac{\partial N}{\partial x} = -\frac{\partial}{\partial x}(e^x)$

$\frac{\partial M}{\partial y} = 4xy + e^x$ and $\frac{\partial N}{\partial x} = -e^x$.

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ The given diff eqn is not exact

consider $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{2xy^2 + e^x y}\left[-e^x - 4xy - e^x\right]$

$= \frac{1}{2xy^2 + e^x y}\left[-2e^x - 4xy\right]$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-2[\cancel{e^x + 2xy}]}{y[\cancel{e^x + 2xy}]} = -\frac{2}{y} = f(y)$

$I.F = e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log y^{-2}}$

$I.F = y^{-2}$ $\quad \{ e^{\log_e a} = a \}$

$I.F = 1/y^2$

Multiplying eq① with $1/y^2$, we get

$\frac{1}{y^2}[2xy^2 + e^x y]dx - \frac{e^x}{y^2}dy = 0$

$\left[2x + \frac{e^x}{y}\right]dx - \frac{e^x}{y^2}dy = 0 \longrightarrow ①$

This is of the form $M_1 dx + N_1 dy = 0$

Here $M_1 = 2x + \frac{e^x}{y}$, $N_1 = -\frac{e^x}{y^2}$

we have $\frac{\partial M_1}{\partial y} = 0 + e^x \frac{\partial}{\partial y}\left(\frac{1}{y}\right)$, $\frac{\partial N_1}{\partial x} = \frac{-e^x}{y^2}$

$$\frac{\partial M_1}{\partial y} = \frac{-e^x}{y^2}$$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ eq ② is exact diff eqn.

The general solution is given by

$\int M_1 dx + \int$ (terms not containing x in N_1) $dy = c$
(y constant)

$$\int \left(2x + \frac{e^x}{y}\right) dx + \int 0 . dy = c$$

$$2\int x\,dx + \int \frac{e^x}{y} dx = c$$

$$\cancel{2}\left[\frac{x^2}{\cancel{2}}\right] + \frac{1}{y}\int e^x dx = c$$

$$x^2 + \frac{e^x}{y} = c$$

This is the general solution of eq ② and hence of eq ①

Solve: the differential equation $y(xy+e^x)dx - e^x dy = 0$ (16)

Solution: Given eqⁿ is $y(xy+e^x)dx - e^x dy = 0 \longrightarrow (1)$

Comparing the given D.E with $Mdx + Ndy = 0$

we have $M = xy^2 + e^x y$ and $N = -e^x$

$\therefore \frac{\partial M}{\partial y} = 2xy + e^x$ and $\frac{\partial N}{\partial x} = -e^x$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given D.E is not exact

Now $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{xy^2 + e^x y}\left[-e^x - 2xy - e^x\right]$

$$= \frac{1}{y[xy+e^x]}\left[-2e^x - 2xy\right]$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-2[e^x + xy]}{y[e^x + xy]} = \frac{-2}{y} = f(y)$$

$$\therefore I.F = e^{\int f(y)dy} = e^{-\int \frac{2}{y}dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y}$$

$$I.F = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Multiplying the given D.E with $\frac{1}{y^2}$, we get

$$\frac{1}{y^2}\left[y(xy+e^x)dx - e^x dy\right] = 0$$

$$\frac{1}{y}[xy + e^x]dx - \frac{e^x}{y^2}dy = 0 \longrightarrow (2)$$

This is of the form $M_1 dx + N_1 dy = 0$ where.

$M_1 = \frac{1}{y}[xy + e^x]$, $N_1 = \frac{-e^x}{y^2}$

$\therefore M_1 = x + \frac{e^x}{y}$, $N_1 = \frac{-e^x}{y^2}$

$\therefore \frac{\partial M_1}{\partial y} = -\frac{e^x}{y^2}$, $\frac{\partial N_1}{\partial x} = -\frac{e^x}{y^2}$

Since $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\therefore$ eq (2) is exact D.E

Hence the general Solution is

$\int M_1 dx + \int$ (terms independent of x in N_1) $dy = 0$

(y constant)

$\int \left(x + \frac{e^x}{y}\right) dx + \int 0 . dy = c$

$$\boxed{\frac{x^2}{2} + \frac{e^x}{y} = c}$$

which is the general Solution.

Linear Differential equations of first Order

(A)

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where P and Q are either constants or function of y only is called a linear differential eqn of first order in y.

Working Rule:—

To solve the linear eqn $\frac{dy}{dx} + P(x)y = Q(x)$

(i) Write the integrating factor $(I.F) = e^{\int P(x)dx}$.

(ii) The General Solution is given by.

$$\boxed{y \times (I.F) = \int Q(x) \times (I.F)\,dx + C} \text{ or } \boxed{y.e^{\int P(x)dx} = \int Q(x).e^{\int P(x)dx}.dx + C}$$

Note: An eqn of the form $\frac{dx}{dy} + P(y)x = Q(y)$ where P and Q are either constants or function of x only is called a linear differential eqn of first order in x.

Working Rule: To solve the linear eqn $\frac{dx}{dy} + P(y)x = Q(y)$

(i) Write the integrating factor $(I.F) = e^{\int P(y)dy}$

(ii) The general solution is given by

$$\boxed{x \times (I.F) = \int Q(y) \times (I.F)\,dy + C}$$

find the integrating factor for the following equations

① $\frac{dy}{dx} = e^{2x} + y - 1$

Sol Given eqn is $\frac{dy}{dx} = e^{2x} + y - 1$

This can be written as $\frac{dy}{dx} - y = e^{2x} - 1$

This is of the for $\frac{dy}{dx} + P(x)y = Q(x)$

where $P = -1$ and $Q = e^{2x} - 1$

$I.F = e^{\int P(x)dx} = e^{\int -1 dx} = e^{-\int 1.dx} = e^{-x}.$

$\boxed{I.F = e^{-x}}$

Ass ① $\frac{dy}{dx} + xy = x$

② $\frac{dy}{dx} - \frac{1}{x+1} y = e^{3x}(x+1)$

IMP

*** ① Solve $(1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

Sol Given eqn is $(1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

This can be written as $\frac{dx}{dy}(1+y^2) + x - e^{\tan^{-1}y} = 0$

Dividing with $(1+y^2)$ on both sides.

$$\frac{dx}{dy} \frac{\cancel{(1+y^2)}}{\cancel{1+y^2}} + \frac{x}{1+y^2} - \frac{e^{\tan^{-1}y}}{1+y^2} = 0$$

$$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{e^{\tan^{-1}y}}{1+y^2} \longrightarrow ①$$

This is of the form $\frac{dx}{dy} + Px = Q$ where P and Q are function of y and it is a first order linear eqn in x

Here $P = \frac{1}{1+y^2}$ and $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ General Solution is given by

$$x \cdot (I.F) = \int Q \times (I.F) dy + C$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} dy + C \longrightarrow (2)$$

Now put $\tan^{-1} y = t$ so that $\frac{1}{1+y^2} dy = dt$ $\left\{ \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \right\}$

Now eq (2) $\Rightarrow$ $x e^{\tan^{-1} y} = \int e^t \cdot e^t \cdot dt + C$

$$x e^{\tan^{-1} y} = \int e^{2t} dt + C \qquad \left\{ \int e^{ax} dx = \frac{e^{ax}}{a} \right\}$$

$$x e^{\tan^{-1} y} = \frac{e^{2t}}{2} + C$$

$$\boxed{x e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C}$$

This is the general solution of the given eqn.

(2) Solve: $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{\sin 2x}{\log x}$

Sol given eqn is $\frac{dy}{dx} + \frac{1}{x \log x} \cdot y = \frac{\sin 2x}{\log x}$

This is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{x \log x}$ and $Q = \frac{\sin 2x}{\log x}$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} \cdot \frac{1}{\log x} dx} = e^{\log_e (\log x)} = \log x$$

$$\left\{ \int \frac{f'(x)}{f(x)} dx = \log f(x) + C \right\}$$

$I.F = \log x$

~~Multiplying eq① with I.F i.e log x, we get~~

~~log x {~~ The General Solution is given by

$$y \times (I.F) = \int Q.(I.F)\,dx + c$$

$$y.\log x = \int \frac{\sin 2x}{\cancel{\log x}}.\cancel{\log x}.dx + c$$

$$y.\log x = \int \sin 2x\,dx + c$$

$$y.\log x = -\frac{\cos 2x}{2} + c \qquad \left\{ \int \sin ax\,dx = -\frac{\cos ax}{a} + c \right\}$$

$$\boxed{y.\log x = -\frac{\cos 2x}{2} + c}$$

Bernoulli's equation :—

An eqn of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's equation if P and Q are constants or functio[n] of x alone and n is a real constant.

(2) Solve: $x\frac{dy}{dx}+y=x^3y^6$ (19) (16)

Sol: Given eqn is $x.\frac{dy}{dx}+y=x^3y^6$

This can be written as $\frac{dy}{dx}+\frac{y}{x}=x^2y^6 \longrightarrow (1)$

This is of the form $\frac{dy}{dx}+P.y=Qy^n$

where $P=1/x$, $Q=x^2$ and $n=6$

Dividing with y^6, we get

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{y^5}\frac{1}{x}=x^2 \longrightarrow (2)$$

Let $\frac{1}{y^5}=u$ Then $-\frac{5}{y^6}\frac{dy}{dx}=\frac{du}{dx}$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx}=-\frac{1}{5}\frac{du}{dx}$$

Substituting in eq (1), we get

$$-\frac{1}{5}.\frac{du}{dx}+u.\frac{1}{x}=x^2$$

$$\frac{du}{dx}-\frac{5u}{x}=-5x^2 \longrightarrow (3)$$

This is a linear eqn in u.

Now $I.F=e^{\int-\frac{5}{x}dx}=e^{-5\int\frac{1}{x}dx}=e^{-5\log x}$

$$I.F=e^{\log x^{-5}}=x^{-5}=\frac{1}{x^5}$$

The general solution of eq (3) is given by

$$u(I.F)=\int Q(I.F)dx+c$$

$$u.\frac{1}{x^5}=-5\int x^2.\frac{1}{x^5}dx+c$$

Q Solve $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

Sol: Given eqn is $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

This can be written as $\frac{dx}{dy}(1+y^2)+x-e^{\tan^{-1}y}=0$

$$\frac{dx}{dy}+\frac{x}{1+y^2}-\frac{e^{\tan^{-1}y}}{1+y^2}=0$$

$$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{e^{\tan^{-1}y}}{1+y^2}$$

Repeat

This is of the form $\frac{dx}{dy}+Px=Q$ where P and Q are functions of y and it is a first order linear equation in x.

Here $P=\frac{1}{1+y^2}$ and $Q=\frac{e^{\tan^{-1}y}}{1+y^2}$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\left\{ \because \int \frac{1}{1+x^2}dx = \tan^{-1}(x)+c \right\}$

General solution is

$$x \times I.F = \int Q \times (I.F)\,dy + c$$

$$x.e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2}.e^{\tan^{-1}y}\,dy+c$$

put $\tan^{-1}y = t$ so that $\frac{1}{1+y^2}.dy = dt$

$\left\{ \int e^{ax}dx = \frac{e^{ax}}{a}+c \right\}$

$$xe^{\tan^{-1}y} = \int (e^t)^2.dt + c$$

$$xe^{\tan^{-1}y} = \frac{e^{2t}}{2}+c \Rightarrow \boxed{xe^{\tan^{-1}y} = \frac{e^{2\tan^{-1}y}}{2}+c}$$

This is the general solution of the given eqn.

M. IMP

*Q Solve: $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ (20)

Solution:

Given eqn is $\sec^2 y \frac{dy}{dx} + 2x\tan y = x^3 \longrightarrow (1)$

put $\tan y = u$ So that $\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$

Substituting these values in eq (1), we get

$$\frac{du}{dx} + 2ux = x^3 \longrightarrow (2)$$

This is a linear equation in u.

Here $P = 2x$ and $Q = x^3$

Now $I.F = e^{\int 2x dx} = e^{2\int x dx} = e^{2 \cdot \frac{x^2}{2}} = e^{x^2}$

The general solution of eq (2) is

$$u \cdot e^{x^2} = \int x^3 \cdot e^{x^2} dx + c$$

$$u e^{x^2} = \int x^3 \cdot e^{x^2} dx + c$$

Put $x^2 = t$ so that $x dx = \frac{1}{2} dt$

$$u e^{x^2} = \int x^2 e^{x^2} \cdot x dx + c$$

$$\Rightarrow u e^{x^2} = \int t \cdot e^t \cdot \frac{1}{2} dt + c \quad \{\int u dv = uv - u'v_1 - \cdots\}$$

$$u e^{x^2} = \frac{1}{2}[t e^t - e^t] + c$$

$$u e^{x^2} = \frac{1}{2} e^t (t-1) + c$$

$$\tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c \quad \{\because t = x^2\} \{u = \tan y\}$$

which is the required Solution of eq (1)

$$u \cdot \frac{1}{x^5} = -5 \int \frac{x^2}{x^5} dx + c$$

$$u \cdot \frac{1}{x^5} = -5 \int \frac{1}{x^3} dx + c$$

$$\frac{u}{x^5} = -5 \left[\frac{x^{-2}}{-2} \right] + c$$

$$u \cdot \frac{1}{x^5} = \frac{5}{2x^2} + c.$$

$$\frac{1}{y^5} \cdot \frac{1}{x^5} = \frac{5}{2x^2} + c \quad \left\{ \because u = \frac{1}{y^5} \right\}$$

$$\frac{1}{y^5} = \frac{5x^5}{2x^2} + cx^5$$

$$\frac{1}{y^5} = \frac{5 \cdot x^3}{2} + cx^5$$

which is the Required Solution

Method: Differential equations Reducible to linear equation by Substitution. method

(21)

(2) Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

Sol Given eqn is $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y \longrightarrow (1)$

This can be written as $\frac{1}{\cos^2 y}\cdot\frac{dy}{dx} + \frac{x\cdot 2\sin y\cos y}{\cos^2 y} = x^3$

$$\sec^2 y\cdot\frac{dy}{dx} + x\cdot 2\tan y = x^3$$

$$\sec^2 y\cdot\frac{dy}{dx} + 2x\tan y = x^3 \longrightarrow (2)$$

put $\tan y = u$ so that $\sec^2 y\cdot\frac{dy}{dx} = \frac{du}{dx}$

Substituting in eq (2), we get $\frac{du}{dx} + 2u\cdot x = x^3$

This is a linear eqn in u.

Now $I.F = e^{\int P dx} = e^{\int 2x dx} = e^{2\int x dx} = e^{2\left[\frac{x^2}{2}\right]} = e^{x^2}$

$$I.F = e^{x^2}$$

∴ General Solution is given by.

$$u\cdot I.F = \int Q\cdot I.F\, dx + c.$$

$$u\cdot e^{x^2} = \int x^3 e^{x^2} dx + c$$

put $x^2 = t$ so that $2x\,dx = dt \Rightarrow x\,dx = \frac{dt}{2}$

$$u e^{x^2} = \int x^2\cdot e^{x^2}\cdot x\,dx + c$$

$$u e^{x^2} = \int t\cdot e^t\cdot\frac{dt}{2} + c.$$

$$u e^{x^2} = \frac{1}{2}\int t e^t dt + c$$

$$u e^{x^2} = \frac{1}{2}\left[t e^t - 1\cdot e^t\right] + c$$

$$u e^{x^2} = \frac{1}{2} e^t(t-1) + c$$

$$\tan y\cdot e^{x^2} = \frac{1}{2}e^{x^2}(x^2-1) + c$$

This is the general solution of eq (1)

Newton's Law of cooling

Statement :—

The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let θ be the temperature of the body at time t. and θ_0 be the temperature of its surrounding medium (usually air)

By the Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto (\theta-\theta_0)$$

i.e $\frac{d\theta}{dt} = -K(\theta-\theta_0)$ where K is a positive constant

$$\frac{d\theta}{\theta-\theta_0} = -K\,dt \quad \text{(variable separable)}$$

Integrating on both sides we get

$$\int \frac{1}{\theta-\theta_0}\,d\theta = \int -K\,dt$$

$$\log(\theta-\theta_0) = -Kt + C \longrightarrow (1)$$

If initially $\theta = \theta_1$ is the temperature of the body at time $t=0$, then eq (1) gives

$$\log(\theta_1-\theta_0) = 0 + C \Rightarrow C = \log(\theta_1-\theta_0) \longrightarrow (2)$$

Substituting eq (2) in eq (1) we get

$$\log(\theta-\theta_0) = -Kt + \log(\theta_1-\theta_0)$$

$$\log(\theta-\theta_0) - \log(\theta_1-\theta_0) = -Kt$$

$$\log\left(\frac{\theta-\theta_0}{\theta_1-\theta_0}\right) = -Kt$$

$$\frac{\theta-\theta_0}{\theta_1-\theta_0} = e^{-Kt} \Rightarrow \theta-\theta_0 = (\theta_1-\theta_0)e^{-Kt}$$

$$\theta = \theta_0 + (\theta_1-\theta_0)e^{-Kt}$$

which gives the temperature of the body at any time t.

① IMP ** A body is originally at 80°c and cools down to 60°c in 20 minutes. If the temperature of the air is 40°C. find the temperature of the body after 40 minutes. (22)

Sol: Let θ be the temperature of the body at time t.
and θ_0 be the temperature of the air $(\because \theta_0 = 40°c)$

By Newton's law cooling, we have.

$$\frac{d\theta}{dt} = -K(\theta-\theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta-40)$$

$$\frac{d\theta}{\theta-40} = -Kdt$$

Integrating on both sides, we get

$$\int \frac{1}{\theta-40}d\theta = -K\int dt$$

$$\log(\theta-40) = -Kt + \log c \Rightarrow \log\left(\frac{\theta-40}{c}\right) = -Kt$$

$$\frac{\theta-40}{c} = e^{-Kt}$$

$$\theta-40 = ce^{-Kt} \longrightarrow ①$$

When $t=0, \theta=80°c \Rightarrow 80-40 = ce^{-K(0)}$

$$40 = c(1)$$

$$\boxed{c=40}$$

and when $t=20, \theta=60°c \Rightarrow$ eq ① $\Rightarrow 60-40 = ce^{-20K}$

$$20 = ce^{-20K}$$

$$e^{-20K} = \frac{20}{c}$$

$$e^{-20K} = \frac{20}{40_2} \Rightarrow e^{-20K} = \frac{1}{2}$$

$$-20K = \log\frac{1}{2}$$

$$-20K = \log 1 - \log 2$$
$$-20K = 0 - \log 2$$
$$20K = \log 2$$
$$\boxed{K = \frac{1}{20}\log 2}$$

eq ① becomes $\theta - 40 = 40 e^{[-\frac{1}{20}\log 2]t}$

$$\theta = 40 + 40 e^{(-1/20 \log 2)t} \longrightarrow ②$$

when $t = 40 \Rightarrow$ eq ② $\rightarrow \theta = 40 + 40 e^{-(1/20 \log 2) 40}$

$$\theta = 40 + 40 e^{-2\log 2}$$
$$\theta = 40 + 40 e^{\log_e 2^{-2}} \quad \{ e^{\log_e a} = a \}$$
$$\theta = 40 + 40 \cdot 2^{-2}$$
$$\theta = 40 + 40 \left(\frac{1}{4}\right)$$
$$\boxed{\theta = 50^\circ C}$$

Hence the temperature of the body after 40 minutes is $50^\circ C$.

② If the air temperature of the air is $20^\circ C$ and the temperature of the body drops from $100^\circ C$ to $80^\circ C$ in 10 minutes. what will be its temperature after 20 minutes. when will be the temperature $40^\circ C$.

Sol Let θ be the temperature of the body at time t. and θ_0 be the temperature of the air $(\theta_0 = 20^\circ C)$

By Newton's law of cooling.

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\frac{d\theta}{\theta-\theta_0} = -k\,dt$$

$$\frac{d\theta}{\theta-20} = -k\,dt$$

$$\int \frac{d\theta}{\theta-20} = -k\int 1.dt$$

$$\log(\theta-20) = -kt+\log c$$

$$\log(\theta-20)-\log c = -kt$$

$$\log\left(\frac{\theta-20}{c}\right) = -kt$$

$$\frac{\theta-20}{c} = e^{-kt}$$

$$\theta-20 = ce^{-kt} \longrightarrow (1)$$

Originally when $t=0$, temperature of $100^\circ C$

eq (1) $\Rightarrow$ $100-20 = ce^{-k(0)}$

$$80 = c(1)$$

$$\boxed{c=80}$$

when $t=10$ min $\theta = 80^\circ C$ $\Rightarrow$ eq (1) $\rightarrow$ $80-20 = ce^{-k(10)}$

$$60 = ce^{-k(10)}$$

$$60 = 80(e^{-k10})$$

$$e^{-k(10)} = \frac{60}{80}\;\frac{3}{4}$$

$$\boxed{e^{-10k} = \frac{3}{4}} \longrightarrow (2)$$

when $t=20$ min $\rightarrow$ eq (1) $\Rightarrow$ $\theta-20 = ce^{-k(20)}$

$$\theta-20 = ce^{-20k}$$

$$\theta-20 = 80(e^{-10k})^2$$

$$\theta-20 = 80(3/4)^2$$

$$\theta = 80\left(\frac{9}{16}\right)+20$$

$$\theta = 45^\circ+20^\circ$$

$$\boxed{\theta = 65^\circ C}$$

Thus the temperature of the body will be 65°C after 20 minutes

when $\theta = 40°C \Rightarrow$ eq 2 $\Rightarrow$ $40 - 20 = Ce^{-kt}$

$$20 = 80e^{-kt}$$

$$\frac{20}{80_4} = (e^{-k})^t$$

$$1/4 = (e^{-k})^t \longrightarrow 4$$

from eq 2 and 4, we have

eq 3 $= e^{-10k} = 3/4 \Rightarrow (e^{-k}) = (3/4)^{1/10}$

$$(e^{-k})^t = 1/4$$

$$\left((3/4)^{1/10}\right)^t = 1/4$$

$$(3/4)^{t/10} = 1/4$$

Taking log on both sides

$$\log(3/4)^{t/10} = \log 1/4$$

$$\frac{t}{10}\log(3/4) = \log(1/4)$$

$$t = \frac{10\log(1/4)}{\log(3/4)}$$

$$\boxed{t = 48.2 \text{ minutes}}$$

The temperature of the body will be 40°C after 48.2 minutes.

SMP ** (21)

(1) The rate of cooling of an object is proportional to the difference b/w the temperature of the object and the surrounding air. If their air temperature is 30°c and the water at a temperature 100°c cools at 80°c in 10 minutes, find when the temperature of water will become 40°c. (2M)

Sol: Let θ be the temperature of the water at time t. and θ_0 be the temperature of air ($\theta_0 = 30°c$)

By Newton's law of cooling

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - 30)$$

$$\frac{d\theta}{\theta - 30} = -K\,dt \longrightarrow (1)$$

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - 30} = -K\int 1\,dt$$

$$\log(\theta - 30) = -Kt + \log c$$

$$\log\left(\frac{\theta - 30}{c}\right) = -Kt$$

$$\frac{\theta - 30}{c} = e^{-Kt}$$

$$\theta - 30 = ce^{-Kt} \longrightarrow (2)$$

when $t = 0$, $\theta = 100° \Rightarrow 100 - 30 = ce^{-K(0)}$

$$70 = c(1)$$

$$\boxed{c = 70°}$$

when $t = 10$ minutes, $\theta = 80°c$

eq (2) $\Rightarrow$ $80 - 30 = ce^{-K(10)}$

GSO

$$50 = c e^{-k10}$$

$$c e^{-k10} = 50$$

$$e^{-k10} = \frac{50}{70}$$

$$e^{-k10} = 5/7 \Rightarrow (e^{-k})^{10} = 5/7$$

$$\boxed{\begin{aligned} -k10 &= \log(5/7) \\ k &= -1/10 \log(5/7) \\ k &= \log(5/7)^{1/10} \end{aligned}}$$

$$e^{-k} = (5/7)^{1/10}$$

eq ② becomes $\theta - 30 = c e^{-kt}$

when $\theta = 40°c$

$$40 - 30 = c(5/7)^{t/10}$$

$$10 = 70(5/7)^{t/10}$$

$$1/7 = (5/7)^{t/10}$$

$$\boxed{\cancel{(1/7)^{10} = (5/7)^{t}}}$$ Taking log on both sides.

$$\log(1/7) = \log(5/7)^{t/10}$$

$$\log 1 - \log 7 = \frac{t}{10}\log(5/7)$$

$$\frac{10(\log 1/7)}{\log(5/7)} = t$$

$$\frac{-8.4509}{-0.1461} = t$$

$$t = 57.84$$

$$t = 57.84 \text{ minutes}$$

∴ The Temperature of water will be 40°c after 57.84 minutes

$50 = Ce^{-K10}$

25

$$Ce^{-10K} = 50$$

$$e^{-10K} = \frac{50}{C} \Rightarrow \frac{50}{70}$$

$$e^{-10K} = 5/7$$

$$(e^{-K})^{10} = 5/7$$

$$(e^{-K}) = (5/7)^{1/10} \longrightarrow (3)$$

eq (2) $\Rightarrow$ $\theta - 30 = Ce^{-kt}$

when $\theta = 40°C$

$$40 - 30 = Ce^{-kt}$$

$$10 = 70(e^{-K})^t$$

$$\frac{10}{70} = \left((5/7)^{1/10}\right)^t$$

$$\left(\frac{5}{7}\right)^{t/10} = 1/7$$

Taking log on both sides we get

$$\log(5/7)^{t/10} = \log 1/7$$

$$\frac{t}{10}\log(5/7) = \log(1/7)$$

$$\frac{t}{10} = \frac{\log(1/7)}{\log(5/7)}$$

$$t = \frac{10 \cdot \log(1/7)}{\log(5/7)}$$

$$t = 57.84 \text{ minutes}$$

$\therefore$ The temperature of water will be 40°C after 57.84 min

① Method 2:

If $M(x,y)dx + N(x,y)dy = 0$ is a homo

Date: 18-5-2022

① D ~~5~~, ~~6~~, ~~7~~, ~~8~~, ~~9~~
E ~~0~~, ~~1~~, ~~2~~, 3, 4, 5, 6, ~~7~~, 8, ~~9~~
F 0, 1, ~~2~~, ~~3~~, 4, 5, ~~6~~, ~~7~~, ~~8~~, ~~9~~
G ~~0~~, 1, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, 7, ~~8~~, 9
H 0, 1, 2, 3, 4, 5, 6, ~~7~~, 8, 9
I ~~0~~, 1, 2, ~~3~~, ~~4~~, 5, 6, 7, ~~8~~, ~~9~~
~~J~~ 0, ~~1~~, ~~2~~, ~~3~~, ~~4~~.
2~~0~~1, 20~~2~~, 203, 204, 205, 206, 207, 2~~0~~8. (34)

Date: 19-5-2022.

D ~~5~~ ~~6~~ ~~7~~ ~~8~~ 9
E 0 ~~1~~, ~~2~~, ~~3~~, ~~4~~, 5, 6, ~~7~~, ~~8~~, ~~9~~
F 0, 1, 2, ~~3~~, 4, 5, ~~6~~, ~~7~~, ~~8~~, ~~9~~
G 0 ~~1~~, ~~2~~, ~~3~~, ~~4~~, 5, ~~6~~, ~~7~~, ~~8~~, 9
H ~~0~~, 1, ~~2~~, ~~3~~, 4, ~~5~~, 6, ~~7~~, ~~8~~, 9 Dinesh P
I ~~0~~, 1, 2, ~~3~~, 4, 5, ~~6~~, 7, ~~8~~, ~~9~~
~~J~~ ~~0~~, ~~1~~, 2, ~~3~~, ~~4~~
201, ~~2~~, 3, 4, ~~5~~, 6, 7, ~~8~~.

Rate of Decay of RadioActive materials

The disintegration at any instant is proportional to the amount of material present

If x is the amount of the material at any time t, then $\frac{dx}{dt} = -kx$, where k is a constant.

IMP

*** (1) If 30% of a radio active Substance disappears in 10 days, how long will it take for 90% of it to disappear?

Sol

The differential equation of the diffusing radio active material is, $\frac{dm}{dt} = -km \longrightarrow (1)$

$$\frac{dm}{m} = -k\,dt$$

Integrating on both Sides

$$\int \frac{dm}{m} = -k\int dt$$

$$\log m = -kt + \log c$$

$$\log m - \log c = -kt$$

$$\log(m/c) = -kt$$

$$\frac{m}{c} = e^{-kt}$$

$$m = ce^{-kt} \longrightarrow (2)$$

when $t = 0$, let $m = m_1$

$\Rightarrow$ eq (2) $\Rightarrow$ $m_1 = ce^{-0k}$

$$\boxed{m_1 = c}$$

By Given data when $t = 10$, $m = \frac{70m_1}{100}$

eq (2) $\Rightarrow$ $\frac{7\cancel{0}\cancel{m_1}}{10\cancel{0}} = \cancel{m_1}e^{-k10}$

$$e^{-10k} = \frac{7}{10} \Rightarrow -10k = \log(7/10)$$

$$-10K = \log\left(\frac{7}{10}\right)$$

$$K = \frac{1}{-10}\log\left(\frac{7}{10}\right)$$

$$K = \frac{1}{10}\log\left(\frac{7}{10}\right)^{-1}$$

$$\boxed{K = \frac{1}{10}\log\left(\frac{10}{7}\right)}$$

Required time at t is

when $m = \frac{10m_1}{100}$

of ②
$$\frac{10m_1}{100} = ce^{-Kt}$$

$$\frac{\cancel{10}\cancel{m_1}}{10\cancel{0}} = m_1 e^{-Kt}$$

$$\frac{1}{10} = e^{-Kt}$$

$$e^{-Kt} = \frac{1}{10}$$

$$-Kt = \log\left(\frac{1}{10}\right)$$

$$t = \frac{-1}{K}\log\left(\frac{1}{10}\right)$$

$$t = \frac{1}{K}\log\left(\frac{1}{10}\right)^{-1}$$

$$t = \frac{1}{K}\log(10)$$

$$t = \frac{\log(10)}{1/10\log\left(\frac{10}{7}\right)}$$

$$t = \frac{10.\log(10)}{\log\left(\frac{10}{7}\right)}$$

$$\boxed{t = 64.5 \text{ days}}$$

═ * ─

Law of Natural Growth or Decay

Let $x(t)$ be the amount of a substance at time t. and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance at that time.

$$\frac{dx}{dt} \propto x$$

i.e $\frac{dx}{dt} = Kx$ where K is a constant of proportionality

$\Rightarrow$ For growth $\frac{dx}{dt}$ should be positive.

$$\boxed{\therefore \frac{dx}{dt} = Kx, \text{ where } K>0}$$

$\Rightarrow$ For decay, $\frac{dx}{dt}$ should be Negative.

$$\boxed{\therefore \frac{dx}{dt} = -Kx, \text{ where } K>0}$$

① A bacterial culture, growing exponentially, increases from 200 to 500 grams in the period from 6 am to 9 a.m. How many grams will be present at noon.

Sol Let N be the number of bacteria in a culture at any time $t>0$, Then

The differential eqn is $\frac{dN}{dt} = KN$

$$\frac{dN}{N} = K\,dt$$

Integrating on both sides, we get

$$\int \frac{dN}{N} = \int K\,dt$$

$$\log N = Kt + \log c$$

$$\log N - \log c = Kt$$

$$\log\left(\frac{N}{c}\right) = Kt$$

$$\frac{N}{c} = e^{Kt}$$

$N = ce^{Kt}$ where c is a Const → ① and K, the Rate constant

Given that $N = 200$ grams when $t = 0$

$$200 = ce^{+K(0)}$$

$$\boxed{c = 200}$$

Thus, we have $N = 200e^{+Kt} \rightarrow$ ②

But when $t = 3$ hours (from 6am to 9am) $N = 500$ grams

Substituting in eq ②, we get

$$500 = 200e^{+3K}$$

$$e^{3K} = \frac{500}{200} \Rightarrow e^{3K} = 5/2$$

$$3K = \log(5/2) \text{ or } K = \frac{1}{3}\log(5/2)$$

$$\boxed{K = 0.3054}$$

Hence the number of bacteria in the culture at any ~~int~~ instant of time $t > 0$ is given by

$$N = 200e^{0.3054t}$$

when $t = 6$ hours (from 6am to 12am)

∴ After 3 hours, the no. of bacteria present will be

$$N = 200e^{0.3054(6)}$$

$N = 1249.8$ grams.

= * =

2. Differential Equations of first order But not of the first Degree

(28)

~~Equations solvable for p:-~~

Def: The differential eqn will involve $\frac{dy}{dx}$ in higher degree and $\frac{dy}{dx}$ is denoted by 'p'. So the diff eqn is of the form $f(x, y, p) = 0$ is called first order non-linear diff eqn such eqns can be solved by following methods.

1) Equation solvable for p.
2) Equation solvable for y
3) Equation solvable for x.

Problems Equation solvable for p

① $p^2 - 5p + 6 = 0$, where $p = \frac{dy}{dx}$

Sol Given eqn $p^2 - 5p + 6 = 0$

$$p^2 - 3p - 2p + 6 = 0$$

$$p(p-3) - 2(p-3) = 0$$

$$(p-2)(p-3) = 0$$

$$p = 2 \text{ and } p = 3$$

So $\frac{dy}{dx} = 2$ and $p = \frac{dy}{dx} = 3$

$$dy = 2dx \text{ and } dy = 3dx$$

Integrating, $\int dy = \int 2dx$ and $\int dy = \int 3dx$

$$y = 2x + c \text{ and } y = 3x + c.$$

$$y - 2x - c = 0, \quad y - 3x - c = 0$$

$\therefore$ The required solution is $(y-2x-c)(y-3x-c) = 0$

Solve

(2) $P(P+y) = x(x+y)$

Sol: Given diff eq^n is $P(P+y) = x(x+y)$

$$P^2 + Py = x^2 + xy$$
$$P^2 + Py - x^2 - xy = 0$$
$$(P^2 - x^2) + Py - xy = 0$$
$$(P+x)(P-x) + y(P-x) = 0$$
$$(P-x)[P+x+y] = 0$$

its components are $P-x=0$ and $P+x+y=0$

$$P = x \text{ and } P+y = -x$$

$$\underbrace{\frac{dy}{dx} = x}_{①} \text{ and } \underbrace{\frac{dy}{dx} + y = -x}_{②}$$

eq ① $\frac{dy}{dx} = x \Rightarrow dy = x\,dx$

Integrating, $\int dy = \int x\,dx$

$$y = \frac{x^2}{2} + c$$
$$2y = x^2 + 2c$$
$$2y - x^2 = c$$
$$(2y - x^2 - c) = 0$$

eq ② This is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I.F = e^{\int P dx} = e^{\int 1 dx} = e^x$$

Solution is

$$y \cdot (I.F) = \int Q \cdot (I.F)\,dx + c$$
$$y \cdot e^x = \int (-x) e^x dx + c$$
$$ye^x = -\int x e^x dx + c$$
$$ye^x = -[xe^x - e^x] + c$$
$$ye^x + xe^x - e^x - c = 0$$
$$e^x(y+x-1) - c = 0$$
$$(y+x-1) - ce^{-x} = 0$$

Hence the general solution is

$$\boxed{(2y - x^2 - c)(y + x - 1 - ce^{-x}) = 0}$$

29

① Solve $p^2 + 2py\cot x = y^2$

Sol: Given $p^2 + 2py\cot x = y^2$

(or) $p^2 + (2y\cot x)p - y^2 = 0$, which is quadratic in p

Here $a = 1$, $b = 2y\cot x$, $c = -y^2$

Solving it for p, we get

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4\cdot 1\cdot(-y^2)}}{2(1)}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x) + 4y^2}}{2} \qquad \{\cot^2 x + 1 = \operatorname{cosec}^2 x\}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2} = \frac{-2y\cot x \pm 2y\sqrt{\operatorname{cosec}^2 x}}{2}$$

$$p = \cancel{-2y\cot \pm 2y} \quad p = \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$p = \frac{\cancel{2}[-y\cot x \pm y\operatorname{cosec} x]}{\cancel{2}}$$

$$p = -y\cot x \pm y\operatorname{cosec} x$$

Its components are

$p = -y\cot x + y\operatorname{cosec} x$ and $p = -y\cot x - y\operatorname{cosec} x$

$p = -y(\cot x - \operatorname{cosec} x) \rightarrow ①$ and $p = -y(\cot x + \operatorname{cosec} x) \rightarrow ②$

from eq ①, we have

$$p = \frac{dy}{dx} = -y\left[\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right]$$

$$\frac{dy}{y} = -\left[\frac{\cos x - 1}{\sin x}\right]dx$$

$\left\{ \begin{array}{l} 1 - \cos\theta = 2\sin^2\theta/2 \\ \text{and } \sin\theta = 2\sin\theta/2\cos\theta/2 \end{array} \right.$

$$\frac{dy}{y} = \left[\frac{1 - \cos x}{\sin x}\right]dx \Rightarrow \frac{dy}{y} = \left[\frac{\cancel{2}\sin^{\cancel{2}}(x/2)}{\cancel{2}\cancel{\sin(x/2)}\cos(x/2)}\right]dx$$

$$\frac{dy}{y} = \tan(x/2)\,dx$$

Integrating, $\int \frac{1}{y}dy = \int \tan(x/2)\,dx$ $\quad \{ \int \tan x\,dx = \log(\sec x)+c$

$$\log y = \frac{\log(\sec x/2)}{1/2} + \log c \quad \{ \int \tan ax\,dx = \frac{\log(\sec ax)}{a}+c$$

$$\log y = 2\log(\sec x/2) + \log c$$

$$\log y = \log(\sec x/2)^2 + \log c$$

$$\log y = \log \sec^2 x/2 \cdot c$$

$$\boxed{y = c \cdot \sec^2 x/2} \longrightarrow (3)$$

from eq (2), we have

$$\frac{dy}{dx} = -y[\cot x + \operatorname{cosec} x]$$

$$\frac{dy}{y} = -\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right]dx$$

$$\frac{dy}{y} = -\left[\frac{\cos x + 1}{\sin x}\right]dx \quad \left\{ \begin{array}{l} \cos\theta + 1 = 2\cos^2\theta/2 \\ \sin\theta = 2\sin\theta/2\cos\theta/2 \end{array} \right.$$

$$\frac{dy}{y} = \left[\frac{-[\cancel{2}\cos^{\cancel{2}} x/2]}{\cancel{2}\sin x/2 \cancel{\cos x/2}}\right]dx$$

$$\frac{dy}{y} = -\cot(x/2)\,dx$$

Integrate $\int \frac{1}{y}dy = -\int \cot x/2\,dx$ $\quad \left\{ \int \cot ax\,dx = \frac{\log(\sin ax)}{a} + c \right\}$

$$\log y = \frac{-\log(\sin x/2)}{1/2} + \log c$$

$$\log y = -2\log \sin x/2 + \log c$$

$$\log y = \log[\sin x/2]^{-2} + \log c$$

$$\log y = \log[\operatorname{cosec} x/2]^2 + \log c$$

$$\log y = \log c \cdot \operatorname{cosec}^2 x/2$$

$$\boxed{y = c \cdot \operatorname{cosec}^2 x/2} \longrightarrow (4)$$

Equation ③ and ④ ~~gam~~ constitute the required solution.

The required solution can be written as

$$\boxed{\left(y - c\sec^2 \tfrac{x}{2}\right)\left(y - c\cdot \operatorname{cosec}^2 \tfrac{x}{2}\right) = 0}$$

① Solve : $(p^2 - 7p + 12) = 0$

② Solve : $yp^2 + (x-y)p - x = 0$

Given problem is

Initially

at $t=0$, N – is the originally Substance conside

originall 100% we have.

then disapper 30% in how many day

$t=10$ day it is in the obvesily $\underline{30\% N}$

$t=?$ 10% N is available.

$$N = ce^{-kt}$$

$$N = ce^{-0}$$

$$\boxed{c = N_1}$$

101, 103, 108, 303, 2, 7.

① Solve $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$.

This is of the form $\frac{dy}{dx} + Py = Q$

here $P = \frac{1}{x}$, and $Q = x^3 - 3$.

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log_e^x} = x$$

$$I.F = x \qquad \{ e^{\log_e^a} = a \}$$

The general solution is given by

$$y(I.F) = \int Q \cdot (I.F) \cdot dx + C.$$

$$y \cdot x = \int (x^3 - 3) x \, dx + C.$$

$$yx = \int [x^4 - 3x] dx + C.$$

$$= \int x^4 dx - 3 \int x^1 dx + C$$

$$= \frac{x^{4+1}}{4+1} - 3 \cdot \frac{x^{1+1}}{1+1} + C \quad \left\{ \int x^n dx = \frac{x^{n+1}}{n+1} + C \right.$$

$$\boxed{yx = \frac{x^5}{5} - \frac{3x^2}{2} + C}$$

This is the general solution of the given eqn.

Ans

① Solve $x\frac{dy}{dx} + y = \log x \Rightarrow xy = x(\log x) - x + 1$

② $(x+2y^3)\frac{dy}{dx} = y \Rightarrow 1/y = I.F$

③ $\frac{dy}{dx} + y\sec x = \tan x \Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + c$

① Solve $x \cdot \frac{dy}{dx} + y = \log x$ Ans $xy = x\log x - x + C$

② $\frac{dy}{dx} + y\sec x = \tan x$ Ans $y(\sec x + \tan x) = \sec x + \tan x - x + c.$

~~Sthud~~ the I.F for the ~~eq~~ n

③ $(x+2y^3)\frac{dy}{dx} = y$ Ans $I.F = 1/y.$

$(x^2+y^2+2x)dx + 2y\,dy = 0 \longrightarrow ①$

$M = x^2+y^2+2x \qquad N = 2y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Consider $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2y}[2y-0] = \frac{2y}{2y} = 1 = f(x)$

$I.F = e^{\int f(x)dx} = e^{\int 1\cdot dx} = e^{x}$

Multiplying eq ① with e^x

$(e^x x^2 + e^x y^2 + e^x 2x)\,dx + e^x 2y\,dy = 0$

$M_1 dx + N_1 dy = 0$

$M_1 = e^x x^2 + e^x y^2 + e^x 2x \qquad N_1 = e^x 2y$

$\frac{\partial M_1}{\partial y} = e^x(2y) = 2ye^x \qquad \frac{\partial N_1}{\partial x} = 2ye^x$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}.$$

Generial soln

① Solve $xy(1+xy^2)\frac{dy}{dx}=1.$

Sol: Given equation is $[xy+x^2y^3]\frac{dy}{dx}=1.$

This is can be written as $xy+x^2y^3=\frac{dx}{dy}$

$$\frac{dx}{dy}-xy=x^2y^3 \longrightarrow ①$$

This is a Bernoulli's eqn in x.

and dividing with x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{\cancel{x}y}{x^{\cancel{2}}}=\frac{\cancel{x^2}y^3}{\cancel{x^2}}$$

$$\frac{1}{x^2}\frac{dx}{dy}-y/x=y^3 \longrightarrow ②$$

put $1/x=u$ so that $-\frac{1}{x^2}\cdot\frac{dx}{dy}=\frac{du}{dy}$

Substituting in eq②, we get

$$-\frac{du}{dy}-u\cdot y=y^3$$

$$\frac{du}{dy}+uy=-y^3$$

This is a linear equation in u. Here $P=y$, $Q=-y^3$

Now $I.F=e^{\int P dy}=e^{\int y dy}=e^{y^2/2}$

The general solution is given by

$$u\cdot(I.F)=\int Q\cdot(I.F)\cdot dy+c$$

$$u\cdot e^{y^2/2}=\int -y^3\cdot e^{\frac{y^2}{2}}dy+c$$

put

$-\int y^3 e^{y^2/2}dy$

$-\int y^3 e^{y^2/2}$ put $y^2=t$

$2y\,dy=dt$

$y\,dy=\frac{dt}{2}$

$-\int y^2 e^{t/2}\cdot\frac{dt}{2}$

$-\frac{1}{2}\int t e^{t/2}dt$

$-\frac{1}{2}\left[t\frac{e^{t/2}}{1/2}-1\cdot\frac{e^{t/2}}{1/2\cdot 1/2}\right]$

$=-\frac{1}{2}[2t-4]e^{t/2}$

$=[2-t]e^{t/2}$

① Solve $xy[1+xy^2]\frac{dy}{dx}=1$

Sol Given eqn is $[xy+x^2y^3]\frac{dy}{dx}=1 \longrightarrow ①$

This is can be written as $xy+x^2y^3=\frac{dx}{dy}$

$$\frac{dx}{dy}-xy=x^2y^3$$

This is a Bernoulli's eqn in x

and dividing with x^2, we get

$$\frac{1}{x^2}\frac{dx}{dy}-\frac{xy}{x^{\cancel{2}}}=\frac{x^{\cancel{2}}y^3}{x^{\cancel{2}}}$$

$$\frac{1}{x^2}\frac{dx}{dy}-y/x=y^3 \longrightarrow ②$$

put $1/x=u$ So that $-1/x^2\frac{dx}{dy}=\frac{du}{dy}$

Substituting in eq ②, we get

$$-\frac{du}{dy}-u.y=y^3$$

$$\frac{du}{dy}+uy=-y^3$$

This is a linear equation in u, Here $p=y$, $Q=-y^3$

Now $I.F=e^{\int Pdy}=e^{\int ydy}=e^{y^2/2}$

The general Solution is given by.

$$u(I.F)=\int Q(I.F)\,dy+C.$$

$$u.e^{y^2/2}=\int(-y^3)e^{y^2/2}dy+C.$$

put $y^2=t$ So that $2ydy=dt \Rightarrow ydy=\frac{1}{2}dt$

$$u.e^{y^2/2}=-\int y^2.y\,e^{\frac{y^2}{2}}dy+c$$

$$u.e^{y^2/2}=-\int y^2e^{y^2/2}.ydy+c.$$

$$u e^{y^2/2} = -\int t e^{t/2} \cdot \tfrac{1}{2}\, dt + c$$

$$u e^{y^2/2} = -\frac{1}{2}\int t e^{t/2}\, dt + c \qquad \left\{ \begin{array}{l} \text{using Bernoullis rule} \\ \int u\,dv = uv - u'v_1 + u''v_2 - \cdots \end{array} \right.$$

$$= -\frac{1}{2}\left[t \cdot \frac{e^{t/2}}{1/2} - 1 \cdot \frac{e^{t/2}}{\frac{1}{2}\cdot\frac{1}{2}} \right] + c \qquad \left\{ \int e^{ax} dx = \frac{e^{ax}}{a} + c \right\}$$

$$= -\frac{1}{2}\left[2t e^{t/2} - 4 e^{t/2} \right] + c$$

$$= \frac{-\cancel{2} e^{t/2}}{\cancel{2}}\left[t - 2 \right] + c$$

$$= -e^{t/2}(t-2) + c$$

$$u e^{y^2/2} = e^{t/2}(2-t) + c$$

$$\frac{1}{x} e^{y^2/2} = e^{y^2/2}(2-y^2) + c \qquad \left\{ \begin{array}{l} t = y^2 \\ u = 1/x \end{array} \right\}$$

which is the general solution of eq ①

UNIT - I

① Solve $(x^3+3xy^2)dx+(y^3+3x^2y)dy=0$

② Solve the differential eqn $y(xy+e^x)dx-e^x dy=0$

③ Solve $[xy\sin xy+\cos xy]y\,dx+(xy\sin xy-\cos xy)x\,dy=0$

④ Solve $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$

⑤ Solve $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3$

⑥ A body is originally at 80°c and cools down to 60°c in 20 minutes If the temperature of the air is 40°c, then find the temperature of the body after 40 minutes.

⑦ If 30% of a radioactive substance disappears in 10days. how long will it take for 90% of it to disappears?

⑧ If the air temperature is 30°c and the water at a temperature 100°c cools to 80°c in 10 minutes, find, when the temperature of water will become 40°c.

⑨ Solve $p^2+2py\cot x=y^2$ for $p=\frac{dy}{dx}$.

~~⑩ form the differential equation of LR CIRCUIT~~

⑩ A resistance of 100 ohms, an inductance 0.5 Henry is connected in Series with a battery of 20 volts. Find the current in the circuit If initially there is no current in the circuit

⑪ An R-L circuit has an Emf given (in volts) by 10 sin t. a resistance of 90 ohms, an inductance of 4 henries. find the current at any time t by assuming zero initial current.