

①

M-II

## Advanced calculus

① UNIT-I

First order Ordinary Differential Equations.

② UNIT-II

Ordinary differential Equations of Higher Order.

③ UNIT-III

Multivariable calculus

④ UNIT-IV

Vector differentiation

⑤ UNIT-V

Vector Integration.

### Differential Equations of first order and first degree

An eqn of the form  $\frac{dy}{dx} = f(x, y)$  is called a differential eqn of first order and first degree.

### Exact differential Equation: -

Let  $M(x, y)dx + N(x, y)dy = 0$  be a first order and first degree differential Equations. where  $M$  and  $N$  are real valued func for some  $x, y$ . Then the differential eqn  $Mdx + Ndy = 0$  is said to be an exact differential

equation if it satisfies the condition

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

The general solution of the exact differential eqn.  
 $M dx + N dy = 0$  is

$$\int M dx + \int (\text{terms not containing } x \text{ in } N) dy = c$$

(y constant)

where  $c$  is a constant.

①  $(x^3 + 3xy^2) dx + (y^3 + 3x^2y) dy = 0$

Sol Given eqn is  $(x^3 + 3xy^2) dx + (y^3 + 3x^2y) dy = 0 \rightarrow ①$

This is of the form  $M dx + N dy = 0$

Here  $M = x^3 + 3xy^2$  and  $N = y^3 + 3x^2y$ .

$$\therefore \frac{\partial M}{\partial y} = 3x(2y) = 6xy$$

$$\frac{\partial N}{\partial x} = 3y(2x) = 6xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\text{The given differential eqn is exact})$$

Hence the general solution is

$$\int M dx + \int (\text{terms not containing } x \text{ in } N) dy = c$$

(y constant)

(2)

$$\int x^3 + 3xy^2 dx + \int y^3 dy = C$$

$$\frac{x^4}{4} + 3y^2 \frac{x^2}{2} + \frac{y^4}{4} = C$$

$$x^4 + 6x^2y^2 + y^4 = 4C$$

$$\boxed{\therefore x^4 + 6x^2y^2 + y^4 = C}$$

This is the general solution of cf ①

$$(e^y + 1) \cos x dx + (e^y) \sin x dy = 0$$

Non Exact differential equation:

Let  $M(x,y)dx + N(x,y)dy = 0$  be not an exact differential eqn. if  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  then

make that eqn exact by finding

Integrating factor of  $Mdx + Ndy = 0$

Integrating factor:

In general, for differential eqn  $Mdx + Ndy = 0$  is not an exact.

In such situation, we find a func  $\lambda$  such that by multiplying  $\lambda$  to the eqn, it becomes an exact eqn

so  $\lambda M(x,y)dx + \lambda N(x,y)dy = 0$  becomes exact diff

Here the func  $\lambda = \lambda(x,y)$  is then called an Integrability factor.

Methods to find an integrating factor (I.F) :-

for given non exact-diff eqn  
 $Mdx + Ndy = 0$

Cases    case I

If  $M(x,y)dx + N(x,y)dy = 0$  is a homogeneous differential equation and  $Mdx + Ndy = 0$  then  $\frac{1}{Mx+Ny}$  is an integrating factor of  $Mdx + Ndy = 0$

① find the integrating factor of

$$(x^2+y^2)dx - 2xydy = 0$$

Sol Given eqn is  $(x^2+y^2)dx - 2xydy = 0$

This is of the form  $Mdx + Ndy = 0$

Here  $M = x^2+y^2$  and  $N = -2xy$

$$\frac{\partial M}{\partial y} = 2y \text{ and } \frac{\partial N}{\partial x} = -2y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  The given eqn is not exact-

But it is homogeneous eqn.

$$\text{Thus I.F} = \frac{1}{Mdx+Ndy}$$

$$\begin{aligned} \text{Now } Mdx+Ndy &= M \cdot x + N \cdot y \\ &= (x^2+y^2)x + (-2xy)y \\ &= x^3+xy^2-2y^2x \end{aligned}$$

$$Mx+Ny = x^3-y^2x$$

$$\text{I.F} = \frac{1}{x^3-y^2x} = \frac{1}{x(x^2-y^2)}$$

① Solve the following differential eqn:

(3)

$$① y(xy + e^n) dn - e^n dy = 0$$

Sol Given eqn is  $y(xy + e^n) dn - e^n dy = 0 \rightarrow ①$

This is of the form  $M dn + N dy = 0$

Here  $M = y(xy + e^n)$  and  $N = -e^n$

$$\frac{\partial M}{\partial y} = 2xy + e^n \text{ and } \frac{\partial N}{\partial n} = -e^n$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial n}$$

The given diff eqn is not exact diff eqn.

Dividing with  $y^2$  in eq ①, we get

$$\frac{y(xy + e^n)}{y^2} dn - \frac{e^n}{y^2} dy = 0$$

$$\left[ \frac{xy^2}{y^2} + \frac{ye^n}{y^2} \right] dn - \left[ \frac{e^n}{y^2} \right] dy = 0$$

$$\left[ n + \frac{e^n}{y} \right] dn - \left[ \frac{e^n}{y^2} \right] dy = 0 \rightarrow ②$$

This is of the form  $M_1 dn + N_1 dy = 0$

$$M_1 = n + \frac{e^n}{y} \text{ and } N_1 = -\frac{e^n}{y^2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{e^n}{y^2} \text{ and } \frac{\partial N_1}{\partial n} = -\frac{e^n}{y^2}$$

$$\therefore \frac{\partial N_1}{\partial y} = \frac{\partial M_1}{\partial n}$$

eq ② is an exact differential eqn.

The general solution is

$$\int M dx + \int (\text{Terms not containing } x \text{ in } N) dy = c$$

(y constant)

$$\int \left( a + \frac{e^x}{y} \right) dx + \int 0 dy = c$$

$$\boxed{\frac{x^2}{2} + \frac{e^x}{y} = c}$$

Which is the required general solution.

(4) ①

**Ques** Exact differential eqn: —

$$(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$$

**Sol** Given eqn is  $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$

This is of the form  $Mdx + Ndy = 0$

Here  $M = x^3 + 3xy^2$  and  $N = y^3 + 3x^2y$

$$\therefore \frac{\partial M}{\partial y} = 6xy \text{ and } \frac{\partial N}{\partial x} = 6xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential eqn is exact

The general solution is

$$\int M dx + \int (\text{terms independent of } x \text{ in } N) dy = c. \\ (\text{by constant})$$

$$\int [x^3 + 3y^2x] dx + \int y^3 dy = c$$

$$\frac{x^4}{4} + 3y^2 \frac{x^2}{2} + \frac{y^4}{4} = c$$

$$x^4 + 6x^2y^2 + y^4 = 4c$$

$$\boxed{\therefore x^4 + 6x^2y^2 + y^4 = c}$$

**Ques**

Method - I

$$y(xy + e^x)dx - e^x dy = 0$$

**Sol**

$$\text{Given eqn is } y(xy + e^x)dx - e^x dy = 0 \longrightarrow ①$$

This is of the form  $(xy^2 + ye^x)dx - e^x dy = 0$

$Mdx + Ndy$

$$\text{Here } M = xy^2 + ye^x, N = -e^x$$

$$\frac{\partial M}{\partial y} = xy + e^x, \quad \frac{\partial N}{\partial x} = -e^x$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ is not}$$

The given eqn is not exact differential eqn.

Dividing with  $y^2$ , we get —

$$\frac{1}{y^2} [xy^2 + ye^x] dx - \frac{1}{y^2} e^x dy = 0$$

$$\frac{x}{y^2} [ny^2 + e^x] dx - \frac{1}{y^2} e^x dy = 0$$

$$\frac{1}{y} [xy + e^x] dx - \frac{1}{y^2} e^x dy = 0$$

$$\left[ \frac{xy}{y} + \frac{e^x}{y} \right] dx - \frac{1}{y^2} e^x dy = 0$$

$$\left[ x + \frac{e^x}{y} \right] dx - \frac{1}{y^2} e^x dy = 0 \longrightarrow ②$$

This is of the form  $M_1 dx + N_1 dy = 0$

$$\text{Here } M_1 = x + \frac{e^x}{y}, \quad N_1 = -\frac{1}{y^2} e^x$$

$$\frac{\partial M_1}{\partial y} = \frac{-e^x}{y^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{1}{y^2} e^x$$

$$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \text{ of } ② \text{ is an exact diff eqn.}$$

The general solution is

$$\int M_1 dx + \int \text{terms of } N_1 \text{ not containing } x \) dy = C  
(y constant)$$

$$\int \left[ x + \frac{e^x}{y} \right] dx + \int 0 dy = C$$

$$\frac{x^2}{2} + \frac{1}{y} e^x = C$$

$$\boxed{\frac{x^2}{2} + \frac{1}{y} e^x = C}$$

(5)

②

$$\text{Solve: } (e^y + 1) \cos x dx + e^y \sin x dy = 0$$

Solution:- The given equation is  $(e^y + 1) \cos x dx + e^y \sin x dy = 0$

This is of the form  $M dx + N dy = 0$

where  $M = (e^y + 1) \cos x$  and  $N = e^y \sin x$

$$\therefore \frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , The given differential eqn is exact

Terms in  $N$  which do not contain  $x$  are 0

$\therefore$  The general solution is given by

$$\int M dx + \int (\text{terms independent of } x \text{ in } N) dy = C$$

(by constant)

$$\Rightarrow \int (e^y + 1) \cos x dx + \int 0 dy = C$$

(by constant)

$$\Rightarrow (e^y + 1) \int \cos x dx = C \quad \left\{ \therefore \int \cos x dx = \sin x + C \right\}$$

$$\boxed{(e^y + 1) \sin x = C}$$

This is the general solution of eq ①

### Method-I

$$\text{Solve: } (1+xy)x dy + (1-yx)y dx = 0$$

Solution:-

The given equation is  $(1+xy)x dy + (1-yx)y dx = 0$

$$\Rightarrow (1-yx)y dx + (1+xy)x dy = 0$$

$$\Rightarrow (y-y^2x) dx + (x+x^2y) dy = 0$$

This is of the form  $M dx + N dy = 0$

where  $M = y-y^2x$ ,  $N = x+x^2y$ .

$$\therefore \frac{\partial M}{\partial y} = 1-2yx, \quad \frac{\partial N}{\partial x} = 1+2xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So that  $M dx + N dy = 0$  is not an exact differential equation.

Dividing the given equation with  $x^2y^2$ , we get

$$\left( \frac{y-y^2x}{x^2y^2} \right) dx + \left( \frac{x+x^2y}{x^2y^2} \right) dy = 0$$

$$\left[ \frac{y}{x^2y^2} - \frac{y^2x}{x^2y^2} \right] dx + \left[ \frac{x}{x^2y^2} + \frac{x^2y}{x^2y^2} \right] dy = 0$$

$$\frac{x dy + y dx}{x^2y^2} + \frac{1}{y} dy - \frac{1}{x} dx = 0$$

$$\frac{d(xy)}{(xy)^2} - \frac{1}{x} dx + \frac{1}{y} dy = 0$$

Integrating, we get-

(3)

$$\int \frac{d(xy)}{(xy)^2} - \int \frac{1}{n} dn + \int \frac{1}{y} dy = 0$$

$$\int \frac{1}{(xy)^2} d(xy) - \int \frac{1}{n} dn + \int \frac{1}{y} dy = 0$$

$$\therefore \int \frac{1}{n^2} dn = -\frac{1}{n} + C$$

$$-\frac{1}{xy} - \log n + \log y = \log c$$

which is the required general solution.

\* Solve: Find the integrating factor of

$$(x^2+y^2)dx - 2xydy = 0$$

Solution: The given eqn is  $(x^2+y^2)dx - 2xydy = 0 \rightarrow ①$

This is of the form  $Mdx+Ndy = 0$

$$\text{where } M = x^2+y^2, N = -2xy$$

$$\text{then } \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

So that  $Mdx+Ndy = 0$  is not an exact differential eqn

But eqn ① is of the form a homogeneous differential eqn

$$\text{and } Mx+Ny = (x^2+y^2)x + (-2xy)y$$

$$Mx+Ny = x^3+xy^2-2xy^2 \neq 0$$

$$\text{Thus } I.F = \frac{1}{Mx+Ny} = \frac{1}{x^3+xy^2-2xy^2} = \frac{1}{x^3-xy^2}$$

$$I.F = \frac{1}{x(x^2-y^2)}$$

Solve:  $(xy \sin xy + \cos xy) y dx + (xy \sin xy - \cos xy) x dy = 0$

Sol: Given equation is

$$[xy \sin xy + \cos xy] y dx + [xy \sin xy - \cos xy] x dy = 0$$

This is of the form  $M dx + N dy = 0$

$$\text{where } M = xy^2 \sin xy + y \cos xy$$

$$N = x^2 y \sin xy - x \cos xy$$

$$\frac{\partial M}{\partial y} = x [2y \sin xy + y^2(-\cos xy)] + [\cos xy \cdot 1 + y \sin xy]$$

$$(y - y^2 n) dx + (n + n^2 y) dy = 0$$

$$M = y - y^2 n \quad N = n + n^2 y$$

$$\frac{\partial M}{\partial y} = 1 - 2ny \quad \frac{\partial N}{\partial x} = 1 + 2ny$$

Divide with  $n^2 y^2$ , we

$$\frac{1}{n^2 y^2} (y - y^2 n) dx + \frac{1}{n^2 y^2} (n + n^2 y) dy = 0$$

$$\left[\frac{1}{n^2 y} - \frac{n}{n^2 y}\right] dx + \left[\frac{1}{n^2 y^2} + \frac{n^2 y}{n^2 y^2}\right] dy = 0$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{n^2 y^2} \quad \frac{\partial N_1}{\partial x} = -\frac{1}{n^2 y^2}$$

$$\frac{1}{y} \int \frac{1}{n^2} dx - \int \frac{1}{n} dy + \int y dy = 0$$

$$\cancel{\int \frac{1}{n^2} dx} + \frac{1}{y} \int n^2 dx$$

$$\frac{5y^3}{3}$$

$$\frac{1}{y} \int \frac{n^2+1}{-2+1} dy$$

$$-\frac{1}{ny}$$

Method 2

(4)



$$\text{Solve: } x^2y \, dx - (x^3 + y^3) \, dy = 0$$

Solution: Given efn is  $x^2y \, dx - (x^3 + y^3) \, dy = 0 \longrightarrow ①$

This is of the form  $M \, dx + N \, dy = 0$

where  $M = x^2y$ ,  $N = -(x^3 + y^3)$

$$\text{we have } \frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = -3x^2$$

$\therefore$  We have  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  so that the given differential efn is not exact

But efn ① is a homogeneous differential efn

$$\text{Now } Mx + Ny = x^3y - x^3y - y^4 = -y^4 \neq 0$$

$$\therefore \text{Integrating factor} = \frac{1}{M \, dx + N \, dy} = -\frac{1}{y^4}$$

Multiplying efn ① with  $-\frac{1}{y^4}$ , we get

$$-\frac{1}{y^3}(x^2y) \, dx - \frac{1}{y^4}[-(x^3 + y^3)] \, dy = 0$$

$$-\frac{x^2}{y^3} \, dx + \frac{x^3 + y^3}{y^4} \, dy = 0 \longrightarrow ②$$

This is of the form  $M_1 \, dx + N_1 \, dy = 0$

$$\text{where } M_1 = -\frac{x^2}{y^3} \text{ and } N_1 = \frac{x^3 + y^3}{y^4}$$

$$\text{we have } \frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4} \text{ and } \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}$$

$\therefore$  Eq ② is an exact differential equation.

General solution of eq ② is given by

$$\int M_1 dx + \int (\text{terms independent of } x \text{ in } N_1) dy = C$$

(y constant)

$$\int -\frac{x^2}{y^3} dx + \int \frac{1}{y} dy = C$$

$$-\frac{1}{y^3} \int x^2 dx + \log y = C$$

$$-\frac{x^3}{3y^3} + \log y = C$$

The general solution of ② is same as the  
general solution of ①

Solve:

Method 3:

(5)

If the equation  $Mdx + Ndy = 0$  is of the form

(8)

$yf(xy)dx + xg(xy)dy = 0$  and  $Mx - Ny \neq 0$  then

$\frac{1}{Mx - Ny}$  is an integrating factor of  $Mdx + Ndy = 0$

Ex:

① Solve  $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$

Sol: Given eqn is

$$(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0 \rightarrow ①$$

This is of the form  $Mdx + Ndy = 0$

$$\text{where } M = xy^2\sin xy + y \cos xy$$

$$N = x^2y\sin xy - x \cos xy$$

$$\text{we have } \frac{\partial M}{\partial y} = x[2y\sin xy + y^2\cos xy \cdot x] + [1 \cdot \cos xy + y(-\sin xy)]$$

$$\frac{\partial M}{\partial y} = 2xy\sin xy + x^2y^2\cos xy + \cos xy - xy\sin xy$$

$$\frac{\partial M}{\partial y} = xy\sin xy + x^2y^2\cos xy + \cos xy.$$

$$\text{and } \frac{\partial N}{\partial x} = y[2x\sin xy + x^2\cos xy \cdot y] - [1 \cdot \cos xy + x(-\sin xy) \cdot y]$$
$$= 2xy\sin xy + x^2y^2\cos xy - \cos xy + xy\sin xy$$

$$\frac{\partial N}{\partial x} = 3xy\sin xy + x^2y^2\cos xy - \cos xy$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  The given differential eqn is exact  
not an exact diff eqn

Since eq① is of the form  $y \cdot f(x,y)dx + x \cdot g(x,y)dy = 0$  and

$$\begin{aligned} Mx - Ny &= (xy^2 \sin xy + y \cos xy)x - (x^2 y \sin xy - x \cos xy)y \\ &= x^2 y^2 \sin xy + xy \cos xy - x^2 y^2 \sin xy + xy \cos xy \end{aligned}$$

$$Mx - Ny = 2xy \cos xy \neq 0$$

$$\text{Thus } Q.F = \frac{1}{Mx - Ny} = \frac{1}{2xy \cos xy}$$

Multiplying eq① with Q.F i.e  $\frac{1}{2xy \cos xy}$ , we get-

$$\begin{aligned} &= \frac{1}{2xy \cos xy} [xy^2 \sin xy + y \cos xy] dx + \frac{1}{2xy \cos xy} [x^2 y \sin xy - x \cos xy] dy = 0 \\ &= \left[ \frac{xy^2 \sin xy}{2xy \cos xy} + \frac{y \cos xy}{2xy \cos xy} \right] dx + \left[ \frac{x^2 y \sin xy}{2xy \cos xy} - \frac{x \cos xy}{2xy \cos xy} \right] dy = 0 \end{aligned}$$

$$\left[ \frac{y}{2} \tan xy + \frac{1}{2x} \right] dx + \left[ \frac{x}{2} \tan xy - \frac{1}{2y} \right] dy = 0 \longrightarrow ②$$

This is of the form  $M_1 dx + N_1 dy = 0$  and it is exact

$$\text{where } M_1 = \frac{y}{2} \tan xy + \frac{1}{2x} \text{ and } N_1 = \frac{x}{2} \tan xy - \frac{1}{2y}$$

$$\frac{\partial M_1}{\partial y} = \frac{1}{2} [y \sec^2 xy \cdot x + \tan xy (1)] + 0$$

$$\boxed{\frac{\partial M_1}{\partial y} = \frac{xy}{2} \sec^2 xy + \frac{1}{2} \tan xy}$$

$$\frac{\partial N_1}{\partial n} = \frac{1}{2} [n \sec^2 ny \cdot y + \tan ny (1)] - 0$$

(9) <sup>6</sup>

$$\boxed{\frac{\partial N_1}{\partial n} = \frac{ny}{2} \sec^2 ny + \frac{1}{2} \tan ny}$$

: cf. ② is an exact differential eqn.

The general solution is given by.

$$\int M_1 dn + \int \text{Terms independent of } n \ln N_1 dy = C$$

(y constant)

$$\int \frac{1}{2} [y \tan ny + \frac{1}{n}] dn + \int -\frac{1}{2y} dy = C$$

$$\frac{1}{2} \int y \tan ny dn + \frac{1}{2} \int \frac{1}{n} dn - \frac{1}{2} \int \frac{1}{y} dy = C$$

$$\frac{y}{2} \int \tan ny dn + \frac{1}{2} \log(n) - \frac{1}{2} \log y = \log C_1$$

$\left\{ \begin{array}{l} \int \tan ny dn = \log |\sec ny| \\ \log(n) = \log n - \log 1 \end{array} \right.$

$$\frac{y}{2} \log |\sec ny| + \frac{1}{2} \log n - \frac{1}{2} \log y = \log C_1$$

$\left\{ \begin{array}{l} \log \frac{m}{n} = \log m - \log n \\ \log m^n = n \log m \end{array} \right.$

$$\frac{1}{2} [\log(\sec ny) + \log(n) - \log y] = \log C_1$$

$$\log \left( \frac{n \sec ny}{y} \right) = \log C_1$$

$$\log \left( \frac{n \sec ny}{y} \right) = \log C_1^2$$

$$\frac{n \sec ny}{y} = C_1^2$$

$$\frac{n \sec ny}{y} = C$$

This is the required general solution.

$$\textcircled{1} \quad y(1+xy)dx + x(1-xy)dy = 0$$

\textcircled{2} Solve the

$$\textcircled{1} \quad y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$$

$$\textcircled{2} \quad (y - xy^2)dx - (x + x^2y)dy = 0.$$

\textcircled{1} Solve the following differential eqns

$$\textcircled{1} \quad (y - xy^2)dx - (x + x^2y)dy = 0$$

$$\textcircled{2} \quad y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$$

Method-3

Solve:  $y(1+xy)dx + x(1-xy)dy = 0$

(7)

(P)

Sol: Given D.E is  $y(1+xy)dx + x(1-xy)dy = 0 \rightarrow ①$

This is of the form  $Mdx + Ndy = 0$

where  $M = y+xy^2$ ,  $N = x-x^2y$

then  $\frac{\partial M}{\partial y} = 1+2xy$  and  $\frac{\partial N}{\partial x} = 1-2xy$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , The given eqn is not exact-D.E

The given D.E is of the form  $y \cdot f(xy) + x \cdot g(xy)dy = 0$

Now  $Mx-Ny = (y+xy^2)x - (x-x^2y)y$

$$Mx-Ny = xy + x^2y^2 - xy + x^2y^2$$

$$Mx-Ny = 2x^2y^2 \neq 0$$

$$\therefore P.F = \frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$$

Multiplying eq ① with  $\frac{1}{2x^2y^2}$ , we get—

$$\frac{1}{2x^2y^2} y(1+xy)dx + \frac{1}{2x^2y^2} x(1-xy)dy = 0$$

$$\left[ \frac{1}{2x^2y} + \frac{xy}{2x^2y} \right] dx + \left[ \frac{1}{2y^2x} - \frac{xy}{2x^2y} \right] dy = 0$$

$$\left[ \frac{1}{2x^2y} + \frac{1}{2x} \right] dx + \left[ \frac{1}{2y^2x} - \frac{1}{2y} \right] dy = 0 \rightarrow ②$$

This is of the form  $M_1dx + N_1dy = 0$

Here  $M_1 = \frac{1}{2x^2y} + \frac{1}{2x}$ ,  $N_1 = \frac{1}{2y^2x} - \frac{1}{2y}$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{2n^2y^2} \text{ and } \frac{\partial N_1}{\partial n} = -\frac{1}{2n^2y^2}, \quad \left\{ \frac{d}{dn}(1/n) = -1/n^2 \right\}$$

$\therefore \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial n}$ , the eqn ② is an exact differential eqn

∴ The general solution of ① is

$$\int M_1 dn + \int N_1 dy = C$$

(y constant)      (Terms not containing n)

$$\int \left[ \frac{1}{2n^2y} + \frac{1}{2n} \right] dn + \int -\frac{1}{2y} dy = C$$

$$\frac{1}{2y} \left[ \frac{x^{2+1}}{2+1} \right] + \frac{1}{2} \log n - \frac{1}{2} \log y = C$$

$$\frac{1}{2y} \left[ \frac{x^1}{1} \right] + \frac{1}{2} [\log n - \log y] = C$$

$$\left\{ \int n^n dn = \frac{x^{n+1}}{n+1} + C \right.$$

$$-\frac{1}{2ny} + \frac{1}{2} [\log(\frac{n}{y})] = C$$

$$\left\{ \int \frac{1}{n} dn = \log n + C \right.$$

$$\frac{1}{2} \left[ \log(\frac{n}{y}) - \frac{1}{ny} \right] = C$$

$$\log(\frac{n}{y}) - \frac{1}{ny} = 2C$$

$$\log(\frac{n}{y}) - \frac{1}{ny} = C_1$$

which is the required solution.

## Method : 4

(ii)

If there exists a continuous single variable function  $f(m)$  such that  $\frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = f(m)$  then  $e^{\int f(m) dm}$  is an integrating factor of  $M dx + N dy = 0$ .

$$① \text{ Solve } 2xy dy - (x^2 + y^2 + 1) dx = 0$$

Sol: Given eqn is  $-(x^2 + y^2 + 1) dx + 2xy dy = 0 \rightarrow ①$

This is of the form  $M dx + N dy = 0$  where

$$M = -(x^2 + y^2 + 1) \text{ and } N = 2xy.$$

$$\text{we have } \frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = 2y$$

$$\text{So that } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$  The given eqn is not exact

$$\text{we have } \frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{2xy} [-2y - 2y] = -\frac{4y}{2xy} = -\frac{2}{x} = f(m)$$

$$\therefore I.F = e^{\int f(m) dm} = e^{-2 \int \frac{1}{x} dm} = e^{-2 \log x}$$

$$I.F = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$$I.F = \frac{1}{x^2}$$

Multiplying eq ① with  $\frac{1}{x^2}$ , we get-

$$-\left[\frac{x^2 + y^2 + 1}{x^2}\right] dm + \frac{2xy}{x^2} dy = 0$$

$$-\left[\frac{x^2}{x^2} + \frac{y^2}{x^2} + \frac{1}{x^2}\right] dm + \frac{2y}{x} dy = 0$$

$$-\left[1 + \frac{y^2}{n^2} + \frac{1}{n^2}\right]dx + \frac{dy}{n} = 0 \longrightarrow ②$$

This is of the form  $M_1 dx + N_1 dy = 0$  where

$$M_1 = -\left[1 + \frac{y^2}{n^2} + \frac{1}{n^2}\right] \text{ and } N_1 = \frac{dy}{n}$$

$$\text{we have } \frac{\partial M_1}{\partial y} = -\left[0 + \frac{2y}{n^2} + 0\right] \text{ and } \frac{\partial N_1}{\partial x} = 2y\left(-\frac{1}{n^2}\right)$$

$$\frac{\partial M_1}{\partial y} = -\frac{2y}{n^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{2y}{n^2}$$

Since  $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$  ∴ eq ② is exact D.E.

Hence the general solution is given by.

$$\int M_1 dx + \int (\text{terms not containing } x \text{ in } N) dy = c \\ (\text{y constant})$$

$$\int -\left[1 + \frac{y^2}{n^2} + \frac{1}{n^2}\right] dx + \int 0 \cdot dy = c$$

$$-\int 1 \cdot dx - y^2 \int \frac{1}{n^2} dx \rightarrow \int \frac{1}{n^2} dx = c$$

$$-x - y^2 \left[ \frac{x^{-2+1}}{-2+1} \right] - \left[ \frac{x^{-2+1}}{-2+1} \right] = c$$

$$\left\{ \int n^2 dx = \frac{x^{n+1}}{n+1} + c \right\}$$

$$-x - y^2 \left[ \frac{x^{-1}}{-1} \right] - \left[ \frac{x^{-1}}{-1} \right] = c$$

$$\boxed{-x + \frac{y^2}{n} + \frac{1}{n} = c}$$

This is the general solution of eq ② and hence of eq ①

(12)

$$\text{Solve: } 2xy \, dy - (x^2 + y^2 + 1) \, dx = 0$$

Solution: Given eqn is  $2xy \, dy - (x^2 + y^2 + 1) \, dx = 0 \rightarrow ①$

This is of the form  $M \, dx + N \, dy = 0$ , where

$$M = -(x^2 + y^2 + 1), N = 2xy$$

$$\text{we have } \frac{\partial M}{\partial y} = -2y, \quad \frac{\partial N}{\partial x} = 2y$$

so that  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ . The given eqn is not exact.

$$\text{we have } \frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{2xy} [-2y - 2y] = \frac{-4y}{2xy} = -\frac{2}{x}$$

$$\frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = -\frac{2}{x} = f(x)$$

$$\therefore I.F = e^{\int f(x) \, dx} = e^{-2 \int \frac{1}{x} \, dx} = e^{-2 \log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$$I.F = \frac{1}{x^2}$$

Multiplying eqn ① with  $\frac{1}{x^2}$ , we get

$$\frac{1}{x^2} [2xy \, dy - (x^2 + y^2 + 1) \, dx] = 0$$

$$\frac{2y}{x} \, dy - \left[ 1 + \frac{y^2}{x^2} + \frac{1}{x^2} \right] \, dx = 0 \rightarrow ②$$

This is of the form  $M_1 \, dx + N_1 \, dy = 0$ , where

$$M_1 = -\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right), \quad N_1 = \frac{2y}{x}$$

$$\text{we have } \frac{\partial M_1}{\partial y} = -\frac{2y}{x^2}, \quad \frac{\partial N_1}{\partial x} = -\frac{2y}{x^2}$$

Since  $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$   $\therefore$  eq(2) is exact D.E

Hence the general solution is given by

$$= \int M_1 dx + \int (\text{terms independent of } x \text{ in } N_1) dy = c.$$

(y constant)

$$\Rightarrow \int \left[ -1 - \frac{y^2}{x^2} - \frac{1}{x^2} \right] dx + \int 0 \cdot dy = c$$

$$-x + \frac{y^2}{x} + \frac{1}{x} = c$$

This is the general solution of eq(2) and hence of eq(1)

$$② \text{ Solve } (3y^2 + 4xy - n)dn + n(x+2y)dy = 0 \quad (13) \quad (10)$$

Sol: The given D.E is  $(3y^2 + 4xy - n)dn + (n^2 + 2ny)dy = 0 \rightarrow ①$   
 This is of the form  $Mdn + Ndy = 0$

where  $M = 3y^2 + 4xy - n$  and  $N = n^2 + 2ny$

we have

$$\frac{\partial M}{\partial y} = 3 \frac{\partial}{\partial y} y^2 + 4x \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial y}(n) \text{ and } \frac{\partial N}{\partial n} = \frac{\partial}{\partial n}(n^2) + 2y \frac{\partial}{\partial n}(n)$$

$$\frac{\partial M}{\partial y} = 3(2y) + 4x(1) - 0$$

$$\frac{\partial N}{\partial n} = 2n + 2y$$

$$\frac{\partial M}{\partial y} = 6y + 4x$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial n}$ , the given D.E is not exact

$$\begin{aligned} \text{consider } \frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial n} \right] &= \frac{1}{n^2 + 2ny} [6y + 4x - (2n + 2y)] \\ &= \frac{1}{n^2 + 2ny} [6y + 4x - 2n - 2y] \\ &= \frac{1}{n(n+2y)} [4x + 2n] \Rightarrow \frac{2(x+2y)}{n(n+2y)} = \frac{2}{n} = f(n) \end{aligned}$$

$$\therefore I.F = e^{\int f(n)dn} = e^{\int \frac{2}{n} dn} = e^{2 \int \frac{1}{n} dn} = e^{2 \log n}$$

$$Q.F = e^{\log n^2} = n^2.$$

Multiplying eq ① with  $n^2$ , we get

$$n^2 [3y^2 + 4xy - n]dn + n^2 [n^2 + 2ny]dy = 0$$

$$[3n^2y^2 + 4n^3y - n^3]dn + [n^4 + 2n^3y]dy = 0 \rightarrow ②$$

This is of the form  $M_1dn + N_1dy = 0$

$$M_1 = 3x^2y^2 + 4x^3y - x^3 \text{ and } N_1 = x^4 + 2x^3y$$

we have  $\frac{\partial M_1}{\partial y} = 6x^2y + 4x^3 - 0$  and  $\frac{\partial N_1}{\partial x} = 4x^3 + 6x^2y$

$$\frac{\partial M_1}{\partial y} = 6x^2y + 4x^3 \text{ and } \frac{\partial N_1}{\partial x} = 4x^3 + 6x^2y$$

Since  $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$  cf ② is exact-diff eqn.

The general solution is given by

$$\int M_1 dx + \int (\text{terms not containing } x \text{ in } N) dy = C$$

(y constant)

$$\int (3x^2y^2 + 4x^3y - x^3) dx + \int 0 dy = C$$

$$3y^2 \int x^2 dx + 4y \int x^3 dx - \int x^3 dx = C$$

$$3y^2 \left[ \frac{x^3}{3} \right] + 4y \left[ \frac{x^4}{4} \right] - \left[ \frac{x^4}{4} \right] = C$$

$$x^3y^2 + 4x^4y - \frac{x^4}{4} = C$$

~~which is the~~ This is the general solution of eq ②  
and hence of eq ①

Ast ① Solve the following differential eqns.

①  $(x^2 + y^2 + 2x) dx + 2y dy = 0$

②  $(x^2 + y^2) dx - 2xy dy = 0$

### Method 5:

If there exists a continuous and differentiable single variable function  $g(y)$  such that  $\frac{1}{M} \left[ \frac{\partial N}{\partial n} - \frac{\partial M}{\partial y} \right] = g(y)$   
 then  $e^{\int g(y) dy}$  is an integrating factor of  $M dn + N dy = 0$

$$\textcircled{1} \quad \text{Solve the differential equation } y(xy + e^n) dn - e^n dy = 0$$

Sol: Given eqn is  $y(xy + e^n) dn - e^n dy = 0 \rightarrow \textcircled{1}$

This is of the form  $M dn + N dy = 0$

$$\text{where } M = xy^2 + ye^n \quad N = -e^n$$

$$\text{we have } \frac{\partial M}{\partial y} = 2xy + e^n \quad \frac{\partial N}{\partial n} = -e^n$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial n}$  The given D.E is not exact

$$\text{Now } \frac{1}{M} \left[ \frac{\partial N}{\partial n} - \frac{\partial M}{\partial y} \right] = \frac{1}{xy^2 + ye^n} \left[ -e^n - 2xy - e^n \right]$$

$$= \frac{1}{xy^2 + ye^n} [-2e^n - 2xy]$$

$$= \frac{1}{y(xy + e^n)} (-2(e^n + xy))$$

$$\frac{1}{M} \left[ \frac{\partial N}{\partial n} - \frac{\partial M}{\partial y} \right] = -\frac{2}{y} = f(y)$$

$$\therefore \text{I.F} = e^{\int f(y) dy} = e^{-2 \int \frac{1}{y} dy} = e^{-2 \log y} = e^{\log y^{-2}} = y^{-2}$$

$$\text{I.F} = \frac{1}{y^2}$$

Multiplying the given D.E with  $\frac{1}{y^2}$ , we get

$$\frac{1}{y^2} [xy^2 + ye^n] dn - \frac{e^n}{y^2} dy = 0$$

$$\left[ \frac{xy^2}{y^2} + \frac{e^y y'}{y^2} \right] dx - \frac{e^y}{y^2} dy = 0$$

$$\left[ x + \frac{e^y}{y} \right] dx + \frac{e^y}{y^2} dy = 0 \quad \rightarrow ②$$

This is of the form  $M_1 dx + N_1 dy = 0$

$$M_1 = x + \frac{e^y}{y} \quad N_1 = -\frac{e^y}{y^2}$$

$$\text{we have } \frac{\partial M_1}{\partial y} = 0 + e^y \left( -\frac{1}{y^2} \right) \text{ and } \frac{\partial N_1}{\partial x} = -\frac{1}{y^2} (e^y)$$

$$\frac{\partial M_1}{\partial y} = -\frac{e^y}{y^2} \text{ and } \frac{\partial N_1}{\partial x} = -\frac{e^y}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \text{ if } ② \text{ is exact-diff eqn}$$

The general solution is given by.

$$\int M_1 dx + \int (\text{terms not containing } x \text{ in } N) dy = C \\ (\text{y constant})$$

$$\int \left( x + \frac{e^y}{y} \right) dx + \int 0 dy = C$$

$$\int x dx + \frac{1}{y} \int e^y dy = C$$

$$\frac{x^2}{2} + \frac{e^y}{y} = C$$

which is the general solution.

Ans ① Solve: ①  $y(2xy + e^y) dx - e^y dy = 0$

$$② (y^4 + 2y) dx + (2y^3 + 2y^4 - 4y) dy = 0.$$

$$③ (xy^2 - x^2) dx + (3x^2y^2 + x^2y - 2x^3) dy = 0.$$

$$② \text{ Solve } y(2xy + e^n)dx - e^n dy = 0$$

(12)

Sol Given  $ef^n$  is  $y(2xy + e^n)dx - e^n dy = 0$

(15)

This is of the form  $Mdx + Ndy = 0$

where  $M = y(2xy + e^n)$  and  $N = -e^n$

We have  $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2xy^2 + ye^n)$  and  $\frac{\partial N}{\partial x} = -\frac{\partial}{\partial x}(e^n)$

$$\frac{\partial M}{\partial y} = 4xy + e^n \text{ and } \frac{\partial N}{\partial x} = -e^n.$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  The given diff  $ef^n$  is not exact

$$\text{consider } \frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{2xy^2 + ye^n} [-e^n - 4xy - e^n]$$

$$= \frac{1}{2xy^2 + ye^n} [-2e^n - 4xy]$$

$$\frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = -\frac{2[e^n + 2xy]}{y[e^n + 2xy]} = -\frac{2}{y} = f(y)$$

$$I.F = e^{\int f(y)dy} = e^{\int -\frac{2}{y} dy} = e^{-2 \int \frac{1}{y} dy} = e^{-2 \log y} = e^{\log y^{-2}}$$

$$I.F = y^{-2} \quad \left\{ e^{\log a} = a \right\}$$

$$I.F = \frac{1}{y^2}$$

Multiplying  $ef^n$  with  $\frac{1}{y^2}$ , we get

$$\frac{1}{y^2} [2xy^2 + e^n]dx - \frac{e^n}{y^2} dy = 0$$

$$\left[ 2x + \frac{e^n}{y} \right] dx - \frac{e^n}{y^2} dy = 0 \longrightarrow ①$$

This is of the form  $M_1dx + N_1dy = 0$

$$\text{Here } M_1 = 2x + \frac{e^y}{y}, \quad N_1 = -\frac{e^y}{y^2}$$

$$\text{we have } \frac{\partial M_1}{\partial y} = 0 + e^y \frac{\partial}{\partial y} \left( \frac{1}{y} \right), \quad \frac{\partial N_1}{\partial x} = -\frac{e^y}{y^2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{e^y}{y^2}$$

since  $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \therefore \text{eqn ② is exact-diff eqn.}$

The general solution is given by

$$\int M_1 dx + \int (\text{terms not containing } x \text{ in } N_1) dy = C$$

(y constant)

$$\int \left( 2x + \frac{e^y}{y} \right) dx + \int 0 \cdot dy = C$$

$$2 \int x dx + \int \frac{e^y}{y} dy = C$$

$$2\left[\frac{x^2}{2}\right] + \frac{1}{y} \int e^y dy = C$$

$$x^2 + \frac{e^y}{y} = C$$

This is the general solution of eqn ② and hence of eqn ①

Solve: the differential equation  $y(xy + e^x)dx - e^x dy = 0$  16

Solution: Given eqn is  $y(xy + e^x)dx - e^x dy = 0 \longrightarrow ②$

Comparing the given D.E with  $Mdx + Ndy = 0$

we have  $M = xy^2 + e^x y$  and  $N = -e^x$

$$\therefore \frac{\partial M}{\partial y} = 2xy + e^x \text{ and } \frac{\partial N}{\partial x} = -e^x$$

Since  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ , the given D.E is not exact.

$$\text{Now } \frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{1}{xy^2 + e^x y} [-e^x - 2xy - e^x]$$

$$= \frac{1}{y(e^x + xy)} [-2e^x - 2xy]$$

$$\frac{1}{M} \left[ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] = \frac{-2(e^x + xy)}{y(e^x + xy)} = -\frac{2}{y} = f(y)$$

$$\therefore P.F = e^{\int f(y) dy} = e^{-\int \frac{2}{y} dy} = e^{-2 \int \frac{1}{y} dy} = e^{-2 \log y} = e^{\log y^{-2}}$$

$$Q.F = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Multiplying the given D.E with  $\frac{1}{y^2}$ , we get

$$\frac{1}{y^2} [y(xy + e^x)dx - e^x dy] = 0$$

$$\frac{1}{y} [xy + e^x] dx - \frac{e^x}{y^2} dy = 0 \longrightarrow ②$$

This is of the form  $M_1 dx + N_1 dy = 0$  where

$$M_1 = \frac{1}{y} [xy + e^y], N_1 = -\frac{e^y}{y^2}$$

$$\therefore M_1 = x + \frac{e^y}{y}, N_1 = -\frac{e^y}{y^2}$$

$$\therefore \frac{\partial M_1}{\partial y} = -\frac{e^y}{y^2}, \frac{\partial N_1}{\partial x} = -\frac{e^y}{y^2}$$

Since  $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$   $\therefore$  eq ② is exact D.E

Hence the general solution is

$\int M_1 dx + \int \text{Terms independent of } x \text{ in } N_1 dy = 0$

(y constant)

$$\int \left( x + \frac{e^y}{y} \right) dx + \int 0 \cdot dy = C$$

$$\boxed{\frac{x^2}{2} + \frac{e^y}{y} = C}$$

which is the general solution.

## Linear Differential equations of first order

(1)

An equation of the form  $\frac{dy}{dx} + P(x)y = Q(x)$  where

P and Q are either constants or function of x only is called a linear differential eqn of first order in y.

Working Rule: —

To solve the linear eqn  $\frac{dy}{dx} + P(x)y = Q(x)$

(i) Write the integrating factor (I.F) =  $e^{\int P(x)dx}$ .

(ii) The General Solution is given by.

$$y \times (I.F) = \int Q(x) \times (I.F) dx + C \quad \text{or} \quad y \cdot e^{\int P(x)dx} = \int Q(x) \cdot e^{\int P(x)dx} \cdot dx + C$$

Note: An eqn of the form  $\frac{dy}{dx} + P(y)x = Q(y)$  where p and q are either constants or function of x only is called a linear differential eqn of first order in x.

Working Rule: To solve the linear eqn  $\frac{dy}{dx} + P(y)x = Q(y)$

(i) Write the integrating factor (I.F) =  $e^{\int P(y)dy}$

(ii) The general solution is given by

$$y \times (I.F) = \int Q(y) \times (I.F) dy + C$$

find the integrating factor for the following equations

$$\textcircled{1} \quad \frac{dy}{dx} = e^{2x} + y - 1$$

Sol Given eqn is  $\frac{dy}{dx} = e^{2x} + y - 1$

This can be written as  $\frac{dy}{dx} - y = e^{2x} - 1$

This is of the form  $\frac{dy}{dx} + P(x)y = Q(x)$

where  $P = -1$  and  $Q = e^{2x} - 1$

$$I.F = e^{\int P(x)dx} = e^{\int -1 dx} = e^{-\int 1 dx} = e^{-x}.$$

$$\boxed{I.F = e^{-x}}$$

Ex ①  $\frac{dy}{dx} + xy = x$

②  $\frac{dy}{dx} - \frac{1}{x+1}y = e^{3x}(x+1)$

H.S.M.P

\* \* \* ① Solve  $(1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

Sol Given eqn is  $(1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

This can be written as  $\frac{dx}{dy} (1+y^2) + x - e^{\tan^{-1}y} = 0$

Dividing with  $(1+y^2)$  on both sides.

$$\frac{dx}{dy} \frac{(1+y^2)}{1+y^2} + \frac{x}{1+y^2} - \frac{e^{\tan^{-1}y}}{1+y^2} = 0$$

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{e^{\tan^{-1}y}}{1+y^2} \rightarrow \textcircled{1}$$

This is of the form  $\frac{dx}{dy} + Px = Q$  where P and Q are function of y and it is a first order linear eqn in  $x$ .

Here  $P = \frac{1}{1+y^2}$  and  $Q = \frac{e^{\tan^{-1}y}}{1+y^2}$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

(13)

(18)

∴ General Solution is given by

$$x \cdot (I.F) = \int Q \times (I.F) dy + C$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} dy + C \quad \rightarrow (2)$$

$$\text{Now put } \tan^{-1} y = t \text{ so that } \frac{1}{1+y^2} dy = dt \quad \left\{ \frac{d}{dt} \tan^{-1} t = \frac{1}{1+t^2} \right\}$$

$$\text{Now of (2)} \Rightarrow x e^{\tan^{-1} y} = \int e^t \cdot e^t \cdot dt + C$$

$$m e^{\tan^{-1} y} = \int e^{2t} dt + C \quad \left\{ \int e^{at} dt = \frac{e^{at}}{a} \right\}$$

$$n e^{\tan^{-1} y} = \frac{e^{2t}}{2} + C$$

$$\boxed{n e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C}$$

This is the general solution of the given eqn.

$$(2) \text{ Solve: } \frac{dy}{dm} + \frac{y}{m \log n} = \frac{\sin 2m}{\log n}$$

Sol Given eqn is  $\frac{dy}{dm} + \frac{1}{m \log n} \cdot y = \frac{\sin 2m}{\log n}$

This is of the form  $\frac{dy}{dm} + Py = Q$

$$\text{Here } P = \frac{1}{m \log n} \text{ and } Q = \frac{\sin 2m}{\log n}$$

$$I.F = e^{\int P dm} = e^{\int \frac{1}{m \cdot \log n} dm} = e^{\log(\log n)} = \log n$$

$$\left\{ \int \frac{f(m)}{f(m)} dm = \log f(m) + C \right\}$$

$$Q.F = \log n$$

Multiplying eq ① with I.F i.e.  $\log n$ , we get  
log $n$ . The General solution is given by

$$y \times (I.F) = \int Q \cdot (I.F) dx + C$$

$$y \cdot \log n = \int \frac{\sin 2x}{\log n} \cdot \log n \cdot dx + C$$

$$y \cdot \log n = \int \sin 2x dx + C$$

$$y \cdot \log n = -\frac{\cos 2x}{2} + C$$

$$\left\{ \int \sin ax dx = -\frac{\cos ax}{a} + C \right\}$$

$$y \cdot \log n = -\frac{\cos 2x}{2} + C$$

Bernoulli's equation: —

An eqn of the form  $\frac{dy}{dx} + Py = Qy^n$  is called Bernoulli's equation if P and Q are constants or functions of x alone and n is a real constant.

$$\textcircled{2} \text{ Solve: } x \frac{dy}{dx} + y = x^3 y^6$$

(19) <sup>16</sup>

Sol: Given eqn as  $x \cdot \frac{dy}{dx} + y = x^3 y^6$

This can be written as  $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6 \longrightarrow \textcircled{1}$

This is of the form  $\frac{dy}{dx} + P \cdot y = Q y^n$

where  $P = \frac{1}{x}$ ,  $Q = x^2$  and  $n = 6$

Dividing with  $y^6$ , we get

$$\frac{1}{y^6} \frac{dy}{dx} + \frac{1}{y^5} \frac{1}{x} = x^2 \longrightarrow \textcircled{2}$$

Let  $\frac{1}{y^5} = u$  then  $-\frac{5}{y^6} \frac{dy}{dx} = \frac{du}{dx}$

$$\Rightarrow \frac{1}{y^6} \frac{dy}{dx} = -\frac{1}{5} \frac{du}{dx}$$

Substituting in eq  $\textcircled{2}$ , we get

$$-\frac{1}{5} \cdot \frac{du}{dx} + u \cdot \frac{1}{x} = x^2$$

$$\frac{du}{dx} - \frac{5u}{x} = -5x^2 \longrightarrow \textcircled{3}$$

This is a linear eqn in  $u$ .

$$\text{Now } Q \cdot F = e^{\int -\frac{5}{x} dx} = e^{-5 \int \frac{1}{x} dx} = e^{-5 \log x}$$

$$Q \cdot F = e^{\log x^5} = x^5 = \frac{1}{x^5}$$

The general solution of eq  $\textcircled{3}$  is given by

$$u(Q \cdot F) = \int Q(Q \cdot F) dx + C$$

$$u \cdot \frac{1}{x^5} = -5 \int x^2 \cdot \frac{1}{x^5} dx + C$$

$$Q) \text{ Solve } (1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$$

Sol: Given eqn is  $(1+y^2) + (x - e^{\tan^{-1}y}) \frac{dy}{dx} = 0$

This can be written as  $\frac{dx}{dy} (1+y^2) + x - e^{\tan^{-1}y} = 0$

$$\frac{dx}{dy} + \frac{x}{1+y^2} - \frac{e^{\tan^{-1}y}}{1+y^2} = 0$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

~~Repeal~~ This is of the form  $\frac{dx}{dy} + Px = Q$  where P and Q are functions of y and it is a first order linear equation in x.

$$\text{Here } P = \frac{1}{1+y^2} \text{ and } Q = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$$

$$\therefore \int \frac{1}{1+y^2} dx = \tan^{-1}(x) + C$$

General solution is

$$x \times I.F = \int Q \times (I.F) dy + C$$

$$x \cdot e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y} dy + C$$

$$\text{put } \tan^{-1}y = t \text{ so that } \frac{1}{1+y^2} dy = dt$$

$$xe^{\tan^{-1}y} = \int (e^t)^2 dt + C$$

$$xe^{\tan^{-1}y} = \frac{e^{2t}}{2} + C \Rightarrow xe^{\tan^{-1}y} = \boxed{e^{\frac{2\tan^{-1}y}{2} + C}}$$

$$\therefore \int e^{ax} dx = \frac{e^{ax}}{a}$$

This is the general solution of the given eqn.

M.S.P

\* Q Solve:  $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$

(20)

Solution:

Given eqn is  $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \rightarrow ①$

put  $\tan y = u$  so that  $\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$

Substituting these values in eq ①, we get

$$\frac{du}{dx} + 2ux = x^3 \rightarrow ②$$

This is a linear equation in  $u$ .

Here  $P = 2x$  and  $Q = x^3$

Now I.F =  $e^{\int P dx} = e^{\int 2x dx} = e^{x^2} = e^{x^2}$

The general solution of eq ② is

$$u \cdot e^{x^2} = \int x^3 \cdot e^{x^2} dx + C$$

$$ue^{x^2} = \int x^3 \cdot e^{x^2} dx + C$$

Put  $x^2 = t$  so that  $x dx = \frac{1}{2} dt$

$$ue^{x^2} = \int t^2 e^t \cdot \frac{1}{2} dt + C$$

$$\Rightarrow ue^{x^2} = \int t \cdot e^t \cdot \frac{1}{2} dt + C \quad \left\{ \int uv du = uv - u'v, \dots \right\}$$

$$ue^{x^2} = \frac{1}{2} [te^t - e^t] + C$$

$$ue^{x^2} = \frac{1}{2} e^t(t-1) + C$$

$$\tan y e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + C \quad \left\{ \begin{array}{l} t = x^2 \\ u = \tan y \end{array} \right\}$$

which is the required solution of eq ①

$$u \cdot \frac{1}{n^5} = -5 \int \frac{n^2}{n^8 3} dn + C$$

$$u \cdot \frac{1}{n^5} = -5 \int \frac{1}{n^3} dn + C$$

$$\frac{u}{n^5} = -5 \left[ \frac{x^{-2}}{-2} \right] + C$$

$$u \cdot \frac{1}{n^5} = \frac{5}{2n^2} + C.$$

$$\frac{1}{y^5} \cdot \frac{1}{n^5} = \frac{5}{2n^2} + C \quad \left\{ \therefore u = \frac{1}{y^5} \right\}$$

$$\frac{1}{y^5} = \frac{5x^8 3}{2n^2} + Cn^5$$

$$\frac{1}{y^5} = \frac{5 \cdot n^3}{2} + Cn^5$$

which is the required solution

————— \* —————

Method: Differential equations reducible to linear equation  
by substitution method.

② Solve  $\frac{dy}{dx} + x \sin y = x^3 \cos^2 y$ .

Sol Given eqn is  $\frac{dy}{dx} + x \sin y = x^3 \cos^2 y \rightarrow ①$

This can be written as  $\frac{1}{\cos^2 y} \cdot \frac{dy}{dx} + \frac{x \cdot 2 \sin y \cos y}{\cos^2 y} = x^3$

$$\sec^2 y \cdot \frac{dy}{dx} + x \cdot 2 \tan y = x^3$$

$$\sec^2 y \cdot \frac{dy}{dx} + 2x \tan y = x^3 \rightarrow ②$$

put  $\tan y = u$  so that  $\sec^2 y \cdot \frac{dy}{dx} = \frac{du}{dx}$

Substituting in eq ②, we get  $\frac{du}{dx} + 2u \cdot x = x^3$

This is a linear eqn in  $u$ .

$$\text{Now I.F.} = e^{\int 2x dx} - e^{\int 2x dx} = e^{2 \int x dx} = e^{2 \left[ \frac{x^2}{2} \right]} = e^{x^2}$$

$$\text{I.F.} = e^{x^2}$$

∴ General solution is given by.

$$u \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + C$$

$$u \cdot e^{x^2} = \int x^3 e^{x^2} dx + C$$

put  $x^2 = t$  so that  $2x dx = dt \Rightarrow x dx = \frac{dt}{2}$

$$u e^{x^2} = \int x^2 \cdot e^{x^2} \cdot x dx + C$$

$$u e^{x^2} = \int t \cdot e^t \cdot \frac{dt}{2} + C$$

$$u e^{x^2} = \frac{1}{2} \int t e^t dt + C$$

$$u e^{x^2} = \frac{1}{2} [t e^t - e^t] + C$$

$$u e^{x^2} = \frac{1}{2} e^t (t-1) + C$$

$$\tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + C$$

This is the general solution of eq ①

## Newton's Law of cooling

Statement :-

The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

Let  $\theta$  be the temperature of the body at time  $t$ .

and  $\theta_0$  be the temperature of its surrounding medium (usually air)

By the Newton's law of cooling, we have

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

i.e.  $\frac{d\theta}{dt} = -K(\theta - \theta_0)$  where  $K$  is a positive constant

$$\frac{d\theta}{\theta - \theta_0} = -K dt \quad (\text{variable separable})$$

Integrating on both sides we get

$$\int \frac{1}{\theta - \theta_0} d\theta = \int -K dt$$

$$\log(\theta - \theta_0) = -Kt + C \rightarrow ①$$

If initially  $\theta = \theta_1$  is the temperature of the body at time  $t=0$ , then eq ① gives

$$\log(\theta_1 - \theta_0) = 0 + C \Rightarrow C = \log(\theta_1 - \theta_0) \rightarrow ②$$

Substituting eq ② in eq ① we get

$$\log(\theta - \theta_0) = -Kt + \log(\theta_1 - \theta_0)$$

$$\log(\theta - \theta_0) - \log(\theta_1 - \theta_0) = -Kt$$

$$\log\left(\frac{\theta - \theta_0}{\theta_1 - \theta_0}\right) = -Kt$$

$$\frac{\theta - \theta_0}{\theta_1 - \theta_0} = e^{-Kt} \Rightarrow \theta - \theta_0 = (\theta_1 - \theta_0)e^{-Kt}$$

$$\theta = \theta_0 + (\theta_1 - \theta_0)e^{-Kt}$$

which gives the temperature of the body at any time  $t$ .

Ques. A body is originally at  $80^{\circ}\text{C}$  and cools down to  $60^{\circ}\text{C}$  in 20 minutes. If the temperature of the air is  $40^{\circ}\text{C}$ . find the temperature of the body after 40 minutes.

Sol: Let  $\theta$  be the temperature of the body at time  $t$ .  
and  $\theta_0$  be the temperature of the air ( $\therefore \theta_0 = 40^{\circ}\text{C}$ )

By Newton's law cooling, we have .

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - 40)$$

$$\frac{d\theta}{\theta - 40} = -Kdt$$

Integrating on both sides, we get-

$$\int \frac{1}{\theta - 40} d\theta = -K \int dt$$

$$\log(\theta - 40) = -Kt + \log c \Rightarrow \log\left(\frac{\theta - 40}{c}\right) = -Kt$$

$$\frac{\theta - 40}{c} = e^{-Kt}$$

$$\theta - 40 = ce^{-Kt} \longrightarrow ①$$

$$\text{When } t=0, \theta=80^{\circ}\text{C} \Rightarrow 80-40=c e^{-K(0)}$$

$$40=c(1)$$

$$\boxed{c=40}$$

$$\text{and when } t=20, \theta=60^{\circ}\text{C} \Rightarrow \text{of } ① \Rightarrow 60-40=c e^{-20K}$$

$$20=c e^{-20K}$$

$$e^{-20K} = \frac{20}{c}$$

$$e^{-20K} = \frac{20}{40} \Rightarrow e^{-20K} = \frac{1}{2}$$

$$-20K = \log \frac{1}{2}$$

$$-20K = \log 1 - \log 2$$

$$-20K = 0 - \log 2$$

$$20K = \log 2$$

$$K = \frac{1}{20} \log 2$$

ef ① becomes  $\theta - u_0 = u_0 e^{[\frac{1}{20} \log 2]t}$

$$\theta = u_0 + u_0 e^{(\frac{1}{20} \log 2)t} \rightarrow ②$$

when  $t = 40 \Rightarrow$  ef ②  $\rightarrow \theta = u_0 + u_0 e^{-(\frac{1}{20} \log 2)40}$

$$\theta = u_0 + u_0 e^{-2 \log 2}$$

$$\theta = u_0 + u_0 e^{\log 2^{-2}} \quad \left. \begin{array}{l} \\ \end{array} \right\} e^{\log a} = a$$

$$\theta = u_0 + u_0 \cdot 2^{-2}$$

$$\theta = u_0 + u_0 \left( \frac{1}{4} \right)$$

$$\boxed{\theta = 50^\circ C}$$

Hence the temperature of the body after 40 minutes is  $50^\circ C$ .

- ② If the air temperature of the air is  $20^\circ C$  and the temperature of the body drops from  $100^\circ C$  to  $80^\circ C$  in 10 minutes, what will be its temperature after 20 minutes, when will be the temperature  $40^\circ C$ .

Sol Let  $\theta$  be the temperature of the body at time  $t$ .  
and  $\theta_0$  be the temperature of the air ( $\theta_0 = 20^\circ C$ )  
By Newton's law of cooling.

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

(23)

$$\frac{d\theta}{\theta - \theta_0} = -K dt$$

$$\frac{d\theta}{\theta - 20} = -K dt$$

$$\int \frac{d\theta}{\theta - 20} = -K \int dt$$

$$\log(\theta - 20) = -Kt + \log C$$

$$\log(\theta - 20) - \log C = -Kt$$

$$\log\left(\frac{\theta - 20}{C}\right) = -Kt$$

$$\frac{\theta - 20}{C} = e^{-Kt}$$

$$\theta - 20 = Ce^{-Kt} \rightarrow ①$$

originally when  $t=0$ , temperature  $\theta = 100^\circ C$   
 if ①  $\Rightarrow 100 - 20 = Ce^{-K(0)}$

$$80 = Ce^0$$

$$\boxed{C = 80}$$

$$\text{when } t=10 \text{ min}, \theta = 80^\circ C \Rightarrow \text{if } ① \rightarrow 80 - 20 = Ce^{-K(10)}$$

$$60 = Ce^{-K(10)}$$

$$60 = 80(e^{-K(10)})$$

$$e^{-K(10)} = \frac{60}{80} = \frac{3}{4}$$

$$\boxed{e^{-10K} = \frac{3}{4}} \rightarrow ③$$

$$\text{when } t=20 \text{ min} \rightarrow \text{if } ③ \rightarrow \theta - 20 = Ce^{-K(20)}$$

$$\theta - 20 = Ce^{-20K}$$

$$\theta - 20 = 80(e^{-10K})^2$$

$$\theta - 20 = 80(3/4)^2$$

$$\theta = 80 \cdot \left(\frac{9}{16}\right) + 20$$

$$\theta = 45 + 20$$

$$\boxed{\theta = 65^\circ C}$$

Thus the temperature of the body will be  $65^{\circ}\text{C}$  after 20 minutes

$$\text{when } \theta = 40^{\circ}\text{C} \Rightarrow \text{ef(2)} \Rightarrow 40 - 20 = Ce^{-kt}$$
$$20 = 80e^{-kt}$$

$$\frac{20}{80} = (e^{-k})^t$$

$$\frac{1}{4} = (e^{-k})^t \rightarrow \text{ef(3)}$$

from ef(2) and ef(3), we have

$$\text{ef(3)} = e^{-10k} = \frac{1}{4} \Rightarrow (e^{-k})^{10} = (\frac{1}{4})^{1/10}$$

$$\boxed{(e^{-k})^{10}} \quad (e^{-k})^t = \frac{1}{4}$$

$$((\frac{1}{4})^{1/10})^t = \frac{1}{4}$$

$$(\frac{1}{4})^{t/10} = \frac{1}{4}$$

Taking log on both sides

$$\log(\frac{1}{4})^{t/10} = \log \frac{1}{4}$$

$$\pm \frac{t}{10} \log(\frac{1}{4}) = \log(\frac{1}{4})$$

$$t = \frac{10 \log(\frac{1}{4})}{\log(\frac{1}{4})}$$

$$\boxed{t = 48.2 \text{ minutes}}$$

The temperature of the body will be  $40^{\circ}\text{C}$  after 48.2 minutes.

QMP

21

22

- ① The rate of cooling of an object is proportional to the difference b/wn the temperature of the object and the surrounding air. If the air temperature is  $30^\circ\text{C}$  and the water at a temperature  $100^\circ\text{C}$  cools at  $80^\circ\text{C}$  in 10 minutes, find when the temperature of water will become  $40^\circ\text{C}$ .

Sol: Let  $\theta$  be the temperature of the water at time  $t$ .  
and  $\theta_0$  be the temperature of air ( $\theta_0 = 30^\circ\text{C}$ )

By Newton's law of cooling

$$\frac{d\theta}{dt} = -K(\theta - \theta_0)$$

$$\frac{d\theta}{dt} = -K(\theta - 30)$$

$$\frac{d\theta}{\theta - 30} = -Kdt \quad \rightarrow ①$$

Integrating on both sides, we get

$$\int \frac{d\theta}{\theta - 30} = -K \int dt$$

$$\log(\theta - 30) = -Kt + \log C$$

$$\log\left(\frac{\theta - 30}{C}\right) = -Kt$$

$$\frac{\theta - 30}{C} = e^{-Kt}$$

$$\theta - 30 = Ce^{-Kt} \quad \rightarrow ②$$

$$\text{when } t = 0, \theta = 100^\circ \Rightarrow 100 - 30 = Ce^{-K(0)}$$

$$70 = C(1)$$

$$C = 70^\circ$$

when  $t = 10 \text{ minutes}, \theta = 80^\circ\text{C}$

$$\text{ef } ② \Rightarrow 80 - 30 = Ce^{-K(10)}$$

GPO

$$50 = C e^{-K10}$$

$$C e^{-K10} = 50$$

$$e^{-K10} = \frac{50}{70}$$

$$e^{-K10} = \frac{5}{7} \Rightarrow (e^{-K})^{10} = \frac{5}{7}$$

$$\boxed{-K10 = \log\left(\frac{5}{7}\right)}$$
$$K = -\frac{1}{10} \log\left(\frac{5}{7}\right)$$
$$\text{then } K = \log\left(\frac{5}{7}\right)^{1/10}$$

$$e^{-K} = \left(\frac{5}{7}\right)^{1/10}$$

$$\text{ef (2) becomes } \theta - 30 = C e^{-Kt}$$

when  $\theta = 40^\circ\text{C}$

$$40 - 30 = C \left(\frac{5}{7}\right)^{t/10}$$

$$10 = 70 \left(\frac{5}{7}\right)^{t/10}$$

$$\frac{1}{7} = \left(\frac{5}{7}\right)^{t/10}$$

$$\boxed{\left(\frac{1}{7}\right)^{10} = \left(\frac{5}{7}\right)^t} \text{ taking log on both sides.}$$

$$\log\left(\frac{1}{7}\right) = \log\left(\frac{5}{7}\right)^{t/10}$$

$$\log 1 - \log 7 = \frac{t}{10} \log\left(\frac{5}{7}\right)$$

$$\frac{10(\log \frac{1}{7})}{\log(5/7)} = t$$

$$\frac{-8.4509}{-0.1461} = t$$

$$t = 57.84$$

$$-t = 57.84 \text{ minutes}$$

$\therefore$  The Temperature of water will be  $40^\circ\text{C}$  after 57.84 minutes

$$50 = Ce^{-Kt}$$

$$Ce^{-10K} = 50$$

$$e^{-10K} = \frac{50}{C} \rightarrow \frac{50}{70}$$

$$e^{-10K} = \frac{5}{7}$$

$$(e^K)^{10} = \frac{7}{5}$$

$$(e^K) = \left(\frac{7}{5}\right)^{1/10} \longrightarrow ③$$

$$\text{eq } ② \Rightarrow \theta - 30 = Ce^{-Kt}$$

when  $\theta = 40^\circ C$

$$40 - 30 = Ce^{-Kt}$$

$$10 = 70(e^{-K})^t$$

$$\frac{10}{70} = \left(\left(\frac{7}{5}\right)^{1/10}\right)^t$$

$$\left(\frac{5}{7}\right)^{t/10} = \frac{1}{7}$$

Taking log on both sides we get

$$\log\left(\frac{5}{7}\right)^{t/10} = \log\frac{1}{7}$$

$$\frac{t}{10} \log\left(\frac{5}{7}\right) = \log\left(\frac{1}{7}\right)$$

$$\frac{t}{10} = \frac{\log\left(\frac{1}{7}\right)}{\log\left(\frac{5}{7}\right)}$$

$$t = \frac{10 \cdot \log\left(\frac{1}{7}\right)}{\log\left(\frac{5}{7}\right)}$$

$$t = 57.84 \text{ minutes}$$

∴ The temperature of water will be  $40^\circ C$  after  
57.84 min

① Method 2:

If  $M(x,y)dx + N(x,y)dy = 0$  is a homo

Date: 18-5-2022

① D, E, F, A, B, C

E, F, A, B, C, D, G, H, I, J, K, L

F, O, I, L, K, B, C, D, E, G, H, J, K, L

G, B, I, L, K, A, C, D, E, F, H, J, M, N

H, O, I, L, K, B, C, D, E, F, G, H, J, M, N

I, P, I, L, K, B, C, D, E, F, G, H, J, M, N

J, P, I, L, K, B, C, D, E, F, G, H, M, N

201, 202, 203, 204, 205, 206, 207, 208

201, 202, 203, 204, 205, 206, 207, 208

(34)

Date: 19-5-2022.

D, E, F, G, H, I, J, K, L

E, D, F, I, L, K, B, C, D, G, H, A

F, O, I, L, K, B, C, D, E, G, H, J, M, N

G, B, I, L, K, A, C, D, E, F, H, J, M, N

H, P, I, L, K, B, C, D, E, F, G, H, J, M, N

J, P, I, L, K, B, C, D, E, F, G, H, M, N

Direct P

K, P, I, L, K, B, C, D, E, F, G, H, M, N

201, 202, 203, 204, 205, 206, 207, 208

## Rate of Decay of RadioActive materials

(26)

The disintegration at any instant is proportional to the amount of material present.

If  $m$  is the amount of the material at any time  $t$ ,

then  $\frac{dm}{dt} = -km$ , where  $k$  is a constant.

**Q1** If 30% of a radio active substance disappears in 10 days, how long will it take for 90% of it to disappear?

**Sol** The differential equation of the diffusing radio active material is.  $\frac{dm}{dt} = -km \rightarrow (1)$

$$\frac{dm}{m} = -kdt$$

Integrating on both sides

$$\int \frac{dm}{m} = -k \int dt$$

$$\log m = -kt + \log C$$

$$\log m - \log C = -kt$$

$$\log(\frac{m}{C}) = -kt$$

$$\frac{m}{C} = e^{-kt}$$

$$m = C e^{-kt} \rightarrow (2)$$

when  $t=0$ , let  $m=m_1$ ,

$$\Rightarrow \text{if } (2) \Rightarrow m_1 = C e^{-0K}$$

$$\boxed{m_1 = C}$$

By Given data ~~at~~ when  $t=10$ ,  $m = \frac{70m_1}{100}$

$$\text{if } (2) \Rightarrow \frac{70m_1}{100} = m_1 e^{-10K}$$

$$e^{10K} = \frac{7}{10} \Rightarrow -10K = \log(\frac{7}{10})$$

$$-10K = \log\left(\frac{7}{10}\right)$$

$$K = -\frac{1}{10} \log\left(\frac{7}{10}\right)$$

$$K = \frac{1}{10} \log\left(\frac{10}{7}\right)^{-1}$$

$$K = \frac{1}{10} \log\left(\frac{10}{7}\right)$$

Required time at  $t$  is

when  $m_F = \frac{10m_i}{100}$

$$\text{Q2} \quad \frac{10m_i}{100} = ce^{-Kt}$$

$$\frac{10m_i}{100} = m_i e^{-Kt}$$

$$\frac{1}{10} = e^{-Kt}$$

$$e^{-Kt} = \frac{1}{10}$$

$$-Kt = \log\left(\frac{1}{10}\right)$$

$$t = -\frac{1}{K} \log\left(\frac{1}{10}\right)$$

$$t = \frac{1}{K} \log\left(\frac{10}{7}\right)^{-1}$$

$$t = \frac{1}{K} \log(10)$$

$$t = \frac{\log(10)}{\frac{1}{10} \log\left(\frac{10}{7}\right)}$$

$$t = \frac{10 \cdot \log(10)}{\log\left(\frac{10}{7}\right)}$$

$$t = 64.5 \text{ days}$$

===== \* -----

## (20)

### Law of Natural Growth or Decay

Let  $x(t)$  be the amount of a substance at time  $t$ .  
 and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount  $x(t)$  of a chemically changing substance is proportional to the amount of the substance at that time.

$$\frac{dx}{dt} \propto x$$

$$\text{i.e. } \frac{dn}{dt} = kn \text{ where } k \text{ is a constant of proportionality}$$

$\Rightarrow$  For growth  $\frac{dn}{dt}$  should be positive.

$$\therefore \frac{dn}{dt} = kn, \text{ where } k > 0$$

$\Rightarrow$  For decay,  $\frac{dn}{dt}$  should be Negative.

$$\therefore \frac{dn}{dt} = -kn, \text{ where } k > 0$$

- ① A bacterial culture, growing exponentially, increases from 200 to 500 grams in the period from 6 a.m. to 9 a.m. How many grams will be present at noon.

Sol Let  $N$  be the number of bacteria in a culture at any time  $t > 0$ , then

The differential eqn is  $\frac{dN}{dt} = KN$

$$\frac{dN}{N} = K dt$$

Integrating on both sides, we get -

$$\int \frac{dN}{N} = \int k dt$$

$$\log N = kt + \log c$$

$$\log N - \log c = kt$$

$$\log\left(\frac{N}{c}\right) = kt$$

$$\frac{N}{c} = e^{kt}$$

$N = ce^{kt}$  where  $c$  is a const  
 $\hookrightarrow$  ① and  $k$ , the rate constant

Given that  $N = 200$  grams when  $t=0$

$$200 = c e^{+k(0)}$$

$$C = 200$$

Thus, we have  $N = 200 e^{kt}$  → ②

But when  $t=3$  hours (from 6am to 9am)  $N = 500$  grams  
 Substituting in eq ②, we get

$$500 = 200 e^{+3k}$$

$$e^{3k} = \frac{500}{200} \Rightarrow e^k = 5^{1/2}$$

$$3k = \log(5^{1/2}) \text{ or } k = \frac{1}{3} \log(5^{1/2})$$

$$K = 0.3054$$

Hence the number of bacteria in the culture at any instant of time  $t > 0$  is given by

$$N = 200 e^{0.3054t}$$

when  $t=6$  hours (from 6am to 12am)

∴ After 3 hours, the no. of bacteria present will be

$$N = 200 e^{0.3054(6)}$$

$$N = 1249.8 \text{ grams.}$$

= \* =

## 2. Differential Equations of first order

But not of the first Degree

(28)

### Equations solvable for P

Def: The differential efn will involve  $\frac{dy}{dx}$  in higher degree and  $\frac{dy}{dx}$  is denoted by 'P'. So the diff efn is of the form  $f(x, y, P) = 0$  is called first order non-linear diff efn such efn's can be solved by following methods.

- 1) Equation Solvable for P.
- 2) Equation Solvable for y
- 3) Equation Solvable for x.

### problems, Equation Solvable for P

$$\textcircled{1} \quad P^2 - 5P + 6 = 0, \text{ where } P = \frac{dy}{dx}$$

Sol. Given efn  $P^2 - 5P + 6 = 0$

$$P^2 - 3P - 2P + 6 = 0$$

$$P(P-3) - 2(P-3) = 0$$

$$(P-2)(P-3) = 0$$

$$P=2 \text{ and } P=3$$

$$\text{so } \frac{dy}{dx} = 2 \text{ and } P = \frac{dy}{dx} = 3$$

$$dy = 2dx \text{ and } dy = 3dx$$

Integrating,  $\int dy = \int 2dx$  and  $\int dy = \int 3dx$

$$y = 2x + C \text{ and } y = 3x + C$$

$$y - 2x - C = 0, \quad y - 3x - C = 0$$

$\therefore$  The required solution is  $(y - 2x - C)(y - 3x - C) = 0$

Solve  
 Q)  $P(P+y) = n(n+y)$

Sol: Given diff eqn w/  $P(P+y) = n(n+y)$

$$P^2 + Py = n^2 + ny$$

$$P^2 + Py - n^2 - ny = 0$$

$$(P^2 - n^2) + Py - ny = 0$$

$$(P+n)(P-n) + y(P-n) = 0$$

$$(P-n)[P+n+y] = 0$$

its components are  $P-n=0$  and  $P+n+y=0$

$$P=n \text{ and } P+y=-n$$

$$\frac{dy}{dn} = n \text{ and } \frac{dy}{dn} + y = -n$$

①

②

ef ①  $\frac{dy}{dn} = n \Rightarrow dy = n dn$

Integrating,  $\int dy = \int n dn$

$$y = \frac{n^2}{2} + C$$

$$2y = n^2 + 2C$$

$$2y - n^2 = C$$

$$(2y - n^2 - C) = 0$$

ef ② This is of the form

$$\frac{dy}{dn} + P(n)y = Q(n)$$

$$I.F = e^{\int P(n)dn} = e^{\int 1 dn} = e^n$$

Solution is

$$y \cdot (I.F) = \int Q \cdot (I.F) dn + C$$

$$y \cdot e^n = \int (-n) e^n dn + C$$

$$y e^n = - \int n e^n dn + C$$

$$y e^n = - [n e^n - e^n] + C$$

$$y e^n + n e^n - e^n - C = 0$$

$$e^n (y + n - 1) - C = 0$$

$$(y + n - 1) - C e^{-n} = 0$$

Hence the general solution is

$$(2y - n^2 - C)(y + n - 1 - C e^{-n}) = 0$$

$$\textcircled{1} \quad \text{Solve } P^2 + 2Py \cot n = y^2$$

Sol: Given  $P^2 + 2Py \cot n = y^2$

$$\textcircled{2} \quad P^2 + (2y \cot n)P - y^2 = 0, \text{ which is quadratic in } P$$

$$\text{Here } a=1, b=2y \cot n, c=-y^2$$

Solving it for P, we get

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2y \cot n \pm \sqrt{4y^2 \cot^2 n - 4 \cdot 1 \cdot (-y^2)}}{2(1)}$$

$$P = \frac{-2y \cot n \pm \sqrt{4y^2(\cot^2 n) + 4y^2}}{2} \quad \{ \cot^2 n + 1 = \cosec^2 n \}$$

$$P = \frac{-2y \cot n \pm \sqrt{4y^2(\cot^2 n + 1)}}{2} = \frac{-2y \cot n \pm 2y \sqrt{\cosec^2 n}}{2}$$

$$P = -y \cot n \pm y \cosec n \quad P = \frac{-2y \cot n \pm 2y \cosec n}{2}$$

$$P = \frac{[-y \cot n \pm y \cosec n]}{2}$$

$$P = -y \cot n \pm y \cosec n$$

Its components are

$$P = -y \cot n + y \cosec n \quad \text{and} \quad P = -y \cot n - y \cosec n$$

$$P = -y(\cot n - \cosec n) \quad \text{and} \quad P = -y(\cot n + \cosec n)$$

①    ②

from eq ①, we have

$$P = \frac{dy}{dn} = -y \left[ \frac{\cos n}{\sin n} - \frac{1}{\sin n} \right] \dagger$$

$$\frac{dy}{y} = - \left[ \frac{\cos n - 1}{\sin n} \right] dn$$

$$\frac{dy}{y} = \left[ \frac{1 - \cos n}{\sin n} \right] dn \Rightarrow \frac{dy}{y} = \left[ \frac{2 \sin^2(n/2)}{2 \sin(n/2) \cos(n/2)} \right] dn$$

$$\begin{cases} 1 - \cos \theta = 2 \sin^2 \theta/2 \\ \text{and } \sin \theta = 2 \sin \theta/2 \cos \theta/2 \end{cases}$$

$$\frac{dy}{y} = \tan(\pi/2) dn$$

Integrating,  $\int \frac{1}{y} dy = \int \tan(\pi/2) dn$        $\left\{ \begin{array}{l} \text{Stammfn} = \log(\sec n) + c \\ \text{Stammfn} = \log(\sec n) + c \end{array} \right.$

$$\log y = \frac{\log(\sec \pi/2)}{\pi/2} + \log c \quad \left\{ \begin{array}{l} \text{Stammfn} = \log(\sec n) + c \\ \text{Stammfn} = \log(\sec n) + c \end{array} \right.$$

$$\log y = 2 \log(\sec \pi/2) + \log c$$

$$\log y = \log(\sec \pi/2)^2 + \log c$$

$$\log y = \log \sec^2 \pi/2 \cdot c$$

$$y = c \cdot \sec^2 \pi/2 \quad \rightarrow ③$$

From eq ②, we have

$$\frac{dy}{dn} = -y [\cot n + \operatorname{cosec} n]$$

$$\frac{dy}{y} = - \left[ \frac{\cos n}{\sin n} + \frac{1}{\sin n} \right] dn$$

$$\frac{dy}{y} = - \left[ \frac{\cos n + 1}{\sin n} \right] dn \quad \left\{ \begin{array}{l} \cos \theta + 1 = 2 \cos^2 \theta/2 \\ \sin \theta = 2 \sin \theta/2 \cos \theta/2 \end{array} \right.$$

$$\frac{dy}{y} = \left[ - \frac{2 \cos^2 \pi/2}{2 \sin \pi/2 \cos \pi/2} \right] dn$$

$$\frac{dy}{y} = -\cot(\pi/2) dn$$

Integrating,  $\int \frac{1}{y} dy = - \int \cot \pi/2 dn \quad \left\{ \begin{array}{l} \text{cotandn} = \log \frac{\sin n}{a} + c \\ \text{cotandn} = \log \frac{\sin n}{a} + c \end{array} \right.$

$$\log y = -\frac{\log(\sin \pi/2)}{\pi/2} + \log c$$

$$\log y = -2 \log \sin \pi/2 + \log c$$

$$\log y = \log [\sin \pi/2]^2 + \log c$$

$$\log y = \log [\csc \pi/2]^2 + \log c$$

$$\log y = \log c \cdot \csc^2 \pi/2$$

$$y = c \cdot \csc^2 \pi/2 \quad \rightarrow ④$$

Equation ③ and ④ constitute the required solution.

20

(30)

The required solution can be written as

$$\boxed{(y - c \sec^2 \alpha/2)(y - c \cdot \csc^2 \alpha/2) = 0}$$

① Solve :  $(P^2 - 7P + 12) = 0$

② Solve :  $YP^2 + (m-y)P - n = 0$

Given problem is

South Italy

at  $t=0$ ,  $N_0$  is the originally Substance consider  
digitally 100% we have

then disappear 30% in how many day

$t = 10 \text{ day}$  at  $\frac{3}{10}$  the obviously 70% N

$t = ?$  10% N is available.

$$N = Ce^{-kt}$$

$$N = Ce^0$$

$$\boxed{C = N_0}$$

101, 103, 105, 1303, 2, 7,

(31)

$$\textcircled{1} \quad \text{Solve } \frac{dy}{dx} + \frac{y}{x} = x^3 - 3.$$

This is of the form  $\frac{dy}{dx} + Py = Q$

here  $P = \frac{1}{x}$ , and  $Q = x^3 - 3$ .

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$I.F = x. \quad \boxed{R^{\log x} = a}$$

The general solution is given by

$$y(I.F) = \int Q \cdot (I.F) \cdot dx + C.$$

$$yx = \int (x^3 - 3) x dx + C.$$

$$yx = \int [x^4 - 3x] dx + C.$$

$$= \int x^4 dx - 3 \int x^2 dx + C$$

$$= \frac{x^{4+1}}{4+1} - 3 \cdot \frac{x^{2+1}}{2+1} + C \quad \boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C}$$

$$\boxed{yx = \frac{x^5}{5} - \frac{3x^3}{3} + C}$$

This is the general solution of the given eqn.

Ans

① solve  $x \frac{dy}{dx} + y = \log n \Rightarrow xy = x(\log n) - x + C$

②  $(x+2y^3) \frac{dy}{dx} = y \Rightarrow I.F = e^{\int \frac{1}{x+2y^3} dx}$

③  $\frac{dy}{dx} + y \sec x = \tan x \Rightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$

① solve  $x \cdot \frac{dy}{dx} + y = \log n \Leftrightarrow xy = x \log n - x + C$

②  $\frac{dy}{dx} + y \sec x = \tan x \Leftrightarrow y(\sec x + \tan x) = \sec x + \tan x - x + C$

③ ~~solved the I.F for the first~~  $(x+2y^3) \frac{dy}{dx} = y \Leftrightarrow I.F = \frac{1}{y}$

$$(x^2 + y^2 + 2z)dx + 2y dy = 0 \rightarrow \textcircled{1}$$

(32)

$$M = x^2 + y^2 + 2z \quad N = 2y$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\text{Consider } \frac{1}{N} \left[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right] = \frac{1}{2y} [2y - 0] = \frac{2y}{2y} = 1 = \text{f.m.}$$

$$I.F. = e^{\int p(x) dx} = e^{\int 1 dx} = e^x$$

Multiplying eq \textcircled{1} with  $e^x$

$$(e^x x^2 + e^x y^2 + e^x 2z) dx + e^x 2y dy = 0$$

$$M_1 dx + N_1 dy = 0$$

$$M_1 = e^x x^2 + e^x y^2 + e^x 2z$$

$$\frac{\partial M_1}{\partial y} = e^x (2y) = 2ye^x$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$$N_1 = e^x 2y$$

$$\frac{\partial N_1}{\partial x} = 2ye^x$$

General soln

$$\textcircled{1} \quad \text{Solve } xy(1+xy^2) \frac{dy}{dx} = 1.$$

Sol: Given equation is  $[xy + x^2y^3] \frac{dy}{dx} = 1$ .

This is can be written as

$$xy + x^2y^3 = \frac{dx}{dy}$$

$$\frac{dx}{dy} - xy = x^2y^3 \rightarrow \textcircled{1}$$

This is a Bernoulli's eqn in x.

and dividing with  $x^2$ , we get-

$$\frac{1}{x^2} \frac{dx}{dy} - \frac{xy}{x^2} = \frac{x^2y^3}{x^2}$$

$$\frac{1}{x^2} \frac{dx}{dy} - \frac{y}{x} = y^3 \rightarrow \textcircled{2}$$

$$\text{put } \frac{1}{x} = u \text{ so that } -\frac{1}{x^2} \cdot \frac{dx}{dy} = \frac{du}{dy}$$

Substituting in eq \textcircled{2}, we get-

$$-\frac{du}{dy} - u \cdot y = y^3$$

$$\frac{du}{dy} + uy = -y^3$$

This is a linear equation in u, here  $P = y$ ,  $Q = -y^3$

$$\text{Now I.F.} = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}$$

The general solution is given by

$$u \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) \cdot dy + C$$

$$u \cdot e^{y^2/2} = \int -y^3 \cdot e^{y^2/2} dy + C$$

put

$$\left\{ \begin{array}{l} - \int y^3 e^{y^2/2} dy \\ - \left[ \frac{y^3}{3} e^{y^2/2} \right] \end{array} \right. \quad \begin{array}{l} \text{put } y^2 = t \\ \& dy = dt \\ \& y dy = \frac{dt}{2} \end{array}$$

$$\begin{aligned} & - \left[ y^2 e^{y^2/2} \cdot \frac{dt}{2} \right] \\ & - \frac{1}{2} \left[ t e^{t/2} dt \right] \\ & - \frac{1}{2} \left[ t e^{t/2} - 1 \cdot \frac{e^{t/2}}{t/2} \right] \\ & = -\frac{1}{2} \left[ 2t - 1 \right] e^{t/2} \\ & = [2t - 1] e^{t/2} \end{aligned}$$

$$\textcircled{1} \text{ Solve } xy[1+ny^2] \frac{dy}{dx} = 1$$

Sol Given efn is  $[xy + n^2 y^3] \frac{dy}{dx} = 1 \quad \text{--- } \textcircled{1}$

This is can be written as  $xy + n^2 y^3 = \frac{dx}{dy}$

$$\frac{dx}{dy} - ny = n^2 y^3$$

This is a Bernoulli's efn in x

and dividing with  $n^2$ , we get-

$$\frac{1}{n^2} \frac{dx}{dy} - \frac{ny}{n^2} = \frac{n^2 y^3}{n^2}$$

$$\frac{1}{n^2} \frac{dx}{dy} - y/x = y^3 \quad \text{--- } \textcircled{2}$$

put  $1/x = u$  so that  $-1/n^2 \frac{du}{dy} = \frac{dy}{dx}$

Substituting in of  $\textcircled{2}$ , we get-

$$-\frac{du}{dy} - u \cdot y = y^3$$

$$\frac{du}{dy} + uy = -y^3$$

This is a linear equation in u, Here  $P = y$ ,  $Q = -y^3$

$$\text{Now I.F} = e^{\int P dy} = e^{\int y dy} = e^{y^2/2}$$

The general solution is given by.

$$u(I.F) = \int Q(I.F) dy + C$$

$$u \cdot e^{y^2/2} = \int (C - y^3) e^{y^2/2} dy + C$$

put  $y^2 = t$  so that  $2y dy = dt \Rightarrow y dy = t/2 dt$

$$u \cdot e^{y^2/2} = - \int y^2 \cdot y \cdot e^{y^2/2} dy + C$$

$$u \cdot e^{y^2/2} = - \int y^2 e^{y^2/2} \cdot y dy + C$$

$$ue^{\frac{y^2}{t_2}} = - \int t e^{\frac{t}{t_2}} dt + c$$

$$ue^{\frac{y^2}{t_2}} = -\frac{1}{2} \int t e^{\frac{t}{t_2}} dt + c. \quad \left\{ \begin{array}{l} \text{using Bernoulli's rule} \\ \int u du = uv - u v_1 + u v_2 - \end{array} \right.$$

$$= -\frac{1}{2} \left[ t \cdot \frac{e^{\frac{t}{t_2}}}{\frac{1}{t_2}} - 1 \cdot \frac{e^{\frac{t}{t_2}}}{t_2} \right] + c \quad \left\{ \begin{array}{l} \int e^{ax} dx = \frac{e^{ax}}{a} + c \\ \end{array} \right.$$

$$= -\frac{1}{2} [2t e^{\frac{t}{t_2}} - ue^{\frac{t}{t_2}}] + c$$

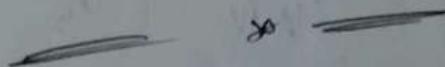
$$= -\frac{2e^{\frac{t}{t_2}}}{2} [t - 2] + c$$

$$= -e^{\frac{t}{t_2}} (t - 2) + c$$

$$ue^{\frac{y^2}{t_2}} = e^{\frac{t}{t_2}} (2 - t) + c$$

$$\frac{1}{t_2} e^{\frac{y^2}{t_2}} = e^{\frac{t}{t_2}} (2 - y^2) + c \quad \left\{ \begin{array}{l} t = y^2 \\ u = \frac{1}{t_2} \end{array} \right\}$$

which is the general solution of eq ①



- ① Solve  $(x^3 + 3xy^2)dx + (y^3 + 3x^2y)dy = 0$
- ② Solve the differential eq?  $y(xy + e^x)dx - e^x dy = 0$
- ③ Solve  $[xy \sin xy + \cos xy] y dx + (xy \sin xy - \cos xy) x dy = 0$
- ④ Solve  $(1+y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$
- ⑤ Solve  $\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$
- ⑥ A body is originally at  $80^\circ\text{C}$  and cools down to  $60^\circ\text{C}$  in 20 minutes. If the temperature of the air is  $40^\circ\text{C}$ , then find the temperature of the body after 40 minutes.
- ⑦ If 20% of a radioactive substance disappears in 10 days, how long will it take for 90% of it to disappear?
- ⑧ If the air temperature is  $30^\circ\text{C}$  and the water at a temperature  $100^\circ\text{C}$  cools to  $80^\circ\text{C}$  in 10 minutes, find, when the temperature of water will become  $40^\circ\text{C}$ .
- ⑨ Solve  $P^2 + 2Py \cot x = y^2$  for  $P = \frac{dy}{dx}$ .
- ⑩ Form the differential equation of LR CIRCUIT
- ⑪ A resistance of 100 ohms, an inductance 0.5 Henry is connected in series with a battery of 20 volts. Find the current in the circuit if initially there is no current in the circuit.
- ⑫ An R-L circuit has an Emf given (in volts) by  $10 \sin t$ . A resistance of 90 ohms, an inductance of 4 henries. Find the current at any time  $t$  by assuming zero initial current.