US

(1) Given,

$$f(z) = \frac{1}{z^2-3z+2} \quad \text{in } 1<|z|<2 \text{ in Laurent's Series}$$

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$1 = -2A - B$$
$$A = -B$$
$$B = 1, A = -1$$

$$1 = A(z-2) + B(z-1)$$
$$1 = -1(z-2) + (z-1)$$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$1 < z$$
$$\frac{1}{z} < 1$$

$$\frac{-1}{z-1} = \frac{-1}{z(1-\frac{1}{z})} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right)$$

$$\frac{1}{z-2} = \frac{1}{-2(1-\frac{z}{2})} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= -1\sum_{n=0}^{\infty}\left(\frac{z^n}{2^{n+1}}\right)$$

$$\therefore \boxed{f(z) = -1\left(\sum_{n=0}^{\infty}\left(\frac{1}{z^{n+1}}\right) + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right)}$$

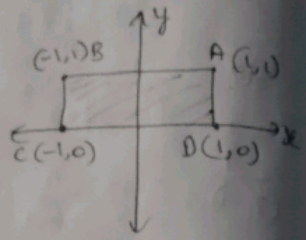
② Given, $f(z) = \int z^3\,dz$ of $\square^{le}$ at $-1, 1, 1+i, -1+i$

$z = x+iy$

Now along AB :- $dz = dx + i\,dy$

$y = 1 \rightarrow dy = 0$

$x \rightarrow 1 : -1$



$\int_{AB} f(z)\,dz = \int_{AB} z^3\,dz$

$= \int_{AB} (x+iy)^3 (dx + i\,dy) = \int_1^{-1} (x+i)^3\,dx = \int_1^{-1} x^3 + i^3 + 3xi^2 + 3ix^2\,dx$

$= \left[\frac{x^4}{4} + i^3 x + \frac{3x^2 i^2}{2} + \frac{3ix^3}{3}\right]_1^{-1}$

$= \left[\frac{x^4}{4} - ix - \frac{3}{2}x^2 + ix^3\right]_1^{-1} = \left[\frac{1}{4} + i - \frac{3}{2} - i\right] - \left[\frac{1}{4} - i - \frac{3}{2} + i\right]$

$= 0$

Along BC :-

$x = -1 \quad dx = 0$

$y : 1 \rightarrow 0$

$\int_{BC} f(z)\,dz = \int_{BC} (x+iy)^3 (dx + i\,dy) = \int_1^0 (-1+iy)^3\, i\,dy$

$= i\int_1^0 (-1 - iy^3 + 3iy + 3(-1)y^2(-1))\,dy$

$= i\left[-y - i\frac{y^4}{4} + \frac{3iy^2}{2} + \frac{3y^3}{3}\right]_1^0 = i\left[(0) - \left(-1 - \frac{i}{4} + \frac{3i}{2} + 1\right)\right]$

$= -\frac{1}{4} + \frac{3}{2} = +\frac{5}{4}$

Along CD :-

$y = 0 \quad dy = 0, \quad x : -1 \text{ to } 1$

$\int_{CD} f(z)\,dz = \int_{CD} (x+iy)^3 (dx + i\,dy) = \int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1}$

$= \frac{1}{4} - \left[\frac{1}{4}\right] = 0$

Along DA :-

$x = 1 \quad dx = 0, \quad y : 0 \text{ to } 1$

$\int_{DA} f(z)\,dz = \int_{DA} (x+iy)^3 (dx + i\,dy) = \int_0^1 (1+iy)^3\, i\,dy$

$= i\int_0^1 (1+iy)^3\,dy = i\int_0^1 (1 - iy^3 + 3iy - 3y^2)\,dy$

$= i\left[y - i\frac{y^4}{4} + \frac{3iy^2}{2} - y^3\right]_0^1 = i\left[1 - \frac{i}{4} + \frac{3i}{2} - 1\right]$

$= \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$

$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$

$= 0 + \frac{5}{4} + 0 - \frac{5}{4}$

$= 0$

$\therefore \int_C f(z)\,dz = 0$

$\therefore$ Cauchy's theorem is verified.

③ $f(z) = \sinh z$ about $z = i\pi$ taylor series

By taylor series,

$\cancel{\int f(z)\,dz = \frac{1}{n!}\sum_{n=0} \frac{a f^n(a)}{(z-z_0)^n}}$

By taylor series, $z = a$

$f(z) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(z-a)^n$

Here $a = i\pi$

$f(z) = \sinh z = \sinh(i\pi) = i\sin\pi = 0$

$f'(z) = \cosh z = \cos\pi = -1$

$f''(z) = \sinh z = i\sinh(\pi) = 0$

$f'''(z) = \cosh z = \cos\pi = -1$

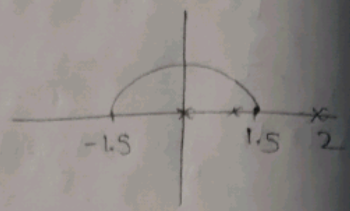
even power's 0,

odd power's -1

$$f(z) = \frac{-1}{1}(z-i\pi) - \frac{1}{3!}(z-i\pi)^3 - \frac{1}{5!}(z-i\pi)^5 - \dots$$

$$\boxed{\therefore \sinh z = -(z-i\pi) - \frac{(z-i\pi)^3}{3!} - \frac{(z-i\pi)^5}{5!} - \dots}$$

(4) $f(z) = \frac{4z-3}{z(z-1)(z-2)}$ $\quad C : |z| = \frac{3}{2}$



i) $z=0$ (inside circle)

Simple pole

ii) $z-1=0$

$z=1$ (inside circle)

Simple pole

iii) $z-2=0$

$z=2$ (outside circle)

Simple pole

we need to find residue w.r.t $z=0,1$

By Cauchy residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \sum Res$$

$$Res_{(z=z_0)} = \frac{1}{(m-1)!} \underset{z \to z_0}{Lt} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Now at $z=0$:

$$Res_{z=0} = \underset{z\to 0}{Lt}\ z \cdot \frac{4z-3}{z(z-1)(z-2)}$$

$$= \underset{z\to 0}{Lt} \frac{4z-3}{(z-1)(z-2)}$$

$$= \frac{-3}{2} = \frac{-3}{2}$$

at $z=1$:

$$Res_{z=1} = \underset{z\to 1}{Lt} (z-1)\frac{4z-3}{z(z-1)(z-2)}$$

$$= \underset{z\to 1}{Lt} \frac{4z-3}{z(z-2)}$$

$$= \frac{1}{-1} = -1$$

Now

$$\sum Res = Res_{z=0} + Res_{z=1}$$

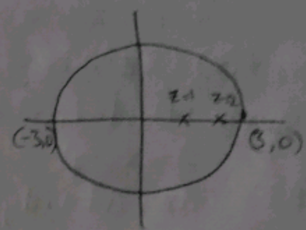
$$= -\frac{3}{2} + (-1) = -\frac{5}{2}$$

$$\int_C f(z)\,dz = 2\pi i \times -\frac{5}{2}$$

$$= -5\pi i$$

$$\boxed{\therefore \int_C \frac{4z-3}{z(z-1)(z-2)}\,dz = -5\pi i}$$

(5) $f(z) = \frac{e^{2z}}{(z-1)(z-2)}$, $\quad C : |z| = 3$



i) $z-1=0$

$z=1$ (inside circle)

Simple pole

ii) $z-2=0$

$z=2$ (inside circle)

Simple pole

we need to find residue w.r.t $z=1,2$

By Cauchy's residue theorem,

$$\int_C f(z)\,dz = 2\pi i \times \sum Res$$

$$Res_{z=z_0} = \underset{z\to z_0}{Lt} (z-z_0) f(z)$$

at $z=1$:

$$\text{Res}_{(z=1)} = \lim_{z\to 1} (z-1)\frac{e^{2z}}{(z-1)(z-2)}$$
$$= \lim_{z\to 1} \frac{e^{2z}}{z-2}$$
$$= -e^2$$

at $z=2$:

$$\text{Res}_{(z=2)} = \lim_{z\to 2} (z-2)\frac{e^{2z}}{(z-1)(z-2)}$$
$$= \lim_{z\to 2}\frac{e^{2z}}{z-1}$$
$$= e^4$$

Now, $\sum \text{Res} = \text{Res}(z=1) + \text{Res}(z=2)$

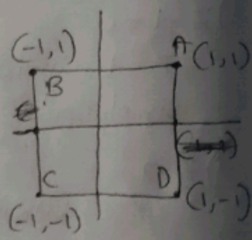
$$= -e^2 + e^4$$

$$\boxed{\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = 2\pi i \times \left(e^2\left(e^2-1\right)\right)}$$

⑥ Given, $f(z) = 3z^2 + iz - 4$

C is square $1+i, 1-i, (-1+i), (-1-i)$

$z = x+iy$



along AB: $dz = dx + i\,dy$

$y=1$, x: 1 to -1

$dy=0$

$$\int_{AB} f(z)\,dz = \int_{AB} 3(z)^2 + iz - 4\, dz$$
$$= \int_1^{-1} 3(x+i)^2 + i(x+i) - 4\, dx$$
$$= \int_1^{-1} 3x^2 + 6xi - 3 + ix - 5\, dx$$
$$= \left[x^3 + \frac{7}{2}x^2 i - 8x\right]_1^{-1} = \left[-1 + \frac{7}{2}i + 8\right] - \left[1 + \frac{7}{2}i - 8\right]$$
$$= 14$$

along BC:

$x=-1$ $dx=0$, y: 1 to -1

$$\int_{BC} f(z)\,dz = \cancel{\int_1^{-1} (-1+iy)\,i\,dy}$$
$$= \int_1^{-1} 3(-1+iy)^2 + i(-1+iy) - 4\; i\,dy$$
$$= i\int_1^{-1} 3 - 3y^2 - 6iy - i - y - 4\, dy$$
$$= i\left[-y^3 - 3iy^2 - iy - y^2 - y\right]_1^{-1}$$
$$= i\left(\left[1 - 3i + i - 1 + 1\right] - \left[-1 - 3i - i - 1 - 1\right]\right)$$
$$= 4i - 2$$

along CD:

$y=-1$, x: -1 to 1

$dy=0$

$$\int_{CD} f(z)\,dz = \int_{-1}^{1} 3(x-i)^2 + i(x-i) - 4\, dx$$
$$= \int_{-1}^{1} 3x^2 - 3 - 6xi + ix + 1 - 4\, dx$$
$$= \left[x^3 - \frac{5}{2}x^2 i - 6x\right]_{-1}^{1} = \left[1 - \frac{5}{2}i - 6\right] - \left[-1 - \frac{5}{2}i + 6\right]$$
$$= -\frac{5}{2}i - 5 + \frac{5}{2}i - 5 = -10$$

along DA:

$x=1$, y: -1 to 1

$$\int_{DA} f(z)\,dz = \int_{-1}^{1} 3(1+iy)^2 + i(1+iy) - 4\; i\,dy$$
$$= i\int_{-1}^{1} 3 - 3y^2 + 6iy + i - y - 4\, dy$$
$$= i\left[-y^3 + 3iy^2 + iy - y^2 - y\right]_{-1}^{1} = i\left(\left[-1 + 3i + i - 1 - 1\right] - \left[1 + 3i - 1 - 1 + 1\right]\right)$$
$$= i\left(4i - 3 - 1 - 2i\right) = -4i - 2$$

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$= 14 + 4i - 2 - 10 - 4i - 2$$

$$= 0$$

$\therefore \int_C f(z)\,dz = 0$

$\therefore$ Cauchy's theorem is verified.

(7) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

i) $(z-1)^2 = 0$

$z = 1$

pole of order 2

ii) $z + 2 = 0$

$z = -2$

simple pole

$$Res_{(z=1)} = \frac{1}{1!} \underset{z\to 1}{Lt} \frac{d}{dz} (z-1)^2 \frac{z^2}{(z-1)^2(z+2)}$$

$$= \frac{1}{1} \underset{z\to 1}{Lt} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= \frac{1}{1} \underset{z\to 1}{Lt} \frac{2z(z+2) - (1)z^2}{(z+2)^2}$$

$$= \frac{1}{1} \times \frac{6-1}{9} = \frac{5}{9}$$

$$Res_{(z=-2)} = \underset{z\to -2}{Lt} (z+2) \frac{z^2}{(z-1)^2(z+2)}$$

$$= \underset{z\to -2}{Lt} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$\therefore$ Residue at $z = -2$ is $\frac{4}{9}$

Residue at $z = 1$ is $\frac{5}{9}$

Poles $z = -2$ (simple), $z = 1$ (order 2).

(8) i) $w = \dfrac{6z-9}{z}$

Set $w = z$

$$z = \frac{6z-9}{z}$$

$z^2 = 6z - 9$

$z^2 - 6z + 9 = 0$

$(z-3)^2 = 0$

$z = 3$

$\therefore z = 3$ is fixed point

ii) $w = \dfrac{z-i}{z+i}$

Set $w = z$ (for fixed pt)

$$z = \frac{z-i}{z+i}$$

$z^2 + iz = z - i$

$z^2 + z(i-1) + i = 0$

$$z = \frac{-(i-1) \pm \sqrt{(i-1)^2 - 4i}}{2}$$

$$= \frac{1-i \pm \sqrt{-1+1-2i-4i}}{2} = \frac{1-i \pm \sqrt{-6i}}{2}$$

$\cancel{\sqrt{6i} = \sqrt{3}(i-1)}$ $(1-i)^2 = -2i$

$(\sqrt{3}(1-i))^2 = 3(-2i) = -6i$

$\sqrt{-6i} = \pm\sqrt{3}(1-i)$

$$= \frac{1-i \pm \sqrt{3}(1-i)}{2}$$

$$z = \frac{(1+i)(1\pm\sqrt{3})}{2}$$

$\therefore z = \dfrac{(1+i)(1+\sqrt{3})}{2}$ & $z = \dfrac{(1+i)(1-\sqrt{3})}{2}$ are fixed points

(9) Given, z-plane : $z_1 = -1, z_2 = 0, z_3 = 1$

w-plane : $w_1 = 0, w_2 = i, w_3 = 3i$

we have,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-3i)}{(w-3i)(i-0)} = \frac{(z+1)(-1)}{(z-1)(1)}$$

$$\frac{-2iw}{i(w-3i)} = \frac{-(z+1)}{z-1}$$

$$\frac{2w}{w-3i} = \frac{z+1}{z-1}$$

$$2wz - 2w = wz - 3zi + w - 3i$$

$$wz - 3w = -3zi - 3i$$

$$\therefore \boxed{w = \frac{-3zi-3i}{z-3}}$$

$\therefore$ Req Mobius transf^{m} is $\boxed{w = \frac{-3zi-3i}{z-3}}$

(10) Given,

z-plane :- $z_1 = 0, z_2 = 1, z_3 = \infty$

w plane : $w_1 = -1, w_2 = -2, w_3 = -i$

we have,

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w+1)(-2+i)}{(w+i)(-2+1)} = \frac{(z-0)}{1}$$

$$\cancel{(w+1)(-2+i) = -z(w+i)}$$

$$\cancel{-4w - 4 + 2iw + 2i = -zw - iz}$$

$$\cancel{w = \frac{-iz-2i+4}{-z+2i+4}}$$

$$\cancel{w(z + 2i - 4) = -iz - 2i + 4}$$

$$\cancel{w = \frac{-iz-2i+4}{z+2i-4}}$$

$$(w+1)(-2+i) = z(w+i)(-1)$$

$$-2w - 2 + iw + i = -zw - iz$$

$$w = \frac{-iz-i+2}{z+i-2} = \frac{iz+i-2}{-z+2-i}$$

① Unit-4

Given, $w = \log z$

In polar form $z = re^{i\theta}$

$\frac{dw}{dz} = \frac{d}{dz}\log(re^{i\theta})$

$= \frac{d}{dz}[\log r + i\theta]$

$= \frac{1}{r} + i$

Unit-4

① Given, $w = \log z = f(z)$

In polar form,

$z = re^{i\theta}$

$\log z = \log re^{i\theta} = \log r + i\theta$

$u = \log r$, $v = \theta$

Cauchy reimann eqn in polar form,

$f'(z) = e^{-i\theta}(u_r + i v_r)$

diff u w.r.t r

$\frac{du}{dr} = \frac{1}{r}$

diff θ w.r.t r

$\frac{d\theta}{dr} = 0$

$f'(z) = e^{-i\theta} \times \frac{1}{r}$

$f'(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

$\therefore \boxed{\frac{dw}{dz} = \frac{1}{z}}$

at the derivates, at $z = 0$, w becomes infinity also in the negative real axis

w is not analytic

$\therefore$ at $z = 0$ & negative real axis, w is not analytic

② Given, $f(z) = u + iv$

$u(x,y) = k_1$, $v(x,y) = k_2$ intersect at right angles at every point.

Two curves are orthogonal if the product of their slopes is -1.

$m_1 \times m_2 = -1$

Diff $u(x,y) = k_1$ w.r.t x

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$

$m_1 = -\frac{\partial u/\partial x}{\partial u/\partial y}$ — ①

Diff $v(x,y) = k_2$ w.r.t y

$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$

$m_2 = -\frac{\partial v/\partial x}{\partial v/\partial y}$ — ②

For analytic,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$m_1 \times m_2 = \left(-\frac{\partial u/\partial x}{\partial u/\partial y}\right) \times \left(-\frac{\partial v/\partial x}{\partial v/\partial y}\right)$

$= \left(-\frac{\partial v/\partial y}{\partial u/\partial y}\right) \times \left(\frac{\partial u/\partial y}{\partial v/\partial y}\right)$

$= -1$

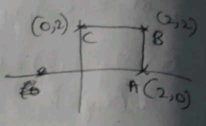
$\therefore k_1, k_2$ form an orthogonal system.

(3) Given, $f(z) = z^2 + 3z$ from $(2,0)$ to $(2,2)$

$(2,2)$ to $(0,2)$

$z = x + iy \Rightarrow dz = dx + i\,dy$

$I = \int_R f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz$



along AB :-

$x = 2 \quad dx = 0, \; y : 0 \text{ to } 2$

$\int_{AB} f(z)\,dz = \int_0^2 [(x+iy)^2 + 3(x+iy)]\,dx + i\,dy$

$= i\int_0^2 (4 + 4iy - y^2 + 6 + i3y)\,dy$

$= i\left[-\frac{y^3}{3} + \frac{7i}{2}y^2 + 10y\right]_0^2 = i\left[-\frac{8}{3} + 14i + 20\right] = I_1$

along BC :-

$y = 2, \; dy = 0, \; x : 2 \text{ to } 0$

$\int_{BC} f(z)\,dz = \int_2^0 (x+2i)^2 + 3(x+2i)\,dx$

$= \int_2^0 x^2 + 4ix - 4 + 3x + 6i\,dx$

$= \left[\frac{x^3}{3} + 2ix^2 - 4x + \frac{3}{2}x^2 + 6ix\right]_2^0$

$= -\left[\frac{8}{3} + 8i - 8 + 6 + 12i\right]$

$= -\frac{8}{3} - 20i + 2 = I_2$

$I = I_1 + I_2$

$= -14 + \frac{52}{3}i - \frac{2}{3} - 20i$

$= -\frac{8}{3}i - \frac{44}{3}$

$\boxed{\therefore \int_C (z^2 + 3z)\,dz = -\frac{44}{3} - i\frac{8}{3}}$

(9)

(4), Given, $f(z) = u + iv$

$F(z) = (1+i)f(z) = u - v + i(u+v)$

$= \cancel{f(z) + if(z)}$

$u - v = U_r, \quad u + v = V_r$

$U_r = e^x(\cos y - \sin y)$ ($\because$ Given)

$\frac{\partial U_r}{\partial x} = e^x(\cos y - \sin y)$

$\frac{\partial U_r}{\partial y} = e^x(-\sin y - \cos y)$

By milne thomson,

$f'(z) = \frac{\partial U_r}{\partial x}(z,0) - i\frac{\partial U_r}{\partial y}(z,0)$

$= e^z + ie^z = e^z(1+i)$

$f(z) = e^z(1+i) + c$

$F(z) = (1+i)f(z)$

$(1+i)f(z) = e^z(1+i) + c$

$\boxed{\therefore f(z) = e^z + \frac{c}{1+i}}$

$\frac{\partial U_r}{\partial y} = -\frac{\partial V_r}{\partial x}$

$\frac{\partial V_r}{\partial x} = -e^x(-\sin y - \cos y)$

$V_r = e^x(\sin y + \cos y) + c'(\phi)$

For analytic,

$\frac{\partial U_r}{\partial x} = \frac{\partial V_r}{\partial y}$

$\frac{\partial V_r}{\partial y} = e^x(\cos y - \sin y)$

$V_r = e^x\int \cos y - \sin y\,dy$

$V_r = e^x[\sin y + \cos y + c'(\phi)]$