Imp

* 1. Show that the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized

Given, matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigen values $[A - \lambda I] = 0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$2-\lambda\left[(2-\lambda)(1-\lambda)+0\right] - 3\{0\} + 4[0] = 0$$

$$2-\lambda\left[2-3\lambda+\lambda^2\right] = 0$$

$$4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

put $\lambda = 1$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda^2 - 2\lambda - 2\lambda + 4 = 0$$

$$(\lambda-2)(\lambda-2) = 0$$

$$\lambda = 2, 2$$

Eigen values $\lambda = 1, 2, 2$

consider the matrix $[A - \lambda I]x = 0$

$$\begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

The eigen vector corresponding to $\lambda = 1$

putting $\lambda = 1$ in eq (1), we get

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{-3-4} = \frac{-y}{-1-0} = \frac{z}{1-0} = k$$

$x = -7k$

$y = k$

$z = k$

$$x_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$$

The eigen vector corresponding to $\lambda = 2$

putting $\lambda = 2$ in eq ①, we get

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank = 2

no. of variable = 3

The rank of coeffecient matrix is 2

This equation will have $3 - 2 = 1$ independent solution

Thus, A is not similar to a diagonal matrix

$\therefore$ the matrix cannot be diagonalized

2. Show that $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ cannot be diagonalized.

2, 2, 3

<u>Sol</u> Eigen ~~vectors~~ values $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$\lambda = 2$

$$3-\lambda\left[(-3-\lambda)(7-\lambda) + 20\right) - 10\left[-14 + 2\lambda + 12\right] + 5\left[-10 + 9 + 3\lambda\right] = 0$$

$$(3-\lambda)\left[-21 + 3\lambda - 7\lambda + \lambda^2 + 20\right] - 10\left[2\lambda - 2\right] + 5\left[3\lambda - 1\right] = 0$$

$$3-\lambda\left[\lambda^2 - 4\lambda - 1\right] - 20\lambda + 20 + 15\lambda - 5 = 0$$

$$3\lambda^2 - 12\lambda - 3 - \lambda^3 + 4\lambda^2 + \lambda - 15 - 5\lambda = 0$$

$$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

$$(\lambda - 2)(-\lambda^2 + 5\lambda - 6)$$

$$(\lambda - 2)(\lambda^2 - 5\lambda + 6) = 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda - 3) = 0$$

$$\begin{array}{c|cccc} 2 & -1 & 7 & -16 & 12 \\ & 0 & -2 & 10 & -12 \\ \hline & -1 & 5 & -6 & \boxed{0} \end{array}$$

The eigen values of are 2, 2, 3

The eigen vector corresponding to $\lambda = 3$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{-40+30} = \frac{-y}{0+10} = \frac{z}{0+20} = k$$

$$\frac{x}{-10} = \frac{-y}{10} = \frac{z}{20} = k \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

The eigen vector corresponding to $\lambda = 2$

putting $\lambda = 2$ ~~in eq~~

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & 4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 15 & 6 \\ 0 & -25 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{R_2}{3}, \frac{R_3}{-5}$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 10 & 5 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of coeffecient matrix is 2

Then the eqn will have $3-2=1$ independent sol^n

$\therefore$ A is not similar to a diagonal matrix

the matrix cannot be diagonalized

3. Diagonalize the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find A^4 using Model matrix P

Sol :

diagonalized where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$,

$$D^4 = \begin{bmatrix} 2^4 & 0 & 0 \\ 0 & 3^4 & 0 \\ 0 & 0 & 6^4 \end{bmatrix}$$

$$A^4 = PD^4P^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 64 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} 3 & 0 & -3 \\ -2 & -2 & -2 \\ -1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 251 & -405 & 235 \\ -405 & 891 & -405 \\ 235 & -405 & 251 \end{bmatrix}.$$

* Most Important *

* Properties of Eigen values and Eigen vectors

1. The sum of the eigen value of a square matrix all equal to its "Trace" and product of the eigen values equal to its determinent

i.e, $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ are eigen values of A an $n \times n$ matrix then, $\lambda_1 + \lambda_2 + \lambda_3 \dots + \lambda_n = \text{Trace}(A)$

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \dots \lambda_n = \det(A$

2. Any square matrix A and its transpose A^T will have the same eigen values

3. If $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$ are eigen values of A,

(i) The eigen values of "KA" are $k\lambda_1, k\lambda_2, k\lambda_3 \dots k\lambda_n$

* (ii) The eigen values of "A^m" are $\lambda_1^m, \lambda_2^m, \lambda_3^m \dots \lambda_n^m$

* (iii) The eigen values of "A^{-1}" are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots \frac{1}{\lambda_n}$

(or) $\lambda_1^{-1}, \lambda_2^{-2}, \dots \lambda_n^{-1}$

** (iv) The eigen values of "Adj A" are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3} \dots \frac{|A|}{\lambda_n}$

4. For symmetric matrix the eigen values are always real number
5. For skew-symmetric matrix the eigen values are ~~at~~ either 0 or pure ~~sym~~ imaginary.
6. For hermitian matrix the eigen values are always real
7. For skew-hermitian matrix the eigen values are 0 or pure imaginary.

*8. ** For triangular matrix [upper or lower] or diagonal matrix the eigen values are the main diagonal element.

*9. For orthogonal matrix the Mod value of eigen values is always unity

*10. ** If λ is an eigen value of A and f(A) is any polynomial then the eigen value of f(A) is $f(\lambda)$

11. The eigen vector corresponding to different eigen values linearly independent.
12. The eigen vector corresponding to repeated eigen values are linearly independent or linearly dependent.

1. Find the sum and product of the eigen value of the matrix

1. $A=\begin{bmatrix}2&1&-1\\3&4&2\\1&0&2\end{bmatrix}$ 2. $A=\begin{bmatrix}2&5&7\\1&4&6\\2&-2&3\end{bmatrix}$ 3. $A=\begin{bmatrix}-1&2&3\\0&3&5\\0&0&-2\end{bmatrix}$ 4. $A=\begin{bmatrix}2&3&-2\\-2&1&1\\1&0&2\end{bmatrix}$

Sol $A=\begin{bmatrix}2&1&-1\\3&4&2\\1&0&2\end{bmatrix}$

(i) Sum of the eigen values = Trace of the matrix
= Sum of the diagonal elements
$= 2+4+2 = 8$

(ii) Product of the eigen values = det A = $\begin{vmatrix}2&1&-1\\3&4&2\\1&0&2\end{vmatrix}$

$2(8-0)-1(6-2)-1(0-4)=0$

$16-\cancel{4}\cancel{+4}=0$

16

ii $A = \begin{vmatrix} 2 & 5 & 7 \\ 1 & 4 & 6 \\ 2 & -2 & 3 \end{vmatrix}$

(i) Sum of the eigen values = Trace of the matrix

= Sum of the diagonal matrix

$= 2+4+3$

$= 9$

(ii) Product of the eigen values = det A

$= 2(12+12) - 5(3-12) + 7(-2-8)$

$= 2(24) - 5(-9) + 7(-10)$

$= 48 + 45 - \cancel{56}\ 70$

$= 93 - 70$

$= 23$

iii $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$

(i) Sum of the eigen values = Trace of the matrix

= Sum of the diagonal matrix

$= -1+3-2$

$= -3+3$

$= 0$

(ii) Product of the eigen values = det A

$= -1(-6-0) - 2(0) + 3(0)$

$= 6$

iv. $A = \begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$

(i) Sum of the eigen values = Trace of the matrix

= Sum of the diagonal matrix

$= 2+1+2$

$= 5$

(ii) Product of the eigen values = det A

$= 2(2-0) - 3(-4-1) - 2(0-1)$

$= 4 + 15 + 2$

$= 21$

2. Find the eigen values of A^{-1} where $A = \begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{vmatrix}$

Given matrix $A = \begin{vmatrix} 2 & 3 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{vmatrix}$ is an upper triangular matrix

So, the eigen values of A are [2, 4, 3] its main diagonal elements

By using the property if λ is a eigen value of A then λ^{-1} is the eigen value of A^{-1}

$\therefore \frac{1}{2}, \frac{1}{4}, ,'$

$\therefore$ The eigen values of A^{-1} are $2^{-1}, 4^{-1}, 3^{-1}$ or $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}$

3. Find the eigen value of adjoint of A and A^{-1} where

$A = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$

$1(6-2) - 1(2-4)$

$4 + 2$

6

$\frac{6}{3}, \frac{6}{2}, 1$

Eigen value $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0$$

$1-\lambda\left[(2-\lambda)(3-\lambda)-2\right] - 0(\ ) - 1(2-4-\lambda) = 0$

$1-\lambda\left[6-5\lambda+\lambda^2-2\right] - 1\{-2-\lambda\} = 0$

$1-\lambda\left[4-5\lambda+\lambda^2\right] - 1[-2-\lambda] = 0$

$4-5\lambda+\lambda^2-4\lambda+5\lambda^2-\lambda^3+2\ \cancel{2}\lambda = 0$

$-\lambda^3+6\lambda^2-8\lambda+6=0$

$\lambda^3-6\lambda^2+8\lambda-6=0$

$\lambda^3-6\lambda^2+11\lambda-6=0$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	$\boxed{0}$

$\lambda^2-5\lambda+6=0$

$\lambda^2-3\lambda-2\lambda+6=0$

$\lambda(\lambda-3)-2(\lambda-3)=0$

$\lambda = 3, 2, 1$

$|A| = 1(6-2) + 0 - 1(2-4) = 0$

$1(4) - 1(-2) = 0$

$= 4 + 2$

$= 6$

1. Eigen values of Adj A $= \frac{6}{1}; \frac{\cancel{6}^{3}}{\cancel{2}}; \frac{\cancel{6}^{2}}{\cancel{3}}$ $\left\{ \because \text{Eigen values of Adj } A = \frac{|A|}{\lambda_1}; \frac{|A|}{\lambda_2} \ldots \right\}$

$= 6; 3; 2$

2. Eigen values of $A^{-1} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}$

3. The eigen value of $A^2 = \lambda^2 = 1^2, 2^2\ 3^2 = 1, 4, 9$

4. Find the eigen values of adj A, A^{-1}, A^2, A^3 where

1. $A = \begin{bmatrix} 2 & 4 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

Sol $A = \begin{bmatrix} 2 & 4 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix}$ Eigen values $[A - \lambda I] = 0$ $\begin{vmatrix} 2-\lambda & 4 & 7 \\ 0 & 1-\lambda & 8 \\ 0 & 0 & 3-\lambda \end{vmatrix}$

$2-\lambda\left[(1-\lambda)(3-\lambda) - 0\right] - 4[0] + 7(0) = 0$

$2-\lambda\left[3 - 4\lambda + \lambda^2\right] = 0$

$6 - 8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 = 0$

$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$\lambda^2 - 5\lambda + 6 = 0$

$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$ $(\lambda-3)(\lambda-2) = 0$

$\lambda = 1, 2, 3$

$\cancel{|A| = 2(3-0) - 4(-4^{0}-1) - 2(0-1) = 0}$

$\cancel{= 4 + 15 + 2}$

$\cancel{= 21}$

$\left\{ \therefore |A| = 2(3) - 4(0) + 7(0) = 6 \right\}$

Eigen value of Adj A $= \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}$

$= \frac{6}{1}, \frac{6}{2}, \frac{6}{3} = 6, 2, 3$

$A^{-1} = \lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1} = 1^{-1}, 2^{-1}, 3^{-1}$ or $1, \frac{1}{2}, \frac{1}{3}$

$A^2 = \lambda_1^2, \lambda_2^2, \lambda_3^3 = 1^2, 2^2, 3^2 = 1, 4, 9$

$A^3 = \lambda_1^3, \lambda_2^3, \lambda_3^3 = 1^3, 2^3, 3^3 = 1, 8, 27$

2. $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ eigen values of upper triangular matrix are its diagonal elements $\{ |A| = 1(-6-0) - 2(0) - 3(0) = -6$

eigen values = $1, 3, -2$

eigen values of Adj A = $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3} = -\frac{6}{1}, -\frac{6}{3}, -\frac{6}{-2}$

$= -6, -2, 3$

$A^{-1} = \lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1} = 1, \frac{1}{3}, -\frac{1}{2}$

$A^2 = \lambda_1^2, \lambda_2^2, \lambda_3^3 = 1^2, 3^2, -2^2 = 1, 9, 4$

$A^3 = \lambda_1^3, \lambda_2^3, \lambda_3^3 = 1^3, 3^3, -2^3 = 1, 27, -8$

3. $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$

Eigen values of (upper or lower) triangular matrix are its diagonal elements

$|A| = 1(6-0) - 0 - 0 = 6$

Eigen values are $1, 2, 3$

$\text{Adj}A = \frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3} = \frac{6}{1}, \frac{6}{2}, \frac{6}{3} = 6, 3, 2$

$A^{-1} = \lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1} = 1, \frac{1}{2}, \frac{1}{3}$

$A^2 = 1^2, 2^2, 3^2 = 1, 4, 9$

$A^3 = 1^3, 2^3, 3^3 = 1, 8, 27$

4. $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ $[A - \lambda I] = 0$ $\begin{vmatrix} 2-\lambda & 3 & -2 \\ -2 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow 2-\lambda(2-3\lambda+\lambda^2-0) - 3(4-2\lambda-1) - 2(0-1+\lambda) = 0$

$2-\lambda(\lambda^2-3\lambda+2) - 3(-5+2\lambda) - 2(-1+\lambda) = 0$

$2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda + 15 - 6\lambda + 2 - 2\lambda = 0$

$\cancel{-\lambda^3 + 5\lambda^2 - 3 = 0}$ $-\lambda^3 + 5\lambda^2 - 16\lambda + 21 = 0$

$\lambda^3 - 5\lambda^2 + 16\lambda - 21 = 0$

5. Find the eigen value of $3A^3+5A^2-6A+2I$ for the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Sol Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

The above matrix is upper triangular matrix = so the eigen values are the main diagonal ~~matrix~~ elements 1, 3, -2)

If λ is a eigen value of A & f(A) is any polynomial then the eigen values of f(A) is $f(\lambda)$

given $f(A) = 3A^3+5A^2-6A+2I$

$f(\lambda) = 3\lambda^3+5\lambda^2-6\lambda+2I$

$f(1) = 3+5-6+2$

$= 8-6+2$

$= 2+2$

$= 4$

$f(3) = 3(3)^3+5(3)^2-6(3)+2(1)$

$= 81+45-18+2$

$= 126-16$

$= 110$

$f(-2) = 3(-2)^3+5(-2)^2-6(-2)+2$

$= -24+20+14$

$= -4+14$

$= 10$

6. Find the eigen values $2A^3+3A^2+5A+3I$ if 2, 3, 5 are the eigen values of matrix A.

Sol: If λ is a eigen value of A & f(A) is any polynomial then the eigen values of f(A) is $f(\lambda)$

Given, $f(A) = 2A^3+3A^2+5A+3I$

$f(\lambda) = 2A^3+3A^2+5A+3I$

$f(2) = 2(2)^3+3(2)^2+5(2)+3(1) = 47$

$f(3) = 2(3)^3+3(3)^2+5(3)+3(1) = 99$

$f(5) = 2(5)^3+3(5)^2+5(5)+3(1) = 353$

1. Find the eigen values and eigen vector of hermitian matrix
$$\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$$

Sol Eigen value $[A-\lambda I]=0$

$$\begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix}=0$$

$$(2-\lambda)(2-\lambda)-(3+4i)(3-4i)=0$$

$$4-4\lambda+\lambda^2-(9+16)=0$$

$$\lambda^2-4\lambda-21=0$$

$$\lambda^2-7\lambda+3\lambda-21=0$$

$$\lambda(\lambda-7)+3(\lambda-7)=0$$

$$(\lambda-7)(\lambda+3)=0$$

$$\lambda=7,-3$$

The eigen values are real
consider the matrix $[A-\lambda I]x=0$

$$\begin{bmatrix} 2-\lambda & 3+4i \\ 3-4i & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

(i) The eigen vector corresponding to $\lambda=-3$
putting $\lambda=-3$ in eq (1), we get

$$\begin{bmatrix} +5 & 3+4i \\ 3-4i & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5x+(3+4i)y=0.$$

$$5x=-(3+4i)y$$

let $y=k$

$$5x=-(3+4i)k$$

$$x=\frac{-(3+4i)k}{5}$$

$$X_1=\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -\frac{(3+4i)}{5}k \\ k \end{bmatrix}=\frac{k}{5}\begin{bmatrix} -(3+4i) \\ 5 \end{bmatrix}$$

$X_1=\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=-3$

The eigen vector corresponding to $\lambda = 7$
putting $\lambda = 7$ in eq ①, we get

$$\begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-5x + (3+4i)y = 0$$

$$\cancel{-}5x = \cancel{-}(3+4i)y$$

$$x = \frac{(3+4i)y}{5}$$

let $y = k$

$$x = \frac{3+4i}{5}k$$

$$x_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3+4i}{5}k \\ k \end{bmatrix}$$

$$x_2 = \frac{k}{5}\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 7$

2. Find the eigen value and eigen vector of skew-hermitian matrix $A = \begin{pmatrix} 2i & 3i \\ 3i & 0 \end{pmatrix}$ eigen value $[A - \lambda I] = 0$

Sol $\cancel{A =} \begin{bmatrix} 2i-\lambda & 3i \\ 3i & 0-\lambda \end{bmatrix} = 0$

$$(2i-\lambda)(-\lambda) - (3i)(3i) = 0$$

$$-2i\lambda + \lambda^2 + 9 = 0$$

$$\lambda^2 - 2i\lambda + 9 = 0$$

then $a = 1$, $b = -2i$, $c = 9$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2i \pm \sqrt{-4-36}}{2} = \frac{2i \pm \sqrt{-40}}{2}$$

$$= \cancel{\frac{i \pm \sqrt{-4/0}}{2}}$$

$$\lambda = \frac{2i \pm \sqrt{4 \times 10 \times i^2}}{2}$$

$$\lambda = \frac{2i \pm 2i\sqrt{10}}{2}$$

$$\lambda = (1 \pm \sqrt{10})i$$

$$\lambda = (1+\sqrt{10})i,\ (1-\sqrt{10})i$$

The eigen value are pure imaginary

consider the matrix $[A - \lambda I]X = 0$

$$\begin{bmatrix} 2i-\lambda & 3i \\ 3i & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow (1)$$

(i) The eigen vector corresponding to $\lambda = (1+\sqrt{10})i$

putting $\lambda = (1+\sqrt{10})i$ in eq (1), we get

$$\begin{bmatrix} 2i - i - \sqrt{10}i & 3i \\ 3i & -(1+\sqrt{10})i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (1-\sqrt{10})i & 3i \\ 3i & -(1+\sqrt{10})i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-\sqrt{10})ix + 3iy = 0$$

$$(1-\sqrt{10})ix = -3iy$$

$$x = \frac{-3i}{(1-\sqrt{10})i} y$$

let $y = k$

$$x = \frac{-3i}{(1-\sqrt{10})i} k$$

$$X_1 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-3i}{(1-\sqrt{10})i} k \\ k \end{bmatrix} = \frac{k}{(1-\sqrt{10})i} \begin{bmatrix} -3i \\ (1-\sqrt{10})i \end{bmatrix}$$

$X_1 = \begin{bmatrix} -3i \\ (1-\sqrt{10})i \end{bmatrix}$ is the eigen vector corresponding

to $\lambda = (1+\sqrt{10})i$.

(ii) The eigen vector corresponding to $\lambda = (1-\sqrt{10})i$

putting $\lambda = (1-\sqrt{10})i$ in eq ①, we get

$$\begin{bmatrix} 2i - i + \sqrt{10}\,i & 3i \\ 3i & -(1-\sqrt{10})i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (1+\sqrt{10})i & 3i \\ 3i & -(1-\sqrt{10})i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1+\sqrt{10})ix + 3iy = 0$$

$$x = \frac{-3i}{(1+\sqrt{10})i}\, y \quad , \text{let } y = k$$

$$x = \frac{-3i}{(1+\sqrt{10})i}\, k$$

$$x_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-3i}{(1+\sqrt{10})i}\, k \\ k \end{bmatrix}$$

$$x_2 = \frac{k}{(1+\sqrt{10})i} \begin{bmatrix} -3i \\ (1+\sqrt{10})i \end{bmatrix}$$

$\therefore x_2 = \begin{bmatrix} -3i \\ (1+\sqrt{10})i \end{bmatrix}$ is the eigen vector corresponding to $\lambda = (1-\sqrt{10})i$ //

3. Show that $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a skew-hermitian matrix and unitary matrix find the eigen values & eigen vectors

$i, i, -i$ purely imm

Sol :- Given $A = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$

Where skew-hermitian matrix condition $(\bar{A})^T = -A$

Where ; $\bar{A} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$ $\quad (\bar{A})^T = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} = -A$

$$(\bar{A})^T = -A$$

A is skew hermitian matrix.

When Unitary matrix condition. $A(\bar{A})^T = (\bar{A})^T A = I$.

$$A(\bar{A})^T = \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

& $$(\bar{A})^T A = \begin{bmatrix} -i & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{bmatrix} \begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

When $A(\bar{A})^T = (\bar{A})^T A = I$

A is unitary matrix

Eigen values $[A - \lambda I] = 0$

$$\begin{vmatrix} i-\lambda & 0 & 0 \\ 0 & 0-\lambda & i \\ 0 & i & 0-\lambda \end{vmatrix} = 0$$

$i-\lambda(+\lambda^2 - (i^2) - (0) + (0) = 0$

$i-\lambda(\lambda^2+1) = 0$

$\lambda^2 i + i - \lambda^3 - \lambda = 0$

$\lambda^3 - i\lambda^2 + \lambda - i = 0$

$(\lambda+i)(\lambda-i)^2 = 0$

$\therefore \lambda = i, i, -i$

Eigen vector corresponding to $\lambda = -i$

putting $\lambda = -i$

$$\begin{bmatrix} 2i & 0 & 0 \\ 0 & i & i \\ 0 & i & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c|ccc} i & 1 & -i & 1 & -1 \\ & 0 & i & 0 & \\ \hline & 1 & 0 & & \end{array}$$

$2ix_1 = 0$,
$x_2 + x_3 = 0$
$x_1 = 0 \quad x_2 = -x_3$

$\therefore$ Eigen vector corresponding to $\lambda = -i$ is $X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Eigen vector corresponding to $\lambda = i$

putting $\lambda = i$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -i & i \\ 0 & i & -i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y + z = 0$
$y - z = 0$
$y = z$
$x = 0$

choose $x_1 = c_1$, where c_1 is arbitrary. Then we have two linearly independent eigen vectors with $x_1 = 0$, $x_2 = 1$ & $x_1 = 1$, $x_2 = 0$

$\therefore$ Eigen vectors corresponding to $\lambda = i$

are $\quad y = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Im
*** Cayley - Hamilton Theorem

Statement :- Every square matrix statisfy its own characteristics equation.

Application :
1. To find the inverse of a matrix
2. To find higher order of a matrix.

Most ...

1. Verify cayley-Hamilton theorem of the matrix. $\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$. Hence find A^{-1} and A^4.

Sol Given matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

The characteristic eqn of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -2 & 2 \\ 1 & -2-\lambda & 3 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left[(-2-\lambda)(2-\lambda)+3\right] + 2\left[2-\lambda\right] + 2\left[-1\right] = 0$$

$$1-\lambda\left[-4+\cancel{2\lambda}-\cancel{2\lambda}+\lambda^2+3\right] + 4-2\lambda-2 = 0$$

$$-\cancel{4}+\lambda^2+3+4\lambda-\lambda^3-3\lambda+\cancel{4}-2\lambda-2 = 0$$

$$-\lambda^3+\lambda^2-\lambda+1 = 0$$

$$\lambda^3-\lambda^2+\lambda-1 = 0$$

By cayley hermiton theorem we have

$$A^3 - A^2 + A - I = 0 \longrightarrow (1)$$

Now $A^2 = A \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

and $A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

$A^3 - A^2 + A - I$

$$= \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ Hence satisfes the cayley-hermilton theorem

To find A^{-1} =

eq① → $A^3 - A^2 + A - I = 0$

$I = A^3 - A^2 + A$

Multiply on both sides A^{-1}, we get

$A^{-1}I = A^{-1}[A^3 - A^2 + A]$

$A^{-1} = A^{-1}A^3 - A^{-1}A^2 + A^{-1}A$

$A^{-1} = A^2 - A + I$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

To find A^4 = ~~Multiplication~~ Multiply with A on both sides in eqn, we get

$A[A^3 - A^2 + A - I] = 0$

$A^4 - A^3 + A^2 - AI = 0$

$\cancel{A^4 - A^3 + AI = 0}$

$A^4 - A^3 + A^2 - AI = 0$

$A^4 = A^3 - A^2 + A \qquad \{AI = IA = A\}$

$$A^4 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**** [5m] last yo

2. Verify cayley - hamilton theorem for the matrix $\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ find A^{-1}

<u>Sol</u>

The charastetic eq^n $|A-\lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$8-\lambda\left[(-3-\lambda)(1-\lambda) - 8\right] + 8\left[4-\cancel{4\lambda}+6\right] + 2\left[-\cancel{8}\,6 + 9 + 3\lambda\right] = 0$$

$$8-\lambda\left[-3+3\lambda-\lambda-\lambda^2-8\right] + 8\left[10-4\lambda\right] + 2\left[-7+3\lambda\right] = 0$$

$$8-\lambda\left[-11+2\lambda-\lambda^2\right] + 80 - 32\lambda - 14 + 6\lambda = 0$$

$$-88+16\lambda-8\lambda^2+11\lambda-2\lambda^2+\lambda^3+80-32\lambda-14+6\lambda = 0$$

$$\cancel{\lambda^3 - 10\lambda^2 + 25\lambda}$$

$$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0$$

Now $A^3 - 6A^2 - A + 22I = 0$

$$A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

64 - 32 + 6

$$A^3 = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 214 & -296 & 206 \\ 68 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$A^3 - 6A^2 - A + 22I$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 68 & -115 & 70 \\ 69 & -100 & 69 \end{bmatrix} - 6\begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + \begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eq^n $\Rightarrow$ $A^3 - 6A^2 - A + 22I = 0$

$$22I = -A^3 + 6A^2 + A$$

Multiply with A^{-1} on b.s we get

$$22A^{-1}I = -A^{-1}A^3 + 6A^{-1}A^2 + A^{-1}A$$

$22A^{-1} = -A^2 + 6A + I$

$A^{-1} = \frac{1}{22}\left[-A^2 + 6A + I\right]$

$$A^{-1} = \frac{1}{22}\left[-\begin{pmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{pmatrix} + 6\begin{pmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{22}\begin{pmatrix} 11 & 0 & -22 \\ 10 & -2 & -24 \\ 7 & -8 & -8 \end{pmatrix}$$

******

3. Verify cayley - hamilton of $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ find A^{-1}

4. $\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$ (5) $\begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ find A^{-1} (6) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1} and A^{+4}.

3 Sol

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characterstic eqn $[A - \lambda I] = 0$

$$\begin{bmatrix} 2-\lambda & -1 & 2 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$2-\lambda\left[(2-\lambda)(2-\lambda)-1\right] + 1\left[-2+\lambda+1\right] + 2\left[1-2+\lambda\right] = 0$

$\left[2-\lambda\right]\left[4-4\lambda+\lambda^2-1\right] + 1\left[-1+\lambda\right] + 2\left[-1+\lambda\right] = 0$

$\times\ 2-\lambda\left[3-4\lambda+\lambda^2\right] + 1\left[-1+\lambda\right] + 2\left[-1+\lambda\right] = 0$

$6 - \cancel{8}8\lambda + 2\lambda^2 - 3\lambda + 4\lambda^2 - \lambda^3 - 1 + \lambda - 2 + 2\lambda = 0$

$\times - \lambda^3$

$\lambda^3 - S_1\lambda^2 - S_2\lambda - S_3 = 0$

$-\lambda^3 + 6\lambda^2 - 8\lambda + 3 = 0$

$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$

$A^3-6A^2+8A-3I=0$

$$A^2=\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}.$$

$$A^3=\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}=\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}$$

$A^3-6A^2+8A-3I=$

$$\begin{bmatrix}29&-28&38\\-22&23&-28\\22&-22&29\end{bmatrix}-6\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}+8\begin{bmatrix}2&-1&2\\-1&2&-1\\1&-1&2\end{bmatrix}-3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$A^3-6A^2+8A-3I=0$

$3I=A^3-6A^2+8A$

Multiple A^{-1} on b.s we get

$3A^{-1}I=A^{-1}A^3-6A^{-1}A^2+8A^{-1}A$

$3A^{-1}=A^2-6A+8I$

$A^{-1}=\frac{1}{3}\left[A^2-6A+8I\right]$

$$A^{-1}=\frac{1}{3}\begin{bmatrix}7&-6&9\\-5&6&-6\\5&-5&7\end{bmatrix}-\begin{bmatrix}12&-6&12\\-6&12&-6\\6&-6&12\end{bmatrix}+\begin{bmatrix}8&0&0\\0&8&0\\0&0&8\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}3&0&-3\\1&2&0\\-1&1&3\end{bmatrix}$$

4
sol

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}$$

Characteristic eqn $|A-\lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 4 & 1 \\ 2 & 1-\lambda & 6 \\ -1 & 4 & 7-\lambda \end{vmatrix} = 0 \qquad (8+(1-4)$$

$$3-\lambda\left((1-\lambda)(7-\lambda)-24\right] - 4\left[14-2\lambda+6\right) + 1\left[8+1\mp\lambda\right] = 0$$

$$3-\lambda\left[7-8\lambda+\lambda^2-24\right] - 4\left[20-2\lambda\right) + 1\left[9\mp\lambda\right] = 0$$

$$21-24\lambda+3\lambda^2-72-80+8\lambda+9\mp\lambda = 0$$

$$-7\lambda+8\lambda^2-\lambda^3+24\lambda$$

$$-\lambda^3+11\lambda^2\mp 122 = 0$$

$$\lambda^3-11\lambda^2+122 = 0$$

$$A^3-11A^2+122I = 0$$

$$A^2 = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{bmatrix}\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{bmatrix}$$

$$A^3-11A^2+122I = \begin{bmatrix} 54 & 220 & 374 \\ 22 & 241 & 550 \\ -22 & 308 & 670 \end{bmatrix} - 11\begin{bmatrix} 16 & 20 & 34 \\ 2 & 33 & 50 \\ -2 & 28 & 72 \end{bmatrix} + \begin{bmatrix} 122 & 0 & 0 \\ 0 & 122 & 0 \\ 0 & 0 & 122 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^3-11A^2+122I$$

$$122I = -A^3+11A^2$$

multiple on b.s A^{-1}

$$122A^{-1}I = -A^{-1}A^3+11A^2A^{-1}$$

$$A^{-1} = \frac{-A^2+11A}{122}$$

$$A^{-1} = \frac{1}{122}\begin{bmatrix} -16 & -20 & -34 \\ -2 & -33 & -50 \\ +2 & -28 & -72 \end{bmatrix} + 11\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 6 \\ -1 & 4 & 7 \end{bmatrix} = \frac{1}{122}\begin{bmatrix} -17 & -24 & 23 \\ -20 & 22 & -16 \\ 9 & -16 & -5 \end{bmatrix}$$

5 $A = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$

Sol The characteristic eq^n of $|A-\lambda I| = 0$

$$\begin{bmatrix} -1-\lambda & -2 & 0 \\ 1 & 0-\lambda & 2 \\ 2 & 3 & 4-\lambda \end{bmatrix} = 0$$

$$-1-\lambda\left((-\lambda)(4-\lambda)-6\right) + 2\left(4-\lambda-4\right] + 0() = 0$$

$$-1-\lambda\left(-4\lambda+\lambda^2-6\right) - 2\lambda = 0$$

$$4\lambda - \lambda^2 + 6 + 4\lambda^2 - \lambda^3 + 6\lambda - 2\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 + 8\lambda + 6 = 0$$

$$\lambda^3 - 3\lambda^2 - 8\lambda - 6 = 0$$

$$-A^3 + 3A^2 + 8A + 6I = 0$$

$$A^3 - 3A^2 - 8A - 6I = 0$$

where

$$A^2 = \begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -4 \\ 3 & 4 & 8 \\ 9 & 8 & 22 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 2 & -4 \\ 3 & 4 & 8 \\ 9 & 8 & 22 \end{bmatrix}\begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -5 & -10 & -12 \\ 17 & 18 & 40 \\ 43 & 48 & 104 \end{bmatrix}$$

$$A^3 - 3A^2 - 8A - 6I = 0$$

$$\begin{bmatrix} -5 & -10 & -12 \\ 17 & 18 & 40 \\ 43 & 48 & 104 \end{bmatrix} - 3\begin{bmatrix} -1 & 2 & -4 \\ 3 & 4 & 8 \\ 9 & 8 & 22 \end{bmatrix} - 8\begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$6I = A^3 - 3A^2 - 8A$$

Multipling on bs A^{-1}

$$6A^{-1}I = A^{-1}A^3 - 3A^{-1}A^2 - 8A^{-1}A$$

$$A^{-1} = \frac{1}{6}\left(A^2 - 3A - 8I\right)$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -1 & 2 & 4 \\ 3 & 4 & 8 \\ 9 & 8 & 22 \end{bmatrix} - 3\begin{bmatrix} -1 & -2 & 0 \\ 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -6 & 8 & -4 \\ 0 & -4 & 2 \\ 3 & -1 & 2 \end{bmatrix}$$

6 Sol: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

The characterstic eqn $[A-\lambda I]=0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$1-\lambda\left((1-\lambda)(1-\lambda)-4\right) - 2\left[2-2\lambda-4\right] + 2\left[4-2+2\lambda\right] = 0$$

$$1-\lambda\left[1-2\lambda+\lambda^2-4\right] - 2\left[-2-2\lambda\right] + 2\left[2+2\lambda\right] = 0$$

$$1-\lambda\left[-3-2\lambda+\lambda^2\right] - 2\left[-2-2\lambda\right] + 2\left[2+2\lambda\right] = 0$$

$$-3-2\lambda+\lambda^2+3\lambda+2\lambda^2-\lambda^3+4+4\lambda+4+4\lambda=0$$

$$-\lambda^3+3\lambda^2+9\lambda+5=0$$

$$\lambda^3-3\lambda^2-9\lambda-5=0$$

$$A^3-3A^2-9A-5I=0$$

$$A^2 = A\cdot A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix}$$

$$A^3-3A^2-9A-5I$$

$$\begin{bmatrix} 41 & 42 & 42 \\ 42 & 41 & 42 \\ 42 & 42 & 41 \end{bmatrix} - 3\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$5I = A^3-3A^2-9A$$

Multiply on b.s A^{-1}

$$5IA^{-1} = A^{-1}A^3 - 3A^{-1}A^2 - 9A^{-1}A$$

$$A^{-1} = \frac{1}{5}\left[A^2-3A-9I\right]$$

$$A^{-1} = \frac{1}{5}\left(\begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 6 \\ 6 & 3 & 6 \\ 6 & 6 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

* Orthogonal Reduction of the matrix:

Statement:

Orthogonal reduction of square matrix:

Let 'A' and 'B' two square matrices of order 'n' then 'B' is said to be orthogonally similar to A if they exists an orthogonal matrix P such that $\boxed{B = P^{-1}AP = P^{T}AP}$

→ Working Rule of orthogonal reduction:

1. Find the eigen values of square matrix A.
2. Find the corresponding eigen vector must be pairwise orthogonal

$$\langle x_1 x_2 \rangle = 0, \quad \langle x_2 x_3 \rangle = 0, \quad \langle x_3 x_1 \rangle = 0$$

3. Find the normalized eigen vector $e_1 = \frac{x_1}{\|x_1\|}$, $e_2 = \frac{x_2}{\|x_2\|}$, $e_3 = \frac{x_3}{\|x_3\|}$ .... $e_n = \frac{x_n}{\|x_n\|}$

Here $\|x_n\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ or $\sqrt{x^2+y^2+z^2}$

4. Find the normalized modal matrix P cointaining the ~~orthogonal~~ normalized eigen vector of A as column matrix

$$P = [e_1 \; e_2 \; e_3 \; \cdots \; e_n] \quad (\text{or}) \quad P = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \cdots \frac{x_n}{\|x_n\|}\right]$$

5. Since, P is orthogonal we have $P^{-1} = P^T$

find $D = P^{-1}AP = P^TAP$

Q. Find the diagonalized matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by orthogonal reduction method.

$3(15-1)+1(-3+3)+1(1-5)$

$42-4$

Sol

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\lambda^2 - 9\lambda + 18 = 0$$

$$\lambda^2 - 3\lambda - 6\lambda + 18 = 0$$

$\lambda(\lambda-3)-6(\lambda-3)=0$

$(\lambda-3)(\lambda-6)=0$

$\lambda = 3, 6, 2$

① eigen vector with $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x-y+z=0$

$-x+y-z=0$

$\frac{x}{1-3} = \frac{-y}{-1+1} = \frac{z}{3-1} = k$

$\frac{x}{-2} = \frac{-y}{0} = \frac{z}{2} = k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

② eigen vector with $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y+z=0,$

$-x+2y-z=0$

$\frac{x}{1-2} = \frac{-y}{0+1} = \frac{z}{0-1} = k$

$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1} = k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

② eigen vector with $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\frac{x}{1+1} = \frac{-y}{3+1} = \frac{z}{3-1} = k$

$\frac{x}{\cancel{2}_1} = \frac{-y}{\cancel{4}_2} = \frac{z}{\cancel{2}_1} = k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

eigen vectors must be pair wise orthogonal

$\left\langle \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\rangle = 0$ $\quad -1+0+1=0$ satisfied

$\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle = 0$ $\quad 1-2+1=0$ satisfied

$$\angle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad -1+0+1=0$$

satisfied

These are pairwise orthogonal

$$\|x_1\| = \sqrt{x^2+y^2+z^2} = \sqrt{(-1)^2+0^2+1^2} = \sqrt{2}$$

$$\|x_2\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|x_3\| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$$

Now normalized eigen values are

$$e_1 = \frac{x_1}{\|x_1\|} \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad e_2 = \frac{x_2}{\|x_2\|} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad e_3 = \frac{x_3}{\|x_3\|} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Normalized modal matrix P is given by

$$P = [e_1 \ e_2 \ e_3]$$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \quad P^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$\therefore$ P is orthogonal $P^{-1} = P^T$

$$D = P^{-1}AP = P^TAP = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

2. Diagonalize $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$ by orthogonal reduction method

64-1
63

Sol :- let $A = \begin{bmatrix} 7-\lambda & 4 & -4 \\ 4 & -8-\lambda & -1 \\ -4 & -1 & -8-\lambda \end{bmatrix} = 0$

$\cancel{7-\lambda}\ (\cancel{(-8-\lambda)(-8-\lambda)})$

$\lambda^3 + 9\lambda^2 - \cancel{39\lambda - 717} = 0$

$$7-\lambda\left((-8-\lambda)(-8-\lambda)-1\right)-4(-32-4\lambda-4)-4(-4+32-4\lambda)=0$$

$$7-\lambda\left(+64+8\lambda+8\lambda+\lambda^2-1\right)-4[-36-4\lambda)-4[28-4\lambda)=0$$

$$7-\lambda\left[63+16\lambda+\lambda^2\right]-4[-36-4\lambda]-4[28-4\lambda]=0$$

$$441+112\lambda+7\lambda^2-63\lambda-16\lambda^2-\lambda^3+144+16\lambda-112+16\lambda=0$$

$$-\lambda^3-9\lambda^2+81\lambda+\overset{729}{\cancel{473}}=0$$

$$\lambda^3+9\lambda^2-81\lambda-\overset{729}{\cancel{473}}=0$$

$$\begin{array}{c|cccc} 9 & 1 & 9 & -81 & -729 \\ & 0 & 9 & 162 & 729 \\ \hline & 1 & 18 & 81 & \boxed{0} \end{array}$$

$$\lambda^2+18\lambda+81=0$$

$$\lambda^2+9\lambda+9\lambda+81=0$$

$$\lambda(\lambda+9)+9(\lambda+9)=0$$

$$(\lambda+9)(\lambda+9)=0$$

$$\lambda=-9,-9$$

Eigen value $\lambda = 9, -9, -9$

(i) Eigen vector corresponding to $\lambda=9$

$$\begin{bmatrix} -2 & 4 & -4 \\ 4 & -17 & -1 \\ -4 & -1 & -17 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{-4-68}=\frac{-y}{2+16}=\frac{z}{34-16}=k$$

$$\frac{x}{\underset{4}{\cancel{-72}}}=\frac{-y}{\cancel{18}},=\frac{z}{\cancel{18}},=k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=k\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

(ii) Eigen vector corresponding to $\lambda=-9$

$$\begin{bmatrix} 16 & 4 & -4 \\ 4 & 1 & -1 \\ -4 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$16x + 4y - 4z = 0$

$4x + y - z = 0$

put $y = k_1$, $z = k_2$

$4x + k_1 - k_2 = 0$

$4x = -k_1 + k_2$

$x = \dfrac{-k_1 + k_2}{4}$

where $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{-k_1+k_2}{4} \\ k_1 \\ k_2 \end{bmatrix} = \frac{k_1}{4}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} + \frac{k_2}{4}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

eigen vectors should be pair wise orthogonally

$x_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$

$\langle x_1 \cdot x_2 \rangle = 0$ $\langle x_2 x_3 \rangle = 0$ $\langle x_3 x_1 \rangle = 0$

$4 - 4 = 0$ $-4 + 4 = 0$

$\|x_1\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-4)^2 + (-1)^2 + 1^2} = \sqrt{16+1+1} = \sqrt{18}$

$\|x_2\| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-1)^2 + (4)^2} = \sqrt{1+16} = \sqrt{17}$

$\|x_3\| = \sqrt{1^2 + 4^2} = \sqrt{17}$

Now normalized eigen values are

$e_1 = \dfrac{x_1}{\|x_1\|} \begin{bmatrix} \frac{-4}{\sqrt{18}} \\ -1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$ $e_2 = \dfrac{x_2}{\|x_2\|} \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix}$ $e_3 = \dfrac{x_3}{\|x_3\|} \begin{bmatrix} \frac{1}{\sqrt{17}} \\ 0 \\ \frac{4}{\sqrt{17}} \end{bmatrix}$

Normalized modal matrix P is given by

$P = [e_1 \ e_2 \ e_3]$

$P = \begin{bmatrix} -\frac{4}{\sqrt{18}} & -\frac{1}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ -1/\sqrt{18} & \frac{4}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{18}} & 0 & \frac{4}{\sqrt{17}} \end{bmatrix}$ $P^{-1} = \begin{bmatrix} -\frac{4}{\sqrt{18}} & \frac{-1}{\sqrt{18}} & \frac{1}{\sqrt{18}} \\ -\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{17}} & 0 & \frac{4}{\sqrt{17}} \end{bmatrix}$

$\therefore P$ is orthogonal $P^{-1} = P^T$

$$D = P^{-1}AP = P^TAP = \begin{bmatrix} -4/\sqrt{18} & -1/\sqrt{18} & 1/\sqrt{18} \\ -\frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{17}} & 0 & \frac{4}{\sqrt{17}} \end{bmatrix} \begin{pmatrix} 7 & -4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix} \begin{bmatrix} \frac{-4}{\sqrt{18}} & \frac{-1}{\sqrt{17}} & \frac{1}{\sqrt{17}} \\ -1/\sqrt{18} & 4/\sqrt{17} & 0 \\ 1/\sqrt{18} & 0 & \frac{4}{\sqrt{17}} \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

continue: These are not pair wise orthogonal

$$x_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix},\ x_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$-4a - b + c = 0 \rightarrow (2)$

$a + 4c = 0 \rightarrow (3)$

let $c = k$

eq 3 $a = -4c$ eq (2) $b = -4(-4k) + k$

$a = -4k$ $b = 17k$

$$x_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} -4k \\ 17k \\ k \end{bmatrix} = k \begin{bmatrix} -4 \\ 17 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix},\ x_2 = \begin{bmatrix} -4 \\ 17 \\ 1 \end{bmatrix}\ x_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$\langle x_1 x_2 \rangle = 0,\ \langle x_2 x_3 \rangle = 0,\ \langle x_3 x_1 \rangle = 0$ pairwise orthogonal

These are required eigen vector and normalized eigen vectors are

$\|x_1\| = \sqrt{(-4)^2 + (-1)^2 + 1^2} = \sqrt{18}$

$\|x_2\| = \sqrt{(-4)^2 + 17^2 + 1^2} = \sqrt{306}$

$\|x_3\| = \sqrt{1^2 + 4^2} = \sqrt{17}$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} -4/\sqrt{18} \\ -1/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}\ e_2 = \begin{bmatrix} -4/\sqrt{306} \\ 17/\sqrt{306} \\ 1/\sqrt{306} \end{bmatrix}\ e_3 = \frac{x_3}{\|x_3\|} \begin{bmatrix} \frac{1}{\sqrt{17}} \\ 0 \\ \frac{4}{\sqrt{17}} \end{bmatrix}$$

Normalized modal matrix P

$$P = [e_1 \;\; e_2 \;\; e_3] = \begin{bmatrix} \frac{-4}{\sqrt{18}} & \frac{-4}{\sqrt{306}} & \frac{1}{\sqrt{17}} \\ \frac{-1}{\sqrt{18}} & 17/\sqrt{306} & 0 \\ 1/\sqrt{18} & \frac{1}{\sqrt{306}} & \frac{4}{\sqrt{17}} \end{bmatrix}$$

P is orthogonal, we have $P^{-1} = P^T$

$$P^T = \begin{bmatrix} -4/\sqrt{18} & -1/\sqrt{18} & 1/\sqrt{18} \\ -4/\sqrt{306} & \frac{17}{\sqrt{306}} & \frac{1}{\sqrt{306}} \\ \frac{1}{\sqrt{17}} & 0 & 4/\sqrt{17} \end{bmatrix}$$

$$D = P^T A P$$

$$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

Q. Determine diagnal mateix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$(9-1) + (18+24)$
$+ (18-4)$
$-8+14+14$
$6(9-1)+2(-6+2)+2(2-8)$
$6(8)+2(-4)+2(-4)$
$48-8-8$
$48-16$

Sol $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{r|rrrr} 8 & 1 & -12 & 36 & -32 \\ & 0 & 8 & -32 & 32 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda^2 - 2\lambda - 2\lambda + 4 = 0$

$\lambda(\lambda - 2) - 2(\lambda - 2) = 0$

$\lambda = 2, 2, 8$

eigen vector putting $\lambda = 8$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{2+10} = \frac{-y}{6} = \frac{z}{6}$$

$$\frac{x}{12} = \frac{y}{-6} = \frac{z}{6} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

eigen vector putting $\lambda = 2$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x - y + z = 0$ put $y = k_1$, $z = k_2$

$2x - k_1 + k_2$

$2x = k_1 - k_2 \qquad x = \frac{k_1 - k_2}{2}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1 - k_2}{2} \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} \\ k_1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \frac{-k_2}{2} \\ 0 \\ k_2 \end{bmatrix}$$

$$x_2 = \frac{k_1}{2}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad x_3 = \frac{k_2}{2}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$\langle x_1 x_2 \rangle = 0$, $\quad \langle x_2 x_3 \rangle \neq 0 \quad \langle x_3 x_1 \rangle = 0$

$2-2=0 \qquad -1 \neq 0 \qquad = 0$

The following vectors are not pair w.r.t orthogonal

consider

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$2a - b + c = 0 \qquad 2a + c = b$

$-a + 0 + 2c = 0 \qquad 2(2k) + k = b$

$\boxed{c = k} \qquad 4k + k = b$

$a = 2k \qquad \boxed{b = 5k}$

$\boxed{a = 2k}$

$$x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2k \\ 5k \\ k \end{bmatrix} = k\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$x_1 x_2 = 0 \qquad x_2 x_3 = 0 \qquad x_3 x_1 = 0$

$4 - 5 + 1 = 0$

$\|x_1\| = \sqrt{(2)^2 + 1^2 + 1^2} = \sqrt{4+1+1} = \sqrt{6}$

$\|x_2\| = \sqrt{2^2 + 5^2 + 1^2} = \sqrt{4+25+1} = \sqrt{30}$

$\|x_3\| = \sqrt{1^2 + 4^2} = \sqrt{5}$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} \frac{2}{\sqrt{6}} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \qquad e_2 = \begin{pmatrix} \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} \end{pmatrix} \qquad e_3 = \begin{bmatrix} -1/\sqrt{5} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$P = [e_1 \; e_2 \; e_3] \quad \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} & -\frac{1}{\sqrt{5}} \\ -1/\sqrt{6} & \frac{6}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$P^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & -1/\sqrt{6} & 1/\sqrt{6} \\ \frac{2}{\sqrt{30}} & 5/\sqrt{30} & 1/\sqrt{30} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \end{bmatrix}$$

$$D = P^T A P$$

$$D = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ \frac{-1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} & -\frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{6}} & \frac{6}{\sqrt{30}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

* **Quadratic form :-** A homogenous expression of second degree at any numbers of variables is called quadratic form

  Ex :- 1) $3x^2 + 5xy + 6y^2$ is a quadratic form in two variables x & y

  2) $x^2 + 2y^2 + 3z^2 - 6xy + 4yz - 2zx$ is a Q.E in three variable x, y & z

* **Quadratic form corresponding to real symmetric matrix :-**

  let $A = [a_{ij}]_{n\times n}$ be a real symmetric matric and let x.

  $x = [x_1 \; x_2 \; x_3 \; -- \; x_n]^T$ be a coloumn matrix then X^TAX

  ~~$A = [a_{ij}]_{n\times n}$~~

will determine a quadratic form $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = X^TAX$ /

on expanding this we seen to the Q.F =

$$Q.F = \sum_{i=1}^{n} a_{ii}x_i x_i + \sum_i \sum_j a_{ij} x_i x_j \; (i \neq j).$$

$$= \sum_{i=1}^{n} a_{ii}x_i^2 + \sum_i \sum_j a_{ij} x_i x_j$$

→ Note : 1) Consider the quadratic form $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$

This can be write $a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$ where $a_{12} = a_{21}$

This is seen to be $X^TAX = [x_1\ x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2) Consider the quadratic form $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 +$

$2a_{23}x_2x_3 + 2a_{31}x_3x_1$ This ~~is~~ can be written as

$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{12}x_1x_2 + a_{23}x_2x_3 + a_{23}x_2x_3 + a_{31}x_3x_1$
$+ a_{31}x_3x_1$

where $a_{12} = a_{21}$, $a_{23} = a_{32}$ & $a_{31} = a_{13}$

$$X^TAX = [x_1\ x_2\ x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Here A is called real symmetric matrix.

* Matrix of the Quadratic form : we know that any quadric form $Q = X^TAX$ the symmetric matrix A is called matrix of the quadratic form Q and det A is called discriminant of the quadratic form.

If det A = 0 the Q.F is called singular.

If det A ≠ 0 " " " " non-singular

The matrix of the Q.F

	x	y	z
x	x^2	$\frac{xy}{2}$	$\frac{xz}{2}$
y	$\frac{yx}{2}$	y^2	$\frac{yz}{2}$
z	$\frac{zx}{2}$	$\frac{zy}{2}$	z^2

} write only their coefficient values

(or)

	x_1	x_2	x_3
x_1	x_1^2	$\frac{x_1x_2}{2}$	$\frac{x_1x_3}{2}$
x_2	$\frac{x_2x_1}{2}$	x_2^2	$\frac{x_2x_3}{2}$
x_3	$\frac{x_3x_1}{2}$	$\frac{x_3x_2}{2}$	x_3^2

Ex:

$3x^2+4y^2+6z^2-2xy+4yz-10zx$

The matrix of the Q.F is

$$A=\begin{pmatrix}3 & -\frac{2}{2} & -\frac{10}{2}\\ -\frac{2}{2} & 4 & \frac{4}{2}\\ -\frac{10}{2} & \frac{4}{2} & 6\end{pmatrix}=\begin{bmatrix}3 & -1 & -5\\ -1 & 4 & 2\\ -5 & 2 & 6\end{bmatrix}$$

****

IM

1. Find the symmetric matrix corresponding to the Q.F $x^2+6xy+5y^2$

The ~~matric of the~~ Q.F is $x^2+6xy+5y^2$

Let A be the symmetric matrix of this Q.F

Then $A=\begin{bmatrix}x^2 & \frac{xy}{2}\\ \frac{yx}{2} & y^2\end{bmatrix}$

$$=\begin{bmatrix}1 & \frac{6}{2}\\ \frac{6}{2} & 5\end{bmatrix}=\begin{bmatrix}1 & 3\\ 3 & 5\end{bmatrix}=\boxed{\begin{matrix}1 & 3\\ 3 & 5\end{matrix}}$$

2. Find the symmetric matric corresponding to the Q.F

$x^2+2y^2+3z^2+4xy+5yz+6zx$

$$A=\begin{pmatrix}1 & \frac{4}{2} & \frac{6}{2}\\ \frac{4}{2} & 2 & \frac{5}{2}\\ \frac{6}{2} & \frac{5}{2} & 3\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\ 2 & 2 & \frac{5}{2}\\ 3 & \frac{5}{2} & 3\end{pmatrix}$$

3. Find the quadratic form corresponding to the symmetric matrix $\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$

Sol Given matrix is $\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$

The Q.F related to the given matrix is X^TAX

Where $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$ $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}$ $X^T=[x\ y\ z]$

$$Q = X^TAX = [x \ y \ z]\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

$$= [x \ y \ z]\begin{bmatrix}x+2y+3z\\ 2x+y+3z\\ 3x+3y+z\end{bmatrix}$$

$$= x(x+2y+3z)+y(2x+y+3z)+z(3x+3y+z)$$

$$= x^2+2xy+3zx+2xy+y^2+3zy+3xz+3yz+z^2$$

$$= x^2+y^2+z^2+4xy+6zx+6yz$$

4. Find the real symmetric matrix corresponding to the Q.F

1. $3x_1^2-2x_2^2-x_3^2-4x_1x_2+12x_2x_3+8x_1x_3$
2. $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$
3. $x^2+2y^2+3z^2+4xy-10yz-9zx$
4. $2x_1x_2+6x_1x_3-4x_2x_3$

5. Find the Q.F corresponding to the matrix

(i) $\begin{bmatrix}1 & 3 & -1\\ 3 & 4 & 5\\ -1 & 5 & 2\end{bmatrix}$ (2) $\begin{bmatrix}1 & 5 & 7\\ 5 & 4 & 6\\ 7 & 6 & 3\end{bmatrix}$ (3) $\begin{bmatrix}1 & 2 & 5\\ 2 & 0 & 3\\ 5 & 3 & 4\end{bmatrix}$

1. $3x_1^2-2x_2^2-x_3^2-4x_1x_2+12x_2x_3+8x_1x_3$

Sol The given Q.F is $3x_1^2-2x_2^2-x_3^2-4x_1x_2+12x_2x_3+8x_1x_3$

A be the symmetric matrix of this Q.F

Then $A = \begin{bmatrix}3 & -\frac{4}{2} & \frac{8}{2}\\ -\frac{4}{2} & -2 & 12/2\\ \frac{8}{2} & 12/2 & -1\end{bmatrix}$

$$A = \begin{bmatrix}3 & -2 & 4\\ -2 & -2 & 6\\ 4 & 6 & -1\end{bmatrix}$$

2. $x_1^2+2x_2^2-7x_3^2-4x_1x_2+8x_1x_3+5x_2x_3$

Sol

$A = \begin{bmatrix}1 & -4/2 & 8/2\\ -4/2 & 2 & 5/2\\ 8/2 & 5/2 & -7\end{bmatrix}$ $A = \begin{bmatrix}1 & -2 & 4\\ -2 & 2 & 5/2\\ 4 & 5/2 & -7\end{bmatrix}$

3. $x^2+2y^2+3z^2+4xy-10yz-9zx$

Sol

$$A = \begin{bmatrix} 1 & \frac{4}{2} & -9/2 \\ 4/2 & 2 & -10/2 \\ -9/2 & -10/2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -9/2 \\ 2 & 2 & -5 \\ -9/2 & -5 & 3 \end{bmatrix}$$

4. $2x_1x_2+6x_1x_3-4x_2x_3$

Sol A be the symmetric matrix of this Quadratic form

$$A = \begin{bmatrix} 0 & 2/2 & 6/2 \\ 2/2 & 0 & -4/2 \\ 6/2 & -4/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{bmatrix}$$

5. (i) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix}$

Sol

$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix}$ $\quad x^T = \begin{bmatrix} x & y & z \end{bmatrix}$

$$Q = x^TAx = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 3 & 4 & 5 \\ -1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x+3y-z \\ 3x+4y+5z \\ -x+5y+2z \end{bmatrix}$$

$$= x[x+3y-z] + y[3x+4y+5z] + z[-x+5y+2z]$$

$$= x^2+3xy-zx+3xy+4y^2+5zy-xz+5yz+2z^2$$

$$= x^2+4y^2+2z^2+6xy+10yz-2xz$$

2. $\begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix}$ $\quad x^T = \begin{bmatrix} x & y & z \end{bmatrix},$

$$Q = X^T A X = [x\ y\ z] \begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x\ y\ z] \begin{bmatrix} x+5y+7z \\ 5x+4y+6z \\ 7x+6y+3z \end{bmatrix}$$

$$= x[x+5y+7z] + y[5x+4y+6z] + z[7x+6y+3z]$$

$$= x^2+5xy+7xz+5xy+4y^2+6zy+7xz+6yz+3z^2$$

$$= x^2+4y^2+3z^2+10xy+14xz+12zy$$

3. $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$ $X^T = [x\ y\ z]$

$$Q = X^T A X$$

$$= [x\ y\ z] \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x\ y\ z] \begin{bmatrix} x+2y+5z \\ 2x+0+3z \\ 5x+3y+4z \end{bmatrix}$$

$$= x(x+2y+5z) + y[2x+3z] + z[5x+3y+4z]$$

$$= x^2+2xy+5xz+2xy+3yz+5xz+3yz+4z^2$$

$$\cancel{= x^2+4z^2+4xy+10xz+6x}$$

$$= x^2+4z^2+4xy+6yz+10xz$$

* Rank of the Quadratic form
Let X^TAX be a quadratic form & the rank of 'r' of A is called the rank of the quadratic form

* Canonical form (or) Normal form of a Q.F :
Let X^TAX be a ~~tran~~ Q.F in 'n' variable then there exist real non-singular transformation $\boxed{X = PY}$ which transformes X^TAX to another Q.F of type Y^TDY

$$\boxed{\therefore Y^TDY = \lambda_1 Y_1^2 + \lambda_2 Y_2^2 + \cdots + \lambda_n Y_n^2}$$

then Y^TDY is called canonical form or normal form of a Q.F X^TAX. Here D is called diagonal of $dia(\lambda_1, \lambda_2 \ldots \lambda_n)$

* Index of the Quadratic forms : The no. of positive term in canonical form of normal form of a Q.F is known as the index of the Quadratic form
It is denoted by 's'.

* Signature of the Q.F : If r is the rank of the quadratic form and 's' is the number of positive term in its normal form then the number of positive term over the no. of negative term a normal form of X^TAX.

$$s - (r - s) = 2s - r$$

*** Imp
Nature of the Q.F :- The Q.F X^TAX in 'n' variables is said to be :

1) Positive definite :- If $r = n$ and $s = n$ (or) if all the eigen values are +ve
2) Negative definite :- If $r = n$ and $s = 0$ (or) if all the eigen values are -ve
3) Positive semidefinite : If $r < n$ and $s = r$ (or) if all the eigen values of $A \geq 0$ and atleast one value zero
4) Negative semi definite : If $r < n$ and $s = 0$ (or) If all the eigen values of $A \leq 0$ and atleast one value zero
5) Indefinite :- In all other cases

* Working rule for Q.F :-

1. Reduction of Q.F into canonical form by orthognalization

Step 1 :- Given Q.F X^TAX.

Step 2 : Write the symmetric A from the given quadratic form

Step 3 : Find the eigen values of A

Step 4 : Find the eigen vector corresponding to eigen values and must be pair wise orthogonal $\langle x_1 x_2\rangle = 0$, $\langle x_2 x_3\rangle = 0$, $\langle x_3 x_1\rangle = 0$

Steps 5 : Find the normalized eigen vector

$$e_1 = \frac{x_1}{\|x_1\|}, \quad e_2 = \frac{x_2}{\|x_2\|}, \quad e_3 = \frac{x_3}{\|x_3\|}$$

Step 6 :- Write the normalized modal matrix $P = [e_1\ e_2\ e_3]$

Step 7 : Since P is orthogonal thus $P^{-1} = P^T$

$D = P^{-1}AP = P^TAP$ = where D is diagonalization

$D = dia(\lambda_1\ \lambda_2\ \lambda_3 \cdots \lambda_n)$

Step 8 : Write the canonical form is Y^TDY

$$Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots \lambda_n y_n^2$$

1. Reduce the Q.F $3x^2 + 3y^2 + 3z^2 - 2yz + 2xy + 2zx$ into canonical form by orthogonal reduction and find rank, index, signatore & natu of Q.F

Sol :- Given, QF is $3x^2 + 3y^2 + 3z^2 - 2yz + 2xy + 2zx$

Write the symmetric matrix of Q.F is

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

characteristic eq $= |A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$3-\lambda[9 - 6\lambda + \lambda^2 - 1] - 1(3 - \lambda + 1] + 1[-1 - 3 + \lambda] = 0$$

$$3-\lambda[8 - 6\lambda + \lambda^2] - 1(4-\lambda] + 1[-4 + \lambda] = 0$$

$$24 - 18\lambda + 3\lambda^2 - 8\lambda + 6\lambda^2 - \lambda^3 - 4 + \lambda - 4 + \lambda = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 16 = 0$$

$$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline & 1 & -8 & 16 & \boxed{0} \end{array}$$

$\lambda^2 - 8\lambda + 16 = 0$

$\lambda^2 - 4\lambda - 4\lambda + 16 = 0$

$\lambda(\lambda - 4) - 4(\lambda - 4) = 0$

$(\lambda - 4)(\lambda - 4) = 0$

$\lambda = 1, 4, 4$ The natux of the Q.F is positive definite

eigen vector putting $\lambda = 1$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{-1-2} = \frac{-y}{-2-1} = \frac{z}{4-1} = k$$

$$\frac{x}{-\cancel{3}_{1}} = \frac{\cancel{-}y}{\cancel{-3}_{1}} = \frac{z}{\cancel{3}} = k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

eigen vector putting $\lambda = 4$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + y + z = 0$

$x - y - z = 0$

put $y = k_1$

$z = k_2$

$-x + k_1 + k_2 = 0$

$x = k_1 + k_2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1 + k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$= k_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\langle x_1 x_2 \rangle = 0 \qquad \langle x_2 x_3 \rangle \neq 0 \qquad \langle x_3 x_1 \rangle = 0$

$-1+1=0$

Not pair wise orthogonal

$x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$-a+b+c=0$

$a+0+c=0$

put $\boxed{c=k}$

$-a+b+k=0$

$\boxed{a=-k}$

$+k+b+\cancel{+k}=0$

$\boxed{\cancel{b=0}}$ $b=-2k$

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\langle x_1 x_2 \rangle = 0 \qquad \langle x_2 x_3 \rangle = 0 \qquad \langle x_3 x_1 \rangle = 0$

$1-2+1=0 \qquad -1+1=0 \qquad -1+1=0$

$\|x_1\| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{1+1+1} = \sqrt{3}$

$\|x_2\| = \sqrt{1+4+1} = \sqrt{6}$

$\|x_3\| = \sqrt{1+1} = \sqrt{2}$

$e_1 = \dfrac{x_1}{\|x_1\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad e_2 = \begin{bmatrix} -1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad e_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$P = [e_1 \; e_2 \; e_3] =$

$P^T = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = P = \begin{bmatrix} -1/\sqrt{3} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$D = P^T A P$$

$$= \begin{bmatrix} -1/\sqrt{3} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ +1/\sqrt{3} & -2/\sqrt{6} & \frac{1}{\sqrt{2}} \\ 1/\sqrt{3} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & -1/\sqrt{3} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The canonical form is $Y^T D Y$. Now $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Y^T D Y = [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$Y^T D Y = 1y_1^2 + 4y_2^2 + 4y_3^2$$

Rank $= r = 3$

Index = no. of +ve terms

$s = 3$

Signature $= 2s - r$

$= 2(3) - 3$

$= 3$

2. Reduce the Q.F $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$ into canonical form find rank, index, signature, nature of index. $\lambda = 0, 3, 3$

(2) $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$ $\lambda = 1, 3, 4$

① $2x^2+2y^2+2z^2-2xy-2yz-2zx$

Sol Given Q.F $2x^2+2y^2+2z^2-2xy-2yz-2zx$

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

characteristic eq^n $[A-\lambda I]=0$

$$A = \begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} = 0$$

$$2-\lambda\left[4-4\lambda+\lambda^2-1\right]+1\left[-2+\lambda-1\right]-1\left[1+2-\lambda\right]=0$$

$$2-\lambda\left[3-4\lambda+\lambda^2\right]+1\left[-3+\lambda\right]-1\left[3-\lambda\right]=0$$

$$6-8\lambda+2\lambda^2-3\lambda+4\lambda^2-\lambda^3-3+\lambda-3+\lambda=0$$

$$-\lambda^3+6\lambda^2-9\lambda=0$$

3	-1	6	-9
	0	-3	9
	-1	3	$\boxed{0}$

$$-\lambda+3=0$$

$$\lambda=3$$

$$\lambda=0,3,3$$

The nature of the Q.F is postive definite

eigen vector putting $\lambda=0$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{1+2} = \frac{-y}{-2-1} = \frac{z}{4-1} = k \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{x}{\cancel{3}\,1} = \frac{\cancel{-}y}{\cancel{-3}\,1} = \frac{z}{\cancel{3}\,1} = k$$

eigen vector putting $\lambda=3$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x-y-z=0$$
$$-x-y-z=0$$

put $y=k_1$
$z=k_2$

$$-x-k_1-k_2=0$$
$$-x=k_1+k_2$$
$$x=-k_1-k_2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1-k_2 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -k_2 \\ 0 \\ k_2 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\langle x_1 x_2 \rangle = 0$ $\quad \langle x_2 x_3 \rangle \neq 0$ $\quad \langle x_3 x_1 \rangle = 0$

$-1+1=0$ $\quad 1$ $\quad -1+0+1=0$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$a+b+c=0$$
$$\underline{-a+c}=0$$

put $\boxed{c=k}$

$$-a+k=0$$
$$-a=-k$$
$$\boxed{a=k}$$

$$k+b+k=0$$
$$\boxed{b=-2k}$$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$\langle x_1 x_2 \rangle = 0$ $\quad \langle x_2 x_3 \rangle = 0$ $\quad \langle x_3 x_1 \rangle = 0$

$1-2+1=0$ $\quad -1+1=0$ $\quad -1+1=0$

$\|x_1\| = \sqrt{3}$

$\|x_2\| = \sqrt{6}$

$\|x_3\| = \sqrt{2}$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad e_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \quad e_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$P = [e_1\ e_2\ e_3] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & -2/\sqrt{6} & 0 \\ \frac{1}{\sqrt{3}} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & \frac{1}{\sqrt{6}} \\ -1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$D = P^T A P$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & -2/\sqrt{6} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The canonical form is $Y^T D Y$ $\quad$ Now $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Y^T D Y = [y_1\ y_2\ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$Y^T D Y = 3y_2^2 + 3y_3^2$

Rank $= r = 2$

Index $=$ no. of +ve terms $= 2 = s$

Signature $= 2s - r$
$= 4 - 2$
$= 2$

Q. $3x^2+2y^2+3z^2-2xy-2yz$

Sol The given Q.F is $3x^2+2y^2+3z^2-2xy-2yz$

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characterstics eqn $[A-\lambda I]=0$

$$A = \begin{bmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = 0$$

$$3-\lambda\left[6-5\lambda+\lambda^2-1\right]+1\left[-3+\lambda+0\right]+0\left[\quad\right]=0$$

$$3-\lambda\left[5-5\lambda+\lambda^2\right]+1\left[-3+\lambda\right]=0$$

$$15-15\lambda+3\lambda^2-5\lambda+5\lambda^2-\lambda^3-3+\lambda=0$$

$$-\lambda^3+8\lambda^2-19\lambda+12=0$$

$$\lambda^3-8\lambda^2+19\lambda-12=0$$

1	1	-8	19	-12
	0	1	-7	12
	1	-7	12	$\boxed{0}$

$$\lambda^2-7\lambda+12=0$$

$$\lambda^2-4\lambda-3\lambda+12=0$$

$$\lambda(\lambda-4)-3(\lambda-4)=0$$

$$(\lambda-4)(\lambda-3)=0$$

$$\lambda=4,3,1$$

The eigen values are $\lambda=1,3,4$

(i) The eigen vector putting $\lambda=1$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{1-0} = \frac{-y}{-2} = \frac{z}{2-1} = k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(ii) Putting $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{1-0} = \frac{y}{0-0} = \frac{z}{0-1} = k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(iii) putting $\lambda = 4$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{1-0} = \frac{-y}{1-0} = \frac{z}{2-1} = k$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\langle x_1 x_2 \rangle = 0 \quad \langle x_2 x_3 \rangle = 0 \quad \langle x_3 x_1 \rangle = 0$

$1 - 1 = 0 \quad 1 - 1 = 0 \quad 1 - 2 + 1 = 0$

This are pairwise orthogonal

$\|x_1\| = \sqrt{6}$

$\|x_2\| = \sqrt{2}$

$\|x_3\| = \sqrt{3}$

$$e_1 = \frac{x_1}{\|x_1\|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad e_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad e_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$P = [e_1\ e_2\ e_3] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} = P^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -1/\sqrt{2} \\ \frac{1}{\sqrt{3}} & -1/\sqrt{3} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$D = P^T A P$

$$= \begin{bmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt2} & 0 & -1/\sqrt2 \\ \frac{1}{\sqrt3} & -1/\sqrt3 & 1/\sqrt3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{2}{\sqrt6} & 0 & \frac{-1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The canonical form $Y^T D Y$ Now $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$Y^T D Y = [y_1 \ y_2 \ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & \cancel{0} \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$Y^T D Y = 1y_1^2 + 3y_2^2 + 4y_3^2$

Rank $= r = 3$

Index = no. of positive terms

$s = 3$

Signature $= 2s - r$

$= 2(3) - 3$

$= 6 - 3$

$= 3$