UNIT-IV

Multivariable calculus

{partial differentiation and Applications}

→ **Partial differentiation** : Let $z = f(x, y)$ be a function of a two variables x, y then partial derivati of $f(x, y)$ w.r.t x & y is given by

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}\right] = f_x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0}\left[\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\right] = f_y$$

→ **Higher order partial derivative** : The first order partial derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and also function x & y and they can be differentited repeadedly to get higher order partial derivative $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Third order :

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3}, \quad \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial y^3}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y \partial x^2}, \quad \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial x \partial y^2} \text{ etc}$$

① Find the first & second order partial derivative of x^3+y^3-3axy and ~~...~~ also to verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Sol Given $f(x,y) = x^3+y^3-3axy \rightarrow ①$

Differentiation eq① partially w.r.to x & y

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^3+y^3-3axy)$$

$$\frac{\partial f}{\partial x} = 3x^2-3ay \rightarrow ②$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3+y^3-3axy]$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax \rightarrow ③$$

differetiate eq2 partially w.r to x & y

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2-3ay] = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{x^2} = 6x$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2-3ay)$$

$$= -3a$$

differetiate eq 3 partially w.r to x & y

$$\frac{\delta}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}[3y^2-3ax) = -3a$$

$$\frac{\delta}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[3y^2-3ax) = 6y$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

$3a = 3a$

2. Find the second order partial derivative

① $\log(x^2+y^2)$

② $\tan^{-1}\left(\frac{x}{y}\right)$

① $\log(x^2+y^2)$

Sol Given

$\cancel{f(x)=}$ $f(x,y) = \log(x^2+y^2) \rightarrow ①$

Differentiation w.r.to x & y eq① partially

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\log(x^2+y^2)\right)$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}\left[2x+0\right]$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \rightarrow ②$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2+y^2)\right) = \frac{1}{x^2+y^2}\left[0+2y\right]$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} \rightarrow ③$$

Differentiate eq② w.r.to x & y.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) \qquad \left[\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right]$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{\left[x^2+y^2\right]2 - 2x\left[2x+0\right]}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2} \qquad \frac{\partial^2 f}{\partial x^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(2x) - 2x\frac{d}{dy}(x^2+y^2)'}{(x^2+y^2)^2} = \frac{x^2+y^2-2x[2y]}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{x^2+y^2-4xy}{(x^2+y^2)^2}$$

Differentiate eq (3) w.r to x & y

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2) - 2y[2x]}{(x^2+y^2)^2} \Rightarrow \boxed{\frac{x^2+y^2-4xy}{(x^2+y^2)^2} = \frac{\partial^2 f}{\partial x \partial y}}$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{2y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}[2y] - 2y\frac{d}{dy}[x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2) - 2y[2y]}{(x^2+y^2)^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$\boxed{\frac{\partial^2 f}{\partial y^2} = \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

2 ② $\tan^{-1}\left(\frac{x}{y}\right)$

Sol $f(x,y) = \tan^{-1}\left[\frac{x}{y}\right] \rightarrow ①$

Diff eq ① parially w.r to x & y

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\ \frac{y - x\frac{dy}{dx}(0)}{y^2} = \frac{1}{1+\frac{x^2}{y^2}}\,2x = \frac{2xy^2}{y^2+x^2}$$

$$= \frac{2y^3}{y^2+x^2}$$

$$= \frac{\cancel{y^2}}{y^2+x^2} \cdot \frac{y - \cancel{x\frac{dy}{dx}(0)}}{\cancel{y^2}}$$

$$= \frac{y}{x^2+y^2} \rightarrow ②$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= \frac{1}{1+\left(\frac{x}{y}\right)^2}\left[\frac{-x}{x^2}\right]$$

$$= \frac{-x}{x^2+y^2} \rightarrow ③$$

Diff eq ② w.r to x & y

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = y\left(\frac{1}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = y - \frac{1}{(x^2+y^2)^2}(2x)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{x^2+y^2(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

Diff eq ③ w.r to x & y

$$\frac{\partial}{\partial x}\left[\frac{-x}{x^2+y^2}\right] = \frac{(x^2+y^2)(-1)+x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial y}\left[\frac{-x}{x^2+y^2}\right] = \frac{x^2+y^2(0)+x(2y)}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$* * * * *$

2((1) $U = \log(x^3+y^3+z^3-3xyz)$ Prove that $\left[\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right]^2$

$$U = -\frac{9}{(x+y+z)^2}$$

Sol Given;

$$U = \log(x^3+y^3+z^3-3xyz) \rightarrow (1)$$

diff eq(1) partially w.r to x.

$$\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right] = \frac{\partial}{\partial x}\left[\log(x^3+y^3+z^3-3xyz)\right)$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\frac{\partial}{\partial x}\left[x^3+y^3+z^3-3xyz\right]$$

$$\frac{\partial U}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\left[3x^2-3yz\right]$$

$$\frac{\partial U}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

diff eq(1) partially w.r to y

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\log(x^3+y^3+z^3-3xyz)$$

$$= \frac{1}{x^3+y^3+z^3-3xyz}\left[3y^2-3xz\right]$$

$$\frac{\partial U}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}.$$

$\\|$ly $\frac{\partial U}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

Now

$$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+\frac{\partial U}{\partial z} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}+\frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}+\frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-yz-xz-xy)}}{(x+y+z)\cancel{(x^2+y^2+z^2-yz-zx-xy)}} = \frac{3}{(x+y+z)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

1) If $w = (y-z)(z-x)(x-y)$, then find the value of $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z}$

Sol:- Given;

$w = (y-z)(z-x)(x-y) \longrightarrow ①$

diff eq ① partial w.r to x & y

$$\frac{d}{dx}\left[\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left((y-z)(z-x)(x-y)\right)$$

$$= (y-z)\frac{\partial}{\partial x}\left((z-x)(x-y)\right)$$

$$= (y-z)\left[(0-1)(x-y) - (z-x)(1-0)\right)$$

$$= (y-z)(-x+y) + (z-x)$$

$$= (y-z)(-x+y+z-x)$$

$$= (y-z)(y+z-2x)$$

$$\frac{\partial w}{\partial x} = y^2 - z^2 - 2x(y-z)$$

Similarly $\frac{\partial w}{\partial y} = z^2 - x^2 - 2y(z-x)$

$$\frac{\partial w}{\partial z} = x^2 - y^2 - 2z(x-y)$$

Now

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = \cancel{y^2} - \cancel{z^2} - 2x(y-z) + \cancel{z^2} - \cancel{x^2} - 2y(z-x) + \cancel{x^2} - \cancel{y^2} - 2z(x-y)$$

$$= -2xy + 2xz - 2yz + 2xy - 2zx + 2zy$$

$$\boxed{\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0}$$ Hence the result

★★★★★

① $U = \log(x^3+y^3+z^3-3xyz)$ Prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2$

$U = -\frac{9}{(x+y-}$

Sol Given; $U = l($

diff eq(

$\frac{\partial}{\partial x}\left[\frac{\partial}{}\right.$

$\frac{\partial U}{\partial}$

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$\frac{\partial}{}$

diff eq①

$\frac{\partial U}{\partial y}$

$\frac{\partial U}{\partial y}$

lly :

Now

$\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}+$

2. If $U = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove that $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2} = 0$

Sol Given,

$U = \frac{1}{\sqrt{x^2+y^2+z^2}} \rightarrow ①$; $x^2+y^2+z^2 \neq 0$

$= (x^2+y^2+z^2)^{-1/2} \rightarrow ①$

p diff eq ① paritially w.r to x

$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2}$

chain Rule ⇒ $\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$

$\frac{d}{dx} x^n = nx^{n-1}$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1} \frac{\partial x}{\partial x}(x^2+y^2+z^2)$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x+0+0)$

$\frac{\partial U}{\partial x} = -x(x^2+y^2+z^2)^{-3/2} \longrightarrow ②$

Again diff w.r to x eq ②

$\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(-x(x^2+y^2+z^2)^{-3/2}\right)$

$= -\left(\frac{\partial}{\partial x}x(x^2+y^2+z^2)^{-3/2}\right)$

$= -\left[1(x^2+y^2+z^2)^{-3/2} + x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-3/2-1}\frac{\partial}{\partial x}(x^2+y\right.$

$= -\left[(x^2+y^2+z^2)^{-3/2} - \frac{3x}{2}(x^2+y^2+z^2)^{-5/2}(2x)\right]$

$= -\left[(x^2+y^2+z^2)^{-3/2} - 3x^2(x^2+y^2+z^2)^{-5/2}\right]$

$= -\left[(x^2+y^2+z^2)^{-3/2-1+1} - 3x^2(x^2+y^2+z^2)^{-5/2}\right]$

$\frac{\partial^2 U}{\partial x^2} = -\left[(x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2)^1 - 3x^2(x^2+y^2+z^2)^{-5/2}\right]$

$= -(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2-3x^2\right]$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$= -(x^2+y^2+z^2)^{-5/2}(-2x^2+y^2+z^2)$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{5/2}(2y^2-z^2-x^2)$$

$$\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2)$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$= (x^2+y^2+z^2)^{-5/2}[2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2]$$

$$= (x^2+y^2+z^2)^{-5/2}(0)$$

$$= 0$$

3. If $z = \log(e^x+e^y)$ show that $rt - s^2 = 0$
where $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

Sol Given;

$$z = \log(e^x+e^y) \rightarrow ①$$

diff eq① parially w.r to x

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\log(e^x+e^y)]$$

$$= \frac{1}{e^x+e^y}\frac{\partial}{\partial x}(e^x+e^y)$$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y} \rightarrow ②$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\log(e^x+e^y)]$$

$$\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y} \rightarrow ③$$

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① $U = \log(x^3+y^3+z^3-3xyz)$ Prove that $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2$

$U = -\frac{9}{(x+y+\dots}$

Sol Given; $U = \log\dots$

diff eq(

$\frac{\partial u}{\partial x}$

diff eq①

$\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial y}$

Illy

Now

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \dots$

Sol

differentiate eq② partially w.r to x

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{e^x}{e^x+e^y}\right)$$

$$= \frac{(e^x+e^y)\frac{\partial}{\partial x}e^x - e^x\frac{\partial}{\partial x}(e^x+e^y)}{(e^x+e^y)^2}$$

$$= \frac{(e^x+e^y)e^x - e^x(e^x+e^y)}{(e^x+e^y)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\cancel{e^{2x}} + e^{xy} - \cancel{e^{2x}}}{(e^x+e^y)^2} \qquad r = \frac{d^2 u}{dx^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

diff eq ③ partially w.r to x & y

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{e^y}{e^x+e^y}\right)$$

$$= \frac{(e^x+e^y)\frac{\partial}{\partial x}e^y - e^y\frac{\partial}{\partial x}(e^x+e^y)}{(e^x+e^y)^2} = \frac{(e^x+e^y)(0)-e^y\dots}{(e^x+e^y)^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{-e^{x+y}}{(e^x+e^y)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{e^y}{e^x+e^y}\right) = \frac{(e^x+e^y)e^y - e^y(0+\dots}{(e^x+e^y)^2}$$

$$= \frac{e^{x+y}+e^{2y}-e^{2y}}{(e^x+e^y)^2} \qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{e^{x+y}}{(e^x+e^y)^2}$$

Now $rt - s^2 = 0$

$$\frac{e^{x+y}}{(e^x+e^y)^2}\left(\frac{e^{x+y}}{(e^x+e^y)^2}\right) - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 = 0$$

$$\left(\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 - \left(-\frac{e^{x+y}}{(e^x+e^y)^2}\right)^2 = 0$$

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \dots$

Homogene...

A ...

degree n inva...

Ex:- f(x...

f(kx...

∴ f(x, ...

Euler's Th...

let z = f(...

x.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$\cancel{\frac{\partial u}{\partial x} + \frac{\partial u}{} \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{3}{x+y+z}\right)^2}$$

$$\cancel{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{9}{(x+y+z)^2}}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right]$$

$$= \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right]\left[\frac{3}{x+y+z}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} \qquad \left\{\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}\right\}$$

$$= \frac{-9}{(x+y+z)^2}$$

Homogeneous function :-

A $f(x,y)$ is said to be homogeneous function of degree 'n' in variable x, y if $f(kx, ky) = k^n f(x,y)$ {n is a real num?

Ex :- $f(x,y) = x^3 + y^3$

$$f(kx, ky) = (kx)^3 + (ky)^3$$
$$= k^3x^3 + k^3y^3$$
$$= k^3(x^3+y^3)$$

$$f(kx, ky) = k^3(f(x,y))$$

$\therefore f(x,y)$ is a homogeneous function of degree $n = 3$

Eulers Theorem on Homogeneous function :-

let $z = f(x,y)$ is a homogeneous function of order 'n' then

$$x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = n \cdot z$$

Similarly $u = f(x, y, z)$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \cdot \frac{\partial u}{\partial z} = n \cdot u$$

1. Verify Euler's theorem for the function $x \cdot y + yz + zx$

Sol Given

Let $f(x, y, z) = xy + yz + zx \rightarrow (1)$

$$f(kx, ky, kz) = (kx)(ky) + (ky)(kz) + (kz)(kx)$$
$$= k^2 xy + k^2 yz + k^2 zx$$
$$= k^2 (xy + yz + zx)$$

$$f(kx, ky, kz) = k^2 [f(x, y, z)]$$

$\therefore$ $f(x, y, z)$ is a homogenous function of order $n = 2$

By Euler's theorem;

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = n \cdot f \cdot$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = 2f$$

Verification:

diff eq(1) partially w.r to x, y & z

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [xy + yz + zx]$$
$$= y + z$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} [xy + yz + zx]$$
$$= x + z$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} [xy + yz + zx]$$
$$= x + y$$

$$x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = x(y+z) + y(x+z) + z(x+y)$$
$$= xy + xz + xy + zy + xz + yz$$
$$= 2xy + 2yz + 2zx$$
$$= 2[xy + yz + zx]$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z.\frac{\partial f}{\partial z}=2f(x,y,z)$$

Hence Euler's theorem is verified.

***** 2. Verify Euler's theorem for the function $U=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left[\frac{y}{x}\right]$

(or) if $U=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)$, S.T $x\cdot\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=0$

Sol

$$f(x,y)=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)\rightarrow ①$$

$$f(kx,ky)=\sin^{-1}\left[\frac{kx}{ky}\right)+\tan^{-1}\left(\frac{ky}{kx}\right)$$

$$=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{ky}{kx}\right)$$

$$f(kx,ky)=f(x,y)$$

$\therefore f(x,y)$ is a homogenous function of order $n=0$

By Euler's theorem

$$x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=n.f$$

Verification:

diff eq ① partially w.r to x, y

$$\frac{\partial U}{\partial x}=\frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)\right]$$

$$\frac{\partial U}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{1}{y}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{-y}{x^2}$$

$$=\frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}}\frac{1}{y}+\frac{1}{\frac{x^2+y^2}{x^2}}\frac{-y}{x^2}$$

$$=\frac{\cancel{y}}{\sqrt{y^2-x^2}}\frac{1}{\cancel{y}}+\frac{x^2}{x^2+y^2}\frac{-y}{x^2}$$

$$\frac{\partial U}{\partial x}=\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}$$

diff eq ① partially w.r to y

$$\frac{\partial U}{\partial y}=\frac{\partial}{\partial y}\left[\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2} + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

1.

So

$$= \frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} \cdot \frac{-x}{y^2} + \frac{1}{\frac{x^2+y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^{\cancel{2}}} + \frac{x^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Now;

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= x\left[\frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}\right] + y\left[\frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}\right]$$

$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{xy}{y\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$= 0$

Hence proved

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

* Reduction of Euler's Theorem :

let z is a homogenous function of x, y of order n and $z\ f(0)$ then

1. $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(0)}{f'(0)}$$

2. $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(0)\left[g'(0) - 1\right]$$

where $g(u) = \dfrac{n f(0)}{f'(0)}$

If $u = \log\left[\frac{x^3+y^3}{x+y}\right]$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$

since u is not a homogenuous function we write.

$$e^u = \frac{x^3+y^3}{x+y} = f(x, y) \text{ say}$$

Now $f(x,y) = \frac{x^3+y^3}{x+y}$

$$f(kx, ky) = \frac{(kx)^3+(ky)^3}{kx+ky} = \frac{k^{3}}{k}\left[\frac{x^3+y^3}{x+y}\right]$$

$$f(kx, ky) = k^2 f(x,y)$$

$\therefore f(x,y)$ is a homogenous function of order $n=2$

and also e^u is a homogenous function of order $n=2$

By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$$

$$x\frac{\partial}{\partial x}e^u + y\frac{\partial}{\partial y}e^u = 2e^u$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 2e^u$$

$$\cancel{e^u}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\cancel{e^u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$$ Hence the result.

① If $u = \log\left[\frac{x^4+y^4}{x+y}\right]$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

② If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

① S:- since u is not a homogenuous function we write

$$e^u = \frac{x^4+y^4}{x+y} = f(x,y) \text{ say}$$

Now $f(x,y) = \frac{x^4+y^4}{x+y}$

$$f(kx, ky) = \frac{(kx)^4+(ky)^4}{kx+ky} = \frac{k^{4}}{k}\left[\frac{x^4+y^4}{x+y}\right]$$

$f(kx, ky) = \cancel{k^{2/3}} \, k^3 f(x, y)$

$\therefore f(x,y)$ is a homogenous function of order $n = 3$

and also e^U is a homogenous function of order $n = 3$

1. By Euler's Theorem

Sol

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

$$x \frac{\partial f}{\partial x} e^U + y \frac{\partial f}{\partial y} = 3 e^U$$

$$\cancel{e^U}\left[x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right] = 3\cancel{e^U}$$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 3$$ Hence the result.

2 If $u = \log\left[\frac{x^2+y^2}{x+y}\right]$ show that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 1$

Sol so u is not a homogenous function we write

$$e^u = \cancel{\log}\left[\frac{x^2+y^2}{x+y}\right] = f(x,y) \text{ say}$$

Now

$$f(x,y) = \left[\frac{x^2+y^2}{x+y}\right]$$

$$f(kx, ky) = \left(\frac{kx^2+ky^2}{kx+ky}\right]$$

$$= \frac{k^{2/2}[x^2+y^2]}{K[x+y]}$$

$f(kx, ky) = k' f(x,y)$

$\therefore f(x)$ is a homogeneous function of order $n = 1$

and also e^u is a homogeneous fun^c of order $n = 1$

By Euler's theorem;

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf$$

$$x \frac{\partial \theta}{\partial x} e^U + y \frac{\partial \theta}{\partial y} e^U = 1 \, e^U$$

$$e^{U}\left[x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial x}\right]=1e^{U}$$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial x}=1$$ Hence result

1. If $U=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then prove that $x\frac{du}{dx}+y\frac{du}{dy}=\operatorname{Tan}U$

Sol $U=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

since U is not a homogenous func, we write

$$\sin U=\frac{x^2+y^2}{x+y}=f(x,y)\text{ say}\rightarrow ①$$

$$f(x,y)=\frac{x^2+y^2}{x+y}$$

$$f(kx,ky)=\frac{(kx)^2+(ky)^2}{kx+ky}=\frac{k^2}{k}\left[\frac{x^2+y^2}{x+y}\right]$$

$$f(kx,ky)=k^1f(x,y)$$

$\therefore$ $f(x,y)$ is a homogenous func of order $n=1$
and also $\sin U$ is a homogenous function of order $n=1$

By Euler's Theorem

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=n.f$$

$$x\frac{\partial}{\partial x}\sin u+y\frac{\partial}{\partial y}\sin u=1.\sin u$$

$$\cos u\left[x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}\right]=\sin u$$

$$x\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=\frac{\sin u}{\cos u}$$

$$\boxed{x\frac{\partial U}{\partial x}+y\frac{\partial u}{\partial y}=\operatorname{Tan}U}$$

2. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

Sol $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

sin u is not a homogenous function, we write

$$\tan u = \frac{x^3+y^3}{x+y} = f(x,y) \text{ say}$$

$$f(x,y) = \frac{x^3+y^3}{x+y}$$

$$f(kx,ky) = \frac{kx^3+ky^3}{kx+ky} \rightarrow \frac{k^3}{k}\left[\frac{x^3+y^3}{x+y}\right]$$

$$f(kx,ky) = k^2 f(x,y)$$

$\therefore$ f(x, y) is a homogeneous function of order n=2

and also tan u " " " " " n= 2

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = n.f$$

$$x\frac{\partial}{\partial x}\tan u + y\frac{\partial}{\partial y}\tan u = 2\tan u$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u\cos^2 u}{\cos u}$$

$$x\frac{du}{dx} + y\frac{du}{dy} = \sin 2u$$

1. If $u = \sec^{-1}\left[\frac{x^3-y^3}{x+y}\right]$
2. $u = \sec^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$

③ $u = e^{x^2+y^2}$

④ $\log u = \dfrac{x^3+y^3}{3x+4y}$ } P.T $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$

⑤ $u = \sin^{-1}\left[\dfrac{x^2 \cdot y^2}{x^2+y^2}\right]$ S.T $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\operatorname{Tan} u$

1. Find $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$ ① $u = \operatorname{Tan}^{-1}\left(\dfrac{x^3+y^3}{x+y}\right) \rightarrow 2\cos 3u \sin u$

② $u = \sec^{-1}\left(\dfrac{x^3-y^3}{x+y}\right)$

Sol Given

$$u = \sec^{-1}\left[\frac{x^3-y^3}{x+y}\right]$$

since u is not a homogenous function, we write

$$\sec u = \frac{x^3-y^3}{x+y} = f(x,y) \text{ say} \rightarrow ①$$

$$f(x,y) = \frac{x^3-y^3}{x+y}$$

$$f(kx,ky) = \frac{(kx)^3-(ky)^3}{kx+ky} = \frac{k^{3^2}(x^3-y^3)}{k(x+y)}$$

$$f(kx,ky) = k^2 f(x,y)$$

$\therefore f(x,y)$ is a homogeneous function of order $n=2$

and also $\sec u$ " " " " " $n=2$

By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} = n\cdot f$$

$$x\frac{\partial}{\partial x}\sec u + y\frac{\partial}{\partial y}\sec u = 2\sec u$$

$$x\sec u \operatorname{Tan} u\frac{\partial u}{\partial x} + y\sec u \cdot \operatorname{Tan} u\frac{\partial u}{\partial y} = 2\sec u$$

$$\sec u \cdot \operatorname{Tan} u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\sec u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{\operatorname{Tan} u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u = g(u)$$

By second order of partial derivative of Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$$

$$=2\cot u\left[-2\operatorname{cosec}^2 u-1\right]$$

So

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-2\cot u\left[2\operatorname{cosec}^2 u+1\right]$$

① $u=\tan^{-1}\left[\dfrac{x^3+y^3}{x+y}\right]$

since u is not a homogenous function, we write

$$\tan u=\frac{x^3+y^3}{x+y}=f(x,y)\text{ say}\rightarrow ①$$

$$f(kx,ky)=\frac{kx^3+ky^3}{kx+ky}=\frac{k^{\cancel{3}2}}{\cancel{k}}\left[\frac{x^3+y^3}{x+y}\right]\quad f(kx,ky)=k^2f(x,y)$$

$\therefore f(x,y)$ is a homogenous function of order $n=2$

and also $\tan u$ " " " " " $n=2$

By Euler's Theorem:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

$$x\frac{\partial}{\partial x}\tan x+y\frac{\partial}{\partial y}\tan y=2\,\cancel{f(x,y)}\tan x$$

$$\sec^2 u\left[x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right]=\tan u$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}=\frac{\tan u}{\sec^2 u}$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}=\frac{2\sin u\cos^2 u}{\cos u}$$

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}=\sin 2u=g(u)$$

By second order of partial derivative by Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\left[g'(u)-1\right]$$

$$=\sin 2u\left(2\sin 2u-1\right)$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u\left[2\quad 2u-1\right)$$

① If $u = \sec^{-1}\left[\frac{x^3-y^3}{x+y}\right]$ & ② If $u = \sec^{-1}\left[\frac{x^3+y^3}{x+y}\right]$

Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$

Sol Given; $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$ | $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

Since u is not a homogenous function, we write

$\sec u = \frac{x^3-y^3}{x+y} = f(x,y)$ say $\to$ ①

$f(kx,ky) = \frac{kx^3-ky^3}{kx+ky} = \frac{k^{3^2}}{k}\left(\frac{x^3+y^3}{x+y}\right)$

$f(kx,ky) = k^2 f(x,y)$

$\sec u = \frac{x^3+y^3}{x+y} = f(x,y)$ say $\to$ ②

$f(kx,ky) = \frac{kx^3+ky^3}{kx+ky} = \frac{k^3}{k}\left(\frac{x^3+y^3}{x+y}\right)$

$f(kx,ky) = k^2 f(x,y)$

$\therefore f(x,y)$ is a homogenous function of order $n=2$

& also sec " " " " " $n=2$

By Euler's Theorem

$$\boxed{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf}$$

$x\frac{\partial}{\partial x}\sec u + y\frac{\partial}{\partial y}\sec u = 2\sec u$

$\cancel{\sec u}\tan u\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] = 2\cancel{\sec u}$

$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = \frac{2}{\tan u}$

$$\boxed{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 2\cot u}$$

$x\frac{\partial}{\partial x}\sec u + y\frac{\partial}{\partial y}\sec u = 2\sec u$

$\cancel{\sec u}\tan u\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] = 2\cancel{\sec u}$

$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = \frac{2}{\tan u}$

$$\boxed{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 2\cot u}$$

---

③ $u = e^{x^2+y^2} \to$ S.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$

S:- Given, $u = e^{x^2+y^2}$

$\log u = x^2+y^2 = f(x,y)$

$f(kx,ky) = kx^2 + ky^2$

$= k^2[x^2+y^2] = k^2 f(x,y)$

$\therefore f(x,y)$ is a homogenous fun of order $n=2$

& also $\log u$ " " " " " $n=2$

By Euler's theorem: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$x\frac{\partial}{\partial x}\log u + y\frac{\partial}{\partial y}\log u = 2\log u$

② $\log u = \frac{x^3+y^3}{3x+4y}$

S:- Given,

$\log u = \frac{x^3+y^3}{3x+4y}$

$f(kx,ky) = \frac{kx^3+ky^3}{3kx+4ky}$

$= \frac{k^{3^2}}{k}\left[\frac{x^3+y^3}{3x+4y}\right] = k^2 f(x,y)$

continue

⑤ $U = \sin^{-1}\left(\frac{x^2y^2}{x^2+y^2}\right)$ S.T $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 2\operatorname{Tan} U$

S:- Given:

$\sin u = \frac{x^2y^2}{x^2+y^2} \rightarrow f(x,y)$ say $\rightarrow$ ①

u is not a homogeneous function, we write

$(kx, ky) = \frac{kx^2ky^2}{kx^2+ky^2} \rightarrow \frac{k^{4}}{k^2}\left[\frac{x^2y^2}{x^2+y^2}\right] = k^2 f(x,y)$

$\therefore f(x,y)$ is a homogeneous function of order $n = 2$

& sin u " " " " $n = 2$

By Euler's Theorem:

$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

$x\frac{\partial}{\partial x}\sin u + y\frac{\partial}{\partial y}\sin u = 2\sin u$

$\cos u\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right] = 2\sin u$

$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 2\frac{\sin u}{\cos u}$

$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = 2\operatorname{Tan} U$

* The chain Rule of partial differentation

Theorem: Let $z = f(u,v)$ where $U = p(x,y)$, $V = g(x,y)$ then

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$

and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$

Total differential coefficient:

Let $z = f(x,y)$ where $x = \phi(t)$ & $y = g(t)$ then the total derivative of z is

$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$ or $\boxed{dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy}$

1] ~~de~~If $u = x^2+y^2$ where $x = at^2$, $y = 2at$ find $\frac{du}{dt}$

Sol :- Given that $u = x^2+y^2$ $x = at^2$, $y = 2at$ $U = x^2+y^2$

partial derivative w.r.to x & y

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x$$

$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

$$x = at^2 \quad \frac{dx}{dt} = a\frac{d}{dt}t^2 = 2at$$

$$y = 2at$$

$$\frac{dy}{dt} = 2a\frac{d}{dt}(t) = 2a$$

The total derivative of u $\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

$$\frac{du}{dt} = 2x(2at) + 2y.2a$$

$$\frac{du}{dt} = 4atx + 4ay$$

$$\frac{du}{dt} = 4a[t.x+y]$$

$$\frac{du}{dt} = 4a[t.at^2+2at]$$

$$\frac{du}{dt} = 4a^2[t^3+2t]$$

1. If $u = x^2+y^2+z^2$ where $x = e^t$, $y = e^t\sin t$, $z = e^t\cos t$
2. $U = x^2y^3$ where $x = \log t$, $y = e^t$
3. $U = xy+yz+zx$ where $x = \frac{1}{t}$, $y = e^t$, $z = e^{-t}$
4. $U = \sin\left(\frac{x}{y}\right)$ where $x = e^t$, $y = t^2$
5. $u = \log(x+y+z)$ where $x = e^{-t}$, $y = \sin t$, $z = \cos t$

HW

1. If $u = f(y-z, z-x, x-y)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Sol $u = f(y-z, z-x, x-y)$

let $r = y-z$, $s = z-x$, $t = x-y$ then $u = f(r,s,t)$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(y-z) = 0 \qquad \frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(z-x) = -1 \qquad \frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(y-z) = 1 \qquad \frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(z-x) = 0 \qquad \frac{\partial t}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(y-z) = -1 \qquad \frac{\partial s}{\partial y} = \frac{\partial}{\partial z}(z-x) = 1 \qquad \frac{\partial t}{\partial z} = \frac{\partial}{\partial z}(x-y) = 0$$

By chain rule of partial differential

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1) \qquad \boxed{\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}}$$

Sol

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\boxed{\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{-\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} + \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$ Hence the proof

(2) If $u = f(r,s,t)$ where $r = x/y$, $s = y/z$, $t = z/x$ then S.T

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = 0$$

Sol $u = f(r,s,t)$ $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y}\right] = \frac{1}{y} \qquad \frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left[\frac{y}{z}\right] = 0 \qquad \frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left[\frac{z}{x}\right] =$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{y}\right] = -\frac{x}{y^2} \qquad \frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y}{z}\right] = \frac{1}{z} \qquad \frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left[\frac{z}{x}\right] = 0$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x}{y}\right] = 0 \qquad \frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left[\frac{y}{z}\right] = \frac{y}{z^2} \qquad \frac{\partial t}{\partial z} = \frac{\partial}{\partial z}\left[\frac{z}{x}\right] = \frac{1}{x}$$

By chain rule of partial differentiation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(-\frac{z}{x^2}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\boxed{\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\boxed{\frac{\partial u}{\partial z} = \frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} =$$

$$x\left[\frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}\right] + y\left[-\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}\right] + z\left[-\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}\right]$$

$$= \frac{x}{y}\cancel{\frac{\partial u}{\partial r}} - \frac{z}{x}\cancel{\frac{\partial u}{\partial t}} - \frac{x}{y}\cancel{\frac{\partial u}{\partial r}} + \frac{y}{z}\cancel{\frac{\partial u}{\partial s}} - \frac{y}{z}\cancel{\frac{\partial u}{\partial s}} + \frac{z}{x}\cancel{\frac{\partial u}{\partial t}}$$

$$= 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Hence the proof.

Given,

(1) ~~If~~ $u = x^2 + y^2 + z^2$

$x = e^t$

$y = e^t \sin t$

$z = e^t \cos t$

partial derivative w.r to x, y, z

~~$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^t) = e^t$~~

~~$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^t \sin t) = e^t \cos t$~~

~~$\frac{\partial u}{\partial z}$~~

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^2 + y^2 + z^2] = 2y$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}[x^2 + y^2 + z^2) = 2z$$

$x = e^t \quad \frac{dx}{dt} = e^t$

$y = e^t \sin t \quad \frac{dy}{dt} = e^t \frac{d}{dt}\sin t = e^t(\cos t + \sin t)$

$z = e^t \cos t \quad = e^t(\sin t + \cos t)$

The total derivative of u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

~~$\left[\frac{du}{dt} = 2xe^t + 2ye^t \sin t + 2ze^t \sin t\right.$~~

~~$\left.= 2e^t[x + y\sin t + z\sin t]\right]$~~

$$\frac{du}{dt} = 2xe^t + 2y(e^t \cos t + e^t \sin t) + 2z(e^t(-\sin t) + e^t \cos t)$$

$$\frac{du}{dt} = 2xe^t + 2y[e^t(\cos t + \sin t)] + 2ze^t[\cos t - \sin t)$$

2
Sol :- Given; $u = x^2y^3$

$x = \log t$

$y = e^t$

$$\frac{\partial u}{\partial x} = \frac{d}{dx} x^2y^3 = 2xy^3$$

$$\frac{\partial u}{\partial y} = \frac{d}{dy} x^2y^3 = 3y^2x^2$$

The total derivative of u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\frac{du}{dt} = 2xy^3\frac{1}{t} + 3y^2x^2e^t$$

$$= xy^2\left[\frac{2y}{t} + 3xe^t\right]$$

3 Given $u = xy + yz + zx$
Sol

$x = \frac{1}{t}$

$y = e^t$

$z = e^{-t}$

partial der w.r to x, y & z

$$\frac{du}{dx} = \frac{d}{dx}[xy + yz + zx] = y + z$$

$$\frac{du}{dy} = \frac{d}{dy}(xy + yz + zx] = x + z$$

$$\frac{du}{dz} = \frac{d}{dz}(xy + yz + zx) = y + x$$

$x = \frac{1}{t}$ $\quad \frac{dx}{dt} = -\frac{1}{t^2}$

$y = e^t$ $\quad \frac{dy}{dt} = e^t$

$z = e^{-t}$ $\quad \frac{dz}{dt} = -e^{-t}$

The total derivative of u

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= (y+z)\left(-\frac{1}{t^2}\right) + x + z(e^t) + y + x(-e^{-t})$$

① If $u = f(r)$ and $x = x\cos\theta$, $y = r\sin\theta$ then P.T $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)$.

Sol $u = f(r)$

partial derivative w.r to x & y

1. $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(x)$

Sol $\frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$

Again partial derivative w.r to x

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r)\frac{\partial r}{\partial x}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\frac{\partial r}{\partial x}\frac{\partial r}{\partial x} + f'(r)\frac{\partial^2 r}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2}$$

Similarly $\frac{\partial u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$

Now; $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r)\frac{\partial^2 r}{\partial x^2} + f''(r)\left(\frac{\partial r}{\partial y}\right)^2 + f'(r)\frac{\partial^2 r}{\partial y^2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = f''(r)\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + f'(r)\left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right] \rightarrow ①$$

given that

$x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2[\cos^2\theta + \sin^2\theta] = r^2$

$r^2 = x^2 + y^2 \rightarrow ②$

differentiate eq ② partially w.r to x

$\frac{\partial}{\partial x}(r^2) = x^2 + y^2$

$2r \cdot \frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$ similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

Again differentiate eq ② partially w.r to x & y

$$\frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r\frac{\partial}{\partial x}x - x\frac{\partial r}{\partial x}}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\left(\frac{x}{r}\right)}{r^2} = \frac{r - x^2/r}{r^2} = \frac{r^2 - x^2}{r^3}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r^2-x^2}{r^3} = \frac{y^2}{r^3} \quad \left\{ \text{from eq ② } x^2+y^2=r^2 \quad r^2 = x^2 - y^2 \right\}$$

lly $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$

Substitute these values in eq①

$$\text{eq①} \rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f''(r)\left[\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right] + f'(r)\left[\frac{y^2}{r^3} + \frac{x^2}{r^3}\right]$$

$$= f''(r)\left[\frac{x^2+y^2}{r^2}\right] + f'(r)\left[\frac{x^2+y^2}{r^3}\right]$$

$$= f''(r)\left[\frac{r^2}{r^2}\right] + f'(r)\left[\frac{r^2}{r^3}\right]$$

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)}$$ Hence the proof.

$x = r\cos\theta$, $y = r\sin\theta$ then prove that $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ and $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$

Sol Given $x = r\cos\theta \rightarrow ①$ $y = r\sin\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2(1)$$

$$x^2+y^2 = r^2$$

$r^2 = x^2+y^2$ diff partially w.r.to x

$$\frac{d}{dx} r^2 = \frac{d}{dx}(x^2+y^2)$$

$$\cancel{2}r\frac{dr}{dx} = \cancel{2}x$$

$$\frac{dr}{dx} = \frac{x}{r} \overset{①}{=} \frac{r\cos\theta}{r}$$

$$x = r\cos\theta$$

$$\frac{x}{r} = \cos\theta$$

$$\frac{dx}{dr} = \cos\theta = \frac{x}{r} \rightarrow \frac{dx}{dr} = \frac{x}{r} \rightarrow ②$$

from ① & ②

$$\boxed{\frac{dr}{\partial x} = \frac{\partial x}{dr}}$$

Sol

$x = r\cos\theta \rightarrow (3)$

$y = r\sin\theta \rightarrow (4)$

eq (4) ÷ eq (3)

$$\frac{y}{x} = \frac{\cancel{r}\sin\theta}{r\cos\theta} = \tan\theta$$

$$\tan\theta = y/x$$

$$\theta = \tan^{-1} y/x$$

$$\frac{\partial}{\partial x}(\theta) = \frac{\partial}{\partial x}\tan^{-1} y/x$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{-y}{x^2}$$

$$= \frac{-\cancel{x^2}y}{x^2+y^2} \cdot \frac{1}{\cancel{x^2}}$$

$$= \frac{-y}{x^2+y^2} \quad \text{wheu } x^2+y^2=r^2$$

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} \rightarrow r\frac{d\theta}{dx} = \frac{-y}{r} \rightarrow (5)$$

$$x = r\cos\theta$$

$$\frac{\partial x}{\partial\theta} = -r\sin\theta$$

$$\frac{\partial x}{\partial\theta} = -r\, y/r$$

$$\frac{1}{r}\frac{dx}{d\theta} = -y/r \rightarrow (6)$$

eq (6) & eq 5

$$\boxed{\frac{1}{r}\frac{dx}{d\theta} = r\frac{d\theta}{dx}}$$

$$\frac{-y}{\cancel{r}} = -y/\cancel{r}$$

* Jacobian method :-

1. If $u = U(x+y)$ and $v = V(x, y)$ then the jacobian of u, v w.r to x, y is given by

$$J\left[\frac{u, v}{x, y}\right] = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

2. If $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ then the Jacobian of x, y, z w.r to u, v, w is given by

$$J\left[\frac{x, y, z}{u, v, w}\right] = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

property:

(1) $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$, (or)

$J = \frac{\partial(u, v)}{\partial(x, y)}$ & $J' = \frac{\partial(x, y)}{\partial(u, v)}$ then $JJ' = 1$

1. * If $x = u(1+v)$, $y = v(1+u)$ then Prove that $\frac{\partial(x, y)}{\partial(u, v)} = 1+u+v$

Sol) $x = u + uv$, $y = v + vu$

differentially partially w.r to u & v

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u + uv) = 1 + v$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u + uv) = 0 + u = u$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(v + vu) = v$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(v + vu) = 1 + u$$

$$J\left[\frac{x, y}{u, v}\right] = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} \qquad = (1+v)(1+u) - uv$$

$$= 1+u+v+\cancel{uv}-\cancel{uv}$$

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v}$$

1.

Sol

2. $u = x^2-2y,\ v = x+y+z,\ w = x-2y+3z$ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Sol:- $u = x^2-2y,\ v = x+y+z,\ w = x-2y+3z$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2-2y) = 2x \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(\cancel{x^2-2y}\ x+y+z) = 1$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\overset{x^2-2y}{\cancel{x+y+z}}\right] = y-2 \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} = 1$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left[\overset{x^2-2y}{\cancel{x-2y+3z}}\right] = 0 \qquad \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}(x+y+z) = 1$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x-2y+3z) = 1$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(x-2y+3z) = -2$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(x-2y+3z) = 3$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 2x(3+2) - (-2)(3-1) + 0$$

$$= 2x(5) + 2(2)$$

$$= 10x + 4.$$

***** 5m

3. If $x+y+z = u,\ y+z = uv,\ z = uvw$ then evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ and $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol $x+y+z = u \rightarrow ①$

$y+z = uv \rightarrow ②$

$z = uvw \rightarrow ③$

eq① $\Rightarrow x = u-(y+z) = u-uv$

eq② $\Rightarrow y = uv - z = uv - uvw$

$x = u-uv,\ y = uv-uvw,\ z = uvw$

$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u-uv) = 1-v$ $\quad$ $\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(uv-uvw) = v-vw$

$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}(u-uv) = -u$ $\quad$ $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(uv-uvw) = u-uw$

$\frac{\partial x}{\partial w} = \frac{\partial}{w}(u-uv) = 0$ $\quad$ $\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}(uv-uvw) = -uv$

$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}(uvw) = vw$

$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}(uvw) = uw$

$\frac{\partial z}{\partial v} = \frac{\partial}{\partial w}(uvw) = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix} = (1-v)\left[(u-uw)(uv)-(-uv)(vw)\right] - (-u)\left[(v-vw)(uv)-(-uv)(vw)\right]+0$$

$$= (1-v)\left[u^2v - u^2\cancel{vw} + u^2\cancel{vw}\right] + u\left[uv^2 - v^2uw + v^2uw\right]$$

$$= (1-v)\left[u^2v\right] + u^2v^2$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$ and

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}} = \frac{1}{u^2v}$$

Ass

① $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol: $x = \frac{u^2}{v}$

$$\frac{dx}{du} = \frac{v\,2u + u^2(0)}{v^2} = \frac{2u\cancel{v}}{v\cancel{2}} = \frac{2u}{v}$$

$$\frac{dx}{dv} = \frac{-1}{v^2}(u^2) = -\frac{u^2}{v^2}$$

$$\frac{dy}{du} = v^2\left(-\frac{1}{u^2}\right) = \frac{-v^2}{u^2}$$

$$\frac{\partial y}{\partial v} = \frac{2v}{u} \qquad = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{2u}{v} & \frac{-u}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix}$$

$$\left(\frac{2u}{v}\right)\left(\frac{2v}{u}\right) - \left(-\frac{v^2}{u^2}\right)\left(\frac{u^2}{v^2}\right)$$

$= 4 - 1$

$= 3 \qquad = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}} = \dfrac{1}{3}$

(2) $x = u(1-v)$, $y = uv$. P.T $JJ' = 1$

Sol: $x = u(1-v)$

$\frac{dx}{du} = \frac{d}{du} u(1-v) = \frac{d}{du} u - uv = 1 - v$

$\frac{dx}{dv} = \frac{d}{dv}(u - uv) = \frac{d}{dv} - u = -u$

$\frac{dy}{du} = \frac{d}{du}(uv) = v$

$\frac{dy}{dv} = \frac{d}{dx}(uv) = u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{du} & \frac{\partial x}{dv} \\ \frac{dy}{du} & \frac{\partial y}{dv} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u - u\cancel{v} + u\cancel{v}$

$= u$

$x = u(1-v) \Rightarrow x = u - uv \Rightarrow \cancel{x = y - v}$

$x = u - y$

$\boxed{x + y = u}$

$y = uv \rightarrow y = (x+y)v \Rightarrow \boxed{v = \frac{y}{x+y}}$

$\frac{du}{dx} = \frac{d}{dx}(x+y) = 1 \qquad \frac{dv}{dx} = \frac{d}{dx}\frac{y}{x+y} = \frac{-y}{(x+y)^2}$

$\frac{du}{dy} = \frac{d}{dy}(x+y) = 1 \qquad \frac{dv}{dy} = \frac{d}{dy}\frac{y}{x+y} = \frac{x}{(x+y)^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{\cancel{x+y}}{(x+y)^{\cancel{2}}} = \frac{1}{x+y} = \frac{1}{u}$$

$JJ' = \partial\left(\frac{x,y}{u,v}\right)\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = u \cdot \frac{1}{u} = 1 \qquad \boxed{JJ' = 1}$

① If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$ then S.T $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

② If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, S.T $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

③ $x = \frac{u^2}{v}$, $y = \frac{v^2}{u}$ find $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{3}$ or 3

④ $x = uv$, $y = u/v$ find $\frac{\partial(x,y)}{\partial(u,v)}$

③ $u = \frac{y^2}{x}$; $v = \frac{x^2}{y}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Sol $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{y^2}{x}\right] = \boxed{-\frac{y^2}{x^2} = \frac{x\,2y}{x^2} = \frac{2y}{x}} =$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{y^2}{x}\right] = \boxed{-\frac{y^2}{x^2}} =$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2}{y}\right] = \frac{y\,2x}{y^2} = \frac{2x}{y}$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial xy}\left[\frac{x^2}{y}\right] = -\frac{x^2}{y^2}$

① If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $\omega = \frac{xy}{z}$ then show that $\frac{\partial(u,v,\omega)}{\partial(x,y,z)} =$

or find $J\left(\frac{u,v,\omega}{x,y,z}\right)$

Sol Given;

$u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

partial derivative w.r to x, y & z

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{yz}{x}\right) = -\frac{yz}{x^2}$

$\frac{du}{dy} = \frac{d}{dy}\left[\frac{yz}{x}\right] = \frac{z}{x}$

$\frac{du}{dz} = \frac{d}{dz}\left[\frac{yz}{x}\right] = y/x$

$\frac{dv}{dx} = \frac{\partial}{\partial x}\left[\frac{xz}{y}\right] = \frac{z}{y}$

$\frac{dv}{dy} = \frac{\partial}{\partial y}\left[\frac{xz}{y}\right] = -\frac{xz}{y^2}$

$\frac{dv}{dz} = \frac{\partial}{\partial z}\left[\frac{xz}{y}\right] = \frac{x}{y}$

$\frac{dw}{dx} = \frac{\partial}{\partial x}\left(\frac{xy}{z}\right) = \frac{y}{z}$

$\frac{dw}{dy} = \frac{\partial}{\partial y}\left[\frac{xy}{z}\right] = \frac{x}{z}$

$\frac{dw}{dz} = \frac{\partial}{\partial z}\left[\frac{xy}{z}\right] = -\frac{xy}{z^2}$

$$\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix}$$

$$\frac{\partial(u,v,\omega)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & y/x \\ z/y & -\cancel{x^2/y^2}\,\frac{xz}{y^2} & \frac{x}{y} \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

Multipling & Dividing c_1 with by x, c_2 by y, c_3 by z

$$\frac{d(u,v,\omega)}{d(x,y,z)} = \frac{1}{xyz}\begin{vmatrix} -\frac{\cancel{x}yz}{x^{\cancel{2}}} & \frac{yz}{x} & zy/x \\ xz/y & -\frac{x\cancel{y}z}{y^{\cancel{2}}} & xz/y \\ \frac{xy}{z} & \frac{xy}{z} & -\frac{xy\cancel{z}}{z^{\cancel{2}}} \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} -\frac{yz}{x} & \frac{yz}{x} & \frac{yz}{x} \\ xz/y & -xz/y & xz/y \\ xy/z & \frac{xy}{z} & -\frac{xy}{z} \end{vmatrix}$$

$= \frac{1}{xyz} \cdot \frac{yz}{x} \cdot \frac{xz}{y} \cdot \frac{xy}{z} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= \frac{\cancel{x^2y^2z^2}}{\cancel{x^2y^2z^2}} \left(-1(1-1) - 1(-1-1) + 1(1+1)\right]$

$= 1\left[-(-2)+2\right]$

$= 2+2$

$= 4$

$$\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4}$$

***

2. $U = x^2 - y^2$, $v = 2xy$, where $x = r\cos\theta$, $y = r\sin\theta$ then P.T $\frac{\partial(U,V)}{\partial(r,\theta)} = 4r^3$

Sol $U = x^2 - y^2$

$U = r^2\cos^2\theta - r^2\sin^2\theta$

$U = r^2(\cos^2\theta - \sin^2\theta)$

$U = r^2\cos 2\theta \Rightarrow \frac{U}{r} = r\cos 2\theta \Rightarrow \frac{U}{r} = 2x$

$V = 2xy$

$V = 2r\cos\theta \; r\sin\theta$

$= 2r^2\cos\theta\sin\theta$

$V = r^2\sin 2\theta \rightarrow \frac{V}{r} = r\sin 2\theta \rightarrow \frac{V}{r} = 2y$

$\frac{du}{dr} = \frac{d}{dr}(r^2\cos 2\theta) = 2r\cos 2\theta$

$\frac{du}{dr} = \frac{\partial}{\partial\theta}(r^2\cos 2\theta) = r^2(-2\sin 2\theta) = -2r^2\sin 2\theta$.

$\frac{dv}{dr} = \frac{d}{dr}(r^2\sin 2\theta) = 2r\sin\theta$.

$\frac{dv}{d\theta} = \frac{d}{d\theta}(r^2\sin 2\theta) = 2r^2\cos 2\theta$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$= 2r \cdot 2r^2 \begin{vmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{vmatrix}$$

$$= 4r^3(\cos^2 2\theta + \sin^2 2\theta)$$

$$= 4r^3(1)$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3$$

Assignment

① If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$ then show that $\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = 1$

Sol:- Given; $x = r\cos\theta$

$y = r\sin\theta$

$\frac{dx}{d\theta} = \frac{d}{dr}(r\cos\theta) = \cos\theta$ $\quad$ $\frac{dy}{dr} = \frac{d}{dr}(r\sin\theta) = \sin\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}(r\cos\theta) = -r\sin\theta$ $\quad$ $\frac{dy}{d\theta} = \frac{d}{d\theta}(r\sin\theta) = r\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$= r(\cos^2\theta + \sin^2\theta)$

$= r(1)$

$= r$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

$$\therefore \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} \times \frac{\partial(r,\theta)}{\partial(x,y)} = r \cdot \frac{1}{r} = 1}$$

(2) If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ S.T $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

Sol: partial diff w.r.to x, y, z

$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta\cos\phi) = \sin\theta\cos\phi$ $\frac{dy}{dr} = \frac{\partial}{\partial r}(r\sin\theta\sin\phi) = \sin\theta\sin\phi$

$\frac{\partial x}{\partial\theta} = \frac{\partial}{\partial\theta}(r\sin\theta\cos\phi) = r\cos\theta\cos\phi$ $\frac{dy}{d\theta} = \frac{\partial}{\partial\theta}(r\sin\theta\sin\phi) = r\cos\theta\sin\phi$

$\frac{\partial x}{d\phi} = \frac{\partial}{\partial\phi}(r\sin\theta\cos\phi) = -r\sin\theta\sin\phi$ $\frac{dy}{d\phi} = \frac{\partial}{\partial\phi}(r\sin\theta\sin\phi) = r\sin\theta\cos\phi$

$\frac{\partial z}{dr} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$

$\frac{\partial z}{\partial\theta} = \frac{\partial}{\partial\theta}(r\cos\theta) = -r\sin\theta$

$\frac{\partial z}{d\phi} = \frac{\partial}{d\phi}(r\cos\theta) = 0$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{dx}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

from C_2 & C_3 taken r common & from C_3 $\sin\theta$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$= r^2\sin\theta\Big[\sin\theta\cos\phi(+\sin\theta(\cos\phi)) - \cos\theta\cos\phi[-\cos\theta(\cos\phi)] - \sin\phi$
$[-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)\Big]$

$= r^2\sin\theta\,[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\phi(\sin^2\theta + \cos^2\theta)]$

$= r^2\sin\theta\,[\cos^2\phi[\sin^2\theta + \cos^2\theta] + \sin^2\phi[1]]$

$= r^2\sin\theta\,[\cos^2\phi(1) + \sin^2\phi]$

$= r^2\sin\theta$

4 Sol $x = uv$; $y = \frac{u}{v}$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$$

$$\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{u}{v^2} \end{vmatrix}$$

$$= -\frac{u}{v} - \frac{u}{v}$$

$$= \frac{-2u}{v}$$

* Functional Dependence :- If the Function u & v of the independent variable x and y are functionally depedent then the jacobian $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

Functional Independence: If the Jacobian $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then u and v are said to be functionally independent

1. Find whether the function $u = e^x \sin y$, $v = e^x \cos y$ are functionally dependent or not?

Sol:- Given:

$u = e^x \sin y$

$v = e^x \cos y$

$$\frac{\partial u}{\partial x} = \frac{d}{dx}(e^x \sin y) = e^x \sin y$$

$$\frac{\partial u}{\partial y} = \frac{d}{dy}(e^x \sin y) = e^x \cos y$$

$$\frac{\partial v}{\partial x} = \frac{d}{dx}(e^x \cos y) = e^x \cos y$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(e^x \cos y) = -e^x \sin y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ e^x \cos y & -e^x \sin y \end{vmatrix}$$

$$= e^x . e^x \begin{vmatrix} \sin y & \cos y \\ \cos y & -\sin y \end{vmatrix}$$

$$= e^x e^x (-\sin^2 y - \cos^2 y)$$

$$= -e^{2x} (\sin^2 y + \cos^2 y)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = -e^{2x} \neq 0$$

u,v are functionally independent.

2. Verify if $u = 2x - y + 3z$, $v = 2x - y - z$; $w = 2x - y + z$ are functionally dependent and find the relation b/w them!

Sol

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x - y + 3z) = 2 \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2x - y - z) = 2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x - y + 3z) = -1 \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2x - y - z) = -1$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}(2x - y + 3z) = 3 \qquad \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}(2x - y - z) = -1$$

$$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(2x - y + z) = 2$$

$$\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(2x - y + z) = -1$$

$$\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(2x - y + z) = 1$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2 & -1 & 3 \\ +2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$\cancel{\left[= 2(-1-1) + 1(-2+2) + 3(2+2) = 2(-2) + 1(0) + 3(4) = -4 + 12 = 8\right]}$$

$$= 2(-1-1) + 1(2+2) + 3(-2+2)$$

$$= 2(-2) + 4 + 0$$

$$= -4 + 4$$

$$= 0 \quad \therefore u,v \text{ are functionally dependent}$$

Now $U+V = 2x-y+3z+2x-y-z$

$= 4x-2y+2z$

$= 2(2x-y+z)$

$U+V = 2w$

$U+V=2w$ is the functional relationship between U, V & w

(2) P.T $U = \dfrac{x^2-y^2}{x^2+y^2}$, $V = \dfrac{2xy}{x^2+y^2}$ are functionally dependent & find the relationship

Sol

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x^2-y^2}{x^2+y^2}\right] = \frac{\cancel{2x}}{\cancel{2x}}\ \frac{x^2+y^2(2x) - x^2-y^2(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^3+2xy^2-2x^3+2xy^2}{(x^2+y^2)^2}$$

$$= \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x^2-y^2}{x^2+y^2}\right] = \frac{(x^2+y^2)(-2y)-(x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2y-\cancel{2y^3}-2x^2y+\cancel{2y^3}}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left[\frac{2xy}{x^2+y^2}\right] = \frac{x^2+y^2[2y]-2xy[2x]}{(x^2+y^2)^2}$$

$$= \frac{2x^2y+2y^3-4x^2y}{(x^2+y^2)^2} = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

$$= \frac{2y(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[\frac{2xy}{x^2+y^2}\right] = \frac{(x^2+y^2)2x-(2xy)(2y)}{(x^2+y^2)^2} = \frac{2x^3+2xy^2-4xy^2}{(x^2+y^2)^2}$$

$$= \frac{2x^3-2xy^2}{(x^2+y^2)^2} = \frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4x^2y}{(x^2+y^2)^2} \\ \frac{2y(y^2-x^2)}{(x^2+y^2)^2} & \frac{2x(x^2-y^2)}{(x^2+y^2)^2} \end{vmatrix}$$

$$= \frac{4xy^2}{(x^2+y^2)^2} \cdot \frac{2x(x^2-y^2)}{(x^2+y^2)^2} - \left[\frac{-4x^2y}{(x^2+y^2)^2} \cdot \frac{2y(y^2-x^2)}{(x^2+y^2)^2}\right]$$

$$= \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4} - \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4}$$

$= 0$ $\quad \therefore u, v$ are functionally dependent

Now $u^2+v^2 = \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 + \left(\frac{2xy}{x^2+y^2}\right)^2 = \left(\cancel{\frac{x^2-y^2}{(x^2+y^2)}}\right)^2$

$$= \frac{(x^2-y^2)^2}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2} = \frac{(x^2-y^2)^2+4x^2y^2}{(x^2+y^2)^2}$$

$u^2+v^2 = \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1 \qquad \{(a-b)^2+4ab = (a+b)^2\}$

3. Show that $u = \sin^{-1}x + \sin^{-1}y$, $v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ then S.T u and v functionally related and also find relationship.

Sol:- $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(\sin^{-1}x + \sin^{-1}y) = \frac{1}{\sqrt{1-x^2}}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(\sin^{-1}x + \sin^{-1}y) = \frac{1}{\sqrt{1-y^2}}$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sqrt{1-y^2} + y\frac{1}{2\sqrt{1-x^2}}(-2x)$

$= \sqrt{1-y^2} + y\frac{-x}{2\sqrt{1-x^2}}$ $\qquad \frac{\partial}{\partial x}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$

$= x\frac{1}{2\sqrt{1-y^2}}(-2y) + \sqrt{1-x^2}$

$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \end{vmatrix}$$

$$= \frac{1}{\sqrt{1-x^2}}\left[\frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}\right] - \frac{1}{\sqrt{1-y^2}}\left[\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right]$$

$$= \cancel{\frac{-xy}{\sqrt{1-x^2}\sqrt{1-y^2}}} + \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{\sqrt{1-y^2}}{\sqrt{1-y^2}} + \cancel{\frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 1-1 = 0$$

$\therefore$ u,v are functionally dependent

Now $u = \sin^{-1}x + \sin^{-1}y$

$u = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

$\sin u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\sin u = v$

$\therefore$ $\sin u = v$ is the functionally relationship b/w u & v

$w^2 = v + 2u$

1) $u = xy+yz+zx$, $v = x^2+y^2+z^2$, $w = x+y+z \Rightarrow w$

2) $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$ $\quad v = \frac{u+1}{u-1}$

3) $x = u\sqrt{1-v^2} + v\sqrt{1-u^2}$, $y = \sin^{-1}u + \sin^{-1}v$

① Given;

Sol $u = xy+yz+zx$

$v = x^2+y^2+z^2$

$w = x+y+z$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[xy+yz+zx] = y+z \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2+z^2) = 2x$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[xy+yz+zx] = x+z \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2+z^2) = 2y$

$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}[xy+yz+zx] = y+x \qquad \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}(x^2+y^2+z^2) = 2z$

$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}(x+y+z) = 1 \qquad \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}(x+y+z) = 1 \qquad \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}(x+y+z) = 1$

$$\frac{\partial(U,V,W)}{\partial(x,y,z)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ \cancel{y+}1 & 1 & 1 \end{vmatrix}$$

$= y+z(2y-2z) - x+z(2x-2z) + y+x(2x-2y)$

$= 2y^2 - \cancel{2yz} + \cancel{2yz} - \cancel{2z^2} - \cancel{2x^2} + \cancel{2zx} - 2xz + \cancel{2z^2} + \cancel{2xy} - \cancel{2y^2} + \cancel{2x^2} - 2xy$

$= 0$

$\therefore$ U, V are functionally dependent

Now

$V + 2U = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$= (x+y+z)^2$

$= w^2$

$V + 2U = w^2$ is the functionally relationship between u, v, w

2. $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

Sol $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y}\right] = \frac{1}{y}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{y}\right] = -\frac{x}{y^2}$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right) = \frac{x-y(1) - x+y(1)}{(x-y)^2}$

$= \frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2} = \frac{-2y}{(x-y)^2}$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x+y}{x-y}\right] = \frac{x-y(1) - x+y(-1)}{(x-y)^2}$

$= \frac{x-y-(-x-y)}{(x-y)^2} = \frac{x-y+x+y}{(x-y)^2}$

$$\frac{\partial(U,V)}{\partial(x,y)} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ -\frac{2y}{(x-y)^2} & \frac{2x}{(x-y)^2} \end{vmatrix}$$

$= \frac{1}{y}\frac{2x}{(x-y)^2} - \frac{x}{y^{\cancel{2}}}\left(\frac{2\cancel{y}}{(x-y)^2}\right)$

$= \frac{2x}{y(x-y)^2} - \frac{2x}{y(x-y)^2} = 0$

U, V are functionally dependent

Now

$$u=\frac{x}{y} \qquad v=\frac{x+y}{x-y}$$

$$\frac{u+1}{u-1} = \frac{\frac{x}{y}+1}{\frac{x}{y}-1} = \frac{\frac{x+y}{\cancel{y}}}{\frac{x-y}{\cancel{y}}} = \frac{x+y}{x-y} = v$$

$\boxed{v=\frac{u+1}{u-1}}$ is the functional relationship b/w u&v

(3) $x = u\sqrt{1-v^2}+v\sqrt{1-u^2}$, $y=\sin^{-1}u+\sin^{-1}v$

Sol:- $\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(u\sqrt{1-v^2}+v\sqrt{1-u^2}\right) = \sqrt{1-v^2}+v\frac{1}{\cancel{2}\sqrt{1-u^2}}(-\cancel{2}u)$

$$= \sqrt{1-v^2} - \frac{uv}{\sqrt{1-u^2}}$$

$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(u\sqrt{1-v^2}+v\sqrt{1-u^2}\right)$ ~~$= u\sqrt{1-v}$~~

$$= u\frac{1}{\cancel{2}\sqrt{1-v^2}}(-\cancel{2}v) + \sqrt{1-u^2}$$

$$= -\frac{\cancel{2}uv}{\sqrt{1-v^2}}+\sqrt{1-u^2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\sin^{-1}u+\sin^{-1}v\right) = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(\sin^{-1}u+\sin^{-1}v\right) = \frac{1}{\sqrt{1-v^2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{1-v^2}-\frac{uv}{\sqrt{1-u^2}} & -\frac{\cancel{2}uv}{\sqrt{1-v^2}}+\sqrt{1-u^2} \\ \frac{1}{\sqrt{1-u^2}} & \frac{1}{\sqrt{1-v^2}} \end{vmatrix}$$

$$= \sqrt{1-v^2}\times\frac{-uv}{\sqrt{1-u^2}}\times\frac{1}{\sqrt{1-v^2}} + \frac{1}{\sqrt{1-u^2}}\times\frac{uv}{\sqrt{1-v^2}}\left(\sqrt{1-u^2}\right)$$

$$= \frac{\cancel{\sqrt{1-v^2}}}{\cancel{\sqrt{1-v^2}}}\times\frac{-uv}{\sqrt{1-u^2}\sqrt{1-v^2}} + \frac{\cancel{\sqrt{1-u^2}}}{\cancel{\sqrt{1-u^2}}}\times\frac{uv}{\sqrt{1-u^2}\sqrt{1-v^2}}$$

$= 0$ ∴ u v are functionally dependent

Now $y = \sin^{-1} u + \sin^{-1} v$

$y = \sin^{-1}\left(u\sqrt{1-v^2} + v\sqrt{1-u^2}\right)$

$\sin y = u\sqrt{1-v^2} + v\sqrt{1-u^2}$

$\sin y = x$: is the functionally relationship between x & y

* Maximum and minimum of functions of two variables :

$f(x,y)$ be a function of two variables x, y.

Let $x = a$, $y = b$, $f(x,y)$ is said have maximum or minimum value, If $f(a,b) > f(a+h, b+k)$ and $f(a,b) < f(a+h, b+k)$ respectively h and k are small values.

→ working Rules to maximum & minimum value of $f(x,y)$

1. find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equal each to zero, solve these equations for x and y

   Let (a_1, b_1) (a_2, b_2) be the pts of values

2. find $l = \frac{\partial^2 f}{\partial x^2}$, $m = \frac{\partial^2 f}{\partial x \partial y}$, $n = \frac{\partial^2 f}{\partial y^2}$

3. 1) If $ln - m^2 > 0$ and $l < 0$ at (a,b), the (a,b) is a point of maximum and $f(a,b)$ is maximum value

   2) If $ln - m^2 > 0$ and $l > 0$ at (a,b) then (a,b) is a point of minimum and $f(a,b)$ is a minimum value.

   3) If $ln - m^2 < 0$ at (a,b) then $f(a,b)$ is not an extreme value.

   4) If $ln - m^2 = 0$ ~~then~~ $f(x,y)$ fails to have maximum or minimum value & it needs ~~for~~ further investigation

① Find the maximum & minimum values of the function

① $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

② $f(x,y) = x^4 + y^4 - 3x^2 + 4xy - 2y^2$

③ $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72y$

Sol Given; $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

then $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + 3xy^2 - 3x^2 - 3y^2 + 4)$

$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^3 + 3xy^2 - 3x^2 - 3y^2 + 4)$

$= 6xy - 6y$

$\frac{\partial f}{\partial y} = 6[xy - y]$

$\cancel{\frac{\partial f}{\partial x} = 0}$

$\cancel{3x^2 + 3y^2 - 6x = 0}$

and $l = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[3x^2 + 3y^2 - 6x]$

$= 6x - 6$

$m = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}[6xy - 6y]$

$= 6y$

$n = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[6xy - 6y] = 6x - 6$

for maximum or minimum then $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 6x = 0$

when $x = 1$

$3(1)^2 + 3y^2 - 6(1) = 0$

$-3 + 3y^2 = 0$

$-1 + y^2 = 0$

$y^2 = 1$

$\boxed{y = \pm 1}$

$6xy - 6y = 0$

$6y(x-1) = 0$

$x - 1 = 0, \ y = 0$

$x = 1, \ y = 0$

when $y = 0$

$3x^2 + 3(0) - 6x = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0, x - 2 = 0$

$x = 0 ; x = 2$

This points are $(1,1)(1,-1) ; (0,0)(2,0)$

① at $(1,1)$

$$ln - m^2 = (6x-6)(6x-6) - (6y)^2$$
$$= 0.0 - 62$$
$$ln - m^2 = -36 < 0$$

$f(1,1)$ is not an extreme value

② at $(1,-1)$

$$ln - m^2 = (6x-6)(6x-6) - (6y)^2$$
$$= 0.0 - 36$$
$$ln - m^2 = -36 < 0$$

$f(1,-1)$ is not an extreme value.

③ at $(0,0)$

$$ln - m^2 = (6x-6)(6x-6) - (6y)^2$$
$$= (-6)(-6) - 0$$
$$ln - m^2 = 36 > 0$$

and $l = (6x-6) = 0-6 = -6 < 0$

$\therefore (0,0)$ is a maximum point

the maximum value is

$$f(0,0) = 0^3 + 3(0)(0)^2 - 3(0)^2 - 3(0)^2 + 4$$
$$f(0,0) = 4$$

④ at $(2,0)$

$$ln - m^2 = (6(2)-6)(6(2)-6) - (6(0))^2$$
$$= 6.6 - 0$$
$$ln - m^2 = 36 > 0$$

and $l = 6(2) - 6 = 12 - 6 = 6 > 0$

$\therefore (2,0)$ is a minimum point

The minimum value is

$$f(2,0) = 2^3 + 3(2)(0)^2 - 3(2)^2 - 3(0)^2 + 4$$
$$= 8 + 0 - 12 - 0 + 4$$
$$f(2,0) = 0$$

2. Find the maximum & minimum value of the function

$f = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Sol Given;

$f = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[3x^4 - 2x^3 - 6x^2 + 6x + 1]$

$= 12x^3 - 6x^2 - 12x + 6$

$l = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12x^3 - 6x^2 - 12x + 6)$

$= 36x^2 - 12x - 12$.

$\frac{\partial f}{\partial x} = 0$

$12x^3 - 6x^2 - 12x + 6 = 0$

$$\begin{array}{c|cccc} 1 & 12 & -6 & -12 & 6 \\ & 0 & 12 & 6 & -6 \\ \hline & 12 & 6 & -6 & \boxed{0} \end{array}$$

$12x^2 - 6x - 12 = 0 \Rightarrow \cancel{2x^2 - x - 2 = 0}$

$\cancel{2x^2 x}$

$\cancel{6x(2x - 1 - 2) = 0}$

$12x^2 + 6x - 6 = 0$

$2x^2 + x - 1 = 0$

$2x^2 + 2x - x - 1 = 0$

$2x(x+1) - 1(x+1) = 0$

$(x+1)(2x-1) = 0$

$x = -1,\ x = 1/2,\ x = 1$

$x = -1$ sub in l

$36(-1)^2 - 12(-1) - 12$

$36 + \cancel{12} - \cancel{12}$

$36 > 0$

minimum pt $(-1, 0)$

$x = 1$ sub in l.

$= 36 - 12 - 12$

$= 36 - 24$

$= 12 > 0$

$x = 1/2$ sub in l

$= \overset{9}{\cancel{36}}\left(\frac{1}{4}\right) - \overset{6}{\cancel{12}}\left(\frac{1}{2}\right) - 12$

$= 9 - 6 - 12$

$= 9 - 18$

$= -9 < 0$

~~minimum pt (1/2~~

maxmimum point.

max value $= 3\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 6\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) + 1$

$= 3\left(\frac{1}{16}\right) - 2\left(\frac{1}{8}\right) - 6\left(\frac{1}{4}\right) + \frac{6}{2} + 1$

$= \frac{3 - 4 - 24 + 48 + 16}{16} = \frac{39}{16}$

① ② $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\underline{\underline{\text{Sol}}}$ $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4 + y^4 - 2x^2 + 4xy - 2y^2)$

$= 4x^3 - 4x + 4y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4 + y^4 - 2x^2 + 4xy - 2y^2)$

$= 4y^3 + 4x - 4y$

And $l = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(4x^3 - 4x + 4y)$

$= 12x^2 - 4$

$m = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(4y^3 + 4x - 4y)$

$= 4$

$n = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(4y^3 + 4x - 4y)$

$= 12y^2 - 4$

for maximum or minimum then $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$4x^3 - 4x + 4y = 0$ $\qquad$ $4y^3 + 4x - 4y = 0$

$4(x^3 - x + y) = 0$ $\qquad$ $4(y^3 + x - y) = 0$

$x^3 - x + y = 0 \rightarrow ①$ $\qquad$ $y^3 + x - y = 0 \rightarrow ②$

$x^3 - \cancel{x} + \cancel{y} = 0$

$y^3 + \cancel{x} - \cancel{y} = 0$

$x^3 + y^3 = 0$

$x^3 + y^3 = 0$

$x^3 = -y^3$

$\boxed{x = -y}$ sub in eq ①

$x^3 - x - x = 0$

$x^3 - 2x = 0$

$x(x^2 - 2) = 0$

$x = 0$; $x^2 = 2$

$x = 0$; $x = \pm\sqrt{2}$

because $\boxed{x=-y}$

If $x=0$ then $y=0$

$x=\pm\sqrt{2}$

$\therefore (0,0)\ (\sqrt{2},-\sqrt{2})\ (-\sqrt{2},\sqrt{2})$

(i) at $(0,0)$

$ln-m^2=(12x^2-4)(4)\cdot(12y^2-4)^2-(4)^2$

$\cancel{=0(4)\ (0-4)}\quad =-4(4)-(-4)^2$

$=\cancel{0-4}<0\quad =-16+16\quad =0$

At $(0,0)$, f has

② at $(\sqrt{2},-\sqrt{2})$

$ln-m^2=(12x^2-4)(4)-(12y^2-4)^2$

$=(12(\sqrt{2})^2-4)(4)-(12(-\sqrt{2})^2-4)^2$

$=(24-4)(4)-(12(2)-4)^2$

$=(20)(4)-(24-4)^2$

$\cancel{=24}=80-(20)^2$

② at $(\sqrt{2},-\sqrt{2})$

$ln-m^2=(12x^2-4)(12y^2-4)-(4)^2$

$=(12(2)-4)(12(2)-4)-16$

$=(20)(20)-16$

$=400-16$

$=384>0$

and $l=(12x^2-4)$

$=24-4$

$=20>0\quad (\sqrt{2},-\sqrt{2})$ is a minimum point

The minimum value is

$f(\sqrt{2},-\sqrt{2})=x^4+y^4-2x^2+4xy-2y^2$

$=(\sqrt{2})^4+(-\sqrt{2})^4-2(\sqrt{2})^2+4(\sqrt{2})(-\sqrt{2})-2(-\sqrt{2})^2$

$=\cancel{4}+\cancel{4}-\cancel{4}-8-4$

$=-8$

③ $(-\sqrt{2}, \sqrt{2})$

$ln - m^2 = (12x^2 - 4)(12y^2 - 4) - 4^2$

$= (12(-\sqrt{2})^2 - 4)(12(\sqrt{2})^2 - 4) - 16$

$= (24-4)(24-4) - 16$

$= 20(20) - 16$

$= 400 - 16$

$= 384 > 0$

where $l > 0$

The function $(-\sqrt{2}, \sqrt{2})$ is a minimum point.

3. $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Sol $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x)$

$= 3x^2 + 3y^2 - 30x + 72$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x)$

$= 6xy - 30y$

$l = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 3y^2 - 30x + 72)$

$= 6x - 30$

$m = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(6xy - 30y)$

$= 6y$

$n = \frac{\partial}{\partial xy}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(6xy - 30y)$

$= 6x - 30$

for maximum or minimum $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$3x^2 + 3y^2 - 30x + 72 = 0$

put $x = 5$

$3(5)^2 + 3y^2 - 30(5) + 72 = 0$

$75 + 3y^2 - 150 + 72 = 0$

$-3 + 3y^2 = 0$

$3y^2 = 3$

$\boxed{y = \pm 1}$

$6xy - 30y = 0$

$6y(x-5) = 0$

$6y = 0$; $x = 5$

$y = 0$; $x = 5$

$(5, 1)\ (5, -1)$

put $y=0$

$3x^2+3y^2-30x+72=0$

$3x^2-30x+72=0$

$3x^2-18x-12x+72=0$

$3x(x-6)-12(x-6)=0$ $\quad x-6=0$ $\quad 3x-12=0$

$\boxed{x=6}$ $\quad \cancel{3}x=\cancel{12}\,4$

$\boxed{x=4}$

$(6,0)\ (4,0)$

pts $(5,1)(5,-1)(6,0)(4,0)$

① $(5,1)$

$ln-m^2=(6x-30)(6x-30)-36y^2$

$=(30-30)(30-30)-36(1)^2$

$=-36<0$

$(5,1)$ is not an extreme value

② $(5,-1)$

$ln-m^2=(6x-30)(6x-30)-36y^2$

$=0.0-36(-1)^2$

$=36\not<0$ $\quad (5,-1)$ is not an extreme value

③ $(6,0)$

$ln-m^2=(6x-30)(6x-30)-36y^2$

$=(6)(6)$

$=36>0$

$l=6>0$

$(6,0)$ is a minimum point

$f(6,0)=x^3+3xy^2-15x^2-15y^2+72x$

$=6^3+0-15(36)-0+72(6)$

$=108$ is minimum value

④ $(4,0)=(6x-30)(6x-30)-36y^2$

$=(24-30)(24-30)-0$

$=(-6)(-6)-0$

$=36>0$

$l=-6<0$ $\quad (4,0)$ is maximum point

$(4,0)=4^3+0-15(4)^2+0+72(4)=112$ is maximum value

★★★★ Imp. Find the maximum and minimum value of function $\sin x + \sin y + \sin(x+y)$.

Sol $f(x,y) = \sin x + \sin y + \sin(x+y)$

then $\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$

$l = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$

$m = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$

$n = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

for maximum (or) minimum = $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$\cos x + \cos(x+y) = 0 \rightarrow (1)$, $\cos y + \cos(x+y) = 0 \rightarrow (2)$

eq (1) - eq (2)

$\cos x + \cancel{\cos(x+y)} = \cos y + \cancel{\cos(x+y)}$

$\cos x = \cos y$

$x = y \rightarrow (3)$

∴ eq (1) $\cos x + \cos(x+x) = 0$

$\cos x + \cos 2x = 0$

$\left[\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right.$ $\{\cos(-\theta) = \cos\theta\}$

$2\cos\frac{3x}{2}\cos\frac{x}{2} = 0$

$\cos\frac{3x}{2} = 0$ & $\cos\frac{x}{2} = 0$

$\cos\frac{3x}{2} = \cos(\pm\pi/2)$, $\cos\frac{x}{2} = \cos(\pm\pi/2)$

$\frac{3x}{\cancel{2}} = \pm\pi/\cancel{2}$ $\quad$ $\frac{x}{\cancel{2}} = \pm\pi/\cancel{2}$

$\boxed{x = \pm\pi/3}$ $\quad$ $\boxed{x = \pm\pi}$

eq (3) $x = y$

$y = \pm\pi/3$, $y = \pm\pi$

The points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)\left(-\frac{\pi}{3}, -\frac{\pi}{3}\right)(\pi, \pi)(-\pi, \pi)$

① at $(\pi/3, \pi/3)$, $ln - m^2 = [\sin x - \sin(x+y)][-\sin y - \sin(x+y)] - (-\sin(x+y))^2$

$ln - m^2 = [-\sin\pi/3 - \sin\frac{2\pi}{3}][-\sin\frac{\pi}{3} - \sin\frac{2\pi}{3}] - [-\sin\frac{2\pi}{3}]^2$

$ln - m^2 = [-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}][-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}] - [-\frac{\sqrt{3}}{2}]^2$

② ~~at $(-\pi/3, -\pi/3)$~~ $= \left(\frac{-2\sqrt{3}}{2}\right)\left(\frac{-2\sqrt{3}}{2}\right) - \frac{3}{4}$

$= 3 - \frac{3}{4} = \frac{9}{4} > 0$

and $l = -\sin x - \sin(x+y) = -\sin\frac{\pi}{3} - \sin\left(\frac{2\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$

$l = -\sqrt{3} < 0$

$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a maximum point.

The maximum value is, $f = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{\pi}{3} + \frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}$ //.

② at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$ $ln - m^2 = \left(-\sin\left(\frac{-\pi}{3}\right) - \sin\left(\frac{-2\pi}{3}\right)\right)\left(-\sin\left(\frac{-\pi}{3}\right) - \sin\left(\frac{-2\pi}{3}\right)\right) - \left(-\sin\left(\frac{-2\pi}{3}\right)\right)^2$

$ln - m^2 = \left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right]\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{2\sqrt{3}}{2} \times \frac{2\sqrt{3}}{2} - \frac{3}{4}$

$= 3 - \frac{3}{4} = \frac{9}{4} > 0$

and $l = -\sin x - \sin(x+y)$.

$l = -\sin\left(\frac{-\pi}{3}\right) - \sin\left(\frac{-2\pi}{3}\right)$

$l = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3} > 0$

$\therefore \left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$ is a minimum point

The minimum value.

$f\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right) = \sin\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3} - \frac{\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$

at $(\pm\pi, \pm\pi)$, $ln - m^2 = 0$. there is a need for further investigation.

****

1. Find the maximum and minimum value of the function $f(x, y) = x^3y^2(1-x-y)$

Sol $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y^2 - x^4y^2 - x^3y^3)$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$= x^2y^2[3-4x-3y]$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3y^2 - x^4y^2 - x^3y^3)$$

$$= 2x^3y - 2x^4y - 3x^3y^2$$

$$= x^3y[2-2x-3y]$$

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(x^2y^2[3-4x-3y]$$

$$= \frac{\partial}{\partial x}[3x^2y^2 - 4x^3y^2 - 3x^2y^3]$$

$$= 6xy^2 - 12x^2y^2 - 6xy^3$$

$$= 6xy^2[1-2x-y]$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}[2x^3y - 2x^4y - 3x^3y^2]$$

$$= 6x^2y - 8x^3y - 9x^2y^2$$

$$= x^2y[6-8x-9y]$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2x^3y - 2x^4y - 3x^3y^2)$$

$$= 2x^3 - 2x^4 - 6x^3y$$

$$= 2x^3(1-x-3y)$$

For maximum & minimum

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$$

$x^2y^2[3-4x-3y) = 0$; $x^3y(2-2x-3y) = 0$

$x^2 = 0, y^2 = 0, 3-4x-3y = 0 \rightarrow (1)$ $x^3 = 0; y = 0; 2-2x-3y = 0 \rightarrow (2)$

Solve eq ① & ② mul with 2 eq②

$3-4x-3y=0$

$4-4x-6y=0$

$-1+3y=0$

$3y=1$

$\boxed{y=1/3}$ in eq ①

$3-4x-3(\frac{1}{3})=0$

$2-4x$

$\boxed{x=1/2}$

Now put $x=0$ to

eq ① ⇒ $3-4(0)-3y=0$

$3-3y$

$\boxed{y=1}$

when $y=0$

eq ① $3-4x-3(0)=0$

$3=4x$

$\boxed{x=3/4}$

when $x=0$

eq ② $2-2(0)-3y=0$

$2=3y$

$\boxed{y=2/3}$

when $y=0$

eq ② $2-2x-3(0)=0$

$2=2x$

$\boxed{x=1}$

The points are $(0,1)$ $(3/4, 0)$ $(0, 2/3)$ $(1,0)$ $(1/2, 1/3)$

At all these points except $(\frac{1}{2},\frac{1}{3})$, $ln-m^2=0$, there is not an extreme value At $(\frac{1}{2},\frac{1}{3})$

$$ln-m^2 = 6xy^2[1-2x-y]\,2x^3[1-x-3y]-[x^2y(6-8x-9y)]^2$$

$$= 12x^4y^2[1-2x-y][1-x-3y]-x^4y^2(6-8x-9y)^2$$

$$= 12\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{3}\right)^2\left[1-2\left(\tfrac{1}{2}\right)-\tfrac{1}{3}\right]\left[1-\tfrac{1}{2}-3\left(\tfrac{1}{3}\right)\right]-\left[\tfrac{1}{2}\right]^4\left[\tfrac{1}{3}\right]^2\left(6-8\left(\tfrac{1}{2}\right)-9\left(\tfrac{1}{3}\right)\right)^2$$

$$= 12\left(\tfrac{1}{16}\right)\left(\tfrac{1}{9}\right)[-1/3][-1/2]-\tfrac{1}{16}\tfrac{1}{9}[-1]^2$$

$$= \tfrac{1}{16}\tfrac{1}{9}[2-1]$$

$$ln-m^2 = \tfrac{1}{144}>0$$

and

$$l = 6xy^2(1-2x-y)$$

$$= 6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)^2\left[1-2\left(\tfrac{1}{2}\right)-\tfrac{1}{3}\right]$$

$$= 6\left(\tfrac{1}{2}\right)\left(\tfrac{1}{9}\right)[-1/3]$$

$$= -1/9<0$$

$\therefore \left(\frac{1}{2}, \frac{1}{3}\right)$ is a maximum point

The maximum value is

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = x^3y^2(1-x-y)$$
$$= \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right)$$
$$= \frac{1}{8}\left(\frac{1}{9}\right)\left(\frac{1}{6}\right)$$
$$= \frac{1}{72}\left(\frac{1}{6}\right) = \frac{1}{432}$$

② Find three +ve number whose sum is 100 and whose product is maximum

S: Let x, y, z be the required 3 numbers given that

$$x+y+z = 100 \rightarrow k \text{ (say)} \rightarrow ①$$
$$f(x,y,z) = xyz \rightarrow ②$$

eq ① $z = k - x - y$

sub the value of z in eq ②, we get

$$f = xy(k-x-y)$$
$$= xyk - x^2y - xy^2$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xyk - x^2y - xy^2) \; ; \; \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xyk - x^2y - xy^2)$$
$$= yk - 2xy - y^2 \qquad = xk - x^2 - 2xy$$

$$l = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(yk - 2xy - y^2) = -2y$$

$$m = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}(xk - x^2 - 2xy) = \cancel{2}\; k - 2x - 2y$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}(xk - x^2 - 2xy) = -2x$$

For maximum and minimum $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$xyk - x^2y - xy^2 = 0 \quad ; \quad xk - x^2 - 2xy = 0$$
$$xy(k-x-y) = 0 \quad ; \quad x(k-x-2y) = 0$$
$$x = 0 \quad y = 0 \qquad\qquad x = 0$$
$$k - x - y = 0 \rightarrow ③ \qquad k - x - 2y \rightarrow ④$$

$$\cancel{k}-\cancel{x}-y=0$$
$$\cancel{k}-\cancel{x}-2y=0$$
$$\underline{-\quad+\quad+\qquad}$$
$$y=0$$

$k=2x+y \to (3)$ $\qquad x=k-2y \to (4)$

Sub the value of x in eq (3)

$k=2(k-2y)+y$

$k=2k-4y+y \rightarrow k=2k-3y \rightarrow 3y=2k-k$

$$\boxed{y=k/3}$$

eq (4) $x=k-2\left(\frac{k}{3}\right)$

$$\boxed{x=k/3}$$

The point is $(k/3, k/3)$

at $(k/3, k/3)$

at $\left(\frac{k}{3},\frac{k}{3}\right)$, $ln-m^2=(-2y(-2x)-[k-2x-2y]^2$

$=4xy-[k-2x-2y]^2$

$=4\frac{k}{3}\frac{k}{3}-\left[k-2\frac{k}{3}-\frac{2k}{3}\right]^2$

$=\frac{4k^2}{9}-\left[-\frac{k}{3}\right]^2$

$ln-m^2=\frac{4k^2}{9}-\frac{k^2}{9}=\frac{3k^2}{9}=\frac{k^2}{3}>0$

and $l=-2y=-\frac{2k}{3}<0$

$\therefore \left(\frac{k}{3},\frac{k}{3}\right)$ is maximum point

Hence $f(x,y)$ has a maximum at $\left(\frac{k}{3},\frac{k}{3}\right)$

from eq (1) $z=k-x-y$

$z=k-\frac{k}{3}-\frac{k}{3}$

$z=k/3$

$\therefore x=\frac{k}{3}, y=\frac{k}{3}, z=\frac{k}{3}$

The required numbers are $\frac{k}{3}, \frac{k}{3}, \frac{k}{3}$ i.e $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$

Thus the product is maximum when all the number are equal.

① A rectangular box open at the top is in have volume of 32 cube ~~fts~~. Find the dimension of the box required least material for its construction

Sol Let x ft, y ft, z ft and 's'

and 's' be the surface area of the box

then $S = nxy + 2yz + 2zx$ { n=1 the box is open at the top
n=2 The box is closed }

$V = xyz = 32$ cube ft.

$S = xy + 2yz + 2zx \rightarrow ①$

$xyz = 32$

$z = \frac{32}{xy}$

Substituting the value of z in eq ①

$S = xy + 2\cancel{y}\left(\frac{32}{x\cancel{y}}\right) + 2\left(\frac{32}{\cancel{x}y}\right)\cancel{x}$

$S = xy + \frac{64}{x} + \frac{64}{y}$

$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(xy + \frac{64}{x} + \frac{64}{y}\right) = y \quad \frac{64}{x^2}$

$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(xy + \frac{64}{x} + \frac{64}{y}\right) = x - \frac{64}{y^2}$

$l = \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x}\left(y \cdot \frac{64}{x^2}\right) = \frac{128}{x^3}$

$m = \frac{\partial^2 s}{\partial x \partial y} = \frac{\partial}{\partial x}\left(x - \frac{64}{y^2}\right) = 1$

$n = \frac{\partial^2 s}{\partial y^2} = \frac{\partial}{\partial y}\left(x - \frac{64}{y^2}\right) = \frac{128}{y^3}$

maximum or minimum

$\frac{\partial s}{\partial x} = 0, \quad \frac{\partial s}{\partial y} = 0$

$y = \frac{64}{x^2} = 0 \rightarrow (2), \quad x - \frac{64}{y^2} = 0$

$x = \frac{64}{y^2}$

Substitution the value of x in eq (2)

$y - \frac{64}{\left(\frac{64}{y^2}\right)^2} = 0$

$y - y^4 \frac{64}{(64)^{\cancel{x}}} = 0$

$\frac{64y - y^4}{64} = 0 \Rightarrow y(64 - y^3) = 0$

$64 - y^3 = 0$

$y^3 = 64$

$\therefore (x, y) = (4, 4) \quad \boxed{y = 4}$

at $(4, 4)$, $ln - m^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 1^2$

$ln - m^2 = \frac{\cancel{128}}{\cancel{64}} \cdot \frac{\cancel{128}}{\cancel{64}} - 1$

$= 4 - 1$

$= 3$

$ln - m^2 = 3 > 0$ and

$l = \frac{128}{x^3} = \frac{128}{\cancel{64}} = 2 > 0$

$(4, 4)$ is the minimum point

Now, $z = \frac{32}{4 \cdot 4} = 2$

$\boxed{z = 2}$

The dimensions of the rectanglar box for least material for its condition are $4, 4, 2$.

* Langrange's Method of undetermined Multiplier

Working rule:

let $f(x, y, z)$ is a given function, $\phi(x, y, z) = 0$ is any condition from lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

where λ is called langrange's multiplier

find $\frac{\partial F}{\partial x} = 0$; i.e $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\frac{\partial F}{\partial y} = 0$; i.e $\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

$\frac{\partial F}{\partial z} = 0$; i.e $\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

solving them find x, y & z

we find the value of $f(x, y, z)$ using this 's' we find critical point for $f(x, y, z)$

(1) Find the minimum value of $x^2+y^2+z^2$ given $x+y+z=3a$

sol let $f(x, y, z) = x^2+y^2+z^2$

$\phi(x, y, z) = x+y+z-3a = 0 \longrightarrow (1)$

then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$ 1 Mark

$\frac{\partial \phi}{\partial x} = 1$, $\frac{\partial \phi}{\partial y} = 1$, $\frac{\partial \phi}{\partial z} = 1$

~~By~~ ~~By~~ consider langrange's function

$$F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$$

$\frac{\partial f}{\partial x} = 0$; i.e $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow 2x + \lambda(1) = 0 \rightarrow x = -\lambda/2$

$\frac{\partial F}{\partial y} = 0$; i.e $\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow 2y + \lambda(1) = 0 \rightarrow y = -\lambda/2$

$\frac{\partial F}{\partial x} = 0$; $\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \rightarrow 2z + \lambda(1) = 0 \Rightarrow z = -\lambda/2$

Substituting the values of x, y & z eq (1)

$-\frac{\lambda}{2}-\frac{\lambda}{2}-\frac{\lambda}{2}-3a=0$

$-\frac{3\lambda}{2}=3a$

$\boxed{\lambda=-2a}$

Now $x=-\frac{\lambda}{2}=-\frac{(-2a)}{2}=a$

$y=-\frac{\lambda}{2}=-\frac{(-2a)}{2}=a$

$z=-\frac{\lambda}{2}=-\frac{(-2a)}{2}=a$

$\boxed{\therefore x=y=z=a}$

The minimum values of $f(x,y,z)$ is $a^2+a^2+a^2=3a^2$

② Find the maximum & minimum value of $x+y+z$ ~~giv~~ subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol Let $f(x,y,z)=x+y+z$

$\phi(x,y,z)=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0 \rightarrow (1)$

then $\frac{\partial f}{\partial x}=1, \ \frac{\partial f}{\partial y}=1, \ \frac{\partial f}{\partial z}=1$

$\frac{\partial \phi}{\partial x}=-\frac{1}{x^2}, \ \frac{\partial \phi}{\partial y}=-\frac{1}{y^2}, \ \frac{\partial \phi}{\partial z}=-\frac{1}{z^2}$

consider lagrange's function

$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$

$\frac{\partial f}{\partial x}=0$; i.e, $\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0 \rightarrow 1+\lambda\left(-\frac{1}{x^2}\right)=0 \Rightarrow \lambda=x^2 \rightarrow x=\pm\sqrt{\lambda}$

$\frac{\partial F}{\partial y}=0$; i.e, $\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial x}=0 \rightarrow 1+\lambda\left(-\frac{1}{y^2}\right)=0 \rightarrow \lambda=y^2 \rightarrow y=\pm\sqrt{\lambda}$

$\frac{\partial F}{\partial z}=0$; i.e $\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0 \Rightarrow 1+\lambda\left(-\frac{1}{z^2}\right)=0 \rightarrow \lambda=z^2 \rightarrow z=\pm\sqrt{\lambda}$

Substitution the values of x, y, & , eq (1)

$\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}-1=0$

$\frac{3}{\sqrt{\lambda}}=1 \rightarrow 3=\sqrt{\lambda} \rightarrow \lambda=3^2 \rightarrow \boxed{\lambda=9}$

Now $x = \pm\sqrt{\lambda} = \pm\sqrt{9} = \pm 3$

$y = \pm\sqrt{\lambda} = \pm\sqrt{9} = \pm 3$

$z = \pm\sqrt{\lambda} = \pm\sqrt{9} = \pm 3$

$\boxed{\therefore x = y = z = \pm\sqrt{3}}$

$\therefore$ The maximum value of $f(x,y,z)$ is $3+3+3 = 9$
The minimum value of $f(x,y,z)$ is $-3-3-3 = -9$.

(3) Find the dimension of the rectangular parallelo pipe box open at the top of maximum capacity whose surface area is

① 108 sq.cm → 6, 6, 3

② 256 sq.cm

③ 432 sq.cm → 12, 12, 6

④ ~~5~~

Let x, y, z be the dimensions of the rectangular box
then volume of box $= V = xyz$

Say $f(x,y,z) = xyz$

It is given that surface area of the rectangular box $s = 256$ sq.cm

$f(x,y,z) = xy + 2yz + 2zx = 256$ {The box is open at the top}

$\phi(x,y,z) = xy + 2yz + 2zx - 256 = 0 \rightarrow ①$

consider lagrange's function

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

then $\frac{\partial f}{\partial x} = yz$, $\frac{\partial f}{\partial y} = xz$, $\frac{\partial f}{\partial z} = xy$

$\frac{\partial \phi}{\partial x} = y + 2z$, $\frac{\partial \phi}{\partial y} = x + 2z$, $\frac{\partial \phi}{\partial z} = 2y + 2x$

$\frac{\partial f}{\partial x} = 0$; ie $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0$

$-\lambda(y + 2z) = yz$

$-\lambda = \frac{yz}{y+2z}$

$-\frac{1}{\lambda} = \frac{y+2z}{yz} = \frac{y}{yz} + \frac{2z}{yz}$

$\boxed{-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{y}} \rightarrow ②$

$\frac{\partial F}{\partial y}=0 ; \quad \frac{\partial f}{\partial y}+\lambda \frac{\partial(\phi)}{\partial y}=0$

$x^2+\lambda(x+2z)=0$

$-\lambda(x+2z)=xz$

$\lambda=\frac{xz}{x+2z}$

$-\frac{1}{\lambda}=\frac{x+2\lambda}{xz}=\frac{x}{xz}+\frac{2z}{xz}$

$\boxed{-\frac{1}{\lambda}=\frac{1}{z}+\frac{2}{x}} \rightarrow ③$

$\frac{\partial F}{\partial z}=0 ; \quad \frac{\partial f}{\partial z}+\lambda \frac{\partial \phi}{\partial z}=0$

$\cancel{\partial y} \quad xy+\lambda(2x+2y)=0$

$-\lambda(2x+2y)=xy$

$-\frac{1}{\lambda}=\frac{2x+2y}{xy}=\frac{2\cancel{x}}{\cancel{x}y}+\frac{2y}{xy}$

$\boxed{-\frac{1}{\lambda}=\frac{2}{x}+\frac{2}{y}} \rightarrow ④$

Solving eqⁿ ② & ③

$\cancel{\frac{1}{z}}+\frac{2}{y}=\cancel{\frac{1}{z}}+\frac{2}{x}$

$\cancel{2}x=\cancel{2}y$

$\boxed{x=y}$

Solving eq ③ & ④ $\quad \frac{1}{z}+\cancel{\frac{2}{x}}=\cancel{\frac{2}{x}}+\frac{2}{y}$

$\boxed{y=2z}$

$\boxed{\therefore x=y=2z}$

Substituting the value of x, y in eq ①

$(2z)(2z)+2(2z)z+2z(2z)=256$

$4z^2+4z^2+4z^2=256$

$12z^2=256$

$z^2=\frac{256}{12}$

$z=\sqrt{\frac{256}{12}}=\frac{16}{\sqrt{12}}=\frac{\cancel{16}\,8}{\cancel{2}\sqrt{3}} \qquad \boxed{z=\frac{8}{\sqrt{3}}}$

and $x = 2z = \frac{16}{\sqrt{3}}$

$y = \frac{16}{\sqrt{3}}$

The dimensions of the rectangular $\frac{16}{\sqrt{3}}, \frac{16}{\sqrt{3}}, 8/\sqrt{3}$

(1) Find the dimensions of rectangular box of maximum capacity whose surface are is 's'

$x = nxy + 2yz + 2zx$ { $n=1$ the box is open
$n=2$ " " " closed }

(2) Find maximum & minimum distance of the 3, 4, 12 from the sphere ~~from~~ + $x^2+y^2+z^2=1$

S:- let $P(x, y, z)$ be any point on the ~~surf~~ sphere & $A(3,4,12)$ be the given point

Then we have

$$PA^2 = (x-3)^2 + (y-4)^2 + (z-12)^2 = f(x,y,z) \text{ say}$$

$$\phi(x,y,z) = x^2+y^2+z^2-1=0$$

$$\frac{\partial f}{\partial x} = 2(x-3); \quad \frac{\partial f}{\partial y} = 2(y-4); \quad \frac{\partial f}{\partial z} = 2(z-12)$$

$$\frac{\partial \phi}{\partial x} = 2x; \quad \frac{\partial \phi}{\partial y} = 2y, \quad \frac{\partial \phi}{\partial z} = 2z$$

consider lagranges function

$$F(x,y,z) = f(x,y,z) + \lambda \phi(x,y,z)$$

$$\frac{\partial f}{\partial x} = 0; \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$$2(x-3) + \lambda(2x) = 0$$

$$2(x-3+\lambda x] = 0$$

$$x(1+\lambda) - 3 = 0$$

$$x = \frac{3}{1+\lambda}$$

$$\frac{\partial F}{\partial x} = 0; \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$2(y-4) + \lambda(2y) = 0$$

$$y + \lambda y - 4 = 0$$

$$y(1+\lambda) = 4$$

$$\boxed{y = \frac{4}{1+\lambda}}$$

$\frac{\partial f}{\partial z}=0$; $\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$

$$2(z-12)+\lambda(2z)=0$$

$$z+\lambda z-12=0$$

$$z(1+\lambda)=12$$

$$\boxed{z=\frac{12}{1+\lambda}}$$

Substituting the value of x, y, z in eq (1)

$$\left(\frac{3}{1+\lambda}\right)^2+\left(\frac{4}{1+\lambda}\right)^2+\left(\frac{12}{1+\lambda}\right)^2-1=0$$

$$\frac{9+16+144}{(1+\lambda)^2}=1$$

$$169=(1+\lambda)^2$$

$$1+\lambda=\sqrt{169}$$

$$1+\lambda=\pm 13$$

$$\lambda=\pm 13-1$$

$$\lambda=+13-1 \; ; \; \lambda=-13-1$$

$$\boxed{\lambda=12} \quad \boxed{\lambda=-14}$$

when $\lambda=12$

$$x=\frac{3}{13},\ y=\frac{4}{13},\ z=\frac{12}{13}$$

when $\lambda=-14$

$$x=-\frac{3}{13},\ y=-\frac{4}{13},\ z=-\frac{12}{13}$$

Thus, we get two stationary points

$$R\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right),\ S=\left(-\frac{3}{13},-\frac{4}{13},-\frac{12}{3}\right)$$

The maximum distance is

$$AS^2=\left(3+\frac{3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2=196$$

$$AS=\sqrt{196}=14$$

The minimum distance is

$$AR^2=\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2=144$$

$$AR=\sqrt{144}=12$$

③

① ~~256 sqr~~ 108 sq.cm

Let x, y, z be dimension of the rectangular box

Volume of box $V = xyz$

$f(x, y, z) = xyz$

Surface area of rectangular box = 108 sq.cm

$f(x, y, z) = xy + 2yz + 2zx = 256$

$\phi(x, y, z) = xy + 2yz + 2zx - 256 = 0$

consider lagrange's function

$F(x, y, z) = f(x, y, z) + \phi(x, y, z)$

then

$\frac{\partial f}{\partial x} = yz$, $\frac{\partial f}{\partial y} = zx$, $\frac{\partial f}{\partial z} = xy$

$\frac{\partial f}{\partial \phi} = y + 2z$, $\frac{\partial \phi}{\partial y} = x + 2z$, $\frac{\partial \phi}{\partial z} = 2y + 2x$

$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$yz + (y + 2z)\lambda = 0$

$yz = -\lambda(y + 2z)$

$-\lambda = \frac{yz}{y + 2z}$

$\lambda = -\frac{yz}{y + 2z}$

$-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{y}$

$\frac{\partial f}{\partial y} = 0$; $\frac{\partial f}{\partial y} + \lambda \frac{\partial(\phi)}{\partial y} = 0$

$x^2 + \lambda(x + 2z) = 0$

$-\lambda(x + 2z) = xz$

$\lambda = \frac{xz}{x + 2z}$

$-\frac{1}{\lambda} = \frac{x + 2z}{xz}$

$-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{x}$

$\frac{\partial f}{\partial z} = 0$ $\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

$xy + \lambda(2x + 2y) = 0$

$-\frac{1}{\lambda} = \frac{2x + 2y}{xy}$

$= \frac{2x}{xy} + \frac{2y}{xy}$

$-\frac{1}{\lambda} = \frac{2}{x} + \frac{2}{y}$

Solving eqn ② & ③

$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$

$2x = 2y$

$\boxed{x = y}$

sol eq ③ & ④

$\frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$

$\boxed{y = 2z}$

$\therefore x = y = 2z$

sub the value

$(2z)(2z) + 2(2z)z + 2z(2z) = 108$

$4z^2 + 4z^2 + 4z^2 = 108$

$12z^2 = 108$

$z^2 = \frac{108}{12} 9$

$z^2 = 9$

$\boxed{z = 3}$

$x = 2z$

$= 2(3)$

$\boxed{x = 6}$ $\boxed{\because x = y}$

$x = 6, y = 6, z = 3$

(3) 432 sq.cm.

same as before problems

Sub the values

$1\cancel{2}z^2 = 432$

$12z^2 = \cancel{18}\ 432$

$z^2 = \frac{\cancel{432}\ 108\ 36}{\cancel{12}\ 3}$

$\boxed{z=6}$

$x = 2z$

$x = 12,\ y = 12,\ z = 6$

(4) Find the point on the plane $x+2y+3z=4$ which is nearest to the orgin

S:- let $P(x,y,z)$ be any point on the sphere &

& $A(0,0,0)$ be the given point

Then we have

$PA^2 = (x-0)^2+(y-0)^2+(z-0)^2 = f(x,y,z)$ say

$\phi(x,y,z) = x+2y+3z = 4$

$\frac{\partial f}{\partial x} = 2(x)$; $\frac{\partial f}{\partial y} = 2y$; $\frac{\partial f}{\partial z} = 2z$; $\frac{\partial \phi}{\partial x} = 1$, $\frac{\partial \phi}{\partial y} = 2$, $\frac{\partial \phi}{\partial z} = 3$

consider lagranges function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$\frac{\partial f}{\partial x} = 0$ $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$ $2x+\lambda = 0$ $\boxed{x = -\lambda/2}$

$\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$ $2y+2\lambda = 0$ $\boxed{y = -\lambda}$

$\frac{\partial f}{\partial z} = 0$; ie $\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$ $2z+3\lambda = 0$ $\boxed{z = \frac{-3\lambda}{2}}$

Sub x,y,z in eq(1) $x+2y+3z-4=0$

$-\frac{\lambda}{2} + 2(-\lambda) + 3\left(\frac{3\lambda}{2}\right) - 4 = 0 \rightarrow -\frac{14\lambda}{2} - 4 = 0$

$\cancel{\frac{-\lambda - 4\lambda - 9\lambda}{2} - 8 = 0}$

$\cancel{\frac{-14\lambda}{2} + 8}$

$-\frac{14\lambda}{2} = 4$

$-14\lambda = 8$

$\lambda = \frac{\cancel{8}\ 4}{\cancel{-14}\ 7}$ $\lambda = \frac{-4}{7}$

Sub in x,y,z $x = \lambda/2$, $y = -\lambda$ $z = \frac{-3\lambda}{2}$

$x = \frac{-\cancel{4}\ 2}{\cancel{14}\ 7}$ $y = 4/7$ $z = \frac{-3\left(\cancel{-4}^{2}/7\right)}{\cancel{2}} = +\frac{6}{7}$ $\Rightarrow \frac{2}{7}, \frac{4}{7}, \frac{6}{7}$