US

(1) Given,

$f(z) = \frac{1}{z^2-3z+2}$ in $1<|z|<2$ in Laurent's Series

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$1 = -2A - B$

$A = -B$

$B = 1, A = -1$

$$1 = A(z-2) + B(z-1)$$

$$1 = -1(z-2) + (z-1)$$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$1 < z$

$\frac{1}{z} < 1$

$$\frac{-1}{z-1} = \frac{-1}{z(1-\frac{1}{z})} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

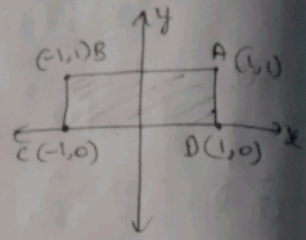
$$= -1\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right)$$

$$\frac{1}{z-2} = \frac{1}{-2(1-\frac{z}{2})} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{z^n}{2^{n+1}}\right)$$

$$\therefore \boxed{f(z) = -1\left(\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right) + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right)}$$

② Given, $f(z) = \int z^3 dz$ of ▭le at $-1, 1, 1+i, -1+i$



Now along AB :- $z = x+iy$, $dz = dx + idy$

$y = 1 \rightarrow dy = 0$

$x \rightarrow 1 : -1$

$\int_{AB} f(z)\,dz = \int_{AB} z^3 dz$

$= \int_{AB} (x+iy)^3 (dx+idy) = \int_1^{-1} (x+i)^3 dx = \int_1^{-1} x^3 + i^3 + 3xi^2 + 3ix^2\, dx$

$= \left[\frac{x^4}{4} + i^3 x + \frac{3x^2 i^2}{2} + \frac{3ix^3}{3}\right]_1^{-1}$

$= \left[\frac{x^4}{4} - ix - \frac{3}{2}x^2 + ix^3\right]_1^{-1} = \left[\frac{1}{4} + i - \frac{3}{2} - i\right] - \left[\frac{1}{4} - i - \frac{3}{2} + i\right]$

$= 0$

Along BC :-

$x = -1$ $dx = 0$

$y : 1 \rightarrow 0$

$\int_{BC} f(z)\,dz = \int_{BC} (x+iy)^3 (dx+idy) = \int_1^0 (-1+iy)^3\, idy$

$= i\int_1^0 (-1 - iy^3 + 3iy + 3(-1)y^2(-1))\, dy$

$= i\left[-y - i\frac{y^4}{4} + \frac{3iy^2}{2} + \frac{3y^3}{3}\right]_1^0 = i\left[(0) - \left(-1 - \frac{i}{4} + \frac{3i}{2} + 1\right)\right]$

$= -\frac{1}{4} + \frac{3}{2} = +\frac{5}{4}$

Along CD :-

$y = 0$ $dy = 0$, $x : -1$ to 1

$\int_{CD} f(z)\,dz = \int_{CD} (x+iy)^3 (dx+idy) = \int_{-1}^{1} x^3 dx = \left[\frac{x^4}{4}\right]_{-1}^{1}$

$= \frac{1}{4} - \left[\frac{1}{4}\right] = 0$

Along DA :-

$x = 1$ $dx = 0$, $y : 0$ to 1

$\int_{DA} f(z)\,dz = \int_{DA} (x+iy)^3 (dx+idy) = \int_0^1 (1+iy)^3\, idy$

$= i\int_0^1 (1+iy)^3 dy = i\int_0^1 (1 - iy^3 + 3iy - 3y^2)\, dy$

$= i\left[y - i\frac{y^4}{4} + \frac{3iy^2}{2} - y^3\right]_0^1 = i\left[1 - \frac{i}{4} + \frac{3i}{2} - 1\right]$

$= \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$

$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$

$= 0 + \frac{5}{4} + 0 - \frac{5}{4}$

$= 0$

$\therefore \int_C f(z)\,dz = 0$

$\therefore$ Cauchy's theorem is verified.

③ $f(z) = \sinh z$ about $z = i\pi$ taylor series

~~By taylor series,~~ $\cancel{\int f(z)\,dz = \frac{1}{n!}\sum_{n=0} \frac{a f^n(a)}{(z-z_0)^n}}$

By taylor series, $z = a$

$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

Here $a = i\pi$

$f(z) = \sinh z = \sinh(i\pi) = i\sin\pi = 0$

$f'(z) = \cosh z = \cos\pi = -1$

$f''(z) = \sinh z = i\sinh(\pi) = 0$

$f'''(z) = \cosh z = \cos\pi = -1$

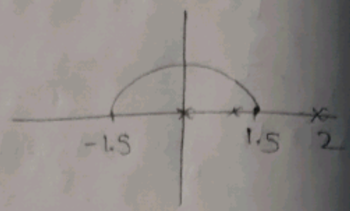
even power's 0,

odd power's -1

$$f(z) = \frac{-1}{1}(z-i\pi) - \frac{1}{3!}(z-i\pi)^3 - \frac{1}{5!}(z-i\pi)^5 - \dots$$

$$\boxed{\therefore \sinh z = -(z-i\pi) - \frac{(z-i\pi)^3}{3!} - \frac{(z-i\pi)^5}{5!} - \dots}$$

(4) $f(z) = \dfrac{4z-3}{z(z-1)(z-2)}$ $\quad C : |z| = \frac{3}{2}$



i) $z=0$ (inside c'le)

Simple pole

ii) $z-1=0$

$z=1$ (inside c'le)

Simple pole

iii) $z-2=0$

$z=2$ (outside c'le)

Simple pole

we need to find residue w.r.t $z=0,1$

By Cauchy residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \sum Res \ \{$$

$$Res_{(z=z_0)} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Now at $z=0$:

$$Res_{z=0} = \lim_{z\to 0} z \cdot \frac{4z-3}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{4z-3}{(z-1)(z-2)}$$

$$= \frac{-3}{2} = \frac{-3}{2}$$

at $z=1$:

$$Res_{z=1} = \lim_{z\to 1} (z-1)\frac{4z-3}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{4z-3}{z(z-2)}$$

$$= \frac{1}{-1} = -1$$

Now

$$\sum Res = Res_{z=0} + Res_{z=1}$$

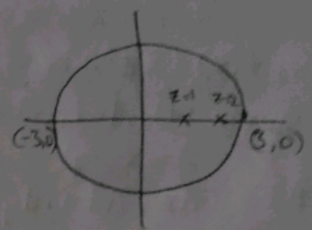
$$= -\frac{3}{2} + (-1) = -\frac{5}{2}$$

$$\int_C f(z)\,dz = 2\pi i \times -\frac{5}{2}$$

$$= -5\pi i$$

$$\boxed{\therefore \int_C \frac{4z-3}{z(z-1)(z-2)}\,dz = -5\pi i}$$

(5) $f(z) = \dfrac{e^{2z}}{(z-1)(z-2)}$, $\quad C : |z| = 3$



i) $z-1=0$

$z=1$ (inside c'le)

Simple pole

ii) $z-2=0$

$z=2$ (inside c'le)

Simple pole

we need to find residue w.r.t $z=1,2$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \sum Res$$

$$Res_{z=z_0} = \lim_{z\to z_0} (z-z_0) f(z)$$

at $z=1$:-

$$Res_{(z=1)} = \lim_{z\to 1} (z-1)\frac{e^{2z}}{(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{e^{2z}}{z-2}$$

$$= -e^2$$

at $z=2$:-

$$Res_{(z=2)} = \lim_{z\to 2} (z-2)\frac{e^{2z}}{(z-1)(z-2)}$$

$$= \lim_{z\to 2} \frac{e^{2z}}{z-1}$$

$$= e^4$$

Now, $\sum Res = Res_{(z=1)} + Res_{(z=2)}$

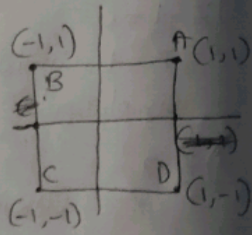
$$= -e^2 + e^4$$

$$\boxed{\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = 2\pi i \times \left(e^2\left(e^2-1\right)\right)}$$

⑥ Given, $f(z) = 3z^2 + iz - 4$

C is square $1+i, 1-i, (-1+i), (-1-i)$

$z = x+iy$



along AB :- $dz = dx + i\,dy$

$y=1$, x: 1 to -1

$dy = 0$

$$\int_{AB} f(z)\,dz = \int_{AB} 3(z)^2 + iz - 4 \; dz$$

$$= \int_{1}^{-1} 3(x+i)^2 + i(x+i) - 4 \; dx$$

$$= \int_{1}^{-1} 3x^2 + 6xi - 3 + ix - 5 \; dx$$

$$= \left[x^3 + \frac{7}{2}x^2 i - 8x\right]_1^{-1} = \left[-1+\frac{7}{2}i+8\right] - \left[1+\frac{7}{2}i-8\right]$$

$$= 14$$

along BC :-

$x=-1$ $dx=0$, y: 1 to -1

$$\int_{BC} f(z)\,dz = \cancel{\int (-1+iy)\,i\,dy}$$

$$= \int_{1}^{-1} 3(-1+iy)^2 + i(-1+iy) - 4 \; i\,dy$$

$$= i\int_{1}^{-1} 3 - 3y^2 - 6iy - i - y - 4 \; dy$$

$$= i\left[-y^3 - 3iy^2 - iy - y^2 - y\right]_1^{-1}$$

$$= i\left(\left[1-3i+i-1+1\right] - \left[-1-3i-i-1-1\right]\right)$$

$$= 4i - 2$$

along CD :-

$y=-1$, x: -1 to 1

$dy=0$

$$\int_{CD} f(z)\,dz = \int_{-1}^{1} 3(x-i)^2 + i(x-i) - 4 \; dx$$

$$= \int_{-1}^{1} 3x^2 - 3 - 6xi + ix + 1 - 4 \; dx$$

$$= \left[x^3 - \frac{5}{2}x^2 i - 6x\right]_{-1}^{1} = \left[1 - \frac{5}{2}i - 6\right] - \left[-1 - \frac{5}{2}i + 6\right]$$

$$= -\frac{5}{2}i - 5 + \frac{5}{2}i - 5 = -10$$

along DA :-

$x=1$, y: -1 to 1

$$\int_{DA} f(z)\,dz = \int_{-1}^{1} 3(1+iy)^2 + i(1+iy) - 4 \; i\,dy$$

$$= i\int_{-1}^{1} 3 - 3y^2 + 6iy + i - y - 4 \; dy$$

$$= i\left[-y^3 + 3iy^2 + iy - y^2 - y\right]_{-1}^{1} = i\left(\left[-1+3i+i-1-1\right] - \left[1+3i-1-1+1\right]\right)$$

$$= i(4i - 3 - 1 - 2i) = -4i - 2$$

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= 14 + 4i - 2 - 10 - 4i - 2$$

$$= 0$$

$\therefore \int_C f(z)\,dz = 0$

$\therefore$ Cauchy's theorem is verified.

(7) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

i) $(z-1)^2 = 0$

$z = 1$

pole of order 2

ii) $z+2=0$

$z=-2$

simple pole

$$Res_{(z=1)} = \frac{1}{1!} \underset{z\to 1}{Lt} \frac{d}{dz} (z-1)^2 \frac{z^2}{(z-1)^2(z+2)}$$

$$= \frac{1}{1} \underset{z\to 1}{Lt} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= \frac{1}{1} \underset{z\to 1}{Lt} \frac{2z(z+2) - (1)z^2}{(z+2)^2}$$

$$= \frac{1}{1} \times \frac{6-1}{9} = \frac{5}{9}$$

$$Res_{(z=-2)} = \underset{z\to -2}{Lt} (z+2) \frac{z^2}{(z-1)^2(z+2)}$$

$$= \underset{z\to -2}{Lt} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$\therefore$ Residue at $z=-2$ is $\frac{4}{9}$

Residue at $z=1$ is $\frac{5}{9}$

Poles $z=-2$ (simple), $z=1$ (order 2).

(8) i) $w = \dfrac{6z-9}{z}$

Set $w = z$

$z = \dfrac{6z-9}{z}$

$z^2 = 6z - 9$

$z^2 - 6z + 9 = 0$

$(z-3)^2 = 0$

$z = 3$

$\therefore z=3$ is fixed point

ii) $w = \dfrac{z-i}{z+i}$

Set $w=z$ (for fixed pt)

$z = \dfrac{z-i}{z+i}$

$z^2 + iz = z - i$

$z^2 + z(i-1) + i = 0$

$$z = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4i}}{2}$$

$$= \frac{1-i \pm \sqrt{-1+1-2i-4i}}{2} = \frac{1-i \pm \sqrt{-6i}}{2}$$

$\cancel{\sqrt{-6i} = \sqrt{3}(i-1)}$ $(1-i)^2 = -2i$

$(\sqrt{3}(1-i))^2 = 3(-2i) = -6i$

$\sqrt{-6i} = \pm\sqrt{3}(1-i)$

$$= \frac{1-i \pm \sqrt{3}(1-i)}{2}$$

$$z = \frac{(1+i)(1\pm\sqrt{3})}{2}$$

$\therefore z = \dfrac{(1+i)(1+\sqrt{3})}{2}$ & $z = \dfrac{(1+i)(1-\sqrt{3})}{2}$ are fixed points

(9) Given, Z-plane : $z_1 = -1,\ z_2 = 0,\ z_3 = 1$

ω-plane : $w_1 = 0,\ w_2 = i,\ w_3 = 3i$

we have,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{-2iw}{i(w-3i)} = \frac{-(z+1)}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2wz - 2w = wz - 3zi + w - 3i$$

$$wz - 3w = -3zi - 3i$$

$$\therefore \boxed{w = \frac{-3zi-3i}{z-3}}$$

∴ Req Mobius transf^{m} is $\boxed{w = \frac{-3zi-3i}{z-3}}$

(10) Given,

Z-plane :- $z_1 = 0,\ z_2 = 1,\ z_3 = \infty$

ω plane :- $w_1 = -1,\ w_2 = -2,\ w_3 = -i$

we have,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-2+i)}{(w+i)(-2+1)} = \frac{(z-0)}{1}$$

$$2(w+1)(-2+i) = -z(w+i)$$

$$-4w - 4 + 2iw + 2i = -zw - iz$$

$$\cancel{w = \frac{-iz-2i+4}{z+2i+4}}$$

$$w(z + 2i - 4) = -iz - 2i + 4$$

$$w = \frac{-iz-2i+4}{z+2i-4}$$

$$(w+1)(-2+i) = z(w+i)(-1)$$

$$-2w - 2 + iw + i = -zw - iz$$

$$w = \frac{-iz-i+2}{z+i-2} = \frac{iz+i-2}{-z+2-i}$$

① Unit-4

Given, $w = \log z$

In polar form $z = re^{i\theta}$

$$\frac{dw}{dz} = \frac{d}{dz}\log(re^{i\theta})$$

$$= \frac{d}{dz}\left[\log r + i\theta\right]$$

$$= \frac{1}{r} + i$$

Unit-4

① Given, $w = \log z = f(z)$

In polar form,

$$z = re^{i\theta}$$

$$\log z = \log re^{i\theta} = \log r + i\theta$$

$u = \log r$, $v = \theta$

Cauchy Reimann eqⁿ in polar form,

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

diff u w.r.t r

$$\frac{du}{dr} = \frac{1}{r}$$

diff θ w.r.t r

$$\frac{d\theta}{dr} = 0$$

$$f'(z) = e^{-i\theta} \times \frac{1}{r}$$

$$f'(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

$$\therefore \boxed{\frac{dw}{dz} = \frac{1}{z}}$$

at the derivates, at $z = 0$, w becomes infinity

also in the negative real axis

w is not analytic

$\therefore$ at $z = 0$ & negative real axis, w is not analytic

② Given, $f(z) = u + iv$

$u(x,y) = k_1$, $v(x,y) = k_2$ intersect at right angles at every point.

Two curves are orthogonal if the product of their slopes is -1.

$$m_1 \times m_2 = -1$$

Diff $u(x,y) = k_1$ w.r.t x

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$m_1 = -\frac{\partial u/\partial x}{\partial u/\partial y} \quad ①$$

Diff $v(x,y) = k_2$ w.r.t y

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$m_2 = -\frac{\partial v/\partial x}{\partial v/\partial y} \quad ②$$

For analytic,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$m_1 \times m_2 = \left(-\frac{\partial u/\partial x}{\partial u/\partial y}\right) \times \left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right)$$

$$= \left(-\frac{\partial v/\partial y}{\partial u/\partial y}\right) \times \left(\frac{\partial u/\partial y}{\partial v/\partial y}\right)$$

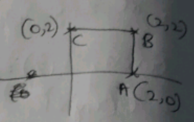
$$= -1$$

$\therefore k_1, k_2$ form an orthogonal system.

(3) Given, $f(z) = z^2 + 3z$ from $(2,0)$ to $(2,2)$, $(2,2)$ to $(0,2)$

$z = x + iy \Rightarrow dz = dx + i\,dy$

$I = \int_R f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz$



along AB :-

$x = 2 \quad dx = 0, \quad y : 0 \text{ to } 2$

$\int_{AB} f(z)\,dz = \int_0^2 [(x+iy)^2 + 3(x+iy)]\,dx + i\,dy$

$= i\int_0^2 (4 + 4iy - y^2 + 6 + i3y)\,dy$

$= i\left[-\frac{y^3}{3} + \frac{7i}{2}y^2 + 10y\right]_0^2 = i\left[-\frac{8}{3} + 14i + 20\right] = I_1$

along BC :-

$y = 2, \; dy = 0, \; x : 2 \text{ to } 0$

$\int_{BC} f(z)\,dz = \int_2^0 (x+2i)^2 + 3(x+2i)\,dx$

$= \int_2^0 x^2 + 4ix - 4 + 3x + 6i\,dx$

$= \left[\frac{x^3}{3} + 2ix^2 - 4x + \frac{3}{2}x^2 + 6ix\right]_2^0$

$= -\left[\frac{8}{3} + 8i - 8 + 6 + 12i\right]$

$= -\frac{8}{3} - 20i + 2 = I_2$

$I = I_1 + I_2$

$= -14 + \frac{52}{3}i - \frac{2}{3} - 20i$

$= -\frac{8}{3}i - \frac{44}{3}$

$\boxed{\therefore \int_C (z^2 + 3z)\,dz = -\frac{44}{3} - i\frac{8}{3}}$

(4), (9) Given, $f(z) = u + iv$

$F(z) = (1+i)f(z) = u - v + i(u+v)$

$= \cancel{f(z) + i f(z)}$

$u - v = U_r, \quad u + v = V_r$

$U_r = e^x(\cos y - \sin y)$ ($\because$ Given)

$\frac{\partial U_r}{\partial x} = e^x(\cos y - \sin y)$

$\frac{\partial U_r}{\partial y} = e^x(-\sin y - \cos y)$

By milne thomson,

$f'(z) = \frac{\partial U_r}{\partial x}(z,0) - i\frac{\partial U_r}{\partial y}(z,0)$

$= e^z + i e^z = e^z(1+i)$

$f(z) = e^z(1+i) + c$

$F(z) = (1+i)f(z)$

$(1+i)f(z) = e^z(1+i) + c$

$\boxed{\therefore f(z) = e^z + \frac{c}{1+i}}$

$\frac{\partial U_r}{\partial y} = -\frac{\partial V_r}{\partial x}$

$\frac{\partial V_r}{\partial x} = -e^x(-\sin y - \cos y)$

$V_r = e^x(\sin y + \cos y) + c'(y)$

For analytic,

$\frac{\partial U_r}{\partial x} = \frac{\partial V_r}{\partial y}$

$\frac{\partial V_r}{\partial y} = e^x(\cos y - \sin y)$

$V_r = e^x \int \cos y - \sin y\,dy$

$V_r = e^x[\sin y + \cos y + c'(x)]$

$$v = \int \left(e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)\right)dx +$$

$$\int e^{2x}(2x\cos 2y + \cos 2y - 2y\sin 2y)\,dy$$

$$= \sin 2y\left[\frac{e^{2x}}{2}\cdot 2x + x^2 e^{2x}\right] + \sin 2y\,\frac{e^{2x}}{2} + 2y\cos 2y\cdot\frac{e^{2x}}{2}$$

$$+c$$

$$= \sin 2y\left[xe^{2x} + x^2e^{2x} + \frac{e^{2x}}{2}\right] + y\cos 2y\cdot e^{2x} + c$$

$$\boxed{\int uv = u\int v - \int u'\int v}$$

$$v = \sin 2y\left[e^{2x}\,2\,\frac{x^2}{2} - \int 2.e^{2x}\cdot 2\frac{x^2}{2}\right] + \sin 2y\cdot\frac{e^{2x}}{2} +$$

$$2y\cos 2y\cdot\frac{e^{2x}}{2}$$

$$v = \sin 2y\left[x^2e^{2x} - 2\left[e^{2x}\frac{x^3}{3} - \int 2e^{2x}\cdot\frac{x^3}{3}\right]\right.$$

$\sin 2y$

v

U-3

1. Given, $n = 960$

$x = 184$

$\alpha = 0.017$

$H_0 :$ $p = \frac{1}{6}$, $q = \frac{5}{6}$

$H_1 :$ $p \neq \frac{1}{6}$

$\hat{p} = \frac{x}{n} = \frac{184}{960} = 0.1917$

Now, $Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}}$

$= \frac{0.1917 - 0.1667}{\sqrt{\frac{\frac{1}{6} \times \frac{5}{6}}{960}}}$

$Z = 2.08$

$Z_\alpha = 2.58$

$Z < Z_\alpha$

$\therefore$ H_0 is accepted

(2) $n_1 = 1000$ $\quad$ $n_2 = 2000$

$\bar{x}_1 = 67.5$ $\quad$ $\bar{x}_2 = 68$

$\sigma = 2.5$

$H_0 :$ $\mu_1 = \mu_2$ (Sample from same)

$H_1 :$ $\mu_1 \neq \mu_2$ (" not " ")

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2}}} = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} - \frac{1}{2000}}} = -5.16$$

at 5%, $Z_\alpha = 1.96$

$Z_\alpha < |Z|$

$\therefore$ we reject H_0.

(3) $n = 1000$

$x = 540 = 0.54 = \hat{p}$

$y = 460 = 0.46 = \hat{q}$

$H_0 :$ $\mu_1 = \mu_2$, $p = 0.5$, $q = 0.5$

$H_1 :$ $\mu_1 \neq \mu_2$, " $\neq$ " , " $\neq$ "

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} = \frac{0.54 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{1000}}}$$

$Z = 2.53$

at 1%

$Z_\alpha = 2.58$

$Z_\alpha > Z$

$\therefore$ H_0 is accepted.

(4) $n = 25$

$\bar{x} = 47.5$

$S = 8.4$

$\mu = 42.5$

$H_0 : \mu = 42.5$

$H_1 : \mu \neq 42.5$

$$t \cancel{z} = \frac{\bar{x} - \mu}{S/\sqrt{n}} = 2.976$$

at degree of freedom,

$n-1 = 24$

at $t_{0.05} = 2.064$

$\cancel{t}$ $2.064 < 2.976$

$\therefore$ we reject H_0.

(5) $n = 10$

$$\bar{x} = \frac{41}{10} = 4.1$$

$H_0 : \mu = 4$

$H_1 : \mu \neq 4$

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

$$s = \sqrt{\frac{\Sigma(x-\bar{x})^2}{n-1}}$$

$$S^2 = \frac{12.42}{9} = 1.38$$

$S = 1.1747$

$$t = \frac{4.1 - 4}{1.17/\sqrt{10}}$$

$t = 0.269$.

$dof = 9$

at $\cancel{t}_{5\%} = 2.262$

$t_{5\%} > t$

$\therefore$ Ho is accepted

(6) $\bar{x}_1 = 210$ $\quad$ $\bar{x}_2 = 220$

$S_1 = 10$ $\quad$ $S_2 = 12$

$n_1 = 100$ $\quad$ $n_2 = 150$

$\sigma = 11$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{-10}{11\sqrt{\frac{1}{100} + \frac{1}{150}}} = -7.04$$

at 5%,

$Z_\alpha = 1.96$

$|Z| > Z_\alpha$

$\therefore$ Ho is rejected

(7) $n_1 = 400$ $\qquad$ $n_2 = 600$

$x_1 = 200$ $\qquad$ $x_2 = 325$

$\hat{P}_1 = \frac{1}{2}$ $\qquad$ $\hat{P}_2 = 0.54$

$H_0 : P_1 = P_2$

$H_1 : P_1 \neq P_2$

$$P = \frac{x_1 + x_2}{n_1 + n_2} = 0.525$$

$$Q = 0.475$$

$$Z = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{-0.04}{\sqrt{(0.525)(0.475)\left(\frac{1}{400} + \frac{1}{600}\right)}}$$

$$= -1.24$$

at 5%

$Z_{5\%} = 1.96$

$Z_{5\%} > |Z|$

$\therefore$ we accept H_0

(8) $\bar{x}_1 = 55$ $\qquad$ $\bar{x}_2 = 57$

$S_1 = 10$ $\qquad$ $S_2 = 15$

$n_1 = 400$ $\qquad$ $n_2 = 100$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

$\cancel{\hat{P}_1 = 0.1375}$ $\qquad$ $\cancel{\hat{P}_2 = 0.57}$

$$P = \frac{\bar{x}_1 + \bar{x}_2}{n_1 + n_2} = 0.224$$

$$Q = 0.776$$

$$Z = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{PQ\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = -9.27$$

at 5%

$Z_{5\%} = 1.96$

$Z_{5\%} < |Z|$

$\therefore$ we reject H_0

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = -1.26$$

at 5%,

$Z_{5\%} = 1.96$

$Z_{5\%} > |Z|$

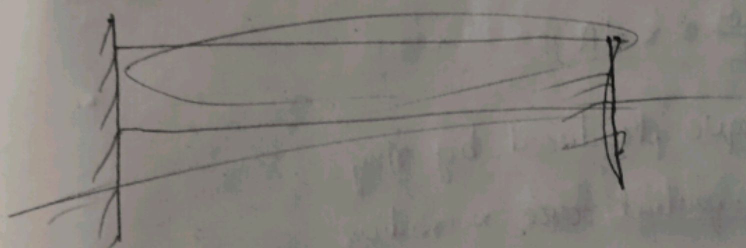
$\therefore$ we accept H_0

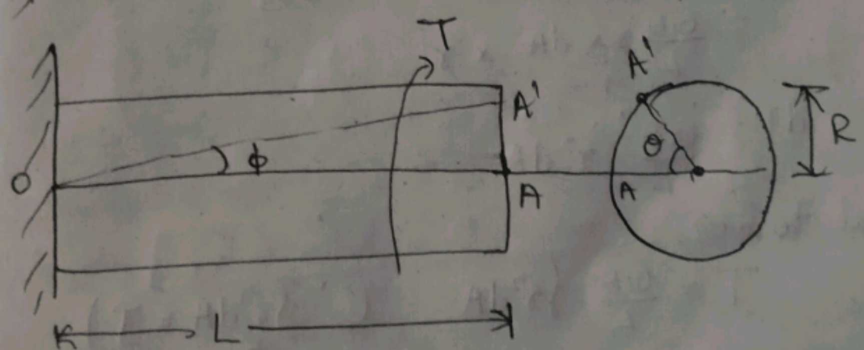
U-5

* Assumptions of Torsion

- The material is homogeneous and isotropic
- The shaft is circular in cross section
- plane cross sections remain plane after twisting.
- plane cross sections remain $\perp^r$ to longitudinal axis of the shaft.
- Twist is uniform along the length.
- shear stress vary linearly from zero at the centre to maximum at the outer radius.
- Material obeys Hooke's law in shear
- only shear stresses are produced.

* Torsional eq^n $\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$





Consider a shaft of length L and radius R.

$\Delta OAA'$,

$$\tan\phi = \frac{AA'}{L}$$

Since ϕ is very small

$$\boxed{\phi = \frac{AA'}{L}}$$

from cross section

$$AA' = R\theta$$

$$\boxed{\phi = \frac{R\theta}{L}} \quad (1)$$

we know,

$$\frac{\tau}{\phi} = G$$

(1) becomes, $\frac{\tau}{G} = \frac{R\theta}{L}$ (or) $\boxed{\frac{\tau}{R} = \frac{G\theta}{L}}$ (A)

$\therefore \tau \alpha R$

Now considering an elemental ring of radius r

$\therefore$ Resistive force developed by ring,

$$dF = \tau \times dA$$

$$dF = \frac{G\theta}{L} r \times dA$$

$\therefore$ Resistive torque produced by ring,

$$dT = \text{Tangential force} \times \text{radius}$$

$$= \frac{G\theta}{L} r \times dA \times r$$

$$dT = \frac{G\theta}{L} r^2 dA$$

Total torque,

$$T = \frac{G\theta}{L} \int r^2 dA \qquad (\because \int r^2 dA = J) \text{ Polar moment}$$

$$T = \frac{G\theta}{L} \times J$$

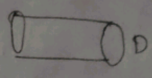
$$\boxed{\frac{T}{J} = \frac{G\theta}{L}} \quad (B)$$

By A & B

$$\boxed{\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}}$$

(Pr) The avg Torque transmitted by a shaft is 2255 N.m. The maximum torque is 146% of avg torque. If the allowable shear stress in the shaft is 45 N/mm², determine suitable diameter of the shaft.

Sol:- Given,



$$T_{avg} = 2255 \text{ N.m}$$

$$T_{max} = 146\% \text{ of } T_{avg}$$

$$\tau_{max} = 45 N/mm^2 = 45 \times 10^6 N/m^2$$

$$T_{max} = \frac{146}{100} \times 2255 = 3292.3 \text{ N.m}$$

$$J = \frac{\pi D^4}{32}, \quad R = \frac{D}{2}$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\frac{3292.3}{\frac{\pi D^{4\,3}}{32}} = \frac{45 \times 10^6}{\frac{\cancel{D}}{2}}$$

$$\therefore \boxed{D = 0.0719 m}$$

$$\therefore \boxed{D = 72 mm}$$

(Pr) A solid shaft is subjected to a maximum torque of 15 M N-cm. Determine the dia of shaft, if the allowable shear stress and the twist are liminted to 1 KN/cm² & 1°, L = 210 cm, G = 8 MN/cm².

Sol:- Given, $T_{max} = 15 \times 10^6$ N-cm

$$\tau_{max} = 1 KN/cm^2, \quad \theta = 1^\circ = \frac{\pi}{180}, \quad L = 210 \text{ cm}$$

$$G = 8 M N/cm^2$$

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

$$J = \frac{\pi}{32} D^4 \qquad \frac{T}{J} = \frac{\tau}{R}$$

$$\frac{15\times10^6}{\frac{\pi}{32} D^4} = \frac{1K}{\frac{D}{2}}$$

$$D = \sqrt[3]{\frac{15\times10^6 \times 32}{\pi \times 10^3 \times 2}}$$

$$\boxed{D = 42.425\, cm} \text{ (1)}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}$$

$$\frac{15\times10^6}{\frac{\pi}{32} D^4} = \frac{8\times10^6 \times \frac{\pi}{180}}{210}$$

$$D^4 = \frac{15\times10^6 \times 210 \times 32 \times 180}{8\times10^6 \times \pi^2}$$

$$\boxed{D = 21.89\, cm} \text{ (2)}$$

for dimension always consider large

i.e. $\boxed{D = 42.425}$

Note :- For strength take D smaller value

* Torsional moment of resistance

$$\frac{T}{J} = \frac{\tau}{R} \Rightarrow \tau = T\frac{R}{J}$$

$$T = \frac{J}{R}\tau$$

at $r = 0$, $\tau = 0$

$r = R$, $\tau = \tau_{max}$ $\qquad \frac{J}{R} = Z_p$

$$\therefore \boxed{T_{max} = Z_p \cdot \tau_{max}}$$

* polar section modulus

$$\boxed{Z_p = \frac{J}{R}}$$

* Power Transmitted by shafts

$$\boxed{P = \frac{2\pi NT}{60}} \qquad \boxed{P = T\omega}$$

P = power in (watt)

N = no. of rev (r.p.m)

T = Avg Torque (N-m)

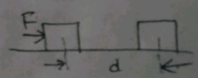
$\omega = \frac{2\pi N}{60}$

ω = angular speed of shaft

* Derive $P = \frac{2\pi NT}{60}$

we have, Power = $\frac{\text{work done}}{\text{time}}$

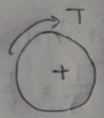
For linear motion,



$$P = \frac{W}{t} = \frac{F\times d}{t} = F \times \frac{d}{t}$$

$$= F \times V$$

For rotary motion,

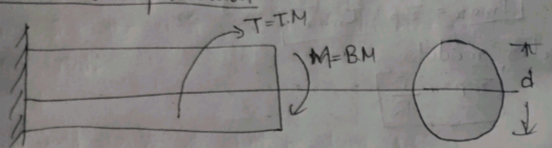


$P = T \times \omega$

$P = T \times \frac{2\pi N}{60}$ $\left(\because \omega = \frac{2\pi N}{60}\right)$

$\boxed{\therefore P = \frac{2\pi N T}{60}}$ watts

* combined Bending & Torsion



BM eq^n, $\frac{M}{I} = \frac{F}{y} = \frac{E}{R}$

$\frac{M}{I} = \frac{F}{y} \Rightarrow F = \frac{My}{I} = \frac{M\,d/2}{\pi d^4/64}$

$\boxed{F = \frac{32M}{\pi d^3}}$

Torsion eq^n, $\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$

$\frac{T}{J} = \frac{\tau}{R} \Rightarrow \tau = \frac{TR}{J}$

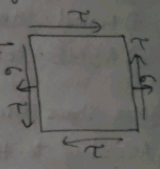
$\tau = \frac{T\,d/2}{\frac{\pi d^4}{32}}$

$\boxed{\tau = \frac{16T}{\pi d^3}}$

equivalent BM:-

$\sigma_1/\sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$\frac{32M_e}{\pi d^3} = \frac{\frac{32M}{\pi d^3}}{2} \pm \sqrt{\left(\frac{32M/\pi d^3}{2}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$



σ_x

$\sigma_y = 0$

$\frac{32M_e}{\pi d^3} = \frac{\frac{32M}{\pi d^3}}{2} \pm \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$

$\frac{32M_e}{\pi d^3} = \frac{16M}{\pi d^3} \pm \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$

$2M_e = M \pm \sqrt{M^2 + T^2}$

$\boxed{M_e = \frac{1}{2}\left[M \pm \sqrt{M^2 + T^2}\right]}$

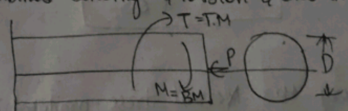
equivalent TM:-

$\tau = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

$\frac{16T_e}{\pi d^3} = \sqrt{\left(\frac{16M}{\pi d^3}\right)^2 + \left(\frac{16T}{\pi d^3}\right)^2}$

$\boxed{T_e = \sqrt{M^2 + T^2}}$

* combined bending & Torsion & end thrust



$\sigma_a = \frac{P}{A} = \frac{P}{\frac{\pi d^2}{4}} = \boxed{\frac{4P}{\pi d^2}}$

$\boxed{\tau = \frac{16T}{\pi d^3}}$

Combined Bending & end thrust

$\sigma_1 = \sigma_b + \sigma_a$

$= \boxed{\frac{32M}{\pi d^3} + \frac{4P}{\pi d^2}}$

* Design of shafts on theory of failure
1) Max shear stress theory
2) Normal stress
3) ASME code for the shaft design.

1) Max shear stress theory
According to this theory, Max shear stress induced in the shaft is equated to allowable / permissible value of shear stress for the shaft material of diameter of shaft is calculated.

Let
S_{sy} = yield strength of shaft in shear (N/mm²)
S_{yt} = " " " " tension (N/mm²)

$$F.O.S = \frac{\text{Max. stress}}{\text{working stress}}$$

$$FOS = \frac{S_{sy}}{\tau_{per}} \Rightarrow \boxed{\tau_{per} = \frac{S_{sy}}{FOS}}$$

$$\boxed{S_{sy} = S_{yt} \times 0.5} \qquad \boxed{\tau_{per} = \frac{0.5\, S_{yt}}{FOS}}$$

2) Normal stress theory
According to this theory, Max Normal stress produced in the shaft is equated with permissible value of normal stress of diameter of the shaft is calculated.

Let σ_{per} = permissible normal stress in N/mm²

$$FOS = \frac{\text{Max. stress}}{\text{Permissible stress}} \Rightarrow F.O.S = \frac{S_{yt}}{\sigma_{per}} \quad \boxed{\sigma_{per} = \frac{S_{yt}}{FOS}}$$

3) ASME code for the shaft design.
According to this theory, the permissible shear stress (τ_{per}) for shaft without key ways is considered as 30% of yield strength in Tension (S_{yt}) or 18% " ultimate tensile strength (S_{ut}) of material.

$$\boxed{\begin{array}{c}\tau_{per} = 0.3\, S_{yt} \\ \text{(or)} \\ \tau_{per} = 0.18\, S_{ut}\end{array}}$$

→ The Min value of τ_{per} is selected
→ If keyways are present

$$\boxed{\tau_{per} = 0.18\, S_{ut} \times 0.75}$$
$$\boxed{\tau_{per} = 0.3\, S_{yt} \times 0.75}$$

→ If shock & fatigue there then $M \times K_b$ & $T \times K_t$

$$~~T_e = M \pm~~ \quad T_e = \sqrt{M^2 + T^2} = \frac{\pi}{16} \times \tau_{max} \times D^3$$

$$\sqrt{(M K_b)^2 + (T K_t)^2} = \frac{\pi}{16} \times \tau_{max} \times D^3 \rightarrow \text{Solid shaft}$$

$$\sqrt{(M K_b)^2 + (T K_t)^2} = \frac{\pi}{16} \times \tau_{max} \times D^3\left(1 - \left(\frac{d}{D}\right)^4\right) \times \tau_{per}$$
hallow

K_b = Combined shock & fatigue for BM
K_t = " " " for TM

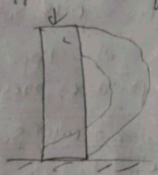
* Euler theory

→ In 1757, swiss mathematician leonhard Euler gave the formalua for stability of long columns.

→ Euler considering only bending of columms as the direct compression was negligible compared to bending.

→ Euler neglected the direct compression of columns

→ Hence, Euler's theory is applicable only for long columns



* Assumptions made in Euler's theory

1) The column is subjected to axial loading
2) Material of the column is homogenous & isotropic.
3) The material of column is elastic & obey's hookes law.
4) The length of the column is very large compared to the other dimensions.
5) Self weight of column was neglected.
6) The columm is straight before loading.

$$\boxed{P_e = \frac{\pi^2 E I_{min}}{(L_e)^2}}$$

* Limitations of Euler's theory

1) Not valid for columns
2) Not valid for intermediate columns.
3) Not " when load is eccentric
4) Not " if material is inelastic
5) " applicable when slenderness ratio is small.
6) " valid for non uniform cross section.
7) Self weight becomes significant for very long columns.
8) Does not consider residual stresses/imperfections

* Effective Length

$L_e = K L$

1) Both Ends hinged $(K=1)$

$\boxed{L_e = L}$

2) one End fixed, other end free $(K=2)$

$\boxed{L_e = 2L}$

3) Both ends fixed $(K=\frac{1}{2})$

$\boxed{L_e = \frac{L}{2}}$

4) one end Fixed, other end Hinged $(K=0.7)$

$\boxed{L_e = 0.707L = \frac{L}{\sqrt{2}}}$

* Rankine's Formula :-

$$\boxed{\frac{1}{P_R} = \frac{1}{P_c} + \frac{1}{P_E}}$$

P_R = rankine safe load P_E = Euler buckling load

P_c = crushing load

$P_c = \sigma_c A$

$$P_E = \frac{\pi^2 E I_{min}}{L_e^2}$$

$$P_R = \frac{\sigma_c A}{1 + \frac{\sigma_c A L_e^2}{\pi^2 E I_{min}}}$$

$$\boxed{P_R = \frac{\sigma_c A}{1 + \alpha\left(\frac{L_e}{K}\right)^2}} \quad , K = \sqrt{\frac{I}{A}}$$

$$\alpha = \frac{\sigma_c}{\pi^2 E}$$

For short column,

$\left(\frac{L_e}{K}\right)^2 = 0 \rightarrow P_R \approx P_c$

For long column,

denominator is large $\rightarrow P_R \approx P_E$

* Secant formula,

when eccentric load (P), acts on column, it bend, BM increases because of deflection

Total BM,

$M = P(e+y)$, e = eccentricity , y = lateral deflection

Secant formula,

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{e}{r}\sec\left(\frac{L}{2r}\sqrt{\frac{P}{EI}}\right)\right]}$$

If $e = 0$

it becomes euler's formula

If $\sec \approx 1$

$$\sigma_{max} = \frac{P}{A} + \frac{Pe}{Z}$$