UNIT-III

Improper Integrals: Consider the integral $\int_a^b f(x)\,dx$ such that an integral for which is not finite, that

(i) either the interval of integral is not finite means (i.e) $a = -\infty$ or $b = \infty$ or both

(ii) The function $f(x)$ is an bounded at one or more points in $[a,b]$, is called an improper integral.

(iii) Integral when satisfy both the condition (i) & (ii) are called improper Integral third kind.

Ex :- (i) $\int_0^\infty \frac{dx}{1+x^4}$ and $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ are improper integral of the first kind

(ii) $\int_0^1 \frac{1}{1+x^2}\,dx$ is improper integral of the second kind

(iii) The Gamma function defined by the integral $\int_0^\infty e^{-x} x^{n-1}\,dx$; where $n>0$ is an improper integral of third kind.

* Beta function :

The difinite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is called beta function and is denoted by $B(m,n)$ and read as "Beta m,n".

- The above integral convergs first $m>0, n>0$

Thus ;

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{where } m>0, n>0$$

*** Properties of Beta functions :-

(i) Symmetry of Beta function (i.e) $B(m,n) = B(n,m)$

Proof:- $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $1-x = y$ so that $-dx = dy$

$dx = -dy$

when $x=0 \Rightarrow y = 1-0$, $y=1$ | when $x=1 \Rightarrow y = 1-1$, $y=0$

$$B(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1} (-dy)$$

$$= -\int_0^1 y^{n-1} (1-y)^{m-1} (-dy)$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx .$$

$$B(m,n) = B(n,m)$$

******* ****

(ii) $\frac{1}{2} B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Proof :- By the definition of Beta function

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \rightarrow (1)$$

put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$
$\sin^2\theta = 0$
$\sin\theta = \sin 0$
$\theta = 0°$

When $x = 1$
$\sin^2\theta = 1$
$\sin^2\theta = \sin^2 \pi/2$
$\theta = \pi/2$

$$\text{eq } (1) \rightarrow B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta\, (\cos^2\theta)^{n-1} \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-2+1}\theta\, d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$$

$$\frac{1}{2} B(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta .$$

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(iii) $B(m,n) = B(m+1,n) + B(m,n+1)$

Proof :- $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

(i) Illy $B(m+1,n) = \int_0^1 x^{m+1-1}(1-x)^{n-1}dx = \int_0^1 x^m(1-x)^{n-1}dx \rightarrow ①$

(ii) $B(m,n+1) = \int_0^1 x^{m-1}(1-x)^n dx \rightarrow ②$

Adding eq ① & ②

(iii) $B(m+1,n) + B(m,n+1) = \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$

$$= \int_0^1 (x^m(1-x)^{n-1} + x^{m-1}(1-x)^n)dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[x+1-x\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$= B(m,n)$$

$\therefore B(m+1,n) + B(m,n+1) = B(m,n)$.

* (iv) $B(m,n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)}$ If m and n are +ve integers

* Other form of Beta function :

(i) $B(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^\infty \dfrac{x^{n-1}}{(1+x)^{m+n}}dx$

** (ii) $B(m,n) = \int_0^\infty \dfrac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$

(iii) $B(m,n) = a^m b^n \int_0^\infty \dfrac{x^{m-1}}{(ax+b)^{m+n}}dx$

(iv) $\dfrac{B(m,n)}{a^n(1+a)^m} = \int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}dx$

(v) $(a-b)^{m+n-1} B(m,n) = \int_a^b (x-b)^{m-1}(a-x)^{n-1}dx \;, m>0, n>0$

1. Prove that $\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5} B\left[\frac{2}{5}, \frac{1}{2}\right]$

S:- $\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx$

put $x^5 = y$

$x = y^{1/5}$

so that $dx = \frac{1}{5} y^{\frac{1}{5}-1} dy$

$dx = \frac{1}{5} y^{-4/5} dy$

when $x=0$, $y=0$ $\downarrow$ when $x=1$, $y=1$

$$\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \int_0^1 \frac{y^{1/5}}{\sqrt{1-y}} \frac{1}{5} y^{-4/5} dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{1}{5}-4/5} (1-y)^{-1/2} dy$$

$$= \frac{1}{5}\int_0^1 y^{-3/5} (1-y)^{-1/2} dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{3}{5}-1} (1-y)^{1/2-1} dy$$

$$\int_0^1 \frac{x}{\sqrt{1-x^5}}\,dx = \frac{1}{5} B\left[\frac{2}{5}, \frac{1}{2}\right],$$

{By ~~po~~ Beta function $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

2. Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ in term of B function.

S:- put $x^5 = y$

$x = y^{1/5}$

so that $dx = \frac{1}{5} y^{1/5-1} dy = \frac{1}{5} y^{-4/5} dy$

when $x=0$, $y=0$ $\downarrow$ $x=1$, $y=1$

$$\int_0^1 \frac{x^2}{\sqrt{1-x5}}\,dx = \int_0^1 \frac{(y^{1/5})^2}{\sqrt{1-y}} \frac{1}{5} y^{-4/5} dx = \frac{1}{5}\int_0^1 \frac{y^{2/5}}{\sqrt{1-y}} y^{-4/5} dx$$

$$= \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2} dy = \frac{1}{5}\int_0^1 y^{3/5-1}(1-y)^{1/2-1} dy = \frac{1}{5} B\left[\frac{3}{5}, \frac{1}{2}\right]$$

(iii) $\int_0^1 \frac{x^3}{\sqrt{1-x^5}} dx$.

S:- put $x^5 = y \Rightarrow x = y^{1/5}$

than $dx = \frac{1}{5} y^{1/5-1} dy$.

$= \frac{1}{5} y^{-4/5} dy$.

$\int_0^1 \frac{y^{3/5}}{\sqrt{1-y}} \frac{1}{5} y^{-4/5} dy = \frac{1}{5}\int_0^1 y^{3/5} \cdot y^{-4/5} (1-y)^{-1/2} dy$

$= \frac{1}{5}\int_0^1 y^{-1/5} (1-y)^{-1/2} dy$

$= \frac{1}{5}\int_0^1 y^{\frac{4}{5}-1} (1-y)^{1/2-1} dy$

$= \frac{1}{5} B\left[\frac{4}{5}, \frac{1}{2}\right]$

(iv) $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$

S:- put $x^2 = y \rightarrow x = y^{1/2}$

$dx = \frac{1}{2} y^{1/2-1} dy$

$= \frac{1}{2} y^{-1/2} dy$

$= \frac{1}{2}\int_0^1 \frac{y^{1/2}}{\sqrt{1-y}} y^{-1/2} dy$

$= \frac{1}{2}\int_0^1 y^{1/2-1/2} (1-y)^{-1/2} dy$

$= \frac{1}{2}\int_0^1 y^0 (1-y)^{-1/2} dy$

$= \frac{1}{2}\int_0^1 y (1-y)^{1/2-1} dy$

$= \frac{1}{2} B\left[\, , \frac{1}{2}\right]$

(v) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$.

S:- put $x^2 = 9y$ $\quad dx = \frac{3}{2} y^{-1/2} dy$

$x = 9^{1/2} y^{1/2}$

$x = 3^{2/2} y^{1/2}$

$x = 3y^{1/2}$

$= \frac{3}{2}\int_0^3 \frac{y^{-1/2}}{\sqrt{9-9y}} dy = \frac{3}{2}\int_0^3 y^{-1/2} (9-9y)^{-1/2}$

$= \frac{3}{2}\int_0^3 y^{-1/2} [9-9y]^{-1/2} dy = \frac{3}{2}\int_0^3 y^{1/2-1} (9$

$= \frac{3}{2} B\left[\frac{1}{2}, \frac{1}{2}\right]$

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* Gamma function:

The finite integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called the Gamma function. $\Gamma(n)$ and read as "Gamma n". The above integral convergs for only $n > 0$

Thus. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ (or) $\Gamma(m) = \int_0^\infty e^{-\infty} x^{m-1} dx$.

Propeeties of Gamma function:-

(1) $\Gamma(1) = 1$

(2) $\Gamma(n) = (n-1)\Gamma(n-1)$ when $n > 1$

(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(4) $\Gamma(n) = (n-1)!$

(5) $\Gamma(n+1) = n!$

(6) $\Gamma(n+1) = n\Gamma(n)$

(7) $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ (or) {only for -ve values}

(8) $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

P.T $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

By using $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

$\{ \Gamma(\frac{1}{2})\Gamma(1-\frac{1}{2}) = \frac{\pi}{\sin\frac{1}{2}\pi} \}$

$\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}) = \frac{\pi}{\sin\pi/2}$

$(\Gamma(\frac{1}{2}))^2 = \frac{\pi}{1}$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

~~B(m,n)~~ =

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Relation between Beta and gamma function:

Proof:- The definition of gamma function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{where, } m>0,\ n>0$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \Bigg| \quad \Gamma(m) = \int_0^\infty e^{-x} x^{m-1} dx \rightarrow (1)$$

put $x = yt$ so that $dx = y\,dt$

when $x=0$, $t=0$ $\downarrow$ when $x=\infty$, $t=\infty$

Then eq (1) becomes

$$\Gamma(m) = \int_0^\infty e^{-yt} (yt)^{m-1} y\, dt$$

$$= \int_0^\infty e^{-yt} t^{m-1} y^{m-1} y\, dt$$

$$\Gamma(m) = \int_0^\infty e^{-yt} t^{m-1} y^{m-1+1} dt$$

$$\Gamma(m) = y^m \int_0^\infty e^{-yt} t^{m-1} dt$$

$$\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yt} t^{m-1} dt \quad \left\{ \int_a^b f(x)\, dt = \int_a^b f(x).\, dx \right\}$$

$$\boxed{\frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx}. x^{m-1} dx} \rightarrow (2)$$

Integrating on both sides w.r to y from 0 to ∞ and Multiplying on both sides $e^{-y}. y^{m+n-1}$ in eq (2), we get

$$\int_0^\infty \frac{\Gamma(m)}{y^m}. e^{-y} y^{m+n-1} dy = \int_0^\infty \left[\int_0^\infty e^{-yx}. x^{m-1} dx \right] e^{-y} y^{m+n-1} dy$$

$$\int_0^\infty \Gamma(m). e^{-y}. y^{m+n-1} y^{-m} dy = \int_0^\infty \left[\int_0^\infty e^{-yx}. e^{-y}. x^{m-1}. y^{m+n-1} dy\, dx \right.$$

$$\Gamma(m) \cdot \int_0^\infty e^{-y}\, y^{\cancel{m}+n-1-\cancel{m}}\, dy = \int_0^\infty \left[\int_0^\infty e^{-y(1+x)}\, x^{m-1} \cdot y^{m+n-1}\, dx \right] dy$$

{By interchanging the order of integration}

$$\Gamma(m) \int_0^\infty e^{-y}\, y^{n-1}\, dy = \int_0^\infty \left[\int_0^\infty e^{-y(1+x)} \cdot y^{m+n-1}\, dy \right] x^{m-1}\, dx$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty \left[\frac{\Gamma(m+n)}{(1+x)^{m+n}} \right] x^{m-1}\, dx \quad \left\{ \text{by using eq ② model} \right\}$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx \quad \left\{ \text{by using form ① of Beta fun} \right\}$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n) \cdot B(m,n)$$

$$\boxed{B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Find the Beta value of $\Gamma\left(\frac{5}{2}\right)$

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - 1\right)$$

① Find the value of $\Gamma\left(-\frac{7}{2}\right)$

Sol $\Gamma\left(-\frac{7}{2}\right) = \dfrac{\Gamma\left(-\frac{7}{2}+1\right)}{-7/2}$

$= \dfrac{\Gamma\left(-\frac{5}{2}\right)}{-7/2}$

$\cancel{= -\frac{2}{7}\Gamma\left(-\frac{5}{2}\right)}$

$\cancel{= \frac{-10}{7}\Gamma\left(-\frac{1}{2}\right)}$

$\cancel{= \frac{-10}{7}\sqrt{\pi}}$

$= -\frac{2}{7}\left[\dfrac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}}\right]$

$= -\frac{2}{7} \times \frac{2}{-5}\Gamma\left(-\frac{3}{2}\right)$

$= \frac{4}{35}\Gamma\left(\dfrac{-\frac{3}{2}+1}{-3/2}\right)$

$= \frac{4}{35} \times -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right)$

$= \frac{-8}{105}\Gamma\left(\dfrac{-\frac{1}{2}+1}{-1/2}\right)$

$= \frac{-8}{105} \times -\frac{2}{1}\Gamma\left(\frac{1}{2}\right)$

$= \frac{16}{105}\Gamma\left(\frac{1}{2}\right)$

$= \frac{16}{105}\sqrt{\pi}$

Ass

① $\Gamma\left(-\frac{9}{2}\right)$ ② $\Gamma\left(-\frac{11}{2}\right)$ ③ $\Gamma\left(\frac{7}{2}\right)$ ④ $\Gamma(10)$

Sol $\Gamma\left(-\frac{9}{2}\right) = \dfrac{\Gamma\left(-\frac{9}{2}+1\right)}{-9/2} = \dfrac{\Gamma\left(\frac{-9+2}{2}\right)}{-9/2} = -\frac{2}{9}\Gamma\left(-7/2\right)$

$-7/2+1$

$= +\frac{2}{9}\,\frac{2}{5}\Gamma\left(-\frac{5}{2}\right) = -\frac{4}{45}\,\frac{2}{3}\Gamma\left(-\frac{3}{2}\right) = \frac{-8}{45\times 3} \times -\frac{2}{1}\Gamma\left(-\frac{1}{2}\right)$

$= \frac{-16}{135}\sqrt{\pi}$

• $\int_0^1 x^4(1-x)^2 dx$.

Sol: $\int_0^1 x^{5-1}(1-x)^{3-1} dx = B(5,3)$

$= \frac{\Gamma(5)\Gamma(3)}{\Gamma(5+3)} = \frac{\Gamma(4+1)\Gamma(2+1)}{\Gamma(8)} = \frac{4!\,2!}{\Gamma(7+1)}$ $\{\Gamma(n+1) = n!\}$

$= \frac{4!\,2!}{7!} = \frac{4\times3\times2\times1\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{105}$ //.

• Evaluate $\int_0^1 x^3\sqrt{1-x}\, dx$ using B-Γ functions.

Sol:- $\int_0^1 x^3\sqrt{1-x}\, dx = \int_0^1 x^{4-1}(1-x)^{1/2} dx = \int_0^1 x^{4-1}(1-x)^{3/2-1} dx$

$B[4,\frac{3}{2}] = \frac{\Gamma(4)\Gamma(\frac{3}{2})}{\Gamma(4+\frac{3}{2})} = \frac{\Gamma(4)\Gamma(\frac{3}{2})}{\Gamma(\frac{11}{2})} = \frac{3!\,\cancel{(\frac{1}{2})}\,\cancel{\Gamma(\frac{1}{2})}\;\Gamma(\frac{3}{2})}{\frac{9}{2}\Gamma(\frac{9}{2})}$

$= \frac{3!\times\cancel{\sqrt{\pi}}\,\Gamma(\frac{3}{2})}{\frac{9}{2}\times\frac{7}{2}\Gamma(\frac{7}{2})} = \frac{3!\times\cancel{\frac{1}{2}}\times\cancel{\sqrt{\pi}}\,\cancel{\Gamma(\frac{3}{2})}}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\cancel{\Gamma(\frac{3}{2})}} = \frac{3!\times2^4}{9\times7\times5\times3}$

$= \frac{\cancel{3}\times2\times1\times2^4}{9\times5\times7\times\cancel{3}} = \frac{2^5}{9\times5\times7} = \frac{32}{315}$ //.

Assignment

• Evaluate $\int_0^1 x^5(1-x)^3 dx$

Sol: $\int_0^1 x^{6-1}(1-x)^{4-1} dx = B[6,4] = \frac{\Gamma(6)\Gamma(4)}{\Gamma(10)} = \frac{5!\times3!}{9!}$

$= \frac{3!\times\cancel{5!}}{8\times9\times7\times6\times\cancel{5!}} = \frac{\cancel{3}\times\cancel{2}\times1}{8\times9\times7\times\cancel{6}} = \frac{1}{8\times63}$ //.

• $\int_0^1 x^7(1-x)^5 dx$.

Sol: $\int_0^1 x^{8-1}(1-x)^{6-1} dx = B[8,6] = \frac{\Gamma(8)\Gamma(6)}{\Gamma(14)} = \frac{7!\times6!}{13!}$

$\frac{\cancel{7!}\times6!}{13\times12\times11\times10\times9\times8\times\cancel{7!}} = \frac{\overset{3}{\cancel{9}}\times\cancel{8}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1}{13\times\cancel{12}\times11\times\cancel{10}\times\underset{3}{\cancel{9}}\times\underset{4}{\cancel{8}}} = \frac{1}{13\times11\times12}$

=

• Compute

① $\int_0^\infty e^{-x}x^3dx$ ② $\int_0^\infty x^6e^{-2x}dx$ ③ $\int_0^\infty e^{-4x}x^{3/2}dx$

1. Sol:- $\int_0^\infty e^{-x}x^3dx = \int_0^\infty e^{-x}x^{4-1}dx = \Gamma(4) = 3! = 6$ //.

2 Sol:- $\int_0^\infty e^{-2x}x^6dx$

Put $2x = y$
$2dx = dy$
$dx = \frac{dy}{2}$

when $x=0$, $y=0$; $x=\infty$, $y=\infty$

$$\int_0^\infty e^{-y}\left(\frac{y}{2}\right)^6\frac{dy}{2} = \frac{1}{2^7}\int_0^\infty e^{-y}y^6dy = \frac{1}{2^7}\int_0^\infty e^{-y}y^{7-1}dy = \frac{1}{2^7}\Gamma(7)$$

$$= \frac{6!}{2^7} = \frac{6\times5\times4\times3\times2\times1}{2^{7}} = \frac{\cancel{6}^3\times5\times\cancel{4}^2\times3\times1}{2\times2\times2\times\cancel{2}\times\cancel{2}\times\cancel{2}} = \frac{45}{8}\ //.$$

3 Sol:- $\int_0^\infty e^{-4x}x^{3/2}dx$

$4x = y$
$dx = \frac{dy}{4}$

when $x=0$, $y=0$; $y=\infty$, $x=\infty$

$$\int_0^\infty e^{-y}\left(\frac{y}{4}\right)^{3/2}\frac{1}{4}dy = \int_0^\infty e^{-y}\frac{(y)^{3/2}}{(2^2)^{3/2}}\cdot\frac{1}{4}dy$$

$$\frac{1}{2^5}\int_0^\infty e^{-y}y^{3/2}dy = \frac{1}{2^5}\int_0^\infty e^{-y}y^{5/2-1}dy = \frac{1}{2^5}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{1}{2^5}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{2^8}\sqrt{\pi} = \frac{15\sqrt{\pi}}{2^8}\ //.$$

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• Evaluate $\int_0^{\pi/2}\sin^6\theta\cos^7\theta\,d\theta$ using B - Γ function.

Sol:- we know that $\frac{1}{2}B(m,n) = \int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$.

Put $2m-1 = 6$
$\boxed{m = \frac{7}{2}}$

$2n-1 = 7$
$2n = 8$
$\boxed{n=4}$

$$\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta = \frac{1}{2}B\left(\frac{7}{2},4\right) = \frac{1}{2}\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{7}{2}+4\right)}$$

$$\frac{1}{2}\left[\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}\right] = \frac{1}{2}\left[\frac{\cancel{\frac{7}{2}}\times\cancel{\frac{5}{2}}\times\cancel{\frac{3}{2}}\times\cancel{\frac{1}{2}}\cancel{\Gamma\left(\frac{1}{2}\right)}\times 3!}{\frac{15}{2}\times\frac{13}{2}\times\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\times\Gamma\left(\frac{7}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{\Gamma\left(\frac{7}{2}\right)}\Gamma(4)}{\frac{15}{2}\times\frac{13}{2}\times\frac{11}{2}\times\frac{9}{2}\times\frac{7}{2}\,\cancel{\Gamma\left(\frac{7}{2}\right)}}\right] = \frac{1}{2}\left[\frac{3!\times 2^{4}}{15\times 13\times 11\times 9\times 7}\right]$$

$$= \frac{3!\times 2^3}{15\times 13\times 11\times 9\times 7} = \frac{3\times 2\times 1\times 2\times 2\times 2}{13\times 11\times \cancel{9}\times 7} = \frac{16}{3003}$$

Assignment:- 1. Evaluate $\int_0^{\pi/2} \sin^{7/2}\theta \cos^{3/2}\theta \, d\theta$.

2. $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$

3. $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$

4) $\int_0^{\pi/2} \sin^5\theta \, d\theta$

5) $\int_0^{\pi/2} \cos^7\theta \, d\theta$

6) $\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$

Sol 1

$$\int_0^{\pi/2} \sin^{7/2}\theta \cos^{3/2}\theta \, d\theta$$

$$= \cancel{\int_0^{\pi/2} \sin^{9/2-1}\theta \cos^{5/2-1}\theta \, d\theta}$$

$2m-1 = \frac{7}{2}$

$2m = \frac{7}{2}+1$

$2m = \frac{7+2}{2}$

$2m = \frac{9}{2}$

$m = \frac{9}{4}$

$2n-1 = \frac{3}{2}$

$2n = \frac{5}{2}$

$n = \frac{5}{4}$

$$= \frac{1}{2}B\left[\frac{9}{4}, \frac{5}{4}\right]$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{9}{4}+\frac{5}{4}\right)} = \frac{\frac{1}{2}\Gamma\left(\frac{9}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{14}{4}\right)} = \frac{1}{2}$$

(2) $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$

(5)

$2m-1=3 \quad 2n-1=5$

$2m=4 \quad 2n=6$

$m=2 \quad n=3$

$$= \frac{1}{2}B(2,3)$$

$$= \frac{1}{2}\frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)} = \frac{1}{2}\frac{1!\,2!}{4!}$$

$$= \frac{1}{2}\frac{\cancel{2}\times 1}{4\times 3\times \cancel{2}} = \frac{1}{24}$$

④ $\int_0^{\pi/2} \sin^5\theta \, d\theta$

$2m-1=5 \qquad 2n-1=0$

$2m=6 \qquad n=1/2$

$m=3$

$= \frac{1}{2} B\left[3, \frac{1}{2}\right]$

$= \frac{1}{2} \frac{\Gamma(3)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(3+\frac{1}{2}\right)}$

$= \frac{1}{2} \frac{2! \times \sqrt{\pi}}{\Gamma(7/2)}$

$= \frac{1}{2} \frac{2 \times 1 \cancel{\sqrt{\pi}}}{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\cancel{\sqrt{\pi}}}$

$= \frac{2 \times 2}{15 \times \cancel{2}}$

$= \frac{8}{15}$

⑤ $\int_0^{\pi/2} \cos^7\theta \, d\theta$

$m=\frac{1}{2} \qquad 2n-1=7$

$2n=8$

$n=4$

$= \frac{1}{2} B\left[\frac{1}{2}, 4\right)$

$= \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(4)}{\Gamma\left(\frac{1}{2}+4\right)} = \frac{1}{2} \frac{\frac{1}{2}\sqrt{\pi}\times 3!}{\Gamma\left(\frac{9}{2}\right)}$

$= \frac{1}{2} \frac{\frac{1}{2}\sqrt{\pi}\times 3\times 2\times 1}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\cancel{\sqrt{\pi}}}$

$= \frac{1}{\cancel{2}} \frac{\cancel{2}\times\cancel{2}\times 2\times 2\times 2}{\cancel{2}\times 35} = \frac{8}{35}$

⑥ $\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$

$2m-1=2$

$2m=3$

$m=\frac{3}{2} \qquad n=5/2$

$\frac{1}{2} B\left[\frac{3}{2}, \frac{5}{2}\right]$

$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)}$

$= \frac{1}{2} \frac{\frac{1}{2}\times\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{3!}$

$= \frac{1}{2} \frac{\sqrt{\pi}\times\cancel{3}\times\sqrt{\pi}}{8\times\cancel{3}\times 2}$

$= \frac{\pi}{32}$

Q. Show that $\int_0^{\infty} x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$.

Sol:- Put $x^2 = y$

$x = \sqrt{y}$

$dx = \frac{1}{2\sqrt{y}} dy$

when $x=0$, $y=0$; $x=\infty$, $y=\infty$.

$$\int_0^{\infty} (\sqrt{y})^4 e^{-y} \frac{1}{2\sqrt{y}} dy = \int_0^{\infty} y^2 e^{-y} \cdot \frac{1}{2\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^{\infty} y^{2-1/2} e^{-y} dy = \frac{1}{2}\int_0^{\infty} y^{3/2} e^{-y} dy$$

$$\frac{1}{2} \int_0^{\infty} e^{-y} y^{5/2-1} dy = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2} \times \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3\sqrt{\pi}}{8} //.$$

Q. $\int_0^{\infty} \frac{x}{1+x^6} dx$ Evaluate using B-Γ function.

Sol:- Put $x^6 = y$

$x = y^{1/6} \Rightarrow dx = \frac{1}{6} y^{-5/6} dy$

$$\int_0^{\infty} \frac{y^{1/6}}{1+y} \frac{1}{6} y^{-5/6} dy = \frac{1}{6}\int_0^{\infty} y^{-4/6} (1+y)^{-1} dy$$

$$= \frac{1}{6}\int_0^{\infty} y^{-2/3} (1+y)^{-1} dy = \frac{1}{6}\int_0^{\infty} \frac{y^{-2/3}}{(1+y)} dy$$

{ form of beta function. $\int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = B(m,n)$ }

$m-1 = -\frac{2}{3}$

$m = -\frac{2}{3} + 1$

$\boxed{m = \frac{1}{3}}$

$m+n = 1$

$\frac{1}{3} + n = 1$

$n = 1 - \frac{1}{3}$

$\boxed{n = \frac{2}{3}}$

$$B[m,n] = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}$$

$$\int_0^{\infty} \frac{y^{1/3-1}}{(1+y)^{1/3+2/3}} = B\left[\frac{1}{3}, \frac{2}{3}\right]$$

$$= \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{6}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{\frac{1}{6}\times\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{0!} = \frac{1}{6}\times\Gamma\left(\frac{1}{3}\right)\times\Gamma$$

$$= \frac{1}{6}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{6}\frac{\pi}{\sin \pi\times\frac{1}{3}} = \frac{\pi}{6\cdot\sin\frac{\pi}{3}} = \frac{\pi}{6\times\frac{\sqrt{3}}{2}} = \frac{2\pi}{\frac{6\sqrt{3}}{2}} = \frac{\pi}{2\sqrt{3}} //.$$

- Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}dx$ using B & Γ functions (or) S.T $4\int_0^\infty \frac{x^2}{1+x^4}dx = \sqrt{2}\,\pi$

Sol:- Put ~~$x^4 = y$~~ $x^2 = \tan\theta \Rightarrow x = (\tan\theta)^{1/2}$

~~$x = y^{1/4}$~~ ~~$2x\,dx = \sec^2\theta\,d\theta$~~ $dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\cdot d\theta$

$$= \frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta$$

$x = 0$ | $x = \infty$

$\tan\theta = 0$ | $\tan\theta = \infty$

$\tan\theta = \tan 0$ | $\tan\theta = \tan\pi/2$

$\boxed{\theta = 0}$ | $\theta = \pi/2$

$$\int_0^{\pi/2}\frac{\tan\theta}{\sec^2\theta}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}}\,d\theta = \frac{4}{2}\int_0^{\pi/2}\tan\theta\cdot(\tan\theta)^{-1/2}d\theta$$

$$\frac{4}{2}\int_0^{\pi/2}(\tan\theta)^{1-1/2}d\theta = \frac{4}{2}\int_0^{\pi/2}(\tan\theta)^{1/2}d\theta = 2\int_0^{\pi/2}\tan^{1/2}\theta\,d\theta$$

$$2\int_0^{\pi/2}\frac{\sin^{1/2}\theta}{\cos^{1/2}\theta}\,d\theta = 2\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$2m-1 = 1/2$ | $2n-1 = -1/2$

$2m = 1/2+1$ | $2n = 1-1/2$

$2m = \frac{3}{2}$ | $2n = \frac{1}{2}$

$\boxed{m = \frac{3}{4}}$ | $\boxed{n = \frac{1}{4}}$ $\boxed{n = \frac{1}{4}}$

$$B\left[\frac{3}{4},\frac{1}{4}\right] = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}$$

$$\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{1} = \frac{\pi}{\sin\pi\cdot\frac{1}{4}} = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \pi\sqrt{2} //.$$

UNIT-3

Calculus

1. Mean Value theorems

→ **Rolle's theorem :-**

Let $f(x)$ be a function such that

(i) It is continuous is closed interval $[a,b]$

(ii) It is differentiable in open interval (a,b) and

(iii) $f(a) = f(b)$

Then there exists atleast one point 'c' in open interval (a,b) such that $f'(c)=0$

1. Verify Rollers theorem for function $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{2}, \sqrt{2}]$

Sol $f(x) = 2x^3 + x^2 - 4x - 2$ is a polynomial in x

(i) $f(x)$ is continoous for every value of x in particular $f(x)$ is continuous in $[-\sqrt{2}, \sqrt{2}]$

(ii) $f(x) = 2x^3 + x^2 - 4x - 2$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$f'(x) = 2(3x^2) + 2x - 4(1) - 0$

$$\frac{d}{dx}(x) = 1$$

$= 6x^2 - 2x - 4$

$f'(x) = 6x^2 - 2x - 4$ exists for every $x \in (-\sqrt{2}, \sqrt{2})$ $\frac{d}{dx}(k) = 0$

$\therefore f'(x)$ is differentiable in $(-\sqrt{2}, \sqrt{2})$

(iii) $f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (\sqrt{2})^2 - 4(-\sqrt{2}) - 2$

$= -4\sqrt{2} + \cancel{2} + 4\sqrt{2} - \cancel{2}$

$= 0$

$f(-\sqrt{2}) = f(\sqrt{2})$

Thus all the three conditions of Rolle's theorem are satisfies

Then there exists one point 'c' in (a,b), such that

$f'(c) = 0$

$f'(x) = 6x^2 + 2x - 4$

$f'(c) = 6c^2 + 2c - 4 = 0$

$= 2(3c^2 + c - 2) = 0$

$3c^2+3c-2c-2=0$

$3c(c+1)-2(c+1)=0$

$(c+1)(3c-2)=0$

$c=-1, 2/3$

Both $c=2/3$ & $c=-1$ in $(-\sqrt{2}, \sqrt{2})$

Hence the Rolle's theorem is verified.

② $f(x)=2x^3+x^2-4x-2$ in $[-\sqrt{3}, \sqrt{3}]$

Sol (i) $f(x)$ is a continous forevery value of x in particular

$f(x)$ is continous in $[-\sqrt{3}, \sqrt{3}]$

(ii) $f(x)=\cancel{6x^2+2x-4}\ 2x^3+x^2-4x-2$

$f'(x)=6x^2+2x-4$

* Note :-

Polynomial, exponential, logs, sin, cos } continous function

(iii) $f(-\sqrt{3}) = 2(-\sqrt{3})^3+(-\sqrt{3})^2-4(-\sqrt{3})-2$

$= 2(-3\sqrt{3})+3+4\sqrt{3}-2$

$= -6\sqrt{3}+3+4\sqrt{3}-2$

$= -2\sqrt{3}+1$

$(\sqrt{3})^3 = 3\sqrt{3}$

$f(\sqrt{3}) = 2(\sqrt{3})^3+(\sqrt{3})^2-4(\sqrt{3})-2$

$= 6\sqrt{3}+3-4\sqrt{3}-2$

$= 2\sqrt{3}+1$

Since $f(-\sqrt{3}) \neq f(\sqrt{3})$ is

Hena the Rolle's theorem is not applicable

*** 1M

③ Verify Rolle's theorem for function $f(x)=\frac{1}{x^3}$ in $[-1,1]$

(ii) $f(x)=\tan x$ in $(0,\pi)$

Sol (i) $f(x)=\frac{1}{x^2}$ in $[-1,1]$

$f(x)$ is discontinuous at $x=0$

∴ The Rollers theorem is not applicable for $f(x)$ in $[-1,1]$

(ii) $f(x)=\tan x$ in $[0,\pi]$

$f(x)$ is discontinous at $x=\pi/2$ Hence roller's th is not applicable

(5) $f(x) = x^3$ in $[1,3]$

Sol $f(x) = x^3$ $[1,3]$

(i) $f(x)$ is a continous in $f(x) = x^3$ in $[1,3]$

(ii) $f(x) = x^3$

$f' = 3x^2$

(iii) $f(x) = x^3$ $[1,3]$

$f(1) = 1$

$f(3) = 3^3$

$= 27$

$f(1) \neq f(3)$

$\therefore$ Hence Rolle's theorem is not applicable $f(x) = x^3$ in $[1,3]$

Note :-

$[a,b] \Rightarrow$ include a, b also

$(a,b) \Rightarrow$ don't include a & b.

6. Verify rolle's theorem for the function $f(x) = 8x^3 - 6x^2 - 2x + 1$ in $[0,1]$

2. $f(x) = x^2 - 2x - 3$ in $[-1,3]$

3. $f(x) = e^x \sin x$ in $[0,\pi]$

4. $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

5. $f(x) = (x+2)^3 (x-3)^4$ in $[-2,3]$

$\frac{d}{dx} = e^x \sin x = \sin x \frac{d}{dx} e^x + e^x \frac{d}{dx} \sin x$

$= \sin x e^x + e^x \cos x$

$f(0) = 0$

$f(\pi) = 0$

(i) $f(x) = 8x^3 - 6x^2 - 2x + 1$

Sol Given $f(x) = 8x^3 - 6x^2 - 2x + 1$

(i) $f(x)$ is a continous for every value of x in particular

$f(x)$ is continous in $[0,1]$

(ii) $f(x) = 8x^3 - 6x^2 - 2x + 1$

$= 8(3x^2) - 12x - 2(1) + 0$

$= 24x^2 - 12x - 2$

(iii) $f(x) = 8x^3 - 6x^2 - 2x + 1$

~~$f(-1) = 8(-1)^3 - 6(-1)^2 - 2(-1)$~~ $+1 = -8 + 6 + 2 + 1 = 1$

$f(3) = 8(3)^3 - 6(3)^3 - 2(3) + 1 =$

$f(0) = 8(0) - 6(0) - 2(0) + 1$
$= 1$

$f(1) = 8 - 6 - 2 + 1$
$= 8 - 8 + 1$
$= 1$

$\therefore f(0) = f(1)$

Hence $\therefore$ All the conditions of Rolle's theorem are satisfied.

$f'(c) = 0$

$f'(x) = 24x^2 - 12x - 2$

$f'(c) \Rightarrow 24c^2 - 12c - 2 = 0$

$\Rightarrow 12c^2 - 6c - 1 = 0$

$12c^2$.

$= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{6 \pm \sqrt{36 + 48}}{2 \times 12} = \dfrac{6 \pm \sqrt{84}}{24} = \dfrac{\cancel{6}^{3} \pm \cancel{2}^{1}\sqrt{21}}{\cancel{2} \times 12}$

$= \dfrac{3 \pm \sqrt{21}}{12}$

(2), $f(x) = x^2 - 2x - 3$ $[-1, +3]$

Sol (i) $f(x)$ is a continous for every value of x in particular $f(x)$ is a continous in $[-1, 3]$

(ii) $f(x) = x^2 - 2x - 3$
$\Rightarrow 2x - 2(1)$
$= 2x - 2$

(iii) $f(x) = x^2 - 2x - 3$
$f(-1) = (-1)^2 - 2(-1) - 3$
$= 1 + 2 - 3$
$= 0$

$f(3) = 3^2 - 2(3) - 3$
$= 9 - 6 - 3 = 0$

$f(-1) = f(3)$

Hence: All the conditions of Rolle's theorem are satisfied on $[-1, 3]$

$f'(c) = 0$

$f'(x) = 2x - 2$

$= 2(c) - 2$

$2c = 2$

$\boxed{c = 1}$

Hence Rolle's Theorem is verified

3. $f(x) = e^x \sin x$ in $[0, \pi]$

Sol (i) $f(x)$ is continous for every value of x

$\therefore f(x)$ is continuous in $[0, \pi]$

(ii) $f(x) = e^x \sin x$

$f'(x) = \cancel{e^x \cos}\ \sin x \frac{d}{dx} e^x + e^x \frac{d}{dx} \sin x$

$= \sin x\, e^x + e^x \cos x$

$= e^x \sin x + e^x \cos x$

(iii) $f(x) = e^x \sin x$

$f(0) = e^0 \sin 0 = 0$

$f(\pi) = e^\pi \sin \pi = 0 \quad f(0) = f(\pi)$

Hence all the conditions of Rolle's theorem are satisfied on $[0, \pi]$

$f'(c) = 0$

$f'(x) \Rightarrow e^x(\sin x + \cos x) = 0$

$f'(c) \Rightarrow e^c(\sin c + \cos c) = 0$

$\sin c = -\cos c$

$\frac{\sin c}{\cos c} = -1$

$\tan c = -1$

$\tan c = -\tan \frac{\pi}{4}$

$\tan ($

4. $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

Sol (i) $f(x)$ is continuous functions in $[0,\pi]$

(ii) $f(x) = \frac{\sin x}{e^x}$

$= \frac{e^x \cos x - e^x \sin x}{(e^x)^2}$

(iii) $f(x) = \frac{\sin x}{e^x}$

$f(0) = \frac{\sin 0}{e^0} = 0$

$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$ $\quad$ $f(0) = f(\pi)$

all three conditions of Rolle's theorem are satisfied

$f'(c) = 0$

$f'(x) = \frac{e^x \cos x - e^x \sin x}{(e^x)^2}$

$= \frac{\cos x - \sin x}{e^x}$

$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0 \Rightarrow \cos c = \sin c$

$\tan c = 1$

$c = \pi/4$

Hence Rolle's theorem is verified

5. $f(x) = (x+2)^3(x-3)^4$ in $[-2,3]$

(i) $f(x)$ is a continuous in $[-2,3]$

(ii) $f(x) = (x+2)^3(x-3)^4$

$= 3(x+2)^2(x-3)^4 + (x+2)^3 \cdot 4(x-3)^3$

$= (x+2)^2(x-3)^3[3x-9+4x+8]$

$= (x+2)^2(x-3)^2(7x-1)$

1. Verify Roller's theorem for $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$

Sol: let $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

(1) $f(x)$ is continous for every value of x, particulas $f(x)$ is continous in (a,b).

(2) $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$

$$f(x) = \log(x^2+ab) - \log[x(a+b)]$$
$$= \log(x^2+ab) - [\log x + \log(a+b)]$$
$$= \log(x^2+ab) - \log x - \log(a+b)$$

$$f(x) = \frac{1}{x^2+ab}(2x) - \frac{1}{x} - \frac{1}{a+b}(0)$$
$$= \frac{2x}{x^2+ab} - \frac{1}{x}$$
$$= \frac{2x^2-x^2-ab}{x(x^2+ab)} = \frac{x^2-ab}{x(x^2+ab)}$$

$\because f(x)$ is direivable in $[a,b]$

(iii) $f(a) = \log\frac{(x^2+ab)}{a(a+b)} = \log\frac{(a^2+ab)}{a^2+ab} = \log 1 = 0$

$f(b) = \log\frac{(b^2+ab)}{ab+b^2} = \log 1 = 0$

$\therefore f(a) = f(b)$

$f'(c) = 0$

$$f'(x) = \frac{x^2-ab}{x(x^2+ab)} = 0$$

$$f'(c) = \frac{c^2-ab}{c(c^2+ab)} = 0$$

$$c^2 - ab = 0$$
$$c^2 = ab$$
$$c = \pm\sqrt{ab}.$$

Hena Roller's theorem is verified.

V.V.Imp

2Q. $f(x) = (x-a)^m (x-b)^n$ where m, n are +ve integer in $[a, b]$.
Verify Rollers theorem.

Sol :- Given fun $f(x) = (x-a)^m (x-b)^n$
every polynomial is continous for all values of x
$\because$ $f(x)$ is continous in $[a, b]$

Let $f(x) = (x-a)^m (x-b)^n$

$$f'(x) = \frac{d}{dx}\left[(x-a)^m (x-b)^n\right]$$

$$= (x-a)^m \frac{d}{dx}(x-b)^n + (x-b)^n \frac{d}{dx}(x-a)^m$$

$$= (x-a)^m \left[n(x-b)^{n-1}(1)\right] + (x-b)^n \left[m(x-a)^{m-1}(1)\right]$$

$$= (x-a)^m \left[n(x-b)^{n-1}\right] + (x-b)^n \left(m(x-a)^{m-1}\right]$$

$$= n(x-a)^{m-1+1}(x-b)^{n-1} + n(x-b)^n (x-a)^{m-1}$$

$$= (x-a)^{m-1}(x-b)^{n-1}\left[n(x-a) + m(x-b)\right]$$

$$= (x-a)^{m-1}(x-b)^{n-1}\left[nx - an + mx - mb\right]$$

$$f'(x) = (x-a)^{m-1}(x-b)^{n-1}\left[x(m+n) - (na+mb)\right]$$

$\because$ $f(x)$ is derivable in (a, b)

(ii) $f(a) = (a-a)^m (a-b)^n = 0$
$f(b) = (b-a)^m (b-b)^n = 0$
$\because$ $f(a) = f(b)$

This the three conditions of Roller's theorem is satisfied then there exists a point $c \in (a, b)$, such that $f'(c) = 0$

$$(c-a)^{m-1}(c-b)^{n-1}\left[c(m+n) - (na+mb)\right] = 0$$

$$c(m+n) - (na+mb) = 0$$

$$c(m+n) = na + mb$$

$$c = \frac{na+mb}{m+n} \in (a, b)$$

2

$\therefore$ Hence Roller's theorem is verified

3. $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$, Verify Roller's theorem.

Sol Given func $f(x) = x(x+3)e^{-x/2}$
Every polynomial is continous for all values of x
$\because$ $f(x)$ is continous in $[-3, 0]$

Let $f(x) = x(x+3)e^{-x/2}$

$= (x^2+3x)e^{-x/2}$

Here $(x^2+3x)e^{-x/2}$ is continous.

(ii) $f(x) = (x^2+3)e^{-x/2}$

$f'(x) = (x^2+3)(e^{-x/2})(-\frac{1}{2}) + e^{-x/2}[2x]$

$= \frac{(x^2+3)(e^{-x/2})}{2} + 2x e^{-x/2} + 3e^{-x/2}$

$x = -e^{-x/2}\left[\frac{(x^2+3)}{2} - 2x - 3\right]$

$= -e^{-x/2}\left[\frac{x^2+3-4x}{2}\right]$

$= \frac{-e^{-x/2}}{2}(x^2+3-4x)$

$= \frac{-e^{x/2}}{2}[x^2-4x+3]$

$= -\frac{e^{x/2}}{2}[x^2-3x-x+3]$

$= -\frac{e^{x/2}}{2}[x(x-3)-1(x-3)]$

$= -\frac{e^{x/2}}{2}[(x-3)(x-1)]$

$\because f[x]$ is $]x$

$f(x) = (x^2+3x)\frac{d}{dx}e^{-x/2} + e^{-x/2}\frac{d}{dx}(x^2+3x)$

$= x^2+3x\,(e^{-x/2})(-1/2) + e^{-x/2}[2x+3]$

$= e^{-x/2}[x^2+3x(-1/2) + 2x+3]$

$= e^{-x/2}\left[\frac{-x^2-3x}{2} + 2x+3\right]$

$= e^{-x/2}\left[\frac{-x^2-3x+4x+6}{2}\right]$

$= e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$

$\boxed{= \frac{e^{-x/2}}{2}[6-x^2+x]}$

$\because f(x)$ is derivable in $(-3,0)$

(iii) $f(3) = (-3)(-3+3)e^{-3/2} = 0$

$f(0) = 0(0+3)e^{-0/2} = 0$

$f(3) = f(0)$

Hence Roller's theorem is applicable

$f'(c) = 0$

$f'(c) = \frac{e^{-c/2}}{2}[6-c^2+c] = 0$

$\Rightarrow e^{-c/2}[-c^2+6+c] = 0$

$\Rightarrow -c^2+c+6 = 0$

$c^2-c-6 = 0$

$c^2-3c+2c-6 = 0$

$c(c-3)+2(c-3) = 0$

$(c+2)(c-3) = 0$

$c = -2, 3$ only -2 lies within the open interval $(-3, 0)$

$c = -2 \in (-3, 0)$

Assignment :

1. $f(x) = \frac{x^2-x-6}{x-1}$ in $[-2, 3]$

S:- Given

func $f(x) = \frac{x^2-x-6}{x-1}$

every polynomial is continuous for all values of x

$\therefore f(x)$ is continuous in $[-2, 3]$

(ii) $f(x) = \frac{x^2-x-6}{x-1}$

$f'(x) = \frac{(x-1)(2x-1)-(x^2-x-6)(1)}{(x-1)^2}$

$= \frac{2x^2-x-2x+1-x^2+x+6}{(x-1)^2}$

$= \frac{x^2-2x+7}{(x-1)^2}$

$f(x)$ is not derivable at $x=1$ at $(-2, 3)$

(iii) $f(-2) = \frac{(-2)^2-(-2)-6}{-2-1} = \frac{4+2-6}{-3} = \frac{6-6}{-3} = \frac{0}{-3}$

$f(3) = \frac{3^2-3-6}{3-1} = \frac{9-9}{2} = \frac{0}{2}$ Hence Rolle's theorem is not applicable to $f(x)$ in $(-2, 3)$

2. $f(x) = x^{2/3} - 2x^{1/3}$ in $[0,8]$

Sol Given $f(x) = x^{2/3} - 2x^{1/3}$ in $[0,8]$

(i) Every polynomial is a continuous for all values of 'x'.

$\therefore f(x)$ is continous in $[-2,3]$

(ii) $f(x) = x^{2/3} - 2x^{1/3}$

$= \frac{2}{3}x^{-1/3} - 2\left(\frac{1}{3}\right)x^{-2/3}$

$f'(x) = \frac{2}{3}\left[x^{-1/3} - x^{-2/3}\right]$

$\therefore f'(x)$ is derivable at $(0,8)$

(iii) $f(x) = x^{2/3} - 2x^{1/3}$ in $[0,8]$

$f(0) = 0^{2/3} - 2(0)^{1/3} = 0$

$f(8) = 8^{2/3} - 2(8)^{1/3}$

$= (2^3)^{2/3} - 2(2^3)^{1/3}$

$= 4 - 4 = 0$

$\because f(0) = f(8)$

$\therefore$ The function satisfies all the three conditions of Rollees theorem

$f'(c) = 0$

$f'(x) = \frac{2}{3}\left[x^{-1/3} - x^{-2/3}\right]$ $\qquad \frac{2}{3}x^{-1/3}$

$f'(c) = \frac{2}{3}\left(c^{-1/3} - c^{-2/3}\right)$

$c^{-1/3} - c^{-2/3} = 0$

$c^{-1/3} = c^{-2/3}$

$c^{-1} = c^{-2} \Rightarrow \frac{1}{c} = \frac{1}{c^2}$ $\qquad c = 1$

$\frac{1}{c} = \frac{1}{c^2}$ $\qquad c = 1$

$c = 1 \in (0,8)$

Lagrange's Mean Value Theorem

Let $f(x)$ be function such that

(i) It is continous in $[a,b]$

(ii) It is derivable in (a,b)

(iii) Then there exists atleast one value c in open interval (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

1. Verify $f(x) = \log_e^x$ in $[1,e]$ obeys lagrangers mean value Theorem.

Sol :- Given funᶜ $f(x) = \log_e^x$

→ This function is continous in $[1,e]$ and derivable in $(1,e)$

→ Hence lagrangees mean value Theorem is applicable

By this theorem, there exists a point c in open interval $(1,e)$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \text{ ; where } a=1, b=e$$

$$f'(x) = \frac{1}{x} \quad f'(c) = \frac{1}{c}$$

$$\frac{1}{c} = \frac{f(e)-f(1)}{e-1} = \frac{\log_e^e - \log_e^1}{e-1} = \frac{1-0}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$$e-1 = c$$

$$c = e-1 \in (1,e)$$

Hence lagrangees mean value Theorem is applicable.

1. Verify lagrangees mean value theorem for funᶜ

①. $f(x) = x^3 - x^2 - 5x + 3$ in $[0,4]$

②. $f(x) = x(x-2)(x-3)$ in $[0,4]$

1 Sol :- $f(x) = x^3 - x^2 - 5x + 3$

The function is continous in $[0,4]$ and derivable in $(0,4)$.

Hence lagrangees mean value Theorem is applicable

By this theorem, there exists a point c in $(0,4)$

$f(x) = x^3 - x^2 - 5x + 3$

$f'(x) = 3x^2 - 2x - 5$

$f'(c) = 3c^2 - 2c - 5$

$3c^2 - 2c - 5 = \frac{f(0) - f(4)}{0 - 4}$

$3c^2 - 2c - 5 = \frac{0 - 0 - 0 + 3 - (64 - 16 - 20 + 3)}{-4}$

$3c^2 - 2c - 5 = \frac{3 - 31}{-4}$

$3c^2 - 2c - 5 = \frac{-\cancel{28}\,7}{\cancel{-4}}$

$3c^2 - 2c - 5 = 7$

$3c^2 - 2c - 5 - 7 = 0$

$3c^2 - 2c - 12 = 0$

$= \frac{2 \pm \sqrt{4 + 144}}{6}$

$= \frac{\cancel{2} \pm \cancel{2}\sqrt{37}}{\cancel{6}\,3}$

$= \frac{1 \pm \sqrt{37}}{3}$

$\because c = \frac{1 + \sqrt{37}}{3}, \frac{1 - \sqrt{37}}{3}$ lies between $(0, 4)$

2. $f(x) = x(x-2)(x-3)$ in $[0, 4]$

$= (x^2 - 2x)(x - 3)$

$= x^3 - 3x^2 - 2x^2 + 6x$

$= x^3 - 5x^2 + 6x$

given function is continous in $[0, 4]$ and derivable in $(0, 4)$ Hence lagrangeer mean value theorem is applicable

$f(x) = x^3 - 5x^2 + 6x$

$f'(x) = 3x^2 - 10x + 6$

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\cancel{= 4^3 - 5(4)^2}$

$3x^2 - 10x + 6 = \frac{4^3 - 5(4)^2 + 6(4) - 0}{4 - 0}$

$3x^2 - 10x + 6 = \frac{64 - 80 + 24}{4}$

$$= \frac{10 \pm \sqrt{100-48}}{6}$$

$$= \frac{10 \pm \sqrt{52}}{6}$$

$$= \frac{\cancel{10}^{5} \pm \cancel{2}\sqrt{13}}{\cancel{6}_{3}}$$

$$= \frac{5 \pm \sqrt{13}}{3}$$

$$c = \frac{5+\sqrt{13}}{3}, \frac{5-\sqrt{13}}{3}$$

$c = \frac{5+\sqrt{13}}{3}, \frac{5-\sqrt{13}}{3}$ lie between 0 and 4

$$c = \frac{5+\sqrt{13}}{3}, \frac{5-\sqrt{13}}{3} \in (0,4)$$

4 m.m
Imp

- If $a>b$, prove that $\frac{b-a}{1+b^2} < Tan^{-1}(b) - Tan^{-1}(a) < \frac{b-a}{1+a^2}$ using lagranges mean value theorem deduce that

(i) $\frac{\pi}{4} + \frac{3}{25} < Tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(ii) $\frac{5\pi+4}{20} < Tan^{-1}(2) < \frac{\pi+2}{4}$

Sol :- $f(x) = Tan^{-1}(x)$ in $[a,b]$ for $0<a<b<1$

$\because f(x)$ is continuous in closed interval $[a,b]$ and derivable in (a,b)

Hence lagranges mean value theorem is applicable then there exists a point $\vec{c} \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad -①$$

$f'(x) = \frac{d}{dx}(\tan^{-1}x)$

$f'(x) = \frac{1}{1+x^2} \rightarrow f'(c) = \frac{1}{1+c^2}$

eq ① $\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} \rightarrow$ ②

But $c \in (a,b)$

$a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

$\rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$ { from eq ② }

$\rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} < \frac{1}{1+a^2}$

$\rightarrow \frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2} \rightarrow$ ③

Deduction :-

Taking $a = 1, b = 4/3$

we have

eq ③ $\frac{4/3 - 1}{1+\left(\frac{4}{3}\right)^2} < \tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}(1) < \frac{4/3 - 1}{1+(1)^2}$

$\frac{1/\cancel{3}}{25/\cancel{9}_3} < \tan^{-1}\left(\frac{4}{3}\right) - \pi/4 < \frac{4/3}{2}$

$\frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) - \frac{\pi}{4} < \frac{1}{6}$

$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \frac{\pi}{4} + \frac{1}{6}$

(ii) Deduction :-

Taking $a=1$, $b=2$

we have

eq 3 $\frac{2-1}{1+(2)^2} < Tan^{-1}(2) - Tan^{-1}(1) < \frac{2-1}{1+1^2}$

$+\frac{1}{5} < Tan^{-1}2 - Tan^{-1}(1) < \frac{1}{2}$

$+\frac{1}{5} < Tan^{-1}2 - \frac{\pi}{4} < \frac{1}{2}$

$\frac{\pi}{4} + \frac{1}{5} < Tan^{-1}(2) < \pi/4 + 1/2$

$\frac{5\pi+4}{20} < Tan^{-1}(2) < \frac{2\pi+4}{8}$

$\frac{5\pi+4}{20} < Tan^{-1}(2) < \frac{\cancel{2}(\pi+2)}{\cancel{8}4}$

$\frac{5\pi+4}{20} < Tan^{-1}(2) < \frac{\pi+2}{4}$

** *

* Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\frac{3}{5} > \frac{\pi}{3} - \frac{1}{8}$ using lagrange's mean value theorem

$f(x) = \cos^{-1}(x)$ in $[a,b]$

By lagrange's mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \longrightarrow (1)$$

$$f'(x) = \frac{d}{dx}\cos^{-1}(x)$$

$$f'(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$f'(c) = \frac{-1}{\sqrt{1-c^2}}$$

eq (1) $\frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a}$

But $c \in (a,b)$

$$a < c < b$$

$$a^2 < c^2 < b^2$$

$$-a^2 > -c^2 > -b^2$$

$$1-a^2 > 1-c^2 > 1-b^2$$

$$\sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a} > -\frac{1}{\sqrt{1-b^2}}$$

$$-\frac{(b-a)}{\sqrt{1-a^2}} > \cos^{-1}(b) - \cos^{-1}(a) > -\frac{(b-a)}{\sqrt{1-b^2}}$$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}(b) - \cos^{-1}(a) > \frac{a-b}{\sqrt{1-b^2}}$$

let $a = \frac{1}{2}$, $b = \frac{3}{5}$

$$\frac{\frac{1}{2}-\frac{3}{5}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} > \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right) > \frac{\frac{1}{2}-\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}}$$

$$\frac{\frac{5-6}{10}}{\sqrt{\frac{4-1}{4}}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > \frac{\frac{5-6}{10}}{\sqrt{\frac{25-9}{25}}}$$

$$\frac{-\cancel{1/10}\ 5}{\frac{\sqrt{3}}{\cancel{2}}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > \frac{-\cancel{1/10}\ 2}{\frac{\sqrt{16}}{\cancel{5}}}$$

$$\frac{-1/5}{\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) - \pi/3 > -\frac{1}{2}/4$$

$$\frac{-1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) - \frac{\pi}{3} > -\frac{1}{8}$$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$$

1. Using lagranges mean value theorem for $0 < a < b$, and hence show that prove that $1 - \frac{a}{b} < \log\frac{b}{a} < \frac{b}{a} - 1$ and hence show that $\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$

2. Using lagranges mean theorem $|\sin u - \sin v| \le |u-v|$ (or) $|\sin b - \sin a| \le |b-a|$

1

Sol 1 $f(x) = \log \cancel{b}^{x}\ \cancel{\log a}$ in $[a,b]$ in closed interval $[a,b]$

$\therefore f(x)$ is continous and derivable in (a,b) is applicable Then

Hence lagranges mean value theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log(b) - \log(a)}{b-a}$$

$$\frac{1}{c} = \frac{\log(b/a)}{b-a} \rightarrow (1)$$

But $c \in (a,b)$

$a < c < b$

$\frac{1}{a} > \frac{1}{c} > \frac{1}{b}$

$\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

$\frac{1}{b} < \frac{\log(b/a)}{b-a} < \frac{1}{a}$

$\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$

$\frac{\cancel{b}}{\cancel{b}} - \frac{a}{b} < \log\left[\frac{b}{a}\right) < \frac{b}{a} - \frac{\cancel{a}}{\cancel{a}}$

$1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1$ Hence proved

let $a = 5$
$b = 6$

$1 - \frac{5}{6} < \log\left(\frac{6}{5}\right) < \frac{6}{5} - 1$

$\frac{6-5}{6} < \log\frac{6}{5} < \frac{6-5}{5}$

$\frac{1}{6} < \log\frac{6}{5} < \frac{1}{5}$

2
Sol Given $f(x) = |\sin u - \sin v| \leq |u - v|$

$f(x) = \sin x$ in $[v, u]$

$\therefore f(x)$ is continous in closed interval $[v, u]$
and derivable in open interval (v, u)
Hence lagranges mean value theorm
then their exists a point $c \in (v, u)$ such that

$f'(c) = \frac{f(u) - f(v)}{u - v}$

$f'(x) = \frac{d}{dx}\sin x \cancel{\cos x} = \cos x$

$f'(c) = \cos c$

eq ① $\cos c = \frac{\sin u - \sin v}{u - v} \longrightarrow ②$ But $c \in (u, v)$

Now $\cos c \leq 1$

$\left|\frac{\sin u - \sin v}{u - v}\right| \leq 1$

$$\frac{|\sin u - \sin v|}{|u-v|} \le 1$$

$$|\sin u - \sin v| \le |u-v|$$

* Cauchy's Mean value Theorem :-

If $f : [a,b] \to R$, $g : [a,b] \to R$ are such that

(i) f, g are continous on $[a,b]$

(ii) f, g are derivable on (a,b)

(iii) $g'(x) \neq 0, \forall x \in (a,b)$.

then there exist a point $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

1) Find the c of cauchy's Mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ where $0 < a < b$

Sol :- Given that $f(x) = \sqrt{x}$

$g(x) = \frac{1}{\sqrt{x}}$

$$\left[\begin{array}{l} \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \\ \frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{-1}{2x\sqrt{x}} \end{array}\right]$$

(i) $f(x)$ and $g(x)$ are continous on $[a,b]$

(ii) $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$

$f'(x) = \frac{d}{dx}(\sqrt{x})$ $\quad g'(x) = \frac{d}{dx}\frac{1}{\sqrt{x}}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad g'(x) = \frac{-1}{2x\sqrt{x}}$

$f(x), g(x)$ are derivable on (a,b)

(iii) $g'(x) \neq 0$

The three conditions of cauchy's mean value theorem are satisfied

Then there exists a point $c \in (a,b)$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{\frac{-1}{2c\sqrt{c}}}$$

$$\frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}} = \frac{1}{\cancel{2\sqrt{c}}}(-\cancel{2}c\cancel{\sqrt{c}})$$

$$\frac{-\cancel{(\sqrt{a}-\sqrt{b})}}{\frac{\cancel{(\sqrt{a}-\sqrt{b})}}{\sqrt{ab}}} = -c$$

$$\cancel{-}\sqrt{ab} = \cancel{-}c$$

$$\boxed{c=\sqrt{ab}} \qquad c\in(a,b)$$

$a<\sqrt{ab}<b$,

2. Verify generalized mean value theorem for $f(x)=e^x$ and $g(x)=e^{-x}$ in $[3,7]$ and find the value of c.

Sol Given $f(x)=e^x$

$g(x)=e^{-x}$

(i) $f(x)$ and $g(x)$ are continous on $[3,7]$

(ii) $f(x)=e^x$, $g(x)=e^{-x}$

$f'(x)=e^x$, $g(x)=-e^{-x}$

$f(x)$, $g(x)$ are derivable on $(3,7)$

(iii) $g'(x)\neq 0$

The three conditions of cauchy's mean value theorem are satisfied

Then there exists a point $c\in(3,7)$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^b-e^a}{e^{-b}-e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{e^c}{\frac{-1}{e^c}} \Rightarrow \frac{e^7 - e^3}{\frac{1}{e^7} - \frac{1}{e^3}} = \frac{e^c}{-\frac{1}{e^c}}$$

$$\frac{e^b - e^a}{\frac{e^a - e^b}{e^a e^b}} = \frac{e^c}{\frac{-1}{e^c}} \Rightarrow \frac{e^7 - e^3}{\frac{e^3 - e^7}{e^3 e^7}} = \frac{e^c}{\frac{-1}{e^c}}$$

$$\frac{-\cancel{(e^a - e^b)}}{\frac{\cancel{e^a - e^b}}{e^a e^b}} = -e^{2c} \Rightarrow \frac{-\cancel{(e^3 - e^7)}}{\frac{\cancel{e^3 - e^7}}{e^3 e^7}} = -e^{2c}$$

$$-e^3 e^7 = -e^{2c}$$

$$-e^a e^b = -e^{2c}$$

$$-e^3 e^7 = -e^{2c}$$

$$\cancel{-}e^{\cancel{10}\,5} = \cancel{-}e^{\cancel{2}c}$$

$$\boxed{c = 5} \quad c \in (3, 7)$$

$$3 < 5 < 7$$

3. $f(x) = e^{-x}$ and $g(x) = e^{x}$ in $[2, 6]$

Sol Given $f(x) = e^{-x}$
$g(x) = e^{x}$

(i) $f(x)$ and $g(x)$ are continous on $[2, 6]$

(ii) $f(x) = e^{-x}$ $\quad g(x) = e^{x}$

$f'(x) = -e^{-x}$ $\quad g'(x) = e^{x}$

$f(x), g(x)$ are derivable on $[2, 6]$

(iii) $g'(x) \neq 0$

Then there exist a point $[2, 6]$, such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{e^{-b} - e^{-a}}{e^{b} - e^{a}} = \frac{-e^{-c}}{e^{c}}$$

$$\frac{\frac{1}{e^b} - \frac{1}{e^a}}{e^b - e^a} = \frac{-\frac{1}{e^c}}{e^c}$$

$$\frac{\frac{e^a - e^b}{e^b e^a}}{e^b - e^a} = \frac{-1}{e^{2c}}$$

$$\frac{\frac{\cancel{e^a - e^b}}{e^b e^a}}{-(\cancel{e^a - e^b})} = \frac{-1}{e^{2c}}$$

$$-e^{-b}e^{-a} = -e^{-2c}$$

$$-e^{-6}e^{-2} = -e^{-2c}$$

$$\cancel{-}e^{-8} = \cancel{-}e^{-2c}$$

$$\boxed{C = 4}$$

$C \in (2, 6)$

$2 < 4 < 6$

Hence cauchy mean value theorem is verified

* Taylor's Theorem :

If $f : [a, b] \to R$ is such that

(i) f^{n-1} is continuous on $[a, b]$

(ii) f^{n-1} is derivable on (a, b) or f^n is exist on (a, b) and $P \in z^{-1}$

then there exists a point $c \in (a, b)$, such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $R_n = \frac{(b-a)^p (b-c)^{n-p}}{(n-1)!\, p} f^n(c)$

* Maclaurin's Theorem :

if $f : [0, x] \to R$ is such that

(i) f^{n-1} is continuous on $[0, x]$

(ii) f^{n-1} is derivable on $(0, x)$ and $P \in z^+$

Then there exists a point $\theta \in (0,1)$ such

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

where $R_n = \frac{x^n(1-\theta)^{n-p}}{(n-1)!\,p} f^n(\theta x)$

Note:- Lagrange's form of remainder

putting $p=n$, we gen $R_n = \frac{x^n}{n!} f^n(\theta x)$

1. Obtain the maclaurin's theores expansion of the following function 1) e^x 2) $\cos x$ 3) $\sin x$

Sol Let $f(x) = e^x$ then $f'(x) = e^x$ $f''(x) = e^x$ $f'''(x) = e^x$ $f^{iv}(x) = e^x$

$f(0) = e^0 = 1$

$f'(0) = 1$

$f''(0) = 1$

$f'''(0) = 1$ ... etc

By maclaurin's theorem

$$f(x) = f(0) + \frac{x}{1!} f(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

$$e^x = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

(ii) $\cos x$

Let $f(x) = \cos x$ $\quad f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\quad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = 1$

By Maclaurin's theorem

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

(iii) $\sin x$

Sol Let $f(x) = \sin x$ $\qquad f(0) = 0$

$f'(x) = \cos x$ $\qquad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\qquad f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x$ $\qquad f'''(0) = -\cos 0 = -1$

$f^{iv}(x) = \sin x$ $\qquad f^{iv}(0) = \sin 0 = 0$

By Maclaurin's theorem

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + R_n$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) - \cdots$$

$$\sin x = \frac{x}{1!}(1) - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

(2) Expand $f(x) = \log \sin x$ about $x = 3$ by Taylor's series expansion.

Sol $f(x) = \log \sin x$

$$f'(x) = \frac{1}{\sin x}(\cos x) = \frac{\cos x}{\sin x} = \cot x$$

$$f''(x) = -\operatorname{cosec}^2 x$$

$$f'''(x) = -2\operatorname{cosec}(-\operatorname{cosec} x \cot x) = 2\operatorname{cosec}^2 x \cot x$$

By Taylor's theorem

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $a = 3$

$$\log \sin x = f(3) + \frac{(x-3)}{1!}\cot x + \frac{(x-3)^2}{2!}(-\operatorname{cosec}^2 x) + \frac{(x-3)^3}{3!}(2\operatorname{cosec}^2 x \cot x)$$

$$\log \sin x = \log \sin 3 + (x-3)\cot 3 - \frac{(x-3)^2}{2!}\operatorname{cosec}^2 3 + \frac{(x-3)^3}{3!} 2\operatorname{cosec}^2 3 \cot 3.$$

Formulas

$$\frac{d}{dx} f(g(x)) = f'[g(x)] \cdot g'(x)$$

$$\frac{d}{dx} \log x = \frac{1}{x}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$$

② Expand $f(x) = \sin x$ about $x = \pi/4$ by Taylor's series expansion

Sol $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$ By Taylor's theorem

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Here $a = \pi/4$

$$\sin x = f(\pi/4) + \left(\frac{x-\pi/4}{1!}\right) f'(\pi/4) + \left(\frac{x-\pi/4}{2!}\right)^2 f''(\pi/4) + \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \cdots$$

$$\sin x = \sin \pi/4 + \frac{(x-\pi/4)}{1!} \cos \pi/4 + \frac{(x-\pi/4)^2}{2!} (-\sin \pi/4) + \frac{(x-\pi/4)^3}{3!} (-\cos \pi/4)$$

$$\sin x = \sin \pi/4 + \frac{4x-\pi}{4} \cos \pi/4 + \frac{\left(\frac{4x-\pi}{4}\right)^2}{2!} (-\sin \pi/4) + \frac{\left(\frac{4x-\pi}{4}\right)^3}{3!} (-\cos \pi/4)$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{4x-\pi}{4}\left(\frac{1}{\sqrt{2}}\right) + \frac{\left(\frac{4x-\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(\frac{4x-\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{4x-\pi}{4}\left(\frac{1}{\sqrt{2}}\right) - \frac{\left(\frac{4x-\pi}{4}\right)^2}{2!}\left(\frac{1}{\sqrt{2}}\right) - \frac{\left(\frac{4x-\pi}{4}\right)^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

*****

3. Verify Taylors theorem $f(x) = (1-x)^{5/2}$ with lagranges form of remainder upto 2 terms in the interval $(0,1)$

Sol :- Consider $f(x) = (1-x)^{5/2}$

(i) $f(x)$, $f'(x)$ are continous in $[0,1]$
(ii) $f''(x)$ is derivable in $(0,1)$

Thus the Taylor's
consider Taylors theorem with lagranges form of remainder

by $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(\theta x)$ with $0 < \theta < 1 \rightarrow$ (1)

Here $n = p = 2$ and $x = 1$

$f(x) = (1-x)^{5/2}$

$f'(x) = \frac{5}{2}(1-x)^{5/2-1}(0-1) = -\frac{5}{2}(1-x)^{3/2}$

$f''(x) = -\frac{5}{2}\frac{3}{2}(1-x)^{3/2-1}(-1) = \frac{15}{4}(1-x)^{1/2}$

$f'''(x) = -\frac{15}{8}(1-x)^{-1/2}$

By Taylor's theorem :

$f(0) = (1-x)^{5/2} = (1-0)^{5/2} = 1$

$f'(0) = -5/2 (1-0)^{3/2} = -5/2$

$f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2} = \frac{15}{4}(1-\theta)^{1/2}$ Here $x = 1$

and $f(1) = (1-1)^{5/2} = 0$

eq (1), we have

$f(1) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(\theta x)$

$0 = 1 + \frac{1}{1!}(-5/2) + \frac{1^2}{2!} \cdot \frac{15}{4}(1-\theta)^{1/2}$

$0 = 1 - 5/2 + \frac{15}{8}(1-\theta)^{1/2}$

$0 = -\frac{3}{2} + \frac{15}{8}(1-\theta)^{1/2}$

$\frac{3}{2} = \frac{\cancel{15}^{5}}{\cancel{8}_{4}}(1-\theta)^{1/2}$

$\frac{4}{5} = (1-\theta)^{1/2}$

$\frac{16}{25} = 1 - \theta$

$\theta = 1 - \frac{16}{25}$

$\theta = \frac{9}{25}$

$\theta = 0.36$

$\theta \in (0,1)$

θ lies between 0 and 1

Thus taylor's theorem is verified.

1. S.T $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!} + \cdots \infty$ (or)

Expand $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ in power of x

S:- Let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ and $f(0)=0$

$\sqrt{1-x^2}\, f(x) = \sin^{-1}x \rightarrow (1)$

Differentiating eq (1) w.r to x on both sides

$\frac{d}{dx}[\sqrt{1-x^2}\, f(x)] = \frac{d}{dx}\sin^{-1}x$

$\sqrt{1-x^2}\, f'(x) \cdot f(x) \frac{1}{2\sqrt{1-x^2}}(0-2x) = \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, f'(x) - \frac{\cancel{2}x\, f(x)}{\cancel{2}\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\left[\sqrt{1-x^2}\, f'(x) - \frac{x f(x)}{\sqrt{1-x^2}}\right] = 1$

$(\sqrt{1-x^2})^2 f'(x) - \frac{\sqrt{1-x^2}\; x\, f(x)}{\sqrt{1-x^2}} = 1$

$(1-x^2) f'(x) - x f(x) = 1 \rightarrow (2)$

Now $x=0$, $f(0)=0$

$(1-0^2) f'(0) - 0 f(0) = 1$

$\boxed{f'(0) = 1}$

Differentating eq (2) w.r to x

$$\frac{d}{dx}\left[\underset{u}{(1-x^2)}\underset{v}{f'(x)} - \underset{u}{x}\ \underset{v}{f(x)}\right] = \frac{d}{dx}(1)$$

$$\left[(1-x^2)f''(x) + f'(x)(0-2x)\right] - \left[x f'(x) + f(x)(1)\right] = 0$$

$$(1-x^2)f''(x) - 2x f'(x) - x f'(x) - f(x) = 0$$

$$(1-x^2) f''(x) - 3x f'(x) - f(x) = 0 \longrightarrow (3)$$

Now $f'(0) = 1$, $f(0) = 0$, $x = 0$

$$(1-0^2)f''(0) - 3\,0 f'(0) - f(0) = 0$$

$$\boxed{f''(0) = 0}$$

Differentating of (3) w.r to x on

$$\frac{d}{dx}\left[(1-x^2) f''(x) - 3x f'(x) - f(x)\right] = \frac{d}{dx}[0]$$

$$\left[(1-x^2) f'''(x) + f''(x)(0-2x)\right] - 3\left[x\ f''(x) + f'(x)(1)\right] - f'(x) = 0$$

$$(1-x^2) f'''(x) - 2x f''(x) - 3x f''(x) - 3f'(x) - f'(x) = 0$$

$$(1-x^2) f'''(x) - 5x f''(x) - 4f'(x) = 0 \longrightarrow (4)$$

Now $f''(0) = 0$, $f'(0) = 1$, $f(0) = 0$, $x = 0$

$$(1-\cancel{x}^2) f'''(\cancel{x}) - 5(0)(0) - 4(1) = 0$$

$$(1-\cancel{x}^2) f'''(\cancel{x}) - 4 = 0$$

$$\boxed{f'''(0) = 4}$$

lly $f^{iv}(0) = 4$

By Taylor's series expansion

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(4) + \cdots$$

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\cdot\frac{x^3}{3!} + \cdots$$