GATE PEDIA

MECHANICAL ENGINEERING

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Fluid Mechanics

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INDEX

1 Fluid & Its Properties

1.1 Introduction

1.1.1 Fluid

Fluid is the phase of substance which can't resist any external shear force in static condition.
Fluid ⇒ Continuous deformation under the action of shear force.

$$\tau \propto \frac{d\theta}{dt}$$

Here $\frac{d\theta}{dt}$ is deformation rate (or) shear strain rate during steady state.

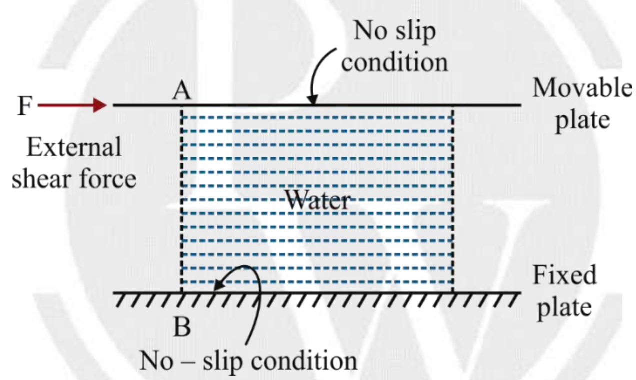


Fig. 1.1 Fluid between two parallel plates (Fixed and Movable)

1.1.2 Fluid Properties

Mass Density / Density:

Density is defined as the mass per unit volume.

Mathematically

Density $(\rho) = \frac{\text{mass (m)}}{\text{Volume(V)}}$

On Increasing Temperature ⇒ Density decreases.

On Increasing Pressure ⇒ Density increases.

Units:

(i) kg/m^3,

(ii) gm/cm^3

> Note:
>
> $$1\frac{gm}{cm^3} = 10^3\frac{kg}{m^3}$$

Dimensional formula: $[\rho] = [ML^{-3}]$

Specific volume:

Specific volume is defined as the volume per unit mass.

Mathematically

$$\text{Specific volume, } (v) = \frac{\text{Volume }(V)}{\text{mass}(m)} = \frac{1}{\rho}$$

Units:

(i) $\frac{m^3}{kg}$,

Dimensional formula: = $[M^{-1} L^3]$

Weight Density/Specific Weight (w):

Weight density / Specific weight is defined as the weight per unit volume.

Mathematically,

$$\text{Specific weight } (w) = \frac{\text{Weight}}{\text{Volume}}$$

$$w = \rho g$$

On increasing Temperature ⇒ Specific weight decreases.

On increasing Pressure ⇒ Specific weight increases.

Units:

(i) $\frac{N}{m^3}$, (ii) $\frac{kg}{m^2 - s^2}$, (iii) $\frac{gm}{cm^2 - s^2}$

Dimensional formula: $[w] = [ML^{-2} T^{-2}]$

Specific Gravity (s):

Specific gravity is defined as the ratio of density of fluid to the density of standard fluid.

$$\text{Specific gravity (s)} = \frac{\text{Density of fluid }(\rho)}{\text{Density of Standard Fluid }(\rho_s)}$$

$$S = \frac{\rho}{\rho_{\text{water}}}$$

- Water is taken as standard fluid.
- Water is Specific gravity is **UNITLESS**.

Note:

Relative density term is used instead of specific gravity if comparison is done with respect to some random fluid instead of standard fluid.

1.2 Viscosity

1.2.1 Viscosity of Liquid (μ)

When the liquid is in motion, the internal resistance offered by one layer of liquid to the adjacent layer of liquid is known as viscosity.

- In any fluid viscosity is because of
  (a) Cohesive force between fluid particles.
  (b) Molecular momentum exchange
- For liquid at rest or relative rest, the concept of viscosity is not valid
- For Ideal fluid, cohesion is zero hence viscosity is also zero.

Newton's Law of Viscosity

$$\tau \propto \frac{d\theta}{dt}$$

$$\tau = \mu \frac{du}{dy}$$

$$\Rightarrow \qquad \mu = \frac{\tau}{du/dy}$$

Here,

μ =Cofficient of viscosity or absolute viscosity or dynamic viscosity

Units:

(i) $\frac{N-s}{m^2}$ (or)$Pa - s$, (ii) $\frac{kg}{m-s}$ (iii) $\frac{gm}{cm-s}$ (poise)

Dimensional formula: $[\mu] = [ML^{-1}T^{-1}]$

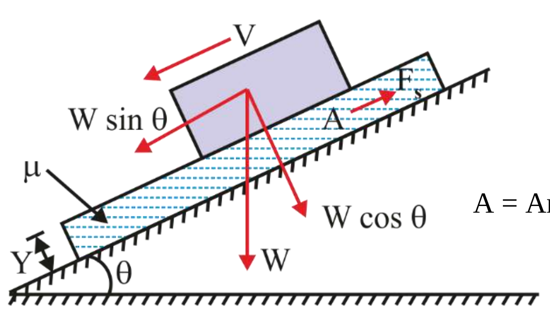
Note:

(i) 1 poise = 10^{-1} Pa-s

(ii) 1 CP = 10^{-2} Poise

1. 2.2 Different Cases of Viscosity

Block Sliding Downward on Inclined Surface Due to Self-Weight:



A = Area of contact between block and fluid

Fig. 1.2 Fluid between two parallel inclined plates (Fixed and movable)

Note:

(i) At $\theta = 0°$, then $V = 0$ (Horizontal surface)

(ii) At $\theta = 90°$, then $V = \frac{WY}{\mu A}$ (Maximum terminal velocity for a given W, Y, μ and A)

At equilibrium

$$F_s = W \sin\theta$$

$$\tau A = W \sin\theta$$

$$\mu \times \frac{V}{Y} A = W \sin\theta$$

$$\Rightarrow \quad V = \frac{(W \sin\theta)Y}{\mu A}$$

Thin Plate Moving in Horizontal Gap:

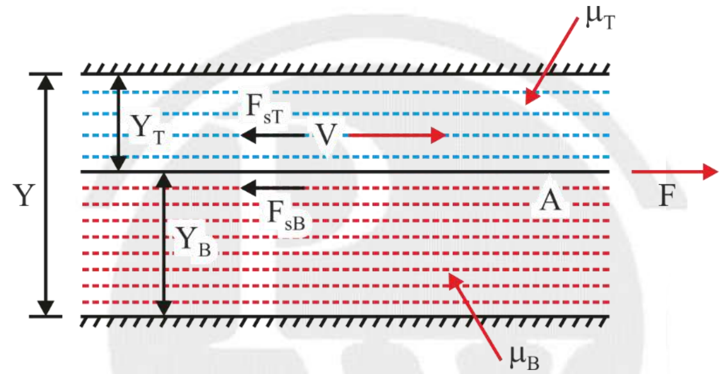


Fig. 1.3 Thin plate moving in horizontal gap

Here,

A → surface area of plate

V → velocity of plate

At equilibrium

$$F = F_{sT} + F_{sB}$$

$$F = \left(\frac{\mu_T}{Y_T} + \frac{\mu_B}{Y_B}\right) VA$$

Special Case:

- For same shear force on both sides $\frac{Y_T}{Y_B} = \frac{\mu_T}{\mu_B}$
- For minimum pulling force $\frac{Y_T}{Y_B} = \sqrt{\frac{\mu_T}{\mu_B}}$

Solid Cylinder Rotating in Hollow Cylindrical Cavity

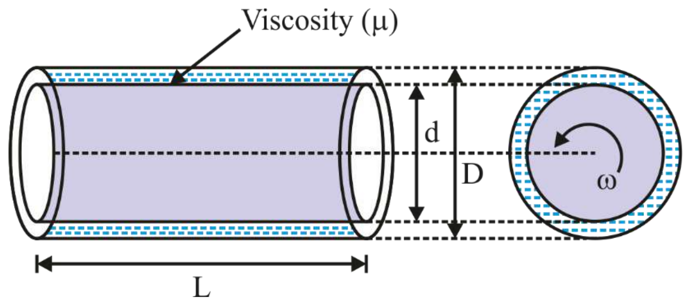


Fig. 1.4 Fluid between two concentric cylinders (One rotating and another fixed)

- Shear stress $\tau = \frac{\mu\omega d}{2Y}$ $\left(Y = \frac{D-d}{2}\right)$
- Shear force applied to rotate solid cylinder $F_s = \frac{\pi\mu\omega d^2 L}{2Y}$
- Torque Applied $T = F_s \times r = \frac{\pi\mu\omega d^3 L}{4Y}$
- Power Required to rotate the solid cylinder $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 d^3 L}{4Y}$

Circular Disk:

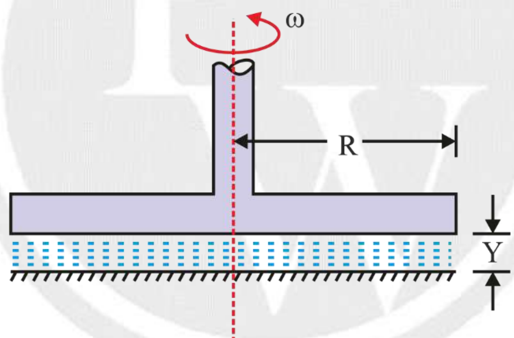


Fig. 1.5 Fluid between two circular disks (One fixed and another rotating)

- Shear force applied to rotate circular disc $F_s = \int_0^R \tau dA = \int_0^R \mu \frac{r\omega}{Y} \times 2\pi r dr = \frac{2\pi\mu\omega R^3}{3Y}$
- Torque Applied $T = \int_0^R \tau dA \times r = \frac{\pi\mu\omega R^4}{2Y}$
- Power Required to rotate to circular disc $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 R^4}{2Y}$

Solid Cone:

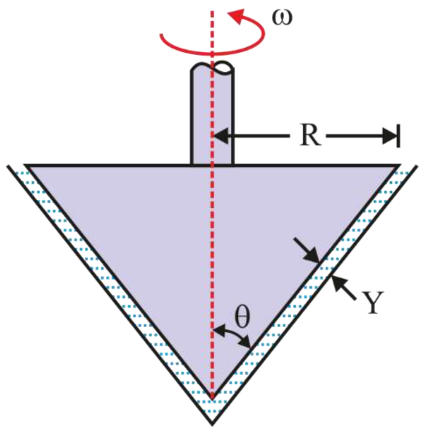


Fig. 1.6 Fluid between a hollow and solid cone

- Shear force applied to rotate solid cone $F_s = \int_0^R \tau dA = \frac{2\pi\mu\omega R^3}{3Y} \times \frac{1}{\sin\theta}$
- Torque Applied $T = \int_0^R \tau dA \times r = \frac{\pi\mu\omega R^4}{2Y} \times \frac{1}{\sin\theta}$
- Power Required to rotate to solid cone $P = T \times \omega = F_s \times V = \frac{\pi\mu\omega^2 R^4}{2Y} \times \frac{1}{\sin\theta}$

Note:
1. For liquid, on increasing temperature cohesion decreases leading to decrease in viscosity
2. For gases, on increasing temperature further increases molecular collisions, so viscosity increases

1. 3 Kinematic Viscosity (ν)

- It is ratio of dynamic viscosity of fluid to density of fluid.

$$\nu = \frac{\mu}{\rho}$$

- $\nu \propto T^{3/2}$ (for ideal gas) and $\mu \propto \sqrt{T}$ (in general) where T = Absolute temperature.

Units: m^2/s

Dimensional formula:

$$[\nu] = [L^2T^{-1}]$$

Note:
1 stoke $=10^{-4}$ m^2/s = $1 cm^2/s$

1. 4 Compressibility & Bulk Modulus

1.4.1 Compressibility (β_c)

Compressibility of a fluid refers to its ability to change its volume and density when subjected to pressure. The coefficient of compressibility β_c is defined as the relative change of volume (or density) per unit pressure and is represented as

$$\beta_c = -\frac{dv/v}{dp} = \frac{d\rho/\rho}{dp}$$

Where,

dp = change in pressure

dv = change in volume

v = volume of the fluid

$d\rho$ = change in density

Note:

The negative sign indicates a decrease in volume v with increase in pressure.

1.4.2 Bulk modulus (*K*)

The bulk modulus of elasticity K is defined as the ratio of compressive stress to volumetric strain. Mathematically it is the reciprocal of compressibility (β_c).

$$K = \frac{1}{\beta_c} = \frac{dp}{\left(-\frac{dv}{v}\right)}$$

Unit: N/m^2 (Pa).

The bulk modulus of liquid and solid are the property of material. In other words, for a given liquid and solid, bulk modulus will be constant.

For gases, bulk modulus is not the property of material but depends on the process.

1.4.3 Compressibility, Bulk Modulus for an Ideal Gas

Process	Bulk Modulus of Elasticity (K)	Compressibility (β_c)
Isobaric process $p = C$	$K = 0$	$\beta_c = \infty$
Isochoric process $V = C$	$K = \infty$	$\beta_c = 0$
Isothermal process $T = C$	$K = p$	$\beta_c = \frac{1}{p}$
Isentropic process $pV^\gamma = C$	$K = \gamma p$	$\beta_c = \frac{1}{\gamma p}$

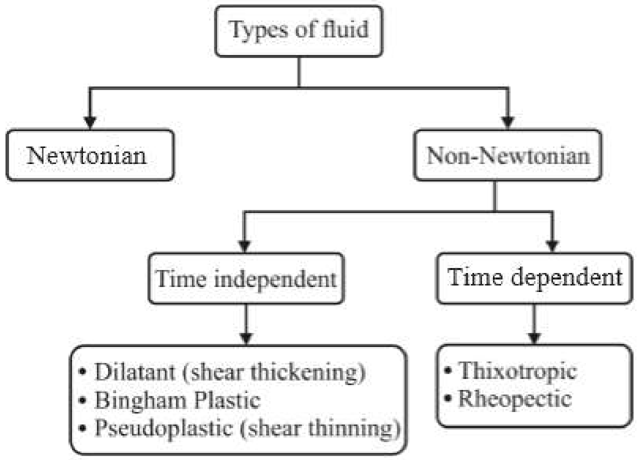
1.5 Types of Fluid

Ideal Fluid:

A fluid which has 0 viscosity, incompressible and zero surface tension is known as Ideal Fluid.

Real Fluid:

A real fluid is a fluid which has viscosity, finite compressibility and surface tension



1.5.1 Power Law Model:

$$\tau = A + B\left(\frac{du}{dy}\right)^n$$

Where

$A \geq 0$, is a constant representing Yield stress.

$B > 0$, is known as consistency Index.

n , is known as flow behavior Index.

$$\tau = A + \mu_{app}\left(\frac{du}{dy}\right), \left(\text{where } \mu_{app} = B\left(\frac{du}{dy}\right)^{n-1}\right)$$

μ_{app} = Apparent Viscosity

1.5.2 Dilatant Fluid (A = 0; n > 1):

- A fluid is said to be dilatant fluid if apparent viscosity increases with deformation rate.
- Dilatant fluid is also known as Shear – thickening fluid

Example:

Quick sand, Butter, Oobleck, Rice starch, Saturated sugar solution etc.

1.5.3 Pseudo Plastic Fluid (A = 0, n < 1):

- A fluid is said to be a pseudo plastic fluid, if apparent viscosity decreases with deformation Rate
- Pseudo plastic fluid is also known as shear – thinning fluid

Example:
Blood, Milk, Paper pulp, Water solution, Polymeric solution, Rubber, Molasses, Slurry etc.

1.5.4 Bingham Plastic Fluid (A > 0; n = 1):

It behaves as a rigid body at low stresses but flows as a viscous fluid at high stress.

- These fluids always have certain minimum shear stress before they yield.
- Shear strain rate increases then μ_{app} remains same.

Examples: Sewage sludge, Drilling mud, Tooth paste and Gel etc.

1.5.5 Rheopectic Fluid (A > 1; n > 1):

- For Rheopectic fluid apparent viscosity increases with time.

  Examples:
  Gypsum, Bentonite slurry, Whipping Cream

1.5.6 Thixotropic Fluid (A > 1; n < 1):

- For thixotropic fluid μ_{app} decreases with time.

  Examples: Lipstick, Printers ink, Hand lotion.

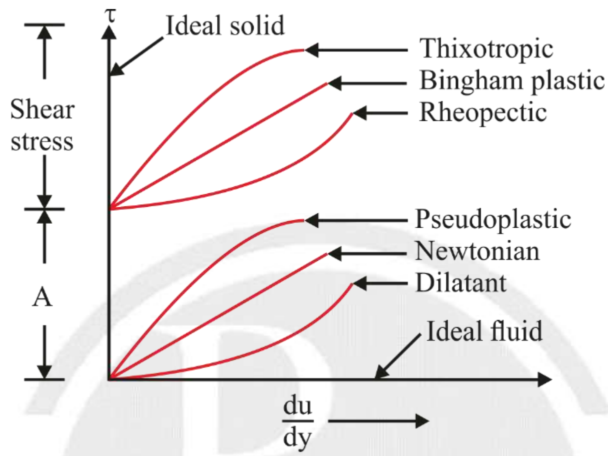


Fig. 1.7 Shear stress vs shear strain rate for difference fluids

1.6 Surface Tension (σ):

It is the tension of the surface film of a liquid caused by the attraction (cohesion) of the particles in the surface layer by the bulk of the liquid, which tends to minimize surface area.

$$\sigma = \frac{\text{Surface Tension Force}\left(F_{st}\right)}{\text{Length}\left(L\right)}$$

Units:

$\frac{N}{m}$ (or) $\frac{J}{m^2}$

Dimensional formula:

$[\sigma] = [MT^{-2}]$

Note:
- For ideal fluid, surface tension is zero due to absence of cohesion force.
- At critical point there is no distinct liquid vapour interface, hence surface tension is zero at critical point.
- On increasing temperature cohesion decreases, leading to decrease in surface tension.
- For water – air interface at 20°C its value is 0.0736 N/m and Air – mercury interface
  $\sigma = 0.480$ N/m.
- It is due to cohesion only.

1.6.1 Gauge Pressure inside Liquid Droplet:

$$\Delta p = \frac{4\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.2 Gauge Pressure inside an Air Bubble

$$\Delta p = \frac{4\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.3 Gauge Pressure inside A Soap Bubble:

$$\Delta p = \frac{8\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$ & $p_{out} = p_{atm} = 0\,\text{gauge}$

1.6.4 Gauge Pressure inside a Liquid Jet:

$$\Delta p = \frac{2\sigma}{d}$$

Here, $\Delta p = p_{in} - p_{out}$

1.6.5 Work Done in Stretching an Interface or Increase in Surface Energy:

$$W_d = \sigma \Delta A$$

$$W_d = \Delta E_s$$

Increase in surface energy $\Delta E_s = W_d = \sigma \Delta A$

$$E_s = \sigma A$$

Where ΔA is increase in Surface area of exposed Surface / interface area.

1.6.6 Work Done in Converting a large Spherical Droplet into 'n' Number of Small Spherical Droplet

$$W_d = \Delta E_s = \sigma\, 4\pi R^2 \left(n^{\frac{1}{3}} - 1\right)$$

Where R = Radius of large spherical droplet.

$r = R/n^{1/3}$

r = Radius of small spherical droplet.

1.7 Angle of contact or contact angle

Surface tension occurs during a gas liquid interface, but if that interface comes in contact with a solid surface such as the walls of a container the interface usually curves up or down near that surface. Such a concave or convex surface shape is known as meniscus.

The contact angle θ is determined as shown in the figure 1.8 and 1.9.

Case I When the meniscus is concave

- Adhesive force > cohesive force
- Angle of contact 'θ' is less than 90^0.

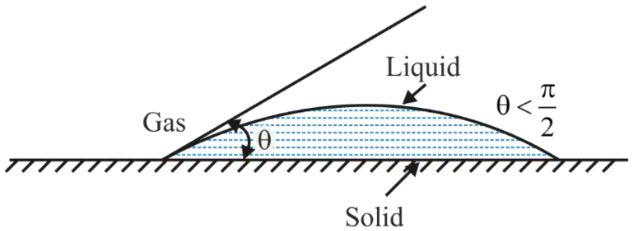


Fig.1.8 Wetting liquid (water) and solid surface interface

Case II When the meniscus is convex

- Adhesive force < cohesive force
- Angle of contact 'θ' is greater than 90°.

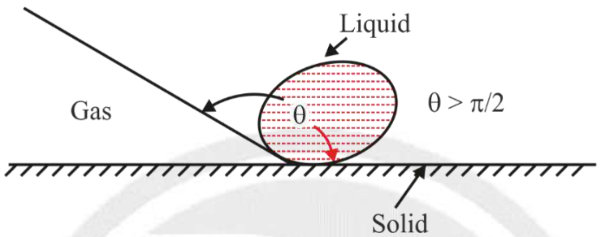


Fig.1.9 Non – wetting liquid (Mercury) and solid surface interface

Note:

(i) **Cohesive Force:** Intermolecular attraction force between same molecules.

(ii) **Adhesive Force:** Intermolecular attraction force between two different molecules.

1.8 Capillarity or Capillary Action

Capillarity or Capillary action is the ability of a narrow tube to rise or fall of a liquid against the force of gravity. Adhesion & cohesion both are responsible for capillary action. The rise of liquid surface is known as capillary rise while the fall of the liquid surface is known as capillary depression.

1.8.1 Capillary Rise

Adhesion > cohesion (Wetting liquid)

Vertically upward component of surface tension force = Weight of liquid rise

Capillary rise stops when vertically downward weight of the liquid rise is exactly balanced by the vertically upward component of surface tension force.

$$h = \frac{4\sigma \cos\theta}{\rho g D}$$

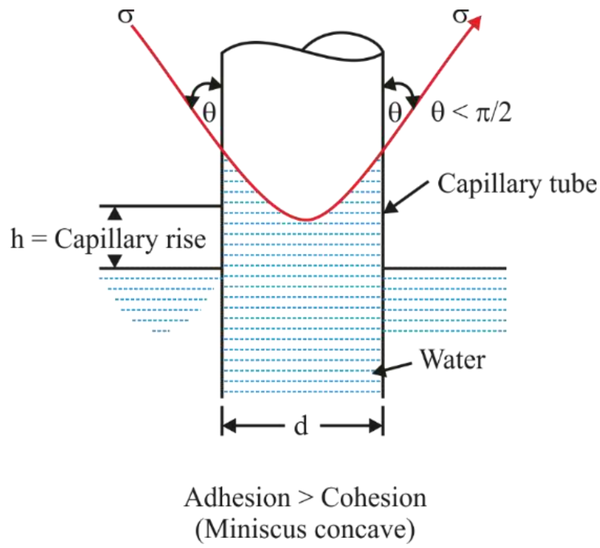


Fig.1.10 Capillary rise in a glass tube

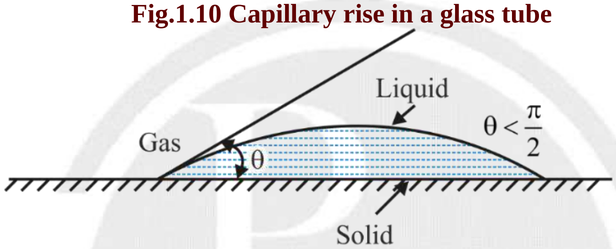


Fig.1.11 Wetting liquid (water) and solid surface interface

The most common interfaces and value of σ for clean interface at $20°C$ are

- $\sigma = 0.073\ \mathrm{N/m}$ for air-water interface
- $\sigma = 0.480\ \mathrm{N/m}$ for air-mercury interface
- $\theta =$ Angle of contact of the water surface
- For water-clean glass surface, $\theta = 0°$ and for mercury-clean glass, $\theta = 130°$

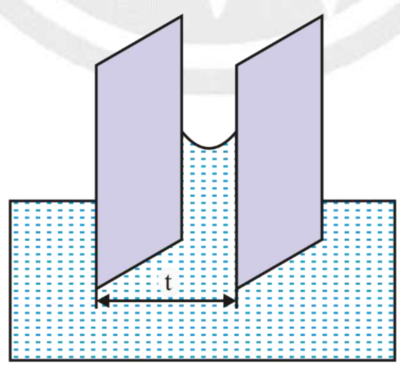


Fig.1.12 Capillary rise between two parallel plates

$$h = \frac{2\sigma \cos\theta}{\rho g t}$$

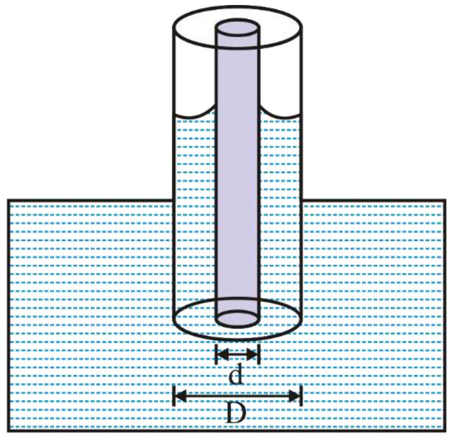


Fig.1.13 Capillary rise in the annulus of two concentric tubes

$$h = \frac{4\sigma \cos\theta}{\rho g(D - d)}$$

1.8.2 Capillary Fall

For Non-wetting Liquid, Cohesion > Adhesion and $\theta > 90°$

Capillary fall stops when vertically upward hydro – static force (pressure force) due to difference of liquid level is exactly balanced by vertically downward component of surface tension force.

$$h = -\frac{4\sigma \cos\theta}{\rho g d}$$

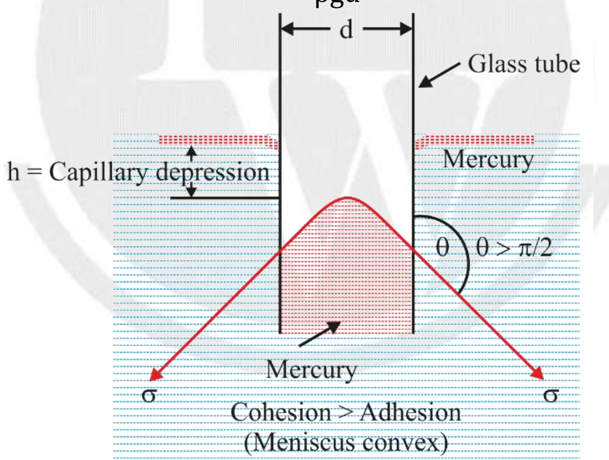


Fig.1.14 Capillary fall in the glass tube

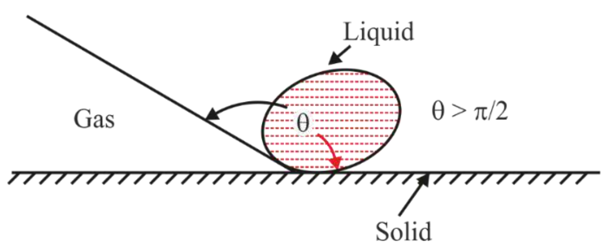


Fig.1.15 Non – wetting liquid (Mercury) and solid surface interface

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2 Pressure and Its Measurement

2.1 Introduction

Pressure is defined as the external force per unit area.

If dF is the force acting on the area dA in the normal direction then,

$$p = \frac{dF}{dA}$$

If the force (F) is uniformly distributed over the area (A), then,

$$p = \frac{F}{A} = \frac{Force}{Area}$$

Note:

(a) If there is no mass of fluid the pressure will be zero and it is known as Absolute Zero/Absolute vacuum.

(b) Characteristics of Pressure Force:

(i) External

(ii) Normal

(iii) Compressive

(c) Pressure is scalar quantity.

Units & Dimension of Pressure

Unit

$\frac{N}{m^2}$ or Pa

$1kPa = 10^3 Pa$

$1bar = 10^5\ Pa = 10^2\ kPa$

$1MPa = 10^6\ Pa = 10^3\ kPa = 10\ bar$

$[P] = [ML^{-1}T^{-2}]$

2.1.3 Absolute, Gauge, Atmospheric and Vacuum Pressures

1 Atmospheric Pressure (p_{atm})

Pressure exerted by atmosphere on a surface.

Atmospheric pressure at mean sea level:

$P_{atm,s}$ = 760 mm of Hg = 76 cm of Hg =760 Torr = 101.325 kPa = 1.01325 bar = 10.336 m of H_2O.

2 Gauge pressure (p_{gauge})

When pressure in a fluid is measured above the atmosphere pressure (p_{atm}) then it is called gauge pressure.

3 Vacuum pressure (p_v)

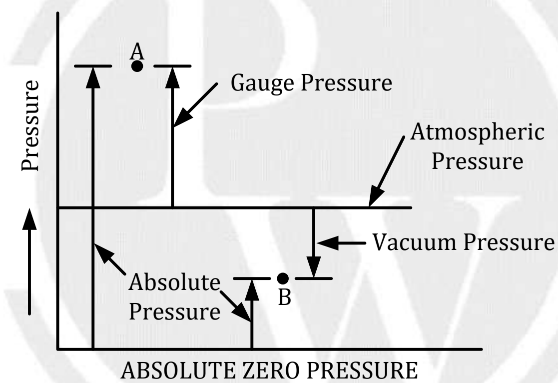
If the pressure at any point is below atmospheric then gauge pressure is negative which is called vacuum pressure.

$$p_v = -p_g$$

4 Absolute Pressure (p_{ab})

When pressure in a fluid is measured above the absolute zero or complete vacuum then it is called absolute pressure. Absolute pressure is always positive.

$$p_{ab} = p_{atm} + p_g$$



RELATIONSHIP BETWEEN PRESSURE

Fig.2.1 Absolute pressure, Gauge Pressure and Vacuum Pressure

2.2 Pascal's Law

Pressure at a point in a static fluid is equal in all directions.

- Pascal's law is valid for any static fluid.
- In the absence of shear forces, Pascal's law is applicable for a fluid in motion also.

2.3 Hydrostatic Law

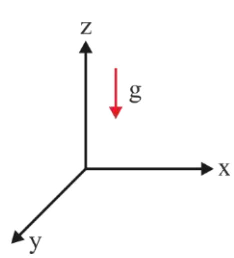


Fig.2.2

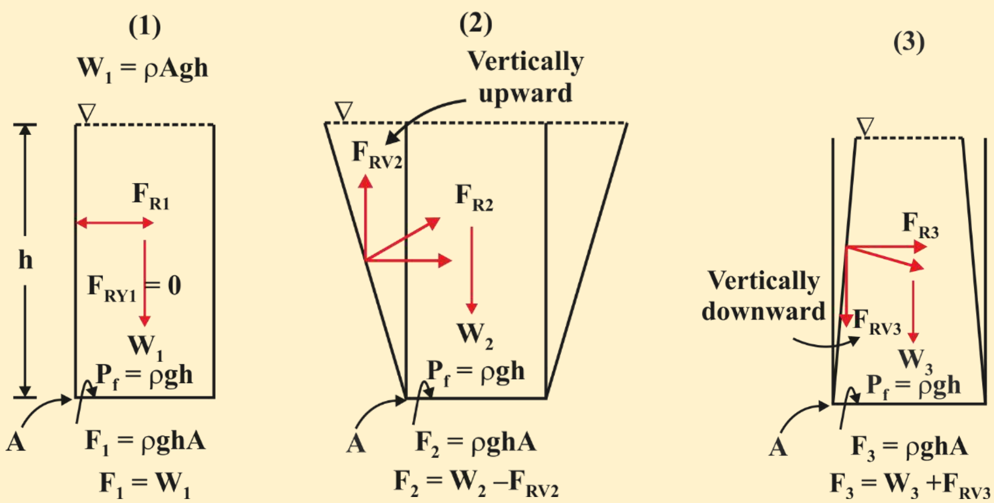
$$P = P(z) \text{ only}$$

$$\frac{dP}{dz} = -\rho g$$

- For static fluid, fluid pressure varies only in vertical direction ($i.e. in\ z$ direction) and magnitude of pressure gradient in vertical direction and is equal to the specific weight of the fluid. $\left(\because \frac{dp}{dz} = -\rho g\right)$
- Negative sign represents that pressure increases in vertically downward direction for a static fluid.

Note:

Fluid pressure is independent of the shape of the container it depends only on the depth of static incompressible fluid. This known as Hydrostatic Paradox.



Hear F = Net force

Fig.2.3 Comparison of pressure at the bottom and weight of the liquid in three containers.

2.4 Conversion of Liquid Column

$$\rho_1 h_1 = \rho_2 h_2$$

$$h_2 = \frac{\rho_1}{\rho_2} h_1$$

- Higher the density of liquid lesser is the depth of liquid required for a given fluid pressure.

Mercury has high density, hence requires lesser depth for a given fluid pressure. This is one of the reasons why mercury is used in manometer.

2.5 Barometer

Barometer is used to measure the local atmospheric pressure.

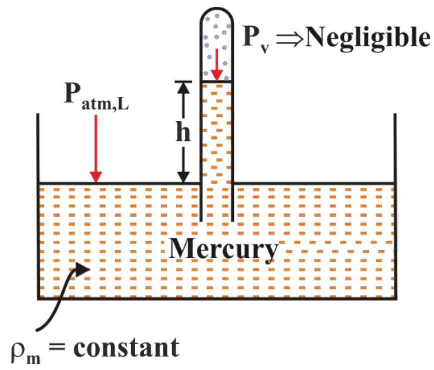


Fig.2.4

Measuring ⇒ Local Atmospheric Pressure

$P_{atm,L} = P_v + \rho_m gh$

Mercury ⇒ Low Vapour Pressure

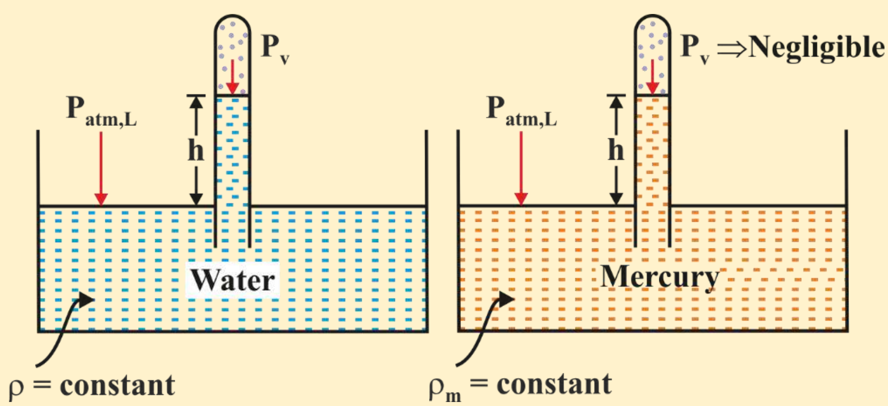
$P_{atm,L} \cong \rho_m gh$

Atmospheric pressure at mean sea level:

$P_{atm,s}$ = 760 mm of Hg = 76 cm of Hg =760 Torr = 101.325 kPa = 1.01325 bar = 10.336 m of H_2O

Note:

Barometer is used to measure local atmospheric pressure, Barometer was given by Torricelli (1608-1647). Local atmospheric pressure is measured by barometer hence local atmospheric pressure is also known as Barometric Pressure.

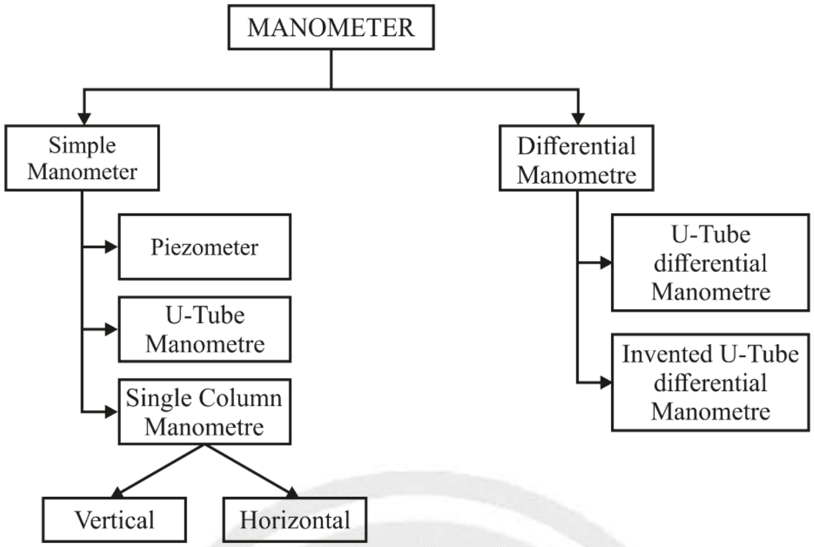


$P_{atm,L} = P_v + \rho gh$

$P_{atm,L} \cong \rho_m gh$

Fig.2.5 Atmospheric pressure measurement by Barometer

2.6 Manometer



2.6.1 Piezometer

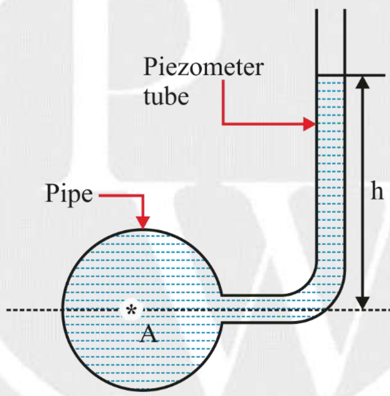


Fig.2.6 Piezometer

$$\left(p_g\right)_A = \rho_f gh$$

Limitations:

- Can't be used for gases
- Can't be used to measure vaccum pressure.
- Can't be used to measure high pressure.

2.6.2 U-Tube Manometer

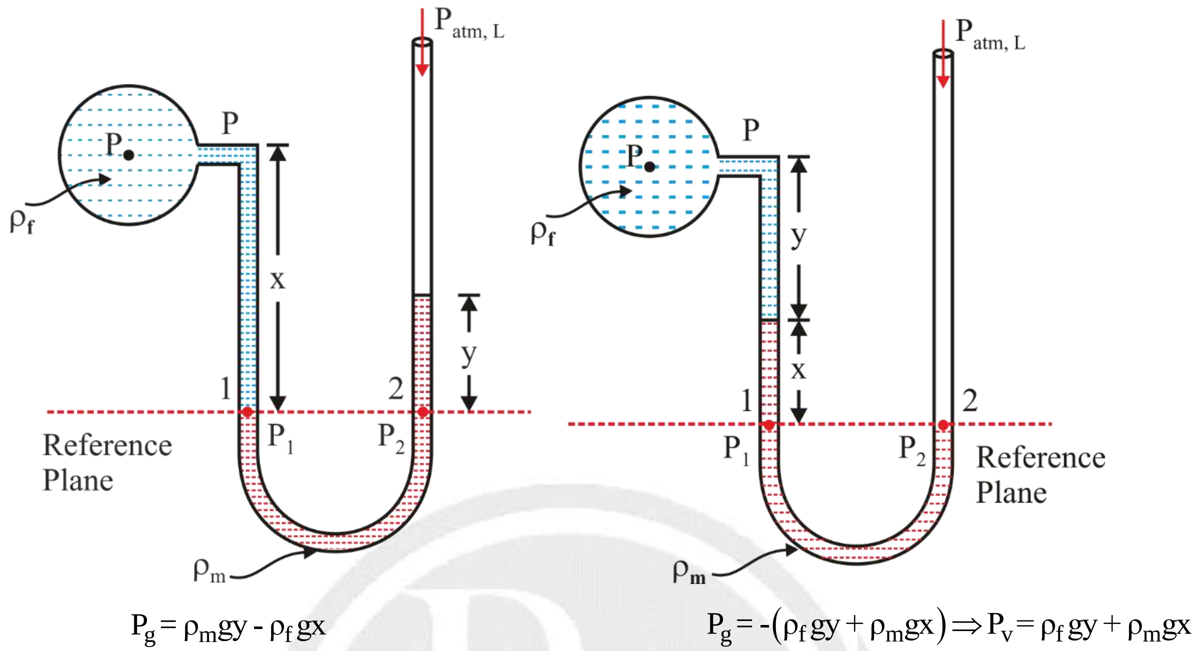


Fig.2.7 Pressure measurement by simple U – Tube Manometer

2.6.3 Vertical Single Column Manometer

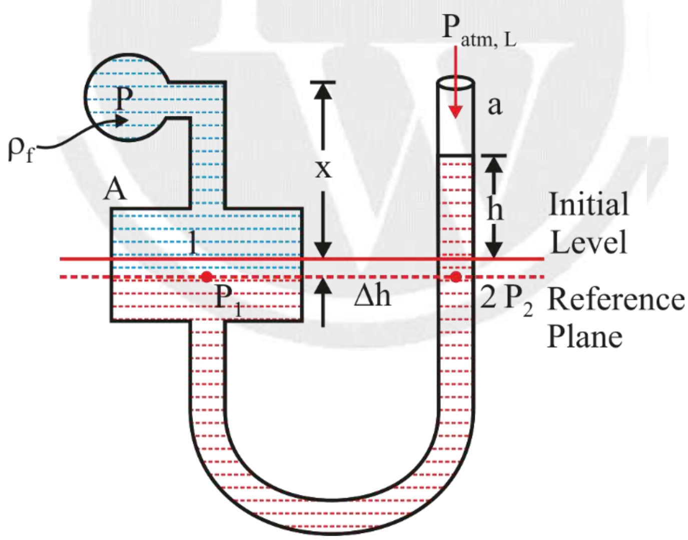


Fig.2.8 Pressure measurement by vertical single column manometer

Assumption: $P > P_{atm,L}$

$V_{fL} = V_{rR}$

$A \times \Delta h = a \times h$

$\Delta h = \frac{a}{A} \times h$

$\because$ $a \lll< A$

$\therefore \quad \Delta h \lll\lll< h$

- $P_g = \rho_m g(h + \Delta h) - \rho_f g(x + \Delta h) \rightarrow$ Exact value of gauge presure
- $P'_g = (\rho_m gh - \rho_f gx) \rightarrow$ Approximate value of gauge pressure
- $\%\ \text{Error} = \frac{(\rho_m - \rho_f)g\Delta h}{\rho_m g(h+\Delta h) - \rho_f g(x+\Delta h)} \times 100 \rightarrow$ Exact value of % error
- $\%\ \text{Error} = \frac{\Delta h}{h} \times 100 \rightarrow$ Approximate value of % error
- $\%\ \text{Error} = \frac{a}{A} \times 100 \rightarrow$ Approximate value of % error

2.6.4 Inclined Single Column Manometer

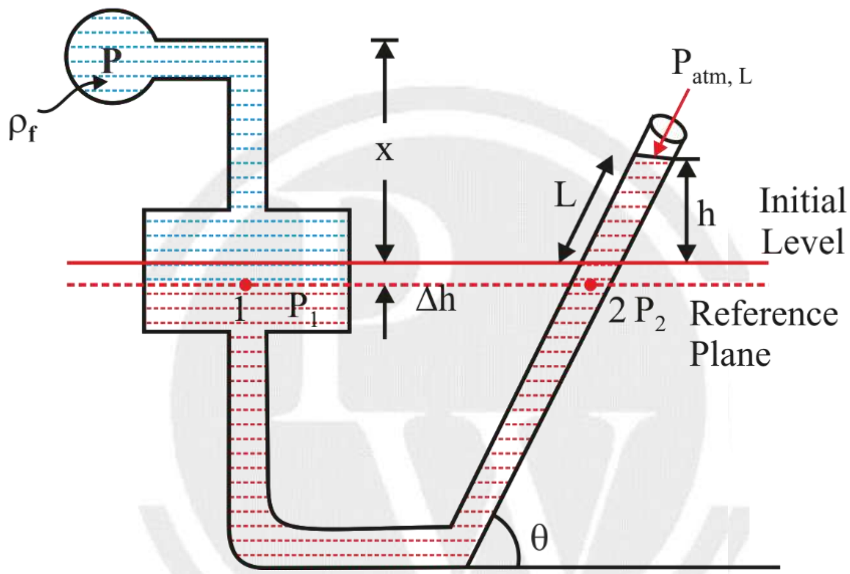


Fig.2.9 Pressure measurement by inclined single column manometer

$P_g = (\rho_m gL \sin\theta - \rho_f gx) \rightarrow$ Approximately

- Inclined single column manometer is used to measure low pressure.
- Using inclined limb, reading comes in terms of ‘L’ which is more than ‘h’ hence error decreases and sensitivity increases by $\frac{1}{\sin\theta}$.
- Sensitivity can also be increased by using lighter manometric fluid.

2.6.5 U-Tube Differential Manometer

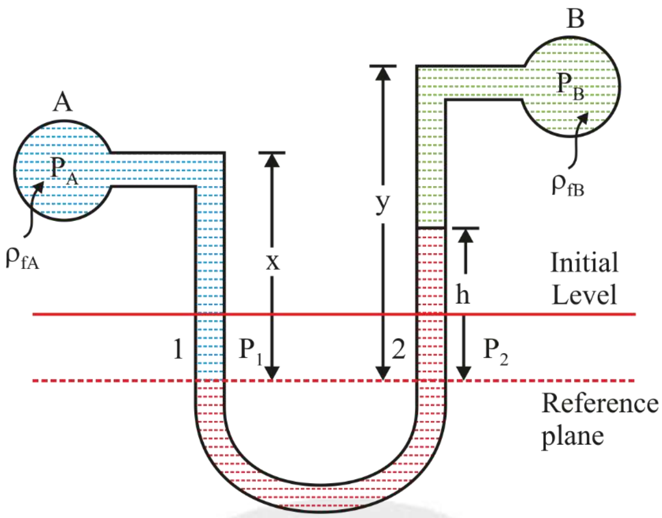


Fig.2.10 Pressure measurement by U – tube differential manometer

- Assumption: $P_A > P_B$
- $P_A - P_B = \rho_m gh + \rho_{fB} g(y - h) - \rho_{fA} gx$

Note:

Both pipes are at same level & Carrying Same Fluid.
$P_A - P_B = (\rho_m - \rho_f)gh$

2.6.6 Inverted U-Tube Differential Manometer

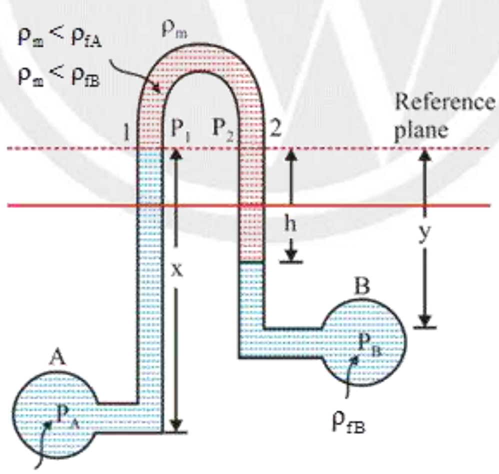


Fig.2.11 Pressure measurement by inverted U-tube differential manometer

- Assumption: $P_A > P_B$
- $\rho_m < \rho_{fA}$ & $\rho_m < \rho_{fB}$
- $P_1 = P_2$
- $P_A - P_B = \rho_{fA} gx - \rho_m gh - \rho_{fB} g(y - h)$

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3 HYDROSTATIC FORCE

3.1 Introduction

Hydrostatic Pressure force

It is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

3.2 Hydrostatic pressure on plane surface

3.2.1 Inclined Plane Surface

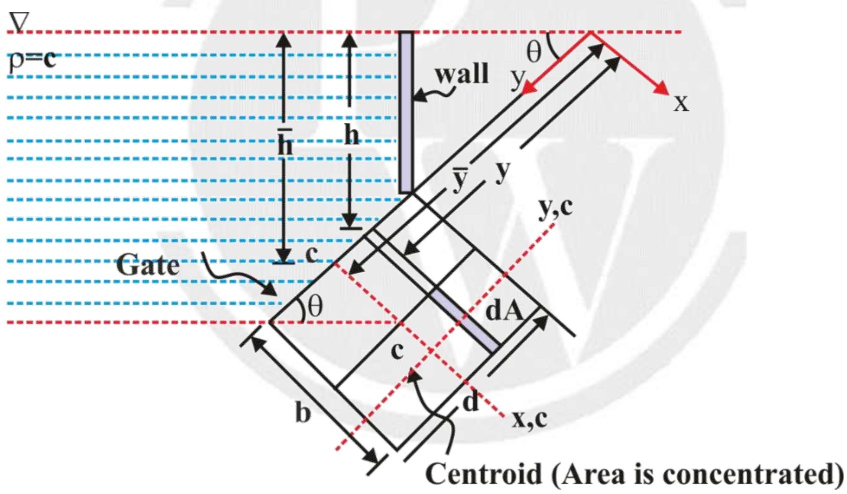


Fig:3.1 Hydrostatic force on a inclined plane surface

$F_p = \rho g \bar{h} A$

Here,

Here,

A = Area of surface which touching fluid

$\bar{h}$ = Vertical distance of C.O.G. of body from free surface.

θ = Angle at which the surface is inclined with horizontal

ρ = Density of liquid

F_p = Total pressure force on the surface

Note:

- Same expression can be used for vertical and horizontal plane surfaces also.
- Magnitude of $\bar{h}$ will depend on magnitude of angle of inclination (θ), hence magnitude of hydrostatic force will vary with angle of inclination.

Vertical Plane Surface ($\theta = 90°$)

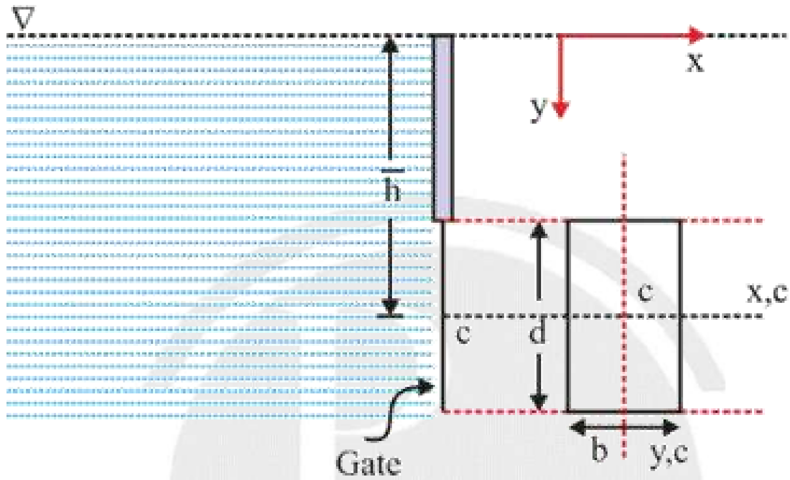


Fig:3.2 Hydrostatic force for a vertical plane surface

$$F_P = \rho g \bar{h} A$$

Horizontal Plane Surface ($\theta = 0°$)

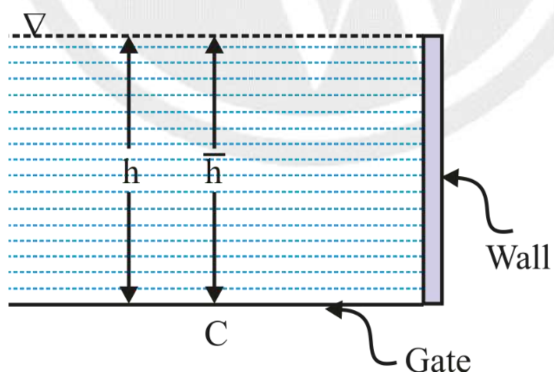


Fig:3.3 Hydrostatic force for a horizontal plane surface

$$F_P = \rho g \bar{h} A = \rho g h A$$

For horizontal plane surface, depth of centroid from the free surface $\bar{h}$ is equal to depth of horizontal plane surface from the free surface (h).

3.2.2 Centre of Pressure (h^*)

Centre of pressure is the point on the surface at which hydrostatic force (total pressure) is assumed to be acting.

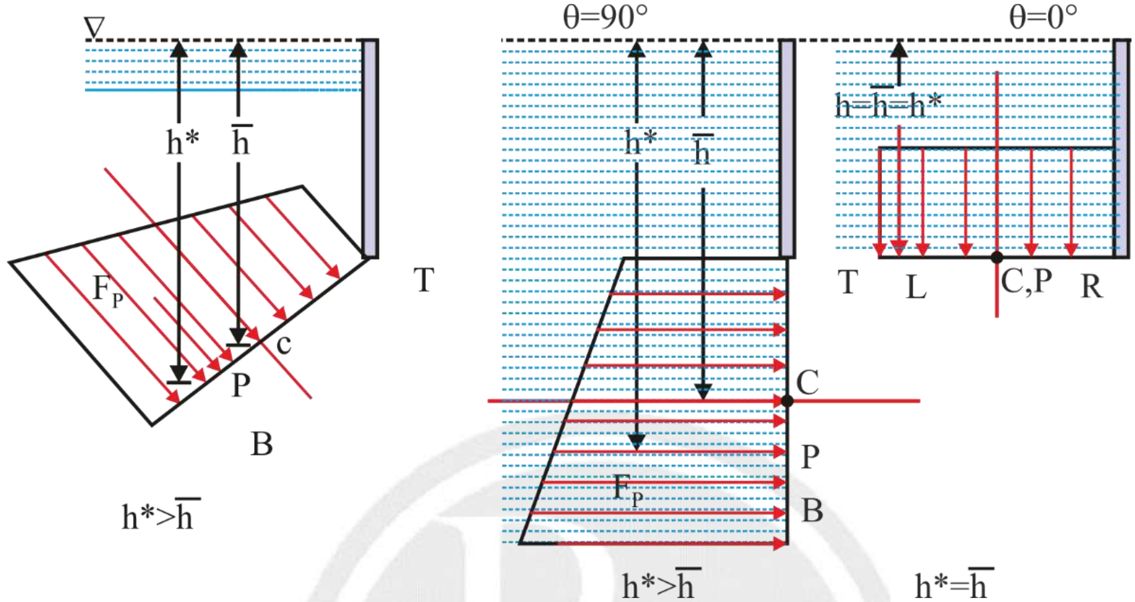


Fig:3.4 Centre of pressure under various conditions

For inclined plane surface

$$h^* = \bar{h} + \frac{I_{xx.c} \sin^2\theta}{A\bar{h}}$$

Here,

$I_{xx.c}$= Area moment of inertia about centroidal axis which is parallel to free surface.

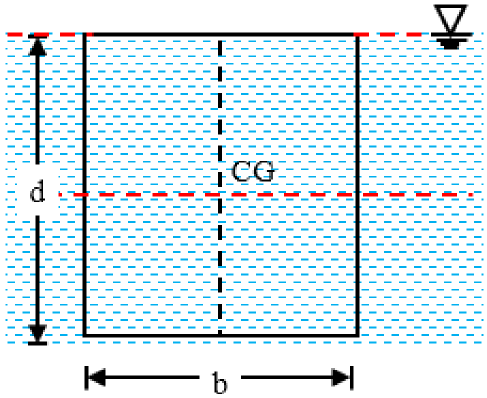
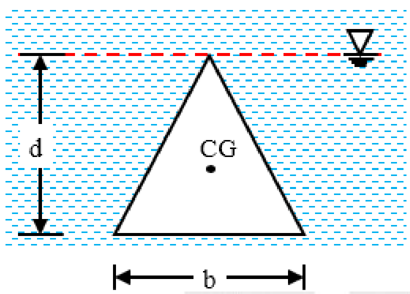
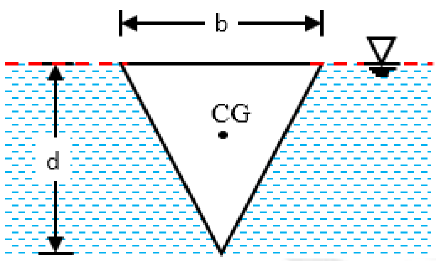
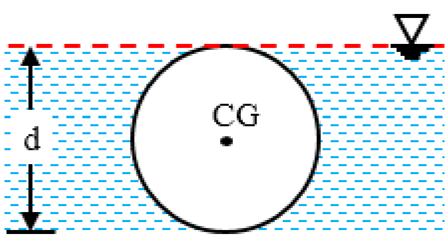
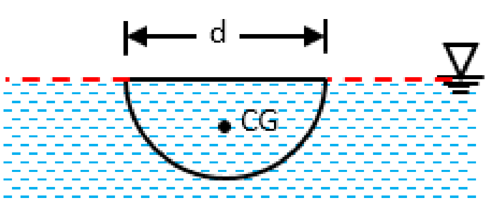
- For Vertical Plane Surface ($\theta = 90°$)

$$h^* = \bar{h} + \frac{I_{xx.c}}{A\bar{h}}$$

- For Horizontal Plane Surface ($\theta = 0°$)

$$h^* = \bar{h}$$

3.2.3 $\bar{h}$, h* AND M.O.I for Different Shapes

Surface	Shape	C.G. $(\bar{h})$	M.O.I $(I_{xx.c})$	C.P (h^*)
Rectangle		$\frac{d}{2}$	$\frac{bd^3}{12}$	$\frac{2d}{3}$
Triangle		$\frac{2d}{3}$	$\frac{bd^3}{36}$	$\frac{3d}{4}$
Inverted Triangle		$\frac{d}{3}$	$\frac{bd^3}{36}$	$\frac{d}{2}$
Circle		$\frac{d}{2}$	$\frac{\pi d^4}{64}$	$\frac{5d}{8}$
Semi-Circle		$\frac{2d}{3\pi}$	$\frac{\pi}{128}d^4 - \frac{\pi}{8}d^2\left(\frac{2d}{3\pi}\right)^2$	$\frac{3\pi d}{32}$

3.3 Force for Curved Surface

3.3.1 Horizontal Component (F_H)

- Horizontal component of hydrostatic force acting on a curved surface is equal to hydrostatic force acting on vertical projection of area.
- It will act at the centre of pressure of the vertical projection of curved area.

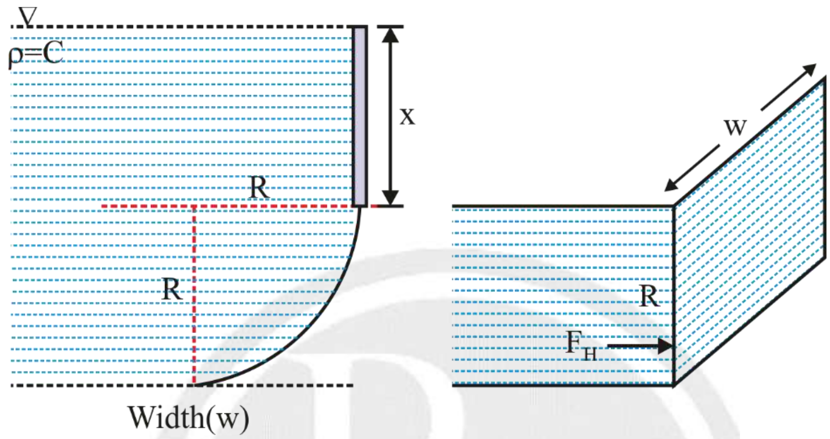


Fig:3.5 Horizontal component of hydrostatic force on curved surface

$$F_H = F_{p.v}$$

$$F_H = \rho g \bar{h}_v A_v$$

Here,

A_v = Projected area in vertical plane = $R \times w$ (for figure 3.5)

$\bar{h}_v$ = Vertical distance of C.O.G. of projected surface on vertical plane from free surface = $x + \frac{R}{2}$ (for figure 3.5)

Hence for figure 3.5,

$$F_H = \rho g\left(x + \frac{R}{2}\right)(Rw)$$

3.3.2 Vertical Component (F_V)

- Vertical component of hydrostatic force acting on a curved surface is equal to weight (W) of the fluid (Real or Imaginary) which can be contained above curved surface when the horizontal projection is taken up to free surface.
- It will be acting at the centre of gravity of the above-mentioned weight.

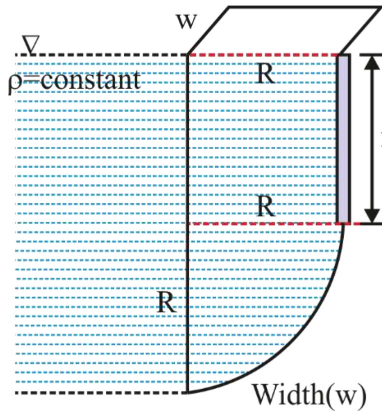


Fig: 3.6 Vertical component of hydrostatic force on curved surface

$$F_V = W$$

$$F_V = \rho V g$$

$$F_V = \rho\left(\frac{\pi}{4}R^2 w + Rxw\right)g$$

$$F_V = \rho\left(\frac{\pi}{4}R^2 + Rx\right)wg$$

3.3.3 Resultant Force (F_R)

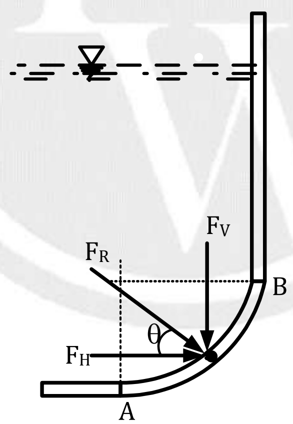


Fig:3.6

$$F_R = \sqrt{F_H^2 + F_V^2}$$

$$\tan\theta = \frac{F_V}{F_H}$$

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4 BUOYANCY AND FLOATATION

4.1 Introduction

- In the presence of gravity all liquids and gases exert an upward force on any object i.e. immersed on it, is known as force.

4.1.1 Buoyancy Force (F_B)

When a body is either partially or completely submerged in a fluid, there is a vertically upward force imposed by the fluid on the body. This force is known as Buoyant Force (F_B)

Buoyant Force ⇒ vertically upward direction

Archimedes Principle

Archimedes Principle helps to calculate the force of buoyancy. It states that the magnitude of force of buoyancy is equal to the weight of the fluid displaced by the body.

$F_b = W_{fd}$

$F_b = \rho_f V_{fd} g$

$F_b = \rho_f V_s g$

Here

V_{fd} = Volume of fluid displaced by the body

V_s = Volume of body immersed in fluid

W_{fd} = Weight of fluid displaced by the body

4.1.2 Centre of Buoyancy (B)

Centre of buoyancy is the point at which buoyant force is supposed to be acting. and it is the centroid of the volume of fluid displaced by the body

4.2 Condition of Floatation

A body will float in a liquid when weight of the body (W_b) is equal to the force of buoyancy & the line of action of force of buoyancy (F_b) & force of gravity must lie in the same vertical line.

Mathematically,

$F_b = W_b$

Note:

Solid body will float on the fluid, if density of the body is less than or equal to the density of fluid

4.3 Tension in Rope for Constrained Body ($\rho_b > \rho_f$)

At Equilibrium

$T + F_b = W_b$

$T = W_b - F_b$

$T = \left(\rho_b V_b - \rho_f V_{f,d}\right) g$

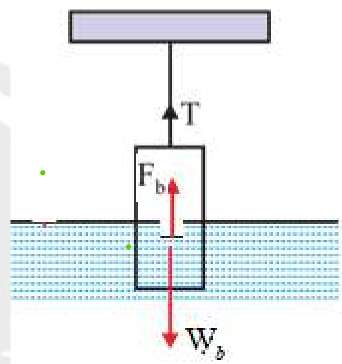

Fig: 4.1 Weight loss due to Buoyancy

4.4 Body Completely Submerged in two Liquids

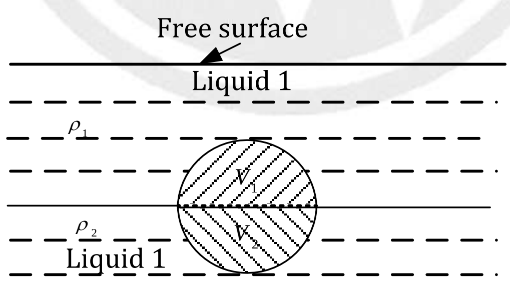

Fig: 4.2 Buoyant force on a body submerged in two liquids

Force of buoyancy,

$$F_B = \rho_1 g V_1 + \rho_2 g V_2$$

Where,

ρ_1 = density of fluid 1

ρ_2 = density of fluid 2

V_1 = Volume of part of body immersed in liquid 1

V_2 = Volume of part of body immersed in liquid 2.

4.5 STABILITY OF SUBMERGED & FLOATING BODIES

4.5.1 Type of equilibrium

(A) Stable Equilibrium

It occurs when a body is tilted slightly by some external force, and then it returns back to its original position due to the weight and the up thrust.

(B) Unstable Equilibrium

It occurs when a body is tilted slightly by some external force & body does not return to its original position from the slightly displaced angular position.

(C) Neutral Equilibrium

It occurs when a body, when given a small angular displacement, occupies a new position and remains at rest.

4.5.2 Rotational Stability of Completely Submerged Body

In case of completely submerged body the position of center of gravity (G) & center of buoyancy is fixed.

Position of G & B determines the stability of submerged bodies:

Case 1: G lies below B ⇒ Stable Equilibrium

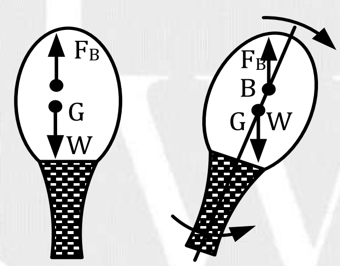

Fig: 4.3 Stable equilibrium

Case 2: G lies above B ⇒ Unstable Equilibrium

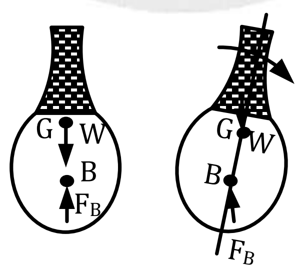

Fig: 4.4 Unstable equilibrium

Case 3: G coincides with B ⇒ Neutral Equilibrium

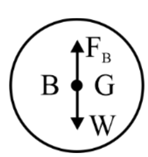

Fig: 4.5 Neutral equilibrium

4.5.3 Stability of Completely Floating Body

Meta-centre

If a body which is floating in liquid is given small angular displacement, it starts oscillating about some point M. This point is called meta-center. It can also be defined as the point at which the line of action of force of buoyancy will meet the normal axis of the body when body is given a small angular displacement.

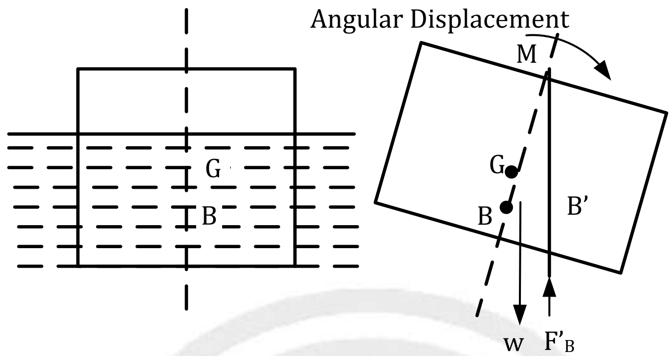

Fig: 4.6 Meta-center

Metacentric height (GM)

Distance between Metacenter of floating body and the center of gravity of the body is called Metacentric Height.

$$GM = BM - BG$$

Expression for calculating metacentric height:

Metacentric height,

$$GM = BM - BG$$

$$GM = \frac{I_{\min}}{V_d} - BG$$

Where,

GM = Metacentric Height

BM = Distance between center of buoyancy under equilibrium (B) & Meta-center (M) = Metacentric radius

BG = Distance between center of buoyancy under equilibrium (B) & center of gravity of body (G)

I = Moment of inertia of top view of floating body about axis of rotation.

= Minimum moment of inertia of top view at the free surface

V_d = Volume of fluid displaced

Rotational stability of floating body

Case I: If the point M is above G (i.e. Metacentric Height is +ve), the body is in stable equilibrium

Case II: If the point M is below G (i.e. Metacentric Height is -ve), the body is in unstable equilibrium

Case III: If the point M coincides with G (i.e. Metacentric Height is 0), the body is in neutral equilibrium.

4.6 Time Period of Oscillation

$$T = 2\pi\sqrt{\frac{k^2}{gGM}}$$

k = radius of gyration of floating body

$$GM_2 > GM_1$$

$$T_2 < T_1$$

Increasing the metacentric height gives greater stability to a floating body and reduces time period of rolling of the body.

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5 FLUID KINEMATICS

5.1 Introduction

Fluid Kinematics deals with the motion of Fluid without considering cause of motion.

5.1.1 Lagrangian approach

Lagrangian approach is for single Fluid particle. (Closed system analysis)

In Lagrangian approach fluid motion is defined by analysing the kinematic behaviour of each and every individual fluid particle consisting the flow.

$\vec{S} = \vec{S}\left(\vec{S_0}, t\right)$

Here, $\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{S_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$, $t = \text{time}$

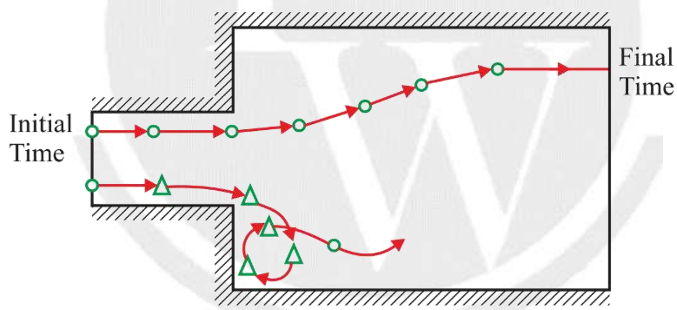

Fig. Lagrangian Approach
(Study of Each Particle With Time)

Fig.5.1 (Lagrangian Approach)

5.1.2 Eulerian approach

Eulerian approach for particular section (or) point. (Open system analysis)

In Eulerian approach fluid motion is defined by analysing the kinematic behaviour of various fluid particles passing through the various fixed points in a control volume.

$\vec{V} = \vec{V}\left(\vec{S}, t\right)$

Here $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, $\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$ and $t = \text{time}$

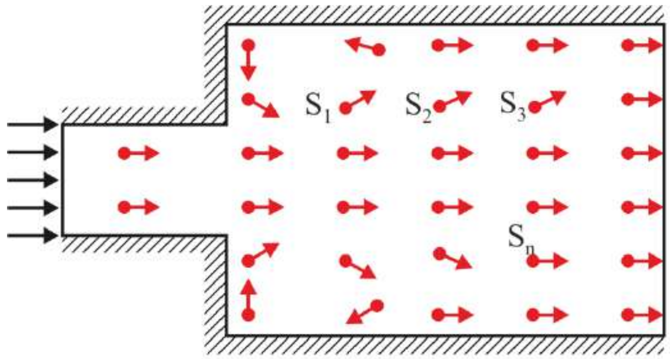

Fig.5.2 Eulerian Approach
(Study of particular section with time)

5.2 Velocity Vector

Velocity Vector in Cartesian Coordinates

Let *u*, *v*, *w* are component of velocity in *x*, *y*, *z* direction.

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$

Here

$$u = u(x, y, z, t)$$
$$v = v(x, y, z, t)$$
$$w = w(x, y, z, t)$$
$$|\vec{V}| = \sqrt{|u|^2 + |v|^2 + |w|^2}$$

Velocity Vector in Cylindrical Coordinates

$$\vec{V} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z}$$

$$v_r = \frac{dr}{dt} \text{ and } v_\theta = \frac{rd\theta}{dt},\ v_z = \frac{dz}{dt}$$

5.3 Acceleration Vector $(\vec{a})$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

$$|a| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

5.3.1 Convective/Advective Acceleration (a_c)

The part of acceleration which is due to change in velocity components with space (x, y, z) is known as Convective/Advective acceleration.

$$a_{c,x} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

5.3.2 Local/Temporal Acceleration (a_l)

The part of acceleration which is due to change in velocity components with time (t) is known as Local/Temporal Acceleration.

$$a_{l,x} = \frac{\partial u}{\partial t}$$

Note:

- Total Acceleration = Convective acceleration + local acceleration

5.3.3 Acceleration Vector in Cylindrical Coordinates

$$a_r = v_r\frac{\partial v_r}{\partial r} + v_\theta\frac{\partial v_r}{r\,\partial\theta} + v_z\frac{\partial v_r}{\partial z} + \frac{\partial v_r}{\partial t} - \frac{v_\theta^2}{r}$$

$$a_\theta = v_r\frac{\partial v_\theta}{\partial r} + v_\theta\frac{\partial v_\theta}{r\,\partial\theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t} + \frac{v_r v_\theta}{r}$$

$$a_z = v_r\frac{\partial v_z}{\partial r} + v_\theta\frac{\partial v_z}{r\,\partial\theta} + v_z\frac{\partial v_z}{\partial z} + \frac{\partial v_z}{\partial t}$$

5.4 Various Types of Fluid Flow

(i) Steady and Unsteady flow
(ii) Uniform and Non-Uniform flow
(iii) Compressible and Incompressible flow
(iv) Laminar and Turbulent flow
(v) Rotational and Irrotational flow
(vi) 1-D, 2-D & 3-D flow.

5.4.1 Steady and Unsteady Flow

A flow is said to be steady flow if fluid properties & flow velocity does not change with time at given location.

The flow which is not steady flow is called unsteady flow.

Steady Flow in Eulerian Approach

For a steady flow, acceleration in a given direction is equal to the convective acceleration in that particular direction, as local acceleration is zero.

$$\vec{V} = \vec{V}(\vec{S}, t) \qquad \text{(In general)}$$

For steady flow

$$\vec{V} \neq \vec{V}(t)$$

$$\therefore\ \vec{V} = \vec{V}(\vec{S}) \qquad \text{only}$$

5.4.2 Uniform and Non-Uniform Flow

A flow is said to be a uniform flow if fluid properties & flow velocity are does not change with space at any given instant of time.

The flow which is not uniform flow is called non-uniform flow.

Uniform Flow in Eulerian Approach

For a uniform flow, acceleration in a given direction is equal to the local acceleration in that particular direction as convective acceleration is zero.

$$\vec{V} \neq \vec{V}(\vec{S}) \text{ and } \vec{V} = \vec{V}(t) \text{ only}$$

Type of fluid flow	Convective Acceleration	Temporal Acceleration	Total Acceleration
Unsteady & Non-uniform	Exists	Exists	Exists
Unsteady & Uniform	Zero	Exists	Exists
Steady & Non-Uniform	Exists	Zero	Exists
Steady & Uniform	Zero	Zero	Zero

5.4.3 Compressible and Incompressible Flow

A flow is said to be incompressible flow if mathematically total derivative/material derivative of density is zero during the flow.

$$\frac{D\rho}{Dt} = 0$$

The flow which is not incompressible flow is compressible flow

Flow of compressible fluid can be incompressible flow, if Mach number must be less than equal to 0.3

5.4.4 Laminar & Turbulent Flow

1. Laminar flow

It is defined as the type of flow in which fluid particle move along well defined path or stream line.

2. Turbulent Flow

It is defined as the type of flow in which fluid particle move in Zig-zag way or random order.

5.4.5 Rotational and Irrotational Flow

A flow is said to be rotational flow if the fluid particles rotate about its own centre of mass during the motion otherwise flow is irrotational.

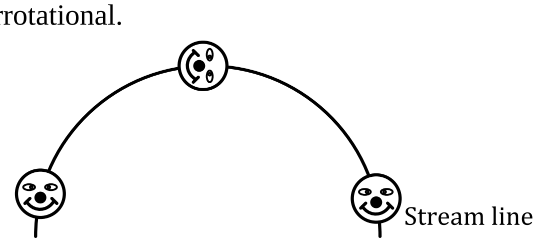

Fig.5.3 Rotaional Flow

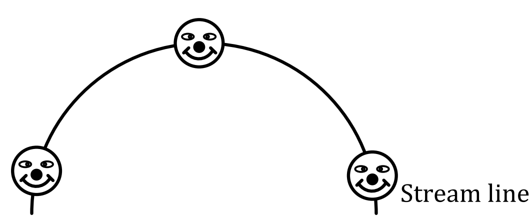

Fig.5.4 Irrotaional Flow

A flow will be Irrotational under the following two conditions:

(i) Ideal fluid flow (inviscid fluid flow)

(ii) Real fluid flow having negligible velocity gradient.

5.4.6 1-D, 2-D & 3-D Flow

A flow is classified as 1-D, 2-D or 3-D flow depending on the number of space coordinates required to specify the velocity field.

5.5 Various Flow Lines

- Streamline
- Path line
- Streak line
- Time line

5.5.1 Streamline

Streamline is an imaginary line or curve in the space such that tangent drawn to it at any point gives the direction of instantaneous velocity of fluid particle present at that point.

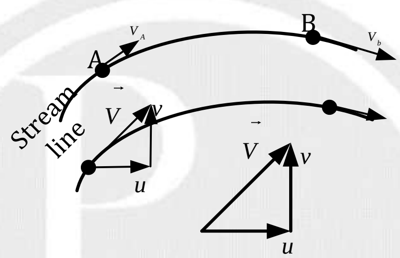

Fig.5.5 Stream lines

Note:

- Component of velocity in normal direction to the streamline is zero, hence there is no flow across the streamline.
- Streamline is drawn for an instant of time.
- Streamlines are based on Eulerian approach.
- Two streamlines cannot intersect each other.
- A given streamline cannot intersect itself.
- Streamlines are defined everywhere except at stagnation point.
- A bundle of neighbouring streamlines may be imagined to form a passage through which the fluid flows. This passage is known as stream tube.

Differential Equation of Streamline in Cartesian Coordinate

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

5.5.2 Path Line

- Path line is the path traced by single fluid particle at different instants of time.
- Path line is based on Lagrangian approach.
- A path line can intersect itself.

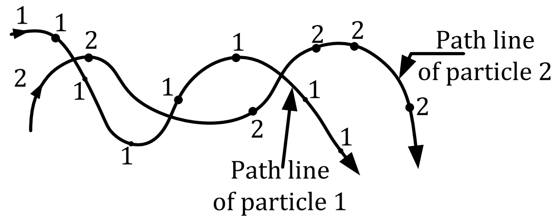

Fig.5.6 Path lines

Stream line	Path line
Direction of Instantaneous velocity	Position / Location
Number of fluid particles	Single fluid Particle
Particular instant of time	Over a period of time
Eulerian approach	Lagrangian approach
Can't intersect itself	Can intersect itself

5.5.3 Streakline

- Streakline is the locus of different fluid particles which passes through a given point.
- Streakline is drawn for a particular instant of time.

Note:
For a steady flow, streamline, path line and streak line are identical.

5.5.4 Timeline

- A timeline is a set of adjacent fluid particles in a flow field that were marked at a given instant of time.

5.6 Volume Flow Rate (Q)

Volume of fluid flowing through a cross section per unit time is known as volume flow rate.

$$Q = \int_{A_c} dA_c \times V_n$$

Units: m^3/s
Dimensional formula:

$$[Q] = [L^3T^{-1}]$$

5.7 Mass Flow Rate

$$\delta\dot{m} = \rho V_n dA$$

$$\dot{m} = \int_A \delta\dot{m}$$

If density is contant,

$$\dot{m} = \rho\dot{Q}$$

5.8 Concept of Average Velocity (V)

Average velocity is the hypothetical uniform velocity across the cross section, which gives same mass flow rate as that of actual mass flow rate through the cross section based on actual velocity.
Average velocity ⇒ Hypothetical uniform velocity

$$\dot{m}_{act} = \dot{m}_{avg}$$

$$\dot{m}_{av} = \rho A_c V$$

- According to the definition of Average Velocity $V = \frac{1}{A_c}\int_{A_C} u dA_c$ (Any cross – section)
- For pipe flow $V = \frac{2}{R^2}\int_0^R urdr$

5.9 Continuity Equation

- It is also known as conservation of mass

$$\dot{m}_i = \dot{m}_o + \dot{m}_{cv}$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla.(\rho\vec{V}) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(For any flow)}$$

5.9.1 Special Case 1: Steady Flow

- For steady flow $\frac{\partial \rho}{\partial t} = 0$

$$\therefore \quad \nabla.(\rho\vec{V}) = 0$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

5.9.2 Special Case 2: Incompressible Flow

$$\nabla.\vec{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- For an incompressible flow divergence of velocity vector is zero.

5.9.3 Special Case: For 1 D Steady flow

$$\rho AV = \text{Constant}$$

- For incompressible fluid flow

$$AV = \text{Constant}$$

5.10 Strain Rate

5.10.1 Linear Strain Rate ($\dot{\varepsilon}_{\forall}$)

It is defined as the rate of change of length per unit length.

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}, \quad \dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}, \quad \dot{\varepsilon}_{zz} = \frac{\partial w}{\partial z}$$

5.10.2 Volume Strain Rate ($\dot{\varepsilon}_{\forall}$)

The rate of change in volume per unit volume

$$\dot{\varepsilon}_{\forall} = \frac{1}{\forall}\frac{D\forall}{Dt} = \nabla \cdot \vec{V}$$

- It is also known as Dilation rate

$$\dot{\varepsilon}_{\forall} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz}$$

$$\dot{\varepsilon}_{\forall} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- For Incompressible flow

$$\dot{\varepsilon}_{\forall} = 0$$

5.10.3 Shear Strain Rate

$$\dot{\varepsilon}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \dot{\varepsilon}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \dot{\varepsilon}_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

5.11 Condition for Pure Rotation (Zero Angular Deformation)

For pure rotation, angular deformation is zero

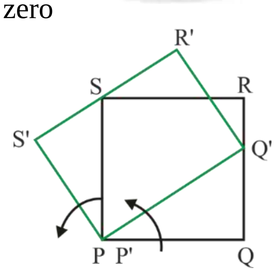

Fig.5.7

For 2 − D (x − y plane) Flow

$$\dot{\varepsilon}_{xy} = 0$$

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

5.12 Angular Velocity Vector

$$\omega = \frac{1}{2}\left(Curl\left(\vec{V}\right)\right) = \frac{1}{2}\vec{\nabla}\times\vec{V}$$

⇒ Angular velocity vector is half of curl of velocity vector.

$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\omega_{xy} = \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_{yz} = \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_{zx} = \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Condition for Irrotational Flow in 2-D (x-y plane)

- For irrotational flow, $\nabla \times \vec{V} = \vec{0}$

$$\omega_z = 0$$

$$\therefore \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

- For irrotational flow, curl of velocity vector is zero vector

5.12.1 Condition for Zero Rotation

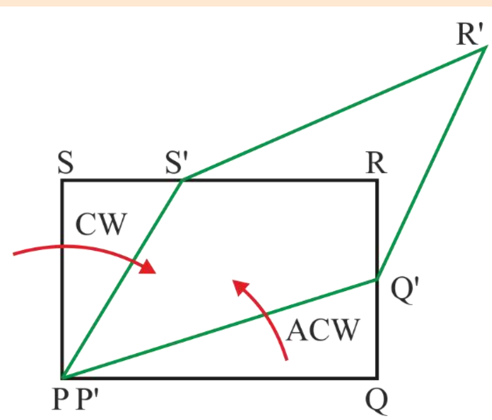

Fig.5.8

- For 2 – D(x – y plane) Flow

$$\omega_{xy} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

5.13 Vorticity Vector $(\vec{\Omega})$

Curl of velocity vector is known as vorticity vector.

$$\vec{\Omega} = 2\vec{\omega}$$

(Vorticity vector is twice of the angular velocity vector of the fluid particle)

$$\vec{\Omega} = \nabla \times \vec{V}$$

- For Irrotational Flow vorticity vector is null vector

$$\vec{\Omega} = 2\vec{\omega} = \vec{0}$$

$$\nabla \times \vec{V} = \vec{0}$$

Vorticity is a measure of rotation of a fluid particle.

5.14 Circulation (Γ)

Circulation is defined as the line integral of the tangential component of the velocity vector taken around a closed curve

Circulation represents the strength of a rotational flow.

$$\Gamma = \oint_C \vec{V} . \overrightarrow{dr}$$

$$\Gamma = \Omega A \text{ (Vorticity x inclosed area)}$$

'C'(anti clock wise) in the flow field.

5.15 Velocity Potential Function (ϕ)

Velocity potential function is defined as the scalar function of space and time such that the partial differentiation with respect to any direction gives the component of velocity in that direction for irrotational flow.

Mathematically,

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}, w = \frac{\partial \phi}{\partial z}$$

Some important point:

(i) Existence of velocity potential function implies that flow is irrotational.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$\omega_z = 0$$

Similarly,

$$\omega_x = 0 \text{ and } \omega_y = 0$$

Since $\omega_x = \omega_y = \omega_z = 0$, hence flow will be irrotational therefore we can conclude that existence of velocity potential function implies that flow is irrotational.

(ii) If velocity potential function satisfies Laplace equation, the incompressible flow is possible.

Differentiating equation 1 with respect to x, y and z direction respectively,

$$\frac{\partial u}{\partial x} = \frac{\partial^2\phi}{\partial x^2}, \frac{\partial v}{\partial x} = \frac{\partial^2\phi}{\partial y^2}, \frac{\partial w}{\partial x} = \frac{\partial^2\phi}{\partial z^2} \qquad \ldots\text{(ii)}$$

Laplace equation is given by

$$\nabla^2\phi = 0$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \ldots\text{(iii)}$$

From equation (ii) and (iii),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Above equation is continuity equation for incompressible fluid. Hence flow will be incompressible possible flow when potential function satisfies Laplace equation.

(iii) Velocity potential function (ϕ) is exact function or point function:

Let, Velocity potential function (ϕ) is two dimensional, i.e.

$$\phi = f(x, y)$$

Since ϕ is exact function, therefore,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

And $d\phi$ will be exact differential, therefore comparing $d\phi$ with exact differential, $d\phi = Mdx + Ndy$,

$$M = \frac{\partial\phi}{\partial x} \text{ and } N = \frac{\partial\phi}{\partial y}.$$

And if $M \& N$ is known then ϕ is determine by the expression given below:

$$\boxed{\phi = \left[\int_{y=\text{const.}} Mdx + \int(\text{terms of } N \text{ nor containg } x)dy\right]}$$

(iv). Equipotential lines

A line along which, velocity potential function (ϕ) remains constant, is called equipotential line.

Slope of equipotential line for 2-D flow $\left(\frac{dy}{dx}\right)$:

For 2D flow,

$$\phi = f(x, y)$$

Since ϕ is exact function, therefore,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

Since, for equipotential line, ,$\phi =$ constant $\Rightarrow d\phi = 0$

Therefore, from above equation

$$0 = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \quad \left[\because u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}\right]$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = -\frac{u}{v}} \quad \text{(Except stagnation point)}$$

5.15.1 Velocity Potential function in Cylindrical Coordinate system

For irrotational flow, the velocity potential function in cylindrical coordinates are.

$$\frac{\partial \phi}{\partial r} = V_r$$

$$\frac{\partial \phi}{r\,\partial \theta} = V_\theta$$

$$\frac{\partial \phi}{\partial z} = V_z$$

5.16 Stream Function Ψ

It is defined as the scalar function of space and time such that its differentiation with respect to any axis gives the component of velocity at right angle to this axis. It exists only for 2-D flow.

Mathematically,

$$u = \frac{\partial \psi}{\partial y} \,\&\, v = -\frac{\partial \psi}{\partial x}$$

Where u, v are component of velocity in x, y direction respectively

Some important point:

(i) Existence of stream function signifies that the flow is incompressible and possible.

From equation (i)

$$\frac{\partial u}{\partial x} = \frac{\partial^2 \psi}{\partial x \partial y}, v = \frac{-\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The above equation is continuity equation for 2-D incompressible flow. Hence, we can conclude that existence of stream function signifies that the flow is incompressible and possible.

(ii) If Stream function satisfies the Laplace equation, then flow will be irrotational.

Laplace equation for 2-D is given by

$$\nabla^2 \psi = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\omega_z = 0$$

Since $\omega_z = 0$ is zero (i.e. flow is irrotational) when stream function satisfying the laplace equation, hence we can conclude that flow will be irrotational if stream function satisfies the laplace equation otherwise flow will be rotational.

(iii) The flow rate per unit depth between two stream line (S_1 & S_2) is constant & given by,

$$Q = \psi_1 - \psi_2$$

Where,

$Q =$ flow rate per unit depth between to stream line (S_1 & S_2)

ψ_1 = equation of stream line S_1

ψ_2 = equation of stream line S_2

(iv) Stream function (ψ) is exact function or point function:

Let, $\psi = f(x, y)$

Since ψ is exact function, therefore,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

And $d\psi$ will be exact differential, therefore comparing $d\psi$ with exact differential, $d\psi = Mdx + Ndy$,

$$M = \frac{\partial \psi}{\partial x} \text{ and } N = \frac{\partial \psi}{\partial y}.$$

And if M and N is known then ψ is determine by the expression given below:

$$\boxed{\psi = \left[\int_{y=\text{const.}} M dx + \int (\text{terms of } N \text{ nor containg } x) \right]}$$

(v) **Stream line**

Line along which stream function is constant is known as stream line.

Slope of stream line for 2-D flow $\left(\frac{dy}{dx}\right)$,

For 2D flow,

$$\psi = f(x, y)$$

Since ψ is exact function, therefore,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

Since, for stream line, $,\psi = \text{constant} \Rightarrow d\psi = 0$

Therefore, from above equation

$$0 = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} \qquad \left[\because u = \frac{\partial \psi}{\partial y} \, \& \, v = -\frac{\partial \psi}{\partial x} \right]$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{v}{u}} \qquad \text{(Except stagnation point)}$$

Some important points regarding velocity potential function and stream function

(i) If stream function & velocity positional function both will exist, then flow will be both incompressible possible flow & irrotational flow.

(ii) Product of slope of Stream line & equi-potential line for two dimensional flow is -1 at all point except stagnation point i.e. Stream line & equipotential line intersect each other orthogonally at all point except stagnation point.

Let,

$m_1 =$ Slope of equipotential line

$m_2 =$ Slope of stream line

Then,

$$m_1 = \frac{-v}{u} \qquad \text{(Except stagnation point)}$$

And,

$$m_2 = \frac{u}{v} \qquad \text{(Except stagnation point)}$$

Therefore,

$$m_1 . m_2 = \frac{-v}{u} . \frac{u}{v}$$

$\Rightarrow$ $m_1 . m_2 = -1$ (Except stagnation point)

(iii) If for two-dimensional steady flow u and v is given, then

(a) $\psi \rightarrow$ exist, only when u and v satisfy continuity equation

(b) $\phi \rightarrow$ exist, only when flow is irrotational i.e. $\omega_z = 0$

(c) Both ψ and ϕ will exist when u and v satisfy continuity equation and flow is irrotational i.e. $\omega_z = 0$.

(iv) If for two-dimensional steady flow ϕ is given, then

(a) $\psi \rightarrow$ exist, only when flow is possible flow and for that ϕ must satisfy Laplace equation.

(v) If for two-dimensional steady flow ψ is given, then

(a) $\phi \rightarrow$ exist, only when flow is irrotational and for that ψ must satisfy Laplace equation.

(vi) If both ψ and ϕ exist, then they satisfy the CR equation as given below:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Points to be Remember				
Type	Existence		Satisfy Laplace Equation	
	Yes	No	Yes	No
Stream Function (ψ)	Possible flow	not possible flow	Irrotational flow	Rotational flow
Velocity Potential Function (ϕ)	Irrotational flow	Rotational flow	Possible flow	not possible flow

5.17 Flow Net

- For a fluid flow graphical representation of streamlines & equipotential lines together is known as flow net.
- Flow net exists for 2-D, steady, incompressible & irrotational flow only.

❑❑❑

6 FLUID DYNAMICS

6.1 Introduction

Fluid dynamics deals with the motion of fluid by considering cause of motion i.e. force. Major forces acting on the fluid particle are pressure force, gravity force and viscous force.

6.1.1 Euler's Equation

- Assumptions
  (i) Flow is inviscid flow
  (ii) Flow is along the stream line

$$\frac{dP}{\rho g} + \frac{ds}{g}\left(v\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t}\right) + dz = 0$$

If flow is steady:

$$\frac{dp}{\rho} + vdv + gdz = 0$$

6.1.2 Bernoulli's Equation

Assumptions:

- Flow is inviscid flow
- Flow is taking place along the streamline direction only
- Flow is steady flow
- Flow is incompressible fluid flow

$$\frac{P}{\rho g} + \frac{V^2}{2g} + z = \text{constant}$$

Pressure head = $\frac{P}{\rho g}$

Velocity Head/Dynamic Head = $\frac{V^2}{2g}$

Potential Head/ Datum Head = z

Piezometric head $h_P = \frac{P}{\rho g} + z$

Stagnation head $h_{stg} = \frac{P}{\rho g} + \frac{V^2}{2g}$

Note:

- Bernoulli's equation is applicable along the streamline only if flow is rotational whereas Bernoulli's equations is applicable across the streamlines, if the flow is irrotational.

6.2 Shortcuts for Applying Bernoulli's Equation

Case 1: Datum head is same at both points

Case 2: Velocity head is same at both points

Case 3: Pressure head is same at both points

6.2.1 Case-1: Datum Head is same at Both Points ($z_1 = z_2$)

Fig.6.1 Flow across horizontal pipe

$z_2 = z_1$

$P_1 - P_2 = \frac{1}{2}\rho(V_2^2 - V_1^2)$

Special case:

If $D_2 = \frac{1}{2}D_1$

$P_1 - P_2 = \frac{15}{2}\rho V_1^2$

6.2.2 Case-2: Velocity head is same at Both Points ($V_1 = V_2$)

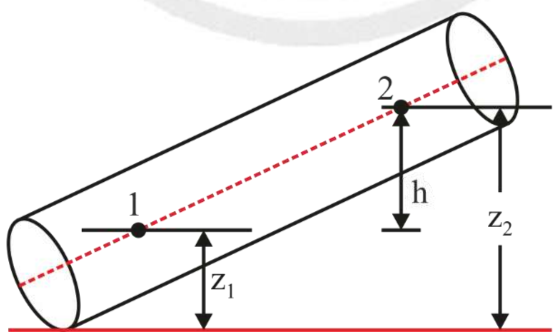

Fig.6.2 Flow across inclined pipe

$P_1 - P_2 = \rho gh$

6.2.3 Case-3: Pressure Head is same at Both Points ($P_1 = P_2$)

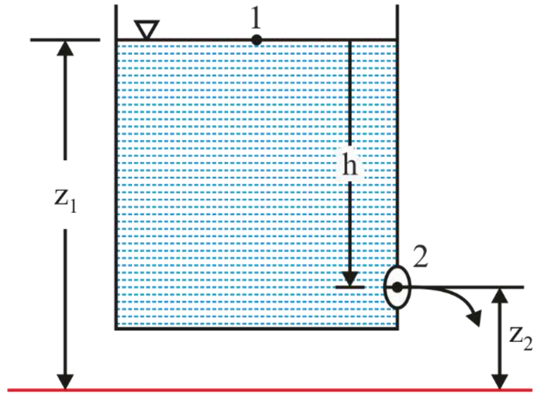

Fig.6.3 Flow through orifice

$$V_2^2 - V_1^2 = 2gh$$

If $A_1 \ggg A_2$, $V_1 \ll V_2$

, $V_2 = \sqrt{2gh}$

6.2.4 Modified Bernoulli's Equation

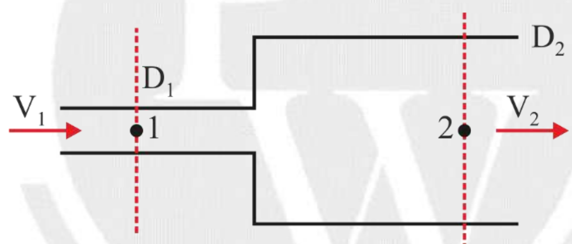

Fig.6.4 Flow across two sections in a horizontal pipe

Considering viscous flow and other losses

$$\frac{P_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{V_2^2}{2g} + z_2 + h_L$$

Where h_L = Head loss

6.3 Time Required to Empty a Rectangular Base Tank

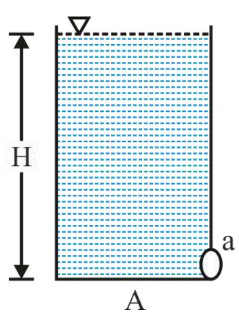

Fig. 6.5 Flow through small hole at the base

A = Tank cross sectional area
a = Tank opening area

$$T = \frac{A}{a}\sqrt{\frac{2}{g}}\sqrt{H}$$

Time required to decrease the level from H_1 to H_2

$$T = \frac{A}{a}\sqrt{\frac{2}{g}}\left[\sqrt{H_1} - \sqrt{H_2}\right]$$

6.4 Discharge Measurement with Help of Inclined Venturimeter

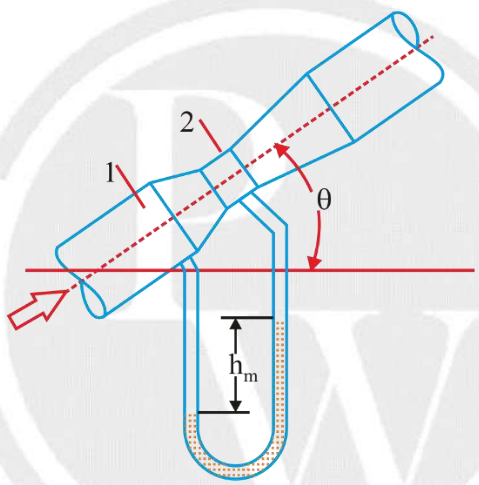

Fig. 6.6

$$Q = \left(\frac{A_1 A_2\sqrt{2g\Delta h_P}}{\sqrt{A_1^2 - A_2^2}}\right)$$

Where Δh_p = Piezometric head difference

For inclined Venturi meter $\Delta h_P = \left(\frac{P_1 - P_2}{\rho g}\right) + (z_1 - z_2)$

For horizontal Venturi meter $(z_1 = z_2)$ $\Delta h_P = \frac{P_1 - P_2}{\rho g}$

A_1 ⇒ Cross-sectional area of pipe

A_2 ⇒ Cross-sectional area of Throat

6.4.1 Measurement of Piezometric Head Difference with the Help of Piezometers

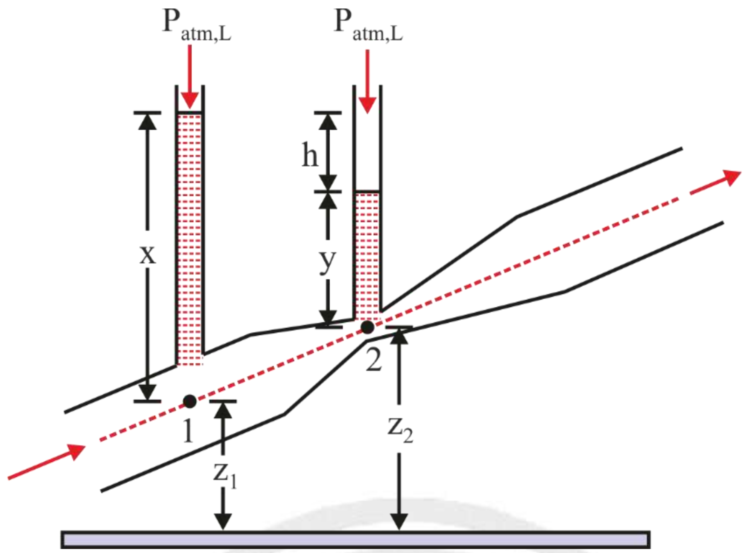

Fig. 6.7 Flow measurement through inclined Venturimeter

$$\Delta h_p = h$$

$$Q = \frac{A_1 A_2 \sqrt{2gh}}{\sqrt{A_1^2 - A_2^2}}$$

6.4.2 Measurement of Piezometric Head Difference with the Help of U-Tube Differential Manometer

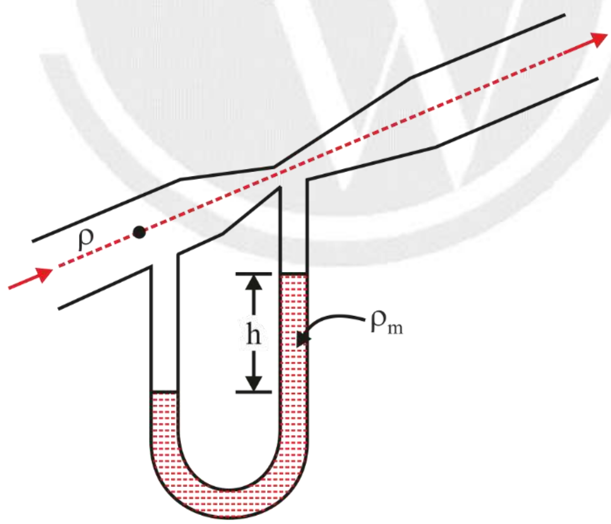

Fig.6.8 Deflection in the height of manometric column

$$\Delta h_p = \left(\frac{\rho m - \rho}{\rho}\right) h$$

$$Q = \frac{A_1 A_2 \sqrt{2g\left(\frac{\rho m - \rho}{\rho}\right) h}}{\sqrt{A_1^2 - A_2^2}}$$

6.4.3 Measurement of Piezometric Head Difference with the Help of Inverted U-Tube Differential Manometer $[\rho_m < \rho]$

$$\Delta h_P = \left(\frac{\rho-\rho_m}{\rho}\right) h$$

6.4.4 Expression for Actual Discharge (Q')

$$Q' = c_d Q$$

Where c_d is coefficient of discharge

h_L = head loss

$$c_d = \sqrt{1 - \frac{h_L}{\Delta h_P}}$$

$$0.95 \le c_d \le 0.99$$

6.5 Velocity Measurement for a Channel

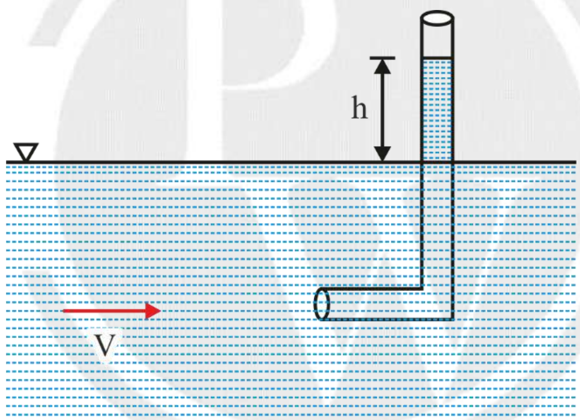

Fig.6.9 Velocity measurement through Pitot tube

$$V' = c_v\sqrt{2g\Delta h_P}\ (0.95 \le c_v \le 0.98)$$

$$\Delta h_P = h$$

$$V' = c_v\sqrt{2gh}$$

6.6 Velocity Measurement for a Pipe Flow

Case 1: with the help of piezometer and pitot tube

Case 2: with the help U - tube differential manometer & pitot tube

Case 3: with the help of Inverted U - tube differential manometer & pitot tube

6.6.1 Case 1: With the Help of Piezometer and Pitot Tube

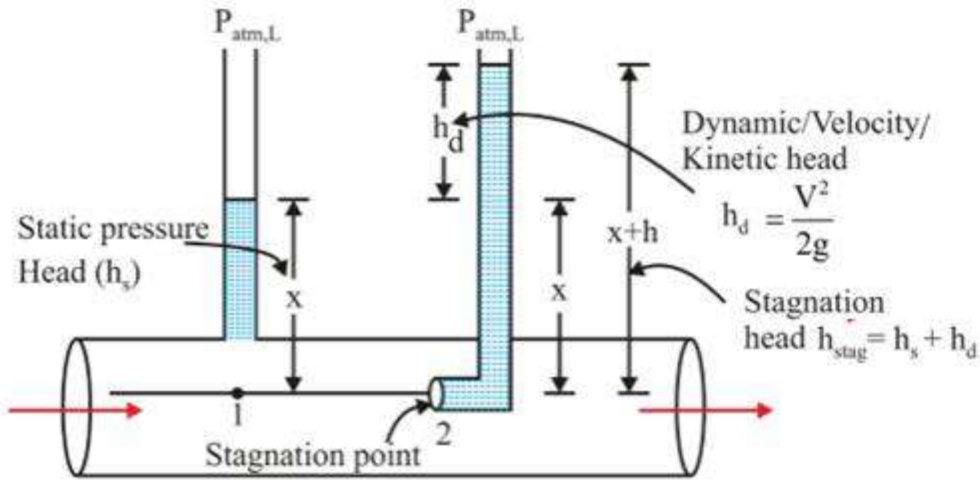

Fig.6.10 Velocity measurement with the help of piezometer & Pitot tube

$$V' = c_v\sqrt{2gh_d}$$

$$h_d = h_{stag} - h_s$$

6.6.2 Case-2: With the Help of U-Tube Differential Manometer & Pitot Tube

$$\rho_m > \rho$$

$$V' = c_V\sqrt{2g\Delta h_P}$$

$$\Delta h_P = \left(\frac{\rho_m - \rho}{\rho}\right) h$$

6.6.3 Case-3: With the Help of Inverted U-Tube Differential Manometer & Pitot Tube.

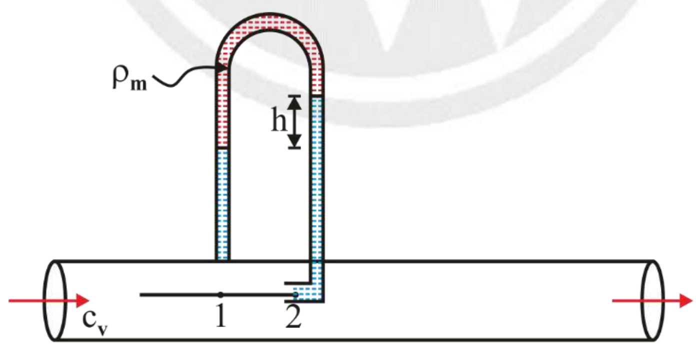

Fig.6.11 Velocity measurement with the help of inverted U-tube differential manometer and Pitot tube

$$V' = c_V\sqrt{2g\left(\frac{\rho - \rho_n}{\rho}\right) h}$$

6.7 Pitot-Static Tube

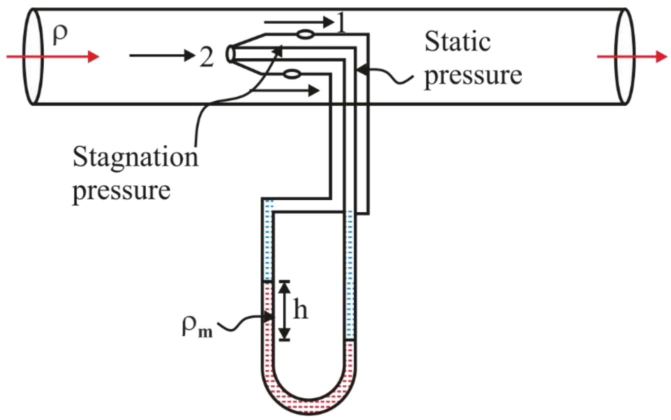

Fig.6.12 Pitot static tube

$$V' = c_v\sqrt{2g\left(\frac{\rho_m-\rho}{\rho}\right)h}$$

6.8 Force Exerted by a Flowing Fluid on a Pipe Bend

Impulse Momentum Equation

It is based on the law of conservation of momentum or on the momentum principle, which states that the net force acting on a fluid mass is equal to the change in momentum of flow per unit time in that direction.

$$F = \frac{d(mv)}{dt}$$

Where F is force acting on fluid of mass m in a short interval of time dt.

6.8.1 Force Exerted by a Flowing Fluid on a Pipe Bend

Assume:

Inviscid, steady & incompressible fluid flow and elbow is horizontal.

In the impulse momentum equation pressure is always the gauge pressure.

Various type of forces acting on the control volume -

1. Impact Force

It is the impact of fluid coming into the pipe and acts in the direction of incoming jet

2. Reaction Force of Leaving Jet

It acts due to the reaction of leaving jet, in the direction opposite to the direction of leaving jet.

3. Pressure force

Pressure force acts in the bend due to the pressure of fluid. To determine the direction of pressure force it can be assumed that the control surface is submerged in fluid and the direction of the hydrostatic force will be the direction of the pressure force.

4. Weight

Always acts downward.

5. Reaction Forces

The control surface is in equilibrium because of the reaction forces.
Consider a fluid on a pipe bend with section 1 and 2 as shown in figure. 6.12

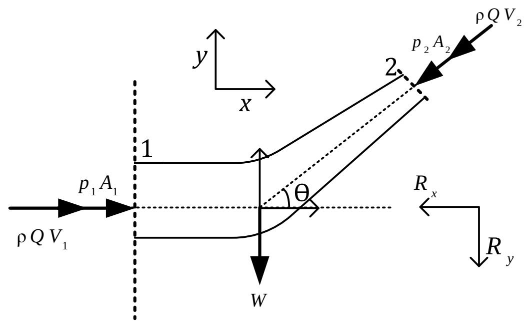

Fig.6.13 Force exerted on pipe bend

Force acting in x direction,

$$F_x = \underbrace{\rho QV_1}_{(\text{impact force})} - \underbrace{\rho QV_2 Cos\theta}_{(\text{reaction force})} + \underbrace{\left(p_1A_1 - p_2A_2Cos\theta\right)}_{(\text{pressure force})}$$

Force acting in y direction,

$$F_y = \underbrace{0}_{(\text{impulse force})} - \underbrace{\rho QV_2 \sin\theta}_{(\text{reaction force})} - \underbrace{\left(p_2A_2\sin\theta\right)}_{(\text{pressure force})} - \underbrace{W}_{(\text{weight})}$$

Reaction force by pipe bend to the fluid for equilibrium is given by,

$$R_x = -F_x$$
$$R_y = -F_y$$

6.9 Force Exerted By Jet, Striking on the Fixed Vertical Plate at the Centre

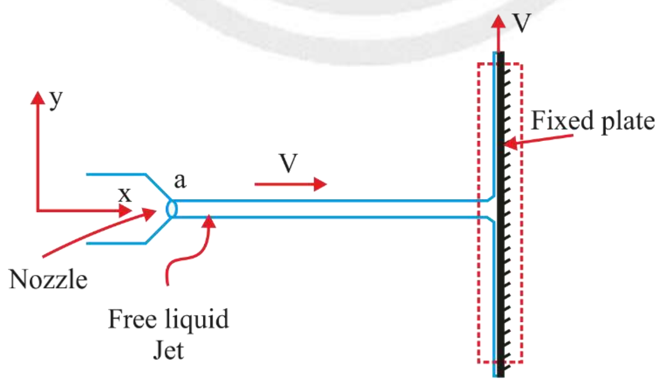

Fig.6.14 – Force Exerted by Jet, Striking on the Fixed Vertical Plate at the Centre

$$F_x = \rho a V^2 = \rho Q V$$

Where Cross-sectional of jet is a and Velocity of Jet is V

$$a = \text{area of jet}\left(\frac{\pi}{4}d^2\right)$$

d = diameter of Jet / diameter of nozzle exit

Q = discharge

6.10 Angular Momentum Equation (Sprinkler Case)

Net torque = Rate of change of angular momentum

= Final angular momentum rate – Initial angular momentum rate

$$\Sigma T = (\dot{m}vr)_f - (\dot{m}vr)_i$$

6.11 Application of Angular momentum equation (sprinkler)

6.11.1 Case 1: Sprinkler is Fixed

$\Sigma T max$, $\omega = 0$

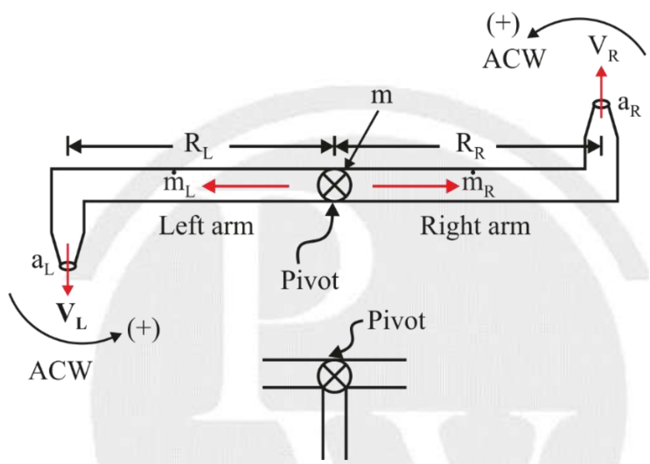

Fig.6.15 Sprinkler is fixed

Net torque acting on fluid element

$$\Sigma T = (\dot{m}_L V_L R_L) + (\dot{m}_R V_R R_R)$$

6.11.2 Case 2: Sprinkler Rotates Freely

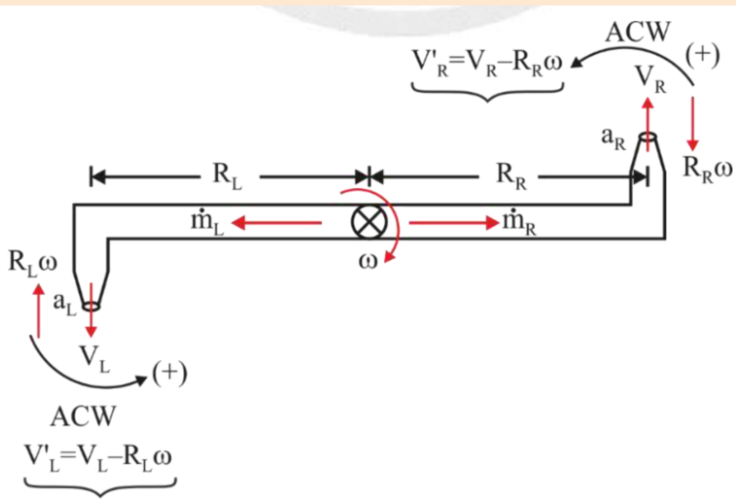

Fig.6.16 Sprinkler rotates freely

$$\dot{m}_L V_L' R_L + \dot{m}_R V_R' R_R = 0$$

6.12 Navier Stokes Equation

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu(\nabla^2\vec{V}) + \frac{1}{3}\nu\{\nabla(\nabla.\vec{V})\} + \vec{g}$$

For incompressible flow

$$\frac{D\overline{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu(\nabla^2\overline{V}) + g$$

For inviscid, incompressible flow $\frac{D\overline{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \overline{g}$

Where

$$\frac{D\vec{V}}{Dt} = \vec{a} = a_x\hat{\imath} + a_y\hat{\jmath} + a_z\hat{k}$$

$$\nabla P = \frac{\partial P}{\partial x}\hat{\imath} + \frac{\partial P}{\partial y} + \hat{\jmath} + \frac{\partial P}{\partial z}\hat{k}$$

$$(\nabla^2\vec{V}) = \nabla^2 u\hat{\imath} + \nabla^2 v\hat{\jmath} + \nabla^2 w\hat{k}$$

$$\text{Here, } \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}$$

$$(\nabla.\vec{V}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Flow is Incompressible Flow

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \nu\nabla^2\vec{V} + \vec{g}$$

- Flow is Inviscid, Incompressible flow

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho}(\nabla P) + \vec{g}$$

$$\frac{\partial P}{\partial x} = -\rho a_x,\ \frac{\partial P}{\partial y} = -\rho a_y,\ \frac{\partial P}{\partial z} = -\rho a_z + g$$

Note:

- For inviscid, incompressible flow

  Navier stoke' s equation in X direction $a_x = -\frac{1}{\rho}\ \frac{\partial P}{\partial x}$

  Y direction $a_y = -\frac{1}{\rho}\ \frac{\partial P}{\partial y}$

  Z direction $a_z = -\frac{1}{\rho}\ \frac{\partial P}{\partial z}$

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7 FLOW THROUGH PIPES

7.1 Loss of energy while flow through pipes

When a fluid is flowing through a pipe, the fluid experiences some resistance due to which some of the energy of fluid is lost. This loss of energy is classified as:

1. Major energy losses
2. Minor energy losses

7.2 Major energy losses

It occurs mainly due to the friction. It is calculated by the following formulae:

1. Darcy-Weisbach Formula
2. Chezy's Formula

7.2.1 Darcy-Weisbach Formula

Frictional head losses are losses due to shear stress on the pipe walls. The general equation for head loss due to friction is the Darcy-Weisbach equation, which is given below,

$$h_f = \frac{fLV^2}{2gD} = \frac{fLQ^2}{12.1d^5}$$

Where,

L = Length of pipe

D = Diameter of pipe

V = mean velocity of flow

$f = 4f'$ = friction factor

f' = coefficient of friction

h_f = Head loss due to friction

Q = Discharge

Note:

For Laminar flow

$$f_{\text{laminar}} = \frac{64}{R_e}$$

7.2.2 Chezy's Relation

Chezy's formula is given below:

$$V = C\sqrt{mi}$$

Where,

V = velocity of fluid

C = chezy's constant

$$m = \frac{A}{P} = \frac{\text{area of flow}}{\text{wetted perimeter of pipe}}$$

i = loss of head per unit length of pipe = hydraulic slope

Loss due to friction h_f is given by,

$$h_f = i.L$$

7.3 Minor energy losses

Minor losses are the losses which are minor in magnitude for very long pipes but not necessarily for shorter pipes. The minor loss of energy includes the flowing cases.

7.3.1 Sudden Expansion Loss

- Boundary layer separation leads to the formation of eddies which causes sudden expansion loss.

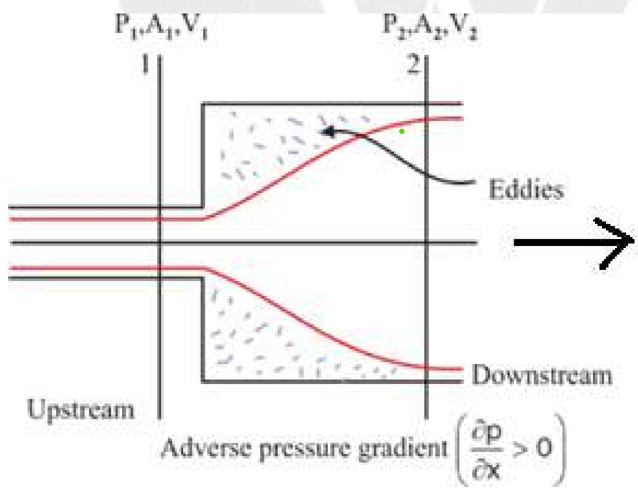


Fig.7.1 Flow representing sudden expansion

$$h_{L,SE} = \frac{(V_1 - V_2)^2}{2g} = \frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2$$

7.3.2 Exit Head Loss

$$h_{L,Ex} = \frac{V_1^2}{2g}$$

7.3.3 Sudden Contraction Loss

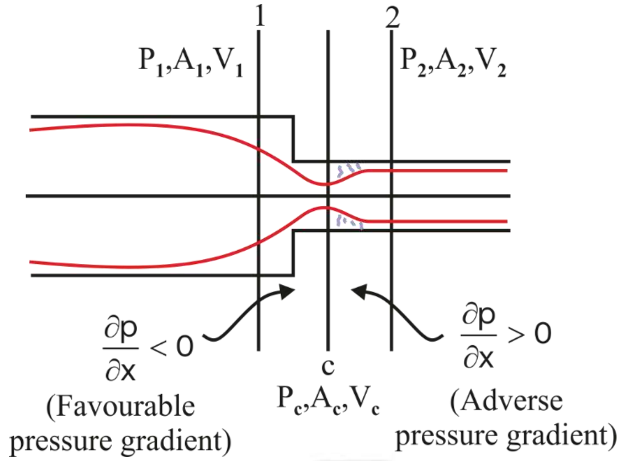


Fig.7.2 Flow representing sudden contraction

Sudden contraction loss is actually due to suddenly expansion from vena contracta to smaller diameter pipe.

$$h_{L,SC} = \left(\frac{1}{c_c} - 1\right)^2 \frac{V_2^2}{2g}$$

Where C_c = coefficient of contraction

$$\left(\frac{1}{C_c} - 1\right)^2 = k_c = \text{Contraction factor}$$

If C_c is not given then $h_{L,SC}$ is given by,

$$h_{L,SC} = \frac{0.5V_2^2}{2g}$$

7.3.4 Entrance Head Loss

$$\mathbf{h_{L,EN} = K_{L,EN} \frac{V_2^2}{2g}}$$

$K_{L,EN}$ = Entrance head loss coefficient

If $\mathbf{K_{L,EN}}$ is not given then take $\mathbf{K_{L,EN} = 0.5}$.

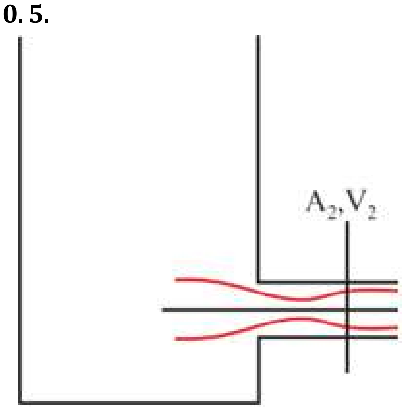


Fig.7.3 Flow representing entrance in a pipe from reservoir

7.3.5 Bend Head Loss

$$h_{L,B} = k\frac{V^2}{2g}$$

$k =$ Coefficient of Bend

7.3.6 Pipe Head Loss

$$h_{L,p} = k\frac{V^2}{2g}$$

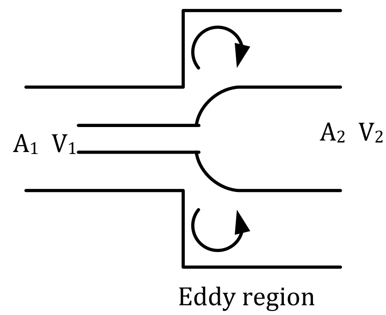
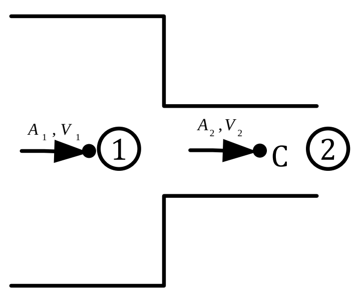
k = Coefficient of valve head loss

7.3.7 Summary

Various Losses (h_L) is calculated by using the formula,

$$h_L = K\frac{u^2}{2g}$$

Where, K is coefficient due to disturbance in pipe flow and u is reference velocity.

Conditions	**K**	**Reference Velocity(u)**	Various losses in Pipe
Major loss	$\frac{fl}{d}$	V	-
Sudden enlargement	$\left(1-\frac{A_1}{A_2}\right)^2$	V_1	
Sudden contraction	$\left(\frac{1}{C_c}-1\right)^2$ or 0.5(if C_c is not given)	V_2	

Loss at Exit	1	V	-
Loss at Entrance	$K_{L,EN}$	V	-
Loss due to bend	K(Coefficient of bend)	V	-
Loss due to various Pipe Fitting	K (Coefficient of pipe fitting)	V	-

7.4 Flow Between Two Reservoirs

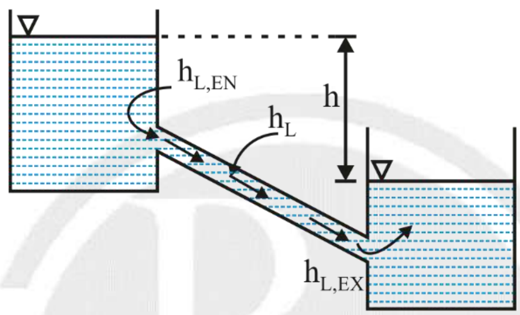


Fig.7.4 Flow between two reservoirs

$$h'_L = h_{L,EN} + h_L + h_{L,EX}$$

$$h = h_{L,EN} + h_L + h_{L,Ex}$$

$$\therefore \ h = h'_L$$

7.5 Power Transmission Through Pipes

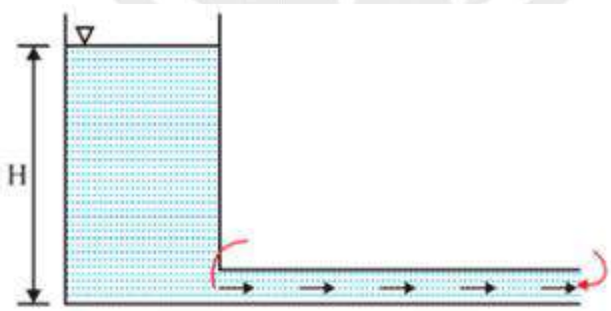


Fig.7.5 Power transmission through pipe

- Power input $P_i = \rho g Q H$
- Power output: $P_o = \rho g Q(H - h_L)$ Watt $\leftarrow$ head loss h_L in pipe
- For maximum power output

$$h_L = \frac{H}{3}$$

$$\therefore \qquad (P_o)_{max} = \frac{2}{3}\rho g Q H \text{ (h-h}_L\text{)}$$

Note:

$$\eta_t = \frac{P_o}{P_i}$$

Maximum power transmission efficiency = 66.67%

7.6 Series Arrangements of Pipes

When two or more pipes of different diameters or roughness are so connected that the full discharge of the fluid from one flow into the others serially, the system represents a series pipeline.

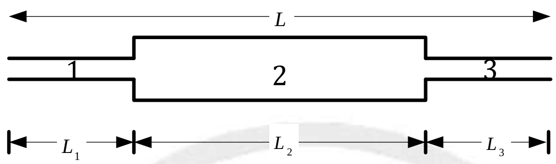


Fig.7.6 Pipes in Series

In series pipeline,

(i) Total head loss (H_L) is the sum of head loss in each Pipeline,

$$H_L = H_{L1} + H_{L2} + H_{L3}$$

Where,

H_{L1}, H_{L2}, H_{L3} are head loss in pipe 1,2 & 3 respectively

(ii) Discharge remains constants in series fitting of pipes

$$Q_1 = Q_2 = Q_3 = Q$$

- $Q_1 = Q_2 = Q$, $h_{L,T} = h_{L_1} + h_{L_2}$

7.7 Parallel Arrangement of Pipes

A combination of two or more pipes connected between two points so that the discharge divides at the first junction and rejoins at the next is known as pipes in parallel (figure 5.6). Here the head loss between the two junctions (M and N in Figure) is same for all the pipes.

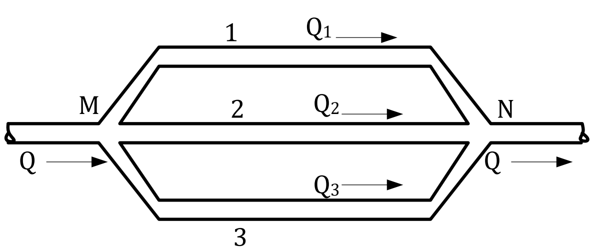


Fig.7.7 Pipes in parallel

In parallel pipeline,

(i) Head loss

$$H_{L1} = H_{L2} = H_{L3} = h_M - h_N$$

Where,

H_{L1}, H_{L2}, H_{L3} are head loss in pipe 1,2 & 3 respectively

(ii) Discharge is the sum of discharge in respective pipes

$Q = Q_1 + Q_2 + Q_3$

7.8 Equivalent Pipe

Equivalent pipe is a method of reducing a combination of pipes into a simple pipe system for easier analysis of a pipe network, such as a water distribution system.

An equivalent pipe is an imaginary pipe in which the head loss and discharge are equivalent to the head loss and discharge for the real pipe system. There are three main properties of a pipe: diameter (D), length (L), and roughness.

(i) A pipe of length L_1, diameter D_1 & friction factor f_1 will be equivalent to another pipe of corresponding parameter L_2, D_2 & f_2 if,

$$\frac{f_1 L_1}{D_1^5} = \frac{f_2 L_2}{D_2^5} \quad \ldots\text{(i)}$$

(ii) If a set of pipes described by (L_1, D_1, f_1), (L_2, D_2, f_2) ,..........are connected in series then equivalent pipe of parameter (L_e, D_e, f_e) is related as:

We know that in series total discharge, $Q = Q_e = Q_1 = Q_2 = \ldots \ldots\text{(ii)}$

And total head loss,

$$h_L = h_{L_e} = h_{L_1} + h_{L_2} + \ldots \ldots\text{(iii)}$$

Now by neglecting minor losses:

$$\Rightarrow \quad \frac{f_e L_e Q^2}{12 D_e^5} = \frac{f_1 L_1 Q_1^2}{12 D_1^5} + \frac{f_2 L_2 Q_2^2}{12 D_2^5} + \ldots \quad [\because Q = Q_e = Q_1 = Q_2 = \ldots]$$

$$\Rightarrow \quad \frac{f_e L_e Q^2}{12 D_e^5} = \frac{f_1 L_1 Q^2}{12 D_1^5} + \frac{f_2 L_2 Q^2}{12 D_2^5} + \ldots$$

$$\Rightarrow \quad \boxed{\frac{f_e L_e}{D_e^5} = \frac{f_1 L_1}{D_1^5} + \frac{f_2 L_2}{D_2^5} + \ldots\ldots} \quad \ldots\text{(iv)}$$

Where L, D, f represents Length, diameter & friction Factor respectively of corresponding pipe.

The above equation is known as Dupuit's equation.

If not specified, then assume friction factor of all the pipes and equivalent pipe same. Then equation (iv) will become,

$$\Rightarrow \quad \boxed{\frac{L_e}{D_e^5} = \frac{L_1}{D_1^5} + \frac{L_2}{D_2^5} + \ldots\ldots} \quad \ldots\text{(v)}$$

(iii) If a set of pipes described by (L_1, D_1, f_1), (L_2, D_2, f_2), ...are connected in parallel between two points then equivalent pipe of parameter (L_e, D_e, f_e) is related as:

We know that in parallel total discharge,

$$Q = Q_e = Q_1 + Q_2 + \ldots \quad \ldots\text{(vi)}$$

And total head loss,

$$h_L = h_{L_e} = h_{L_1} = h_{L_2} = \ldots \qquad \ldots\text{(vii)}$$

$$\left[\because h_L = \frac{fLQ^2}{12D^5} \Rightarrow Q = \left(\frac{12D^5 h_L}{fL}\right)^{\frac{1}{2}}\right]$$

Hence equation (i) can be written as:

$$\left(\frac{12D_e^5 h_L}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{12D_1^5 h_{L_1}}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{12D_2^5 h_{L_2}}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots \left[\because h_L = h_{L_e} = h_{L_1} = h_{L_2} = \ldots\right]$$

$$\Rightarrow \quad \left(\frac{12D_e^5 h_L}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{12D_1^5 h_L}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{12D_2^5 h_L}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots$$

$$\Rightarrow \quad \boxed{\left(\frac{D_e^5}{f_e L_e}\right)^{\frac{1}{2}} = \left(\frac{D_1^5}{f_1 L_1}\right)^{\frac{1}{2}} + \left(\frac{D_2^5}{f_2 L_2}\right)^{\frac{1}{2}} + \ldots\ldots\ldots\ldots} \qquad \ldots\text{(viii)}$$

Where L, D, f represents Length, diameter & friction Factor respectively of corresponding pipe.

If not specified, then assume friction factor of all the pipes and equivalent pipe same. Then equation (viii) will become,

$$\Rightarrow \quad \boxed{\left(\frac{D_e^5}{L_e}\right)^{\frac{1}{2}} = \left(\frac{D_1^5}{L_1}\right)^{\frac{1}{2}} + \left(\frac{D_2^5}{L_2}\right)^{\frac{1}{2}} + \ldots\ldots\ldots\ldots} \qquad \ldots\text{(ix)}$$

Note:

Above three relations are derived by neglecting minor losses.

7.9 Syphon

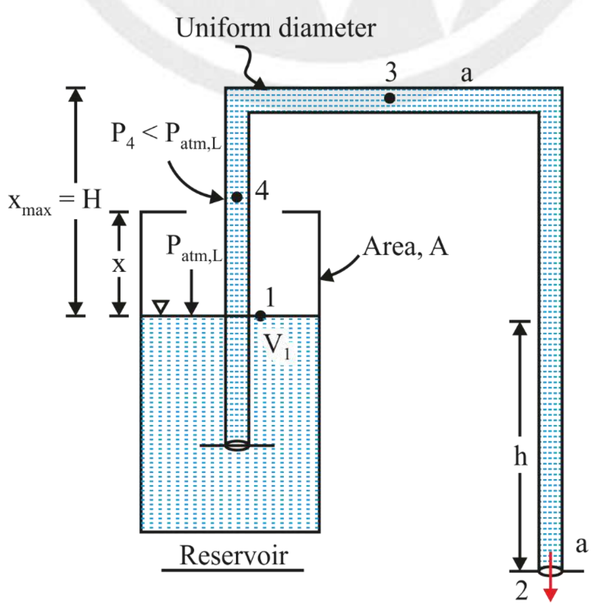


Fig.7.8 Flow through syphon

- Siphon/Syphon is a flexible pipe and Siphon effect result continuous flow
- Siphon $\rightarrow$ uniform diameter pipe
- $V_2 = \sqrt{2gh}$

  $\therefore$ Velocity of liquid coming out of the siphon depends only on the potential head difference between free surface and end of the siphon open to atmosphere.
- $V_3 = V_2 = \sqrt{2gh}$
- The highest point of the siphon is called the SUMMIT. The pressure at the summit is less than the atmospheric pressure
- Pressure $P_4 = P_{atm,L} - \rho g(h + x)$

  $\therefore$ Minimum pressure occurs at top/peak/summit where $x_{max} = H$
- $P_3 = P_{atm,L} - \rho g(h + H)$

7.10 Hydraulic Gradient Line and Total Energy Line

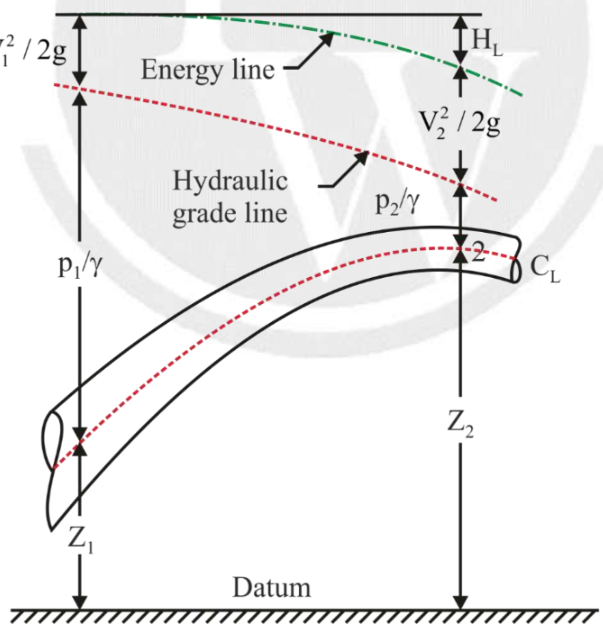


Fig.7.9 HGL & TEL for flow through pipe

- A line joining the piezometric heads at various points in a flow is known as Hydraulic Gradient Line or Piezometric Head Line (HGL).

  $$h = \left(\frac{P}{\gamma} + Z\right)$$
- The Total Energy Line (TEL) at any point in a flow field is

$$H = \left(\frac{P}{\gamma} + Z\right) + \frac{V^2}{2g}$$

$$H = h + \frac{V^2}{2g}$$

- The Hydraulic Gradient Line lies everywhere at a distance equal to the velocity head below the Total Energy Line.

Note:

1. TEL can never be horizontal or slope upward in the direction of flow, if the fluid is real and no energy is being added.
2. TEL and HGL are coincident and lie in the free surface for the body of liquid at rest. (e.g. Large reservoir).
3. HGL is always parallel and lower than TEL.

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8 LAMINAR FLOW

8.1 Introduction

Smooth flow of one layer of fluid over another layer is known as Laminar Flow (or) Viscous Flow.

8.1.1 Reynolds Number

It is defined as ratio of inertia force of flowing fluid and viscous force of fluid.

Inertia force $F_i = \rho A V^2$

Viscous force (F_v) = shear stress × Area

$$F_V = \tau \times A = \left(\mu . \frac{\partial u}{\partial y}\right) \times A$$

$$F_v = \mu . \frac{V}{L} \times A$$

By definition,

Reynolds Number $R_e = \frac{F_i}{F_v}$

$$R_e = \frac{\rho A V^2}{\mu . \frac{V}{L} . A}$$

$$Re = \frac{\rho V L}{\mu} = \frac{V \times L}{\mu / \rho} = \frac{V \times L}{\nu}$$

$$\Rightarrow \quad \boxed{Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu}}$$

$$\frac{\mu}{\rho} = \nu = \text{kinematic viscosity}$$

In case of pipe flow, L is taken as hydraulic diameter d_H

Reynolds Number, $\boxed{R_e = \frac{\rho V d_H}{\mu} = \frac{V d_H}{\nu}}$

Where,

$$d_H = \frac{4A}{P}$$

A = Cross sectional area of pipe,

P = Perimeter of cross section

For circular pipe of diameter d,

$$d_H = \frac{4A}{P} = \frac{4 \times \frac{\pi}{4} d^2}{\pi d} = d$$

Hence Reynolds Number for circular pipe,

$$\boxed{R_e = \frac{\rho V d}{\mu} = \frac{Vd}{\nu}}$$

Critical Reynolds Number

The Reynolds number value below which the flow can be definitely considered to the laminar is known as Critical Reynolds number.

The value depends upon the flow conditions and the geometry of the flow. For circular conduits a value of Recrit = 2000 is generally taken as the Critical Reynolds number.

8.1.2 Fully Developed Flow

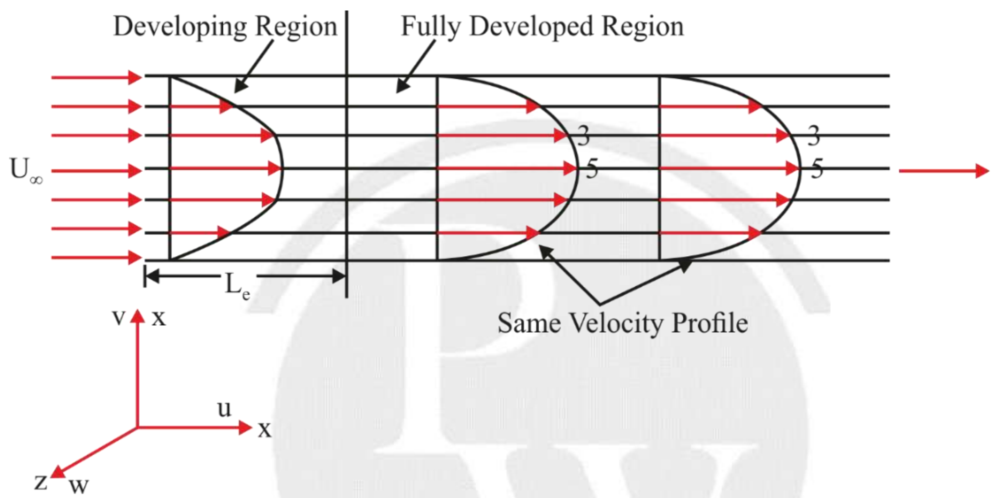

Fig. 8.1 Fully developed flow through pipe

- In fully developed flow each fluid particle moves at constant velocity along a stream line and velocity profile remains unchanged in flow direction.
- There is no motion in radial direction. Hence component of velocity in the direction normal to the axis is zero everywhere.

$$u \neq 0, \frac{\partial u}{\partial x} = 0$$

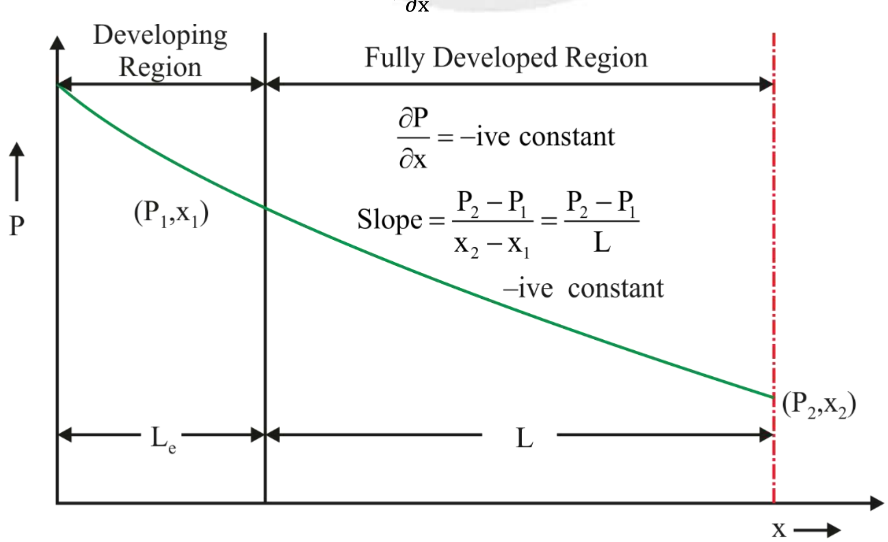

Fig. 8.2 Variation of pressure gradient in developing and fully developed flow

- For steady and incompressible fluid flow, average velocity in developing flow region and fully developed flow region are identical and Re is also same at every cross section.

8.3 Laminar Flow Through Circular Pipe (Hagen Poiseuille Flow)

- Steady, uniform
- Incompressible
- Fully Developed Flow

1. Shear Stress (τ) Distribution

$$\Rightarrow \quad \boxed{\tau = \left(-\frac{\partial p}{\partial x}\right) \cdot \frac{r}{2}} \quad \ldots(i)$$

The negative sign on $\frac{\partial p}{\partial x}$ indicates decrease in pressure in the direction of flow.

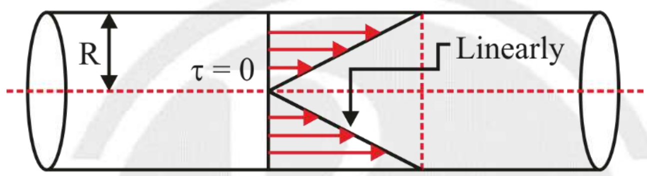

Fig. 8.3 Shear stress variation in flow through pipe

- At center line r = 0 then τ = 0
- At wall r = R then

$$\tau_w = \left(-\frac{dP}{dx}\right)\left(\frac{R}{2}\right)$$

2. Velocity Distribution

$$\boxed{u = -\frac{1}{4\mu}\frac{\partial p}{\partial x}\left[R^2 - r^2\right]} \quad \ldots(ii)$$

3. Maximum velocity (u_{max})

Velocity u will be maximum at $r = 0$, hence putting $r = 0$ in equation (ii) will get maximum velocity,

$$\boxed{u_{max} = \frac{1}{4\mu}\left(-\frac{\partial p}{\partial x}\right)R^2} \quad \ldots(iii)$$

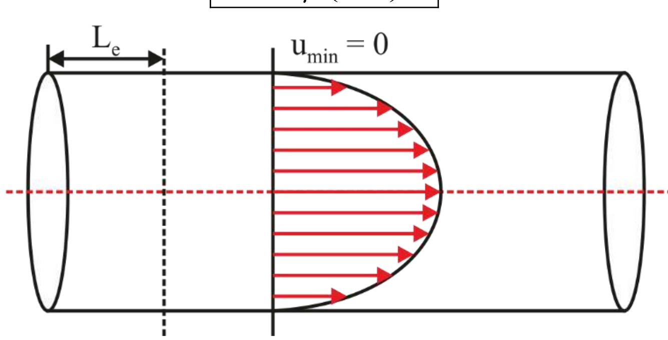

Fig.8.4 Velocity distribution for flow through pipe

4. Average Velocity (u_{avg})

$$\Rightarrow \qquad \boxed{u_{avg} = \frac{u_{\max}}{2}}$$

5. Location of Average Velocity:

$$\Rightarrow \qquad \boxed{r = \frac{R}{\sqrt{2}} = 0.707R}$$

6. Discharge (Q) Through A Pipe

$$\boxed{Q = u_{avg} \times A = \frac{\pi}{8\mu}\left(\frac{-\partial p}{\partial x}\right)R^4}$$

7. Pressure drop ($p_1 - p_2$) between 1 & 2

$$\Rightarrow \qquad \boxed{p_1 - p_2 = \frac{8\mu u_{avg} L}{R^2}}$$

8. Drop in pressure head (h_f) between 1 and 2

$$h_f = \frac{p_1 - p_2}{\rho g}$$

$$\Rightarrow \qquad \boxed{h_f = \frac{8\mu u_{avg} L}{\rho g R^2}} \qquad \left[\because R = \frac{D}{2}\right]$$

$$\Rightarrow \qquad h_f = \frac{32\mu u_{avg} L}{\rho g D^2} \qquad \left[\because u_{avg} = \frac{Q}{A} = \frac{Q}{\frac{\pi}{4}d^2}\right]$$

$$\Rightarrow \qquad \boxed{h_f = \frac{128\mu QL}{\rho g \pi D^4}}$$

Note:

Hence for laminar flow through circular pipe having discharge Q,

loss of head $h_f \propto \frac{1}{D^4}$

9. Power (P) Required to Maintain the Laminar Flow

$$\boxed{P = (p_1 - p_2) \times Q}$$

Note:

In case of inclined plate gravity force will also active, therefore pressure (p) will be replaced by (p + ρgz) i.e. use $d(p + \rho gz)$ in place of dp. Then

$$-\frac{\partial p}{\partial x} \to -\frac{\partial(p+\rho gz)}{\partial x}$$

and, $h_f = \frac{p_1 - p_2}{\gamma}$ will change to

$$h_f = \frac{(p_1+\gamma z_1)-(p_2+\gamma z_2)}{\gamma} = \left(\frac{p_1}{\gamma}+z_1\right)-\left(\frac{p_2}{\gamma}+z_2\right)$$

8.4 Friction Factor

- For steady incompressible Newtonian fluid, fully developed laminar flow through pipe.

$$f = \frac{64}{Re}$$

- Minimum value of Friction factor for laminar flow through pipe.

$$F_{min} \cong 0.032$$

- For laminar flow

$$(Re)_{max} \cong 2000$$

8.5 Momentum Correction Factor

- It is the ratio of the momentum rate based on V_{act} to the momentum rate based on V_{avg}

$$\beta = \frac{\dot{P}_{act}}{\dot{P}_{avg}}$$

$$\beta = \frac{1}{A}\int\left(\frac{u}{u_{av}}\right)^2 dA$$

- For flow through pipe $u = u(r)$ only then

$$\beta = \frac{1}{\pi r^2}\int\left(\frac{u}{u_{av}}\right)^2 2\pi r dr$$

$$\beta = \frac{4}{3} \quad \text{(Hagen Poiseuille Flow)}$$

8.6 Kinetic Energy Correction Factor

- Ratio of kinetic energy rate based on actual velocity to the KE rate based on Average velocity
- Flow through pipe $u = u(r)$ only,

$$\alpha = \frac{1}{A}\int\left(\frac{u}{u_{av}}\right)^3 dA$$

$\alpha = 2$ (Hagen Poiseuille Flow)

- According to Bernoulli's Equation, for steady incompressible Newtonian fluid, fully developed laminar flow through circular pipe.

$$\frac{P_1}{\rho g} + \alpha\frac{V_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \alpha\frac{V_2^2}{2g} + z_2$$

Note:

For uniform velocity β = 1 and α = 1

8.7 LAMINAR FLOW BETWEEN TWO STATIONARY PARALLEL PLATES

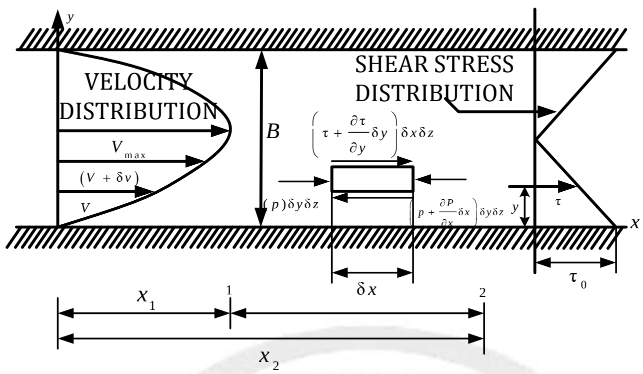

Fig.8.4

Since both the plate is fixed, therefore,

At $y = 0, u = 0$

& at $y = B, u = 0$

Pressure gradient is given by,

$$\frac{\partial p}{\partial x} = \frac{d\tau}{dy} \qquad \left[\because \tau = \mu\frac{du}{dy}\right]$$

$$\frac{\partial p}{\partial x} = \mu\frac{d^2u}{dy^2}$$

1. Velocity Distribution

$$\boxed{u = \left(-\frac{1}{2\mu}\frac{dp}{dx}\right)\left(By - y^2\right)}$$

Above equation is parabolic, hence velocity profile is parabolic in this case.

and $$\frac{du}{dy} = \left(-\frac{1}{2\mu}\frac{dp}{dx}\right)(B - 2y)$$

2. Maximum velocity (u_m)

Point of maximum velocity,

$$\Rightarrow \qquad \boxed{y = \frac{B}{2}}$$

Maximum Velocity,

$$\boxed{u_m = \left(-\frac{1}{8\mu}\frac{\partial p}{\partial x}\right)B^2}$$

3. Average Velocity (u_{avg})

$$\boxed{u_{avg} = \frac{2}{3} u_m}$$

4. Discharge per unit width (Q')

$$\Rightarrow \qquad \boxed{Q = u_{av} \times B = \frac{B^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right)}$$

5. Pressure drop ($p_1 - p$) between 1 & 2

$$\left(-\frac{\partial p}{\partial x}\right) = \frac{12\mu u_{avg}}{B^2}$$

$$\Rightarrow \qquad \frac{p_1 - p_2}{L} = \frac{12\mu u_{avg}}{B^2}$$

$$\Rightarrow \qquad \boxed{p_1 - p_2 = \frac{12\mu u_{avg} L}{B^2}} \qquad \ldots(x)$$

6. Drop in pressure head (h_f) between 1 and 2

$$h_f = \frac{p_1 - p_2}{\rho g}$$

$$\Rightarrow \qquad \boxed{h_f = \frac{12\mu u_{avg} L}{\rho g B^2}} \qquad \ldots(xi)$$

7. Power (P) Required to Maintain the Laminar Flow

$$P = (p_1 - p_2) \times Q$$

8. Shear Stress (τ) Distribution

$$\tau = \mu \frac{du}{dy}$$

$$\tau = \left(-\frac{\partial p}{\partial x}\right)\left(\frac{B}{2} - y\right)$$

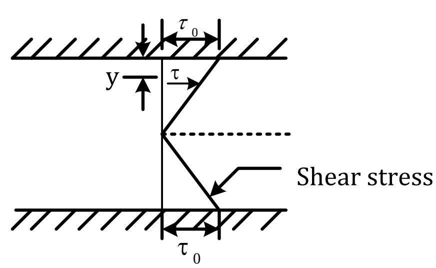

Fig.8.5

Note:

In case of inclined plate gravity force will also active, therefore pressure (p) will be replaced by (p + ρgz) i.e. use $d(p + \rho gz)$ in place of dp. Then

$$-\frac{\partial p}{\partial x} \to -\frac{\partial(p+\rho gz)}{\partial x}$$

and, $h_f = \frac{p_1 - p_2}{\gamma}$ will change to

$$h_f = \frac{(p_1+\gamma z_1)-(p_2+\gamma z_2)}{\gamma} = \left(\frac{p_1}{\gamma}+z_1\right)-\left(\frac{p_2}{\gamma}+z_2\right)$$

8.8 Flow Between Two Plate – One Moving & Other Fixed

The flow between two parallel plates where one plate is moving & other is fixed is known as Coutte Flow as shown in figure 7.7. given below:

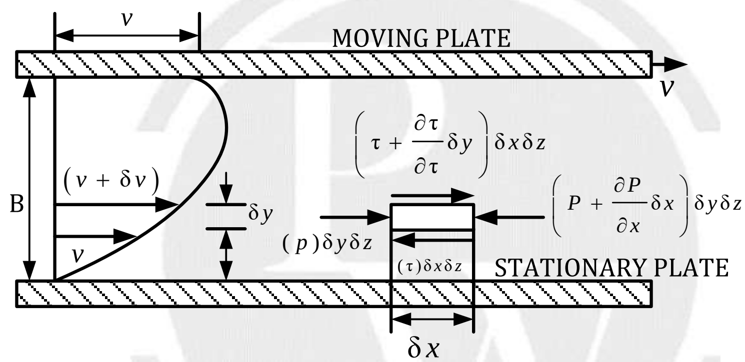

Fig.8.6

The velocity distribution for Couette Flow is given by,

$$u = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)\frac{y^2}{2} + C_1 y + C_2$$

Applying Boundary condition ($u = 0\, at\ y = 0$ & $u = V\ at\ y = B$) in above equation,

$$u = \frac{Vy}{B} - \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(By - y^2\right)$$

If pressure gradient in the direction of flow is zero, then,

$$u = \frac{Vy}{B} \leftarrow \text{linear}$$

The above equation is linear. This is the particular case of Coutte flow which is known as Plain Couette Flow or simply shear flow.

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9 TURBULENT FLOW

9.1 Introduction

9.1.1 Turbulent Flow

- In Turbulent flow, fluid particles move in disorderly manner causing the formation of swirling eddies.
- As a result, Turbulent flow is associated with much higher values of friction, heat & mass transfer coefficients in Turbulent flow, we have larger loss of energy and therefore a greater drop in pressure.

9.1.2 Reynolds Decomposition Principle to specify the velocity field

$$u = \bar{u} + u'$$

Where u = Instantaneous velocity in $x -$ direction

$\bar{u}$ = Time average velocity $x -$ direction

u' = Fluctuations in x-direction (it can be + ive / -ive / 0)

$$\bar{u}' = 0, \bar{u}'^2 \neq 0$$

Note:
$\bar{u}$ will not change with time.

9.1.3 Turbulence Intensity (I_t)

$$I_t = \frac{\sqrt{\overline{u'^2}}}{\bar{u}}$$

9.2 SHEAR STRESS IN TURBULENT FLOW

9.2.1 As Per Boussinesq's Theory

$\tau = \tau_v + \tau_t$

$\tau = \mu \frac{d\bar{v}}{dy} + \eta \frac{d\bar{v}}{dy}$

Where,

τ_v =Shear stress due to viscosity

τ_t =Shear stress due to Turbulence

μ = dynamic coefficient of viscosity (fluid characteristic),

$\bar{v}$ = Time average of velocity at the point of consideration

η = eddy viscosity coefficient (flow characteristic)

$\eta = 0$ for laminar flow.

Eddy viscosity comes in picture due to the turbulence effect.

9.2.2 As per Reynolds Theory

$$\tau = \mu \frac{du}{dy} - \rho \overline{u'v'}$$

Where,

μ = dynamic coefficient of viscosity (fluid characteristic)

$\overline{u'v'}$ = Time average of product of fluctuating velocity $u'\&v'$

9.2.3 As per Prandtl's mixing Length Theory

$$\tau = \mu \frac{d\bar{v}}{dy} + \rho l^2 \left(\frac{d\bar{v}}{dy}\right)^2$$

If viscous effect is neglected,

$$\tau = \rho l^2 \left(\frac{d\bar{v}}{dy}\right)^2$$

Where,

μ = dynamic coefficient of viscosity (fluid characteristic),

$\bar{v}$ = Time average of velocity at the point of consideration

l = mixing length

9.3 VARIOUS LAYERS IN TURBULENT FLOW

Turbulent flow along a wall can be considered to consist of four regions, characterized by the distance from the wall.

(i) Laminar Viscous Sublayer:

The very thin layer next to the wall where viscous effects are dominant is the viscous sublayer. The velocity profile in this layer is very nearly linear, and the flow is streamlined.

(ii) Buffer layer:

Next to the viscous sublayer is the buffer layer, in which turbulent effects are becoming significant, but the flow is still dominated by viscous effects.

(iii) Overlap layer:

Above the buffer layer is the overlap (or transition) layer, also called the inertial sublayer, in which the turbulent effects are much more significant, but still not dominant.

(iv) Outer layer:

Above the overlap layer is the outer (or turbulent) layer in the remaining part of the flow in which turbulent effects dominate over molecular diffusion (viscous) effects.

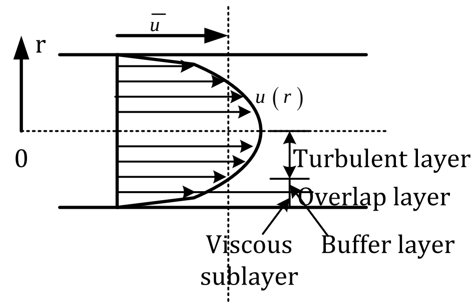

Fig.9.1

9.4 Hydro Dynamically Smooth and Rough Boundaries

If the average height of irregularities (k) is greater than the thickness of laminar sub-layer (δ') then the boundary is called hydro dynamically smooth Boundaries.

If the average height of irregularities (k) is less than the thickness of laminar sub-layer (δ') then the boundary is called hydro dynamically rough Boundaries. In general it can be understood from the following figure 9.2 and 9.3.

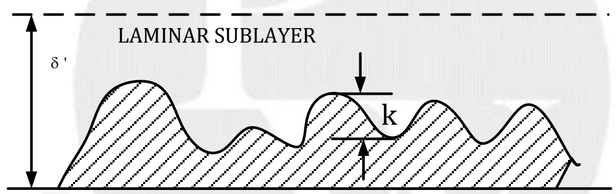

Fig.9.1 Absolute Smooth boundary

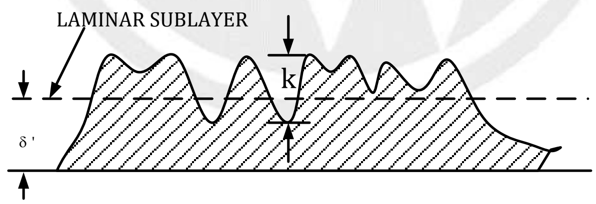

Fig.9.3 Rough boundary

On the basis of NIKURADSES EXPERIMENT the boundary is classified as.

Hydro dynamically smooth Boundaries $\frac{k}{\delta'} < 0.25$

Boundary in transition: $0.25 < \frac{k}{\delta'} < 6.0$

Hydro dynamically Rough Boundaries $\frac{k}{\delta'} > 6.0$

Expression of thickness of laminar sublayer,

$$\delta' = \frac{11.6\upsilon}{u_f} \qquad \ldots(i)$$

Roughness Reynolds Number

It is given by,

$$\frac{u_f k}{\upsilon} \qquad \ldots(ii)$$

Specific Roughness

It is given by,

$$\frac{R}{k} \qquad \ldots(iii)$$

Where,

$u_f = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

$k =$ average height of roughness

υ = kinematic viscosity.

$R =$ radius of the pipe

Rewriting Equation (i)

$$\delta' = \frac{11.6\upsilon}{u_f}$$

$$\Rightarrow \frac{k}{\delta'} = \frac{u_f k}{11.6\upsilon}$$

$$\Rightarrow \frac{u_f k}{\upsilon} = 11.6\frac{k}{\delta'}$$

Now, by using above relation, in terms of roughness Reynolds number $\left(\frac{u_* k}{\upsilon}\right)$ the boundary is classified as:

Hydro dynamically smooth Boundaries $\frac{u_* k}{\upsilon} < 3$

Boundary in transition: $3 < \frac{u_* k}{\upsilon} < 70$

Hydro dynamically Rough Boundaries $\frac{u_* k}{\upsilon} > 70$

9.5 VELOCITY DISTRIBUTION FOR TURBULENT FLOW IN PIPES

Prandtl's velocity distribution equation is given by,

$$u = u_{\max} + 2.5u_* \log_e\left(\frac{y}{R}\right)$$

Where

$u_* = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

$y =$ distance from pipe wall,

ρ = Density of fluid

$u_{\max}$ = Centerline velocity

R = Radius of pipe

Note

The above equation is valid for both smooth and rough pipe boundaries.

9.5.1 Local Velocity

For turbulant flow in a pipe the velocity distribution at a distance y from boundary is given by,

(i) For Hydro Dynamically Smooth Boundaries

$$\frac{u}{u_*} = 5.75\log_{10}\left(\frac{u_* y}{\upsilon}\right) + 5.5$$

(ii) For Hydro Dynamically Rough Boundaries

$$\frac{u}{u_*} = 5.75\log_{10}\left(\frac{y}{k}\right) + 8.5$$

Where,

u = Local velocity at distance y from boundary

$u_* = \sqrt{\frac{\tau_0}{\rho}}$ = Shear or friction velocity

υ = kinematic viscosity

$k =$ average height of roughness

9.5.2 Velocity Distribution in Terms of Mean Velocity

The mean velocity (V) in a pipe of radius R is obtained by integrating the above equation over the entire area of pipe and dividing it by cross sectional area of pipe (i.e. πR^2).

This gives,

(i) For Hydro Dynamically Smooth Boundaries

$$\frac{V}{u_*} = 5.75\log_{10}\left(\frac{u_* R}{\upsilon}\right) + 1.75$$

(ii) For Hydro Dynamically Rough Boundaries

$$\frac{V}{u_*} = 5.75\log_{10}\left(\frac{R}{k}\right) + 4.75$$

Where,

$V =$ Mean velocity

$u_* = \sqrt{\frac{\tau_0}{\rho}} =$ Shear or friction velocity

$y =$ distance from pipe wall,

υ = kinematic viscosity

$k =$ average height of roughness

R = Radius of pipe

Difference in Local velocity (u) & Mean velocity (V) is given by,

$$\frac{u - V}{u_*} = 5.75\log_{10}\left(\frac{y}{R}\right) + 3.75 \text{ (valid for Both smooth and rough pipes)}$$

9.6 Friction Factor

Friction Factor 'f' for Laminar Flow

$f = \frac{64}{R_e}$ where Re = Reynolds number

Friction Factor (f) For Turbulent Flow in Smooth Pipes

1. Blasius Equation

$$f = \frac{0.316}{(R_e)^{1/4}} \quad (\text{for } 4\times10^3 < \text{Re} < 10^5)$$

$$\frac{1}{\sqrt{f}} = 2.0\log_{10}\left(\text{Re}\sqrt{f}\right) - 0.8 (\text{for } 5\times10^4 < \text{Re} < 4\times10^7)$$

Friction Factor (f) For Turbulent Flow in Rough Pipes.

$$\frac{1}{\sqrt{f}} = 2.0\log_{10}\left(\frac{R}{k}\right) + 1.74$$

Friction Factor for Commercial Pipes

$$\frac{1}{\sqrt{f}} - 2.0\log_{10}\left(\frac{R}{k}\right) = 1.74 - 2.0\log\left(1 + 18.7\frac{R/k}{R_e\sqrt{f}}\right)$$

Where,

$f =$ Friction factor

$y =$ Distance from pipe wall,

R_e = Reynolds number

R = Radius of pipe

$k =$ Average height of roughness

Note:

The above equation shows that for smooth pipe friction factor depends upon Reynolds number (R_e) only and for rough pipe it depends upon Reynolds number (R_e) and Relative smoothness ($\frac{R}{k}$) both.

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10 BOUNDARY LAYER THEORY

10.1 Introduction

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the wall will be same as that of the wall. If the wall is stationary, the velocity of fluid at the wall will be zero. Farther away from the wall, the velocity will be higher and as a result of this variation of velocity, the velocity gradient $\frac{du}{dy}$ will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity (U) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory. In this region velocity gradient $\frac{du}{dy}$ exists and hence the fluid exerts a shear stress on the wall in the direction of motion.

The velocity in the remaining fluid, which is outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient $\frac{du}{dy}$ becomes zero. As a result of this the shear stress is zero.

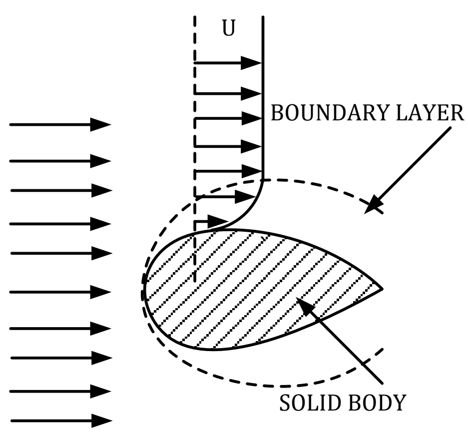

Fig.10.1

10.2 Boundary Layer Thickness

Boundary layer thickness is the distance from solid surface in normal direction at which velocity becomes equal to the 99% of free stream velocity.

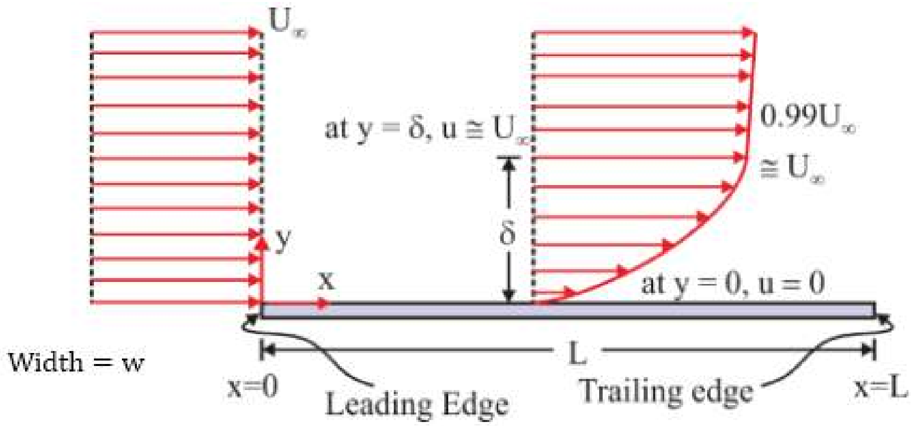

Fig.10.2 Flow over flat plate

10.2.1 Boundary Conditions

At $-$ $x = 0 \Rightarrow \delta = 0$, $\;y = 0 \Rightarrow u = 0$, $\;y = \delta \Rightarrow u \cong U_\infty$, $\;y \geq \delta \Rightarrow \frac{du}{dy} = 0$

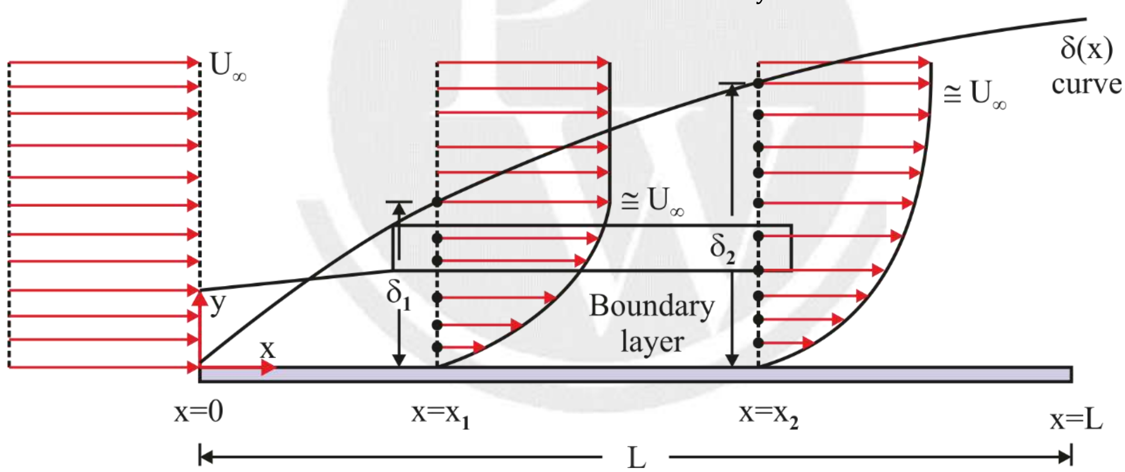

Fig.10.3 Boundary condition for flow over flat plate

Note:

As the flow moves downstream, retarded fluid particles further retard and they retard other fluid particles also, leading to growth of boundary layer. Hence, the thickness of boundary layer keeps on increasing in downstream direction.

10.3 Reynolds Number for Flow Over Plate

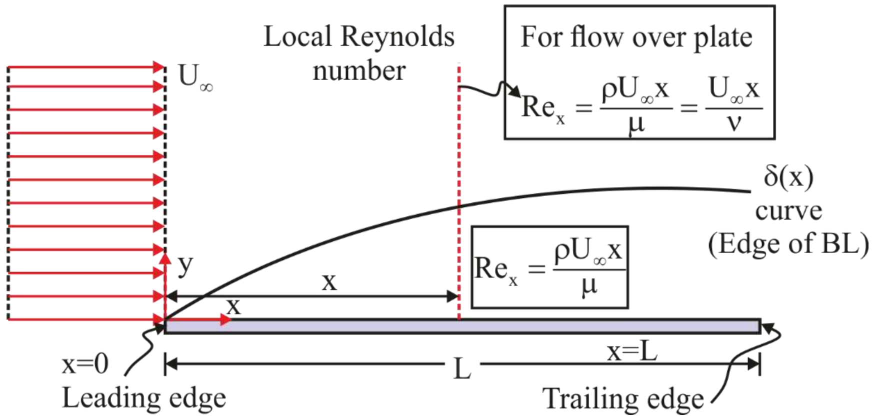

Fig. 10.4 Reynolds number for flow over flat plate

$$Re = \frac{\rho VL}{\mu}$$

$$\mathbf{Re_x = \frac{\rho U_0 x}{\mu}}$$

10.4 Development of a Boundary Layer Over Flat Plate

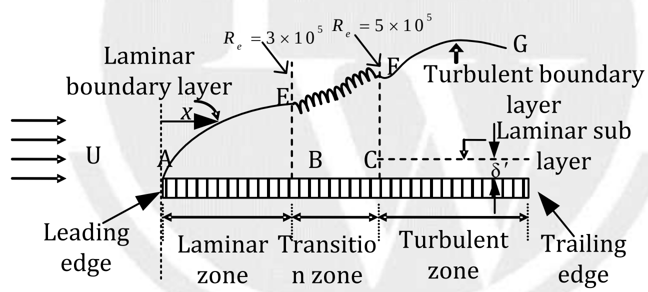

Fig.10.5 Development of boundary layer over flat plate

Near the leading edge the flow is laminar irrespective of whether the flow of incoming streams Laminar (or) Turbulent.

10.4.1 Laminar Boundary Layer

This is defined as the boundary up to which flow is laminar. This is shown by distance AB in figure. 10.5. In a laminar boundary layer any exchange of mass or momentum takes place only between adjacent layers on a microscopic scale which is not visible to the eye. The velocity distribution in Laminar flow is Parabolic.

In figure 10.5 flow from A to B is completely laminar. The Reynold's No. (R_e) at point B is 3×10^5 .

In laminar boundary layer, boundary layer thickness (δ) is given by,

$$\delta = \frac{5x}{\sqrt{R_{e_x}}} \qquad ...(i)$$

Reynold's no. (R_{e_x}) is given by,

$$\left(R_{e_x}\right) = \frac{U \times x}{\upsilon}$$

Where,

$U =$ Free stream velocity

$x =$ Distance from leading edge

$\upsilon =$ Kinematic Viscosity

From above two equations we can conclude that for laminar boundary layer,

$\delta \propto \sqrt{x}$

10.4.2 Turbulent Boundary Layer

If length of plate is more than the distance, x calculated from (i), then the laminar boundary layer becomes unstable and there is transition of motion of fluid from laminar to turbulent boundary.
The velocity distribution in turbulent flow is logarithmic.

The short length over which the boundary flow changes from laminar to turbulent is called transition zone. This is shown by distance BC in figure 10.5.

At point C where Reynold's no. (R_e) is 5×10^5 . transition completes & flow becomes completely turbulent.

A turbulent boundary layer on the other hand is marked by mixing across several layers of it. The mixing is now on a macroscopic scale. Packets of fluid may be seen moving across. Thus, there is an exchange of mass, momentum and energy on a much bigger scale compared to a laminar boundary layer. A turbulent boundary layer forms only at larger Reynolds numbers.

In turbulent boundary layer, boundary layer thickness (δ) is given by,

$$\delta = \frac{0.377x}{\left(R_{e_x}\right)^{\frac{1}{5}}}$$

Reynold's no. (R_{e_x}) is given by

$$\left(R_{e_x}\right) = \frac{U \times x}{\upsilon}$$

Where,

$U =$ Free stream velocity

$x =$ Distance from leading edge

$\upsilon =$ kinematic viscosity

From above two equations we can conclude that for laminar boundary layer,

$\delta \propto x^{\frac{4}{5}}$

10.4.3 Laminar Sub-Layer

In the vicinity of the wall t in turbulent boundary layer there is very thin layer of the fluid where turbulent fluctuations are damped. In this region the velocity variation is influenced only by viscous effect. This region is called laminar sub-layer. Though the velocity distribution would be parabolic in laminar sub-layer, but in view of very small thickness we can assume that velocity distribution is linear in this region, thus velocity gradient can be considered constant. Therefore, shear stress in laminar sub-layer equal to constant & will be equal to boundary shear stress (τ_0). In figure 10.5-line DD' is laminar sub-layer, which is given by,

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu\frac{u}{y}$$

10.5 Various thickness involved in Boundary Layer Theory

10.5.1 Displacement Thickness

It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate due to velocity defect on account of boundary layer formation.

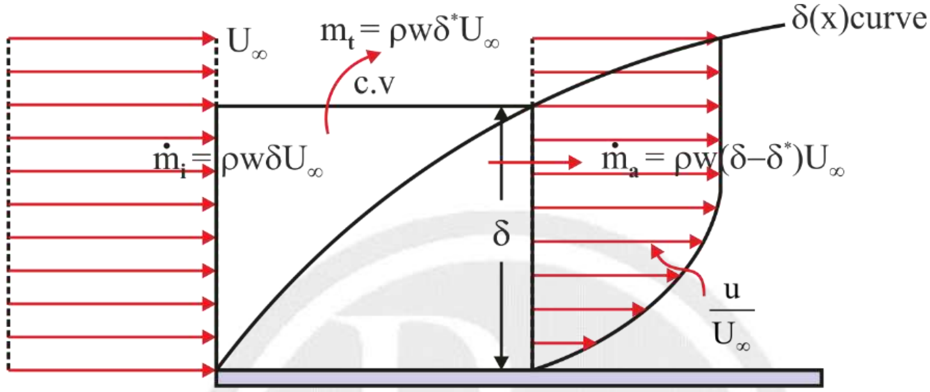

Fig.10.6

It is given by,

$$\delta^* = \int_0^{\delta}\left(1 - \frac{u}{U}\right)dy$$

Where,

$U =$ Free stream velocity

$\delta =$ Boundary layer thickness

$u =$ Velocity distribution

Physical Meaning of Displacement Thickness

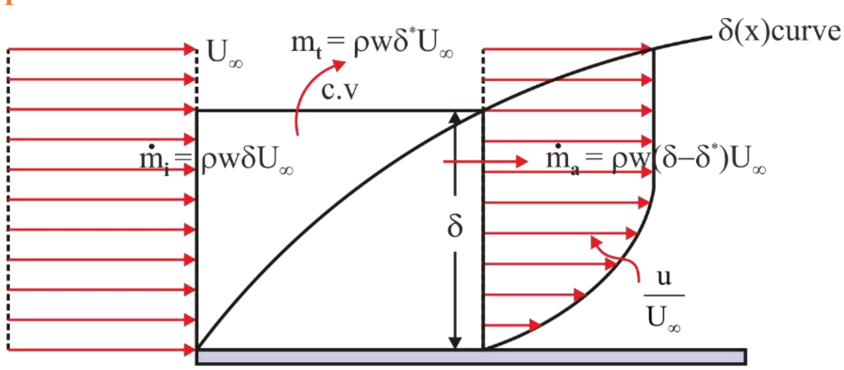

Fig. 10.7 Physical significance of displacement thickness

$$\frac{\dot{m}_0}{\dot{m}_i} = \frac{\delta - \delta^*}{\delta}$$

$$\frac{\dot{m}_t}{\dot{m}_i} = \frac{\delta^*}{\delta}$$

$\dot{m}_t = \rho w \delta^* U_\infty$

$\dot{m}_i = \rho w \delta U_\infty$

$\dot{m}_o = \rho w (\delta - \delta^*) U_\infty$

10.5.2 Momentum Thickness (θ)

- Momentum thickness represents the amount of thickness of solid plate that must be increased (imaginarily) above δ^*, so that ideal, uniform, inviscid flow has the same momentum rate as that of actual momentum rate given by actual velocity distribution.

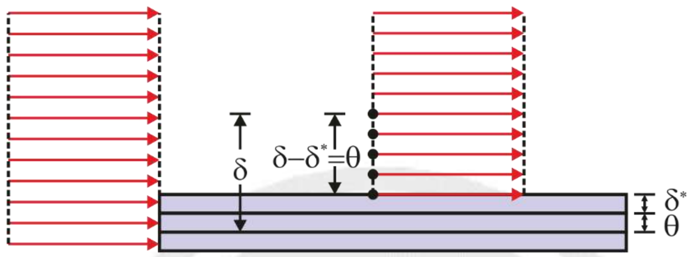

Fig. 10.8 Momentum thickness

$$\theta = \int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right) dy$$

Physical Meaning of Momentum Thickness

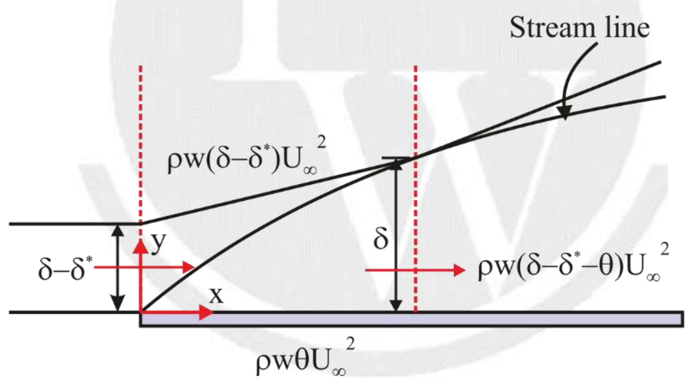

Fig. 10.9 Physical significance of momentum thickness

$$F_{d,s} = \rho \omega \theta U_\infty^2$$

10.5.3 Shape Factor

It is the ratio of displacement thickness to the momentum thickness for a given velocity distribution.

$$H = \frac{\delta^*}{\theta}$$

10.5.4 Energy Thickness (δ_e)

It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is given by,

$$\delta_e = \int_0^\delta \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy$$

Where,

$U =$ Free stream velocity

$\delta =$ Boundary layer thickness

$u =$ Velocity distribution

10.6 Local and Average Co-Efficient of Drag

10.6.1 Local Co-efficient of Drag (C_{fx})

It is defined as,

$$C_{fx} = \frac{\tau_{w,x}}{\frac{1}{2}\rho U^2}$$

Where,

$A =$ Area of the surface (or plate), $U =$ Free-stream velocity

$\rho =$ Mass density of fluid

$F_D =$ Total Drag force on the plate

$\tau_{w,x} =$ Shear stress at distance x from leading edge on the plate

10.6.2 Average Co-efficient of Drag (C_D)

It is defined as,

$$C_D = \frac{F_D}{\frac{1}{2}\rho A U^2}$$

Where,

$A =$ Area of the surface (or plate), $U =$ Free-stream velocity

$\rho =$ Mass density of fluid

$F_D =$ Total Drag force on the plate

10.7 Von-Karman Momentum Integral Equation

Assumptions:

Momentum thickness is the function of x only.

Pressure gradient in x-direction is zero

$$\frac{\tau_{wx}}{\rho U_\infty^2} = \frac{d\theta}{dx}$$

Where,

$\tau_{w,x} =$ shear stress on the plate at distance x from the leading edge

$\rho =$ density of fluid

$U =$ Free stream velocity

$\theta =$ Momentum thickness

10.7.1 Application of Von-Karman Momentum Integral Equation for a Given Velocity Profile

- For a given velocity profile, Von-Karman momentum integral equation is used to find
  (i) Local Wall shear stress (τ_{wx})
  (ii) Local skin friction drag coefficient (c_{fx})
  (iii) Skin friction force acting on plate $(F_{d,s})$
  (iv) Average skin friction drag coefficient $(C_{d,s})$

Velocity Distribution	δ	C_{fx}	C_D
$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$	$5.48x/\sqrt{R_{e_x}}$	$0.73/\sqrt{R_{e_x}}$	$1.46/\sqrt{R_{e_L}}$
$\frac{u}{U}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$	$4.64x/\sqrt{R_{e_x}}$	$0.646/\sqrt{R_{e_x}}$	$1.292/\sqrt{R_{e_L}}$
$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4$	$5.84x/\sqrt{R_{e_x}}$	$0.68/\sqrt{R_{e_x}}$	$1.36/\sqrt{R_{e_L}}$
$\frac{u}{U}=\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	$4.79x/\sqrt{R_{e_x}}$	$0.655/\sqrt{R_{e_x}}$	$1.31/\sqrt{R_{e_L}}$

10.8 Prandtl's Boundary Layer Equation

Assumption: Flow is 2 − D (x − y plane), steady & incompressible flow Gravitational effects are neglected

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

10.9 Blasius Equation

- The classical problem considered by Blasius was a 2D, steady, incompressible fluid flow over a plate at a zero angle of incidence
- In his problem Blasius used Prandtl Boundary layer equation & Continuity equation.

  $$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)=0$$

  Order ⇒ 3

  Non-Linear

  Ordinary Differential equation

- Boundary conditions for Blasius equation

  at $\eta = 0$ $f'(\eta) = 0$

at $\eta = 0$ $f(\eta) = 0$

at $\eta \to \infty$ $f'(\eta) = 1$

10.10 Boundary Layer Separation

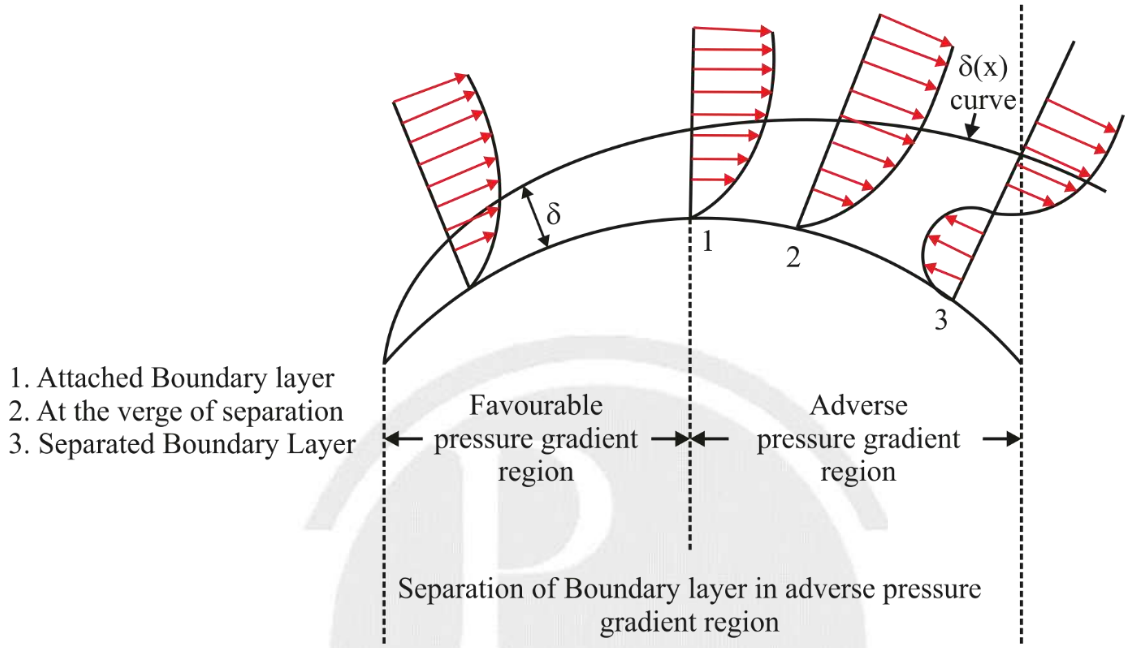

Fig. 10.10 Boundary layer separation

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus, the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking body. In other works, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

10.10.1 Effect of Pressure on Boundary Layer Separation

1. Adverse Pressure Gradient

$$\frac{dp}{dx} > 0$$

When pressure gradient is positive $\left(\frac{dp}{dx} > 0\right)$ the velocity in the direction of flow will decreases. Thus, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Hence chances of flow separation increase therefore positive pressure gradient $\left(\frac{dp}{dx} > 0\right)$ is called adverse pressure gradient.

2. Favorable Pressure Gradient

$$\frac{dp}{dx} < 0$$

When pressure gradient is negative $\left(\frac{dp}{dx} < 0\right)$ the velocity in the direction of flow will increases. Thus, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on increasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Hence reduces the chance of flow separation. Therefore, negative pressure gradient $\left(\frac{dp}{dx} < 0\right)$ is called favorable pressure gradient.

10.10.2 Condition for Boundary Layer Separation

The separation point S is determined from the condition $\left(\frac{\partial u}{\partial y}\right)_{y=0}$

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} < 0$ Flow has separated

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ Flow is on the verge of separation

If $\left(\frac{\partial u}{\partial y}\right)_{y=0} > 0$ Flow is attached with the surface

10.10.3 Methods for Preventing Separation

- Streamlining the body shape.
- Tripping the boundary layer from laminar to turbulent by provision of surface roughness.
- Sucking the retarded flow.
- Injecting high velocity fluid in the boundary layer.
- Providing slots near the leading edge.
- Guidance of flow in a confined passage.
- Providing a rotating cylinder near the leading edge.
- Energizing the flow by introducing optimum amount of swirl in the incoming flow.

Acceleration of Fluid in the Boundary Layer

This method of controlling separation consists of supplying additional energy to particles of fluid which are being retarded in the boundary layer. This may be achieved either by injecting the fluid into the region of boundary layer from the interior of the body with the help of some available device.

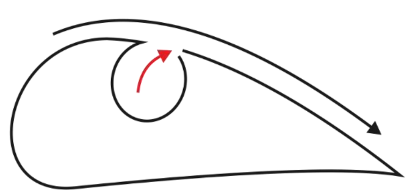

Fig. 10.11 Injecting fluid in boundary layer

By diverting a portion of fluid of the main stream from the region of high pressure to the retarded region of boundary layer through a slot provided in the body.

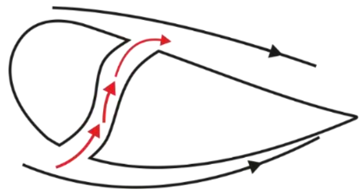

Fig. 10.12 Slotted wing

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11 DRAG AND LIFT

11.1 Drag Force

- Component of force acting on the body parallel to the relative velocity direction.

$$F_d = F_{ds} + F_{dp}$$

Here F_{ds} = Skin friction drag.

F_{dp} = Pressure drags.

$\therefore$ $F_d = \frac{1}{2}\rho A_f U_\infty^2 \times C_d$

Here, C_d = Coefficient of drag

A_f = Area

U_∞ = Free stream velocity

11.2 Analysis for Sphere

- For Creeping flow

$$Re \le 1$$

$$U_\infty = V$$

According to stokes

$$C_d = \frac{24}{Re}$$

- Stokes law, $F_d = 3\pi\mu VD$

(i) If sphere moving downward then terminal velocity, $V = \frac{(\rho_b - \rho_f)D^2 g}{18\mu}$

(ii) If sphere moving upward then terminal velocity, $V = \frac{(\rho_f - \rho_b)D^2 g}{18\mu}$

11.3 Lift Force

Force acting on the body normal to the relative velocity direction.

$$F_l = \frac{1}{2}\rho A_p U_\infty^2 \times C_l$$

C_l = Coefficient of lift

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12 DIMENSIONAL ANALYSIS

12.1 Introduction

In the dimensional analysis of a physical phenomenon the relationship between the dependent and independent variables is studies in terms of their basic dimensions to obtain the information about the functional relationship between the dimensionless parameters that control the phenomenon.

There are several methods of reducing the number of dimensional variables into a smaller number of dimensionless parameters. Two of the commonly used methods are:

(i) Raleigh's method,

(ii) Buckingham Pi theorem method.

Advantages of dimensional analysis

1. It expresses the functional relationship between the variables in dimensionless terms.
2. It enables us in getting up the simplified dimensional form.
3. The conversion of units of quantities from the one system to another is facilitated.

Uses of dimensional analysis

1. To test the dimensional homogeneity of any equation of fluid motion
2. To derive rational formulae for a flow phenomenon
3. To plan model tests and present experimental result in a systematic manner, thus making it possible to analysis the complex fluid flow phenomenon.

12.2 Important Dimensions

Two common system of dimensioning a physical quantity are; M-L-T and F-L-T system of units, where F = force.

Quantity	**Symbol**	**Dimensions in terms of**	
		M-L-T	**F-L-T**
A. Geometric:			
Length (any linear measurement)	L, l	L	L
Area	A	L^2	L^2
Volume	V	L^3	L^3
Curvature	C	1/L	1/L

B Kinematic:			
Discharge per unit width	q	L^2/T	L^2/T
Kinematic viscosity (μ/ρ)	μ	L^2/T	L^2/T
Circulation	Γ	L^2/T	L^2/T
Time	T, t	T	T
Velocity (linear)	V, v	L/T	L/T
Angular velocity	ω	1/T	1/T
Frequency	n	1/T	1/T
Acceleration (linear)	a	L/T^2	L/T^2
Angular acceleration	α	$1/T^2$	$1/T^2$
Discharge	Q	L^3/T	L^3/T
C. Dynamic:			
Mass	M, m	M	FT^2/L
Force	F	ML/T^2	F
Mass density	ρ	M/L^3	FT^2/L^4
Specific weight	ω, γ	M/L^2T^2	F/L^3
Specific gravity	*s*	$M^oL^oT^o$	$F^oL^oT^o$
Pressure intensity	p	M/LT^2	F/L^2
Shear stress	τ	M/LT^2	F/L^2
Dynamic viscosity	μ	M/LT	FTL^2
Surface tension	σ	M/T^2	F/L
D. General:			
Moment of a force	$F \times l$	ML^2/T^2	FL
Moment of Inertia of area	I	L^4	L^4
Moment of Inertia of mass	Ml^2	ML^2	FLT^2

12.3 Raleigh's Method

If A_1 is dependent variable and $A_2, A_3, \ldots An$ are independent variables in a phenomenon then A_1 is expressed as:

$$A_1 = k A_2^a A_3^b \ldots\ldots \qquad \ldots(i)$$

Where k is a dimensionless constant.

The dimensions of each of the quantities $A_2, A_3, \ldots An$ are written and the sum of exponents of each of M, L, and T are equated on both sides. Solution of the equations on simplification gives the value of $a, b, c \ldots$. Putting these values in equation (i) yields dimensionless groups controlling the phenomenon. While this method is simple for a small number of parameters, it becomes rather cumbersome when a large number of parameters are involved.

12.4 Buckingham's Pi Theorem

The Buckingham Pi theorem states that if there are m primary dimensions involved in the n variables controlling a physical phenomenon, then the phenomenon can be described by (n-m) independent dimensionless groups (known as πs). The word Pi here refers to a product of variables and the Greek letter π is used to indicate these products, for example π_1, π_2…

In the application of this method, m numbers of repeating variables are selected and dimensionless groups obtained by each one of the remaining variables one at a time. Raleigh's method is used to solve this part of the operation. This method is also known as the method of repeating variables.

Care is needed in selecting the repeating variables.

- Number of repeated variables must be equal to number of fundamental dimensions
- Repeated variables must be selected from the independent variables only
- Repeated variables must contain all the fundamental dimensions.
- Repeated variables must be selected such that they are not forming dimensionless terms all together.
- Most fundamental quantity must be given preferences.

12.5 MODEL ANALYSIS

Model analysis is the study of the structure or machine by constructing a small-scale replica of the machine or structure known as model and performing the various tests so as to detect and remove various defects in the design of the actual structure called prototype.

For predicting the performance of the hydra

.0ulic structures or hydraulic machine, before actually constructing or manufacturing, models of the structures or machines are made and test are performed on them to obtain the desired information.

Mostly the models are much smaller than the corresponding prototypes, but in some cases the model may be larger than the prototype.

Advantages of Model Analysis

1. The model tests are quite economical and convenient because the design, construction and operation of the model may be altered several times without incurring much expenditure.
2. The performance of the hydraulic structure or hydraulic machines can be easily predicted.
3. Model tests are used to detect and rectify the defects of an existing structure which is not functioning property.
4. With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensionless parameters is obtained. This relationship helps in conducting tests on the model.

Application of the Model Analysis

1. Design of turbine, pumps and compressors.
2. Design of harbors, ships and submarines.
3. Design of dams, spillways, wires, canals etc.

12.5.1 Similitude Principle

In hydraulic and aeronautical engineering valuable results are obtained at a relatively small cost by performing tests on small scale models of full-size systems (prototypes). Similarity laws help us interpret the results of model studies. Similitude is the relation between model and a prototype and is classified into three kinds as follows:

(i) Geometric similarity

(ii) Kinematic similarity

(iii) Dynamic similarity

Geometric Similarity

If the ratios of corresponding length in a model and the prototype are the same, the model is said to be in geometrically similar model.

In such models

If $$\frac{L_m}{L_p} = L_r$$

Then, $$\frac{A_m}{A_p} = \frac{(L_m)^2}{(L_p)^2} = L_r^2,$$

$$\frac{\cancel{V}_m}{\cancel{V}_p} = \frac{(L_m)^3}{(L_p)^3} = L_r^3$$

Where,

L_m = Length of Model

A_m = Area of model

$\cancel{V}_m$ = Volume of Model

Similarly, L_p, A_p & $\cancel{V}_p$ are corresponding parameter for prototype.

Kinematic Similarity

Kinematic similarity means geometric similarity and in addition it also signifies that the ratio of velocities at all corresponding points in the flow for model & prototype is the same.

If $$\frac{L_m}{L_p} = L_r$$

$$\frac{V_m}{V_p} = V_r$$

Then

(i) Time ratio $= \frac{t_m}{t_p} = t_r = \frac{L_r}{V_r}$

(ii) Acceleration ratio $= \frac{a_m}{a_p} = a_r = \frac{V_r^2}{L_r} = \frac{L_r}{T_r^2}$

(iii) Discharge ratio $= \frac{Q_m}{Q_p} = Q_r = \frac{L_r^3}{T_r}$

Dynamic Similarity

Two systems are dynamically similar, if geometric and kinematic similarities exist and further the ratios of all corresponding forces in the two systems are same.

If forces due to

Gravity = F_G

Viscosity = F_v

Elasticity =F_E

Surface tension = F_T

Inertia = F_1

And suffixes m and p stand for model and prototype respectively, strict dynamic similarity means

$$\frac{F_{Gm}}{F_{Gp}} = \frac{F_{vm}}{F_{vp}} = \frac{F_{Em}}{F_{Ep}} = \frac{F_{Tm}}{F_{Tp}} = \frac{F_{1m}}{F_{1p}} = \text{constant}$$

From the above the following relationships can be derived.

$$\frac{(Inertiaforce)_m}{(\text{Viscous force})_m} = \frac{(\text{Inertia force})_p}{(\text{Viscous force})_p} = \text{Constant 1}$$

$$\frac{(\text{Inertia force})_m}{(\text{Gravity force})_m} = \frac{(\text{Inertia force})_p}{(\text{Gravity force})_p} = \text{Constant 2}$$

And so on for all the forces

12.6-Dimensional Formula of Various Forces

- Inertia force $(F_i) = \rho V^2 L^2$
- Viscous force $(F_v) = \mu V L$
- Gravity force $(F_g) = \rho L^3 g$
- Pressure force $(F_p) = pL^2$
- Surface tension force $(F_{st}) = \sigma L$
- Elastic force $(F_e) = KL^2$

  where K is bulk modulus of Elasticity

12.6.1 Dimensionless Numbers

Number	Equation	Significance
Reynolds No	$\frac{\rho VL}{\mu}$	Flow in closed conduit pipe
Froude No.	$\frac{V}{\sqrt{Lg}}$	Where a free surface is present, structure eg. Weirs, spillway, channels, etc. where gravity force is predominant
Euler's No.	$\frac{\rho V^2}{p}$	In cavitation studies
Mach No.	$\frac{V}{C}$	Where fluid compressibility is important.
Weber No.	$\frac{\rho V^2 L}{\sigma}$	In capillary studies.

12.7 Model Law

12.7.1 Reynold's Model Law

For the flows where in Viscous Force is predominant in addition to inertia force, similarity can be established by equating Reynolds Number of models $(R_e)_m$ and prototype $(R_e)_p$. This is known as Reynold's Model Law.

$$(R_e)_m = (R_e)_p,$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

Applications of Reynolds Model Law

- (i) Air planes, (ii) Torpedo's
- Flow through pipes, flow past plates, Fluid drag and lift on body shapes (cars, trains, parachutes… etc.).
- Settling of particles and creeping flow
- Venturi meters, Orifice meters, etc.
- Fans, blowers, propellers, pumps and turbines.

12.7.2 Froude's Model Law

When the force of gravity is predominant in addition to inertia force then similarity can be established by equating Froude's number of model & prototype. This is known as Froude's model law.

$$(F_e)_m = (F_e)_p$$

$$\frac{V_m}{\sqrt{L_m g_m}} = \frac{V_p}{\sqrt{L_p g_p}}$$

Applications of Froude's Model Law

- (i) Motion of ships, boats, (ii) Break waters and harbors
- Flow in canals, streams and rivers
- Spillways, stilling basins, hydraulic/jumps, weirs and notches
- Wave and water flow forces in bridge piers, off-shore structures, jetties and piers

12.7.3 Euler's Model Law

When pressure force is predominant in addition to inertia force, similarity can be established by equating Euler number of model and prototype. This is called Euler's model law.

$$(E_u)_m = (E_u)_p$$

$$\frac{V_m}{\sqrt{P_m / \rho_m}} = \frac{V_p}{\sqrt{P_p / \rho_p}}$$

Applications of Euler's Model Law

- Enclosed fluid system where the turbulence is fully developed so that viscous forces are negligible and also the forces of gravity and surface tension are entirely absent.
- Where the phenomenon of cavitation occurs.

12.7.4 Weber Model Law

If surface tension forces are predominant with inertia force, similarity can be established by equating Weber number of model and prototype

$$(W_e)_m = (W_e)_p$$

$$\frac{V_m}{\sqrt{\sigma_m / \rho_m L_m}} = \frac{V_p}{\sqrt{\sigma_p / \rho_p L_p}}$$

Applications of Weber Model Law

- Capillary rise in narrow passages.
- Capillary movement of water in soil.
- Capillary waves in channels.
- Flow over weirs for small heads.

12.7.5 Mach Model Law

In places where elastic forces are significant in addition to inertia, similarity can be achieved by equating Mach numbers for both the system. This is known as Mach model law.

$$(M)_m = (M)_p$$

$$\frac{V_m}{\sqrt{K_m / \rho_m}} = \frac{V_p}{\sqrt{K_p / \rho_p}}$$

Applications of Mach Model Law

- Aerodynamic testing.
- Under water testing of torpedoes.
- Water-hammer problems.

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13 BASIC COMPRESSIBLE FLOW

13.1 Introduction

13.1.1 Mach Number (Ma)

$$\text{Mach Number (Ma)} = \frac{\text{Velocity of fluid flow (V)}}{\text{Speed of sound in fluid (c)}}$$

$\text{Ma} \le 0.3 \Rightarrow$ Incompressible Flow

$\text{Ma} > 0.3 \Rightarrow$ Compressible Flow

Here $c = \sqrt{\frac{dP}{d\rho}}$

13.1.2 For Isentropic Flow

Speed of sound $c = \sqrt{\frac{k_s}{\rho}}$

Here k_s = Isentropic Bulk Modulus

$$k_s = \rho \left.\frac{\partial P}{\partial \rho}\right|_s$$

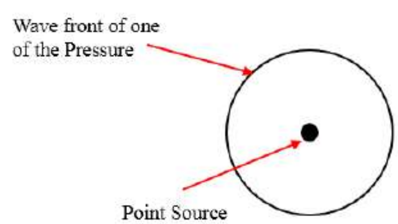
13.2 For Isentropic Flow of Perfect Gas

$$c = \sqrt{\gamma RT}$$

Here T = Absolute temperature in Kelvin.

13.2.1 Point Source

- Point source will emit the radially outward spherical pressure waves at the speed of sound.



- In Stationary point source in Stationary Fluid, a stationary observer will hear the same sound frequency coming from the source.
- In Moving point source ($V < c$) in stationary fluid ($Ma < 1$), a stationary observer will hear the different sound frequency coming from the source.
- In Moving point source ($V = c$) in stationary fluid ($Ma = 1$), formation of Mach wave (Tangent Plane)
- In moving point source ($V > c$) in stationary fluid ($Ma > 1$)
- $\sin\alpha = \frac{1}{Ma}$, Formation of Mach wave (Mach Cone).

13.3 Stagnation Temperature

$$T_o = T + \frac{V^2}{2c_p} \qquad \text{(Adiabatically perfect gas)}$$

$$\frac{T_o}{T} = 1 + \frac{(\gamma-1)}{2}Ma^2 \qquad \text{(Isentropically perfect gas)}$$

13.3.1 Stagnation Pressure

$$\frac{P_o}{P} = \left[1 + \frac{(\gamma-1)}{2}Ma^2\right]^{\frac{\gamma}{(\gamma-1)}} \qquad \text{(Isentropically perfect gas)}$$

For Incompressible fluid flow

$$\frac{P_o}{P} = 1 + \frac{\gamma}{2}Ma^2 \qquad \text{(Incompressible fluid flow)}$$

13.3.2 Stagnation Density

$$\frac{\rho_o}{\rho} = \left[1 + \left(\frac{\gamma-1}{2}\right)Ma^2\right]^{\frac{1}{(\gamma-1)}} \qquad \text{(Isentropically perfect gas)}$$

- Stagnation properties are defined at each & every point of fluid flow.

Note:

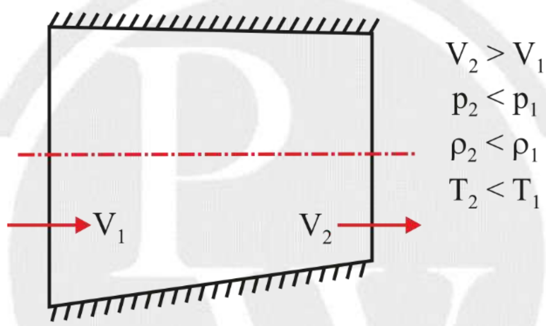
- Stagnation properties (h_o, T_o, P_o, ρ_o) are the combination of thermodynamic properties, hence stagnation properties are defined at each of every point of fluid flow.
- If we want to achieve particular value of stagnation properties then we need to bring the fluid at rest under the discussed conditions.

13.4 Effect of Area Variation on Flow Properties

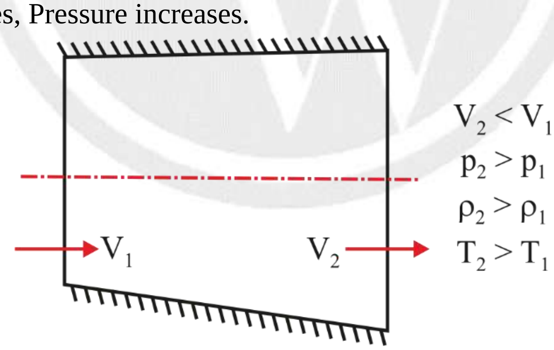
$$\frac{dA}{A} = \frac{dP}{\rho V^2}(1 - Ma^2)$$

$$\frac{dA}{A} = -\frac{dV}{V}(1 - Ma^2)$$

- At throat, Mach number will be either maximum or minimum of converging duct.
- At throat, it is not compulsory to have Ma = 1 of converging duct.
- If in the fluid flow at any section Mach is 1, that has to be throat.
- At $Ma = 1$(Sonic condition), $\frac{dA}{A} = 0 \Rightarrow$ Area is minimum at throat, so V is maximum.

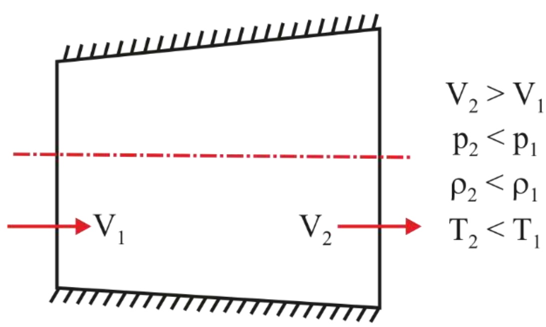


Diffuser Effect Velocity decreases, Pressure increases.

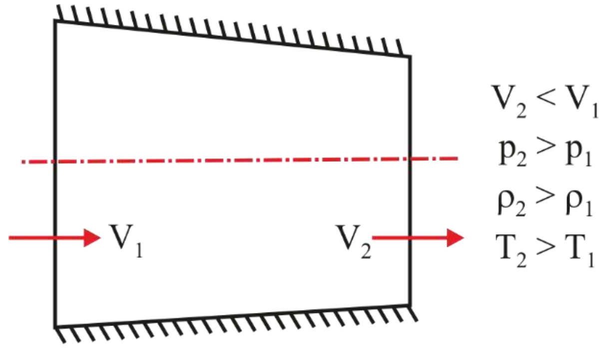


- Super Sonic Flow → $Ma > 1$

Nozzle Effect Velocity decreases, Pressure increases.

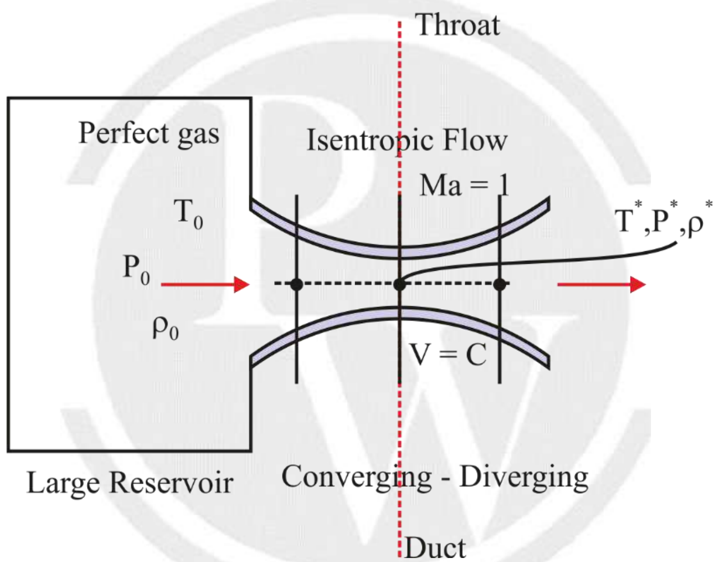


Diffuser Effect Velocity increases, Pressure decreases



13.4.1 Sonic Properties

- Ma = 1, V = c

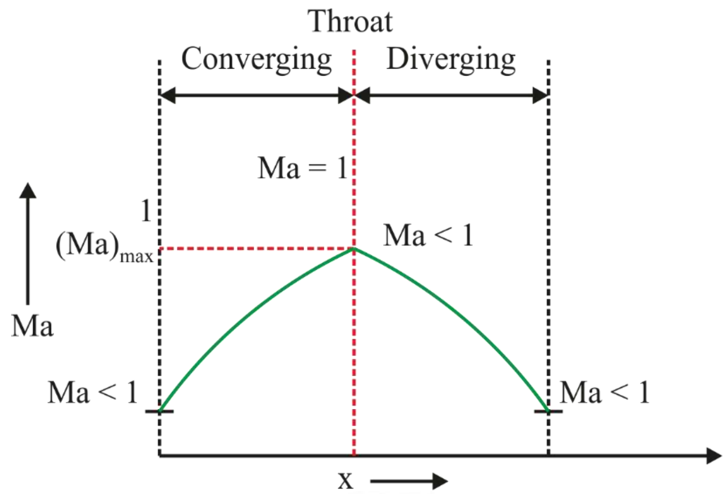


$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2}$$

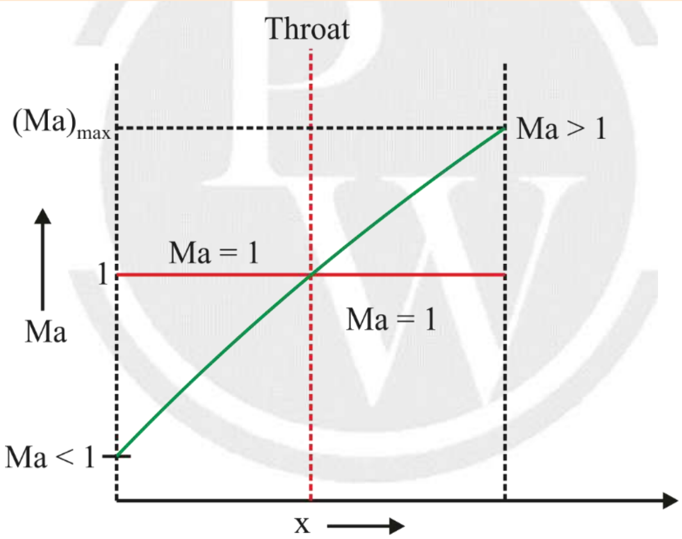
$$\frac{\rho_0}{\rho^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{1}{\gamma - 1}}$$

$$\frac{P_0}{P^*} = \left[\frac{\gamma + 1}{2}\right]^{\frac{\gamma}{\gamma - 1}}$$

13.4.2 Subsonic Flow at Both Inlet & Outlet

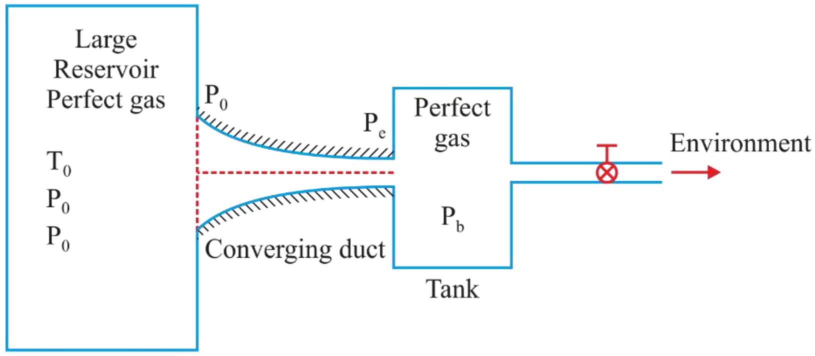


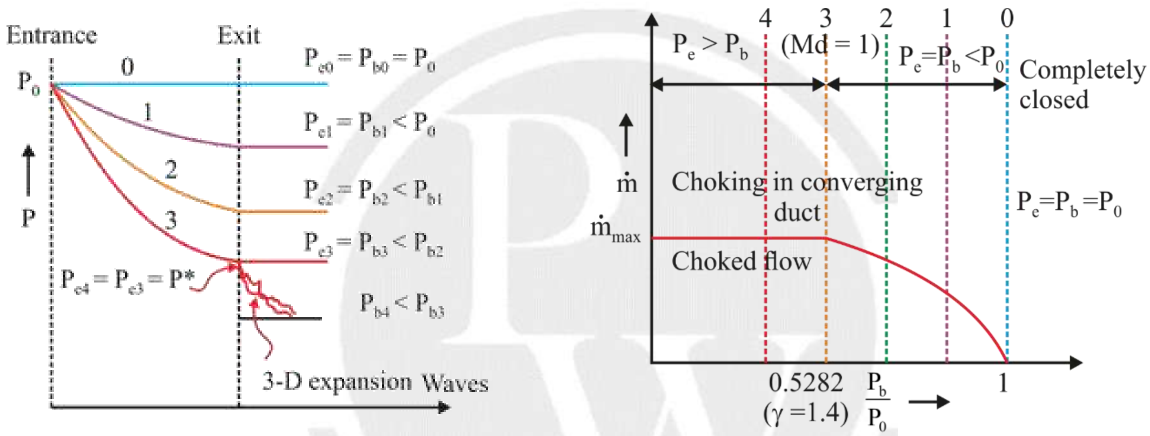
13.4.3 Subsonic Flow at the Inlet & Supersonic Flow at Outlet



Maximum mass flow rate will occur at throat (Ma = 1)

13.5 Pressure distribution in a Converging duct





For perfect gas, $\gamma = 1.4$

$$\frac{P_0}{P^*} = \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_0}{P^*} = 1.8929$$

$$\frac{P^*}{P_0} = 0.5282$$

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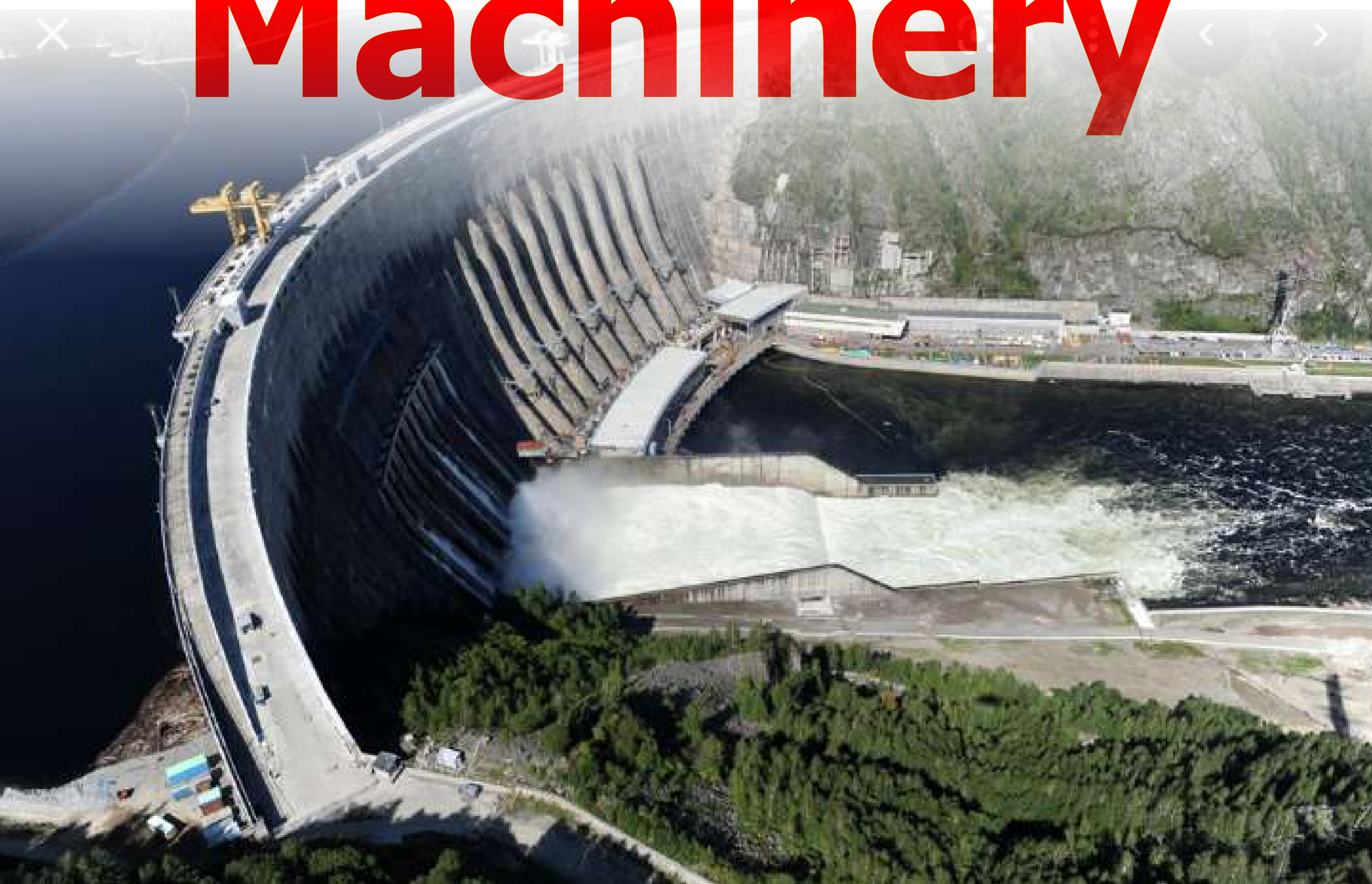
Fluid Machinery



Fluid Machinery

INDEX

1 IMPACT OF JET

1.1 Impact of Jets

Impact of jet means the force exerted by a jet on a surface which may be stationary or moving. This force is obtained from momentum equation.

As per momentum equation

Force = Rate of change of momentum

= (Final momentum rate) – (Initial momentum rate)

$$= \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

If m is constant, $\frac{dm}{dt} = 0$

$$\vec{F} = m\frac{dv}{dt} = m\vec{a}$$

1.2 Force Exerted by Jet on a Stationary Plate

1.2.1 Plate is vertical and jet is normal to the plate

Assumptions:

1. Everywhere pressure is atmospheric pressure.
2. Friction between the jet and plate is neglected.
3. No loss of energy due to impact of jet.
4. Neglect the elevation difference between incoming and outgoing jet.

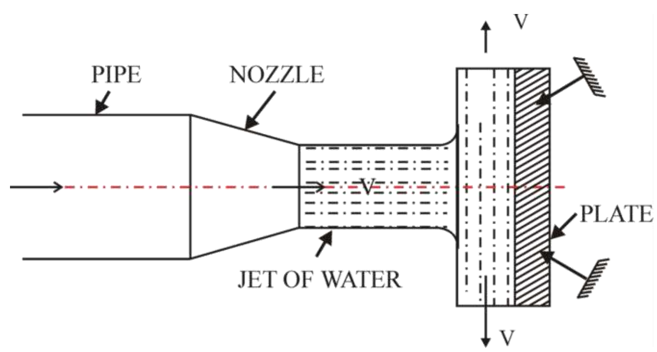


Fig. 1.1 Jet striking normal to fixed flat plate

Force exerted by the jet normal to the plate

$$F_n = \rho a V^2$$

$$F_n = \rho Q V$$

a = area of jet $\left(a = \dfrac{\pi d^2}{4}\right)$

V = Velocity of jet (m/s)

Q = Discharge (m^3/s)

1.2.2 Jet strikes on an inclined stationary plate.

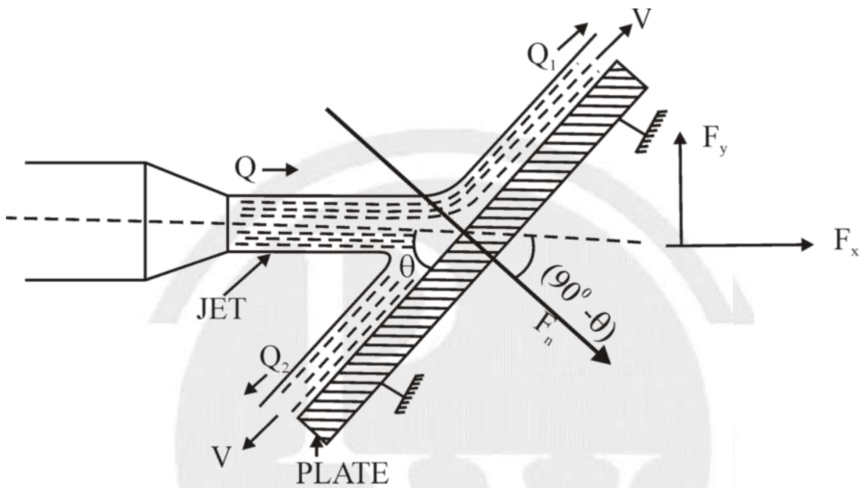


Fig. 1.2 Jet striking on an inclined stationary plate

Force exerted by the jet normal to the plate

$$F_n = \rho a V^2 \cdot \sin\theta = \rho Q V . sin\theta$$

$$F_x = \rho a V^2 sin^2\theta$$

$$F_y = \rho a V^2 \sin\theta\cos\theta$$

$$Q = Q_1 + Q_2$$

$$Q_1 = \frac{Q}{2}(1+\cos\theta)$$

$$Q_2 = \frac{Q}{2}(1-\cos\theta)$$

$$\frac{Q_1}{Q_2} = \frac{1+\cos\theta}{1-\cos\theta}$$

1.2.3 Jet strike on a curved plate

Jet Striking on a symmetrical Stationary curved plate.

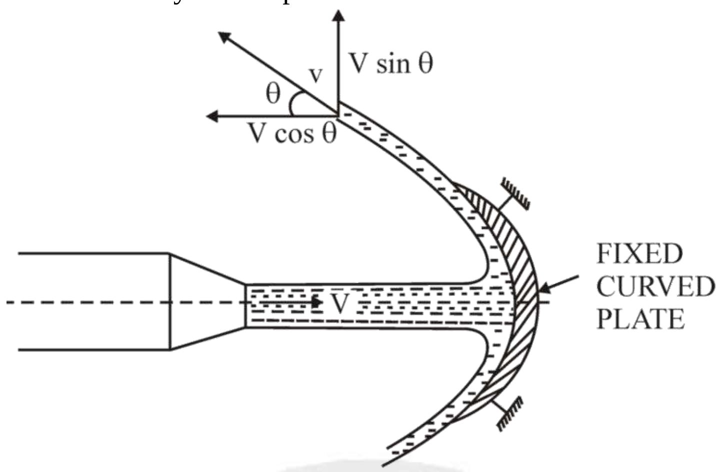


Fig. 1.3 Jet striking on a fixed curved plate at the center

$F_n = \rho a V^2 (1+\cos\theta)$

- Force exerted by a jet in its direction of flow on a curved vane is **always greater** than that exerted on flat plate.
- Angle of deflection = $(180° - \theta)$

1.3 Force Exerted by Jet on a Moving Plate

1.3.1 Plate is vertical to the jet

Force exerted by jet on moving flat plate normal to jet.

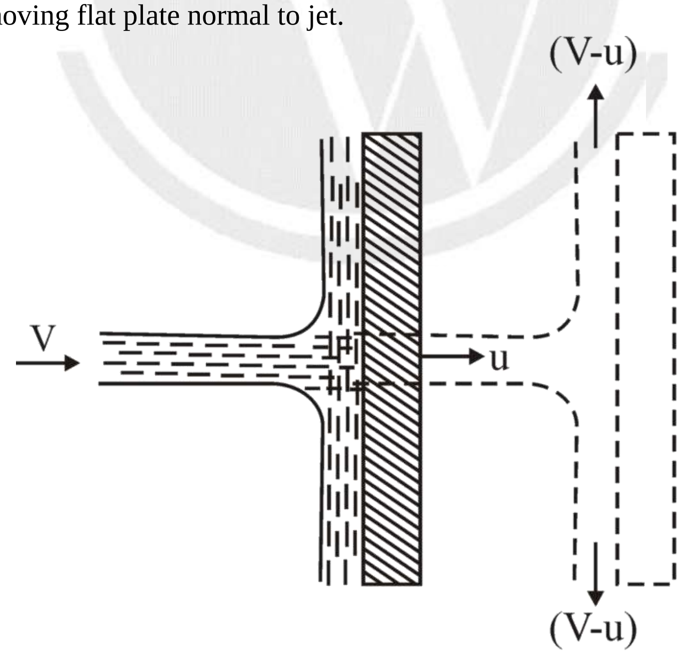


Fig. 1.4 Jet striking a moving flat plate

V= Velocity of jet

u = Plate velocity

$$F_n = \rho a(V-u)^2$$

Work done per second (W) = $F_n \times u = \rho a[V-u]^2 \times u$

Efficiency (η) = Work done by the jet / Kinetic energy of the jet

$$\eta = 2\left(1-\frac{u}{V}\right)^2 \cdot \frac{u}{V}$$

$$\frac{u}{V} = \text{Speed ratio}$$

Note:

η is max when $\frac{u}{V} = \frac{1}{3}$

$\eta_{max} = \frac{8}{27} = 29.63\%$

1.3.2 Plate Mounted on the periphery of Wheel

Jet Strikes on series of flat plate mounted on the periphery of wheel.

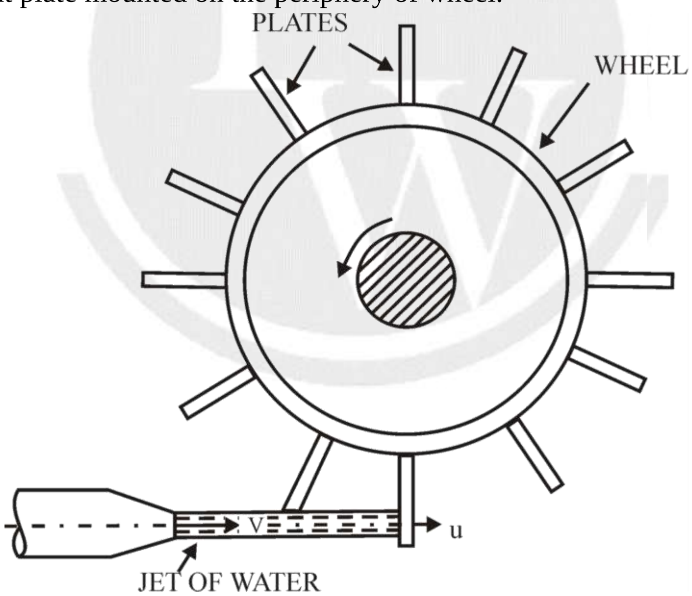


Fig. 1.5 Jet striking on a series of vanes mounted on wheel

V = Velocity of jet

u = Tangential velocity of wheel

$$u = \frac{\pi DN}{60}$$

D= Diameter of wheel; N = rpm

$$F_n = \rho aV(V-U)$$

Work done by the jet = $F_n \times u$

$W = \rho aV(V - U)u$

Efficiency (η) = Work done by the jet / Kinetic energy of the jet

Efficiency (η) = $2\left(1-\frac{u}{V}\right)\frac{u}{V}$

Note:

η is max when $\frac{u}{V} = \frac{1}{2}$

$\eta_{max} = \frac{1}{2} = 50\%$

1.3.3 Curve plate when the plate is moving in the direction of jet

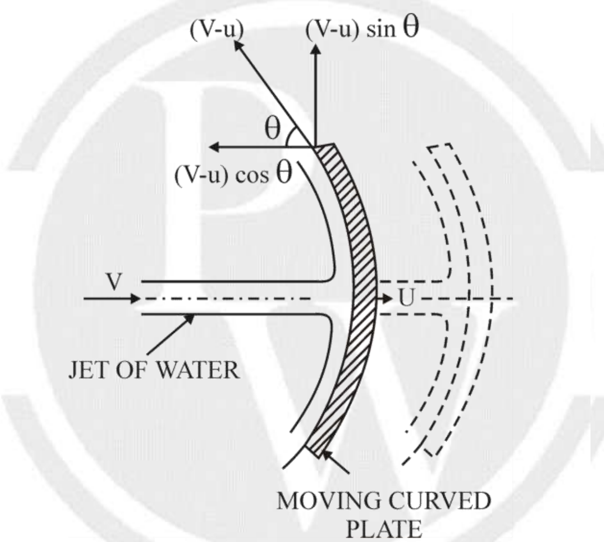


Fig. 1.6 Jet striking on a curved moving plate at the center

Force exerted by the jet of water on the curved plate in the direction of the jet.

$$F_n = \rho a(V-u)^2(1+cos\theta)$$

Work done by the jet on the plate per second

W = $\rho a(V-u)^2[1+\cos\theta]u$

1.4 Force Exerted by a Jet on a Hinged Plate

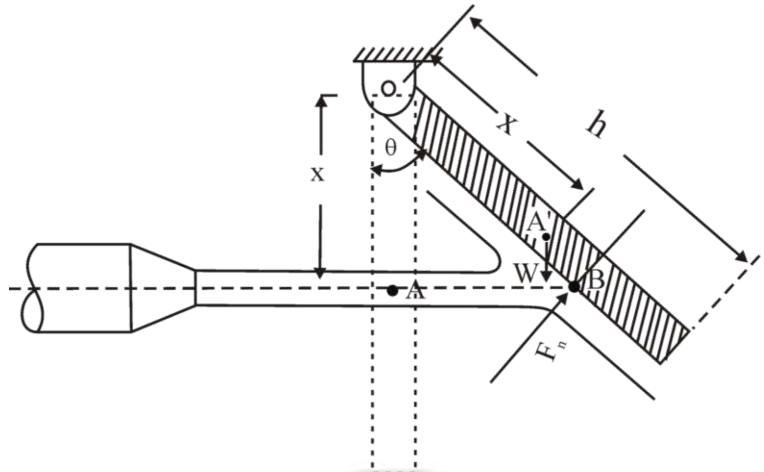


Fig. 1.7 Jet striking on a plate hinged at one end

Force exerted by the jet of water, normal to the plate

$$F_n = \rho a V^2 \sin(90° - \theta) = \rho a V^2 \cos\theta$$

For equilibrium of plate

$$\sin\theta = \frac{\rho a V^2}{W}$$

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2 TURBINES

2.1 Hydraulic Machines

Hydraulic machines are defined as those machines which convert either hydraulic energy into mechanical energy or mechanical energy into hydraulic energy. The hydraulic machines, which convert the hydraulic energy into mechanical energy, are called turbines.

2.2 Classification of Turbines

2.2.1 Turbines on the basis of energy at inlet

- Impulse Turbines:

  Turbines which have only Kinetic Energy at the inlet is known as impulse turbine.

  Ex. Pelton turbine

- Reaction Turbines:
  Turbines which have both Kinetic Energy and Pressure Energy at the inlet are known as reaction turbine.
  Ex. - Francis Turbine, Propeller Turbine, Kaplan Turbine.

2.2.2 Turbines on the basis of flow within turbine

- Tangential Flow Turbines

  Ex. Pelton Turbine.

- Radial Flow Turbines

  Ex. Inward-radial flow turbines is Francis' turbine, whereas outward-radial flow turbine is Fourneyron turbine.

- Axial Flow Turbines

  Ex. Propeller Turbine, Kaplan Turbine.

- Mixed Flow Turbine

  Ex. Modern Francis Turbine.

2.2.3 Turbines on the basis of Discharge

High Discharge Turbine	:	Kaplan, Propeller
Medium Discharge Turbine	:	Francis Turbine
Low Discharge turbine	:	Pelton Turbine

2.2.4 Turbines on the basis of Specific speed

Turbine	Specific speed
Pelton turbine	10 – 60
Francis turbine	60 – 250
Kaplan & propeller turbine	Above 250

2.2.5 Turbines on the basis of Head

Turbine	Head
Pelton turbine	Above 250 m
Francis turbine	60 m – 250 m
Kaplan & propeller turbine	Below 60 m

2.3 Various Efficiencies in Turbine

- Hydraulic efficiency (η_h)

$$\eta_h = \frac{\text{Power developed by the runner}}{\text{Net power supplied at the turbine entrance (Water power)}}$$

- Mechanical efficiency (η_{mech})

$$\eta_{mech} = \frac{\text{Power available at the turbine shaft}}{\text{Power developed by the runner}}$$

- Volumetric efficiency (η_{vol})

$$\eta_{vol} = \frac{\text{Quantity of water actually stricking the runner}}{\text{Quantity of water supplied to the turbine}}$$

- Overall efficiency (η_0)

$$\eta_0 = \frac{\text{Shaft Power}}{\text{Water Power}}$$

$$\therefore \eta_0 = \eta_{hyd} \times \eta_{mech} \times \eta_{vol}$$

2.4 Pelton Turbine

- Tangential Flow Impulse Turbine.
- High head, Low Discharge Turbine.
- Low specific Speed.
- Minimum Pressure is atmospheric pressure; pressure never falls below atmospheric pressure.
- Problem of cavitation never occurs.
- Draft tube not used.
- The bucket of a Pelton wheel is double semi-ellipsoidal in shape. The jet of water impinges at the center of the bucket & deflects through 160 – 170^o.
- The advantage of having double cup-shaped buckets is that the axial thrust neutralizes each other.

2.4.1 Analysis of Pelton Turbine

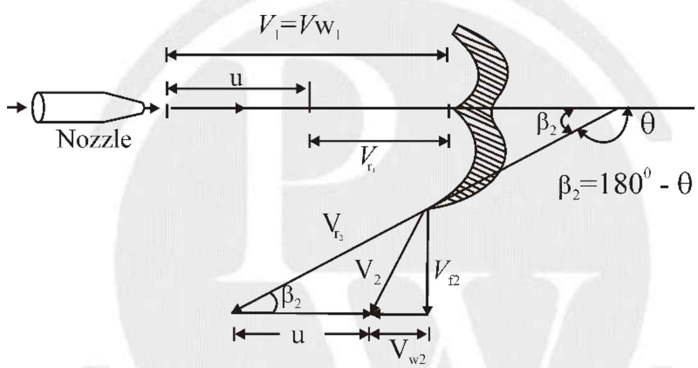

Fig. 2.1 Velocity diagram for Pelton wheel

- Rate of work done $= \rho Q(V_{w_1} + V_{w_2})u$

  Now

  $V_{w_1} = V_1$

  $V_{w_2} = V_{r_2} \cos\beta_2 - u$

  $V_{r_2} = kV_{r_1}$

  $V_{r_1} = V_1 - u$

  Rate of work done $= \rho Q(V_1 - u)(1 + k\cos\beta_2)u = \rho AV_1(V_1 - u)(1 + k\cos\beta_2)u$

  $V_1 =$ Velocity of jet coming out from nozzle

  u = tangential velocity of wheel $\left(u = \frac{\pi DN}{60}\right)$

  D = Wheel diameter, N = Wheel rpm

  A = Jet Area $\left(A = \frac{\pi}{4}d^2\right)$ (d – diameter of the jet)

  k = Blade Friction Coefficient ($k \le 1$)

  β_2 = Blade angle at the exit ($\beta_2 = 180° - \theta$)

  θ = Deflection angle

Note:

If P is power developed by the Pelton wheel when working under head (H) & having one jet only, the power developed by the same wheel will be nP if *n* jets are used under the same head.

- Velocity of jet, $V_1 = C_v\sqrt{2gH}$

  C_v = coefficient of velocity (0.97 – 0.99); H = Net Head Available
- Net Head Available: - Gross Head – Head loss; $H = H_g - h_f$
- Speed Ratio $K_s = \frac{u}{\sqrt{2gH}}$, where $u = \frac{\pi DN}{60}$
- Jet ratio (m) = $\frac{\text{diameter of wheel (D)}}{\text{jet diameter (d)}}$
- A jet ratio of 12 is normally adopted.
- No. of Buckets (z) $= \frac{D}{2d} + 15 = \frac{m}{2} + 15 = 18 \text{ to } 25$
- This formula is called Tygun formula.

2.4.2 Efficiency of Pelton Wheel

- Nozzle efficiency

$$\eta_{Nozzle} = \frac{\text{Rate of Kinetic Energy at the nozzle exit}\left(\frac{1}{2}\dot{m}v_1^2\right)}{\text{Water power}(\gamma QH)}$$

- Wheel Efficiency

$$\eta_{wheel} = \frac{\text{Runner Power}}{\text{Kinetic Energy per sec}}$$

$$\eta_{wheel} = \frac{2(V_1 - u)(1 + k\cos\beta_2)u}{V_1^2}$$

$$(\eta_{wheel})_{max} = \left(\frac{1 + k\cos\beta_2}{2}\right) \quad \left[\text{when, } u = \frac{V_1}{2}\right]$$

k = Friction factor for blades

$k_s = \frac{u}{V_1}$ = Blade speed Ratio

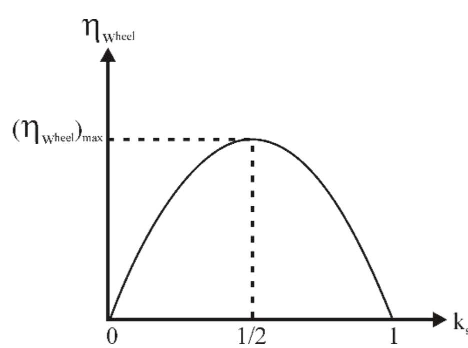

Fig. 2.2 Wheel Efficiency vs blade speed ratio

- Mechanical efficiency

  $$\eta_{mech} = \frac{\text{Shaft Power}}{\text{Runner power}}$$

- Hydraulic efficiency

  $$\eta_{hyd} = \frac{\text{Runner power}}{\text{Water power}}$$

- Overall efficiency

  $$\eta_{overall} = \frac{\text{Shaft power}}{\text{Water power}}$$

  $$\eta_0 = \eta_{Nozzle} \times \eta_{hyd.} \times \eta_{mech}$$

2.5 Francis Turbine (Modern Francis Turbine)

- Mixed Flow Reaction Turbine.
- Medium Head, Medium Discharge Turbine.
- Medium Specific speed.
- Minimum Pressure is below atmospheric pressure at the runner exit.
- Problem of Cavitation Occurs.
- Scroll casing is used to evenly distribute the water along the periphery & maintaining the constant velocity for the water.
- Draft Tube is used.
- The basic function of the draft tube is to convert kinetic energy into pressure energy.

2.5.1 Analysis of Francis Turbine

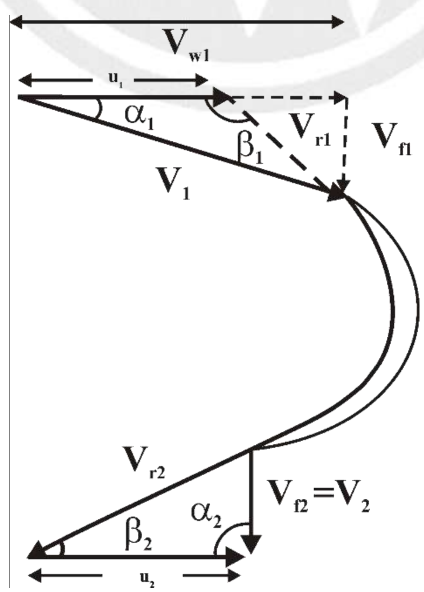

Fig. 2.3 Velocity triangle for Francis turbine

- Work done per sec = $\rho Q\left[V_{w_1}.u_1 - V_{w_2}.u_2\right]$

  For maximum efficiency flow should be radial at outlet, $V_{w_2} = 0$

  i.e., work done per sec = $\rho Q V_{w1} u_1$

- Degree of reaction $= \dfrac{\text{Pressure head drop in the runner}}{\text{Hydraulic work done on the runner / unit weight of water}}$

$$= \frac{\left(\frac{P_1}{\rho g} - \frac{P_2}{\rho g}\right)}{\left(\frac{V_{w_1}.u_1 - V_{w_2}.u_2}{g}\right)}$$

  V_{w_1} & V_{w_2} are whirl velocities at inlet and outlet

  u_1 & u_2 are peripheral velocities of blades at inlet and outlet.

  $u_1 = \dfrac{\pi D_1 N}{60}$ $\quad\quad$ $u_2 = \dfrac{\pi D_2 N}{60}$

  Where, D_1 and D_2 are diameter at inlet and outlet respectively.

- Flow ratio $= \psi = \dfrac{V_{f_1}}{\sqrt{2gH}}$ $(0.15 - 0.30)$

  Speed ratio $= k_s = \dfrac{u_1}{\sqrt{2gH}}$ $(0.60 - 0.90)$

- Discharge through the turbine

  $Q = \pi D_1 B_1 V_{f_1}$

  $= \pi D_2 B_2 V_{f_2}$

  B_1, B_2 are width at inlet & outlet respectively.

  D_1, D_2 are diameter at inlet & outlet respectively.

  V_{f_1}, V_{f_2} are velocities of flow at inlet & outlet respectively.

2.5.2 Various Efficiencies for Francis turbine

- Hydraulic efficiency

  $\eta_{hyd} = \dfrac{\text{Runner Power}}{\text{Water Power } (\gamma QH)}$

- Mechanical efficiency

  $\eta_{mech} = \dfrac{\text{Shaft Power}}{\text{Runner Power}}$

- Overall efficiency

  $\eta_{overall} = \dfrac{\text{Shaft Power}}{\text{Water Power}}$

- $\eta_0 = \eta_{hyd} \times \eta_{mech}$

2.6 Kaplan/Propellor Turbine

- Axial flow reaction turbine.
- Low Head, High Discharge Turbine.
- High Specific speed.
- Minimum Pressure is below atmospheric pressure at the runner exit.
- Problem of Cavitation Occurs.
- Scroll casing is used to evenly distribute the water along the periphery & maintaining the constant velocity for the water.
- Draft Tube is used.
- The basic function of the draft tube is to convert kinetic energy into pressure energy.

2.6.1 Some Important Facts of Different Turbines

Kaplan Turbine

- Its velocity diagram and calculation of efficiency is similar to Francis turbine.
- It is axial flow reaction turbine.
- Runner of Kaplan turbine has four to six blades.
- Runner blades of propeller turbines are fixed but of Kaplan turbine that can be turned about axis, that means it is adjustable.
- Kaplan Turbine has highest part load efficiency.

2.6.2 Analysis of the Kaplan Turbine

Kaplan Turbine Velocity Diagram and calculation of efficiency is similar to the Francis turbine.

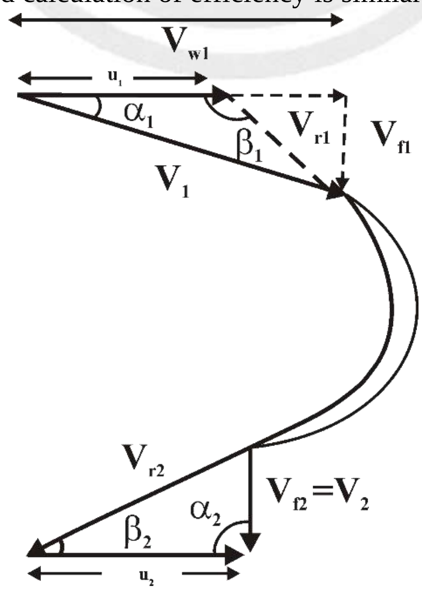

Fig. 2.4 Velocity triangle for Kaplan Turbine

Velocity Diagram

$$V_{w_2} = 0, V_{f_1} \approx V_{f_2} = V_f$$

$$\text{Runner Power} = \dot{m}\left[V_{w_1}.u_1\right] = \rho Q.V_{w_1}.u_1$$

$Q = A_f \times V_f$

$$A_f = \frac{\pi}{4}\left[D_0^2 - D_b^2\right]$$

D_0 = Runner Diameter

D_b = Boss/Hub diameter

$u_1 = u_2 = u$

$$u = \frac{\pi D_m N}{60}$$

D_m = Mean Diameter

$$D_m = \frac{D_0 + D_b}{2}$$

2.7 Specific Speed

It is defined as the speed of a geometrically similar turbine working under unit head (1 m) to produce a unit power output (1 kW).

$$N_s = \frac{N\sqrt{P}}{H^{5/4}}, \quad N \to \text{rpm}$$

$P \to$ Power (kW)

$H \to$ Net head (m)

Specific speed is not a dimensionless parameter, its dimension is $\left[M^{1/2}L^{-1/4}T^{-5/2}\right]$

Dimensionless form of specific speed is called dimensionless specific speed (K_s).

$$K_s = \frac{N'\sqrt{P/\rho}}{(gH)^{5/4}}$$

$N' \to$ rps　　$P \to$ Power (W)

$\rho \to$ density (kg/m^3)　　$H \to$ Net head (m)

2.8 Unit Quantities

Unit quantities are those quantities when the same turbine is allowed to operate under unit head (1m).

- Unit speed (N_u)

$$N_u = \frac{N}{\sqrt{H}}$$

$$\frac{N_1}{\sqrt{H_1}} = \frac{N_2}{\sqrt{H_2}}$$

- Unit discharge (Q_u)

$$Q_u = \frac{Q}{\sqrt{H}}$$

$$\frac{Q_1}{\sqrt{H_1}} = \frac{Q_2}{\sqrt{H_2}}$$

- Unit power (P_u)

$$P_u = \frac{P}{H^{\frac{3}{2}}}$$

$$\frac{P_1}{H_1^{\frac{3}{2}}} = \frac{P_2}{H_2^{\frac{3}{2}}}$$

2.9 Model Laws of Turbines

Similarity of model (m) and Prototype (P) assume that efficiency of model is equal to that of the prototype.

$Q \rightarrow$ discharge, $P \rightarrow$ Power

$N \rightarrow$ rpm $N` \rightarrow$ rps $H \rightarrow$ Net head

- $\left(\frac{Q}{ND^3}\right)_m = \left(\frac{Q}{ND^3}\right)_P$
- $\left(\frac{H}{N^2D^2}\right)_m = \left(\frac{H}{N^2D^2}\right)_P$
- $\left(\frac{P}{D^5N^3}\right)_m = \left(\frac{P}{D^5N^3}\right)_P$
- $\left[\frac{N'\sqrt{P/\rho}}{(gH)^{5/4}}\right]_m = \left[\frac{N'\sqrt{P/\rho}}{(gH)^{5/4}}\right]_P$

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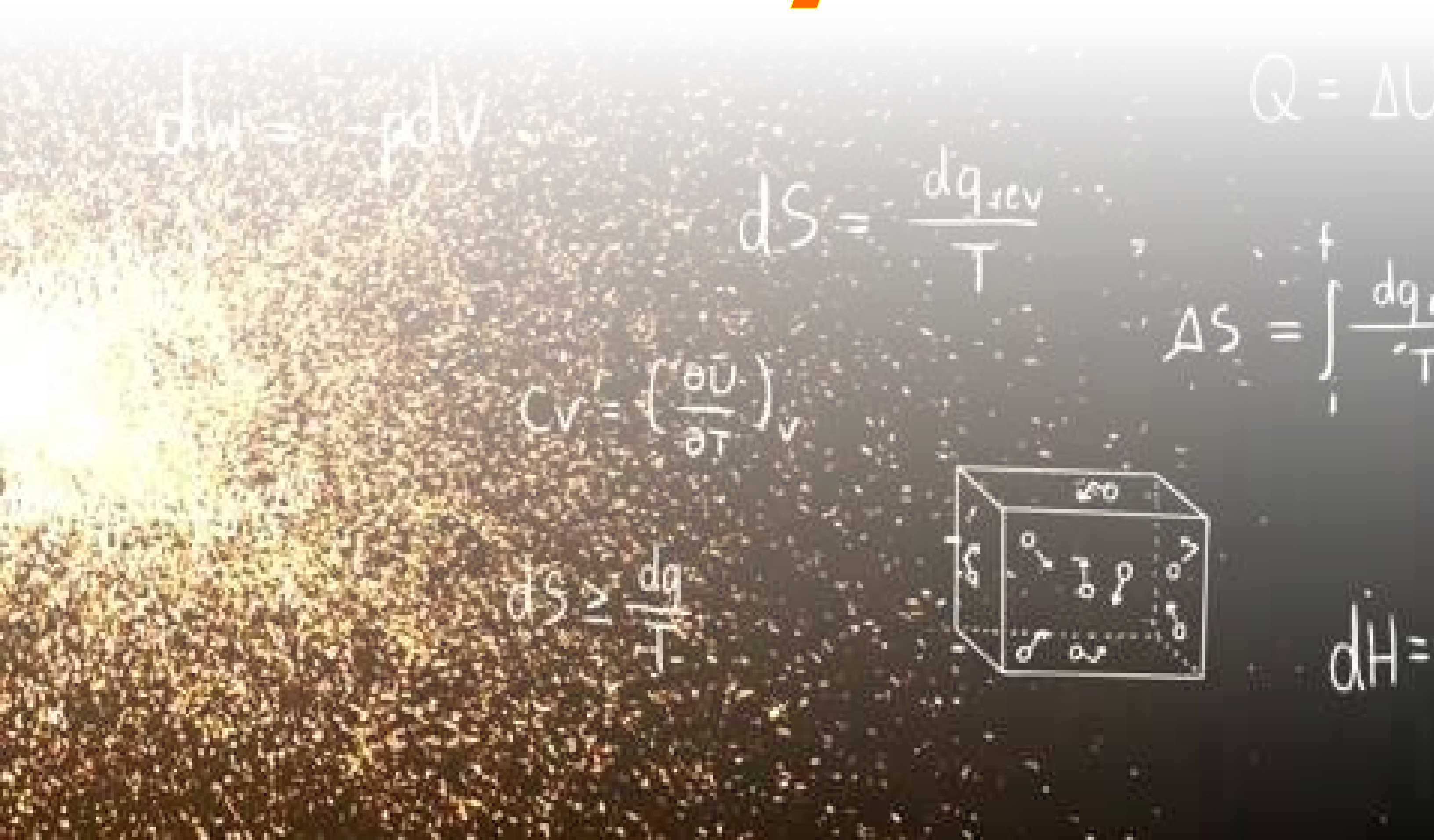
Basic Thermodynamics



Thermodynamics

INDEX

1 BASIC CONCEPTS

1.1 Introduction

- Thermodynamics is the branch of science which deals with the energy interactions (heat & work interactions) and its effect on properties of the system.

1.2 System, Surrounding & Boundary

System: Quantity of matter or region in space which is under investigation.

Surrounding: Anything which is external to the system.

Boundary: Interface shared by system and surrounding.

Boundary can be Rigid (Fixed) or Flexible (Movable)

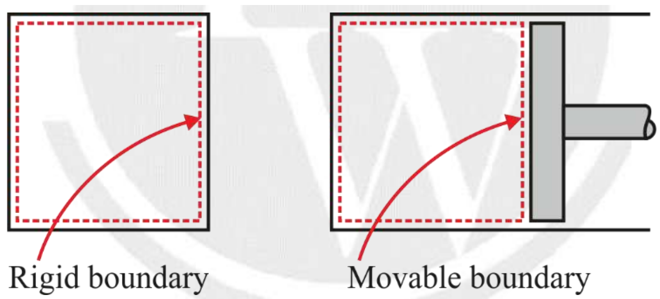


Fig. 1.1 Fixed and movable boundary

Mathematically, thickness of boundary is zero, hence it has neither volume nor mass.

Universe = System + Surroundings

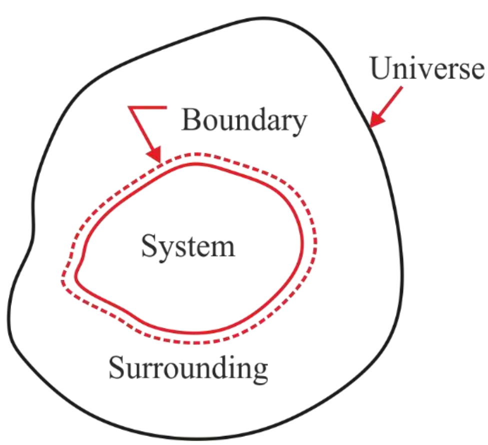


Fig. 1.2 System, boundary, surroundings and universe

1.2.1 Types of System

Closed System:-

- Closed System is the system in which mass interaction can't take place but energy interactions can take place.
- The terms Control Mass is some times used in place of closed system.

eg. Air in rigid container, Air in piston cylinder arrangement (without valves)

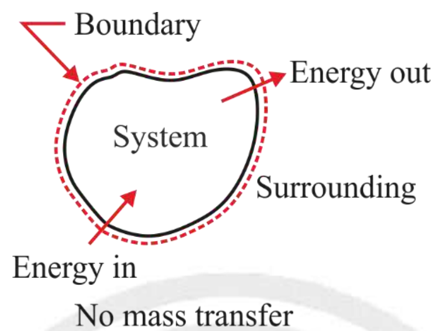


Fig. 1.3 Closed system

Isolated System:-

- Isolated system is the system in which neither mass nor energy interactions can take place.
- In isolated system, energy transfer can't take place, but energy transformation can take place.

eg. Stationary well insulated thermo flask

Universe

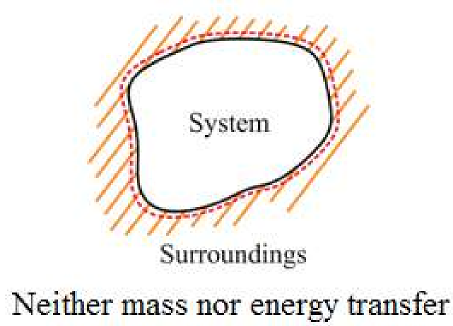


Fig. 1.4 Isolated system

Open System:-

- Open system is the system where both mass and energy interactions can take place.
- An open system is said to be steady flow open system if none of the properties change with time at a particular location.
- For steady flow open system

$$\left.\frac{dm}{dt}\right|_{cv} = 0 \ \& \ \left.\frac{dE}{dt}\right|_{cv} = 0$$

- The term control volume is sometimes used in place of open system.

- eg. Pump, Boiler, Super heater, Turbine, Condenser

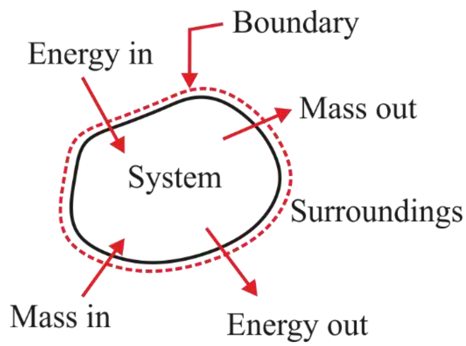


Fig. 1.5 Open system

1.3 Thermodynamic Equilibrium

- Two systems are said to be in thermodynamic equilibrium when they are in thermal, mechanical, chemical and phase equilibrium with each other.

1.4 Property

- Macroscopic characteristic of the system to which a numerical value can be assigned at any given instant of time is known as property.

1.4.1 Types of Property

1. Extensive Property ⇒ depends on the extent (Size) of the system.
2. Intensive Property ⇒ Independent of the extent (Size) of the system.

Mass (m) ⇒ Extensive

Volume (V) ⇒ Extensive

Pressure (P) ⇒ Intensive

Temperature, Viscosity, Thermal conductivity, Pressure, Velocity ⇒ Intensive properties

Density $(\rho) = \frac{\text{Mass (m)}}{\text{Volume (V)}}$

Density ⟶ Intensive

Specific Extensive Properties ⟶ Intensive properties

Kinetic energy, Potential energy, Internal energy (U), Enthalpy (H) and Entropy (S) are Extensive properties

1.4.2 Conclusions

- Density is an Intensive property.
- Ratio of two extensive properties is an intensive property.
- Specific properties are defined as extensive properties per unit mass.
- Specific properties such as specific volume, specific internal energy, specific enthalpy, specific entropy are intensive properties.

- In case of extensive property, its value for the overall system is the sum of its values for the parts in which system is divided, i.e. extensive properties are additive in nature.
- Intensive properties are not additive in nature.

1.5 Process

- A process is said to be occurred if the system undergoes from one equilibrium state to another equilibrium state.

1.5.1 Quasi-static process

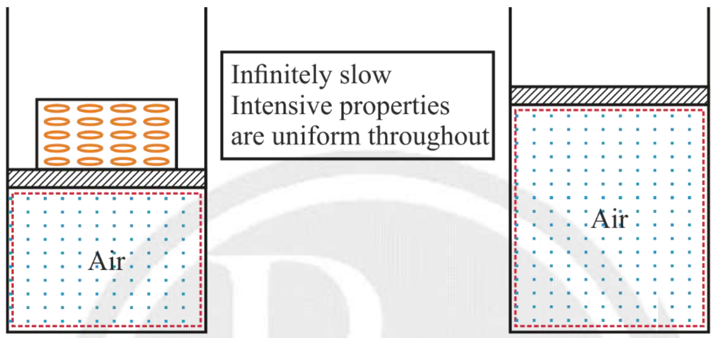


Fig. 1.6 Piston cylinder arrangement representing Quasi – static process

- Quasi-static process is infinitely slow process such that intensive properties are uniform throughout the system.
- All internally reversible processes are quasi-static process but all quasi-static processes need not to be internally reversible.

1.5.2 Internally Reversible Process

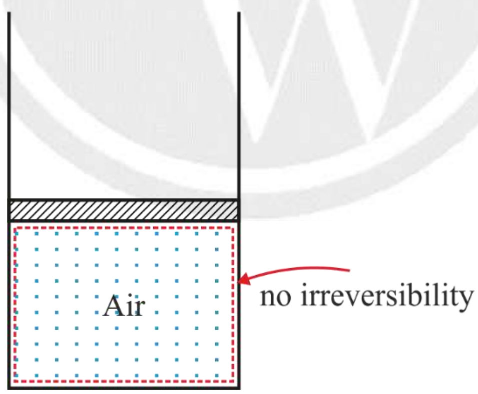


Fig. 1.7 A system undergoing an internally reversible process

- A process is said to be internally reversible process if no irreversibility occurs inside the boundary of the system.
- For closed system, at every intermediate state of an internally reversible process all intensive properties are uniform throughout i.e. closed system passes through series of equilibrium states.
- For closed system, any quasi-static process (neglecting viscous losses) is an internally reversible process.
- If we reverse the process, the system can pass through exactly the same equilibrium states while returning to its initial state i.e. the forward & reverse process can coincide for an internally reversible process.
- In general Phase change process can be considered as internally reversible process.

1.5.3 Externally Reversible Process

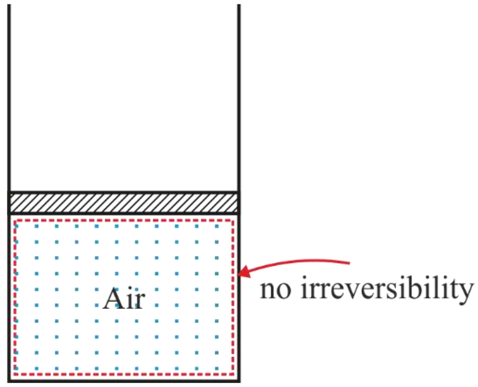


Fig. 1.8 A system undergoing an externally reversible process

- A process is said to be externally reversible process, if no irreversibility occurs outside of the boundary of the system.
- Heat transfer through negligible temperature difference is an externally reversible process whereas heat transfer through finite temperature difference is an externally irreversible process.

1.5.4 Totally Reversible Process

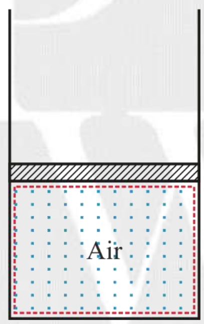


Fig. 1.9 A system undergoing a reversible process

- A process is said to be totally reversible process, if no irreversibility occurs inside & outside of the boundary of the system.
- Reversible Process = Internally Reversible Process + Externally Reversible Process
- For a totally reversible process, the system and its surroundings can be exactly restored to their respective initial states after the process has taken place.

1.6 Cyclic Process

- If a system undergoes number of processes such that the initial and final states are same, then the process is known as cyclic process.
- Cyclic processes are used to get continuous effect.

1.7 Characteristics of Properties

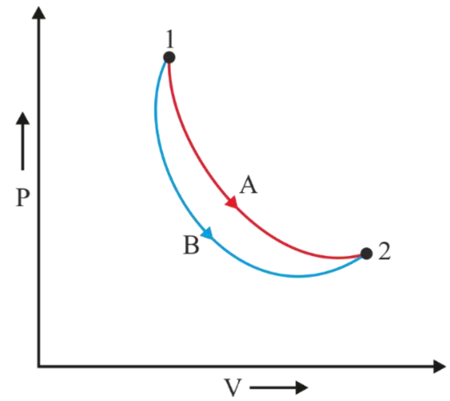


Fig. 1.10 Two processes with same end states

- Properties are independent of path followed by the process. They depend only on the state and hence properties are known as state functions.
- On property diagram state is represented by a point hence properties are also known as point functions.
- Whatever the path, internally reversible or internally irreversible, the difference of properties between two states is exactly same, hence properties are known as exact differentials. In differential form properties are represented by dB.
- Mathematically

  $dB = Mdx + Ndy$

  If B is an Exact differential

  $\left.\frac{\partial M}{\partial y}\right|_x = \left.\frac{\partial N}{\partial x}\right|_y$ (Test of Exactness)
- For a cyclic process change in property is always zero, $\oint dB = 0$
- For Cyclic Process $\Delta U = 0$

1.8 Zeroth Law of Thermodynamics

- When a body A is in thermal equilibrium with a body B, and also separately with a body C, then bodies B and C will be in thermal equilibrium with each other. This is known as the Zeroth law of thermodynamics.
- It is the basis of temperature measurement.

1.8.1 Thermodynamic Temperature Scale

$$\frac{T_I - I_I}{S_I - I_I} = \frac{T_{II} - I_{II}}{S_{II} - I_{II}}$$

Where, T_I = Temperature on first scale (in °C)

T_{II} = Temperature on second scale (in °C)

S_I = Steam point temperature on first scale

S_{II} = Steam point temperature on second scale

I_I = Ice point temperature on first scale

I_{II} = Ice point temperature on first scale

Temperature Conversion Formula:

$$\frac{°C}{5} = \frac{°F - 32}{9} = \frac{T - 273.15}{5}$$

Where, °C = Temperature in degree Celsius

°F = Temperature in degree Fahrenheit

T = Temperature in Kelvin.

Thermometer	Thermometric property	Symbol
1. Constant volume gas thermometer	Pressure	P
2. Constant pressure gas thermometer	Volume	V
3. Electrical resistance thermometer	Resistance	R
4. Thermocouple	Thermal e.m.f	ε
5. Mercury-in-glass thermometer	Length	L

1.9 Universal Gas Constant

$$\overline{R} = \lim_{P \to 0} \frac{PV}{nT}$$

$$\overline{R} = 8.314 \frac{kJ}{kmol - K}$$

$$\overline{R} = 8.314 \frac{J}{mol - K}$$

$$\overline{R} = 8314 \frac{J}{kmol - K}$$

Compressibility Factor (Z)

- Mathematically

$$Z = \frac{PV/nT}{\overline{R}}$$

Where $\overline{R} = \lim_{P \to 0} \frac{PV}{nT}$

Reduced property

- Reduced property at a given state is the value of property at that state divided by the value of property at critical point.
- Mathematically

$$B_r = \frac{B}{B_c}$$

- Reduced pressure $P_r = \frac{P}{P_c}$
- Reduced Temperature $T_r = \frac{T}{T_c}$

1.9.1 Generalized Compressibility Chart

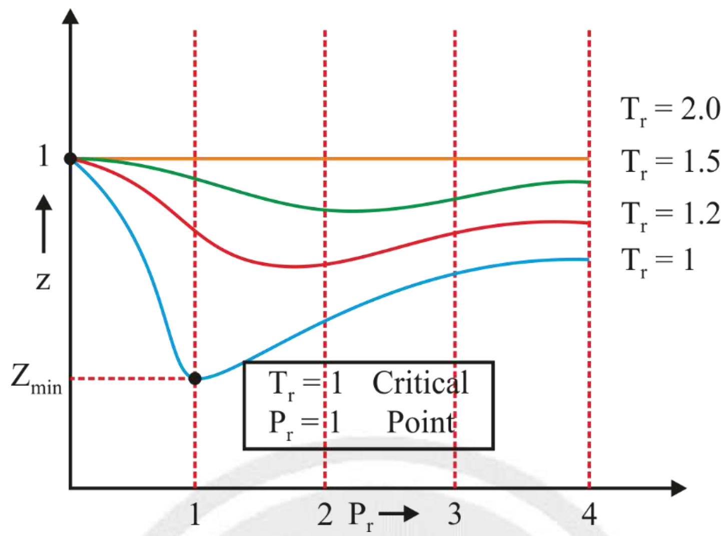


Fig. 1.11 Generalized compressibility chart

- At $P_r \rightarrow 0$ (low pressure) the value of compressibility factor approaches to unity irrespective of temperature.
- At $T_r \geq 2$ (high temperature) the value of compressibility factor approaches to unity up to $P_r = 4$.
- Compressibility factor is minimum at critical point ($P_r = 1$, $T_r = 1$).
- Compressibility factor is approximately same, at same reduced pressure and reduced temperature. This is known as principle of corresponding states.
- Generalised compressibility chart is extremely useful in situation where detailed data on a particular gas are lacking but critical properties are available.

1.10 Ideal Gas

- For Ideal Gas compressibility factor is unity
- Various forms of Ideal gas equation
  - $PV = n\bar{R}T$
  - $PV = mRT$
  - $P = \rho RT$
  - $Pv = RT$
  - $R = \frac{\bar{R}}{M}\left(\frac{kJ}{Kg-K} / \frac{J}{gm-K}\right)$
  - Units of characteristic gas constant is kJ/kg-K or J/kg-K.
  - Characteristic gas constant depends only on the molar mass of the gas and is independent of the temperature of the gases.
  - Characteristic gas constant is different for different gases. Higher the molar mass of the gas, lesser the characteristic gas constant.

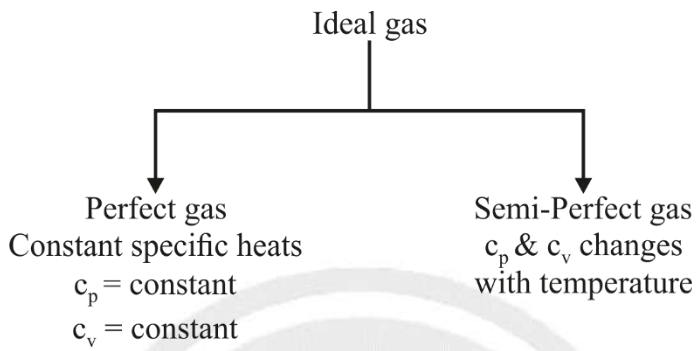
1.10.1 Physical Meaning of Ideal Gas

Ideal gas is having following characteristics

(i) Negligible intermolecular forces of attraction.

(ii) Volume of the gas molecules is negligible as compared to the container volume.

1.10.1.1 Ideal gas vs Perfect Gas



1.11 Analysis of Non-Reacting Ideal Gas Mixture

Case 1: Mole fractions ($\bar{x}$) of gases are given

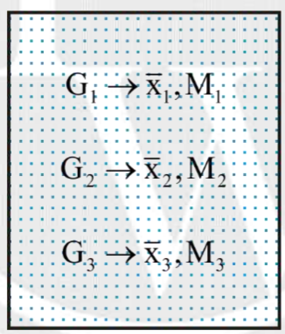


Fig. 1.12 Mole fraction of gases in a mixture

$\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = 1$

$M_{mix} = \bar{x}_1 M_1 + \bar{x}_2 M_2 + \bar{x}_3 M_3$

$R_{mix} = \dfrac{\bar{R}}{M_{mix}}$

Case 2: Mass fractions (x) of gases are given

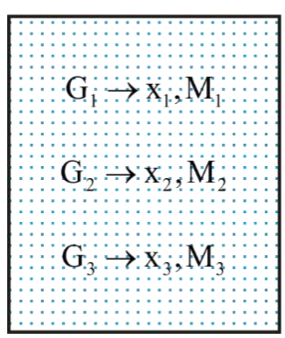


Fig. 1.13 Mass of fraction of gases in a mixture

$x_1 + x_2 + x_3 = 1$

$M_{mix} = \dfrac{1}{\frac{x_1}{M_1} + \frac{x_2}{M_2} + \frac{x_3}{M_3}}$

$R_{mix} = \dfrac{\bar{R}}{M_{mix}}$

1.12 Van der Waals Equation of State

- $\left(P + \frac{a}{v^2}\right)(v - b) = RT \quad (v \text{ in } m^3/kg)$
- $\left(P + \frac{\bar{a}}{\bar{v}^2}\right)(\bar{v} - \bar{b}) = \bar{R}T \quad (\bar{v} = \text{molar specific volume in } m^3/kmol)$
- $\frac{a}{v^2}$ accounts for intermolecular force of attraction.
- b accounts for volume of molecules.

1.12.1 Discussion on Critical Isotherm

- Critical isotherm passes through a point of inflection at the critical point.

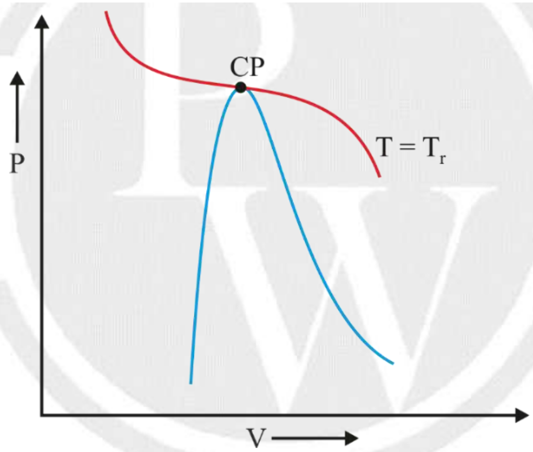


Fig. 1.14 Isotherm on P-v diagram

- $\left.\frac{\partial P}{\partial v}\right|_{cp} = 0$
- $\left.\frac{\partial^2 P}{\partial v^2}\right|_{cp} = 0$
- $\left.\frac{\partial^3 P}{\partial v^3}\right|_{cp} < 0$

1.12.2 Expressions of a & b in Van der Walls equation

$a = 3P_c v_c^2$

$b = \frac{v_c}{3}$

Where P_c & v_c are pressure and specific volume at critical point.

$Z_c = \frac{3}{8} = 0.375$

1.13 Various Process

1.13.1 Constant Volume Process / Isochoric Process / Isometric Process

V = Constant [Any Substance]

dV = 0 [Any Substance]

Physical representation for Isochoric heat addition process

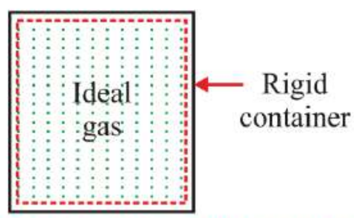


Fig. 1.15 Ideal gas in a rigid container

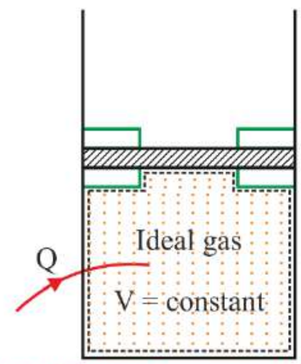
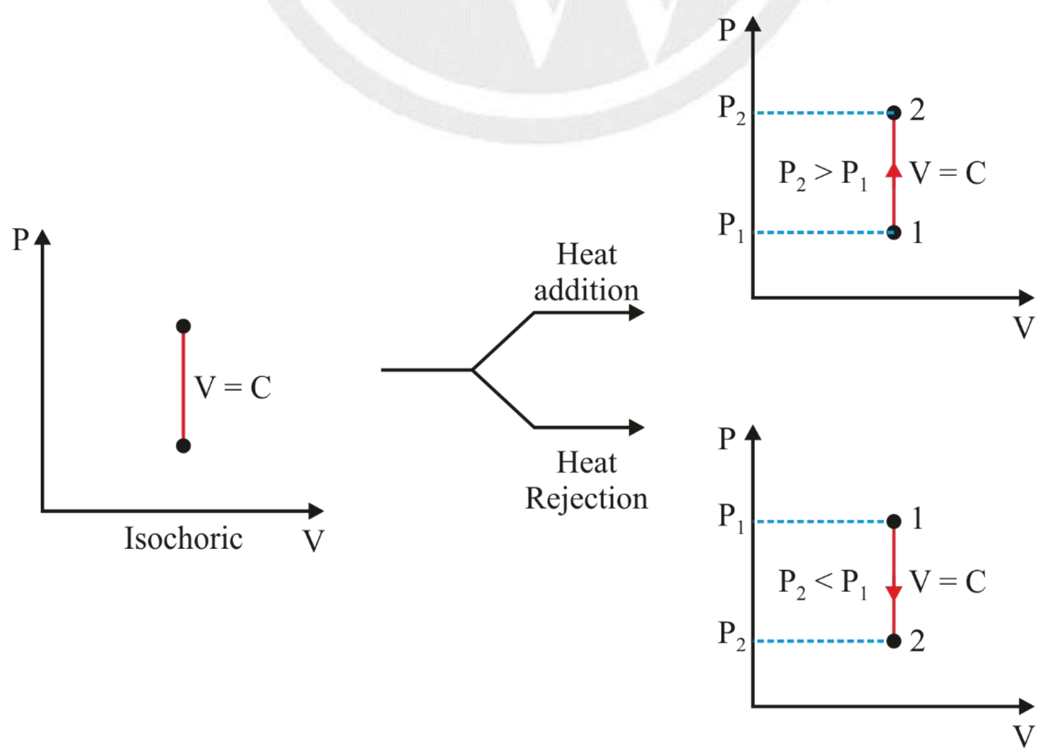


Fig. 1.16 Piston cylinder arrangement with stoppers

- For an ideal gas undergoing Isochoric heat addition process then both temperature and pressure increases.
- For an ideal gas undergoing isochoric process

  $\frac{P}{T} = \frac{mR}{V} = \text{constant}$
- According to Gay Lussac's law, for a given mass of an ideal gas undergoing isochoric process, absolute pressure is directly proportional to absolute temperature.



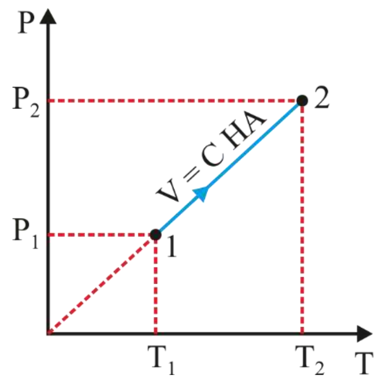


Fig. 1.17 Isochoric process on P – V and P – T diagram

1.13.2 Constant pressure process/ Isobaric Process/ Isopiestic Process

The pressure is constant at equilibrium states.

- P = constant
- dP = 0 [Any substance]

Physical representation for Isobaric heat addition process

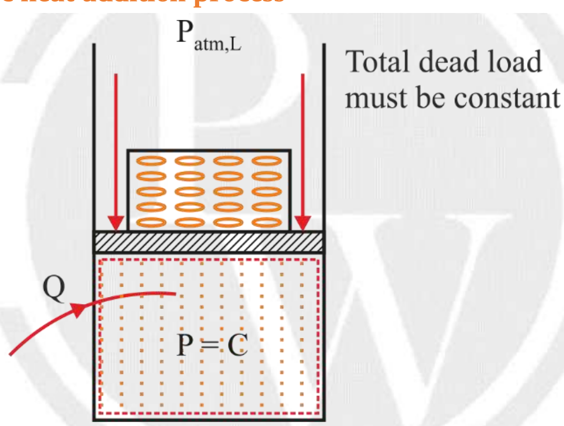


Fig. 1.18 Piston cylinder arrangement have constant total dead load

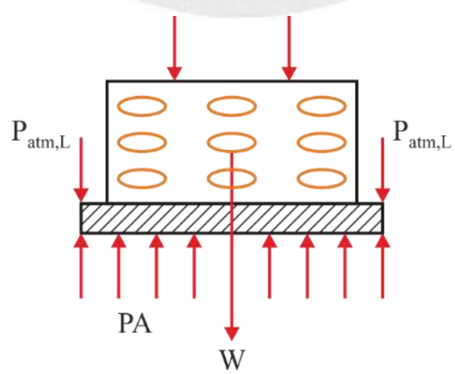


Fig. 1.19 Representation of different pressures on piston

- At equilibrium state for vertical piston cylinder arrangement

$$PA = P_{atm,L} A + W$$

$$\Rightarrow P = P_{atm,L} + \frac{W}{A}$$

- For an ideal gas undergoing isobaric heat addition process then both temperature and volume increases.

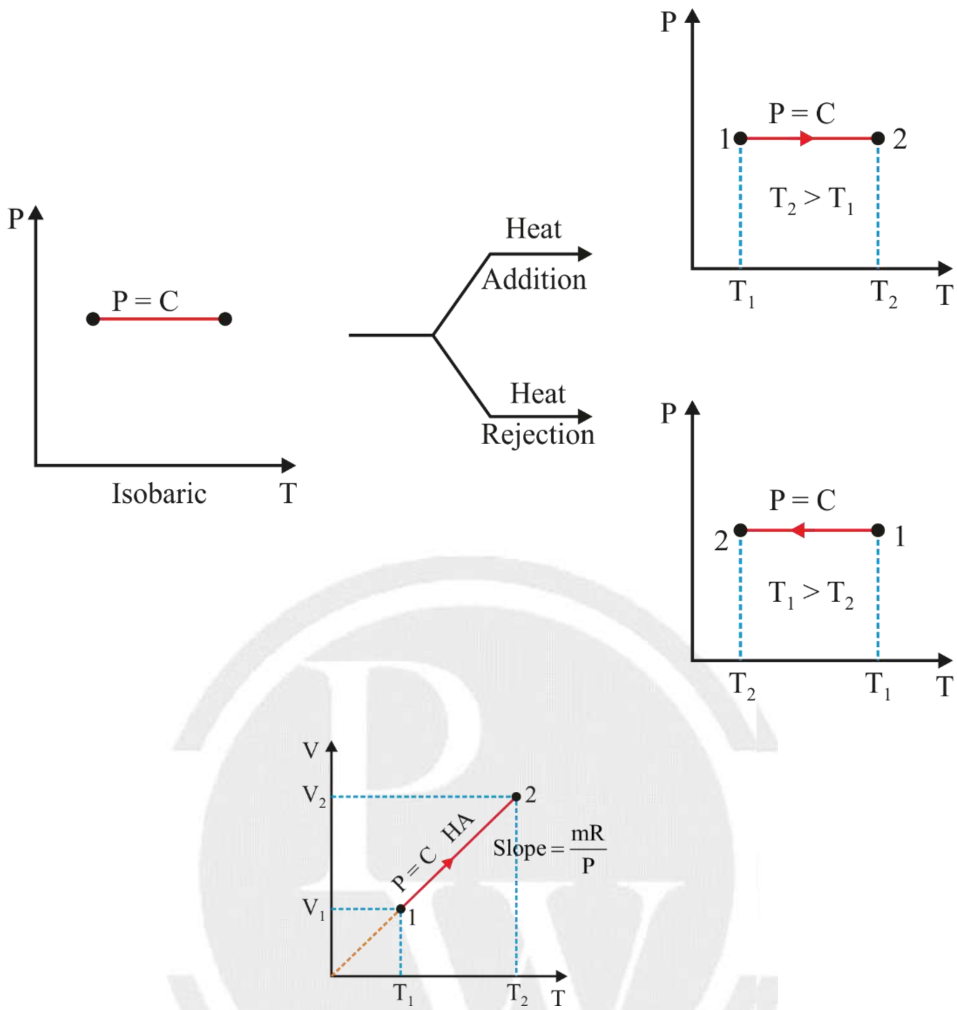


Fig. 1.20 Isochoric process on P – T and V – T diagram

- For an ideal gas, $\frac{V}{T} = \frac{mR}{P} = \text{constant}$
- According to Charles law, for a given mass of an ideal gas undergoing isobaric process, volume is directly proportional to absolute temperature.

1.13.3 Constant temperature process/Isothermal Process

- T = Constant [Any Substance]
  dT = 0 [Any Substance]

Physical representation for Isothermal heat addition process

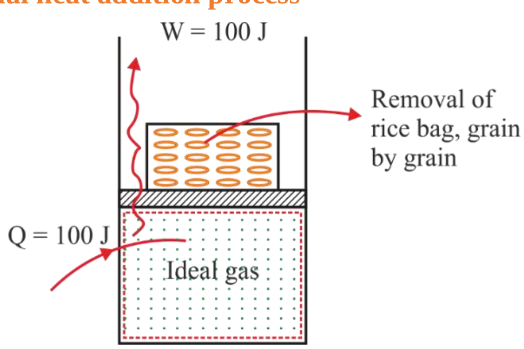


Fig. 1.21 Arrangement for Isothermal process

- For ideal gas, heat can be supplied at constant temperature.
- For an ideal gas undergoing isothermal process heat completely converts into work.
- Isothermal heat transfer to a closed system, neglecting friction can be considered as an internally reversible process.

Analysis for an Ideal gas

For an ideal gas undergoing isothermal heat addition process pressure decreases and volume increases.

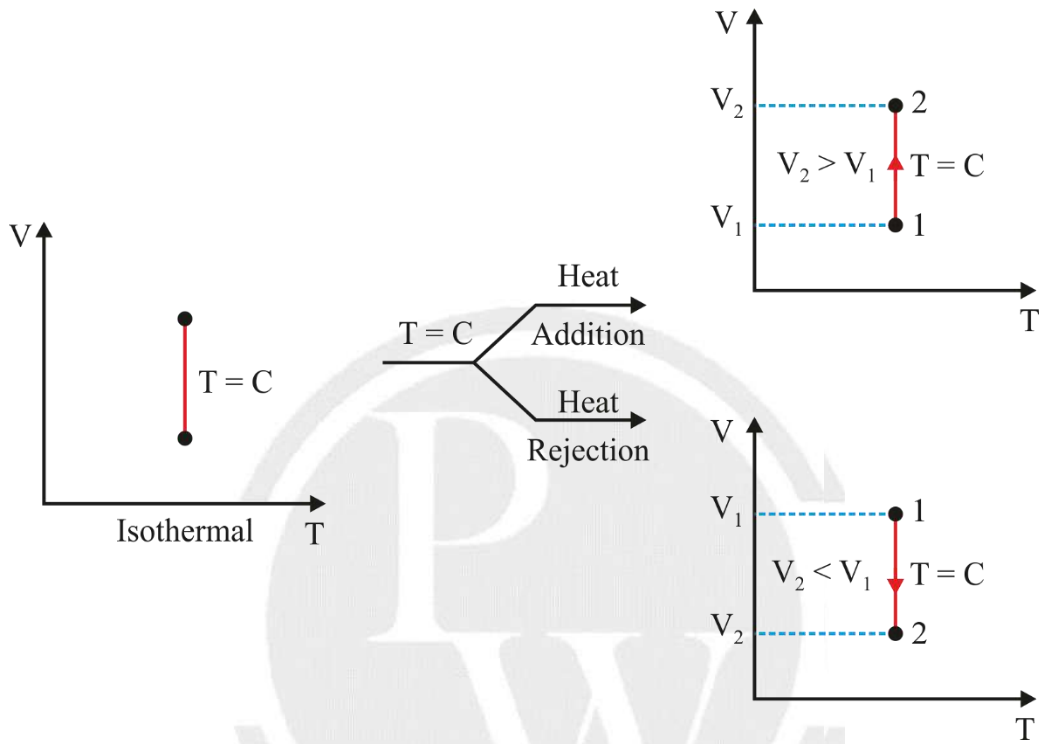


Fig. 1.22 Isothermal process on V – T diagram

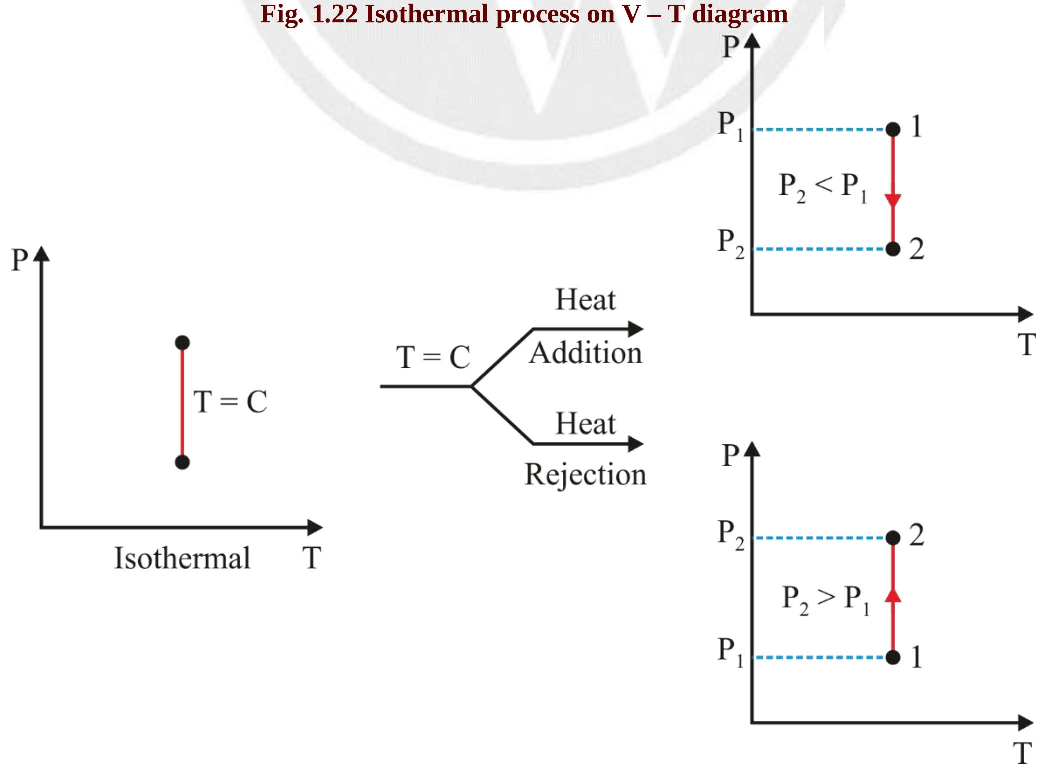


Fig. 1.23 Isothermal process on P-T diagrams

- For an Ideal gas
- PV = mRT = Constant

According to Boyle's law, for a given mass of an ideal gas undergoing isothermal process, absolute pressure is inversely proportional to volume.

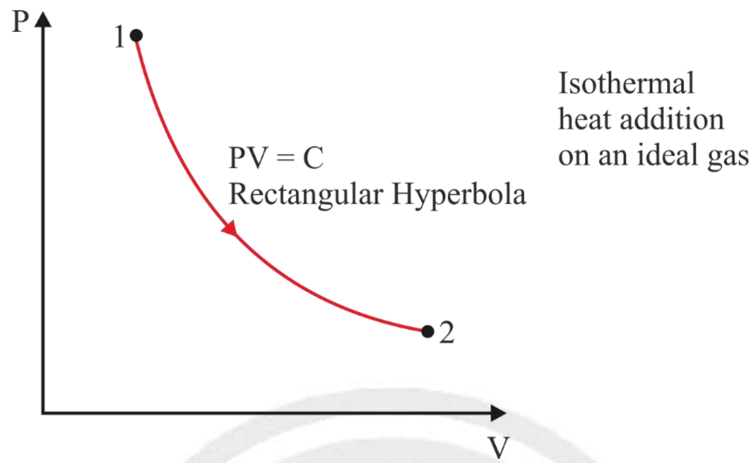


Fig. 1.24 Isothermal heat addition process on P-V diagram

Note:

For an ideal gas, Isothermal process on a P-V diagram is represented by a rectangular hyperbola which is symmetric to both pressure and volume axis.

1.13.4 Adiabatic process

- A process during which there is no heat interaction is called an adiabatic process.

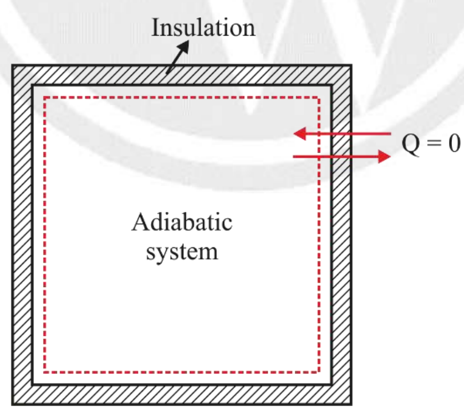


Fig. 1.25 Adiabatic system

- Q = 0 (Any Substance)

- For perfect gas undergoing Isentropic process following relations can be used

  PV^{γ} = Constant

  $TP^{\frac{1-\gamma}{\gamma}}$ = Constant

  $TV^{\gamma-1}$=Constant
- For a perfect gas undergoing Isentropic Expansion process, Pressure decreases, volume increases, temperature decreases.

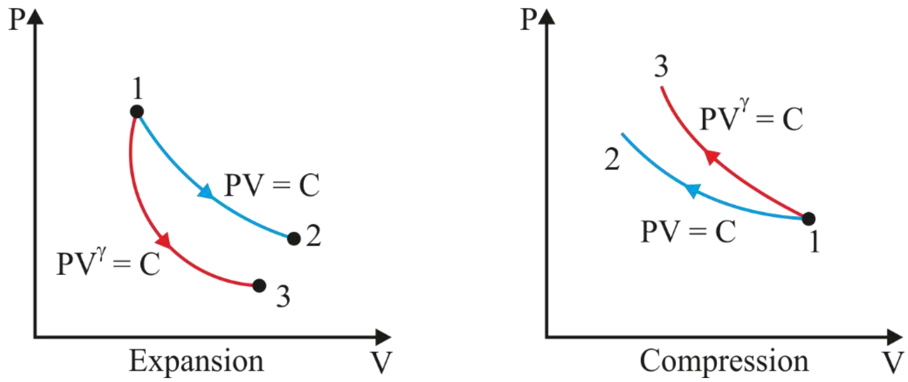


Fig. 1.26 Isothermal and adiabatic processes during expansion and compression

- $\left.\frac{\partial P}{\partial v}\right|_S = \gamma \left.\frac{\partial P}{\partial v}\right|_T$

> **Note:**
> - For perfect gas on P-V diagram, both isothermal and Isentropic (internally reversible adiabatic) processes have negative slopes.
> - For perfect gas on the same P-V diagram, the slope of internally reversible adiabatic process is more (γ times) than the slope of isothermal process.

1.13.5 Polytropic Process

- For Polytropic process following equations can be used

  PV^n = Constant (Any substance)

  $TP^{\frac{1-n}{n}}$ = Constant (Ideal Gas)

  TV^{n-1} = Constant (Ideal Gas)

Where n is known as polytropic index

Representation of various processes on same P-V diagram

- P-V diagram for expansion

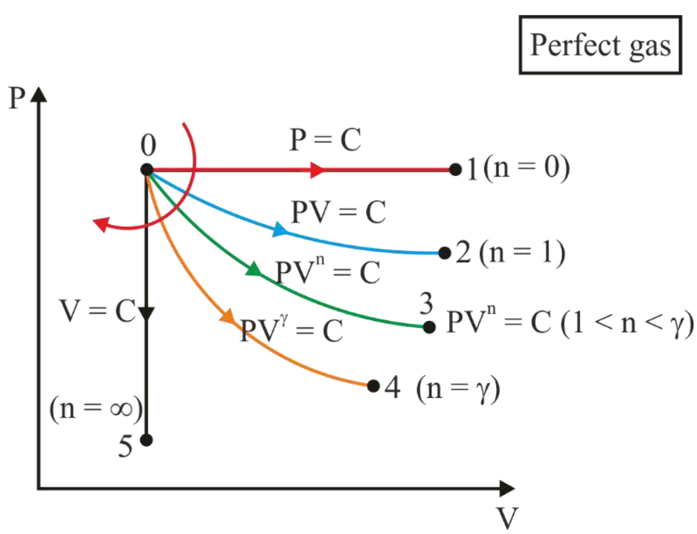


Fig. 1.27 P – V diagram for expansion

- P-V diagram for compression

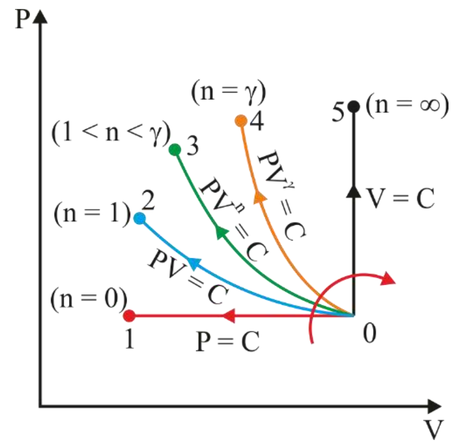


Fig. 1.28 P – V diagram for compression

1.14 Energy of A System

- E = Macroscopic Energy (KE + PE) + Microscopic energy (U)

  E = KE + PE + U

  ΔE = ΔKE + ΔPE + ΔU

- Internal energy depends on the size of the system. Larger the size of the system, higher will be the number of molecules hence higher will be the internal energy. Internal energy is an extensive property.

1.14.1 Stationary System

- A stationary system is a closed system having no change in kinetic energy & potential energy during a process.
- For a stationary system change in energy is equal to the change in internal energy.

❑❑❑

2 WORK INTERACTIONS

2.1 Thermodynamic Work

- Thermodynamic Work is done by the system if the sole effect on the surrounding (everything external to the system) could be raising of weights (real or imaginary).
- Thermodynamic Work is a

1. Form of Energy in transit
2. Boundary Phenomenon

Note:

Thermodynamic work is a form of energy in transit, since it is energy being transferred across the system boundary.

Thermodynamic work is a boundary phenomenon since it is observed only at the boundary.

2.1.1 Sign Convention

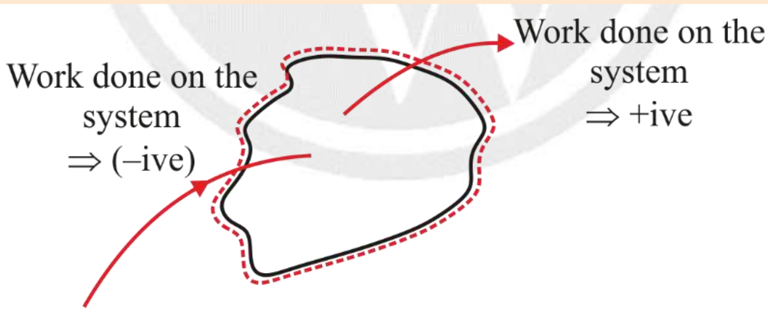


Fig.2.1 Sign convention of work

Work done by the system is taken as positive whereas work done on the system is taken as negative.

2.2 Heat Interaction

- Heat interaction is defined as the form of energy that is transferred across a boundary of a system by virtue of temperature difference between system and surrounding.

2.2.1 Sign Convention

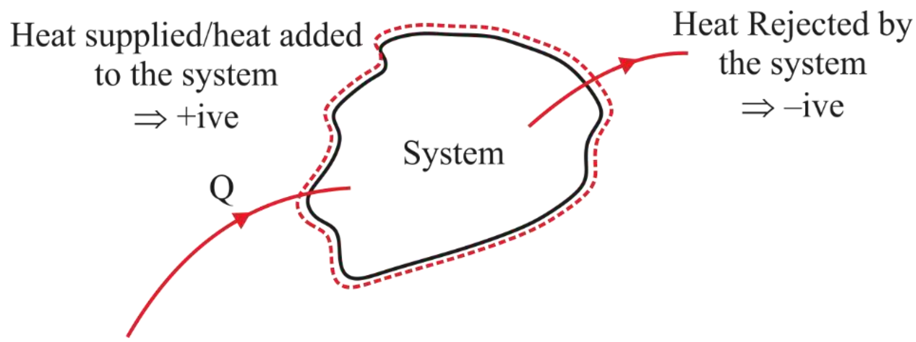


Fig. 2.2 Sign convention of heat interaction

Heat supplied to the system is taken as positive whereas heat rejected by the system is taken as negative.

2.3 Work Done at the moving boundary of a closed system (Displacement Work)

$\delta W_d = PdV$ [Closed system and Int. Rev. Process]

$W_{d_{1-2}} = \int_1^2 PdV$ [System undergoing a process 1 − 2]

2.4 Displacement Work on P-V Diagram

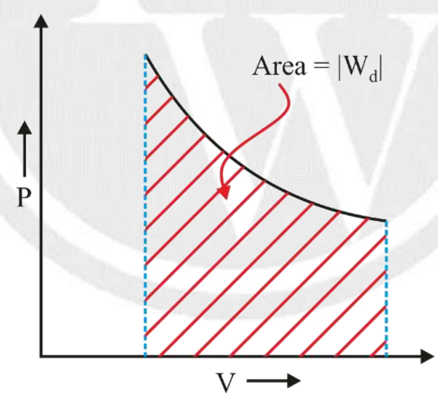


Fig. 2.3 Displacement work on P-V diagram

$\delta W_d = PdV$

- $dV = +ve \rightarrow \delta W_d = +ive$ (For expansion process) (Done by the system)
- $dV = -ve \rightarrow \delta W_d = -ive$ (For compression process) (Done on the system)

Note:
For a closed system undergoing internally reversible process,

- On P-V diagram area under the curve projected on volume axis gives the magnitude of displacement work.
- For expansion displacement work is positive while for compression displacement work is negative.

2.5 Observation for Work Interactions

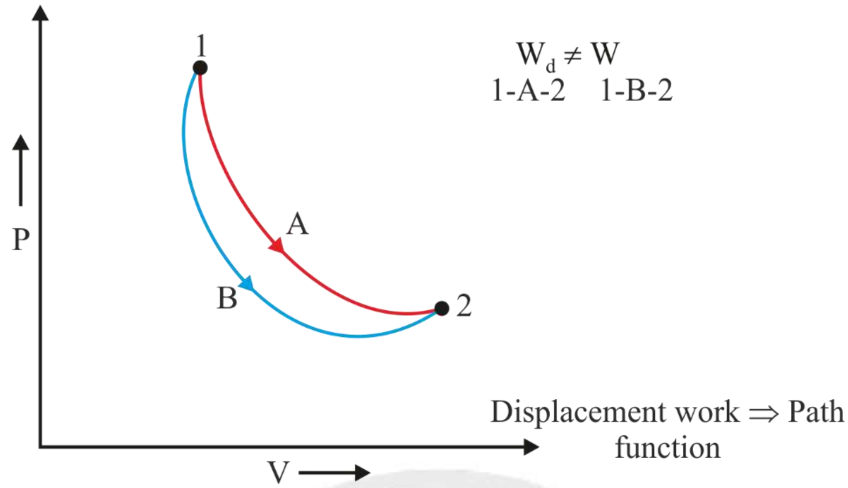


Fig.2.4 Two processes A and B with same end states

- Even though initial and final states are same for both the paths 1-A-2 and 1-B-2, but work interactions are different, hence work interaction is a path function.
- Work interaction is an inexact differential.

Note:

Heat and work are energy transfer mechanisms between a system and its surroundings, and there are many similarities between them:

I. Both are recognized at the boundaries of a system as they cross the boundaries. That is, both heat and work are boundary phenomena.

II. Systems possess energy, but not heat or work.

III. Both are associated with a process, not a state. Unlike properties, heat or work has no meaning at a state.

IV. Both are path functions (i.e., their magnitudes depend on the path followed during a process as well as the end states).

2.6 Displacement Work for Various Process

Isochoric process

Isobaric process

Isothermal process

Adiabatic process

Polytropic process

2.6.1 Isochoric Process

- For any substance consisting closed system undergoing Isochoric process
- $W_{d_{1-2}} = 0$

2.6.2 Isobaric Process

- For any substance, consisting closed system under goes an internally reversible isobaric process.

  $W_{d_{1-2}} = P(V_2 - V_1) = P\Delta V_{1-2}$

- For an ideal gas, consisting closed system undergoing an internally reversible isobaric process

  $W_{d_{1-2}} = mR(T_2 - T_1)$

  $W_{d_{1-2}} = n\bar{R}(T_2 - T_1)$

2.6.3 Isothermal process

- For an ideal gas consisting closed system undergoing internal reversible isothermal process

  $$W_{d_{1-2}} = C\ln\left(\frac{V_2}{V_1}\right) = C\ln\left(\frac{P_1}{P_2}\right)$$

  Where C can be P_1V_1, P_2V_2, mRT or $n\bar{R}T$

2.6.4 Adiabatic Process

- For a perfect gas consisting closed system undergoing isentropic process.

$$W_{d_{1-2}} = \frac{P_1V_1 - P_2V_2}{\gamma - 1}$$

$$W_{d_{1-2}} = \frac{mR(T_1 - T_2)}{\gamma - 1}$$

$$W_{d_{1-2}} = \frac{n\bar{R}(T_1 - T_2)}{\gamma - 1}$$

2.6.5 Polytropic process

- For any substance consisting closed system undergoing Internally reversible polytropic process

$$W_{d_{1-2}} = \frac{P_1V_1 - P_2V_2}{n - 1}$$

- For Ideal gas consisting closed system undergoing Internally reversible polytropic process

$$W_{d_{1-2}} = \frac{mR(T_1 - T_2)}{n - 1} = \frac{n\bar{R}(T_1 - T_2)}{n - 1}$$

- Finding of Polytropic Index (n)

$$PV^n = \text{Constant},\ n = \frac{\ln\left(\frac{P_2}{P_1}\right)}{\ln\left(\frac{V_1}{V_2}\right)}$$

$$TV^{n-1} = \text{Constant} \Rightarrow n-1 = \frac{\ln\left(T_2/T_1\right)}{\ln\left(V_1/V_2\right)}$$

$$TP^{\frac{1-n}{n}} = \text{Constant} \Rightarrow \frac{1-n}{n} = \frac{\ln\left(T_2/T_1\right)}{\ln\left(P_1/P_2\right)}$$

Note:

Don't put –ve sign in front of formulas seeing compression in problem. Automatically answer will come negative for compression.

All formulas are derived with the sign convention that work done by the system is +ve while work done on the system is – ve.

If the sign conventions are reversed, then – ve sign must be placed before the formula.

2.7 Random Process

- If $P \propto V$

$$W_{d_{1-2}} = \frac{1}{2}\frac{P_1}{V_1}(V_2^2 - V_1^2)$$

- If $P \propto D^3$

$$W_{d_{1-2}} = \frac{\pi}{12}\frac{P_1}{D_1^3}(D_2^6 - D_1^6)$$

- $P \propto D$

$$W_{d_{1-2}} = \frac{\pi}{8}\frac{P_1}{D_1}(D_2^4 - D_1^4)$$

2.8 Piston Cylinder Arrangement with Spring

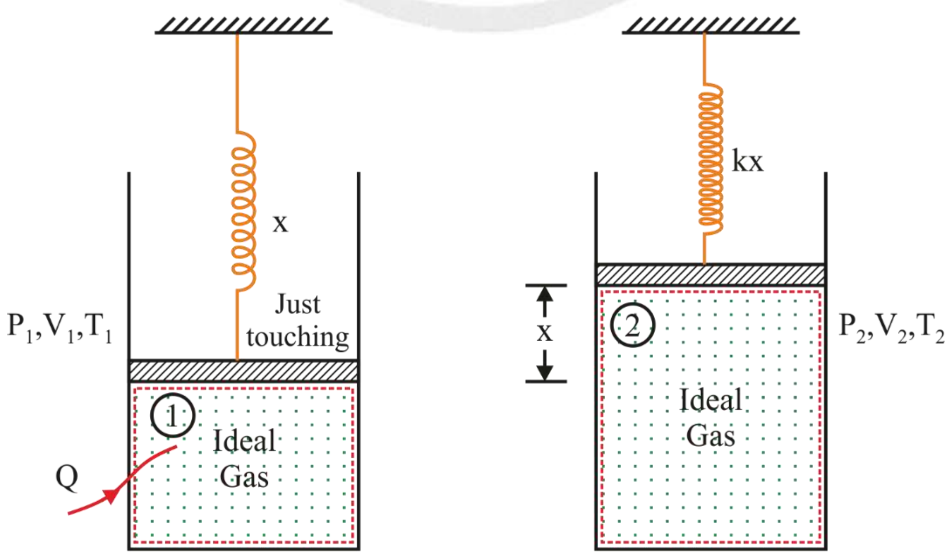


Fig. 2.5 Piston cylinder arrangement with springs

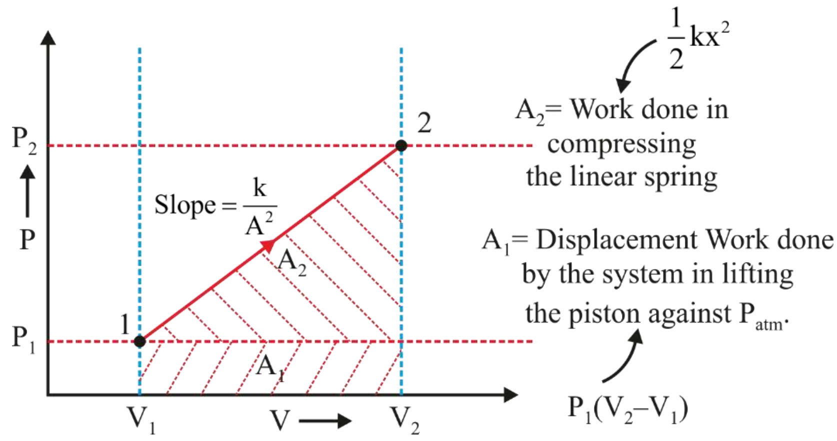


Fig. 2.6 Expansion process against spring force

Note:

- For an ideal gas enclosed by piston cylinder arrangement with spring heat addition leads to increase in pressure, volume and temperature.

2.9 Net Displacement Work for Cyclic Process

Net displacement work for a cyclic process is the sum of displacement work of all the processes consisting the cycle.

$$\sum W_d = W_{d_{1-2}} + W_{d_{2-3}} + W_{d_{3-4}} + W_{d_{4-1}}$$

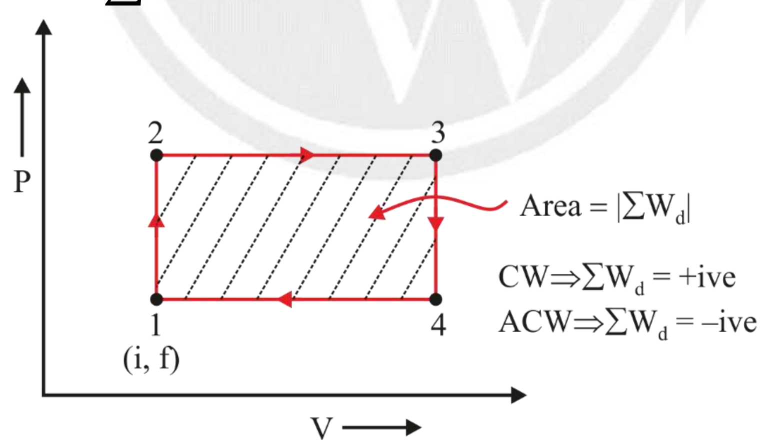


Fig.2.7 Cyclic process on P.V. Diagram

2.9.1 Conclusions

On a P-V diagram, the area enclosed by the cycle represents the magnitude of net displacement work of the cycle.

For clockwise cycles, on P-V diagram net displacement work is +ve. All power producing cycles are clock-wise on P-V diagram e.g. I.C. engine cycle

Otto, Diesel, Dual cycle

Gas power cycle (Brayton cycle)

Vapour power cycle (Rankine cycle)

For anti-clockwise cycles, on P-V diagram, net displacement work is negative. All power consuming cycles are anti-clockwise on P-V diagram.

e.g. Refrigeration cycles (Reversed Carnot cycle, Reversed Brayton cycle, Vapour Compression Refrigeration System cycle)

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3 FIRST LAW OF THERMODYNAMICS

3.1 Introduction

- First law of thermodynamics is based on experimental results.
- First law of thermodynamics is the law of conservation of energy.
- First law of thermodynamics gives the concept of internal energy.

3.2 Statement

- For a closed system undergoing a cyclic process, net heat interaction is equal to the net work interaction.
- For a closed system undergoing cyclic process

$$\Sigma Q = \Sigma W \text{ (Any cycle)}$$

$$\text{or } \oint \delta Q = \oint \delta W \text{ (Any cycle)}$$

Note:

- For power producing cycles, both network interactions and net heat interactions are positive.
- For power consuming cycles, both network interactions and net heat interactions are negative.

3.3 Consequences of First Law of Thermodynamics

(1) Heat interaction is a path function.

(2) Energy of the system is the property of the system.

(3) Energy of an isolated system is constant.

(4) Perpetual motion machine of first kind (PMM-1) is impossible.

3.4 Heat Interaction is a Path Function

Heat interaction is a

- Path function
- Inexact differential
- Form of a energy in transit
- Boundary phenomena

3.5 Energy of the System is the Property of the System

Energy of the system is a

- Property
- State function
- Point function
- Exact differential
- $\oint dE = 0$

3.6 First Law of Thermodynamics for a Process

- $Q = W + \Delta E$

Where $\Delta E = \Delta KE + \Delta PE + \Delta U$ and $W = W_e + W_s + W_d$

- $\delta Q = \delta W + dE$

  Q and W are inexact differential & E is an exact differential

3.6.1 For Simple compressible, Stationary system undergoing internally reversible process

- $Q = \int PdV + \Delta U$

  $\delta Q = PdV + dU$

3.7 Energy of an Isolated System is Constant

- $Q = 0, W = 0$

  $Q = W + \Delta E$

  $\Rightarrow \Delta E = 0$

  $E = \text{Constant}$

3.8 Perpetual Motion Machine of first kind (PMM-1) is impossible

- PMM-1 is fictitious machine which continuously produces the mechanical work without consuming any form of energy.

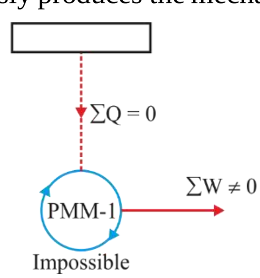

Fig. 3.1 PMM – 1

- PPM-1 is impossible as it violates first law of thermodynamics for a cycle.

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4 HEAT INTERACTIONS

4.1 State Postulate

According to State postulate, state of a simple compressible system can be completely defined by 2 independent intensive properties.

- (P, v) ⇒ Always independent intensive properties
- (T, v) ⇒ Always independent intensive properties
- (T, P) ⇒ In single phase independent intensive properties

4.2 Internal Energy (U)

- Internal energy is an extensive property having unit J or kJ.
- Specific Internal energy is an intensive property having unit J/kg or kJ/kg.
- Internal energy of an Ideal gas depends on temperature only, if temperature of an ideal gas increases internal energy increases & vice versa.
- For an Ideal gas undergoing Isothermal process, change in internal energy is zero.
- For any substance undergoing Isochoric process (or) Ideal gas undergoing any process we can apply the following two equations,

$$du = c_v dT$$

$$\Delta u = \int c_v \, dT$$

- For any substance having constant c_v undergoing isochoric process or perfect gas undergoing any process, we can apply following three equations,

Change in specific Internal Energy is $\Delta u = c_v \Delta T \left(\frac{J}{kg} \text{ or } \frac{kJ}{kg}\right)$

Change in molar specific internal energy is $\Delta \bar{u} = \bar{c}_v \, \Delta T \left(\frac{J}{mol} \text{ or } \frac{kJ}{kmol}\right)$

Total internal energy change is $\Delta U = mc_v \Delta T \ \ (J \text{ or } kJ)$

4.3 Enthalpy (H)

- It was observed that in thermodynamics calculations, U + PV term was coming very frequently.
- For convenience, U + PV was named as "Enthalpy".
- Enthalpy is an extensive property having unit J & kJ.
- Specific enthalpy is an intensive property having unit J/kg & kJ/kg.
- Enthalpy of an Ideal gas depends on temperature only, if temperature of an ideal gas increases enthalpy increases & vice versa.
- For an Ideal gas undergoing isothermal process the change in Enthalpy is zero.
- For any substance undergoing Isobaric process (or) Ideal gas undergoing any process we can apply these two equations

  $\mathrm{dh} = \mathrm{c_p dT}$

  $\Delta \mathrm{h} = \int \mathrm{c_p\, dT}$

- For any substance having constant c_p undergoing isobaric process or perfect gas undergoing any process, we can apply following three equations.

  Change in specific Enthalpy is $\Delta \mathrm{h} = \mathrm{c_p \Delta T} \left(\frac{\mathrm{J}}{\mathrm{kg}} \text{ or } \frac{\mathrm{kJ}}{\mathrm{kg}}\right)$

  Change in molar specific Enthalpy is $\Delta \bar{\mathrm{h}} = \bar{\mathrm{c}}_\mathrm{p}\, \Delta \mathrm{T} \left(\frac{\mathrm{J}}{\mathrm{mol}} \text{ or } \frac{\mathrm{kJ}}{\mathrm{kmol}}\right)$

  Total Enthalpy change is $\Delta \mathrm{H} = \mathrm{mc_p \Delta T}$ (J or kJ)

4.4 Heat Interactions for Various Processes

(a) Isochoric process

(b) Isobaric process

(c) Isothermal process

(d) Adiabatic expansion

(e) Polytropic process

4.4.1 Isochoric process

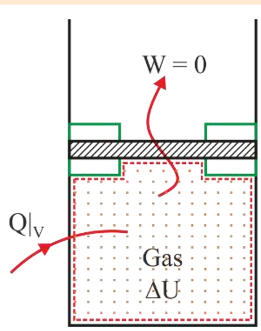


Fig. 4.1 Piston cylinder arrangement with stoppers

- $\mathbf{Q|_v = \Delta U_{1-2}}$
- If c_v is constant

  $Q|_v = m\, c_v \Delta T_{1-2}$
- For a simple compressible stationary system undergoing isochoric process, heat supplied is completely utilized to increase the internal energy.

4.4.2 Isobaric process

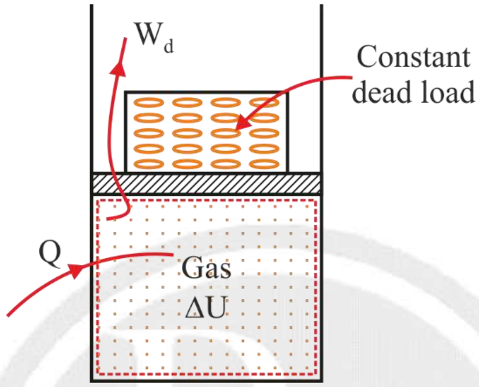


Fig. 4.2 Piston cylinder arrangement with constant dead load

- $Q|_P = \Delta H_{1-2}$
- If c_p is constant

  $Q|_P = m\, c_p \Delta T_{1-2}$
- For a simple compressible stationary system undergoing internally reversible Isobaric process, heat supplied is equal to the increase in enthalpy of the system.

4.4.2.1 Isobaric heat addition to perfect gas

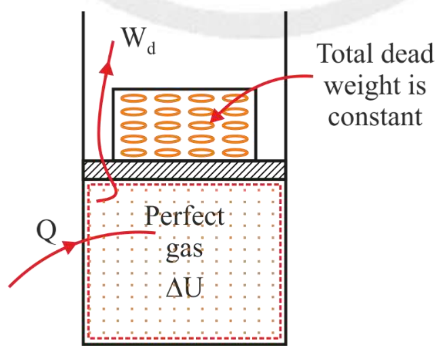


Fig.4.3 Isobaric heat addition to the perfect gas

If Q = +ive

then W_d = +ive, ΔU = +ive

- For simple compressible stationary system consisting of perfect gas, isobaric heat supplied partially converts into displacement work while remaining is stored as internal energy.

4.4.2.2 Analysis for an ideal gas

- $c_p - c_v = R \;\&\; \bar{c}_p - \bar{c}_v = \bar{R}$
- $c_v = \frac{R}{\gamma-1} \;\&\; \bar{c}_v = \frac{\bar{R}}{\gamma-1}$
- $c_p = \frac{R\gamma}{\gamma-1} \;\&\; \bar{c}_p = \frac{\bar{R}\,\gamma}{\gamma-1}$

4.4.2.3 Adiabatic Index

- $\gamma = \frac{c_p}{c_v} = \frac{\bar{c}_p}{\bar{c}_v}$
- For Monoatomic gas $\Rightarrow \gamma = \frac{5}{3} = 1.67$
- For Diatomic gas $\Rightarrow \gamma = \frac{7}{5} = 1.4$
- For Triatomic gas $\Rightarrow \gamma = \frac{4}{3} = 1.33$

Note:

For air, following values can be used for calculation purpose

- $c_p = 1.005$ kJ/kg-K
- $c_v = 0.718$ kJ/kg-K

4.5 Fraction of Isobaric Heat Supplied Converting into Displacement Work

$$\mathbf{W} = \left(1 - \frac{1}{\gamma}\right) Q|_P$$

$$\Delta U = \frac{1}{\gamma} Q|_P$$

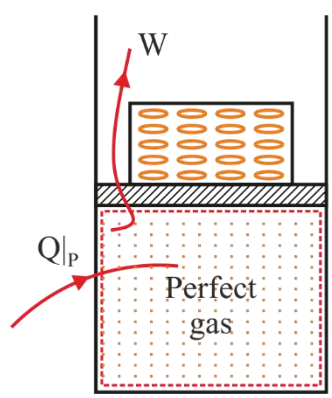


Fig. 4.4 Isobaric heat addition to a system

4.5.1 Relation between Isochoric & Isobaric heat addition for same temperature increase

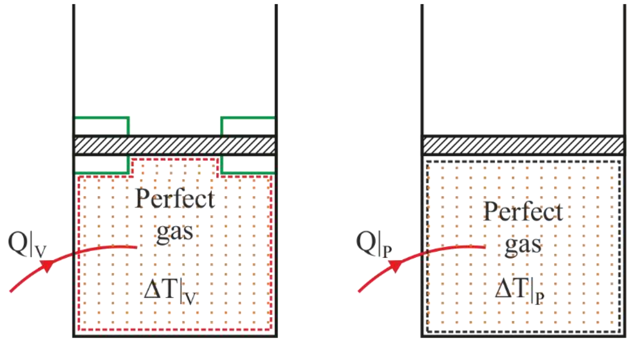


Fig.4.5 Isochoric and isobaric heat addition

- $Q|_P > Q|_v$
- $Q|_P = \gamma Q|_v$

Note:

For same increase in Internal Energy, Isobaric Heat addition will be more than Isochoric Heat addition since in Isobaric heat addition some part of the heat addition will convert into displacement work.

4.6 Isochoric & Isobaric Heat Interactions to Perfect gas in P-V Terms

- $Q|_v = \Delta U = mc_v\Delta T$

  $$Q|_v = \frac{V(P_2 - P_1)}{\gamma - 1}$$

- $Q|_p = \Delta H = mc_p\Delta T$

  $$Q|_p = \frac{\gamma P(V_2 - V)}{\gamma - 1}$$

4.7 Isothermal Process

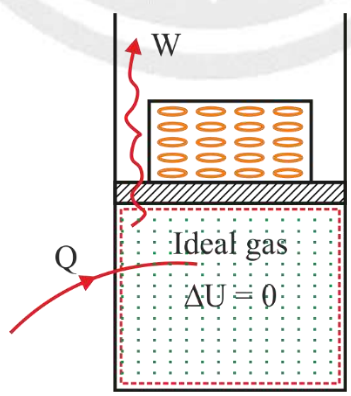


Fig. 4.6 Isothermal heat addition to the system

- $\mathbf{Q = W_d}$

  $$\mathbf{Q = C\ln\left(\frac{V_2}{V_1}\right) = C\ln\left(\frac{P_1}{P_2}\right)}$$

Where C can be $\mathbf{P_1V_1, P_2V_2, mRT, n\bar{R}T}$

- For simple compressible stationary system consisting of Ideal gas undergoing internally reversible isothermal process, heat completely converts into displacement work.

4.8 Adiabatic Process

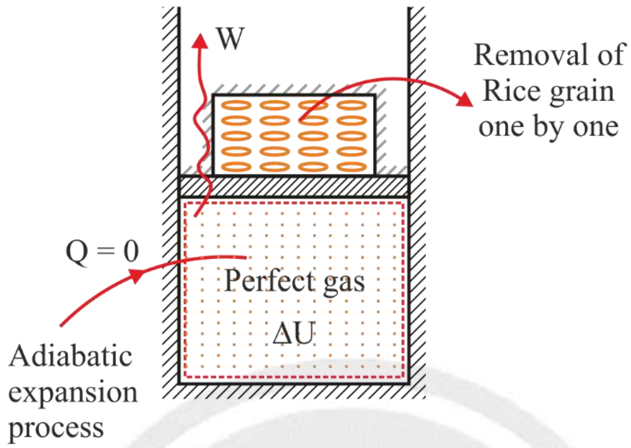


Fig. 4.7 Adiabatic expansion of perfect gas

$\mathbf{Q = 0}$

$\mathbf{W_d + \Delta U = 0}$

- For simple compressible stationary system, undergoing adiabatic expansion, displacement work is done at the expense of internal energy.

4.9 Polytropic Process

- $Q = \left(\frac{\gamma - n}{\gamma - 1}\right) W_d$ (Perfect Gas)
- $C_{po} = -\frac{(\gamma-1)}{(n-1)} c_v$

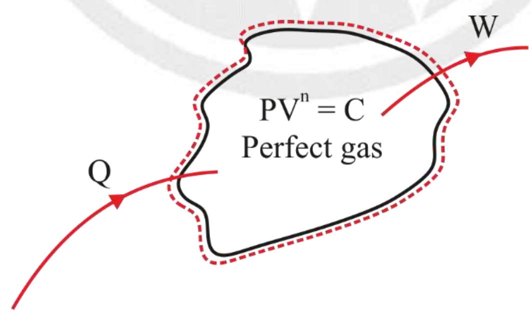


Fig. 4.8 A system undergoing polytropic process

- $W_d = \frac{P_1V_1 - P_2V_2}{n-1}$ (Any Gas)
- For polytropic heat addition to a perfect gas having polytropic index between 1 & γ
  (i) Displacement work is more than the heat supplied.
  (ii) The extra displacement work is done at the expense of internal energy.
- Polytropic heat addition to a perfect gas having polytropic index between 1 & γ leads to increase in volume but decrease in pressure & temperature.

4.10 Free Expansion

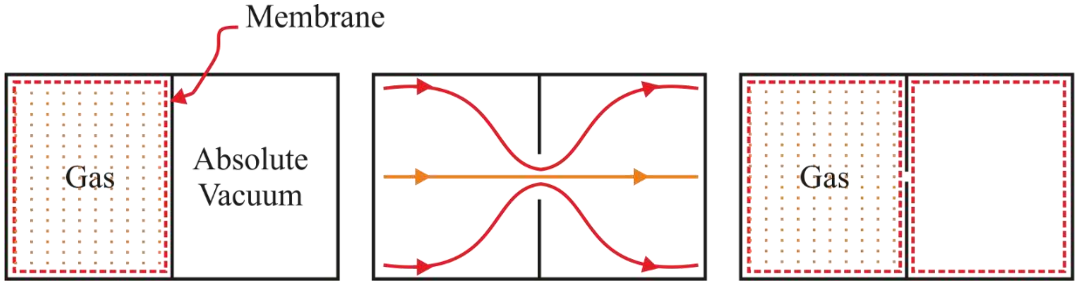
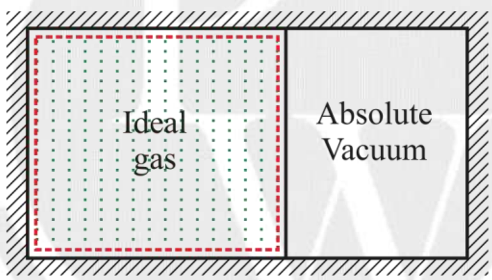


Fig. 4.9 Free expansion

- Expansion of a gas against absolute vacuum is known as free expansion.
- In free expansion, external resistance is zero, hence work interaction is also zero.
- In case of free expansion, although $\int PdV$ is non-zero, but it can't be applied for free expansion, since free expansion is non quasi-static process.

4.10.1 Free expansion of an Ideal gas inside an insulated chamber



$\Delta KE = 0$

$\Delta PE = 0$

Fig. 4.10 Ideal gas vacuum space separated by a membrane in an insulated chamber

- For an Ideal gas undergoing free expansion inside an insulated chamber,
  1. $\Delta U_{1-2} = \Delta T_{1-2} = \Delta H_{1-2} = 0$
  2. $P_1V_1 = P_2V_2$

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5 OPEN SYSTEM ANALYSIS

5.1 Introduction [Open System Overview]

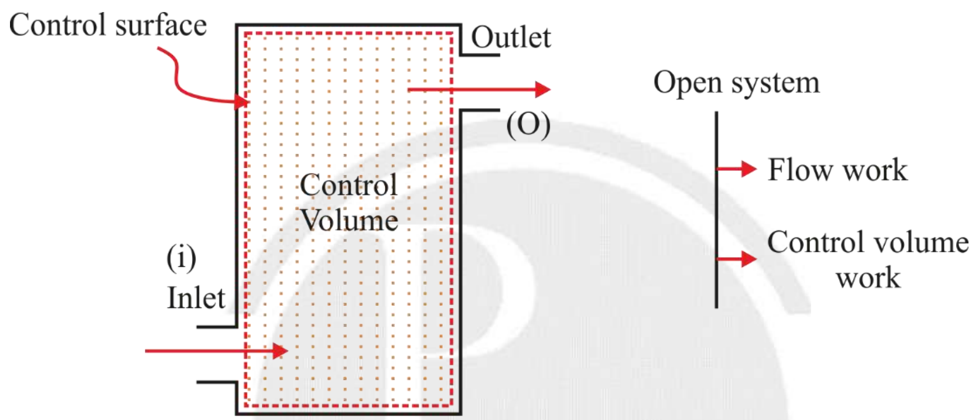


Fig.5.1 Open system

5.2 Flow Work

Flow work is the energy needed to push a fluid into (or) out of a control volume.

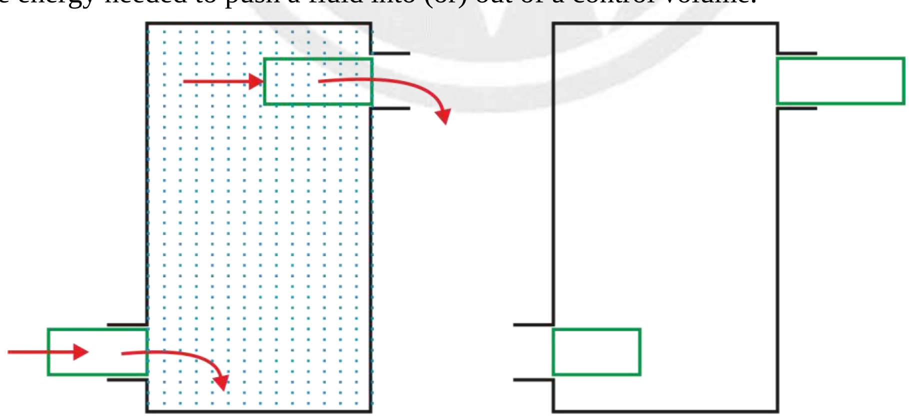


Fig.5.2 Fluid elements crossing the control surface

- The work involved in crossing the fluid element either to enter or to leave the control volume is known as flow work.

5.2.1 Expression of flow work

Specific flow work, $w_f = P_v$

Rate of flow work, $\dot{W}f = \dot{m}Pv$

Note:
Unlike closed systems, control volumes involve mass flow across their boundaries, and some work is required to push the mass into or out of the control volume. This work is known as the flow work, or flow energy, and is necessary for maintaining a continuous flow through a control volume.

5.3 Control Volume Work (W_{cv})

- Closed system ⇒ Shaft Work

  Open System ⇒ Control Volume Work

5.3.1 Control Volume Work

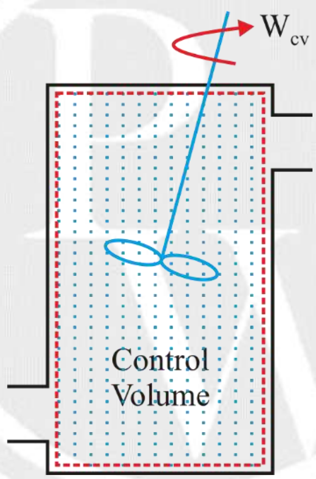


Fig. 5.3 An open system with control volume work

- Rotated from inside of CV ⇒ $W_{cv} = +ive$

  Rotated from outside of CV ⇒ $W_{cv} = -ive$

- Work interaction across the control surface for the open system is known as control volume work.

  For turbine ⇒ $W_{cv} = +ive$

  For compressor ⇒ $W_{cv} = -ive$

  For pump ⇒ $W_{cv} = -ive$

Note:

Expression for control volume work

- $w_{cv} = -\int vdP$ $\begin{bmatrix}\text{Int. Rev. Steady Flow}\\ \Delta ke \cong 0, \Delta pe \cong 0\end{bmatrix}$
- $W_{cv} = -\dot{m}\int vdP$

5.3.1.1 Control Volume Work on P-v diagram

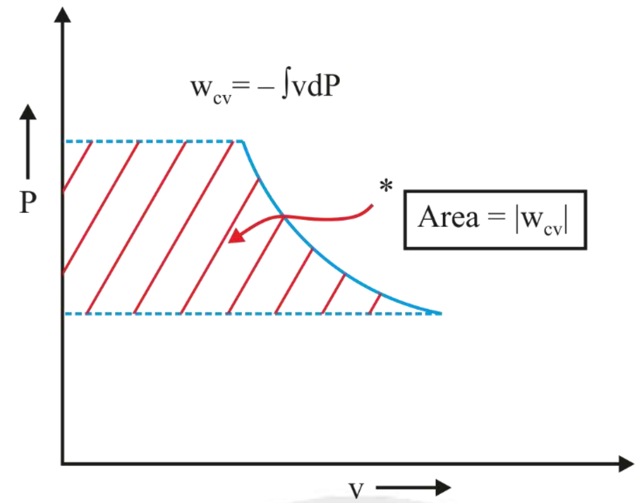


Fig. 5.4 Control volume work on P-v Diagram

5.3.1.2 Analysis for Pump work

- Pump ⇒ Vapour Power Plant
- $w_i = v(P_b - P_c)$

5.4 Conservation of Mass and Energy

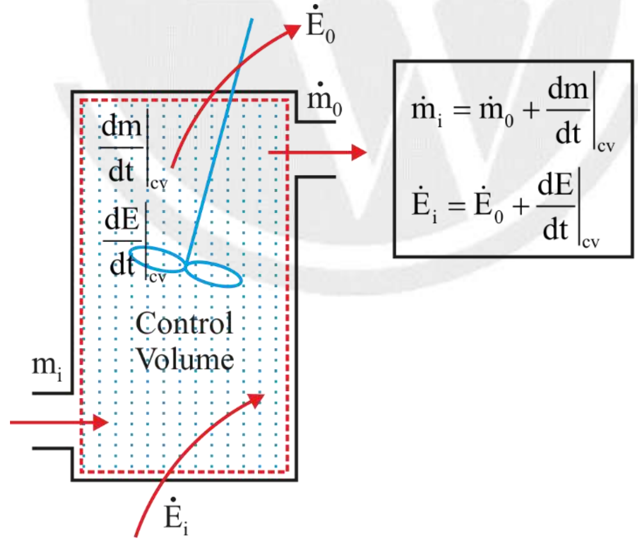


Fig. 5.5 Mass and energy conservation in an open system

5.4.1 Energy equation for an open system

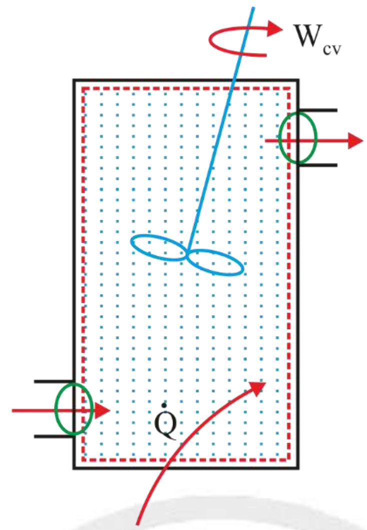


Fig.5.6 Heat and work interaction in open system

- $\dot{Q} = \dot{W}_{cv} + \Delta\dot{KE} + \Delta\dot{PE} + \Delta\dot{H} + \left.\frac{dE}{dt}\right|_{CV}$

Where

$\Delta\dot{KE} = \frac{1}{2}\dot{m}_o c_o^2 - \frac{1}{2}\dot{m}_i c_i^2$

$\Delta\dot{PE} = \dot{m}_O g z_O - \dot{m}_i g z_i$

$\Delta\dot{H} = \dot{m}_o h_o - \dot{m}_i h_i$

- Energy equation for an open system is also known as first law of thermodynamics for an open system.

5.4.1.1 Steady Flow

For steady flow

- A flow is said to be steady flow if thermodynamic properties don't change with time at a particular location.
- Thermodynamic properties may vary along the space coordinate but do not vary with time.
- $\left.\frac{dm}{dt}\right|_{cv} = 0, \dot{m}_i = \dot{m}_o = \dot{m}$
- $\left.\frac{dE}{dt}\right|_{cv} = 0$

5.4.1.2 Steady flow energy equation for an open system

- $\dot{Q} = \dot{W}_{cv} + \Delta\dot{KE} + \Delta\dot{PE} + \Delta\dot{H}$ (in W/kW)

Where

$\Delta\dot{KE} = \frac{1}{2}\dot{m}\left(c_o^2 - c_i^2\right)$

$\Delta\dot{PE} = \dot{m}g\,(z_o - z_i)$

$\Delta\dot{H} = \dot{m}(h_o - h_i)$

$q = w_{cv} + \Delta ke + \Delta pe + \Delta h$ (in J/kg / kJ/kg)

Where

$$\Delta ke = \frac{1}{2}(c_o^2 - c_i^2)$$

$$\Delta pe = g(z_o - z_i)$$

$$\Delta h = (h_o - h_i)$$

5.5 Nozzle

- Nozzle is the steady flow device which increases the kinetic energy. Nozzles are used in jet propulsion system and inlet of impulse turbine.

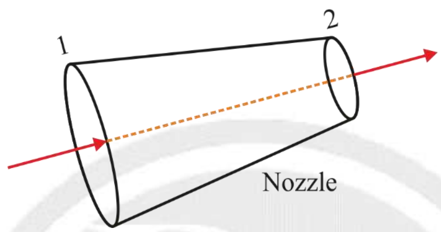


Fig.5.7 Nozzle for subsonic flow

- $c_o > c_i$
- $\dot{Q} = \Delta\dot{KE} + \Delta\dot{PE} + \Delta\dot{H}$

  $q = \Delta ke + \Delta pe + \Delta h$

5.5.1. Well Insulated Horizontal Nozzle

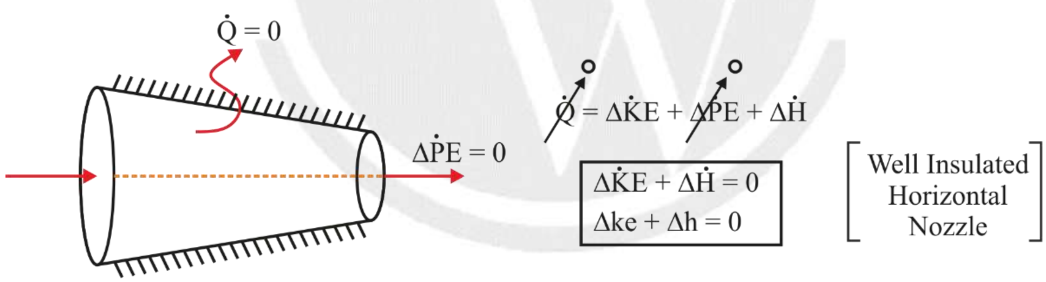


Fig. 5.8 Horizontal nozzle with insulation

5.5.1.2 Perfect gas is the working fluid

- $\Delta H = mc_p\Delta T$

  $\Delta h = c_p\Delta T$

  For perfect gas flowing through well insulated horizontal nozzle

  $\Delta ke + \Delta h = 0$

  $$\frac{1}{2}(c_o^2 - c_i^2) + c_p(T_o - T_i) = 0$$

Note:

- For perfect gas, flowing through perfectly insulated horizontal nozzle, temperature at the outlet is always less than the temperature at the inlet.
- value should be taken in J/kg-K (to write the equation in J/kg)
- $TP^{\frac{1-\gamma}{\gamma}} = C$ can be applied only when perfect gas is having internally reversible adiabatic flow.
  (Internal reversible flow means Ideal flow/Inviscid flow/Frictionless flow)

5.6 Diffuser

- Diffuser is a steady flow device which increases the pressure. It is used in centrifugal compressor.

Note:
Equations for the Diffuser are exactly same as Nozzle.

5.7 Turbine

- Turbine is the steady flow device which is used to produce power. It is used in gas turbine power plant, vapour power plant & hydroelectric power plant.
- For turbine, $\dot{W}_{cv} = +\text{ive}$

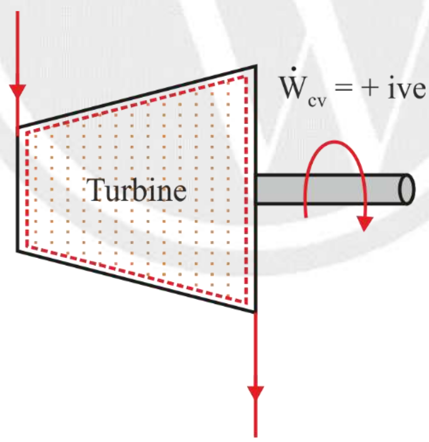
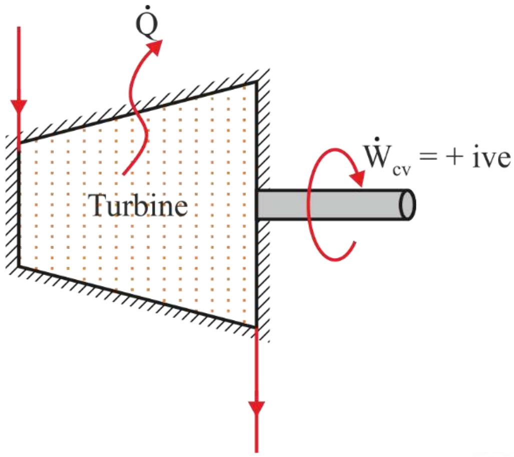


Fig.5.9 Turbine

- $\dot{Q} = \dot{W}_{CV} + \Delta\dot{KE} + \Delta\dot{PE} + \Delta\dot{H}$

5.7.1 Well Insulated turbine having negligible changes in KE & PE



$$\cancel{\dot{Q}} = \dot{W}_{cv} + \cancel{\Delta \dot{K}E} + \cancel{\Delta \dot{P}E} + \Delta \dot{H}$$

$$\boxed{\begin{array}{l} \dot{W}_{cv} + \Delta \dot{H} = 0 \\ \dot{W}_{cv} + \Delta h = 0 \end{array}}$$

Fig. 5.10 Insulated turbine producing work

- For perfectly insulated turbine, having negligible changes in kinetic and potential energies, control volume work is done at the expense of enthalpy.

5.7.1.1 Perfect gas is the working fluid

- $\Delta H = mc_p \Delta T$
  $\Delta h = c_p \Delta T$
- For perfect gas flowing through well insulated turbine having negligible changes in KE & PE.
  $w_{cv} + \Delta h = 0$
  $w_{cv} + c_p \Delta T = 0$

5.8 Compressor

- Compressor is the steady flow device, which is used to increase the pressure of the fluid. It is used in Gas power plant & Vapour compression refrigeration system.

Note:
Equations for Compressor are exactly same as Turbine.

5.9 Throttling Device

- Throttling device is a steady flow restricting device which causes pressure drop of the fluid.
- Throttling device produces a pressure drop without involving any control volume work.

5.9.1 Examples:

1. Orifice
2. Partially opened adjustable valve
3. Porous plug
4. Capillary tube

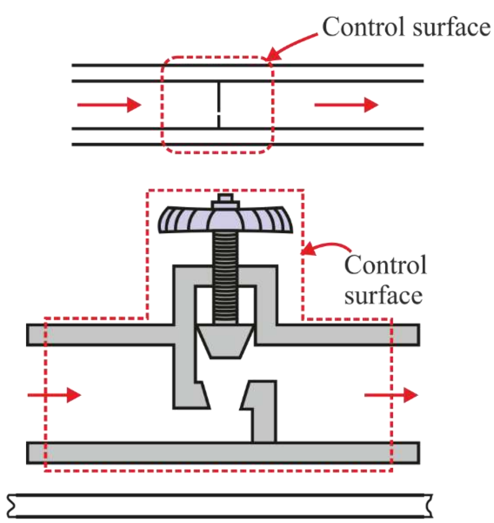


Fig. 5.11 Orifice and partially opened adjustable valve

Note:
- In VCRS throttling valve (throttling device) is used to reduce the pressure of refrigerant from condenser pressure to evaporator pressure while in Domestic refrigerator, same function is performed by capillary tubes.
- Throttling device is used in throttling calorimeter which is used to measure the dryness fraction (quality of two-phase liquid–vapour mixture).

5.9.2 Ideal gas undergoing throttling in small throttling device with negligible changes in KE and PE

- For an ideal gas undergoing throttling in a small throttling device with negligible changes in KE and PE following equations can be used,

$$\Delta h = \Delta T = \Delta u = 0$$

$$P_i v_i = P_o v_o$$

5.10 Mixing Chamber

- Mixing chamber is the steady device where mixing process takes place.

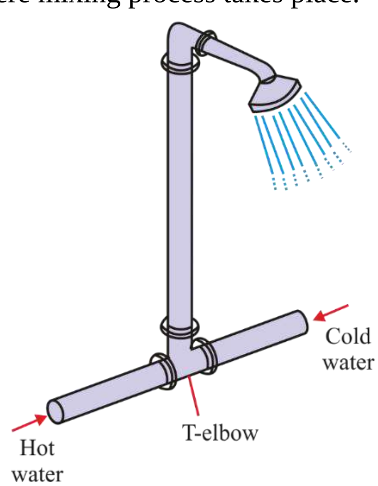


Fig. 5.12 A mixing chamber for hot and cold-water streams

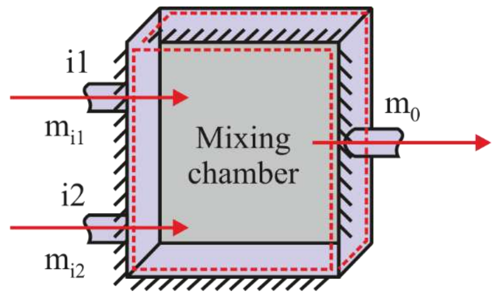


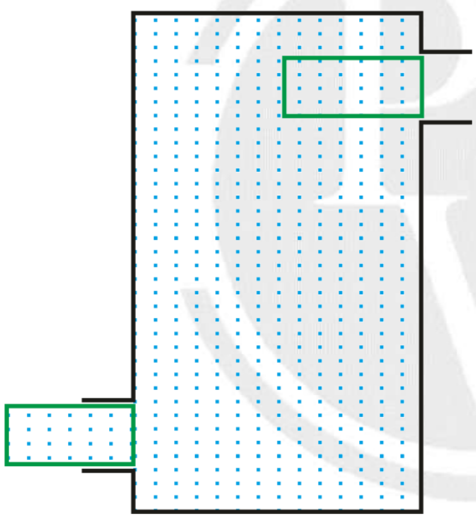
Fig. 5.13 An insulated mixing chamber

- $\dot{m}_{i1} + \dot{m}_{i2} = \dot{m}_0$
- $\dot{Q} = \Delta\dot{KE} + \Delta\dot{PE} + \Delta\dot{H}$
- Well insulated mixing chamber having negligible changes in KE and PE

  $\Delta\dot{H} = 0$

  $$h_0 = \frac{\dot{m}_{i1} h_{i1} + \dot{m}_{i2} h_{i2}}{\dot{m}_{i1} + \dot{m}_{i2}}$$

5.11 Unsteady Flow

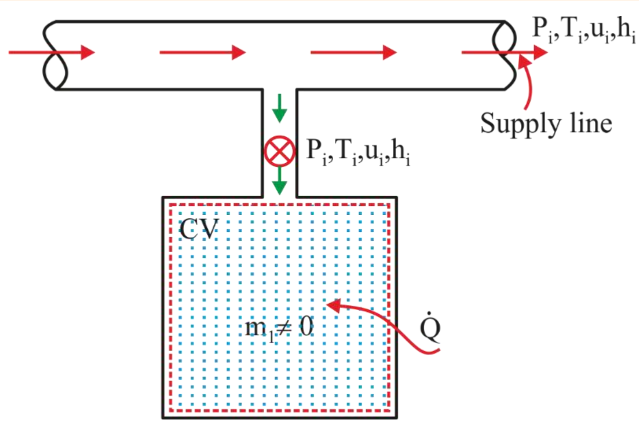


$$\left.\frac{dm}{dt}\right|_{cv} \neq 0 \qquad \int_1^2 dm = m_2 - m_1$$

$$\left.\frac{dE}{dt}\right|_{cv} \neq 0 \qquad \int_1^2 dU = U_2 - U_1$$

Fig. 5.14 Unsteady flow through a control volume

5.11.1 Charging of non-insulated tank having initial mass non-zero



- $U_2 - U_1 = (m_2 - m_1)h_i + Q$ (Any Fluid)
- $u_2 = \frac{m_1}{m_2}u_1 + \left(1 - \frac{m_1}{m_2}\right)h_i + \frac{Q}{m_2}$ (Any Fluid)
- $T_2 = \frac{\frac{m_1}{m_2}T_1 + \left(1 - \frac{m_1}{m_2}\right)\gamma T_i}{1 - \frac{(\gamma-1)Q}{P_2V_2}}$ (Perfect gas)

5.11.2 Charging of an insulated evacuated tank

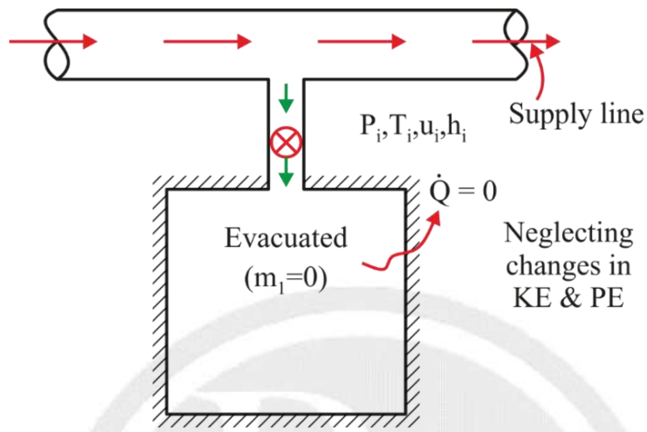


Fig.5.15 An evacuated tank connected to a supply pipeline with $m_1 = 0$

- $U_2 = m_2h_i$ (Any Fluid)
- $u_2 = h_i$ (Any Fluid)
- $T_2 = \gamma T_i$ (Perfect gas)

Conclusions

- Final specific internal energy of the fluid in the tank is equal to the specific enthalpy of the fluid flowing in supply pipeline.
- Final temperature of perfect gas in the tank is γ times of the temperature of the fluid flowing in the supply pipeline.
  (a) Well Insulated
  (b) Evacuated tank

5.11.3 Charging of a non-insulated evacuated tank

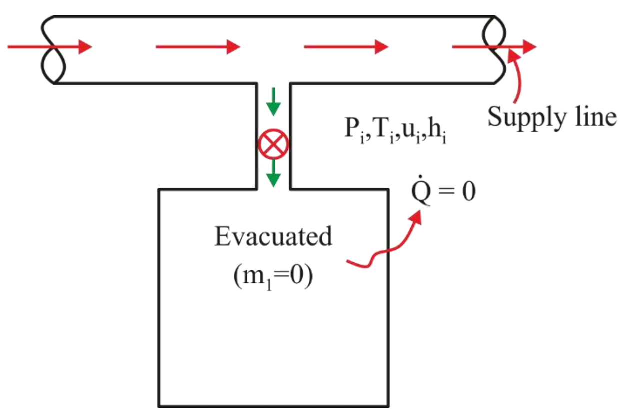


Fig.5.16 Charging of a non-insulated evacuated tank

- $U_2 = m_2h_i + Q$ (Any Fluid)
- $u_2 = h_i + \frac{Q}{m_2}$ (Any Fluid)

- $T_2 = \frac{\gamma T_i}{1-\frac{(\gamma-1)Q}{P_2 V_2}}$ (Perfect gas)

5.11.4 Charging of an insulated tank having initial mass non-zero

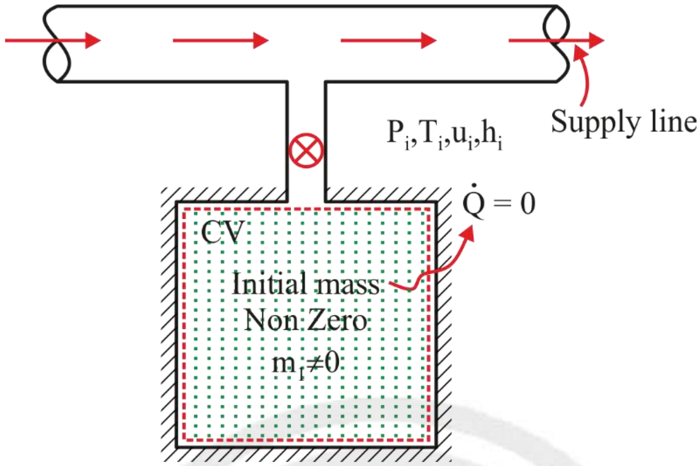
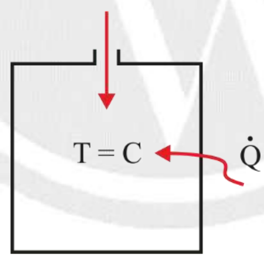


Fig. 5.16 An evacuated tank connected to a supply pipeline with $m_1 = 0$

- $U_2 - U_1 = (m_2 - m_1)h_i$ (Any Fluid)
- $u_2 = \frac{m_1}{m_2}u_1 + \left(1 - \frac{m_1}{m_2}\right)h_i$ (Any Fluid)
- $T_2 = \frac{\gamma T_i}{1+\frac{P_1}{P_2}\left(\frac{\gamma T_i}{T_1} - 1\right)}$ (Perfect gas)

5.11.5 Charging of a non-insulated evacuated tank from tiny hole maintaining constant temperature of fluid in tank



- Total heat transfer during charging
  $Q = -P_2 V_2$

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6 SECOND LAW OF THERMODYNAMICS

6.1 Drawback of First Law of Thermodynamics

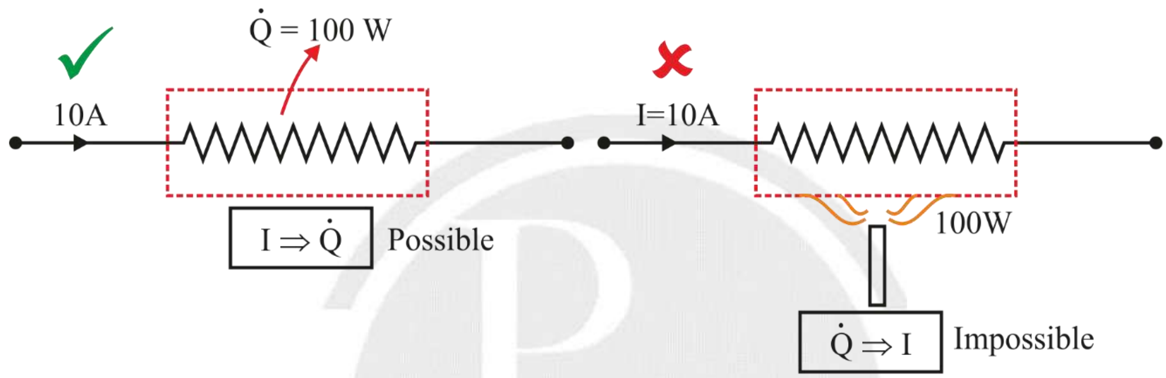


Fig. 6.1 Conversion of Electric energy into heat and vice - versa

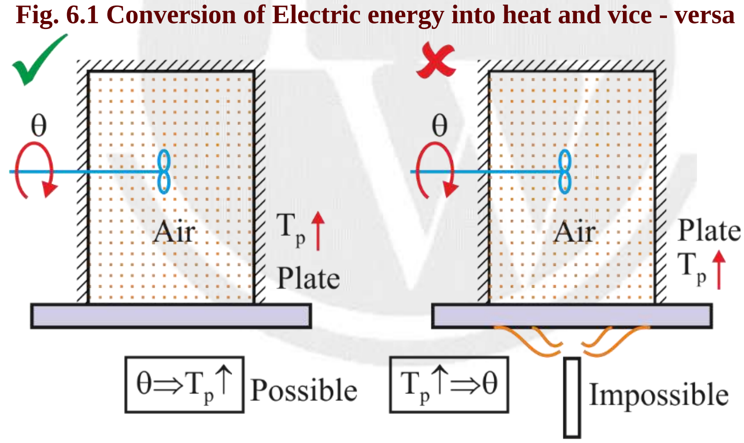


Fig. 6.2 Conversion of paddle work into heat and vice versa

- First law of thermodynamics gives the information about the conservation of energy during a process, but it doesn't give any information about the feasibility & nature of the process.

> Note:
> - Second Law of thermodynamics tells the direction of a process, with the help of concept of entropy & is known as directional law.
> - The process takes place in the direction in which $\Delta S_{universe} \geq 0$.

- $\Delta S_{uni} < 0 \rightarrow$ Impossible
- $\Delta S_{uni} = 0 \rightarrow$ Possible & Reversible

- $\Delta S_{uni} > 0 \rightarrow$ Possible & Irreversible

Second Law of Thermodynamics

↓

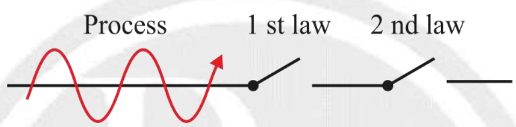
Entropy

↓

ΔS_{uni}

6.2 Remember

- A process cannot occur unless it satisfies both the first and the second laws of thermodynamics.



- The use of the second law of thermodynamics is not limited to identifying the direction of processes. The second law also asserts that energy has quality as well as quantity.

6.3 Thermal Energy Reservoirs

- A hypothetical body with a relatively large thermal energy capacity (mass × specific heat) that can supply or absorb finite amounts of heat without undergoing any change in temperature, such a body is called a thermal energy reservoir.

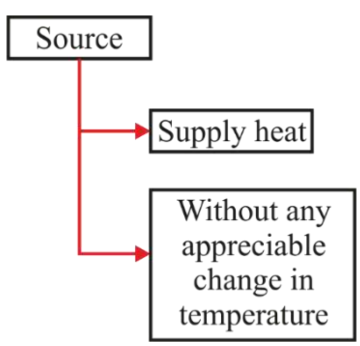
(i) Source

(ii) Sink

6.3.1 Source

- Source is the relatively large heat capacity body, which can supply finite amount of heat without undergoing any appreciable change in temperature.

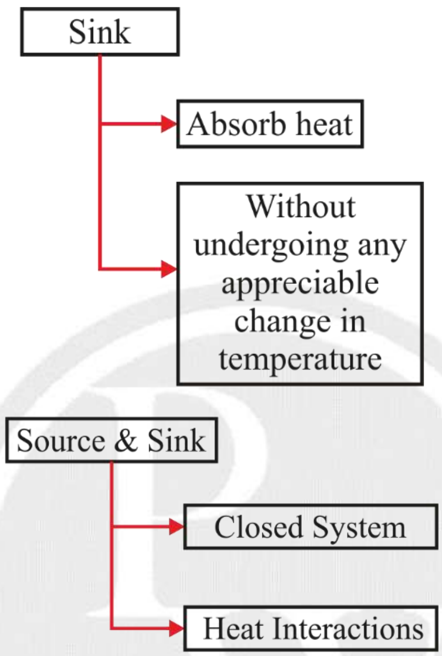
  e.g. Sun, industrial furnace, phase change from vapour to liquid.



6.3.2 Sink

- Sink is the relatively large heat capacity body, which can absorb finite amount of heat without undergoing any appreciable change in temperature.

  e.g. Atmospheric air, lake, river, sea etc.



Note:

- For analysis purpose source and sink can be assumed as closed system having only heat interactions.
- The changes that take place in such large bodies are so slow and so minute that the process within it can be assumed as internally reversible.

6.4 Statements of Second Law of Thermodynamics

(1) Kelvin-Planck Statement

(2) Clausius Statement

6.4.1 Kelvin-Planck statement

- It is impossible for a system to operate in thermodynamic cycle and deliver net work to its surroundings while receiving heat from a single thermal energy reservoir.

6.4.2 Consequences of Kelvin – Planck statement

(1) Perpetual Motion Machine of Second Kind (PMM-2) is impossible

(2) Concept of Heat Engine

6.4.3 Perpetual Motion Machine of Second Kind (PMM-2) is impossible

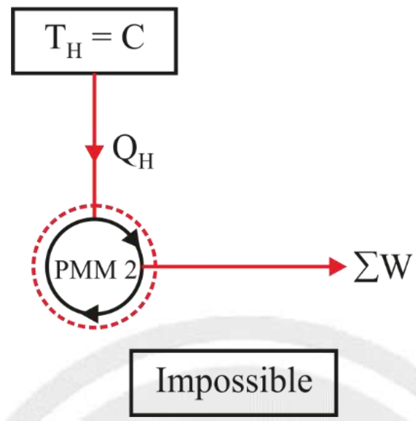


Fig. 6.3 PMM – 2

PMM-2 having 100% thermal efficiency is impossible.

6.5 CONCEPT OF HEAT ENGINE

- Heat engine is the power producing cyclic device which converts a part of heat into work and rejects remaining heat to the sink.

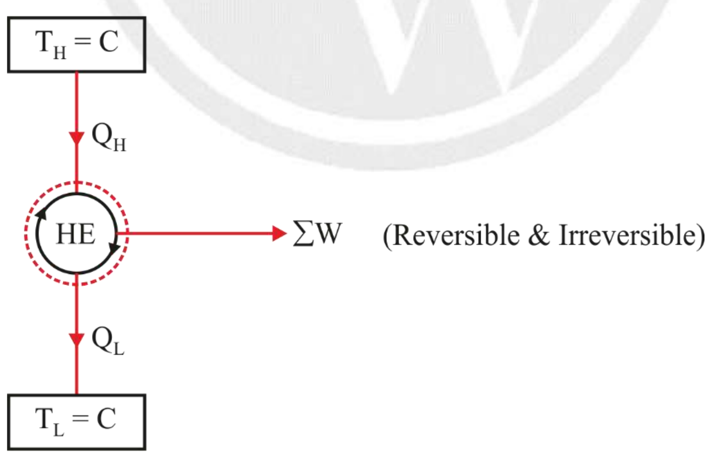


Fig. 6.4 A cyclic heat engine

- The fraction of the heat input that is converted to network output is a measure of the performance of a heat engine and is called the thermal efficiency.

  Thermal efficiency = $\frac{\text{Net work output}}{\text{Total heat supplied}}$

- $\eta_{th} = \frac{Q_H - Q_L}{Q_H}$

Note:
- In a cyclic process, Heat can't be completely converted into work hence "Heat" is known as "Low Grade Energy".

6.6 Heat Engine in Series

Sink of HE-1 = Source of HE-2

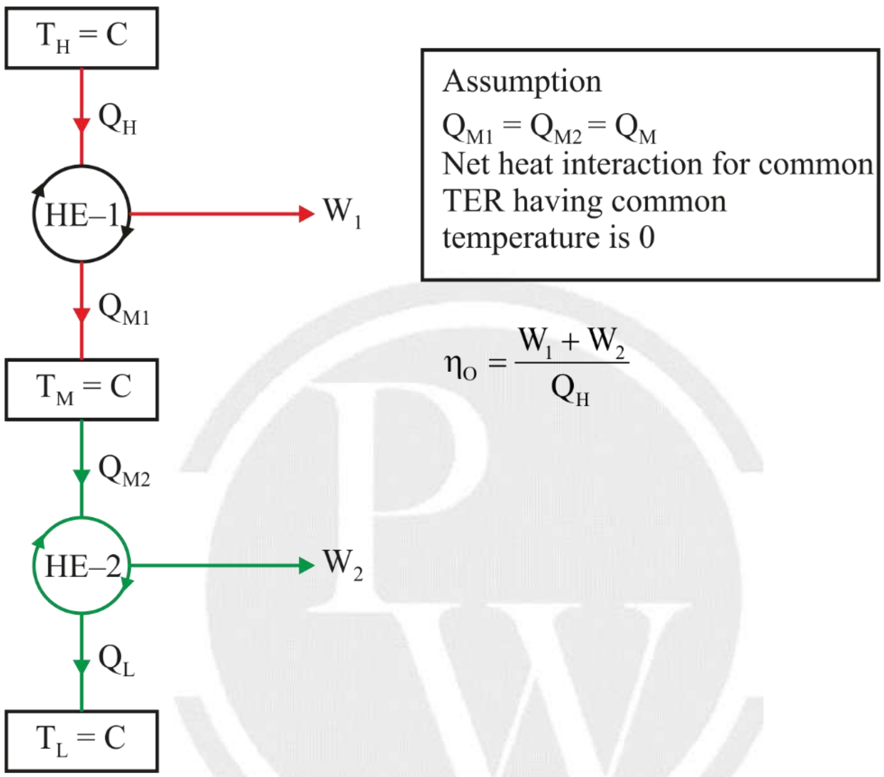


Fig. 6.5 Heat engines in series

- Overall efficiency

$$\eta_o = \frac{W_1 + W_2}{Q_H}$$

$$\eta_o = \eta_1 + \eta_2 - \eta_1\eta_2$$

Above formula valid for any Heat Engine Reversible or Irreversible

6.7 Clausius Statement

- It is impossible for a system to operate in such a way that the sole result could be heat transfer from a cooler body to a hotter body.

6.7.1 Consequences of Clausius Statement

1. Direction of Heat Transfer
2. Concept of Refrigerator
3. Concept of Heat pump

6.7.1.1 Direction of Heat Transfer

Naturally/Automatically

$\downarrow$

Hotter body (High temperature body) To Cooler body (Low temperature body)

1. Naturally heat transfer take place from hotter body to cooler body.
2. Direction of heat transfer is given by the Second Law of Thermodynamics (Clausius statement)

6.7.1.2 Concept of Refrigerator

- The transfer of heat from a low-temperature medium to a high-temperature requires power consuming cyclic devices called refrigerators. Refrigerators, like heat engines, are cyclic devices.

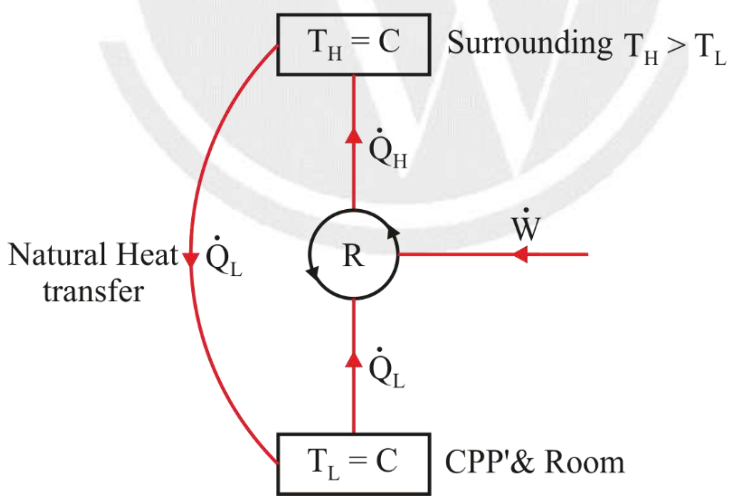


Fig. 6.6 A cyclic Refrigerator

- Here $\dot{Q}_L$ is the magnitude of the heat extracted from the refrigerated space at temperature T_L, $\dot{Q}_H$ is the magnitude of the heat rejected to the warm environment.
- Power consuming cyclic device, aim $\Rightarrow$ to maintain the lower temperature of lower temperature body
- Desired effect $\Rightarrow \dot{Q}_L$

6.7.1.3 Concept of Heat pump

- The device which transfers heat from a low-temperature medium to a high-temperature one is the heat pump.

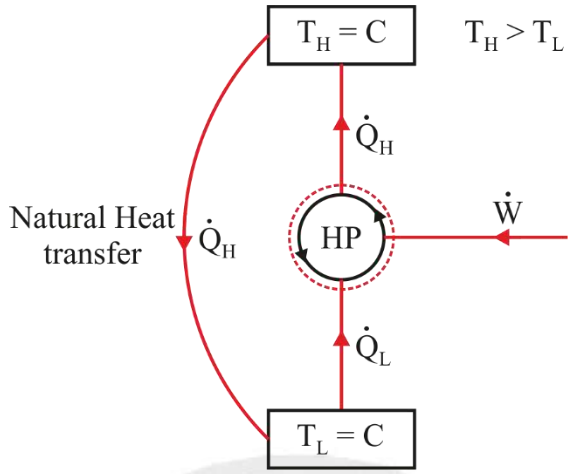


Fig. 6.7 A cyclic heat pump

- Power consuming cyclic device, aim ⇒ to maintain the high temperature of higher temperature body
- Desired effect ⇒ $\dot{Q}_H$

6.8 Coefficient of Performance (COP)

Kelvin-Plank Statement → Power Producing cyclic device → Heat Engine → Thermal Efficiency (η_{th})

Clausius Statement → Power consuming cyclic device → Refrigerator / heat pump → Coefficient of performance

- For power producing cyclic device (Heat Engine), the performance parameter was thermal efficiency (η). For power consuming cyclic devices, (Refrigerator and Heat Pump), the performance parameter was named as Coefficient of Performance (COP).
- $COP_R = \frac{\dot{Q}_L}{\dot{W}} = \frac{\dot{Q}_L}{\dot{Q}_H - \dot{Q}_L} = \frac{Q_L}{Q_H - Q_L}$
- $COP_{HP} = \frac{\dot{Q}_H}{\dot{W}} = \frac{\dot{Q}_H}{\dot{Q}_H - \dot{Q}_L} = \frac{Q_H}{Q_H - Q_L}$

6.8.1 Relation between COP_R & COP_{HP}

Assumption:- Both refrigerator and heat pump are having same amount of heat and work interactions.

- $COP_{HP} = COP_R + 1$
- This relation implies that the coefficient of performance of a heat pump is always greater than unity since COP_R is a positive quantity.

6.9 Relation between Thermal Efficiency (η) & Coefficient of Performance (COP)

Assumption:- All devices Heat Engine, Refrigerator and Heat Pump are having same amount of heat & work interactions.

- $COP_{HP} = \frac{1}{\eta_{HE}}$ and $COP_R = \frac{1}{\eta_{HE}} - 1$

6.10 Carnot Cycle

6.10.1 Various processes of Carnot Cycle

1-2: Reversible Adiabatic compression

2-3: Reversible Isothermal heat addition

3-4: Reversible Adiabatic expansion

4-1: Reversible Isothermal heat rejection

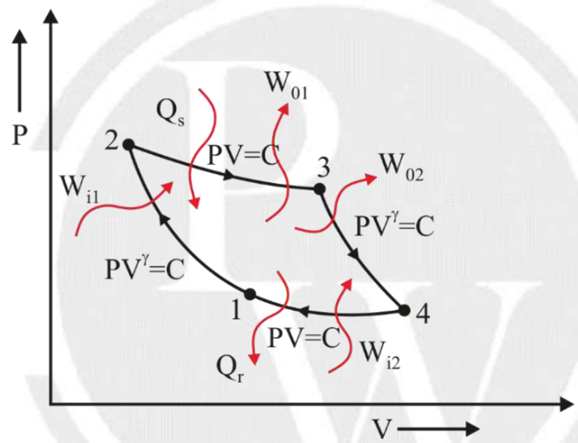


Fig. 6.8 P – V diagram of Carnot cycle

- For Carnot Cycle, all the processes are reversible (both internally & externally), hence Carnot cycle is a reversible cycle.
- For Carnot Cycle

  Heat interactions ⇒ 2 Processes

  Work interactions ⇒ 4 Processes

Analysis for Carnot cycle

- $V_1V_3 = V_2V_4$
- $r = \varepsilon$

  Here r = compression ratio, $r = \frac{V_1}{V_2}$ and ε = expansion ratio, $\varepsilon = \frac{V_4}{V_3}$

- $\eta_c = \frac{T_H - T_L}{T_H}$

  $\because \frac{T_L}{T_H} = \frac{1}{r^{\gamma-1}} = \frac{1}{\varepsilon^{\gamma-1}}$

  $\therefore \eta_c = 1 - \frac{1}{r^{\gamma-1}} = 1 - \frac{1}{\varepsilon^{\gamma-1}}$

Most effective way to increase the thermal efficiency of Carnot cycle

$\eta_c = 1 - \frac{T_L}{T_H}$

There are two ways of increasing the thermal efficiency.

(a) Decreasing T_L keeping T_H constant.

(b) Increasing T_H keeping T_L constant.

Out of the two option (a) is more effective.

6.11 Carnot Theorem

For different heat engines operating between same temperature limits, thermal efficiency of reversible heat engine is more than the efficiency of irreversible heat engine.

- For same Temperature limits

$\eta_{RHE} > \eta_{IRHE}$

6.12 Carnot Corollary

- For different reversible heat engines operating between same temperature limits, thermal efficiency of all reversible heat engines is equal.
- For same temperature limits

$\eta_{RHE1} = \eta_{RHE2}$

- Carnot, Stirling & Ericsson cycles are reversible cycles, hence heat engines operating on these cycles will have same thermal efficiency between same temperature limits.
- For reversible heat engine thermal efficiency is the function of temperature only. It is independent of construction working fluid and processes.

$\eta_{RHE} = \frac{T_H - T_L}{T_H}$

$\eta_{RHE} = f\,(T_H\,,\,T_L)$

6.13 Thermodynamic Temperature Scales

Temperature scales that are independent of the properties of the substances and are used to measure the temperature are known as thermodynamic temperature scales.

6.13.1 Thermodynamic Kelvin Scale

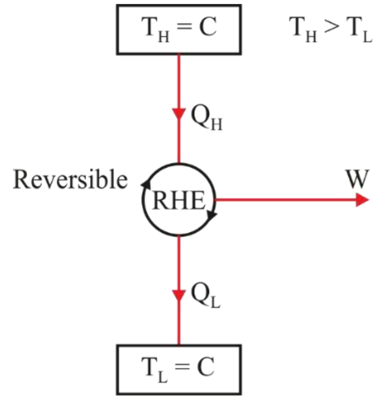


Fig. 6.9 A cyclic reversible heat engine

$$\frac{Q_H}{Q_L} = \frac{T_H}{T_L}$$

Note:
- Thermodynamic Kelvin scale is valid only for reversible cycles having single heat addition and single heat rejection.
- For Thermodynamic Kelvin scale, Temperature must be absolute temperature (in Kelvin) only.

6.14 Thermal Efficiency of Reversible Heat Engine

- $\eta_{HE} = \frac{Q_H - Q_L}{Q_H}$ (Reversible & Irreversible)
- $\eta_{RHE} = \frac{T_H - T_L}{T_H}$ (Reversible)

6.14.1 Reversible heat engines in series

- Sink of RHE − 1 = Source of RHE − 2
- Assumption:-
    - Net heat interaction for common TER having temperature T_M is 0.
    - $Q_{M1} = Q_{M2} = Q_M$.

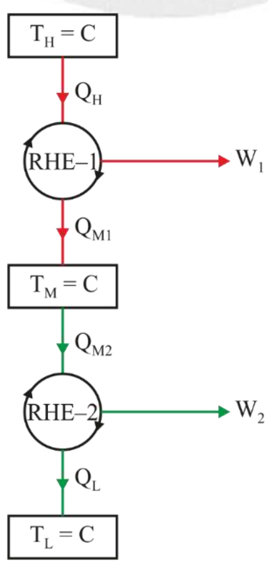


Fig. 6.10 Reversible heat engines in series

- $\frac{Q_H}{T_H} = \frac{Q_M}{T_M} = \frac{Q_L}{T_L}$

6.14.2 Different Cases

Case 1: Same thermal efficiency

- $T_M = \sqrt{T_H T_L}$

Case 2: Same network output

- $T_M = \frac{T_1 + T_2}{2}$

6.15 COP of Reversible Refrigerator

- $COP_R = \frac{Q_L}{Q_H - Q_L}$ [Reversible & Irreversible]
- $COP_{RR} = \frac{T_L}{T_H - T_L}$ [Reversible]

6.16 COP of Reversible Heat Pump

- $COP_{HP} = \frac{Q_H}{Q_H - Q_L}$
- $COP_{RHP} = \frac{T_H}{T_H - T_L}$

6.17 Clausius Inequality

- Whether the cycle is reversible/irreversible can be known with the help of Clausius inequality.
- According to Clausius inequality

$\oint \frac{\delta Q}{T} < 0 \rightarrow$ Possible, Internally Irreversible cycle

$\oint \frac{\delta Q}{T} = 0 \rightarrow$ Possible, Internally Reversible cycle

$\oint \frac{\delta Q}{T} > 0 \rightarrow$ Impossible cycle

6.17.1 Remember

- Clausius Inequality is valid only for cycles.
- Whenever in a given question, more than two thermal reservoirs are involved for a cyclic device we have to apply Clausius inequality.
- $\oint \frac{\delta Q}{T} \leq 0 \rightarrow$ Valid for all thermodynamic cycles, reversible/irreversible, including the power consuming cycles.

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7 ENTROPY

7.1 Mathematical feel of Entropy – Part A

- For a closed system, undergoing an internally reversible process, entropy change is given by the expression

$$\Delta S = \int \frac{\delta Q}{T}\bigg|_{\text{Int.Rev}}$$

- Entropy is a property. It is a state (point) function and an exact differential.
- In differential form entropy change is represented by $dS = \frac{\delta Q}{T}\Big|_{\text{Int.Rev}}$.
- For a closed system, undergoing an internally reversible process, entropy change takes place due to entropy transfer (associated with heat transfer) only.
- Since entropy is a state (point) function hence entropy change between two given states is always same whether the path is internally reversible or internally irreversible.

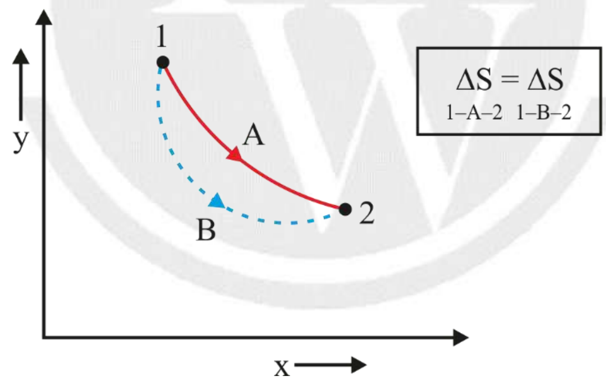


Fig. 7.1 Internally Reversible and Internally Irreversible processes with same initial & final states

7.2 Mathematical feel of Entropy – Part B

- Entropy change for a closed system undergoing internally irreversible process, is greater than entropy transfer for the process.
- $\Delta S > \int \frac{\delta Q}{T}\Big|_{\text{Int.Irr}}$
- For a closed system undergoing internally irreversible process

$$\Delta S = \int \frac{\delta Q}{T} + S_g$$

Here ΔS = Entropy change

$\int \frac{\delta Q}{T}$ = Entropy transfer (due to heat transfer)

S_g = Entropy generation (due to irreversibilities)

- Entropy generation within the system takes place only for internally irreversible process & it is always positive.
- For a closed system undergoing internally irreversible processes, entropy change takes place due to entropy transfer (associated with heat transfer) and entropy generation (associated with Irreversibility).
- For internally reversible processes, entropy generation within the system is "zero".
- Entropy generation depends upon the irreversibilities. Higher the irreversibilities, higher will be the entropy generation.
- Different paths have different irreversibilities, hence different magnitudes of entropy generation.
- Entropy generation is a path function & inexact differential represented by δS_g .

7.3 Entropy Change for a Closed System Undergoing Various Internally Reversible Heat Interactions

$$dS = \frac{\delta Q}{T}$$

- Heat Addition $\Rightarrow$ Entropy increases
- Heat Rejection $\Rightarrow$ Entropy decreases
- Adiabatic Process $\Rightarrow$ Isentropic process

7.4 Entropy Change for a Closed System Undergoing Various Internally Irreversible Heat Interactions

- $dS = \frac{\delta Q}{T} + \delta S_{gen}$
- Heat addition $\Rightarrow$ Entropy Increases
- Adiabatic Process $\Rightarrow$ Entropy Increases
- Heat Rejection $\Rightarrow$ Entropy may increase, decrease or remain same

Note:

- For a closed system undergoing heat addition process, entropy of the system always increases.
- For an insulated closed system, entropy of the system can never decrease. It is constant for internally reversible process and increases for internally irreversible process.
- For a closed system, heat rejection is the only way to decrease the entropy of the system.
- For a closed system, every internally reversible adiabatic process is isentropic process, but every isentropic process is not internally reversible adiabatic process.
- For a closed system an internally irreversible heat rejection process will be isentropic, if entropy decrease due to entropy transfer is exactly equal to the entropy increase due to entropy generation.
- Entropy of a closed system can increase, decrease & remains constant depending upon the type of process (Internally reversible/Internally Irreversible) and the type of heat interaction (heat addition/heat rejection/adiabatic process), but entropy of universe can never decrease.

7.5 Third Law of Thermodynamics

- It is based on low temperature chemical reaction observations.
- The entropy of a pure crystalline substance at nearly absolute zero temperature is “zero”.
- The entropy of a substance that is not pure crystalline (solid solution) is not zero at nearly absolute zero temperature since more than one molecular configuration exists for such substances, which introduces some uncertainty about the microscopic state of the substance.

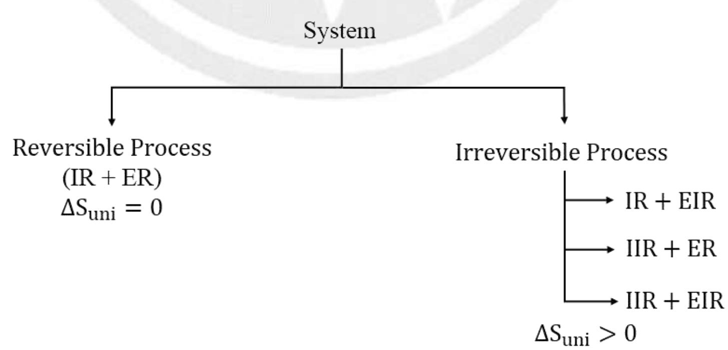
7.6 Discussion on Work Interaction

- Work interaction is organised form of energy. Work is free from disorderness or randomness and thus free of entropy.
- An energy interaction that is accompanied by entropy transfer is heat interaction and energy interaction that is not accompanied by entropy transfer is work interaction.

Note:

- Entropy is an extensive property.
- Entropy generation in the universe is equal to the entropy change of the universe which is further equal to the summation of entropy change of system and entropy change of surrounding.
- $S_{g,uni} = \Delta S_{uni}$
- Entropy change of Universe $\Delta S_{uni} = \Delta S_{sys} + \Delta S_{sur}$

7.7 Entropy change of Universe



Where

- IR:- Internally Reversible process, ER:- Externally Reversible process
- IIR = Internally Irreversible process, EIR = Externally Irreversible process
- Entropy of universe during an actual process (Irreversible process) always increases. In limiting case, for ideal process (Reversible process) entropy of universe remains constant. In other words, Entropy of universe can never decrease. This is known as “Increase of entropy principle”.

- Entropy of universe can never decrease but entropy of system may decrease during a process.
- Entropy is not a conserved property for actual process (Irreversible processes) however, it is conserved for ideal processes (Reversible processes). Mass & Energy are always conserved.

Note:

For a closed system, entropy change will be zero in the following cases

- System undergoing internally reversible adiabatic process.
- System undergoing internally irreversible heat rejection process such that entropy increase due to entropy generation is equal to the entropy decrease due to entropy transfer.
- Cyclic process
- Steady state heat interaction

7.7.1 Physical meaning of Entropy

- Entropy is related to the predictability of the position of the molecules. As the system molecules become more disordered, the position of molecules becomes less predictable and entropy increases.
- Entropy can be defined as the measure of molecular randomness or molecular disorderness.
- Entropy of a substance is minimum in solid phase while it is maximum in gaseous phase.
- When heat is supplied to an incompressible substance, the position of molecules becomes less predictable, hence entropy of the system increases.
- Temperature decrease → Oscillations decreases → Predictability Increases → Entropy decreases

7.7.2 Entropy transfer for closed system undergoing constant temperature heat interactions

- $S_t = \frac{Q}{T}$
- Entropy transfer for a closed system undergoing constant temperature heat interaction can be calculated by evaluating (Q/T).
- Entropy transfer to the system is taken as positive whereas entropy transfer from the system is taken as negative.

7.8 Entropy Change for a Closed System Undergoing Internally Reversible Constant Temperature Heat Interaction

7.8.1 Case 1: Source

(Closed system, T = constant)

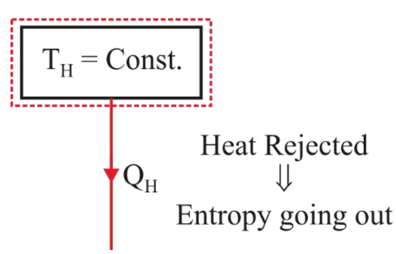


Fig. 7.2 Heat rejection from the source

- $S_{t,source} = -\frac{Q_H}{T_H}$
- $S_{out,source} = \frac{Q_H}{T_H}$

7.8.2 Case 2: Sink

(Closed system, T = constant)

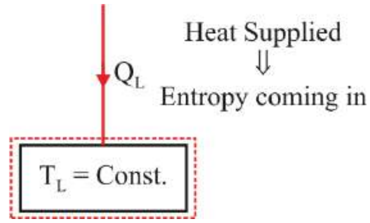


Fig. 7.3 Heat addition to the sink

- $\Delta S_{t,sink} = \frac{Q_L}{T_L}$
- $\Delta S_{in,sink} = \frac{Q_L}{T_L}$

7.8.3 Case 3: Plane wall

Assumption

Steady state heat conduction

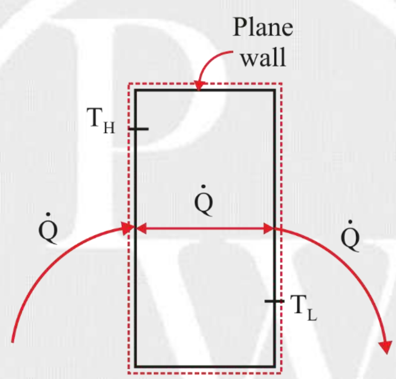


Fig. 7.4 Steady state heat conduction through a plane wall

- $\dot{S}_{g,sys} = \frac{(T_H - T_L)\,\dot{Q}}{T_H\ T_L}$

7.8.4 Phase change process

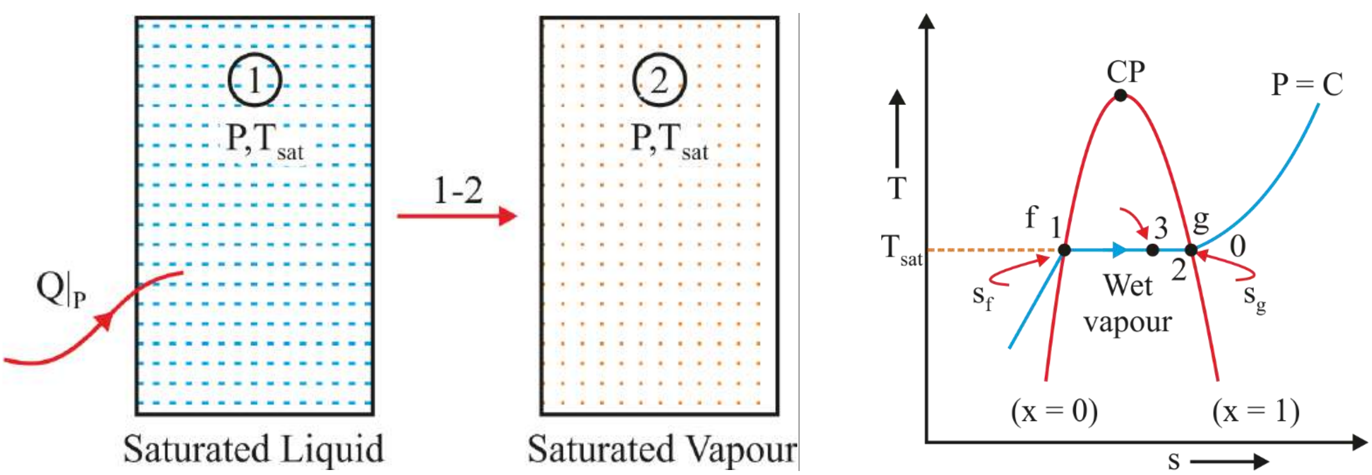


Fig. 7.5 Phase change from saturated liquid to saturated vapour on T-s diagram

Saturated Liquid ⇒ Saturated Vapour
(f) (g)

- $S_{fg} = \frac{H_{fg}}{T_{sat}}, \left[\frac{J}{K} \text{ or } \frac{kJ}{K}\right]$

  H_{fg} = Latent Heat [J or kJ]

  T_{sat} = Saturation Temperature (K)
- $s_{fg} = \frac{h_{fg}}{T_{sat}} \left[\frac{J}{kg-K} \text{ or } \frac{kJ}{kg-K}\right]$

  h_{fg} = Specific latent heat $\left[\frac{J}{kg} \text{ or } \frac{kJ}{kg}\right]$

  T_{sat} = Saturation Temperature (K)

7.8.5 Heat transfer between Source & Sink

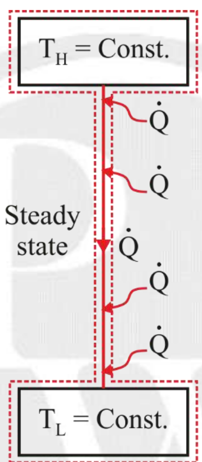


Fig. 7.6 Heat transfer between source and sink

- $\dot{S}_{g,sys} = \frac{(T_H - T_L)}{T_H T_L} \dot{Q}$

7.9 Tds Equations (Gibbs Equations)

- Tds = du + Pdv
- Tds = dh – vdP
- Tds equations are valid for any substance, any system and any process.

7.9.1 Specific entropy change calculation for a perfect gas

- $\Delta s_{1-2} = c_v \ln\left(\frac{T_2}{T_1}\right) + R \ln\left(\frac{v_2}{v_1}\right)$
- $\Delta s_{1-2} = c_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{P_2}{P_1}\right)$
- $\Delta s_{1-2} = c_v \ln\left(\frac{P_2}{P_1}\right) + c_p \ln\left(\frac{v_2}{v_1}\right)$
- Specific entropy change equations for a perfect gas are valid for both internally reversible & internally irreversible processes.
- Entropy of a perfect gas is not the function of temperature only.
- For finding total entropy change, specific entropy change equation must be multiplied by mass.

7.9.2 Entropy change of incompressible substances (Solid, Liquid)

- $\Delta s_{1-2} = c \ln\left(\frac{T_2}{T_1}\right)$; assuming $c = c(T)$ only
- $\Delta S_{1-2} = mc \ln\left(\frac{T_2}{T_1}\right)$
- Entropy change for a perfect incompressible substance depend on temperature only.

7.10 Different Cases

7.10.1 Two finite bodies (having same heat capacities) at different temperatures are in contact with each other in an insulated chamber

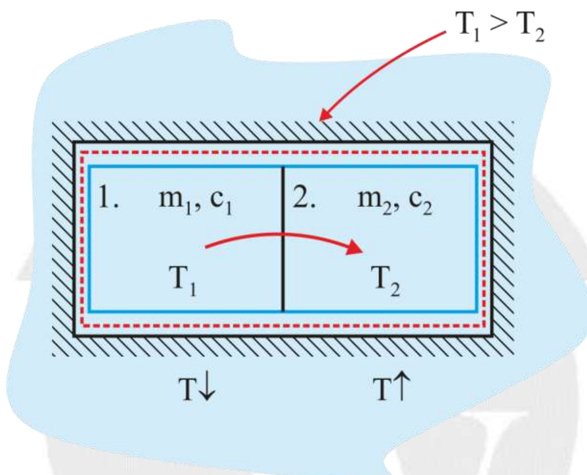


Fig. 7.7 Two finite bodies in an insulated chamber

- Assuming system is two finite bodies
- $T_f = \frac{T_1+T_2}{2}$
- $\Delta S_{sys} = 2\ mc \ln\left(\frac{AM}{GM}\right)$, here $AM = \frac{T_1+T_2}{2}$ and $GM = \sqrt{T_1\ T_2}$
- $\Delta S_{sys} = \Delta S_{uni} = S_{g,uni} = S_{g,sys} = 2\ mc \ln\left(\frac{AM}{GM}\right)$
- $\Delta S_{surr} = S_{g,surr} = 0$

7.10.2 Reversible heat engine is connected between two finite bodies (having same heat capacities) at different temperatures

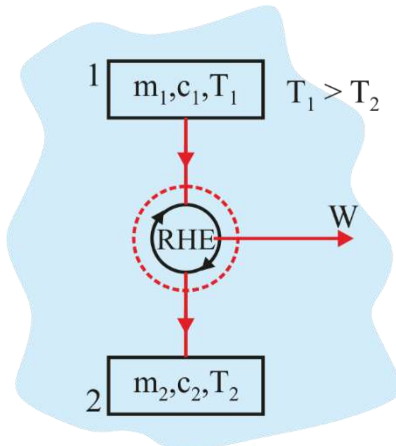


Fig. 7.8 A cyclic reversible heat engine between two finite bodies

- Assuming system is reversible heat engine
- $\Delta S_{sur} = 0$
  $T_f = \sqrt{T_1 T_2}$
- $W_{max} = 2\,mc\,(AM - GM)$; here $AM = \frac{T_1+T_2}{2}$ and $GM = \sqrt{T_1\,T_2}$

7.10.3 Reversible Refrigerator connected between two finite bodies (having same heat capacities) at different temperatures

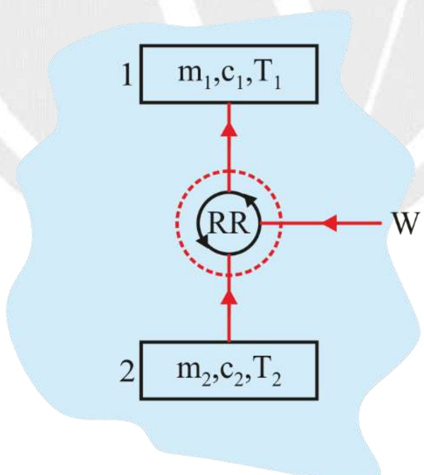


Fig. 7.9 A cyclic reversible refrigerator between two finite bodies

- Assuming system is a reversible refrigerator.
- $\Delta S_{sur} = 0$
- $T_{f1} T_{f2} = T_1 T_2$

  Where T_{f1} & T_{f2} are the final temperature of body 1 and body 2 respectively.

7.10.4 Entropy change of an Open System

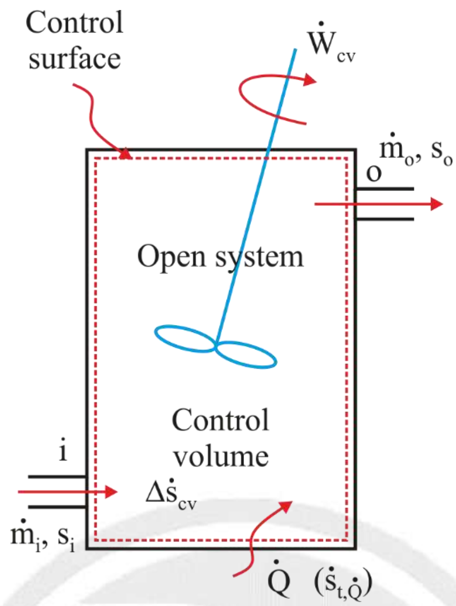


Fig. 7.10 Entropy change in a control volume

- $\Delta\dot{S}_{cv} = \dot{S}_{t,\dot{m}} + \dot{S}_{t,\dot{Q}} + \dot{S}_{g,cv}$
- The rate of entropy change within control volume during a flow is equal to the sum of the rate of entropy transfer into the control volume by mass transfer, the rate of entropy transfer into the control volume by heat transfer and the rate of entropy generation within control volume as a result of irreversibilities.
- For open system the rate of entropy changes of control volume and change of rate of entropy fluid are different things.
  $\Delta\dot{S}_{cv} \neq \Delta\dot{S}_f$

Note:

$\Delta\dot{S}_{cv} + \Delta\dot{S}_f = \Delta\dot{S}_{t,\dot{Q}} + \dot{S}_{g,cv}$

7.10.5 Steady Flow

- For steady flow $\boldsymbol{\Delta\dot{S}_{cv} = 0}$
- For adiabatic flow $\boldsymbol{\dot{S}_{t,\dot{Q}} = 0}$
- For Internally Reversible flow $\boldsymbol{\dot{S}_{g,cv} = 0}$

7.10.6 Throttling of ideal gas in small throttling device

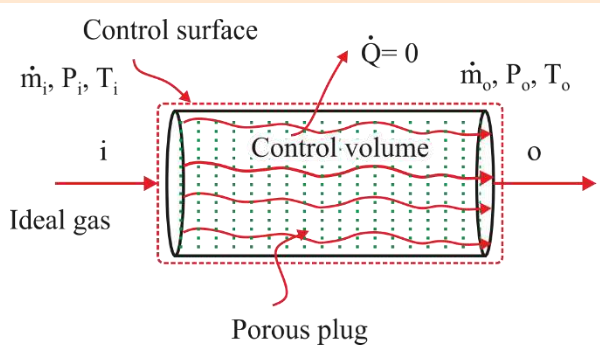


Fig. 7.11 Entropy generation in a throttling device

- $\dot{S}_{g,cv} = \Delta \dot{S}_f$
- $\Delta \dot{S}_f = -\dot{m} R \ln\left(\frac{P_o}{P_i}\right)$

7.10.7 T-s diagram

- On T-s diagram area under the curve projected on specific entropy axis gives the magnitude internally reversible heat interactions per unit mass.
- For heat addition process entropy increase while for heat rejection process entropy decreases.

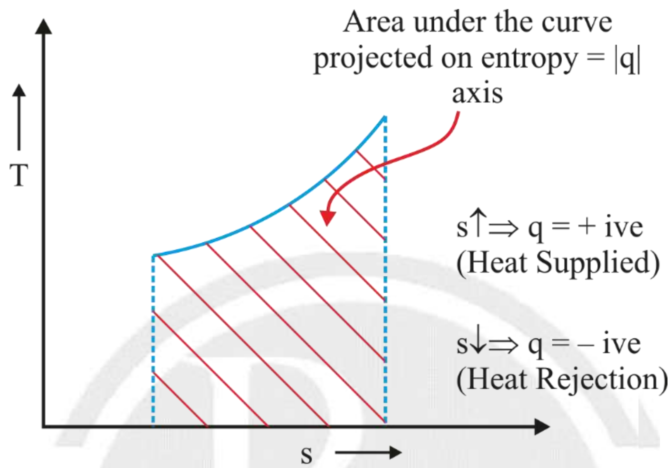


Fig. 7.12 Heat supplied on T-s diagram

7.11 T-s Diagram for Various Internally Reversible Processes

7.11.1 Isochoric process

Slope of Isochore on T-s diagram is given by

$$\left.\frac{\partial T}{\partial s}\right|_v = \frac{T}{c_v}$$

$$\left.\frac{\partial T}{\partial s}\right|_v = \frac{T}{c_v}$$

As absolute temperature increases slope increases

As absolute temperature decreases slope decreases

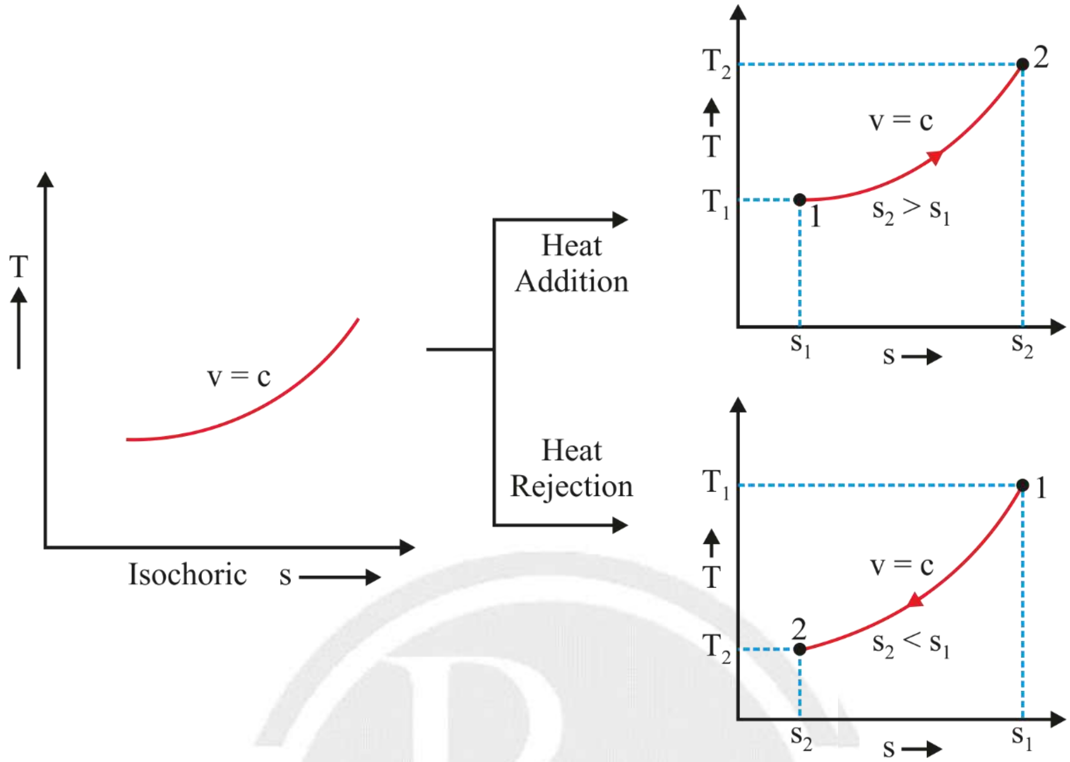


Fig. 7.13 T-s diagram of heat interactions in isochoric process

7.11.2 Isobaric process

Slope of Isobar on T-s diagram is given by

$$\left.\frac{\partial T}{\partial s}\right|_P = \frac{T}{c_P}$$

$$\left.\frac{\partial T}{\partial s}\right|_p = \frac{T}{c_p}$$

As absolute temperature increases slope increases	As absolute temperature decreases slope decreases

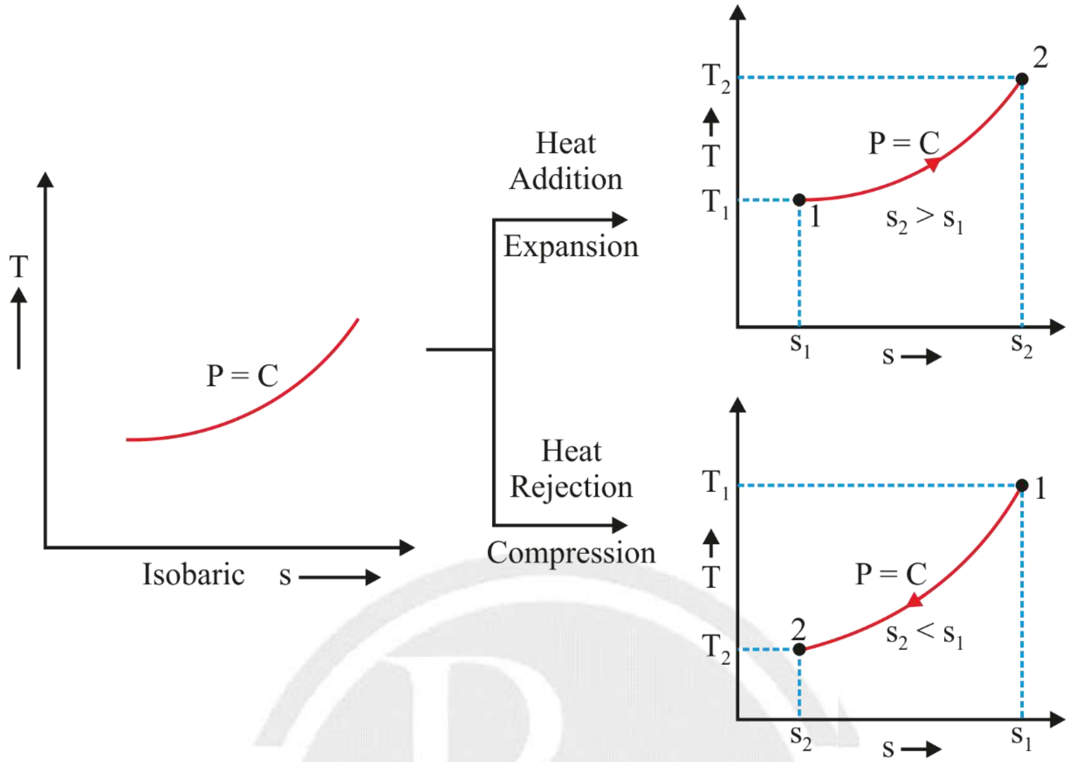


Fig. 7.14 T-s diagram of heat interactions in isobaric process

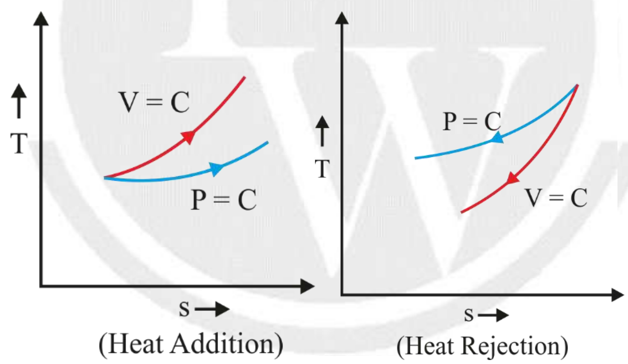


Fig. 7.15 T-s diagram of Isobaric and isochoric processes

- $\left.\frac{\partial T}{\partial s}\right|_V = \gamma \left.\frac{\partial T}{\partial s}\right|_P$

  $\gamma > 1$

  $\left.\frac{\partial T}{\partial s}\right|_V > \left.\frac{\partial T}{\partial s}\right|_P$
- On T-s diagram internally reversible isochoric and isobaric processes have positive slopes.
- On T-s diagram, slope of internally reversible isochoric process is more (γ times) than the slope of internally reversible isobaric process.
- On T-s diagram both internally reversible isochoric and isobaric processes are curves.

7.11.3 Isothermal process

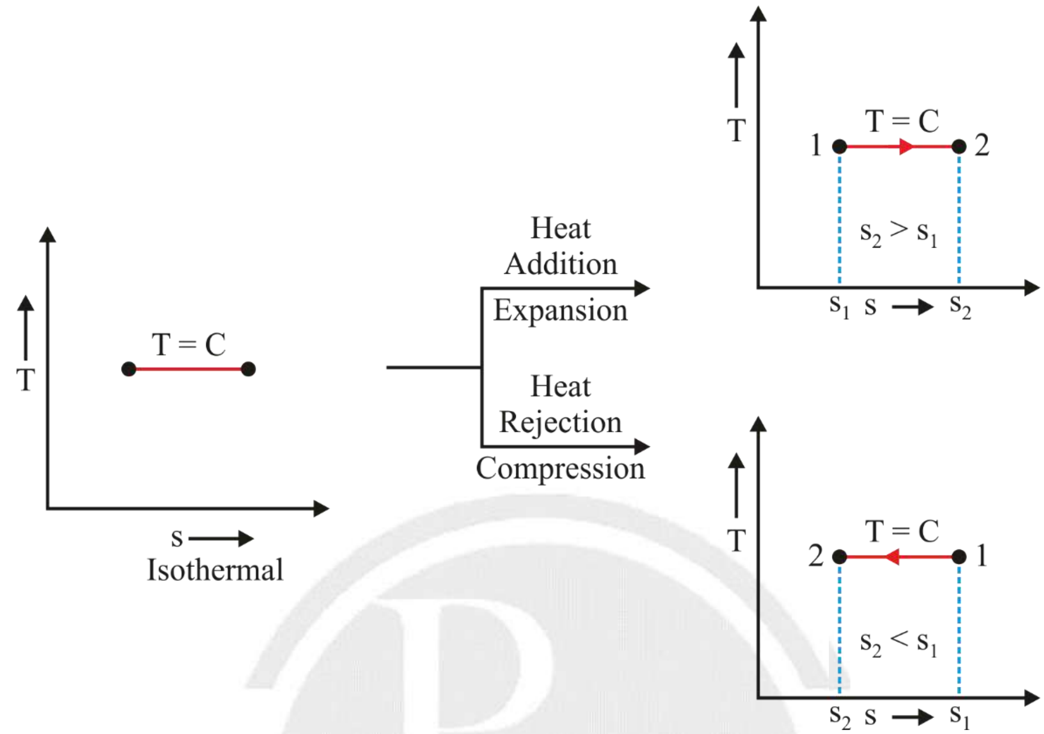
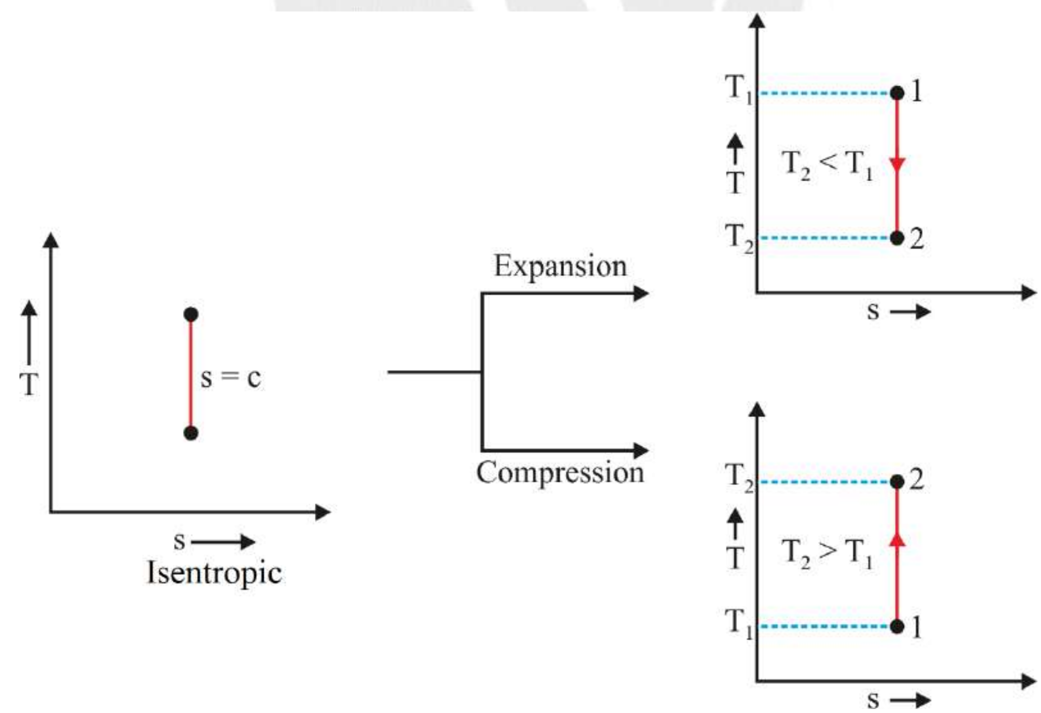


Fig. 7.16 T-s diagram of heat interactions in polytropic process

7.11.4 Adiabatic process



7.11.5 Net heat interaction for a cyclic process

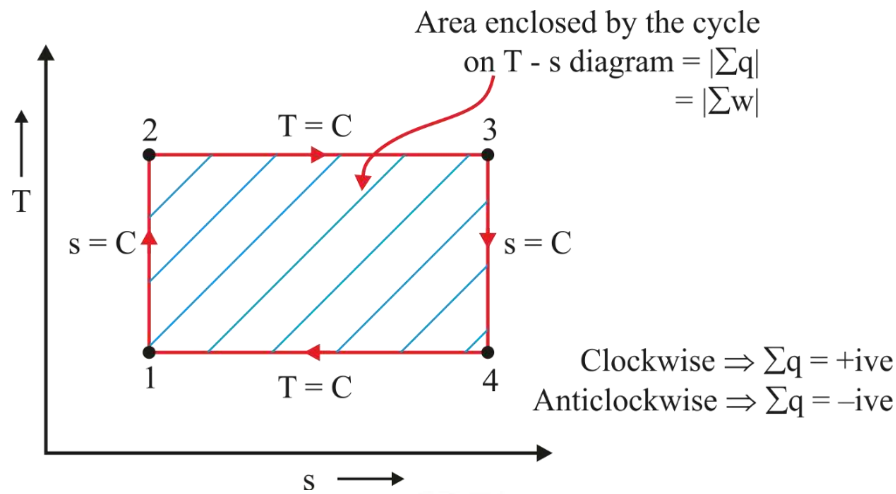


Fig. 7.17 T-s diagram of a cyclic process

- On **T-s** diagram area enclosed by the cycle represents the magnitude of net heat interaction per unit mass for the cycle.
- For clockwise cycle on **T-s** diagram net het interaction is positive while for anticlockwise cycle on **T-s** diagram net heat interaction is negative.
- Power producing cycles are clockwise on both P-V & **T - s** diagrams while power consuming cycles are anticlockwise on both P-V & **T-s** diagrams.
- From **T-s** diagram whatever is net heat interaction per unit mass that will be network interaction per unit mass.

7.11.6 Mixing of perfect gases

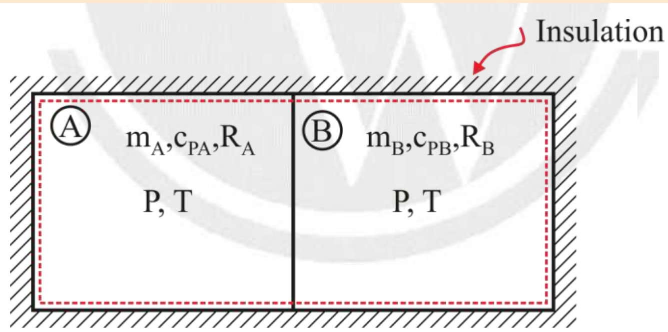


Fig. 7.18 Mixing of gases in an insulated cylinder

Assuming system is perfect gases and pressure after mixing is P only

- $S_{g,sys} = S_{g,uni} = \Delta S_{uni} = \Delta S_{sys} = +X,\ S_{g,sur} = \Delta S_{sur} = 0$
- $\Delta S_{sys} = -m_A R_A \ln\left(\frac{P_{2A}}{P}\right) - m_B R_B \ln\left(\frac{P_{2B}}{P}\right)$
- $\Delta S_{sys} = -\bar{R}[n_A \ln(\bar{x}_A) + n_B \ln(\bar{x}_B)]$

Note:
If two gases are same & initially at same temperature and pressure, then entropy increase is zero since we can't distinguish between the gases.

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8 EXERGY

8.1 Introduction

- Exergy is the maximum possible useful work obtainable from the system.
- In exergy analysis the initial state is specified, and thus it is not a variable.
- Exergy associated with heat is also known as available energy while exergy associated with state of system (closed system / open system) is also known as availability.

8.2 Available Energy Associated with Heat

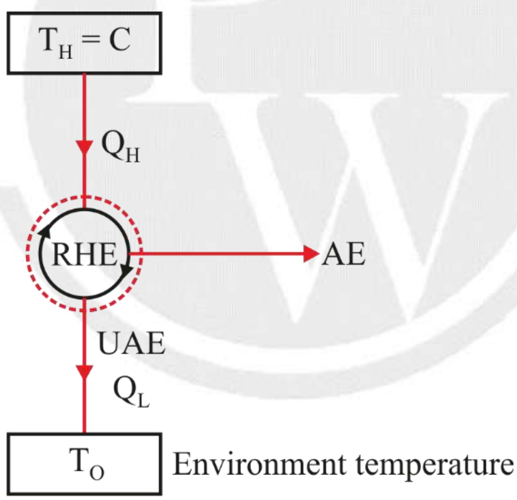


Fig. 8.1 AE and UAE in a reversible heat engine

- Available energy is the maximum possible useful work output obtainable from a certain heat input at a given temperature.
- The minimum heat that must be rejected to the environment is known as unavailable energy
- $AE = \left(\frac{T_H - T_0}{T_H}\right) Q_H$
- $AE + UAE = Q_H$
- $\frac{UAE}{T_0} = \frac{Q_H}{T_H}$

8.3 Representation of AE & UAE on T-S Diagram for Carnot Heat Engine (RHE)

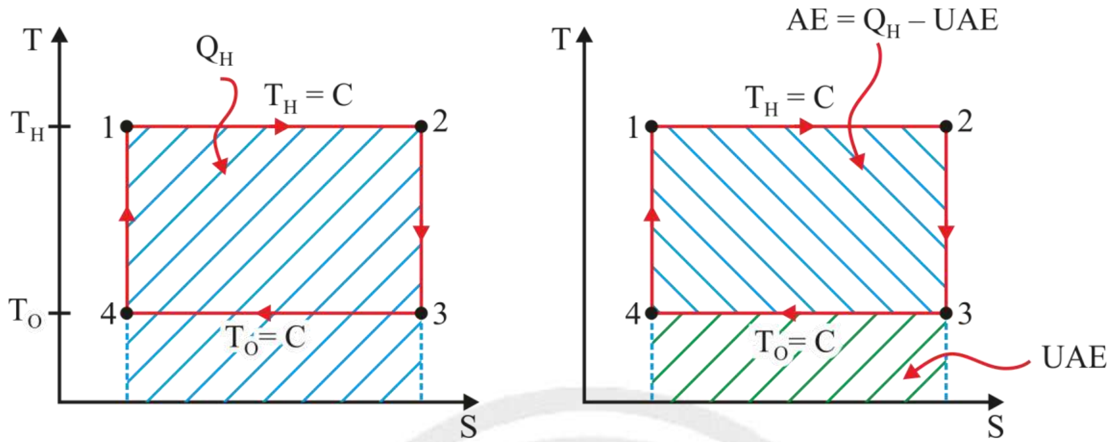
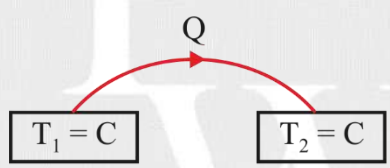


Fig. 8.2 AE and UAE on T-S diagram

8.4 Decrease in Available Energy When Heat is Supplied from A Finite Temperature Difference ($T_1 > T_2$ and T_0 = ambient)



$$\Delta AE = AE_1 - AE_2$$

- As the temperature of heat supplied decreases, available energy associated with the heat decreases leading to increase in unavailable energy.
- Whenever heat transfer takes place from a higher temperature body to lower temperature body, then the AE obtained from the heat provided by the lower temperature body decreases, due to heat transfer.
- If the heat is transferred to the environment then the complete AE is lost.

Note:
Decrease in AE & increase in UAE both are equal in magnitude

8.5 Available Energy Associated with the Heat Loss from Finite Body

- **Assumptions**

  Finite body is perfect incompressible substance.

  Heat is lost to environment.

  - $AE = T_0\left[mc\ln\left(\frac{T_2}{T_1}\right) + \frac{mc(T_1 - T_2)}{T_0}\right]$
  - $AE = T_0 \Delta S_{uni}$

8.6 Irreversibility

- IR = W_{Idal} - W_{act}
- If $T_L = T_o$, then $IR = AE - W_{act}$
- $IR = W_R - W_{IR}$
- The actual work done by a system is always less than the idealised reversible work and the difference between the two is called irreversibility.
- Irreversibility is always positive.
- The smaller the irreversibility associated with a process, the greater is the actual work produced.

8.7 Gouy – Stodola Theorem

- $IR = T_o \Delta S_{uni}$
- $IR = T_o, S_{g,uni}$
- According to Gouy – Stodola theorem irreversibility is directly proportional to the entropy generation in the universe.

8.8 Availability

- Maximum possible useful work output obtained from a given system when it comes in equilibrium with environment is known as Availability.
- When the system is in equilibrium and at rest relative to the environment then the system is said to the in dead state.

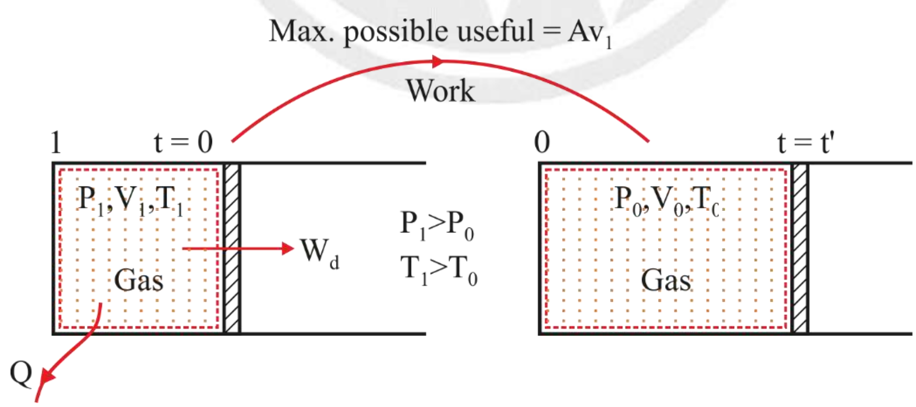


Fig. 8.3 Expansion of gas in a cylinder

- $AV_1 = \phi_1 - \phi_0$

  Where ϕ is availability function for closed system.
- $\phi = E + P_0V - T_0S$

  Neglecting changes in KE & PE, $\phi = U + P_0V - T_0S$
- $Av_1 = (E_1 - E_0) + P_0(V_1 - V_0) - T_0(S_1 - S_0)$

8.8.1 Availability Change for a Closed System

- $\Delta Av_{1-2} = \phi_2 - \phi_1$
- $\Delta Av_{1-2} = (E_2 - E_1) + P_o(V_2 - V_1) - T_O(S_2 - S_1)$

8.9 Specific Availability for Open System

- $av_i = \Psi_i - \Psi_d$
  Where Ψ is specific availability function for an open system
- $\Psi = \frac{C^2}{2} + gz + h - T_o s$
- $av_i = \frac{1}{2}(c_i^2) + g(z_i) + (h_i - h_d) - T_d(s_i - s_d)$

8.9.1 Specific availability changes for an Open System

(1) $\Delta av_{i-o} = \Psi_o - \Psi_i$

(2) $\Delta av_{i-o} = \frac{1}{2}(c_o^2 - c_i^2) + g(z_o - z_i) + (h_o - h_i) - T_d(s_o - s_i)$

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9 THERMODYNAMIC RELATIONS

9.1 Basic Mathematics

1. $dB = Mdx + Ndy$

If B is continuous point function having exact differential then

- $\left.\frac{\partial M}{\partial y}\right|_x = \left.\frac{\partial N}{\partial x}\right|_y$ (Test of Exactness)

2. If $x = x(y, z)$

$y = y(x, z)$ then

- $\left.\frac{\partial x}{\partial y}\right|_z = \frac{1}{\left.\frac{\partial y}{\partial x}\right|_z}$ (Reciprocity relation)
- $\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1$ (Cyclic Relation)

3. If $z = z(x, y, f)$ then

- $\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial f}\right|_z \left.\frac{\partial f}{\partial x}\right|_z = 1$

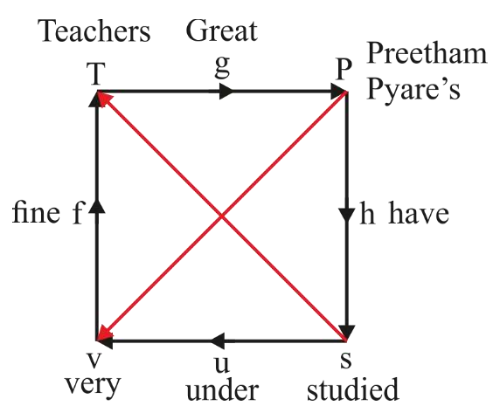
9.2 Gibbs Relations

$du = Tds - Pdv$ (Derived from $Tds = du + Pdv$)

$dh = Tds + vdP$ (Derived from $Tds = dh - vdP$)

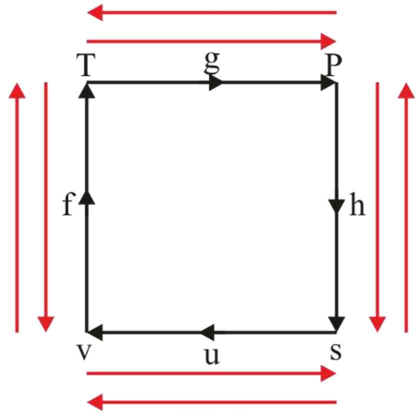
$df = -sdT - Pdv$ (Derived from $f = u - Ts$)

$dg = -sdT + vdP$ (Derived from $g = h - Ts$)



9.3 Maxwell Relations

- Maxwell relations are useful for finding specific change by measuring the changes in pressure, volume, temperature.



$$\left.\frac{\partial T}{\partial P}\right|_s = \left.\frac{\partial v}{\partial s}\right|_P ; \quad \left.\frac{\partial T}{\partial v}\right|_s = -\left.\frac{\partial P}{\partial s}\right|_v$$

$$\left.\frac{\partial P}{\partial T}\right|_v = \left.\frac{\partial s}{\partial v}\right|_T ; \quad \left.\frac{\partial v}{\partial T}\right|_P = -\left.\frac{\partial s}{\partial P}\right|_T$$

9.4 Volume Expansivity (α)

- Volume Expansivity is an indication of the change in volume that occurs when temperature changes while pressure remains constant.

$$\alpha = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_P$$

Note:
- $\alpha_I = \frac{1}{T}$ (For ideal gas)
- For water at 1 atm pr & 4°C
  $\alpha_{H_2O} = 0$

9.5 Isothermal Compressibility (β_T)

- Isothermal compressibility is an indication of the change in volume that occurs when pressure changes while temperature remains constant.

$$\beta_T = -\frac{1}{v}\left.\frac{\partial v}{\partial P}\right|_T$$

Note:
- $\beta_{T,I} = \frac{1}{P}$ (For ideal gas)

9.6 Isentropic Compressibility (β_s)

- Isentropic compressibility is an indication of the change in volume that occurs when pressure changes while entropy remains constant.

$$\beta_s = -\frac{1}{v}\frac{\partial v}{\partial P}\bigg|_s$$

Note:

1. $\beta_{I,G} = \frac{1}{\gamma P}$
2. $\frac{\partial v}{\partial T}\Big|_P = \alpha v$
3. $\frac{\partial P}{\partial T}\Big|_v = \frac{\alpha}{\beta_T}$

9.7 Specific entropy change equations

$$ds = \frac{c_v}{T}dT + \frac{\alpha}{\beta_T}\,dv$$

$$\Delta s_{1-2} = \int_1^2 \frac{c_v}{T}\,dT + \int_1^2 \frac{\alpha}{\beta_T}\,dv$$

$$ds = \frac{c_p}{T}dT - \alpha v\,dp$$

$$\Delta s_{1-2} = \int_1^2 \frac{c_p}{T}\,dT - \int_1^2 \alpha\,v\,dP$$

9.8 Tds Equations

- $Tds = c_v dT + T\frac{\alpha}{\beta_T}dv$
- $Tds = c_p dT - T\alpha v dP$

9.9 Mayer's Equation

$$c_p - c_v = \frac{Tv\alpha^2}{\beta_T}$$

9.9.1 Conclusions

(1) In general c_p is always greater than c_v

(2) The difference between c_p and c_v approaches to zero as the absolute temperature approaches to zero Kelvin.

(3) For incompressible substances, the difference between c_p and c_v is negligible. Hence the two specific heats c_p and c_v are nearly identical for incompressible substance.

(4) For water at 1 atm pressure and 4°C, $c_p = c_v$

(5) For Ideal gas $c_p - c_v = R$

9.10 Specific Internal Energy Change Equation

$$du = c_v dT + \left(T\frac{\alpha}{\beta_T} - P\right) dv$$

$$\Delta u_{1-2} = \int_1^2 c_v dT + \int_1^2 \left(T\frac{\alpha}{\beta_T} - P\right) dv$$

9.11 Specific Enthalpy Change Equation

$$dh = c_p dT - (T\alpha v - v)dP$$

$$\Delta h_{1-2} = \int_1^2 c_p dT - \int_1^2 (T\alpha v - v)dv$$

9.12 Adiabatic Index

$$\gamma = \frac{c_p}{c_v} = \frac{\beta_T}{\beta_s}$$

Where β_T is Isothermal compressibility

β_s is Isentropic compressibility

9.13 Speed of Sound

$$c = \sqrt{\frac{v}{\beta_s}} \quad \text{(Any substance)}$$

$$c = \sqrt{\frac{\gamma v}{\beta_T}} \quad \text{(Any substance)}$$

$$c = \sqrt{\gamma RT} \quad \text{(Perfect gas)}$$

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10 PROPERTIES OF PURE SUBSTANCES

10.1 Introduction

Pure Substance

- Any substance having homogeneous chemical composition throughout is knows as pure substance.
- A mixture of various chemical elements or compounds will behave as pure substance as long as the mixture is homogeneous in chemical composition.
- A mixture of two or more phases of a pure substance will be a pure substance as long as chemical composition of all the phases is same.

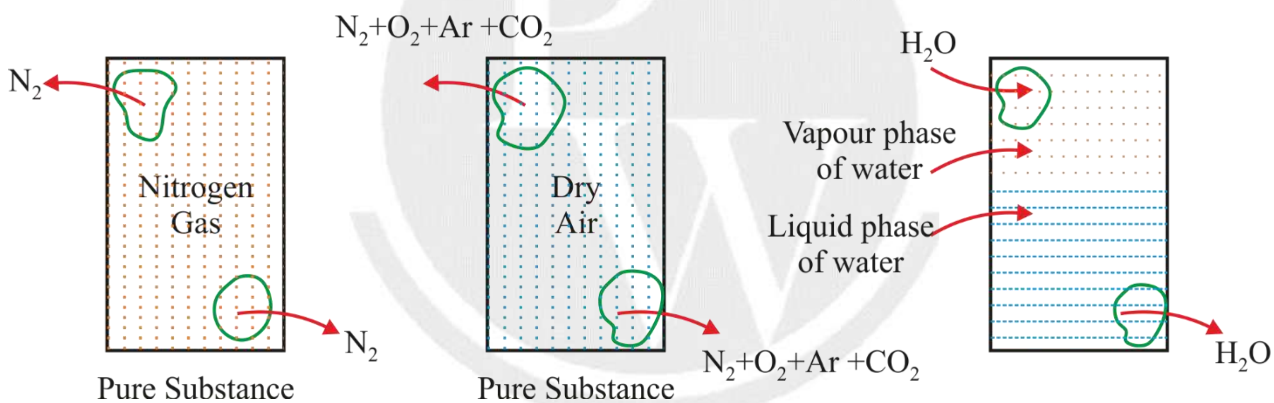

Fig.10.1 Nitrogen, dry air and two phases of water as a pure substance

10.2 Gibbs Phase Rule

F = C – P + 2

Where

C = number of components

P = number of phases (Solid /Liquid /Gas)

F = Degree of freedom

- Degree of freedom represents the number of independent intensive variables required to fix the intensive state of the system.
- For sub cooled liquid/Superheated vapour ⇒ F = 2 (T & P)
- For wet vapour ⇒ F = 1 (T or P)
- For triple phase state ⇒ F = 0

Note:

Triple phase state (Triple point)

It is the state at which all 3 phases (solid, liquid and vapour) can coexist together.

For water

$T_t = 0.01°C = 273.16\ K$

$P_t = 612\ Pa$

10.3 Temperature - Specific Heat Diagram

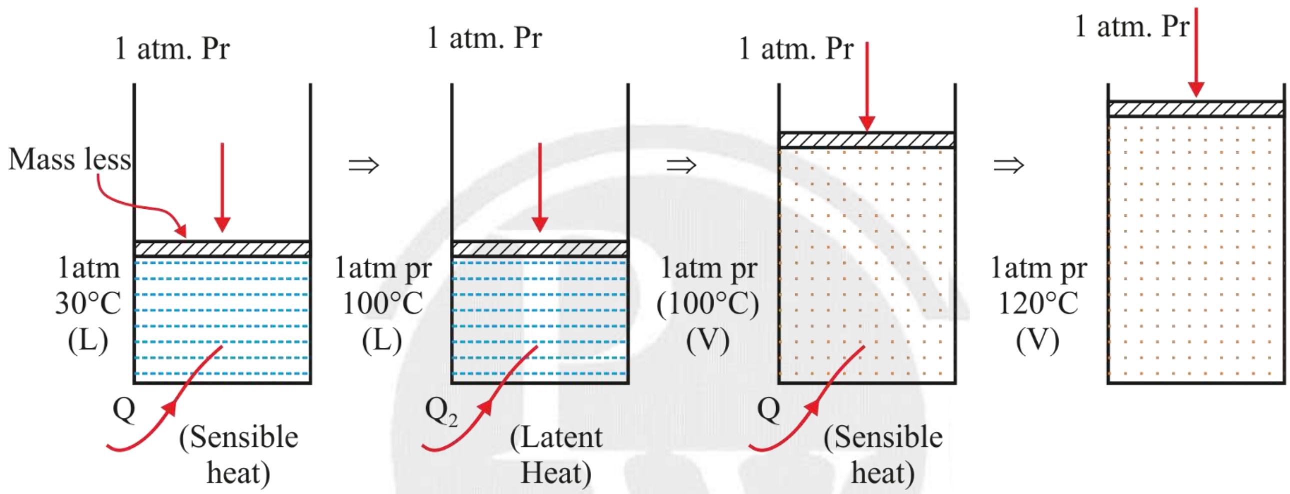

Fig.10.2 Sensible and latent heat addition to the water at 1 atm pr

10.3.1 Sensible heat

The heat which is used to change the temperature without any change in phase is known as sensible heat.

10.3.2 Latent heat

The heat which causes change in phase without any change in temperature is known as latent heat.

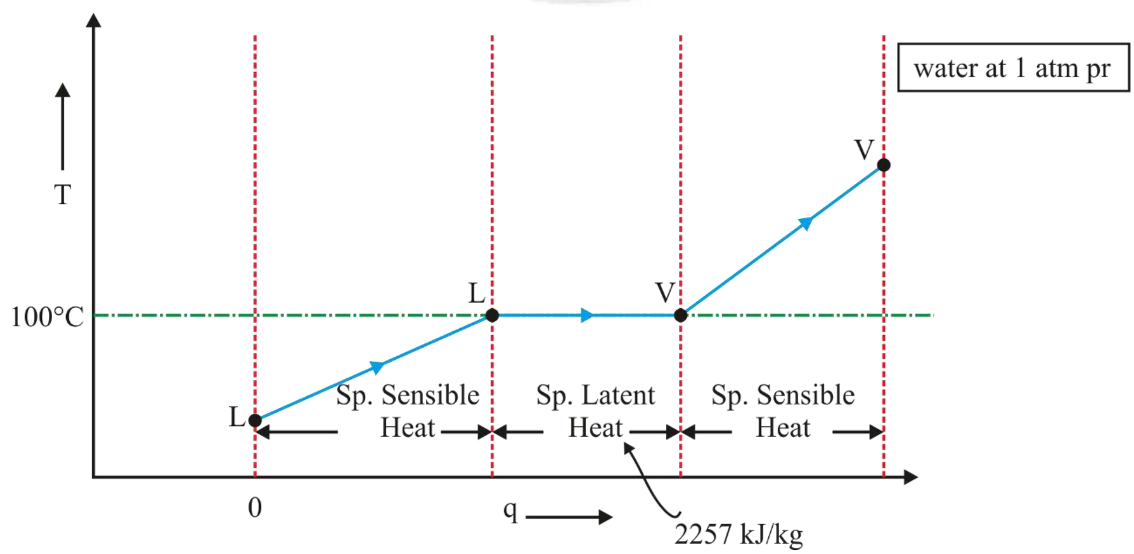

Fig.10.3 T-q diagram of sensible and latent heat addition to the water at 1 atm pr

For water

- Specific latent heat of fusion $(L_f) = 334$ kJ/kg
- Special latent heat of vaporisation $(L_v) = 2257$ kJ/kg

$\}$ at 1 atm pressure

10.4 Saturation Temperature

- Saturation temperature is the temperature at which phase change starts at a particular pressure.
- The saturation temperature of water for liquid to vapour phase change

  Corresponding to 1 atm pressure is 100°C

  Corresponding to 5 bar is 152°C

  Corresponding to 10 bar is 180°C

  Corresponding to 3.17 kPa is 25°C
- As the pressure changes, corresponding saturation temperature also changes, hence saturation temperature is always defined at a particular pressure.
- On increasing pressure, corresponding saturation temperature also increases.

10.5 Saturation Pressure

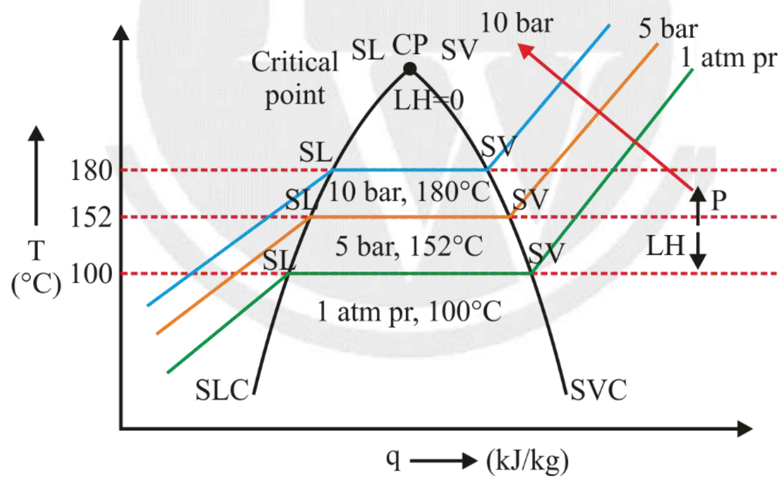

Fig.10.4 T-q diagram per saturation pressure

- Saturation pressure is the pressure at which phase change starts at particular temperature.
- The saturation pressure of water for liquid to vapour phase change

  Corresponding to 100°C is 1 atm pressure

  Corresponding to 152°C is 5 bar

  Corresponding to 180°C is 10 bar

  Corresponding to 25°C is 3.17 kPa

10.6 Saturated Liquid Curve

- It is the locus of all states where liquid starts converting into vapour at temperature equal to saturation temperature.
- A liquid that is about to vaporize is knows as Saturated liquid.

10.7 Saturated Vapour Curve

- It is the locus of all states where liquid completely converts into vapour at temperature equal to saturation temperature.
- A vapour that is about to condense is knows as saturated vapour.

Note:

Critical Point

It is defined as the state at which saturated liquid and saturated vapour states are identical.

At critical point Latent heat of vaporization is zero.

10.8 Subcooled Liquid Region

Subcooled liquid region is the region in which

- Actual temperature (T_{act}) is less than corresponding saturation temperature (T_{sat}) at a particular pressure (P).
- Actual pressure (P_{act}) is more than corresponding saturation pressure (P_{sat}) at a particular temperature (T).
- Subcooled liquid is the liquid which is not about to vaporise.
- For a given temperature subcooled liquid has higher pressure as compared to corresponding saturation pressure, hence it is known as compressed liquid also.

Note:

In Subcooled liquid region, at a given pressure $T_{act} < T_{sat}$ and at a given temperature $P_{act} > P_{sat}$.

10.9 Superheated Vapour Region

Superheated vapour region is the region in which

- Actual temperature (T_{act}) is more than corresponding saturation temperature (T_{sat}) at a particular pressure (P).
- Actual pressure (P_{act}) is less than corresponding saturation pressure (P_{sat}) at a particular temperature (T).
- Superheated vapour is the vapour, which is not about to condense.

Note: In superheated vapour region, at a given pressure $T_{act} > T_{sat}$ and at a given temperature $P_{act} < P_{sat}$.

10.10 Wet Vapour Region (Saturated Liquid – Saturated Vapour Mixture)

Wet vapour region is the region in which

- Actual temperature is exactly equal to the saturation temperature corresponding to a particular pressure (excluding saturated liquid curve and saturated Vapour curve).

- Actual pressure is exactly equal to the saturation pressure corresponding to a particular temperature (excluding saturated liquid curve and saturated Vapour curve).
- In wet region, saturated liquid and saturated vapour are at equilibrium.
- In wet region pressure and temperature are dependent variables. On changing one of the Variable, other variable changes automatically.

Note:

In Wet vapour region, at a given pressure $T_{act} = T_{sat}$ and at a given temperature $P_{act} = P_{sat}$.

10.11 Dryness Fraction

- Dryness fraction represents the fraction of saturated vapour present in wet vapour (saturated liquid – saturated vapour mixture)
- $x = \frac{m_v}{m_L + m_v}$

  Where

  m_v represents mass of saturated vapour

  m_L represents mass of saturated liquid.

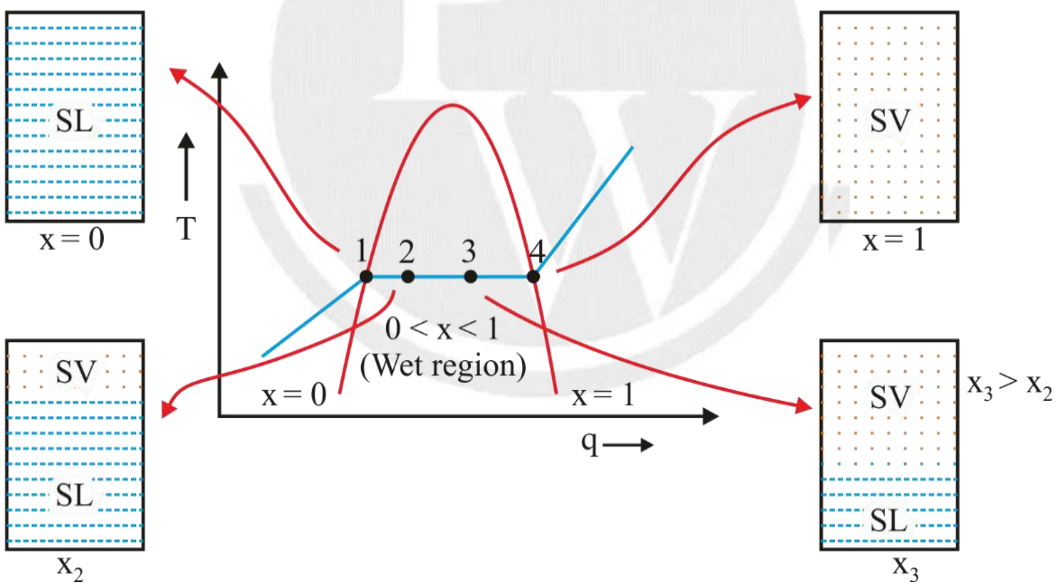

Fig.10.5 Dryness fraction at various states

- At critical point dryness fraction is not defined.

10.12 T-h and T-s Diagrams

Heat supplied at constant pressure increases the specific enthalpy and specific entropy. T-h and T-s diagram will be having same profile as that of T-q diagram for isobar.

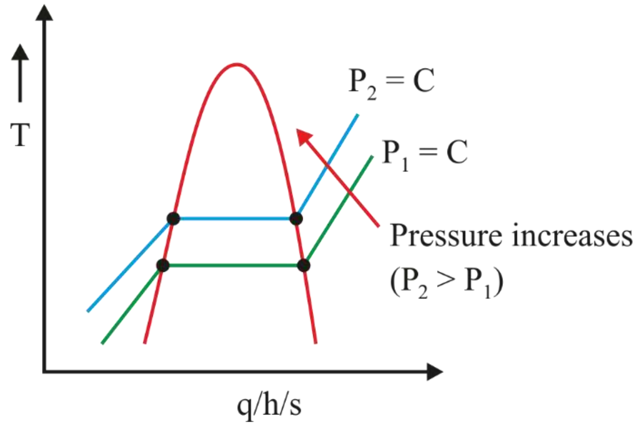

Fig.10.6 T-q/h/s diagrams of isobaric heat addition

10.13 Specific Enthalpy Calculation at Various States

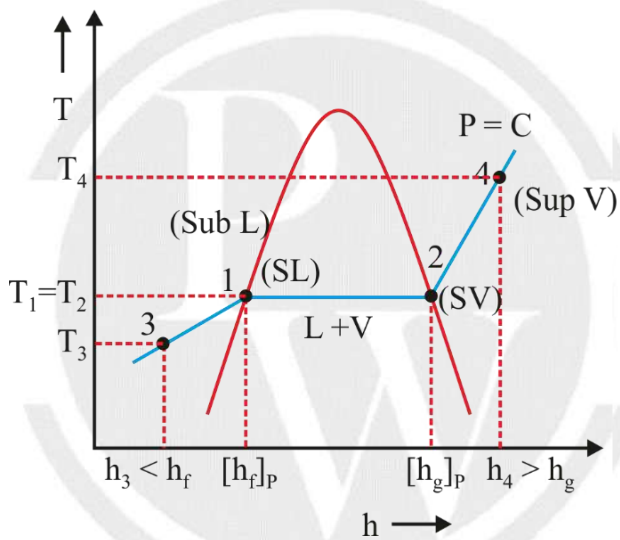

Fig.10.7 Isobar on T-h diagram

- 3For saturated liquid state: $h_1 = [h_f]_P$
- For saturated vapour state $h_2 = \left[h_g\right]_P$
- For subcooled liquid state $h_3 = [h]_{P,T_3}$
- For subcooled liquid state $h_3 = h_f - c(T_{sat} - T_{act})$, assuming subcooled liquid is behaving as perfect incompressible substance having constant specific heat.
- For superheated vapour state $h_4 = [h]_{P,T_4}$
- For superheated vapour state $h_4 = h_g + c_p\,\Delta T_{sup}$, assuming superheated vapour is behaving as perfect gas having constant specific heat.

10.14 Specific Entropy Calculation at Various States

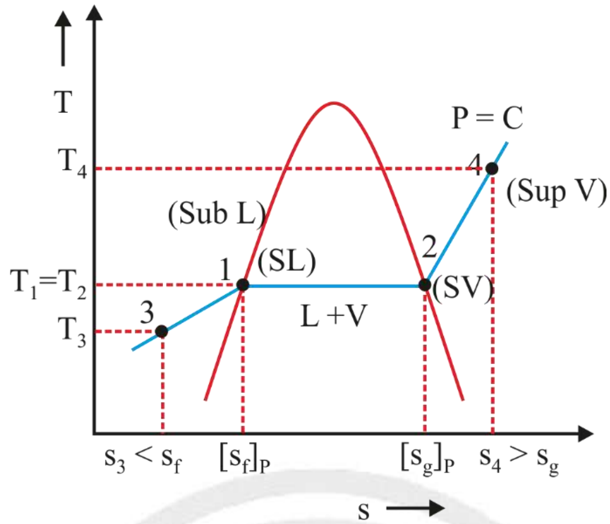

Fig.10.8 Isobar on T-s diagram

- For saturated liquid state: $s_1 = [s_f]_P$
- For saturated vapour state $s_2 = [s_g]_P$
- For subcooled liquid state $s_3 = [s]_{P,T_3}$
- For subcooled liquid state $s_3 = s_g - c \ln\left[\frac{T_{sat}}{T_{act}}\right]$, assuming subcooled liquid is behaving as perfect incompressible substance having constant specific heat.
- For superheated vapour state $s_4 = [s]_{P,T_4}$
- For superheated vapour state $s_4 = [s_g]_p + c_p \ln\left[\frac{T_{act}}{T_{sat}}\right]_p$, assuming superheated vapour is behaving as perfect gas having constant specific heat.

10.15 Properties Calculation in Wet Region

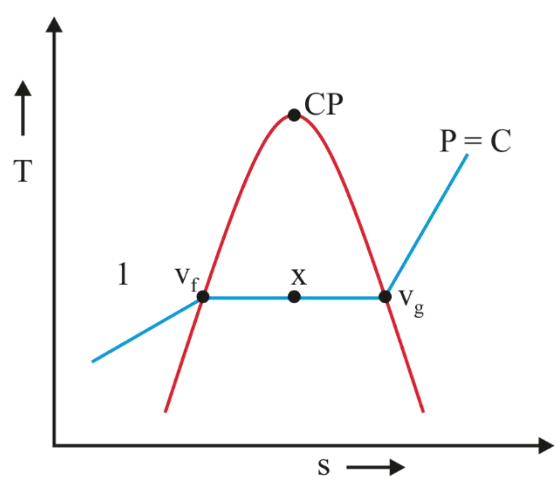

Fig.10.9 Isobar on T-s diagram

- $p = p_f + x\,p_{fg}$

  where p is specific extensive property

$v = v_f + xv_{fg}$, $u = u_f + xu_{fg}$

$h = h_f + xh_{fg}$, $s = s_f + xs_{fg}$

Remember: $\frac{1}{\rho} = \frac{1}{\rho_f} + x\left(\frac{1}{\rho_g} - \frac{1}{\rho_f}\right)$

10.16 Mollier Diagram

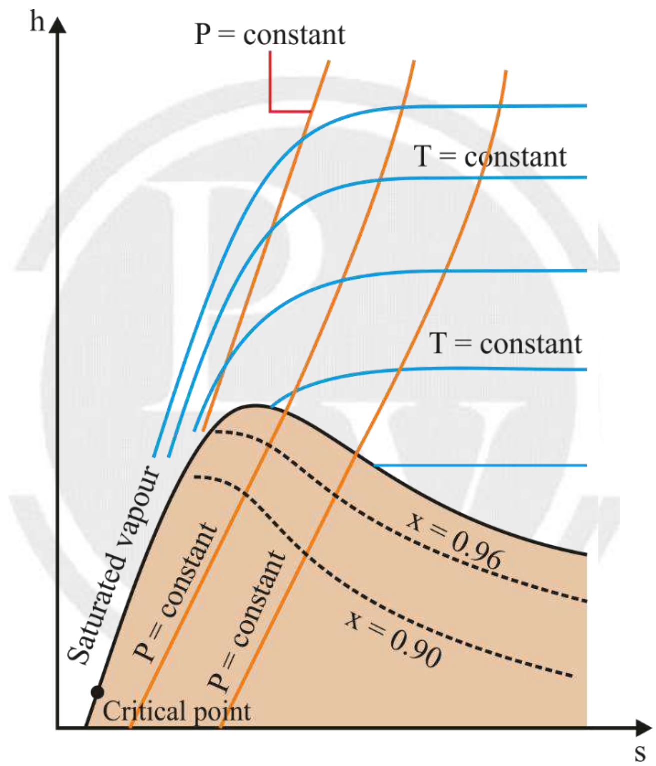

Fig.10.10 Specific enthalpy specific entropy (h-s) diagram

- Mollier diagram was named after Richard Mollier.
- It is specific enthalpy-specific entropy (h-s) diagram.
- Mollier diagram is used in vapour power plant for evaluating properties of superheated vapour region and high-quality wet vapour. Property of low-quality wet vapour data is seldom used.
- Slope of Isobar on Mollier diagram is constant and equal to absolute temperature.

  $\left.\frac{\partial h}{\partial s}\right|_P = T$

- In wet region slope of isobar is constant and equal to saturation temperature corresponding to that constant pressure, hence on Mollier diagram isobars are straight lines in wet region.

- Higher the pressure, higher is the saturation temperature, hence higher the slope of isobar on Mollier diagram in wet region.
- In superheated region as temperature increases slope of isobar on Mollier diagram also increase.
- It is impossible to have negative slope of isobar on Mollier diagram since absolute temperature can never be negative.

10.17 Pressure-Specific Volume (P-v) Diagram

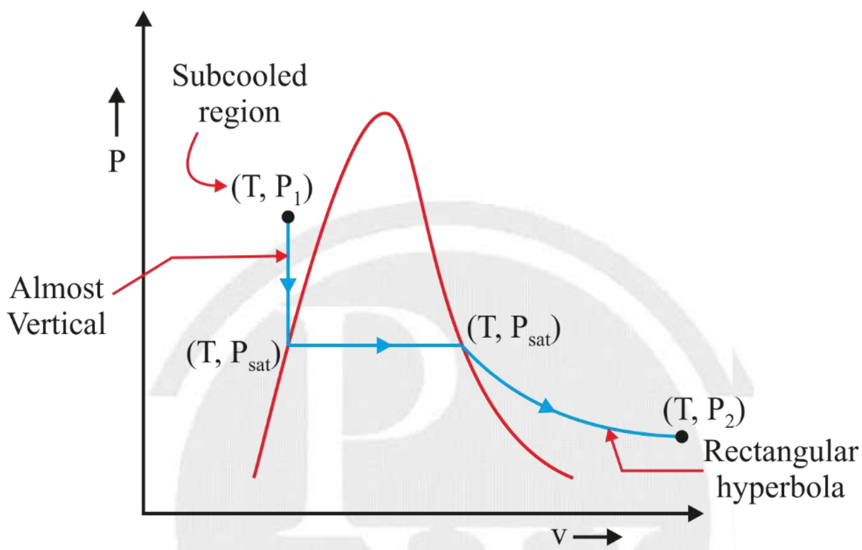

Fig.10.11 Isotherm on P-v diagram

- In subcooled region, liquid behaves as an incompressible substance.
- In subcooled region decrease in pressure, there is negligible increase in specific volume at constant temperature, hence isotherms are almost vertical in subcooled region on P-v diagram.
- In wet region, pressure and temperature are dependent intensive variables hence for constant temperature process pressure is also constant.
- In superheated region, superheated vapour behaves as compressible substance. With small decrease in pressure there is huge increase in volume.
- For sub cooled region, $v_{(T,P)} \cong v_{f(T)}$, $u_{(T,P)} \cong u_{f(T)}$, $s_{(T,P)} \cong s_{f(T)}$

10.18 Analysis for Isochoric Heat Addition

If the specific volume of wet vapour (saturated liquid – saturated vapour mixture) is

1. more than critical specific volume then isochoric heat addition (1-2) leads to the increase in dryness fraction & decrease in liquid level.
2. less than critical specific volume then isochoric heat addition (3-4) leads to the decrease in dryness fraction & increase in liquid level.

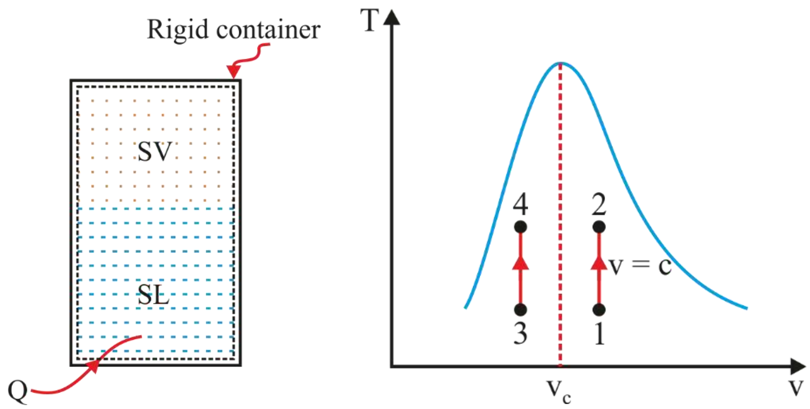

Fig.10.12 Isochoric heat addition to wet vapour

10.19 Pressure - Temperature (P-T) Diagram

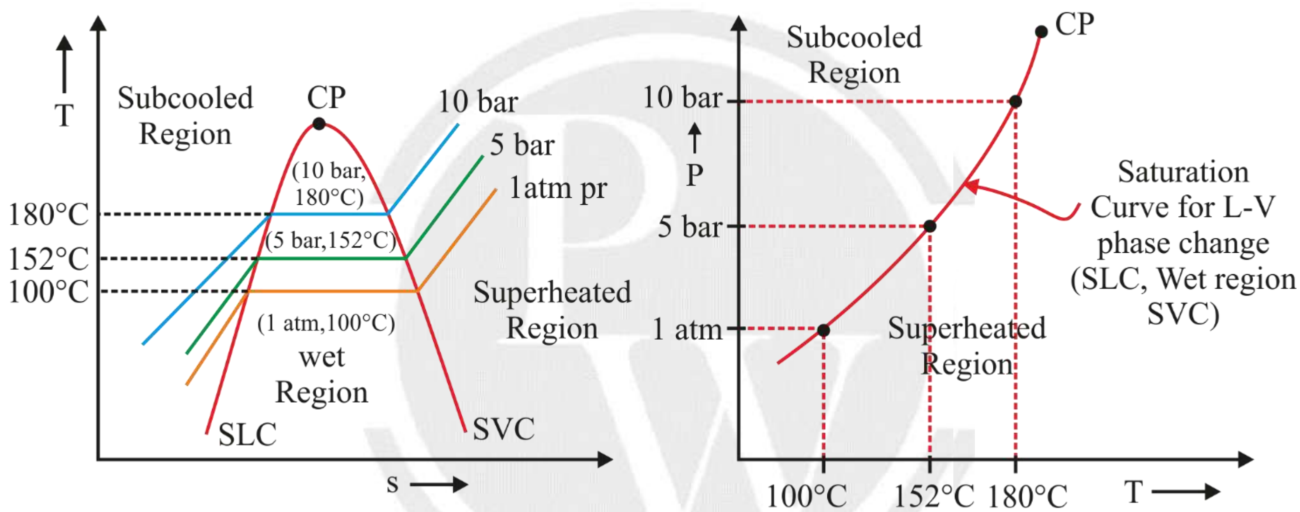

Fig.10.13 T-s and P-T diagrams during phase change

- The slope of fusion curve on P-T diagram for water is negative.
- If the pressure is less than triple pressure, then a pure substance can't exist in liquid phase.

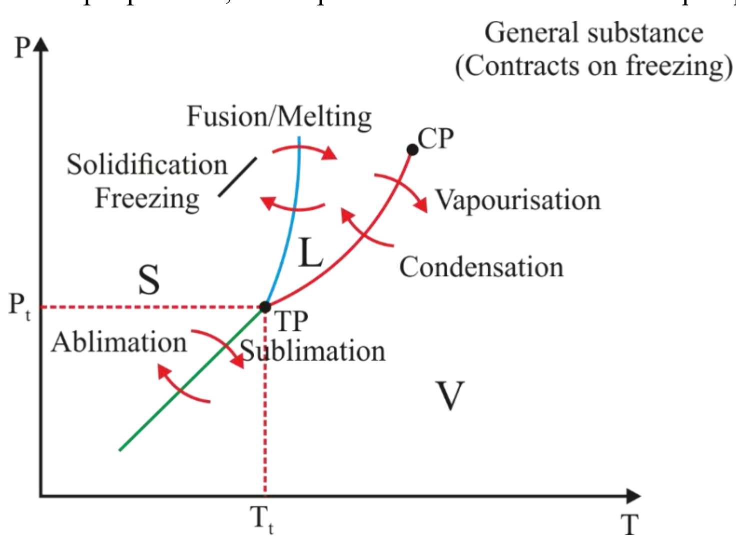

Fig.10.14 P-T diagram for general substance

For general substance

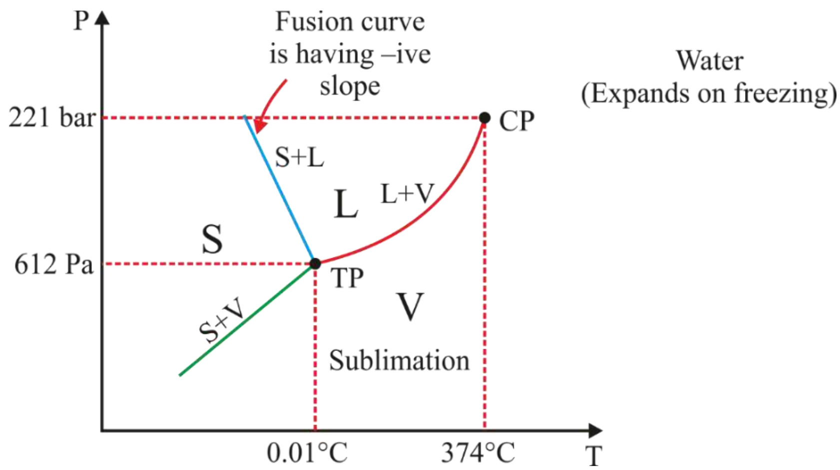

Fig.10.15 Triple point of water on P-T diagram

> **Note:**
> Triple phase state (Triple point). It is the state at which all the three phases, (solid, liquid & vapour) can co-exist together.

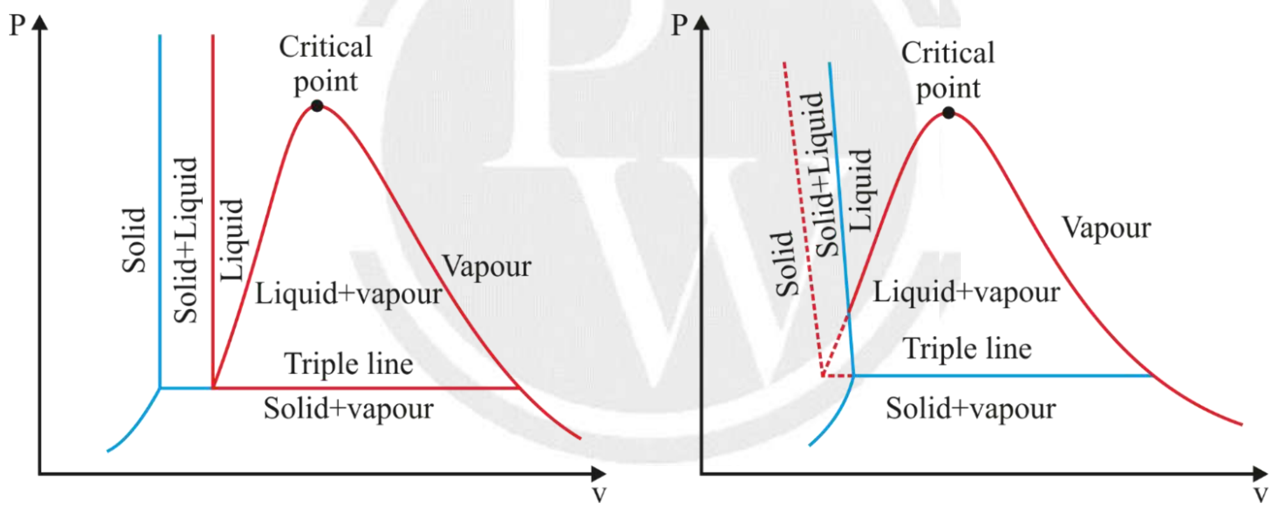

Fig.10.16 Triple phase state on P- v of a pure substance and water

- Triple phase state (Triple point) is a point on P-T diagram but it is line on P-v and T-v diagram.
- The Specific volume of the mixture at triple phase state (triple point) is not fixed and it depends on the composition of the mixture.

10.19.1 Isobaric process on P-T Diagram

- When heat is extracted by solid phase at constant pressure which is more than triple pressure then solid first melts and subsequently evaporates. Example: Heating of Ice at atmospheric pressure.
- When heat is extracted by solid phase at constant pressure, which is less than triple pressure, then solid directly sublimates (evaporates) without undergoing melting. Example: Sublimation of solid CO_2 at atmospheric pressure.

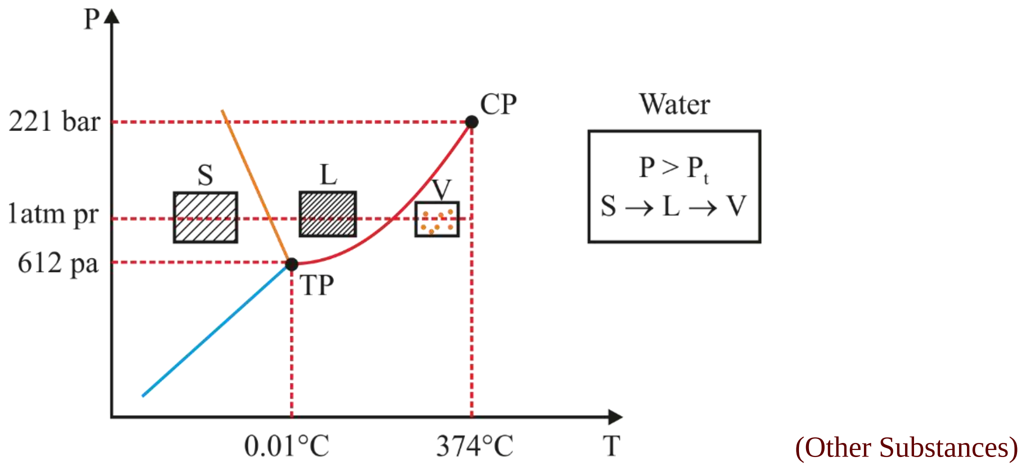

Fig.10.17 Solid, liquid and vapour states above triple point

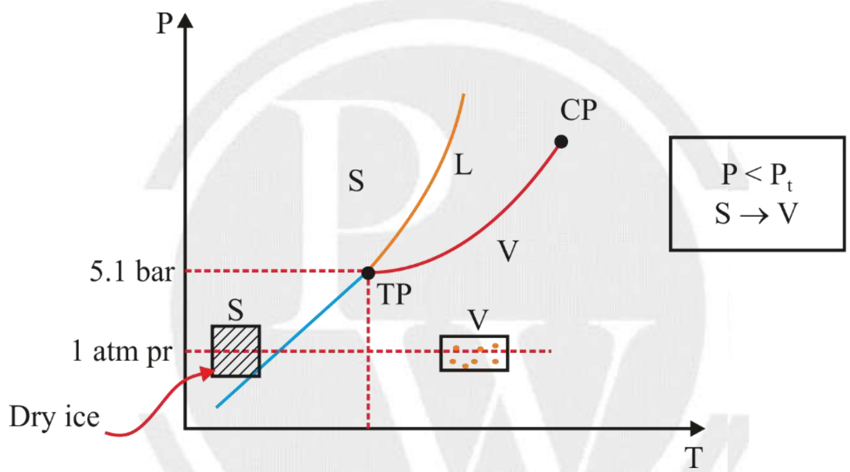

Fig.10.18 Solid and vapour states below triple point on P-T diagram

10.20 Critical Point

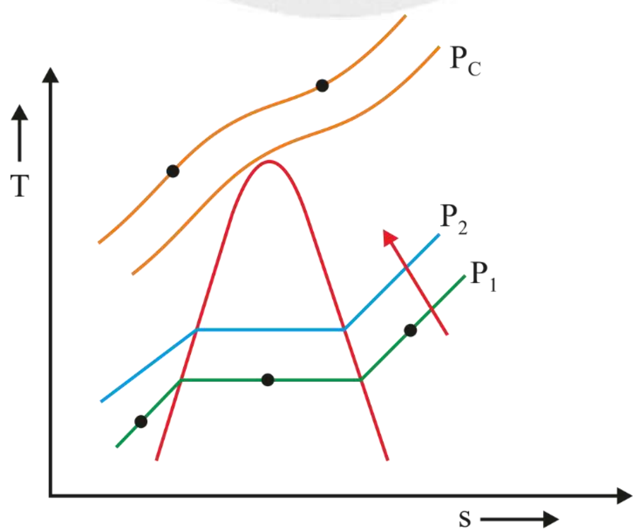

Fig.10.19 Critical point on T-s diagram

- Critical point is the state at which saturated liquid and saturated vapour states are identical.
- At pressure above critical pressure, there is no distinct phase change process. i.e., all the times there is only one phase.

Note:

A liquid is having pressure more than critical pressure doesn't undergo phase change in wet region, rather it directly converts into super critical fluid.
It is impossible to liquify a gas by compression which is having temperature more than critical temperature.

10.21 P-V-T Surface for Substance Which Contracts on Freezing (General Substance)

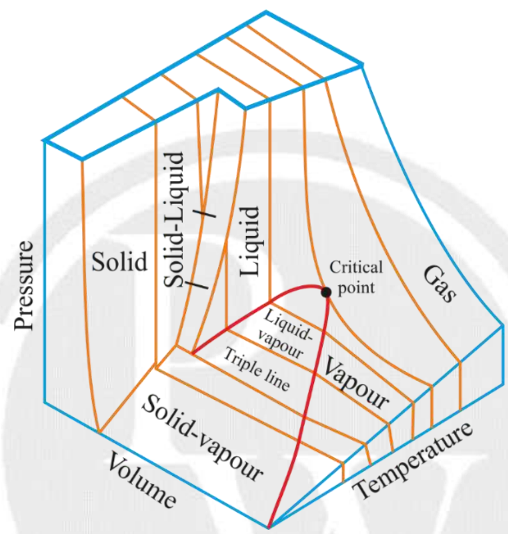

Fig.10.20 P – V – T surface of a substance contracting on freezing

10.22 P-V-T Surface for Substance Which Expands on Freezing (Water)

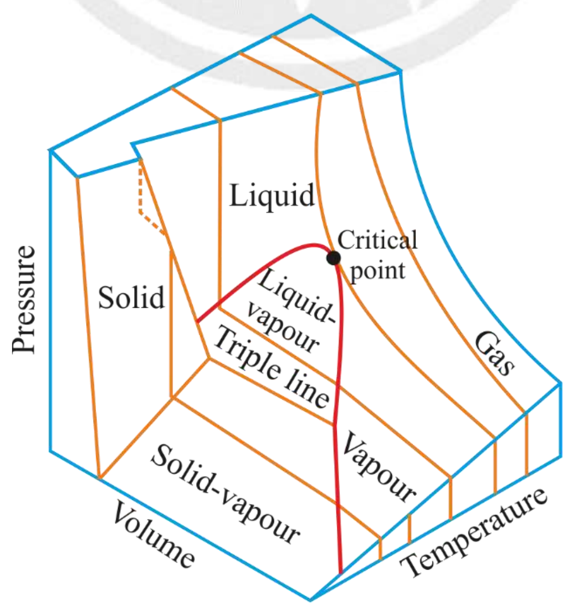

Fig.10.21 P-V-T surface of a substance expanding on freezing

10.23 Clapeyron Equation

- Clapeyron equation is used to find specific enthalpy change associated with phase change.
- The Clapeyron equation is applicable for any phase change process whether it is solid to liquid or liquid to vapour or solid to vapour.
- For liquid to vapour phase change.

$$\left.\frac{dP}{dT}\right|_{sat} = \frac{1}{T_{sat}}\frac{h_{fg}}{v_{fg}}$$

10.23.1 Clausius – Clapeyron equation

Assumptions: $v_f \lll v_g$

Saturated vapour is behaving as an Ideal gas

Differential form

$$\left.\frac{dP}{dT}\right|_{sat} = \frac{P_{sat}}{T_{sat}^2} \times \frac{h_{fg}}{R}$$

Integral form

$$\ln\left(\frac{P_2}{P_1}\right) = \frac{h_{fg}}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

10.24 Joule – Thomson Coefficient (μ)

- Joule – Thomson coefficient is used to evaluate specific heat at constant pressure.

Mathematically, $\mu = \left.\frac{\partial T}{\partial P}\right|_h$

- Slope of isenthalpic curve on T-P diagram.

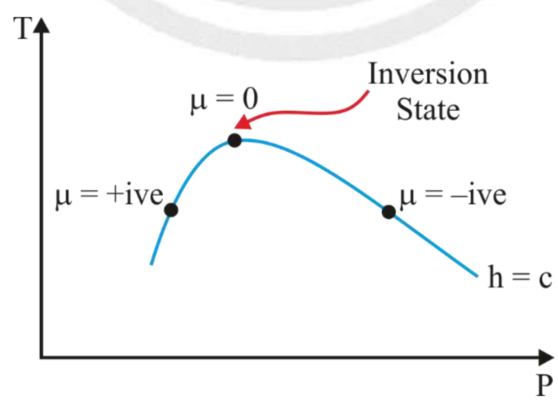

Fig.10.22 Isenthalpic curve on T-P diagram

- $\mu = \frac{v\,(T\alpha - 1)}{c_p}$

Where α is volume expansivity

10.24.1 Experimental evaluation of Joule -Thomson coefficient

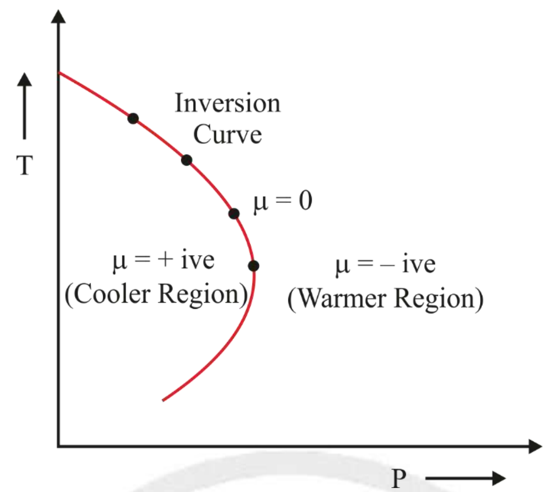

Fig.10.23 Inversion curve on T-P diagram

- The temperature behaviour of a real fluid for a given pressure drop is given by Joule – Thompson coefficient.
  $\mu = -$ive, warmer region, on throttling Temperature increases.
  $\mu = +$ive, cooler region, on throttling Temperature decreases
- For maximum temperature decrease, initial state must be inversion state.

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Applied Thermodynamics

Applied Thermodynamics

INDEX

1 IC ENGINE CYCLES

1.1 Introduction

1.1.1 Application of TD (IC Engine)

(1) Swept volume $V_S = \frac{\pi}{4} D^2 L_S$

Length of stroke $L_S = 2 \times r_C$ ($r_C \rightarrow$ radius of crank)

(2) Compression ratio $r = \frac{V_T}{V_C} = \frac{V_S + V_C}{V_C} = 1 + \frac{V_S}{V_C} \Rightarrow r > 1$

(3) Clearance ratio $C = \frac{V_C}{V_S} \Rightarrow \frac{1}{C} = \frac{V_S}{V_C} = r - 1 \Rightarrow \boxed{r = 1 + \frac{1}{C}}$

Assumptions while dealing with Engines

1. Mass in cylinder is fixed. (Air is perfect gas)
2. Combustion is replaced by heat transfer process from external source.
3. All the processes involved are internally reversible.
4. Specific heats of the working substance are assumed to be constant.

1. 2 Otto cycle

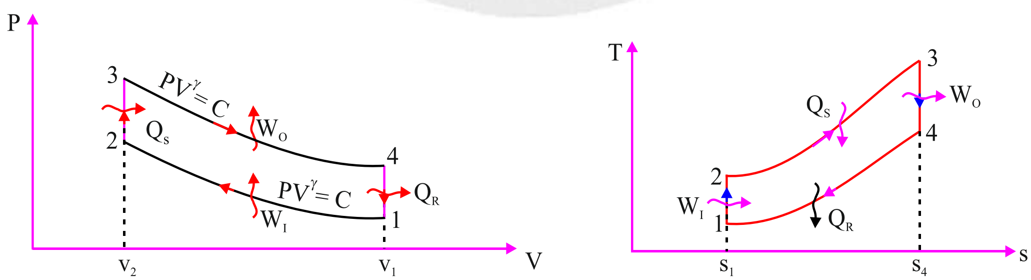


Fig. 1.1. P-V and T-s plots of Otto cycle

Processes involved:
- $1 - 2 \rightarrow$ Internally Reversible Isentropic compression
- $2 - 3 \rightarrow$ Internally Reversible Constant Volume Heat addition
- $3 - 4 \rightarrow$ Internally Reversible Isentropic expansion
- $4 - 1 \rightarrow$ Internally Reversible Constant Volume Heat rejection

Clearance Ratio $C = \frac{v_2}{v_1 - v_2}$; $r = 1 + \frac{1}{C} = 1 + \frac{v_1}{v_2} - 1 = \frac{v_1}{v_2}$

Pressure ratio $r_P = \frac{P_3}{P_2} = \frac{P_4}{P_1} = \frac{T_3}{T_2} = \frac{T_4}{T_1}$

$\boxed{T_1T_3 = T_2T_4}$ and $\boxed{P_1P_3 = P_2P_4}$ and $\boxed{v_1v_3 = v_2v_4}$ and $\boxed{s_1s_3 = s_2s_4}$

$w_I = u_2 - u_1;\ w_O = u_3 - u_4$

$q_S = u_3 - u_2;\ q_R = u_4 - u_1$

$$\eta_{th,otto} = \frac{\sum w}{q_S} = \frac{(u_3 - u_4) - (u_2 - u_1)}{(u_3 - u_2)}$$

$$= 1 - \frac{(u_4 - u_1)}{(u_3 - u_2)} = 1 - \frac{T_1}{T_2} = 1 - \frac{1}{r^{\gamma-1}}$$

$$\Sigma q = q_S - q_R = (u_3 - u_2) - (u_4 - u_1) = u_1 - u_2 + u_3 - u_4$$

$$\Sigma w = w_O - w_I = (u_3 - u_4) - (u_2 - u_1) = u_1 - u_2 + u_3 - u_4$$

$$\Sigma q = \Sigma w$$

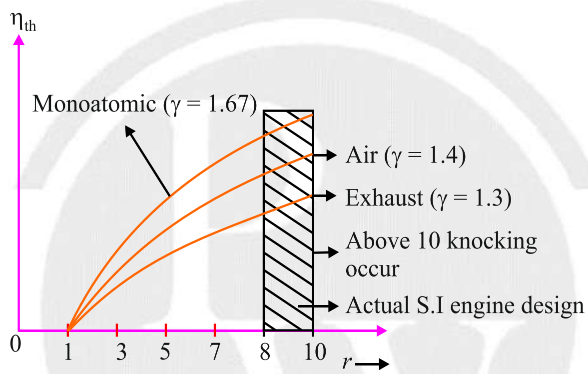


Fig. 1.2. Variation of thermal efficiency with working fluid

In general, η_{th} of spark ignition is from 25% to 30%

- Network output for Otto cycle

$$\Sigma w = w_O - w_I = (u_3 - u_4) - (u_2 - u_1)$$

$$= u_1 - u_2 + u_3 - u_4$$

$$= c_v(T_1 - T_2 + T_3 - T_4)$$

$$\therefore \quad \Sigma w = c_v.T_1(r^{\gamma-1} - 1)(r_P - 1)$$

$$= \frac{P_1v_1}{\gamma - 1}(r^{\gamma-1} - 1)(r_P - 1)$$

$$\boxed{\therefore\ \Sigma w = \frac{P_1v_1}{\gamma - 1}(r^{\gamma-1} - 1)(r_P - 1)}$$

For max. network

$$r^{\gamma-1} = \sqrt{\frac{T_{max}}{T_{min}}}$$

and $T_2 = T_4 = \sqrt{T_{max}.T_{min}}$

$$\Sigma w_{max} = c_v\left[\sqrt{T_{max}} - \sqrt{T_{min}}\right]^2$$

$T_1 = T_{min} \rightarrow$ Environmental constraint

$T_3 = T_{max} \rightarrow$ Material constraint

- Mean Effective Pressure: (P_{mean})

$$\Sigma w = w_O - w_I = P_{mean} \times v_s$$

$$\boxed{P_{mean} = \frac{P_1}{(\gamma - 1)}\left(\frac{r}{r-1}\right)\left[(r^{\gamma-1} - 1)(r_P - 1)\right]}$$

- In general, r of S.I is 7 to 10
  Tetra Ethyl Lead (TEL) can take it to 12; but exhaust + TEL is poisonous. So, limited.

1.3 Diesel Cycle

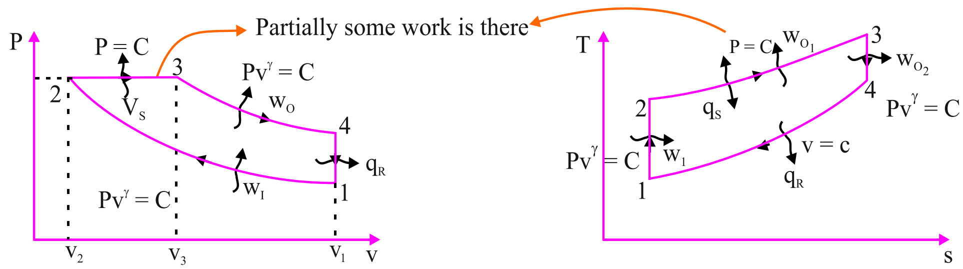


Fig. 1.3. P-v and T-s plots of Diesel cycle

Processes involved: $1-2 \rightarrow$ Internally Reversible Isentropic compression
$2-3 \rightarrow$ Internally Reversible Constant Pressure Heat addition
$3-4 \rightarrow$ Internally Reversible Isentropic expansion
$4-1 \rightarrow$ Internally Reversible Constant Volume Heat rejection

Clearance ratio $'C' = \dfrac{v_2}{v_1 - v_2}$; $r = 1 + \dfrac{1}{C} = \dfrac{v_1}{v_2}$

Pressure ratio $(r_p) = \dfrac{P_2}{P_1}$; cut off volume = $v_3 - v_2$

Cut off Ratio $(\rho) = \dfrac{v_3}{v_2}$; Expansion ratio $\varepsilon = \dfrac{v_4}{v_3}$

$r = \dfrac{v_1}{v_2}$; $\varepsilon = \dfrac{v_4}{v_3} = \dfrac{v_1}{v_3}$

$\because v_3 > v_2$

$\boxed{\Rightarrow r > \varepsilon}$: $\boxed{r = (r_P)^{1/\gamma}}$

$$\boxed{r = \frac{v_1}{v_2} = 1 + \frac{1}{C} = (r_P)^{1/\gamma} = \rho \times \varepsilon}$$

$$\eta_{th,D} = 1 - \frac{(u_4 - u_1)}{(h_3 - h_2)}$$

$$\eta_{th,D} = 1 - \frac{1}{r^{\gamma-1}}[F]$$

Where $F = \dfrac{\rho^{\gamma} - 1}{\gamma(\rho - 1)}$;

For same 'r'

$\rightarrow$ If F > 1 $\Rightarrow \eta_{otto} > \eta_{Diesel}$

As $\rho \uparrow$; $F \uparrow$

If $\rho = 1 \Rightarrow v_3 = v_2$

$\eta_{otto} = \eta_{Diesel}$

In general:

'r' for Diesel engines is 12 to 24 and η is between 35 to 40%

→ **Net Work**

$\Sigma w = \Sigma q = q_S - q_R$

$= (h_3 - h_2) - (u_4 - u_1)$

$$\boxed{\Sigma w = \frac{P_1 v_1}{\gamma - 1}\left[r^{\gamma-1}\{\gamma \,.\, (\rho - 1)\} - (\rho^{\gamma} - 1)\right]}$$

$$\boxed{p_m = \frac{P_1 r}{(\gamma - 1)(r - 1)}\left[r^{\gamma-1}\{\gamma(\rho - 1)\} - (\rho^{\gamma} - 1)\right]}$$

→ For same initial conditions and same pressure ratio,

on increasing Maximum temperature → **η_{th} will decrease**

→ Σ w increases slightly

But q'_s increases significantly.

$\therefore$ η_{th} **will** ↓

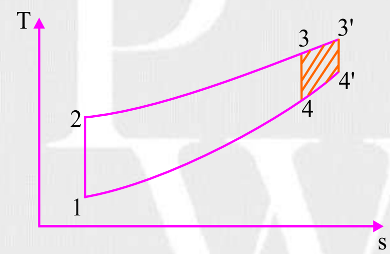
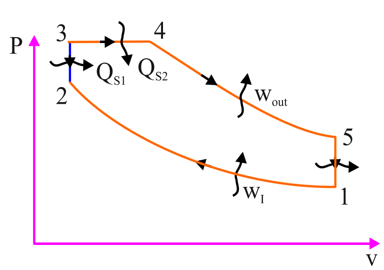


Fig. 1.4. Diesel cycle with increased T_{max}

So, As max. Temp ↑ ; η ↓

Same compression Ratio + any other condition $\Rightarrow \eta_{otto} > \eta_{Diesel}$
Same Maximum Pressure + any other condition $\Rightarrow \eta_{Diesel} > \eta_{otto}$

1.4 Dual cycle (Limited Pressure cycle (or) Mixed cycle)



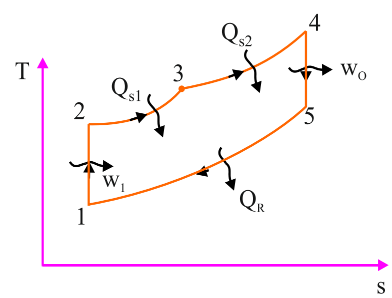


Fig. 1.5. P-v and T-s plots of Dual cycle

Processes involved:

1 – 2 → Internally reversible Isentropic compression
2 – 3 → Internally Reversible Isochoric Heat addition
3 – 4 → Internally Reversible Isobaric Heat Addition
4 – 5 → Internally Reversible Isentropic expansion
5 – 1 → Internally Reversible Isochoric Heat rejection.

$$\eta_{th} = \frac{\sum W}{Q_s} = \frac{\sum Q}{Q_s} = \frac{Q_{s,1} + Q_{s,2} - Q_R}{Q_{s,1} + Q_{s,2}}$$

$$\eta_{th,\ Cycle} = \frac{mc_v(T_3 - T_2) + mc_p(T_4 - T_3) - mc_v(T_5 - T_1)}{mc_v(T_3 - T_2) + mc_p(T_4 - T_3)}$$

When P_3 tends to P_2, $Dual \to$ Diesel

When V_4 tends to V_3, Dual → Otto

For same compression ratio → $\eta_{otto} > \eta_{Dual} > \eta_{Diesel}$
For same Peak Pressure → $\eta_{Diesel} > \eta_{Dual} > \eta_{otto}$

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2 GAS POWER CYCLES

2.1 Introduction

2.1.1 Gas Power Cycles

Gas Turbine

Rotodynamic machine that converts thermal Energy of gas into Mechanical work.

Closed Cycle Gas Turbine

Joule Proposed it. The main components of a power plant working on Joule cycle are

1. Compressor   2. Heat Exchange   3. Turbine   4. Inter Cooler

$\downarrow$ (Inter Cooler) Decrease Specific Volume of Working fluid.

Reason for Intercooling

The lower value of specific volume Decrease the compressor work $(-\int vdp)$

Open Cycle Gas Turbine:

In Brayton cycle, combustion supplies the heat required. (4 –1 Process is heat rejection to atmosphere at constant pressure).

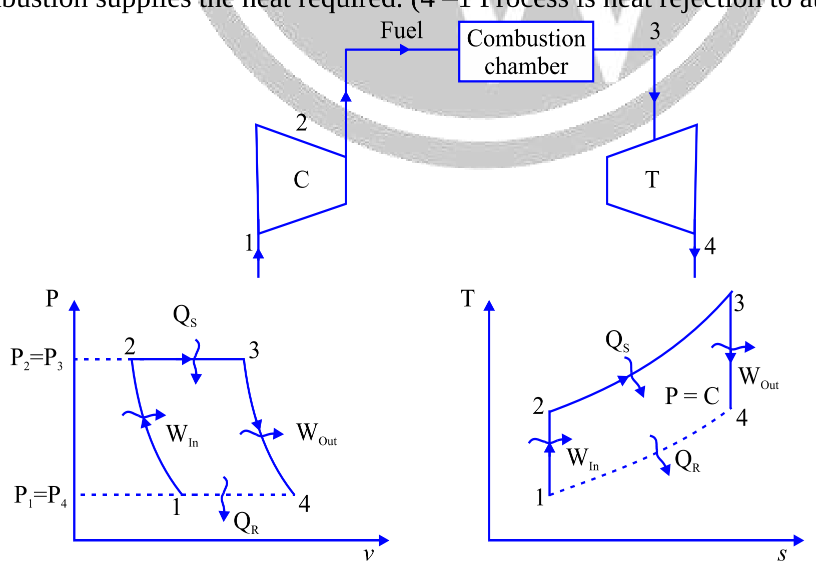

Fig. 2.1. Components of open cycle gas turbine and its graph on P-v and T-s plots

$$W_{In} = C_P(T_2 - T_1); \quad Q_S = C_P(T_3 - T_2); \quad W_{Out} = C_P(T_3 - T_4)$$

$$W_{net} = \Sigma Q = C_P(T_3 - T_4 - T_2 + T_1)$$

$\boxed{P_1P_3 = P_2P_4}$; $\boxed{T_1T_3 = T_2T_4}$;

$$\eta_{th} = 1 - \frac{1}{(r_P)^{\left(\frac{\gamma-1}{\gamma}\right)}} \quad \text{where } r_P = \frac{P_2}{P_1} = \text{Pressure ratio.}$$

$\therefore$ η_{joule} increases when γ increases and r_P increases

2.1.2 Back Work Ratio: (B.W.R)

$$\text{B.W.R} = \frac{W_C}{W_T} = \frac{C_P(T_2 - T_1)}{C_P(T_3 - T_4)} = \frac{T_2 - T_1}{T_3 - T_4} \quad \{\text{Generally around } 0.8\}$$

2.1.3 Work Ratio: (W.R)

$$\text{W.R} = \frac{W_{net}}{W_T} = \frac{W_T - W_C}{W_T} = 1 - \frac{W_C}{W_T} = 1 - \text{B.W.R.}$$

$\Rightarrow$ $\boxed{\text{Work Ratio + Back Work Ratio = 1}}$

2.1.4 Condition for maximum work output in Brayton Cycle for given maximum and minimum temperatures.

$$r_P = \frac{P_2}{P_1} \text{ and } W_{net} = C_P\{T_3 - T_4 - T_2 + T_1\}$$

For max. work: $$\frac{dW_{net}}{dr_P} = O \Rightarrow \frac{T_{max}}{T_{min}} = (r_P)^{2\left(\frac{\gamma-1}{\gamma}\right)}$$

So, for Max. work output.

$$\boxed{\frac{T_{max}}{T_{min}} = (r_P)^{\frac{2(\gamma-1)}{\gamma}}} \quad \text{and} \quad \boxed{(r_P)_{optimal} = \left(\frac{T_{max}}{T_{min}}\right)^{\frac{\gamma}{2(\gamma-1)}}}$$

and $T_2 = T_4 = \sqrt{T_1.T_3}$ and $\boxed{W_{max} = C_P.\left\{\sqrt{T_{max}} - \sqrt{T_{min}}\right\}^2}$

$$\boxed{\eta = 1 - \sqrt{\frac{T_{min}}{T_{max}}}}$$

2.1.4 Actual Gas Turbine Cycle

In Practice, compression & expansion at such high temperature are not Isentropic

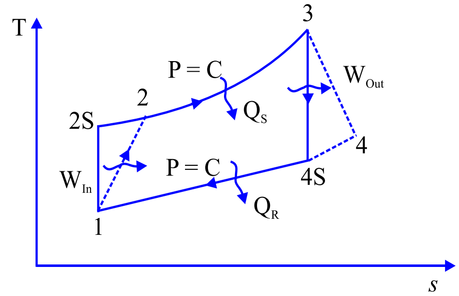

Fig. 2.2 T-s diagram of Actual gas turbine cycle

$$\eta_{compressor} = \frac{T_{2S} - T_1}{T_2 - T_1}; \quad \eta_{turbine} = \frac{T_3 - T_4}{T_3 - T_{4S}}$$

$$W_{net} = C_P \cdot \eta_{turbine}(T_3 - T_{4S}) - \frac{C_P}{\eta_{compressor}}(T_{2S} - T_1)$$

For Max. Work output.

$$\frac{dW_{net}}{dr_P} = O \quad \Rightarrow \quad (r_P)_{optimal} = \left\{\eta_T \cdot \eta_C \left(\frac{T_{max}}{T_{min}}\right)\right\}^{\frac{\gamma}{2(\gamma-1)}}$$

2.2 Methods for improvement in Performance of open cycle Gas Turbine:

2.2.1 Regeneration

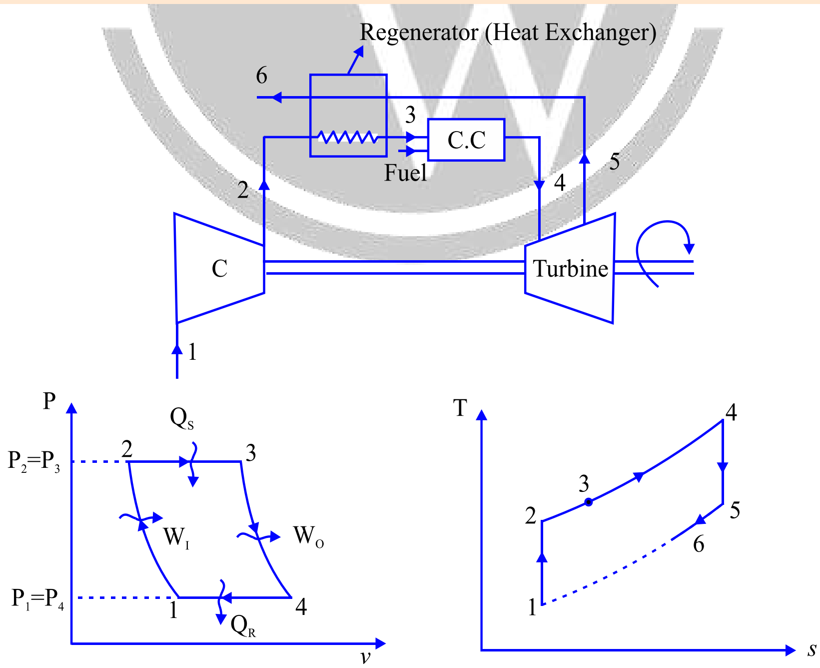

Fig. 2.3. Components of regenerative open cycle gas turbine and its graph on P-v and T-s plot

Temperature of gases leaving the turbine is very high. A counter flow heat exchanger called Regenerator is used to transfer the heat available in exhaust gases to the intake air. This results in decrease in the heat supply in combustion chamber for the same work output from the turbine. So $\eta_{th}\uparrow$.

- Regeneration is possible only when $T_5 > T_2$. {Turbine exit Temp > Compressor exit Temp)

Effectiveness of Regeneration: $\varepsilon = \dfrac{C_P(T_3 - T_2)}{C_P(T_5 - T_2)}$

i.e. $\varepsilon = \dfrac{\text{Actual H.T to air}}{\text{Max. Possible H.T to air}}$

$\therefore$ For regenerative cycle: $\eta_{th} = \dfrac{W_{net}}{Q_{3-4}}$;

For ideal Regeneration case: $\boxed{\eta_{th} = 1 - \dfrac{(r_P)^{\frac{\gamma-1}{\gamma}}}{\left(\dfrac{T_{max}}{T_{min}}\right)}}$

For a given cycle

This is in contradiction to Brayton Cycle.

When $r_P = 1$; $\boxed{\eta_{Regeneration} = \eta_{carnot}}$

In Regeneration, Heat supply to the system decreases for same work output from the cycle. So thermal efficiency increases.

2.2.2 Reheating

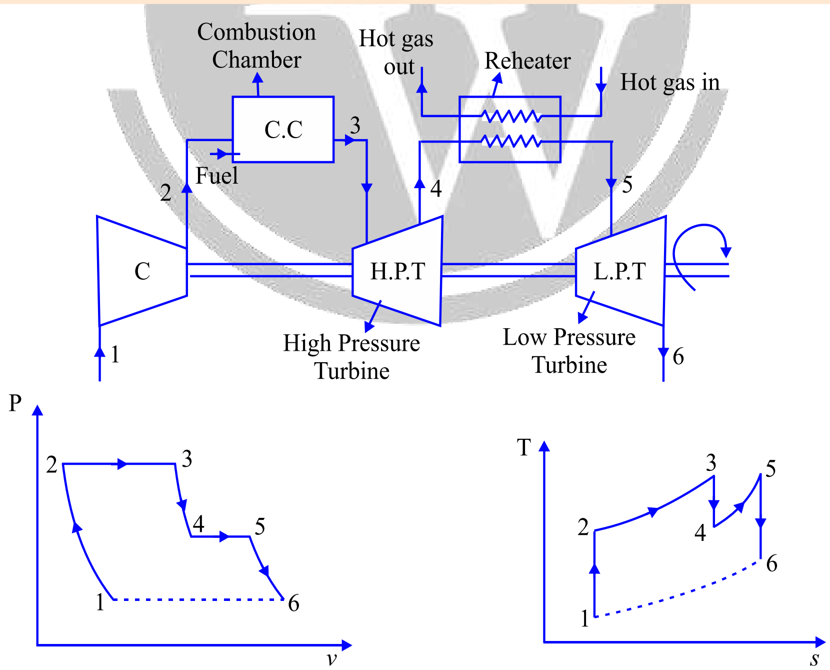

Fig. 2.4. Components of open cycle gas turbine with reheating and its graph on P-v and T-s plot

$$W_{net} = C_P(T_3 - T_4) + C_P(T_5 - T_6) - C_P(T_2 - T_1)$$
$$Q_S = C_P(T_3 - T_2) + C_P(T_5 - T_4).$$
Here $W_{net} \uparrow; Q_s \uparrow; \eta \downarrow$

2.2.3 Intercooling

The cooling of air between two compressors is called Intercooling.

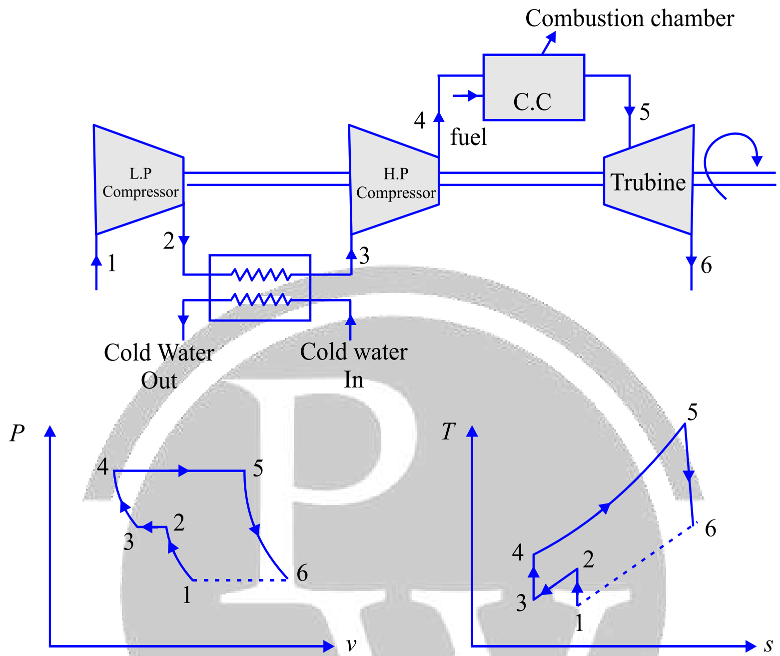

Fig. 2.5 Components of open cycle gas turbine with intercooling and its graph on P-v and T-s plot

$$W_{net} = C_P(T_5 - T_6) - C_P(T_4 - T_5) - C_P(T_2 - T_1)$$
$$Q_S = C_P(T_5 - T_4)$$

Hence, $W_{net} \uparrow$; $Q_s \uparrow$; $\eta \downarrow$ as mean temperature of heat addition decreases.

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3 Vapour Power Cycles

3.1 Introduction

Cycles are required to serve the need of continuous conversion of Heat to Work.

Phase changes some process

- Vapor Power Cycle
- Gas Power Cycle

No Phase change through out

Based on the phase of the working system

3.1.1 Different components of a Thermal Power Plant

Basic Vapour powerplant operates on the principle of Rankine cycle and the basic components of the cycle are

(i) Turbine
(ii) Condenser
(iii) Feed Pump
(iv) Steam Generator (Boiler and Superheater)

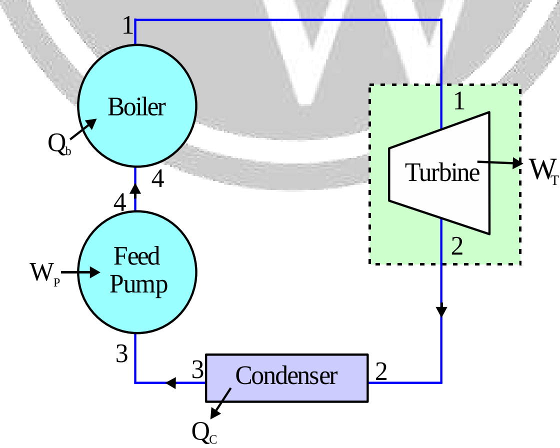


Fig. 3.1. Components of Vapour Power Cycle

(i) Turbine is insulated to avoid unnecessary loss of heat. So (1-2) will be isentropic

(ii) State 1 is at dry saturated state under ideal conditions.

(iii) Isentropic compression is most practically feasible compression process involving minimum work.

So, 3-4 is assumed as isentropic compression.

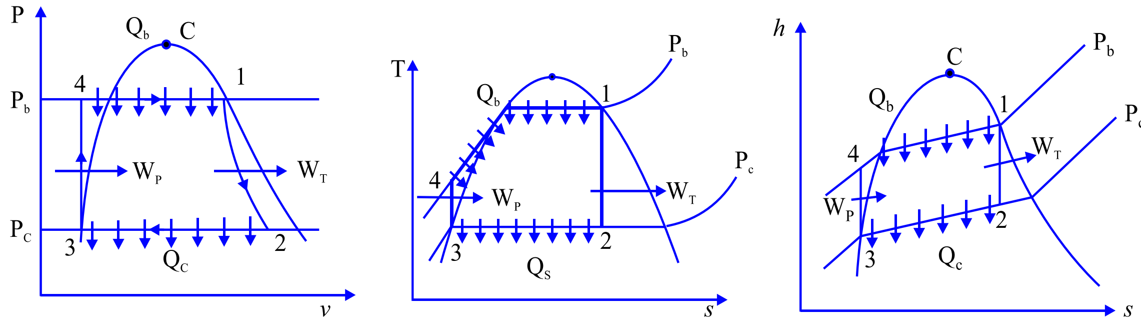


Fig. 3.2 P-v, T-s and h-s diagrams of vapour power cycle with no superheating

The line 3-4 in the P- *v* is almost straight because pumps handle liquids and liquids are practically incompressible (Constant specific Volume).

Applying S.F.E.E. for different components individually.

We find, $W_T = h_1 - h_2; W_P = h_4 - h_3$ (W_P →work required per unit mass of working fluid)

$Q_b = h_1 - h_4$ and $Q_C = h_2 - h_3$ (Q_b and Q_c are heat added and rejected respectively per unit mass of working fluid)

$$\text{So,}\quad \eta_{th} = \frac{W_{net}}{Q_b} = \frac{W_T - W_P}{Q_b} = \frac{(h_1 - h_2) - (h_4 - h_3)}{(h_1 - h_4)} = 1 - \frac{(h_2 - h_3)}{(h_1 - h_4)}$$

$$\eta_{th} = \frac{W_{net}}{Q_b} = 1 - \frac{Q_C}{Q_b}.$$ So, h_1, h_2, h_3, h_4 are to be known from the steam tables.

In process 3 – 4, since the specific volume of liquids is very small, we generally neglect pump work. So, in general $h_3 \approx h_4$.

3.1.2 Specific Steam Consumption

The mass of steam required to produce 1 unit of power.

$$\boxed{\text{S.S.C.} = \frac{3600}{W_{net}} \frac{\text{Kg}}{\text{kW} - \text{hr}}}$$

where W_{net} is in kJ/kg.

3. 2 Practicalities of Rankine Cycle

3. 2. 1 Compression and Expansion Process are not adiabatic in general

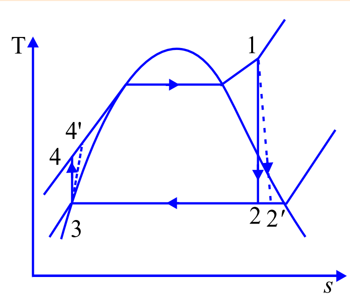


Fig. 3.3 Practical compression and expansion processes in Rankine cycle

Practically 1-2 irreversible and can be approximated to be adiabatic since there is always some possible heat loss from the turbine for a temperature difference with surroundings.

Similarly, there can be irreversibility in feed pump. So, it can end at 4'.

Now considering turbine as control volume.

$$W_T = h_1 - h_2'; \ (W_T)_{ideal} = h_1 - h_2 \ \text{and} \ h_2' > h_1 \text{(always)}.$$

(Because of irreversibility friction increases intermolecular energy and it increases temperature. which increases enthalpy)

$\Rightarrow \quad W_T < (W_T)_{ideal}$ $\quad \boxed{\therefore \ \eta_{\substack{\text{Turbine,}\\ \text{isertropic}}} = \dfrac{h_1 - h_2'}{h_1 - h_2}}$

Actual pump work; $W_P = h_4' - h_3$ (Work needed per unit mass of working fluid)

Ideal Pump work $\Rightarrow (W_P)_{actual} = h_4 - h_3$ and $h_4' > h_4$

$$\boxed{\therefore \ \eta_{\substack{\text{Pump}\\ \text{isertropic}}} = \frac{h_4 - h_3}{h_4' - h_3}}$$

3. 2. 2 Pressure Drops in Boiler

This is due to certain heat loss from the boiler. (But in practice it is very low. So, it can be neglected).

3.3 Methods to increase efficiency of Rankine cycle:

3. 3. 1 Increasing the Mean temperature of Heat Addition

For a heat rejection at temperature T_2 and Mean temperature of Heat addition T_{M_1}, the thermal efficiency of Rankine cycle is given by

$$\eta = 1 - \frac{T_2}{T_{M_1}}$$

So, to increase η we can decrease T_2 (or) increase T_{M1}. But T_2 is fixed by the ambient and condenser operating conditions. ∴ we can increase T_{M1} by super heating.

Increasing Boiler Pressure

On increasing the boiler pressure, the temperature of phase change increases, So the mean temperature of heat addition also increases.

This continuous increase in P_b is restricted because the expansion reduces the dryness fraction of steam.

As P_b increases, dryness fraction of steam after expansion decreases.

But as x decrease, water particles in steam increases, thus increase erosion of blade materials, which can cause drastic damage to turbine. So, x has minimum limitation.

3.3.2 Why condenser is used in steam power plant.

We have $\eta = 1 - \dfrac{T_2}{T_M}$ and as T_2 decrease $\Rightarrow \eta$ increases.

Generally, condensers are operated at pressures lower than the atmospheric pressures. Thus, condensation at lower pressures reduces the phase change (or) heat rejection temperature which would have been normal atmospheric temperature in the absence of the condenser. Thus, condensers decrease the T_2 value and increase the efficiency of the plant.

3.4 Methods to improve the performance of Rankine Cycle

3. 4. 1 Concept of Reheat in a Rankine Cycle:

In reheating the work output is increased without sacrificing dryness fraction.

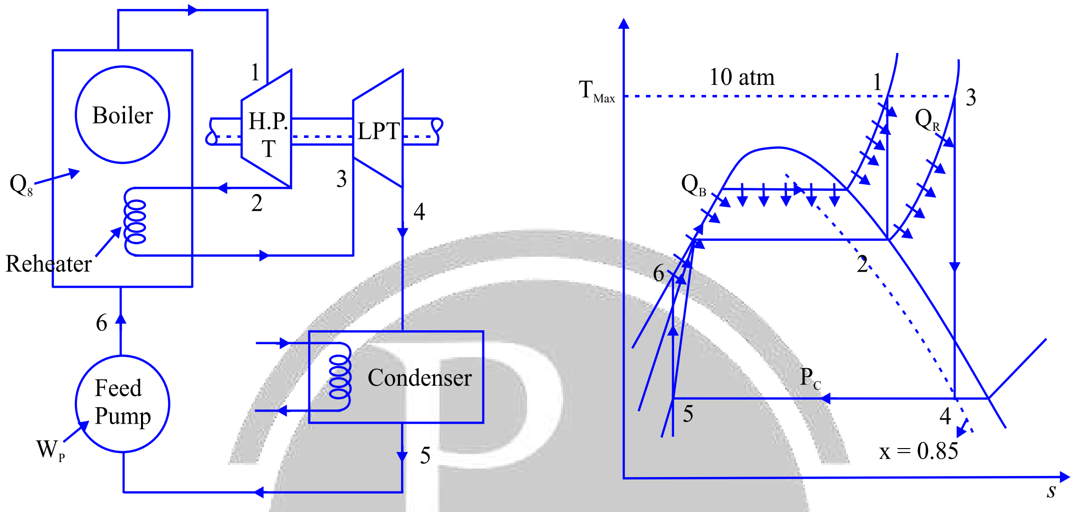


Fig. 3.4. Components of Rankine cycle with reheating and its graph on T-s plot

Heat supply Q increase, Work done also increase

But we can't comment on 'η' (Depends on T_M of 6 to 1) and (2 to 3)

Specific heat added = $(h_1 - h_6) + (h_3 - h_2)$

Specific heat Rejected = $(h_4 - h_5)$

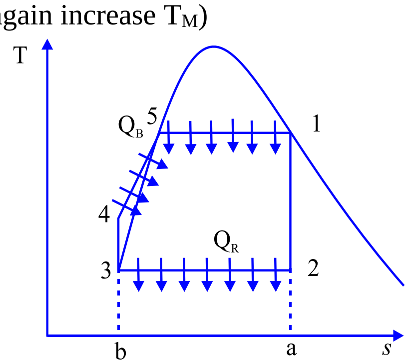
Specific work done by the turbines = $(h_1 - h_2) + (h_3 - h_4)$, specific work input to the pumps can be neglected because of very less specific volume of the liquid.

$$\therefore \eta_{Thermal} = \frac{W_T}{Q_S} = \frac{(h_1 - h_2) + (h_3 - h_4)}{(h_1 - h_6) + (h_3 - h_2)}$$

We cannot comment on 'η' until operating conditions are given.

3. 4. 2 Concept of Regeneration:

Need for Regeneration (Principle is to again increase T_M)



The working substance which is in liquid state is heated to state 5 from 4 using the steam that is bled out from the turbine. So, Heat is added from outer source only from point 5 to 1. So, the Heat addition process is obtained to be a near isothermal Heat addition. So, it gets closer to Carnot cycle analysis.

By regeneration, W_{net} decreases because the turbine work decreases and heat supply also decreases. But efficiency of the cycle increases because of increase in the mean temperature of heat addition.

3.5 Practical Circuit for Regeneration:

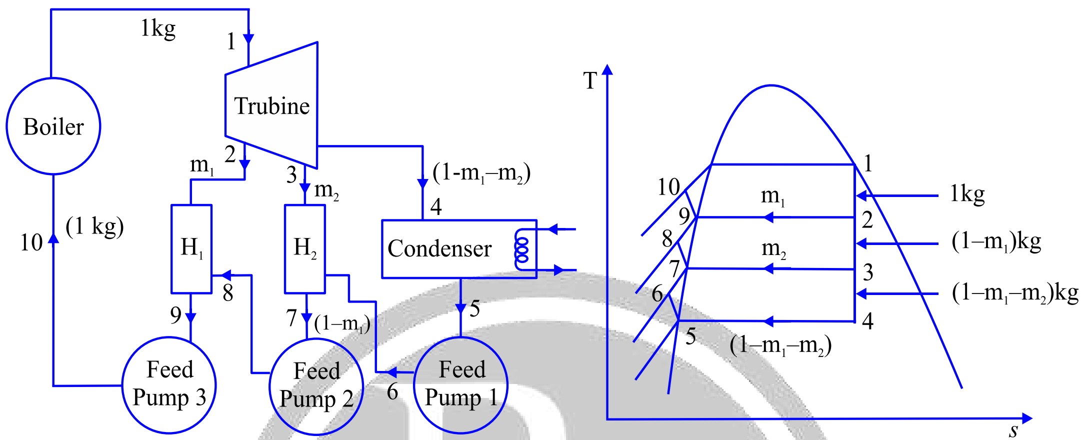


Fig. 3.5 Components of regenerative Rankine cycle and its graph on T-s plot

$$W_T = (h_1 - h_2) + (1 - m_1)(h_2 - h_3) + (1 - m_1 - m_2)(h_3 - h_4)$$

$$Q_{added} = (h_1 - h_{10}); \eta = \frac{W_T - W_P}{Q_{added}}$$

$$W_P = (1 - m_1 - m_2)(h_6 - h_5) + (1 - m_1)(h_8 - h_7) + 1\,.\,(h_{10} - h_9)$$

3.5.1 Calculation of m_1, m_2

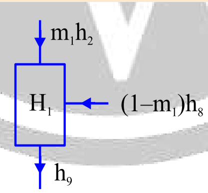


Fig. 3.6 Flow of working fluid through open feed water heater 1

$\Rightarrow \quad m_1h_2 + (1 - m_1)h_8 = h_9$

$\Rightarrow \quad m_1(h_2 - h_8) = h_9 - h_8$

$\Rightarrow \quad \boxed{m_1 = \frac{h_9 - h_8}{h_2 - h_8}}$

$$\boxed{h_8 - h_7 = v_7(P_8 - P_7)}$$

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4 REFRIGERATION AND AIR CONDITIONING

4.1 Introduction

(a) Refrigeration

- The process of producing and maintaining a temperature lower than that of surroundings.

(b) Air-Conditioning

- The process of treating and thus simultaneously controlling the properties of air like temperature, moisture etc.

(c) Cooling by salt solutions

Chemical Reactions
- Exothermic $\rightarrow$ (Liberate Heat)
- $\boxed{\text{Endothermic}}$ $\rightarrow$ (Absorb Heat)

$NaCl + H_2O \longrightarrow$ Capable of Producing – 10°C.

NaCl: Sodium Chloride

$CaCl_2 + H_2O \longrightarrow$ Capable of Producing – 50°C.

$CaCl_2$: Calcium Chloride

(d) Artificial Refrigeration

- Vapour Refrigeration Systems
  - Vapour compression refrigeration system
  - Vapour absorption refrigeration system
- Gas Refrigeration System

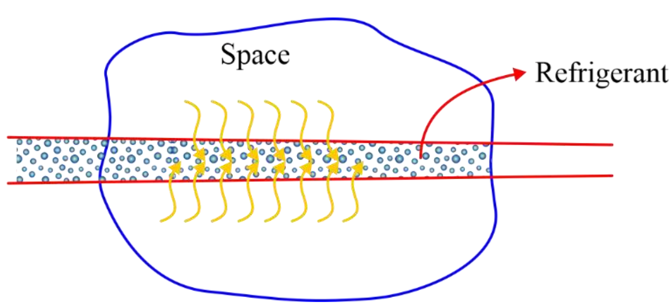


Fig.4.1 Heat absorption by the refrigerant from cooling space

4.2 Refrigeration cycles

The basic components of a refrigeration cycle include-

(a) Evaporator (b) Compressor (c) Condenser (d) Expansion device.

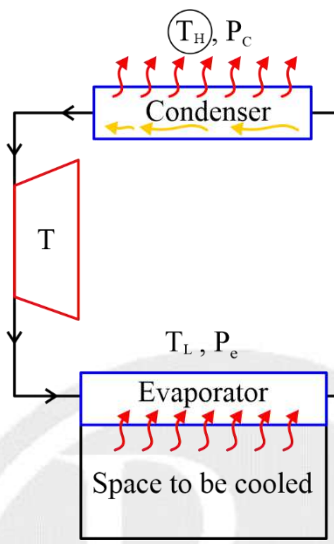


Fig.4.2 Basic components of Refrigeration cycle

4.2.1 Reversed Carnot cycle

Reversed Carnot cycle involves isentropic compression and expansion of the working fluid (refrigerant) and isothermal Heat addition and Heat rejection Processes as shown below.

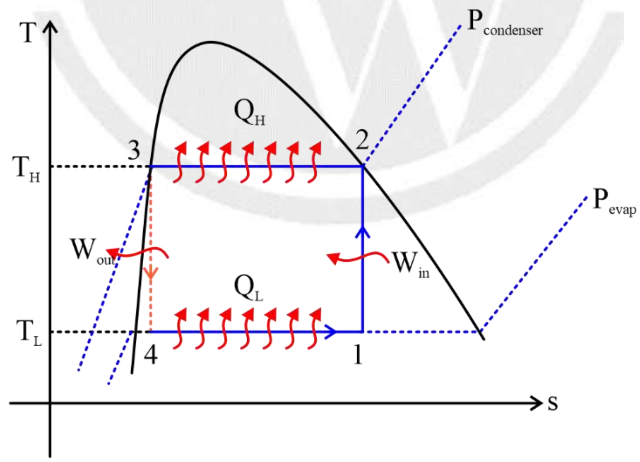


Fig.4.3 T-s diagram of reversed Carnot cycle

4.2.2 Limitations of Reversed Carnot Cycle

Expansion in turbine is practically highly uneconomical because state 3 is saturated liquid for which specific volume is very low and presence of liquid particles will erode turbine blades.

State-1: Wet vapour (Two phase mixture) so compression from 1-2 is difficult.

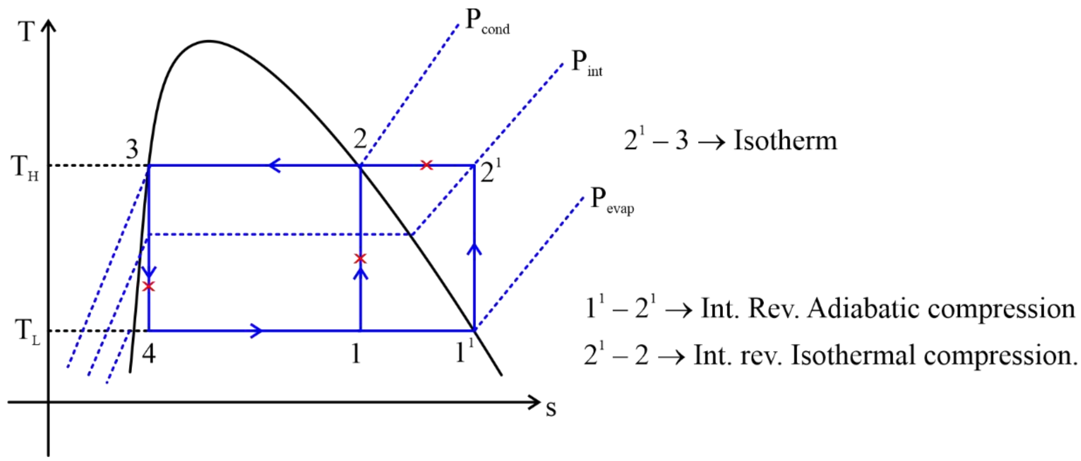
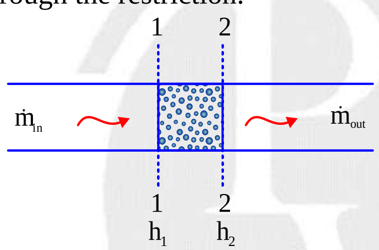


Fig.4.4 T-s diagram of different reversed Carnot cycles

4.3 Throttling Devices

- Throttling devices are used to drop the pressure of the fluid.
  Throttling → Passing through the restriction.



Applying SFEE for the porous plug,

$\dot{E}_{in} = \dot{E}_{out}$

$\Rightarrow \cancel{\dot{Q}_{CV}}^{0} - \cancel{\dot{W}_{CV}}^{0} = \Delta \cancel{\dot{KE}}^{0} + \Delta \cancel{\dot{PE}}^{0} + \Delta \dot{H}$

$\Rightarrow \Delta \dot{H} = 0$

$\Rightarrow \cancel{\dot{m}}(h_2 - h_1) = 0$

Fig.4.5 Throttling device

Throttling through small devices is generally an isenthalpic process.

4.3.1 Joule Thomson Coefficient

- $\mu_{JT} = \left.\dfrac{\partial T}{\partial p}\right]_h$ → Pressure drops

  h = h (T, P) → For any substance

$$dh = cpdT + \left[\upsilon - T\left(\frac{\partial \upsilon}{\partial T}\right)_p\right]dp$$

$$\Rightarrow -c_p dT = \left[\upsilon - T\left(\frac{dv}{\delta T}\right)_p\right]dp$$

$$\Rightarrow \left.\frac{\downarrow dT}{\downarrow dp}\right|_h = \frac{1}{c_p}\left[T\left(\frac{\partial \upsilon}{\partial T}\right)_P - \upsilon\right] = \mu_{JT}$$

In refrigeration cycles, during throttling of refrigerant, the refrigerant temperature should also decrease along with pressure. So (μ_{JT}) of refrigerant (non-ideal fluid) should be positive since both the changes are negative.

4.4 Standard Vapour Compression Refrigeration System (VCRS)

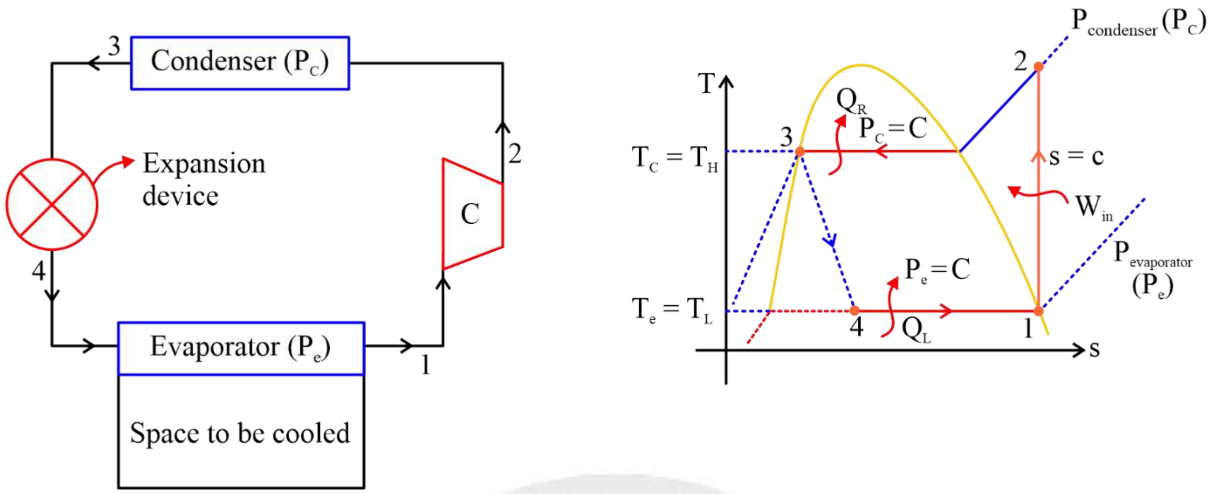


Fig.4.6 Components of standard VCRS cycle and its T-s diagram

- Processes involved in standard VCRS

  1-2: Internally Reversible adiabatic compression

  2-3: Internally Reversible Isobaric heat rejection

  3-4: Irreversible expansion in throttling device

  4-1: Internally reversible isothermal heat addition

  Refrigeration effect = $\dot{m}_{\text{ref}}\left(h_1 - h_4\right) = \dot{V}\dfrac{\left(h_1 - h_4\right)}{v}$

$$= \dot{V}\frac{(h_1 - h_4)}{v} = \dot{V}_1 \boxed{\frac{(h_1 - h_4)}{v_1}} = \text{volumetric refrigeration effect}$$

4.4.1 Coefficient of performance

$$(\text{COP}) = \frac{\text{Desired output}}{\text{Required Input}} = \frac{\text{Refrigeration effect}}{\text{Work input to the compressor}} = \frac{\dot{m}_{ref}(h_1 - h_4)}{\dot{m}_{ref}(h_2 - h_1)}$$

$$since\ h_4 = h_3, COP = \frac{h_1 - h_4}{h_2 - h_1} = \frac{h_1 - h_3}{h_2 - h_1}$$

$$\therefore (COP)_{std.VCRS} = \frac{h_1 - h_3}{h_2 - h_1}$$

4.4.2 P-h diagram of standard VCRS

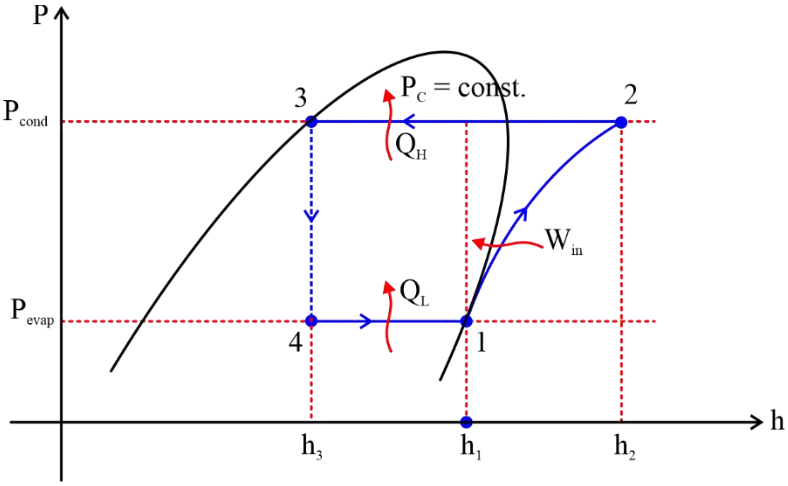


Fig.4.7 P-h diagram of standard VCRS

From Tds equation, $Tds = dh - vdP$

For isentropic process (ds = 0)

$\Rightarrow dh = vdP$

$\Rightarrow \frac{dP}{dh} = \frac{1}{v}$

4.4.3 Reversed Carnot Vs standard VCRS

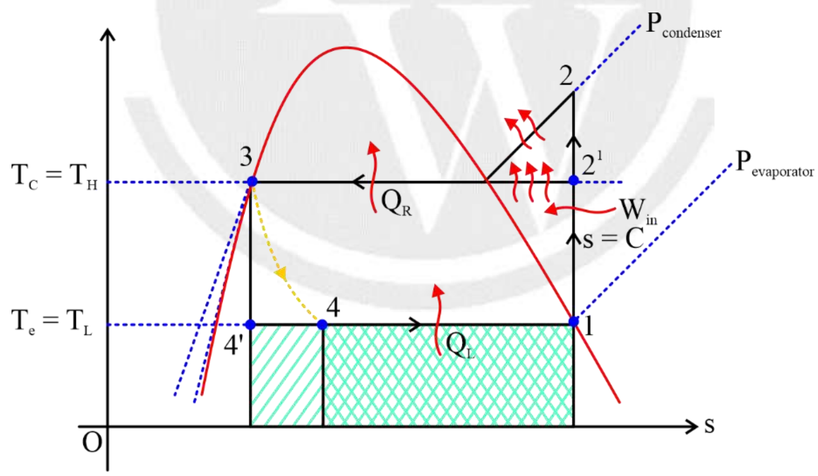


Fig.4.8 Comparison of reversed Carnot and standard VCRS on T-s graph

$(\text{Ref.effect})\big|_{\substack{\text{std.}\\ \text{VCRS}}} < \text{Ref.effect}\big|_{\substack{\text{Rev}\\ \text{Carnot}}}$

Ref. effect in Revised Carnot = $h_1 - h_4, kJ/kg = T_L(s_1 - s_4)$

Ref. effect in Standard VCRS = $h_1 - h_4, kJ/kg = T_L(s_1 - s_4)$

$$(COP)_{\substack{\text{Rev.}\\\text{Carnot}}} = \frac{T_L}{T_H - T_L} = \frac{1}{\left(\frac{T_H}{T_L}\right) - 1}$$

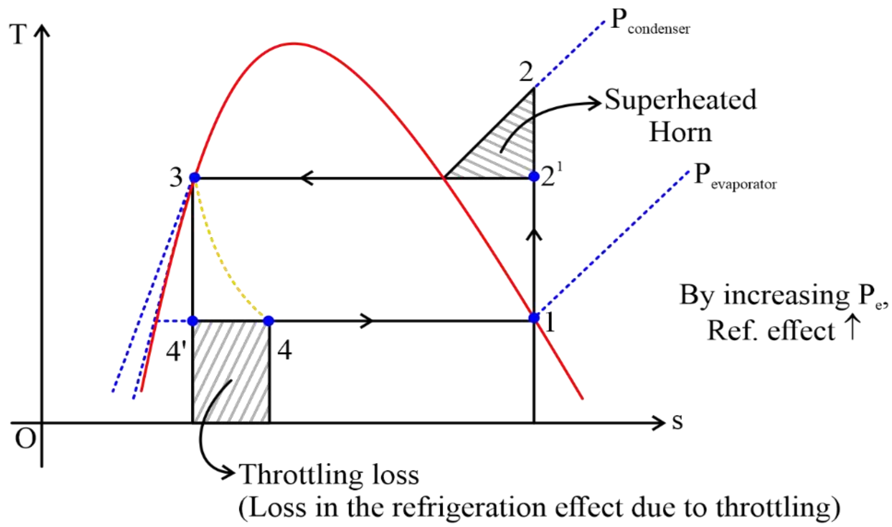


Fig.4.9 Throttling loss and superheated horn in VCRS on T-s graph

4.5 Gas Refrigeration Systems (Rev. Carnot cycle)

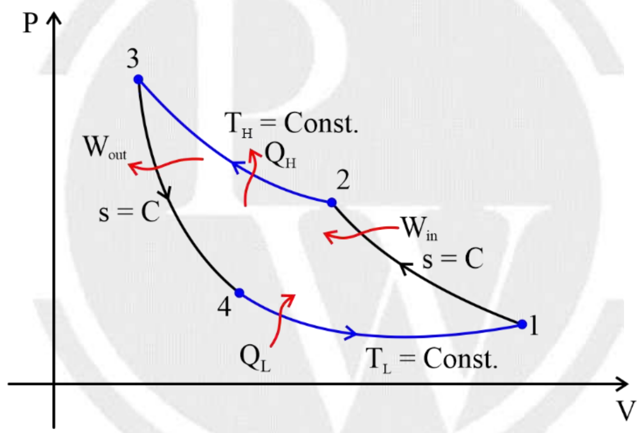


Figure 4.10 P-V diagram of reversed Carnot cycle

- Process:

  1-2: Internally reversible adiabatic compression
  2-3: Internally reversible isothermal heat rejection
  3-4: Internally reversible adiabatic expansion
  4-1: Internally reversible isothermal heat addition.

4.5.1 Reversed Brayton Cycle (or) Joule Cycle (or) Bell-Coleman Cycle

Since isothermal heat additions and heat rejections are practically difficult to achieve at high speeds of compressor, heat interactions are replaced by Isobaric processes with air as working fluid.

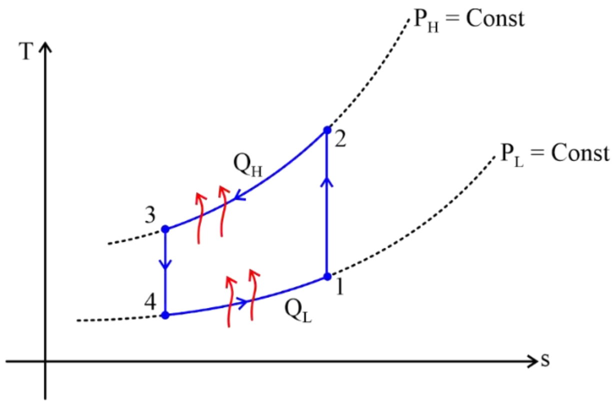


Fig.4.11 Bell- Coleman cycle

Refrigeration effect = $\dot{m}_{ref}(h_1 - h_4)$

$$= \dot{V}_1 \boxed{\left(\frac{h_1 - h_4}{v_1}\right)} \rightarrow \text{volumetric refrigeration capacity}$$

4.6 Refrigerants

Refrigerants

Working fluids in the refrigeration cycle.

Refrigerants → Saturated $\boxed{\text{halogenated}}$ refrigerants (F, Cl) (Alkane refrigerants) C_nH_{2n+2}

Refrigerants → Inorganic refrigerants

4.6.1 Designation of a refrigerant

Any organic refrigerant is generally designated as R-XYZ

Number of carbon atoms = X+ 1

Number of hydrogen atoms = Y – 1

Number of Fluorine atoms = Z

Remaining atoms are chlorine atoms = (2x + 4) – (y – 1) – z

Ex.

(i) R-1 3 4

Number of C atoms = x + 1 = 1 + 1 = 2

Number of H atoms = y – 1 = 3 – 1 = 2

Number of F atoms = 4

R – 134 → $C_2H_2F_4$ (Tetra fluoro ethane)

(ii) R- 0XY

R 012⇒ Number of C atoms = 0 + 1 = 1

Number of H atoms = 1 – 1 = 0

Number of F atoms = 2

Number of Cl atoms = 2

Inorganic Refrigerants:

R-7XY

XY $\rightarrow$ Molecular weight of the refrigerant.

$NH_3 \rightarrow$ R717

$H_2O \rightarrow$ R718

4.7 Unit of Refrigeration

1 Tonn of refrigeration = 211 kJ/min of heat removal from a space.

$$= 211\frac{kJ}{60\sec}$$

$$= \frac{211}{60}kW$$

1 Tonn of refrigeration = 3.516 Kw

4.8 Air Conditioning

- **Air-Conditioning:** The process of treating and thus simultaneously controlling the properties of air like temperature, moisture content etc.

Air
- Dry air $\rightarrow$ Mixture of gases (N_2 + O_2)
  - N_2: 78% by vol
  - O_2: 21% by vol
- Atmospheric air $\rightarrow$ Mixture of gases (N_2 + O_2 + other gases) + $\boxed{\text{Water vapour.}}$
  - Water vapour: Generally available in superheated vapor state

- **Daltons law of partial pressure:**

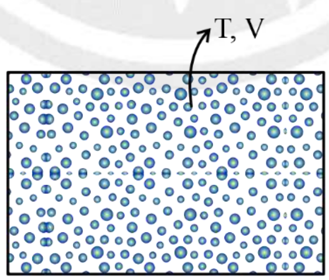


Fig.4.12 A volume containing different gases at temperature T

$$P_{tot}(T,V) = P_1(T,V) + P_2(T,V) + P_3(T,V)$$

$$\boxed{PV = nR_uT}$$

$$P_i = \overline{x}_i.P_{tot}$$

$$n_{tot} = n_1 + n_2 + n_3$$

$$\frac{P_{tot}.V}{P_1.V} = \frac{n_{tot}.R_u.T}{n_1.R_u.T}$$

$$\Rightarrow \frac{P_1}{P} = \frac{n_1}{n_{tot}} \rightarrow \text{Mole fraction of gas- 1}$$

$$\Rightarrow \frac{P_1}{P_{tot}} = \bar{x}_1 \Rightarrow P_i = \bar{x}_1 P_{tot}$$

Atmospheric Air $\rightarrow$ $\underset{\text{Ideal gas}}{\text{Dry air}}$ + $\underset{\text{Ideal gas}}{\text{Water Vapour}}$

$h_a(T°C) = 1.005T(°C)$

$h_a(0°C) \approx 0$ (Reference)

Mol. Wt. of air $= \frac{100}{\left(\frac{78.5}{28}\right)+\left(\frac{21.5}{32}\right)} = 28.9$ kg/kmol

$dh_v = c_P dT$

$\Rightarrow h_v(T°C) - h_v(0°C) = 1.82(T - 0)$

$\Rightarrow h_v(T°C) = 2500.9 + 1.82T(°C)$

Mol. Weight of vapour = 18 kg/kmol.

4.8.1 Enthalpy of moist air

$H = H_a + H_v$

$\Rightarrow H = m_a.h_a + m_v.h_v$

For 1 kg of dry air

$\Rightarrow \frac{H}{m_a} = \frac{m_a}{m_a}.h_a + \frac{m_v}{m_a}.h_v$

$\Rightarrow h = h_a + \left(\frac{m_v}{m_a}\right).h_v$

$h(T°C) = 1.005.T(°C) + \left(\frac{m_v}{m_a}\right)[2500.9 + 1.82.T(°C)]$

$$\Rightarrow \boxed{h(T°C) = 1.005.T(°C) + \omega[2500.9 + 1.82.T(°C)]}$$

4.8.2 Specific Humidity (or) Absolute Humidity

- The mass of vapor present per kg of dry air is called specific humidity and it is denoted by 'ω.'
  It is given by

$\omega = \frac{m_v}{m_a} = \frac{\left(\frac{P_v.V}{R_v.T}\right)}{\left(\frac{P_a.V}{R_a.T}\right)}$

$\Rightarrow \omega = \frac{P_v}{P_a}.\frac{R_a}{(R_v)} \Rightarrow \omega = \frac{\left(\frac{R_u}{MW_a}\right)}{\left(\frac{R_u}{MW_v}\right)} \times \frac{P_v}{P_a}$

$\Rightarrow \omega = \left(\frac{MW_v}{MW_a}\right) \times \frac{P_v}{P_a} = \left(\frac{18}{28.9}\right) \times \frac{P_v}{P_a} = 0.622\frac{P_v}{P_a}$

$\therefore \omega = 0.622\frac{P_v}{P_a}$

$$\Rightarrow \boxed{\omega = 0.622\frac{p_v}{p_{tot} - p_v}}$$

4.8.3 Relative Humidity (ϕ)

- The ratio of mass of water vapour present in the air to the maximum amount of vapor that the air can with stand at the same condition.

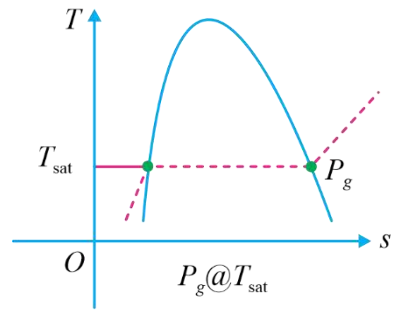


Fig.4.13 Pressure of saturated vapour at T_{sat}

It is denoted by ϕ and is given by

$$\phi = \frac{m_v}{(m_v)_{max}} = \frac{\left(\frac{P_v.V}{R_vT}\right)}{\left(\frac{P_g.V}{R_v.T}\right)}$$

$\Rightarrow$ $\boxed{\phi = \frac{p_v}{p_g}}$ where $P_g = P_{sat}$ at given T°C

As air is heated, ϕ decreases

$$h = 1.005.T(°C) + \frac{m_v}{m_a}[2500.9 + 1.82.T(°C)]$$

$$h = 1.005.T(°C) + \omega[2501 + 1.82.T(°C)]$$

$$\omega = \frac{m_v}{m_a}; \phi = \frac{P_v}{P_g}$$

$0 < \phi \leq 1$

$\boxed{0\% < \phi \leq 100\%}$

4.8.4 DBT, WBT AND DPT

Dry Bulb Temperature: (DBT)

The actual/normal temperature of the air measured with a thermometer.

Wet Bulb Temperature: (T_{WBT})

The temperature measured by the thermometer when the bulb is covered with a wet cotton wick.

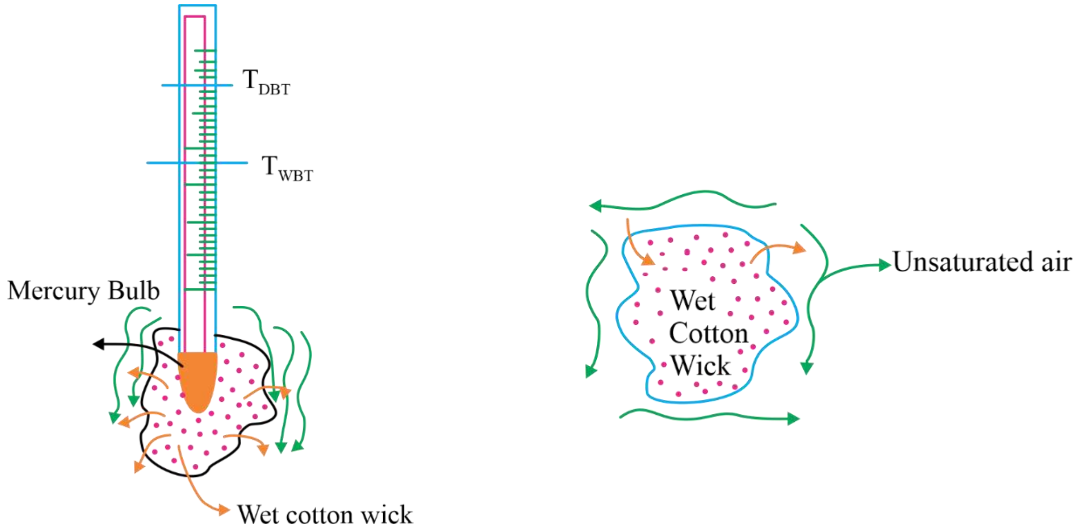


Fig.4.14 Thermometer with wet cotton wick

For unsaturated air, $T_{WBT} < T_{DBT}$

If air is saturated then there is no net Evaporation

In case of saturated air, $T_{DBT} = T_{WBT}$

Dew Point Temperature: (T_{DPT})

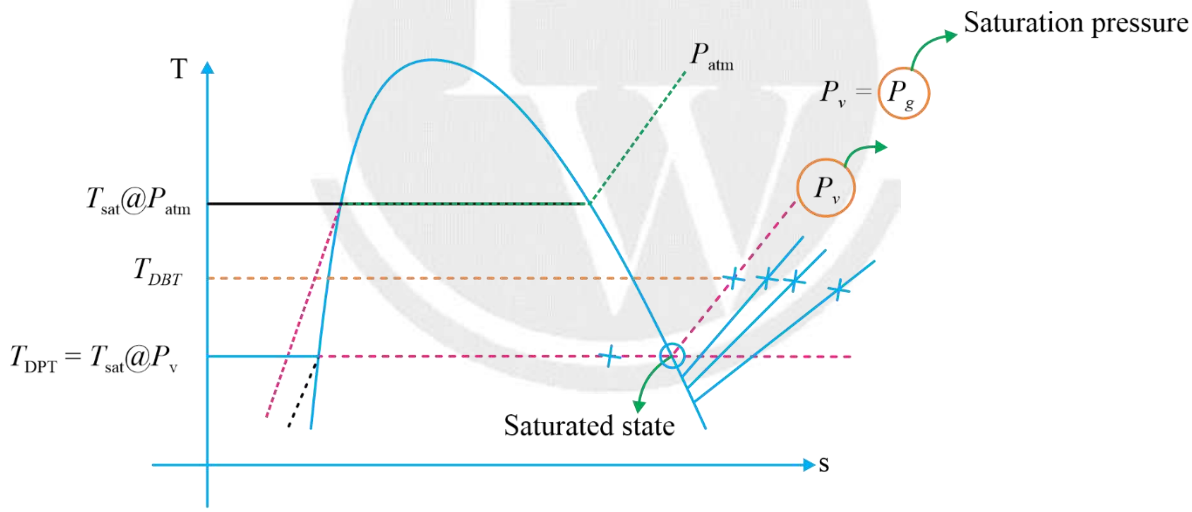


Fig.4.15 Finding Dew point temperature on T-s diagram

During condensation, the air is in its saturated state.

T_{DPT} is the temperature at which the first droplet gets formed.

In case of saturated air,

$T_{DBT} = T_{DPT}$

$\therefore$ In case of saturated air, $T_{DBT} = T_{WBT} = T_{DPT}$

4.8.5 Adiabatic Saturation Temperature

The temperature achieved by the air undergoing saturation under adiabatic condition is called Adiabatic Saturation Temperature.

If $\dot{m}_f \rightarrow$ Rate of evaporation then make up water is also supplied at the rate of $\dot{m}_f$.

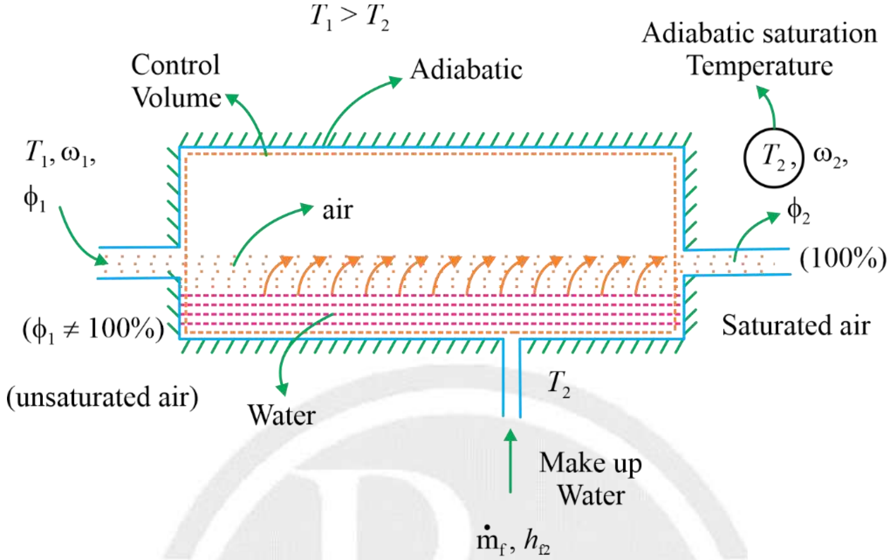


Fig.4.16 An adiabatic control volume to achieve adiabatic saturation temperature

Mass flow rate of water that is getting evaporated.

$\Rightarrow \dot{m}_f = (\omega_2 - \omega_1)\dot{m}_a$

Energy Balance:

$E_{in} = E_{out}$

$\Rightarrow \dot{m}_{in}h_{in} + \dot{m}_f h_{f_2} = \dot{m}_{out}.h_{out}$

$\Rightarrow \dot{m}_a h_{in} = (\omega_2 - \omega_1)\dot{m}_a h_{f_2} = \dot{m}_a.h_{out}$

$\Rightarrow h_{in} + (\omega_2 - \omega_1).h_{f,_2} = h_{out}$ $\qquad h_{out}\big|_{\substack{water\\vapour}} = h_{g,_2}$

$\Rightarrow c_P.T_1 + \omega_1(2500.9 + 1.82T_1) + (\omega_2 - \omega_1).h_{f,_2} = c_P T_2 + \omega_2(2500.9 + 1.82T_2)$

$\Rightarrow c_P T_1 + (\omega_1 = \omega_2)(2500.9) + (\omega_2 - \omega_1)h_{f,_2} = T_2(c_P + 1.82\omega_2)$

$\Rightarrow T_2 = \frac{c_P T_1 + (\omega_2 - \omega_1)(h_{f,2} - 2500.9)}{(c_p + 1.82\omega_2)}$

4.9 Psychrometric Chart

The plot that depicts the variation of specific humidity with variation of dry bulb temperature.

Constant specific volume lines are steeper than the constant T_{WBT} lines.

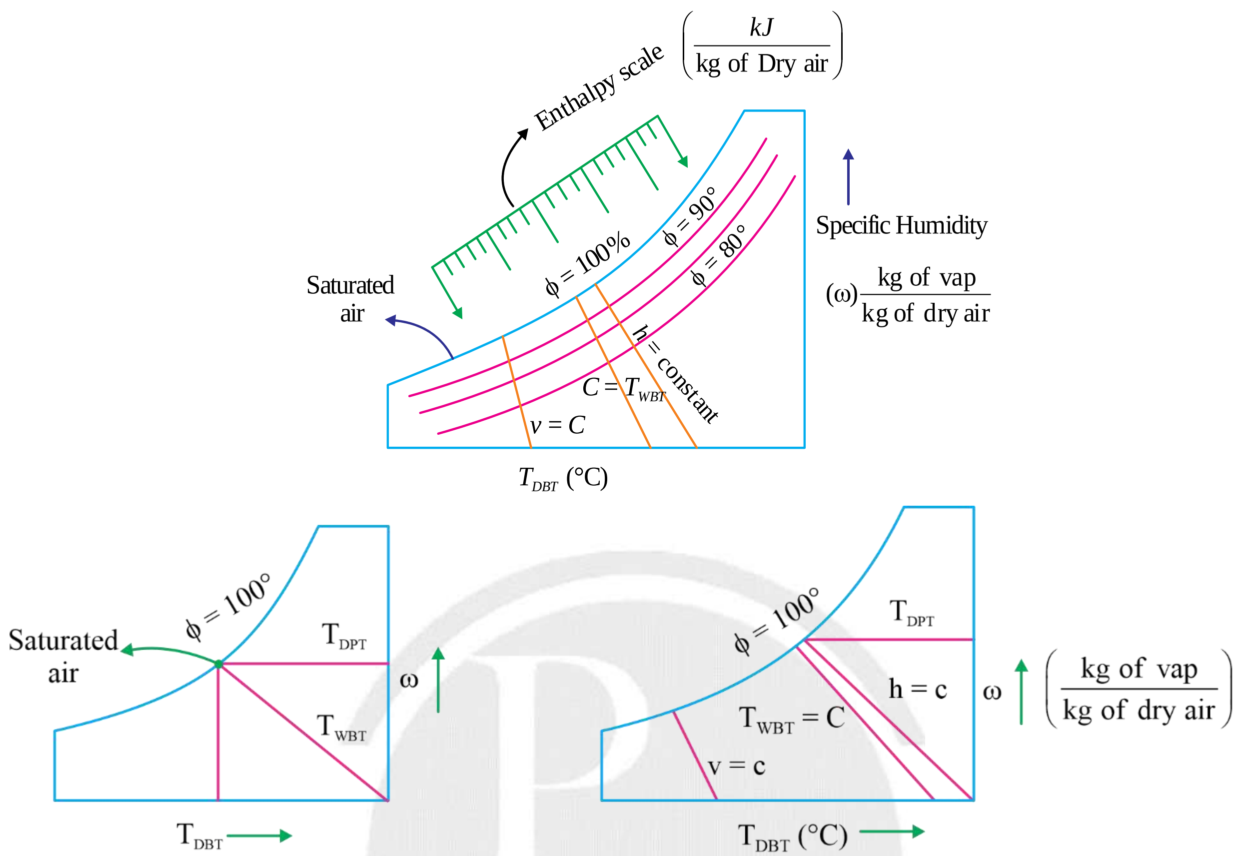


Fig. 4.17 Psychrometric chart

4.9.1 Basic Psychrometric Processes

(i) Sensible Heating:

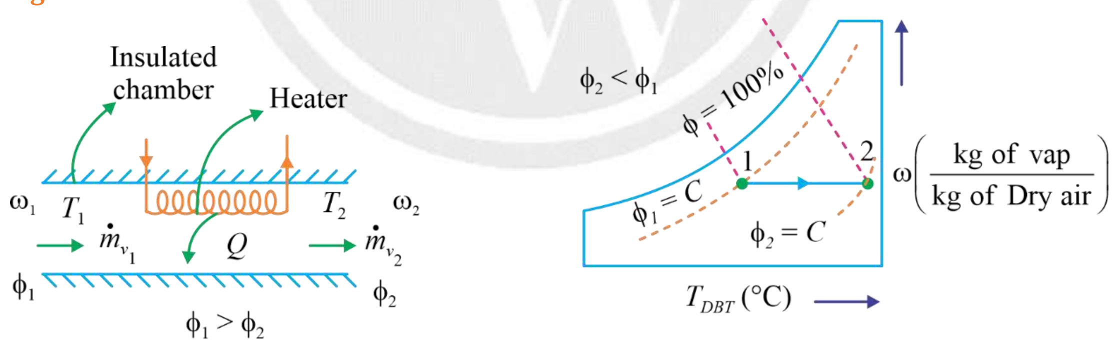


Fig.4.18 Arrangement for sensible heating and its representation on Psychrometric chart

In case of sensible heating,

$\omega_1 = \omega_2$ $\quad\quad\quad$ $\dot{m}_{a_1} = \dot{m}_{a_2} = \dot{m}_a$

$\phi_2 < \phi_1$ $\quad\quad\quad$ $\dot{m}v_1 = \dot{m}v_2 = \dot{m}v$

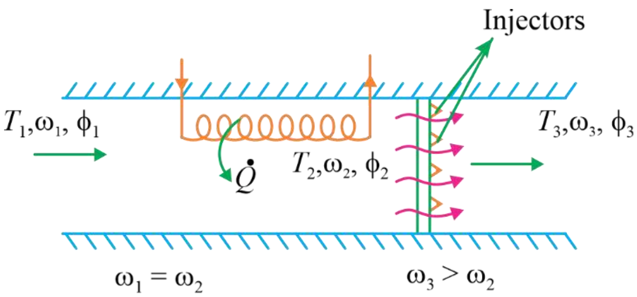
For the sections 1-2:

$$\dot{m}_{\text{in}} h_{\text{in}} + Q = \dot{m}_{\text{out}} \cdot h_{\text{out}}$$

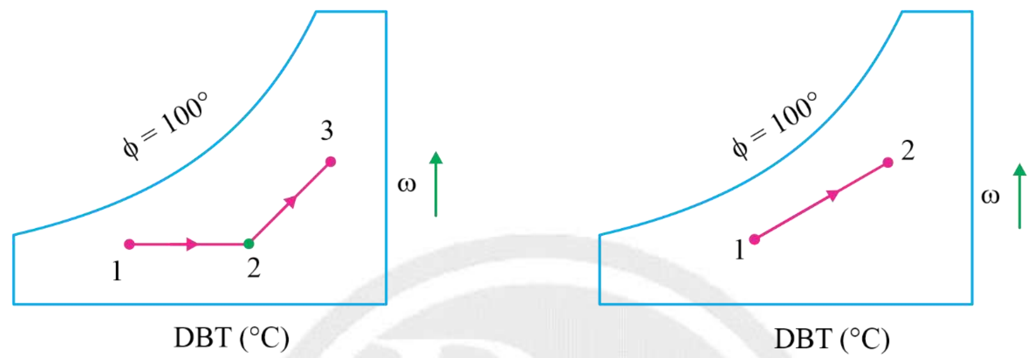
$$\Rightarrow \dot{m}_a . h_{in} + \dot{Q} = \dot{m}_a . h_{out}$$

$$\Rightarrow \dot{Q} = \dot{m}_a (h_{out} - h_{in})$$

(ii) Heating with humidification:



If water is injected, $T_3 < T_2$
If steam is injected, $T_3 > T_2$



Heating and Humidification done separately

Heating with humidification
Both heating and humidification done simultaneously

Fig. 4.19 Arrangement for heating with humidification and its representation on Psychrometric chart

(iii) Sensible Cooling:

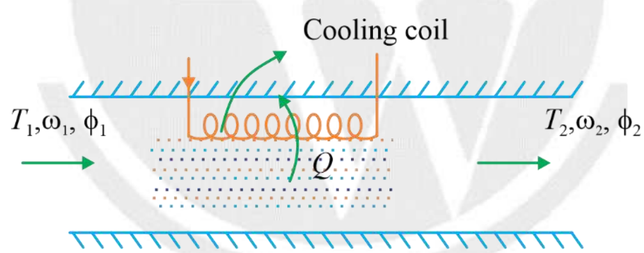


Fig.4.20 Arrangement for sensible cooling

In Sensible cooling
$(\omega_1 = \omega_2),\ (T_2 < T_1)\ and\ (\phi_2 > \phi_1)$

(iv) Cooling with dehumidification:

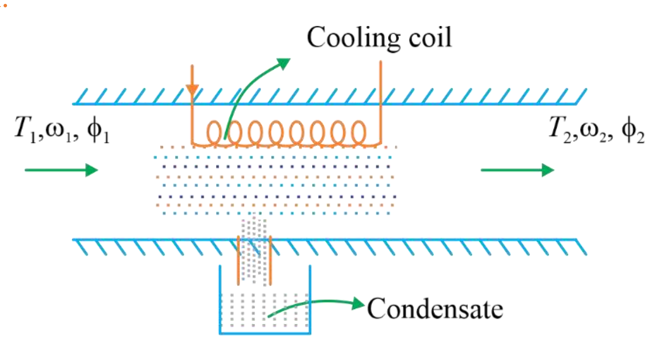


Fig. 4.21 Arrangement for cooling with dehumidification

$T_2 < T_1$.
$\omega_2 < \omega_1$.
In general, $\phi_2 > \phi_1$.
$\dot{m}_{v,1} = \omega_1 \times \dot{m}_a$
$\dot{m}_{v,2} = \omega_2 \times \dot{m}_a$
Rate of condensation $= (\omega_1 - \omega_2)\dot{m}_a$

4.9.2 Representation of all the processes in Psychrometric Chart

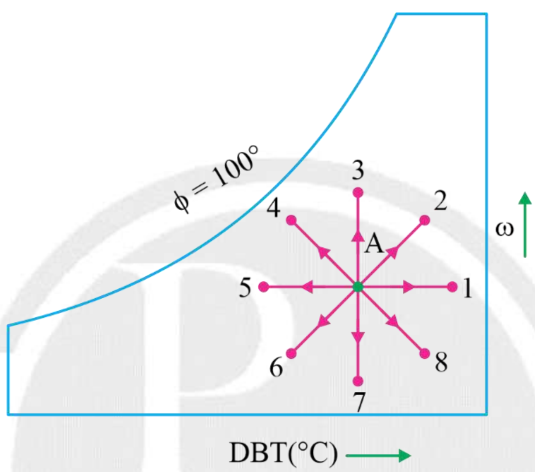


Fig.4.22 Representation of all processes on psychrometric chart

A – 1 → Sensible Heating

A – 2 → Heating with humidification

A – 3 → Humidification

A – 4 → Cooling with humidification (Occurs in air washer)

A – 5 → Sensible cooling.

A – 6 → Cooling with dehumidification

A – 7 → Dehumidification.

A – 8 → Heating with dehumidification.
(Chemical Dehumidification)

4.10 Adiabatic mixing of two moist air streams

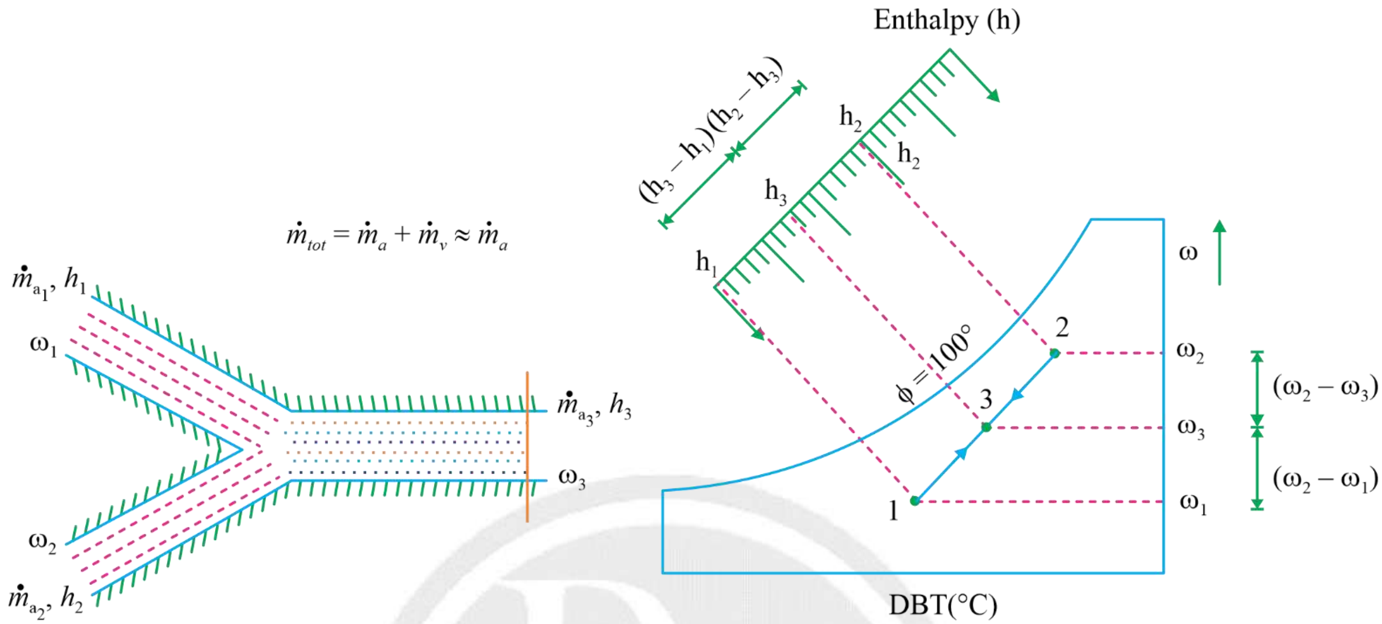


Fig.4.23 Mixing of two streams of moist air and representation of different states on psychrometric chart

For air $\rightarrow \dot{m}_{a_1} + \dot{m}_{a_2} = \dot{m}_{a_3}$

For water vapour $\rightarrow \dot{m}_{v_1} + \dot{m}_{v_2} = \dot{m}_{v_3}$

$\Rightarrow \omega_1 . \dot{m}_{a_1} + \omega_2 . \dot{m}_{a_2} = \omega_3 \left(\dot{m}_{a_1} + \dot{m}_{a_2} \right)$

$\Rightarrow (\omega_1 - \omega_3) \dot{m}_{a_1} . = (\omega_3 - \omega_2) \dot{m}_{a_2}$

$$\boxed{\frac{\dot{m}_{a,1}}{\dot{m}_{a,2}} = \frac{\omega_3 - \omega_2}{\omega_1 - \omega_3} = \frac{h_3 - h_2}{h_1 - h_3}}$$

5 GAS COMPRESSORS

5.1 Introduction

A compressor is a device in which work is done on the gas, to raise its pressure.

Applications of Compressed air: Motor for tools, air brake for vehicles, servo Mechanisms etc.

Compressors
- → +ive displacement machine → Reciprocating, Root's blower, Rotary.
- → Non +ive displacement machine → Centrifugal compressors, Axial flow.

+ive displacement → Possess means to prevent undesired flow reversal. Here, work is transferred by virtue of hydrostatic force on boundary.

In non-+ive displacement→ Work is transferred by virtue of change of momentum of stream of fluid flowing over the blades.

5.1.1 Work of compression

Work for compression is same for both reciprocating and centrifugal compressor. {Expression is same}

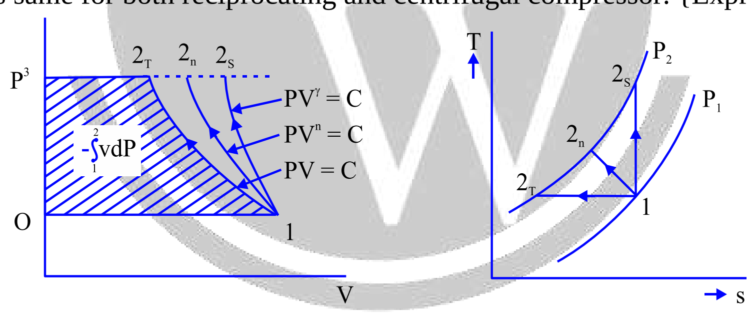
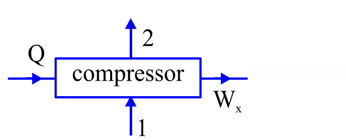


Fig. 5.1. Compression work in different processes

Steady flow energy equation for compressor is.



For reversible process; $Q = \Delta h - \int vdP$

Compression is adiabatic, then Q = O

$\Rightarrow \quad h_1 + Q = h_2 + W_X \rightarrow$ (S.F.E.E for compressor)

$\Rightarrow \quad h_2 - h_1 = -w_x = \int vdp \rightarrow$ Work required for compression

If compression is polytropic, $(PV^n = C) \Rightarrow V^n = \frac{P_1 V_1^n}{P} \Rightarrow V = \frac{P_1^{\frac{1}{n}}.V_1}{P^{\frac{1}{n}}}$

$$\therefore \quad W_X = -\int vdP = \frac{-n}{n-1}.P_1V_1\left(\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right)$$

$$\therefore \quad \text{Work required for compression} = \frac{n}{n-1}.P_1V_1\left[\left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} - 1\right]$$

If process of compression is isothermal, then $W = P_1V_1 \ln\left(\frac{P_2}{P_1}\right)$. For any flow process, $W_{comp.}$ is denoted by Area of curve projected on to P-axis on a P-v diagram.

So, $PV^n = C \Rightarrow \frac{dP}{dV} = -n.\frac{P_1}{V_1} \rightarrow$ Slope of any point.

In general, $1 < n < \gamma$ and

For a given pressure ratio $\left(\frac{P_2}{P_1}\right)$; if 'W' denotes compression work,

$$W_{Isotherma} < W_{Polytropic} < W_{adiabatic}$$

Adiabatic efficiency of compressor; $\eta_s = \frac{h_{2s} - h_1}{W_C}$

Isothermal efficiency of compressor; $\eta_s = \frac{h_{2T} - h_1}{W_C}$

Minimum work of compression, with cooling is isothermal work.

Minimum work of compression, without cooling is Isentropic work.

5.2 Single stage Reciprocating Air Compressor

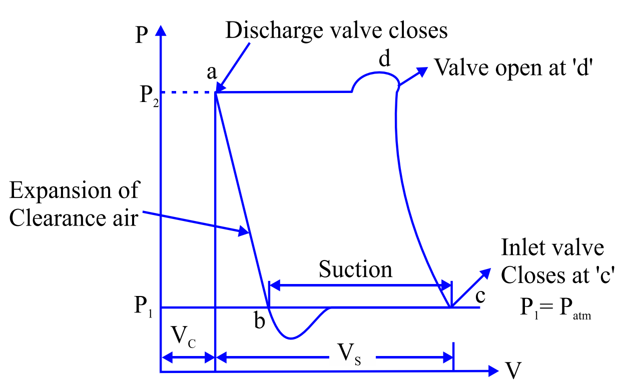


Fig. 5. 2. P-V diagram of single stage reciprocating air compressor

Compressor operates on 2-stroke cycle.

Stroke-1 (a-c): From a to b; air in the clearance volume expands and at 'b' pressure of air inside < $P_{atm.}$ So, suction begins and this suction of air into cylinder continues till 'c'. where $P = P_{atm}$.

Stroke-2 (c-a): Compression follows until the pressure in the cylinder is more than that in the receiver. Outlet valve opens at d and Air is delivered for rest of the stroke.

It is seen, the effect of air in clearance volume is to reduce the quantity of air drawing into piston, during the suction. So, clearance volume is made as small as possible.

Areas above P_2 and below P_1 is the work done for physical pressure drop. This work is called valve loss.

5.2.1 For Idealized machine

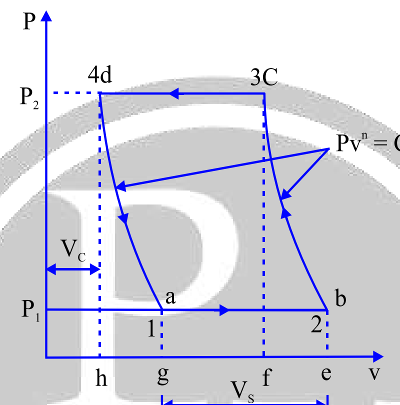


Fig. 5.3. Idealized P-V diagram of single stage reciprocating air compressor

Volume of air drawn during suction = $V_b - V_a$.

Mass of gas in clearance volume doesn't have any effect on compression work.

5.2.2 Volumetric Efficiency of Reciprocating Compressor

The ratio of actual volume of gas taken into cylinder during suction stroke to the swept volume (V_S) of piston is η_{vol}.

$\therefore \eta_{vol} = \frac{mv_1}{V_s}$ where m→ Mass of gas, v_1 → Specific volume at inlet.

$$\therefore \eta_{vol} = \frac{V_2 - V_1}{V_S} = \frac{V_C + V_S - V_1}{V_S} = 1 + \frac{V_C}{V_S} - \frac{V_1}{V_S}$$

Let C = Clearance ratio = $\frac{\text{Clearance Volume}}{\text{Swept Volume}} = \frac{V_C}{V_S}$

$$\therefore \quad \eta_{Vd} = 1 + C - \frac{V_1}{V_C} \times \frac{V_C}{V_S} = 1 + C - C\left(\frac{V_1}{V_C}\right)$$

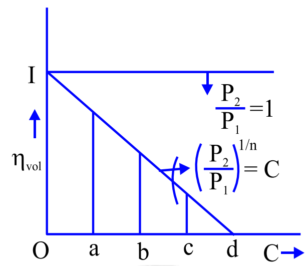
Here $P_1V_1^n = P_2V_4^n \Rightarrow V_1 = V_4\left(\frac{P_2}{P_1}\right)^{1/n} = V_c.\left(\frac{P_2}{P_1}\right)^{1/n}$

$$\therefore\ \eta_{vol} = 1 + C - C.\left(\frac{P_2}{P_1}\right)^{1/n}$$

$$\boxed{\therefore\ \eta_{vol} = 1 + C - C.\left(\frac{P_2}{P_1}\right)^{1/n}}$$

Since $\left(\frac{P_2}{P_1}\right) > 1$; η_{vol} decrease as C increase and η_{vol} decrease as $\left(\frac{P_2}{P_1}\right)$ increase

5.2.3 Effect of Clearance on Volumetric Efficiency



If clearance volume increase,

η_{vol} & *m* decrease

$$W_{comp} \neq f(C)$$

So, for a given pressure ratio, $\eta_{vol} = 0$ when $C_{max} = \dfrac{1}{\left(\frac{P_2}{P_1}\right)^{1/n} - 1}$

$$\therefore\ \eta_{vol} = 0 \text{ when } C = C_{max}\ \&\ C_{max} = \frac{1}{\left(\frac{P_2}{P_1}\right)^{1/n} - 1}$$

5.2.4 Effect of Pressure Ratio on Volumetric Efficiency

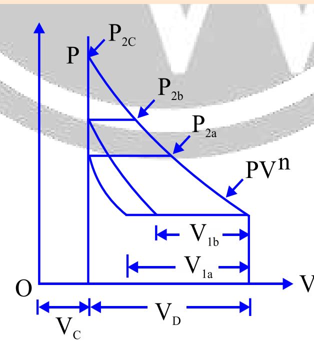


Fig. 5. 4. Effect of pressure ratio on volumetric efficiency

As $\left(\frac{P_2}{P_1}\right)$ increase; η_{vol} decrease

The max. pressure ratio $\left(\frac{P_{2\,max}}{P_1}\right)$ attainable for a reciprocating compressor cylinder is limited by the clearance 'C'

$$\boxed{\therefore\ \left(\frac{P_{2\,max}}{P_1}\right) = \left(1 + \frac{1}{C}\right)^n \ \{\text{When } \eta_{vol} = 0\}}$$

Compressor Displacement Volume, $V = \frac{\pi}{4} d^2 L$

Induction volume rate/volume flow rate = $V = \frac{\pi}{4} d^2 L \cdot \left(\frac{N}{60}\right)$

$\downarrow$ Where N $\rightarrow$ r.p.m.

For single acting compressor. $\boxed{\left(I.P. = \frac{P_m.LANK}{60} kW\right)}$

5.3 Multi Stage Compression

For compressing to high pressure, it is advantageous to do in multi stage.
The compression for min. work requires the compression to be isothermal.

But $T_2 = T_1 \left(\frac{P_2}{P_1}\right)^{\frac{n-1}{n}} \Rightarrow T_2$ increase as $\left(\frac{P_2}{P_1}\right)$ increase and also η_{vol} decrease as $\left(\frac{P_2}{P_1}\right)$ increase

For these factors when $P_2 >> P_1$; mutli-stage is preferred.

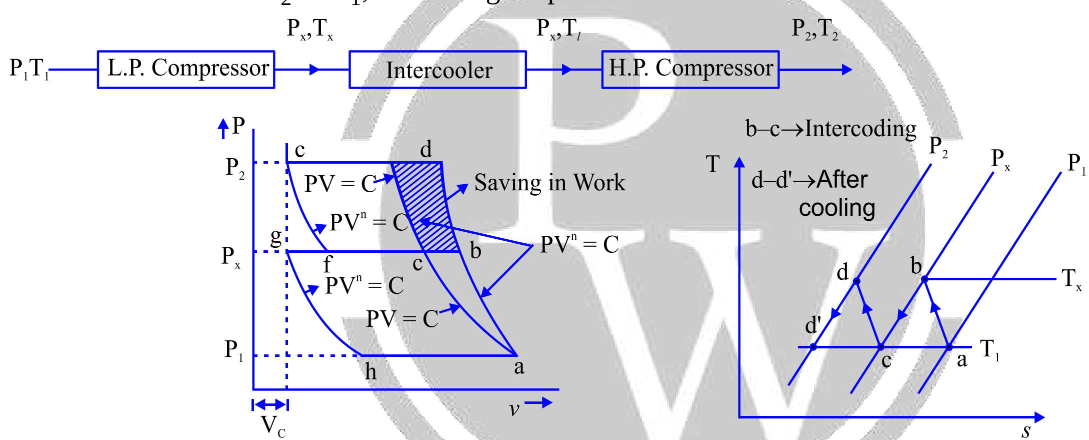


Fig. 5.4 P-v and T-s diagram of multistage compression

5.3.1 Perfect Intercooling

The exiting gas from intercooler at T_x is cooled completely to the original temperature 'T_1'. ($b - c \rightarrow$ Intercooling)

The total work for compression in two stages per kg of gas is given by

$$W_C = \frac{n}{n-1} rT_1 \left[\left(\frac{P_x}{P_1}\right)^{\frac{n-1}{n}} - 1\right] + \frac{n}{(n-1)} rT_1 \left[\left(\frac{P_2}{P_x}\right)^{\frac{n-1}{n}} - 1\right]$$

$$= \frac{n}{(n-1)} rT_1 \left[\left(\frac{P_x}{P_1}\right)^{\frac{n-1}{n}} + \left(\frac{P_2}{P_x}\right)^{\frac{n-1}{n}} - 2\right]$$

Here P_1, T_1 & P_2 are fixed. Only P_x is variable.

$\therefore$ For optimum value of minimum work, $\frac{dW_c}{dP_x} = O$

$\Rightarrow \qquad P_x = \sqrt{P_2 P_2} \quad \Rightarrow \quad T_2 = T_x$

So, for $(W_c)_{min}$; Pressure Ratio in L.P. stage = Pressure Ratio in H.P. Stage.

$$\therefore \qquad (W_c)_{min} = \frac{2.nrT_1}{n-1}\left\{\left[\frac{P_2}{P_1}\right]^{\frac{n-1}{2n}} - 1\right\}$$

Heat rejected in Intercooler, $Q_{bc} = c_P.[T_x - T_1].\frac{kJ}{kg}$

For perfect Intercooling with 'N' stage compression.

Optimum Pressure Ratio in each stage = $\frac{P_x}{P_1} = \left(\frac{P_d}{P_s}\right)^{1/N}$

Minimum work of compression, $\boxed{W_C = \frac{N.n\,rT_1}{n-1}\left\{\left(\frac{P_d}{P_s}\right)^{\frac{n-1}{nN}} - 1\right\}}$

5.3.2 Advantage of Multi-stage Compression

(i) Increased overall η_{vol}.

(ii) Leakage losses are reduced considerably.

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Heat Transfer

Heat Transfer

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1 BASICS OF HEAT TRANSFER

1.1 Introduction

- Heat transfer is the energy transfer due to temperature difference.
- Heat transfer is governed by the 2^{nd} Law of Thermodynamics.

1.2 Mode of Heat Transfer

- There are three modes of heat transfer
  (i) Conduction heat transfer
  (ii) Convection heat transfer
  (iii) Radiation heat transfer

1.2.1 Conduction:

- Conduction heat transfer is governed by "Fourier's law of heat conduction".
- As per Fourier's law of heat conduction (1 D, Steady state, No Internal heat generation)

$$\dot{Q} \propto A\frac{dT}{dx}$$

$$\dot{Q}_{\text{conduction}} = -kA\frac{dT}{dx}$$

$$\frac{\dot{Q}_{\text{conduction}}}{A} = \dot{q} = -k\frac{dT}{dx}$$

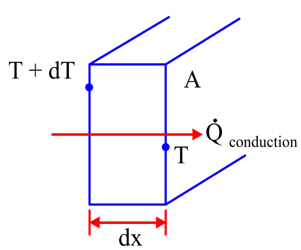

Fig. 1.1. Heat transfer through Plane wall

$\dot{Q}_{\text{conduction}} \Rightarrow$ Rate of conduction heat transfer (W)

k $\Rightarrow$ Thermal conductivity (W/mK) $\left(MLT^{-3}\theta^{-1}\right)$

$A \Rightarrow$ Area normal to direction of conduction heat transfer

$\frac{dT}{dx} \Rightarrow$ Temperature gradient in x-direction

$\dot{q} \Rightarrow$ Conductive heat flux (W/m^2)

Note:
Negative sign shows temperature decreases in the direction of conduction heat transfer.

1.2.2 Convection

- Convection heat transfer is governed by "Newton's law of cooling".
- As per Newton's law of cooling

$$\dot{Q} \propto A(T_s - T_\infty)$$
$$\dot{Q} = hA(T_s - T_\infty)$$
$$\dot{q} = h(T_s - T_\infty)$$

h ⇒ Convective heat transfer coefficient (W/m^2 K) ($MT^{-3}\theta^{-1}$)
A ⇒ Area responsible for convection heat transfer
T_s ⇒ Plate surface temperature
T_∞ ⇒ Fluid temperature

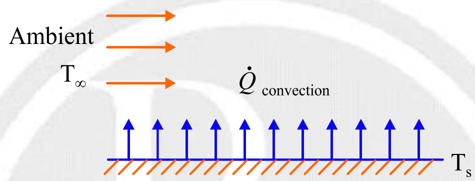

Fig. 1.2. Convection heat transfer from a hot flat plate

- h depends on number of factors
  (a) Free or forced convection
  (b) Laminar or turbulent flow
  (c) Type of fluid (liquid/gas)
  (d) Surface finish of plate (smooth/rough)
  (e) Orientation of plate (horizontal, vertical, inclined)

1.2.3 Radiation

- Radiation heat transfer is governed by "Stefan's Boltzmann Law"
- As per Stefan's Boltzmann Law

$$E_b \propto T^4$$
$$E_b = \sigma_b T^4 \left(\frac{W}{m^2}\right)$$
$$\sigma_b = 5.67 \times 10^{-8} \left(\frac{W}{m^2 K^4}\right)$$

E_b ⇒ Total radiation energy emitted by black body per unit time per unit area. (W/m^2)
T ⇒ Absolute temperature (in K)

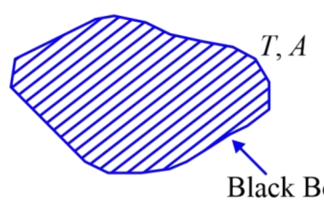

Fig. 1.3. Black body at absolute temperature T

- If body is non-black, then

$$E = \varepsilon E_b$$

$$E = \varepsilon \sigma_b A T^4$$

E $\Rightarrow$ Radiation energy emitted by a non-black body

$\varepsilon \Rightarrow$ Emissivity

- Radiation heat transfer between two black bodies

$$\dot{Q}_{\text{Radiation}} = A_1 F_{12} \sigma_b \left(T_1^4 - T_2^4\right)$$

$F_{12} \Rightarrow$ View factor/shape factor/geometric factor

$$A_1F_{12} = A_2F_{21} \qquad \text{(Reciprocity Theorem)}$$

- Radiation heat transfer between two non-black body

$$\dot{Q}_{\text{Radiation}} = A_1 \left(F_g\right)_{12} \sigma_b \left(T_1^4 - T_2^4\right)$$

$(F_g)_{12} \Rightarrow$ Configuration factor

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2 STEADY-STATE CONDUCTION HEAT TRANSFER

2.1 General heat conduction equation in Cartesian coordinate

$$T(x, y, z, \tau)$$

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)+\dot{q}_g=\rho c \frac{\partial T}{\partial \tau}$$

For homogenous and isotropic material $(k_x = k_y = k_z = k)$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\nabla^2 T+\frac{\dot{q}_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau} \quad \text{(Fourier-Biot equation)}$$

Where $\dot{q}_g \Rightarrow$ Rate of heat generation per unit volume or volumetric heat generation rate $\left(\frac{W}{m^3}\right)$

$\alpha \Rightarrow$ Thermal diffusivity $\left(\frac{m^2}{s}\right)$

$\alpha = \frac{k}{\rho c}$

$\frac{\partial T}{\partial \tau} \Rightarrow$ Change of temperature with respect to time

Case 1. Steady, No Internal Heat Generation

$$\nabla^2 T = 0 \quad \text{(Laplace Equation)}$$

Case 2. Steady, with Internal Heat Generation

$$\nabla^2 T+\frac{\dot{q}_g}{k}=0 \quad \text{(Poisson's Equation)}$$

Case 3. Unsteady, with No Internal Heat Generation

$$\nabla^2 T=\frac{1}{\alpha}\frac{\partial T}{\partial \tau} \quad \text{(Diffusion Equation)}$$

Case 4. 1-D, Steady, with Internal Heat Generation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\dot{q}_g}{k}=0$$

2.2 The Thermal Properties of matter

2.2.1 Thermal Diffusivity (α)

- Ratio of rate of heat energy conducted to heat energy stored. Mathematically,

$$\alpha=\frac{k}{\rho c}$$

c ⇒ Specific heat (J/kg K)
mc ⇒ Heat capacity (J/K)

$\rho c=\frac{mc}{V}\Rightarrow$ Heat capacity per unit volume $\left(J/\mathrm{K\,m^3}\right)$

- S.I. Unit of $\alpha=\frac{m^2}{s}\left(L^2T^{-1}\right)$
- α depends on type of material

$$\alpha_{\text{metal}}>\alpha_{\text{wood}}$$

2. 2. 2 Thermal Conductivity (k)

- Diamond has highest thermal conductivity because it has well-arranged crystal structure.
- In metal, silver (Ag) has highest thermal conductivity
- Descending order of thermal conductivity for some metals

  $\mathrm{k_{Ag}>k_{Cu}>k_{Au}>k_{Al}>k_{Iron}}$

- Material which is good conductor of heat but bad conductor of electricity is diamond.
- In general, purest form of metal will have higher thermal conductivity than its alloy.

  For Example: $k_{\text{copper}}>k_{\text{brass}},\ k_{\text{iron}}>k_{\text{steel}}$

- In case of metals, as temperature increases thermal conductivity decreases except in case of Al and Ur
- In case of Al, as temperature increases, thermal conductivity increases, then constant and then start to decrease.

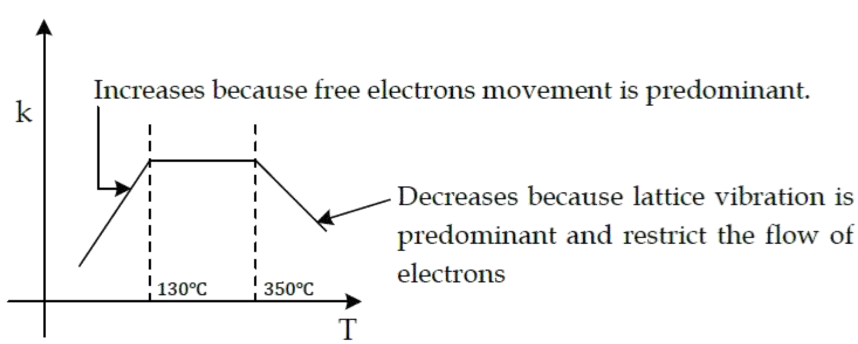

Fig 2.1(a) Temperature variation of thermal conductivity with temperature for Aluminum

- In case of Ur, as temperature increases, thermal conductivity increases.
  (In case of Ur, outer orbit electrons are strongly bonded with nucleus. Hence, lattice vibration is predominant)

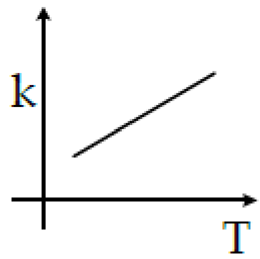

Fig 2.1(b) Temperature variation of thermal conductivity with temperature for Uranium

- In non-metals, as temperature increases, thermal conductivity increases because lattice vibration increases as temperature increases (no free electrons)
- In case of gases:

$$k_{gas} \propto V$$

Where $V \rightarrow$ mean travel velocity, $V = \sqrt{\dfrac{3\overline{R}T}{M}}$

$\overline{R} \rightarrow$ Universal gas constant, $\left\{\overline{R} = 8.314 \dfrac{kJ}{k\,mol.K}\right\}$

$T \rightarrow$ absolute temperature

$M \rightarrow$ Molecular weight of gas

- For a given gas as temperature increases, the thermal conductivity also increases.
- For a given temperature

$$k_{gas} \propto \frac{1}{M}$$

Therefore, $k_{H_2} > k_{N_2} > k_{O_2} > k_{CO_2}$

- For most liquids as temperature increases, thermal conductivity decreases except water, Hg.
- In general, conduction heat transfer is most predominant in solids, then in liquids and least in gases $\left(k_{solid} > k_{liquid} > k_{gas}\right)$

2.3 Thermal Contact Resistance

- When two surfaces are pressed against each other, the peaks will form good contact but the valleys will form voids filled with air.
- As the air thermal conductivity is low, the interface offers some thermal resistance and this resistance per unit interface area is called thermal contact resistance.
- Mathematically we can write

$$\dot{Q} = h_c\, A\, \Delta T_{\text{interface}}$$

Where $h_c \rightarrow$ thermal contact conductance

- For composite wall, we can write

$$\dot{Q} = \frac{T_1 - T_2}{\left(\frac{\delta_1}{k_1 A}\right) + \left(\frac{1}{h_c A}\right) + \left(\frac{\delta_2}{k_2 A}\right)}$$

$$\Rightarrow \dot{q} = \frac{T_1 - T_2}{\left(\frac{\delta_1}{k_1}\right) + R_c + \left(\frac{\delta_2}{k_2}\right)}$$

Where $R_c \rightarrow$ Thermal contact resistance

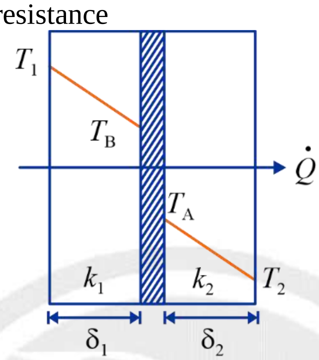

Fig. 2.2 Composite wall with thermal contact resistance

2.4 Conduction heat transfer from plane wall

Assumptions

- Steady
- 1-D (x-direction)
- No internal heat generation
- $k \neq f(T)$

$$\frac{\partial^2 T}{\partial x^2} = 0 \Rightarrow T = C_1 x + C_2$$

$$T = T_1 + (T_2 - T_1)\frac{x}{\delta} \qquad \text{(Linear Variation)}$$

$$\dot{Q}_{\text{conduction}} = -kA\frac{dT}{dx}$$

$$= -kA\left(\frac{T_2 - T_1}{\delta}\right) = \frac{kA(T_1 - T_2)}{\delta}$$

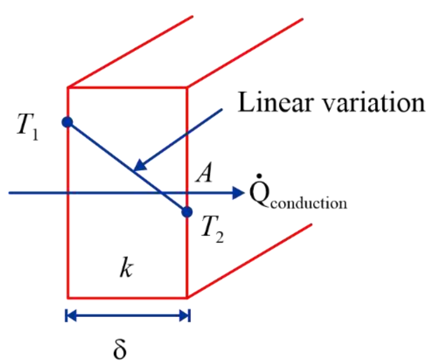

Fig 2.3 Temperature variation in a plane wall

Note:

Consider two identical wall of different material and having same amount of conduction heat transfer. In which there will be higher temperature difference across the wall?

Let $k_1 > k_2$

- From Fourier law of Heat Conduction

$$\dot{Q} = -kA\frac{dT}{dx}$$

$$\Rightarrow \frac{\dot{Q}}{A} = \dot{q} = -k\frac{dT}{dx} = \text{ constant}$$

- If k is high then $\frac{dT}{dx}$ should be low

Since, $k_1 > k_2$, $(\Delta T)_1 < (\Delta T)_2$, $\theta_1 < \theta_2$

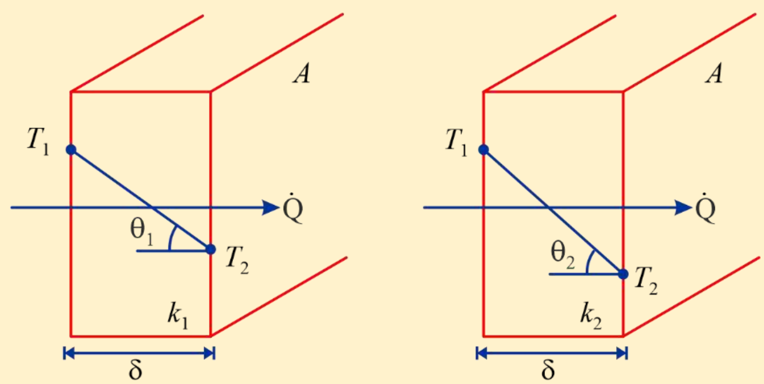

Fig 2.3 Temperature variation in a plane wall with different thermal conductivities

2.4.1 Thermal resistance (R_{th})

- Resistance offered by the material against the heat flow.
- Thermal resistance is analogous to electrical resistance.
- From Ohm's law' $\quad i = \frac{\Delta V}{R}$
- From Fourier's law $\quad \dot{Q} = \frac{\Delta T}{R_{th}}$

$$R_{th} = \frac{\Delta T}{\dot{Q}}\left(\frac{K}{W} / \frac{^\circ C}{W}\right)$$

Conduction Thermal Resistance

- In case of plane wall, heat transfer rate is given by

$$\dot{Q}_{conduction} = kA\frac{\Delta T}{\delta} = \frac{\Delta T}{\delta / kA}$$

$$R_{th} = \delta / kA$$

Convection Thermal Resistance

- Heat transfer rate by convection is given by

$$\dot{Q}_{convection} = hA\Delta T = \frac{\Delta T}{1 / hA}$$

$$R_{th} = 1 / hA$$

Radiation Thermal Resistance

- Radiation heat transfer in terms of Newton's Law of cooling is given by

$$\dot{Q}_{\text{radiation}} = h_r A(T_1 - T_2) = A_1 F_{12} \sigma_b \left(T_1^{\,4} - T_2^{\,4}\right)$$

$$R_{th} = \frac{1}{h_r A}$$

$h_r \Rightarrow$ Radiative heat transfer coefficient

$$h_r = \sigma_b F_{12}\left(T_1^2 + T_2^2\right)(T_1 + T_2)$$

2.4.2 Composite wall/slab:

- Wall is made of more than one material.

(a) Series Connection

$$\dot{Q}_1 = \dot{Q}_2 = \dot{Q}$$

1. **Equivalent thermal resistance or total thermal resistance**

$$(R_{th})_{eq} = R_{th_1} + R_{th_2} + \cdots$$

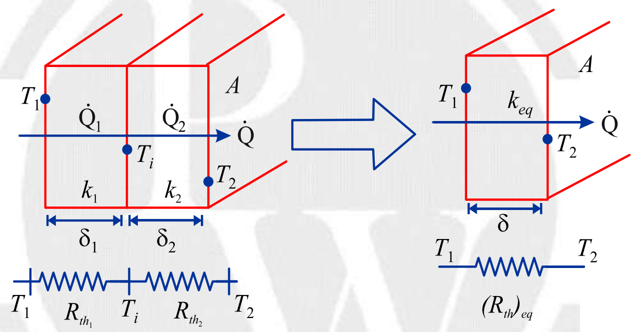

Fig. 2.4 Composite wall in series connection

2. **Equivalent thermal conductivity (k_{eq})**

$$(R_{th})_{eq} = R_{th_1} + R_{th_2}$$

$$\frac{\delta}{k_{eq}A} = \frac{\delta_1}{k_1 A} + \frac{\delta_2}{k_2 A}$$

$$k_{eq} = \frac{\delta}{\dfrac{\delta_1}{k_1} + \dfrac{\delta_2}{k_2}}$$

- If N number of slab or wall connected in series then

$$k_{eq} = \frac{\delta_1 + \delta_2 + \ldots + \delta_N}{\dfrac{\delta_1}{k_1} + \dfrac{\delta_2}{k_2} + \cdots + \dfrac{\delta_N}{k_N}}$$

- If thickness of each wall is same then

$$k_{eq} = \frac{\delta_1 N}{\delta_1\left(\dfrac{1}{k_1} + \dfrac{1}{k_2} + \cdots + \dfrac{1}{k_N}\right)}$$

- Two walls of same thickness connected in series then

$$\boxed{k_{eq} = \frac{2}{1/k_1 + 1/k_2} = \frac{2k_1k_2}{k_1 + k_2}}$$

3. Intermediate temperature (T_i)

$$\dot{Q}_1 = \dot{Q}_2 = \dot{Q}$$

$$\frac{T_1 - T_2}{(R_{th})_{eq}} = \frac{T_1 - T_i}{(R_{th})_1} = \frac{T_i - T_2}{(R_{th})_2}$$

4. If $k_2 > k_1$ then $\theta_1 < \theta_2$

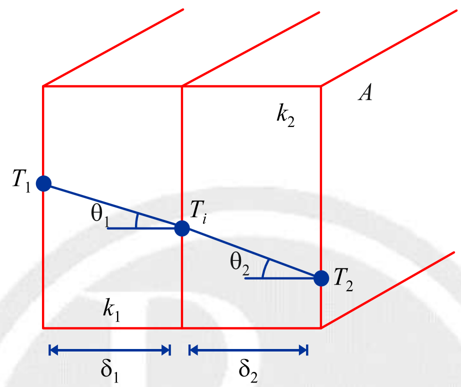

Fig 2.5 Temperature variation for composite wall in series connection

5. Plane wall with convection boundary condition

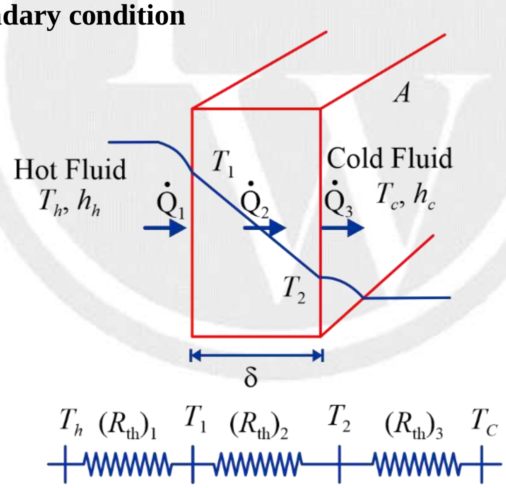

Fig 2.6 Plane wall with convection boundary condition

$$(R_{th})_{eq} = R_{th_1} + R_{th_2} + R_{th_3}$$

$$R_{th_1} = \frac{1}{h_h A}$$

$$R_{th_2} = \frac{\delta}{kA}$$

$$R_{th_3} = \frac{1}{h_c A}$$

$$\dot{Q} = \dot{Q}_1 = \dot{Q}_2 = \dot{Q}_3$$

$$\frac{T_h - T_c}{(R_{th})_{eq}} = \frac{T_h - T_1}{(R_{th})_1} = \frac{T_1 - T_2}{(R_{th})_2} = \frac{T_2 - T_c}{(R_{th})_3}$$

(b) Parallel Connection:

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2$$

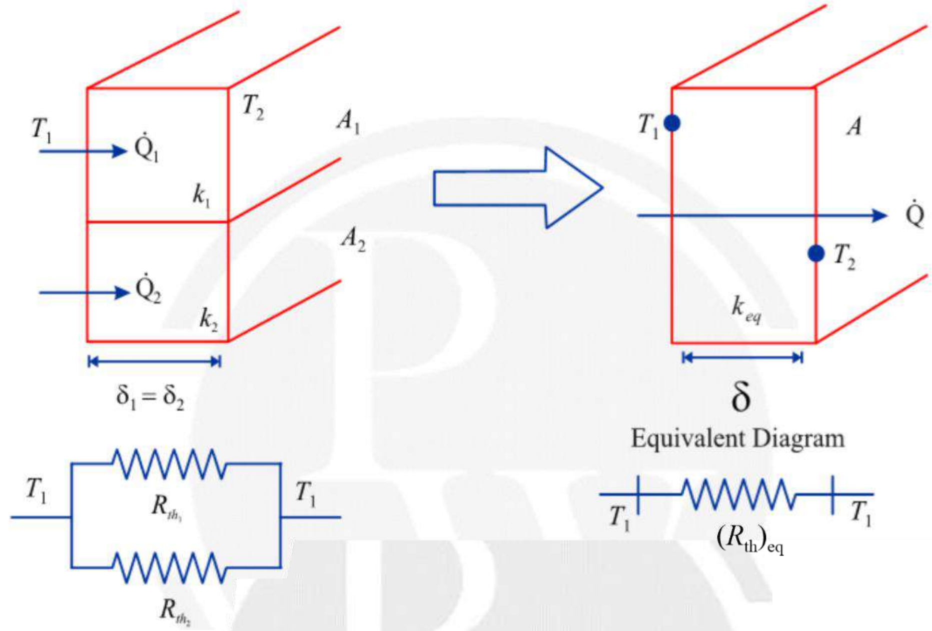

Fig. 2.7 Composite wall in Parallel connection

1. **Equivalent thermal resistance**

$$\frac{1}{(R_{th})_{eq}} = \frac{1}{R_{th_1}} + \frac{1}{R_{th_2}} + \cdots$$

2. **Equivalent thermal conductivity (k_{eq})**

$$\frac{1}{\left(\frac{\delta}{k_{eq}A}\right)} = \frac{1}{\left(\frac{\delta_1}{k_1A_1}\right)} + \frac{1}{\left(\frac{\delta_2}{k_2A_2}\right)}$$

$$\frac{k_{eq}A}{\delta} = \frac{k_1A_1}{\delta_1} + \frac{k_2A_2}{\delta_2}$$

$$k_{eq} = \frac{k_1A_1 + k_2A_2}{A}$$

- If N number of slabs are connected in parallel having same thickness

$$k_{eq} = \frac{k_1A_1 + k_2A_2 + \cdots + k_NA_N}{A_1 + A_2 + \cdots + A_N}$$

Let $A_1 = A_2 = \cdots A_N$

$$k_{eq} = \frac{A_1(k_1 + k_2 + k_3 + \cdots)}{A_1N}$$

$$k_{eq} = \frac{(k_1 + k_2 + k_3 + \cdots)}{N}$$

- If two slabs of same cross-sectional area are connected in parallel then

$$k_{eq} = \frac{k_1 + k_2}{2}$$

- $\theta_1 = \theta_2$ (Slope of temperature variation in both the wall is same)

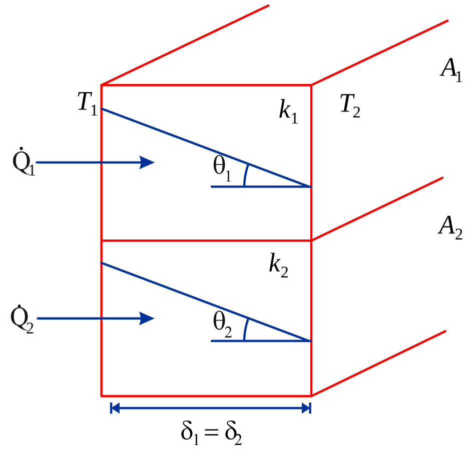

Fig 2.8 Temperature variation for composite wall in Parallel connection

3. Composite wall in parallel connection with convection boundary condition

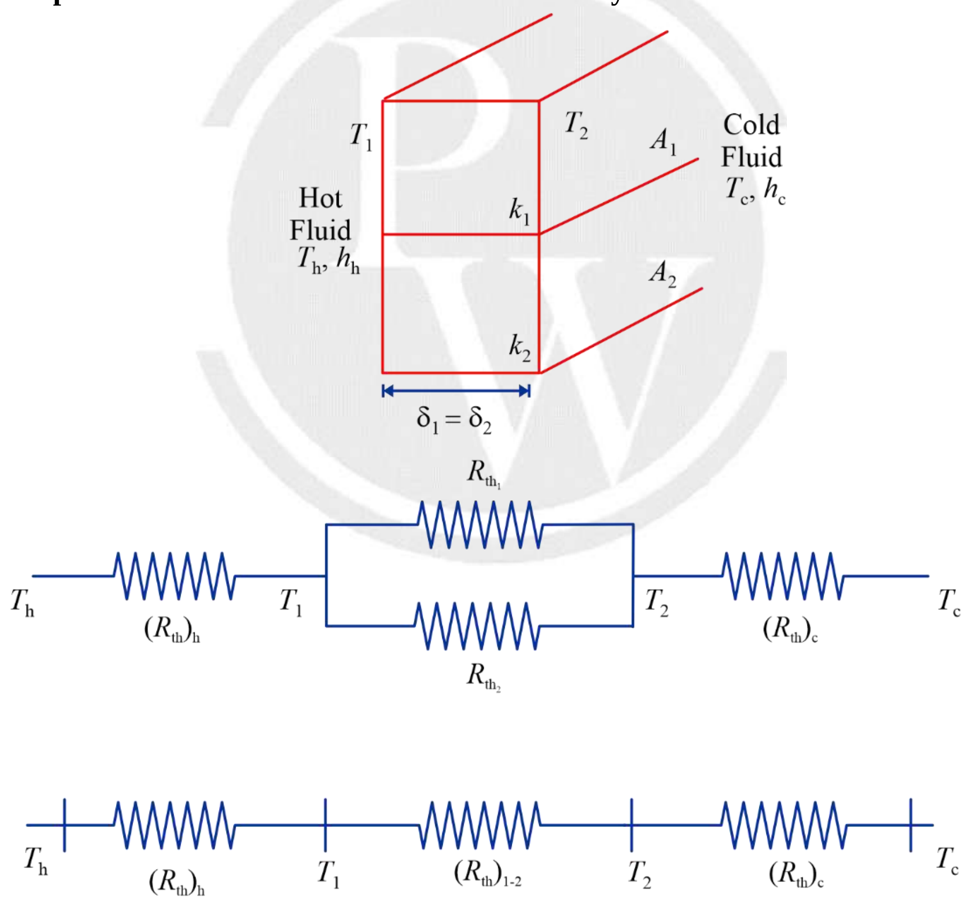

Fig 2.9 Thermal resistance network for composite wall with convection boundary condition

$$\frac{1}{(R_{th})_{1-2}} = \frac{1}{R_{th_1}} + \frac{1}{R_{th_2}}$$

$$(R_{th})_{eq} = (R_{th})_h + (R_{th})_{1-2} + (R_{th})_c$$

$$\dot{Q} = \frac{T_h - T_c}{(R_{th})_{eq}}$$

2. 4. 3 Plane walls with internal heat generation

Assumption

- Steady state
- 1-D
- k ≠ f (T)
- Uniform heat generation rate

Case 1. Wall having different temperature at two faces (Unsymmetric case)

Let $\dot{q}_g \rightarrow$ Uniform rate of heat generation per unit volume $\left(W/m^3\right)$ or volumetric heat generation rate

$$\nabla^2 T + \frac{\dot{q}_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ (General heat conduction equation)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_g}{k} = 0$$

$$T = -\frac{\dot{q}_g}{2k}x^2 + C_1 x + C_2$$

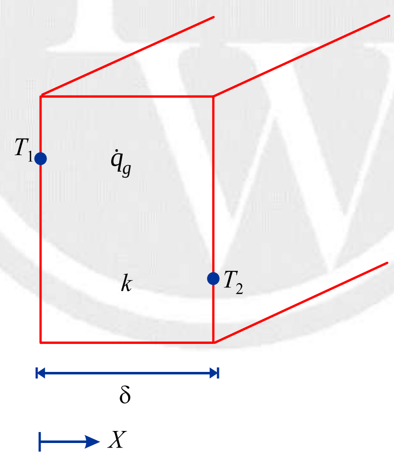

Fig 2.10 Plane wall with uniform heat generation

Where C_1 and C_2 are integration constant and evaluate by using boundary conditions

At $x = 0, T = T_1$

At $x = \delta, T = T_2$

$$\boxed{T = T_1 + \left(T_2 - T_1\right)\frac{x}{\delta} + \frac{\dot{q}_g}{2k}\left(\delta x - x^2\right)}$$

To get location of maximum temperature $\frac{dT}{dx} = 0$

Let

At x $= x', T = T_{max}$ $\qquad x' = \frac{\delta}{2} - \frac{\left(T_1 - T_2\right)k}{\dot{q}_g \times \delta}$

Subcase 1.

$$T_1 > T_2$$
$$x' < \delta / 2$$

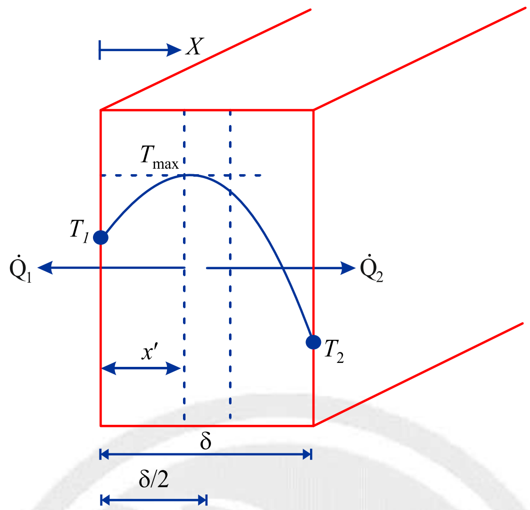

Fig 2.11 Temperature profile for different wall temperatures ($T_1 > T_2$)

Subcase 2. $T_2 > T_1$

$$x' > \delta / 2$$

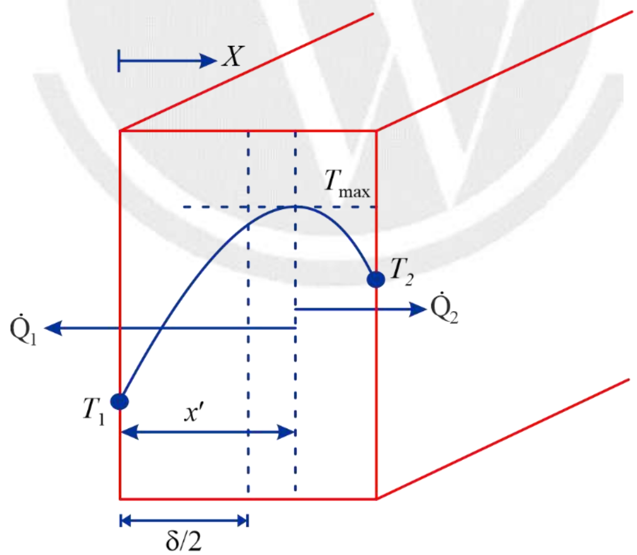

Fig 2.12 Temperature profile for different wall temperatures ($T_2 > T_1$)

Case 2. Wall having same temperature (T_w) at two faces (Symmetric case)

- $T_1 = T_2 = T_w$

$$\boxed{T = T_w + \frac{\dot{q}_g}{2k}\left(\delta x - x^2\right)}$$

Maximum temperature occurs at $x = \delta / 2$

$$\boxed{T_{\max} = T_w + \frac{\dot{q}_g \delta^2}{8k}}$$

- $\dot{Q}_1 = \dot{Q}_2$ (Due to symmetry)

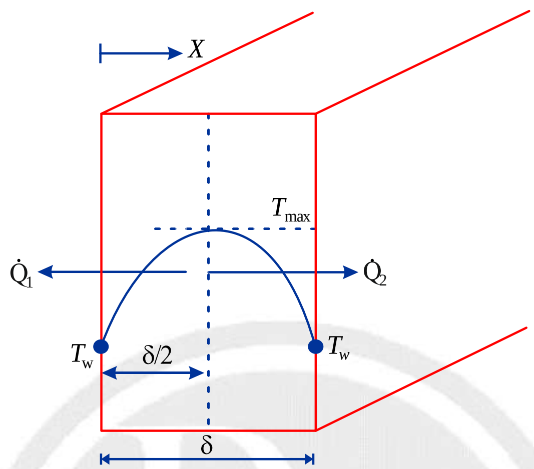

Fig 2.13 Temperature profile for same wall temperatures (T_w)

Case 3. One face is adiabatic and other face has temperature T_w,

Boundary conditions,

- At $x = \delta, T = T_w$
- At $x = 0, \dot{Q}_{\text{conduction}} = 0$

$$-kA\frac{dT}{dx} = 0$$

$$\frac{dT}{dx} = 0$$

$$\boxed{T = T_w + \frac{\dot{q}_g}{2k}\left(\delta^2 - x^2\right)}$$

At $x = 0, T = T_{\max}$

$$\boxed{T_{\max} = T_w + \frac{\dot{q}_g}{2k}\delta^2}$$

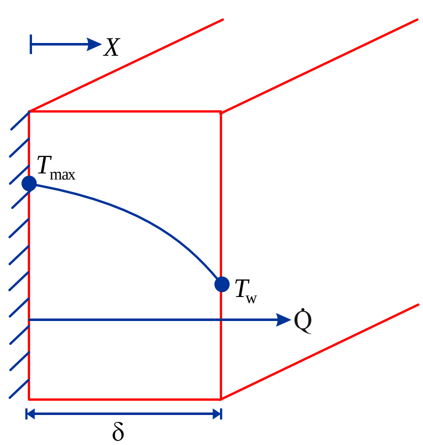

Fig 2.14 Temperature profile for adiabatic wall at one end

2. 4. 4 Effect of Variable Thermal Conductivity

Let

$$k = k_0(1+\beta T)$$

Where

k_0 = Thermal conductivity at 0°C
β = Constant (depends on type of material)

- Value of β may be Positive, may be Negative or zero.

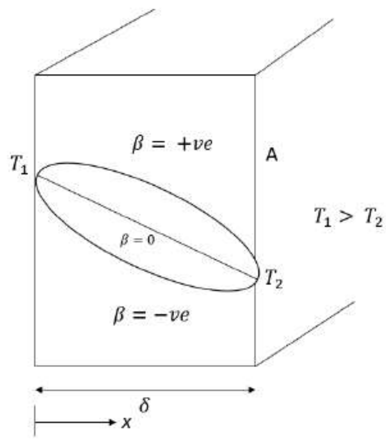

Fig 2.15 Temperature profile for plane wall with variable thermal conductivity

- Rate of conduction Heat Transfer

$$\dot{Q} = k_m A\left[\frac{T_1 - T_2}{\delta}\right] \quad \left(\begin{array}{l}\text{Applicable only when thermal conductivity} \\ \text{varies linearly with temperature}\end{array}\right)$$

k_m = Thermal conductivity at mean temperature

$$k_m = k_0\left[1+\beta T_m\right]$$

$$T_m = \frac{T_1 + T_2}{2}$$

2.5 General Heat Conduction Equation in Cylindrical Coordinate System

T (r, θ , z, τ)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk_r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k_\theta\frac{\partial T}{\partial \theta}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+\dot{q}_g = \frac{\partial}{\partial \tau}(\rho c T)$$

If material properties k, ρ & c are constant then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

Poisson Equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{q}_g}{k}=0$$

Laplace Equation $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=0$

Diffusion Equation $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$

2.6 Conduction Heat Transfer through Hollow cylinder

- At $r = R_1, T = T_1$
- At $r = R_2, T = T_2$

  Let $T_1 > T_2$

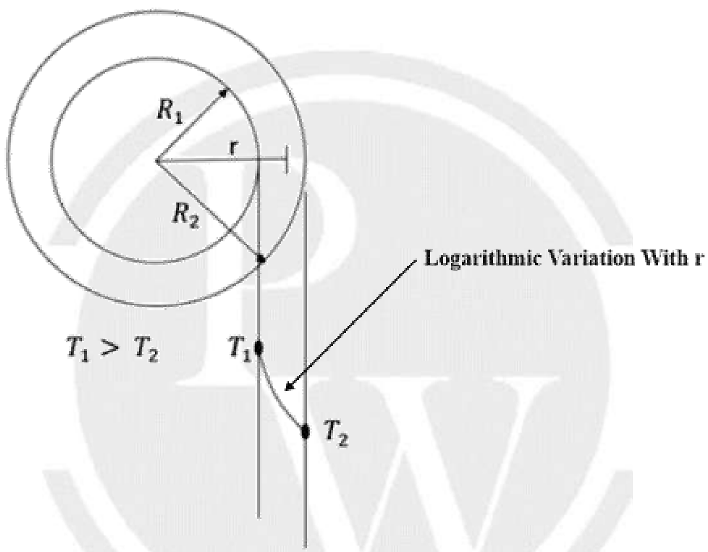

Fig 2.16 Hollow cylinder

- at radial distance r, temperature is T

Assumption –

1. Material is homogenous & isotropic
2. One dimensional (Radial)
3. No internal heat generation
4. Steady State
5. k is Constant. $k \neq f(T)$

- Temperature Distribution

$$\frac{T(r)-T_1}{T_2-T_1} = \frac{\ln(r/R_1)}{\ln(R_2/R_1)} \text{ (Logarithmic temperature variation with } r\text{)}$$

- Thermal resistance in case of a hollow cylinder

$$R_{\text{th}} = \frac{\ln(R_2/R_1)}{2\pi k L}$$

- Conduction heat transfer through hollow cylinder

$$\dot{Q} = \frac{T_1 - T_2}{\dfrac{\ln(R_2/R_1)}{2\pi k L}}$$

2.6.1 Hollow Cylinder with Convection Boundary Condition

- $\dot{Q} = \dfrac{T_h - T_1}{(R_{\text{th}})_1} = \dfrac{T_1 - T_2}{(R_{\text{th}})_2} = \dfrac{T_2 - T_a}{(R_{\text{th}})_3} = \dfrac{T_h - T_a}{(R_{\text{th}})_{\text{eq}}}$

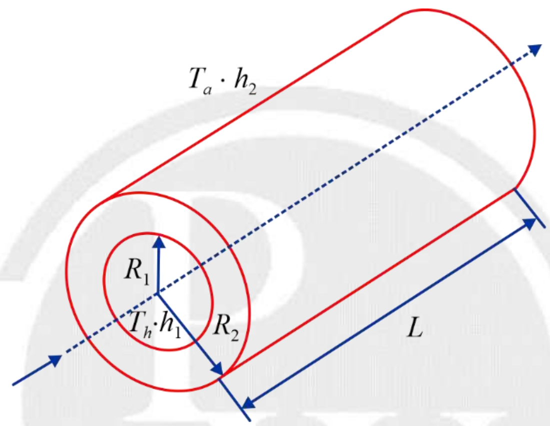

Fig 2.17 Hot fluid carrying hollow cylinder exposed to the ambient

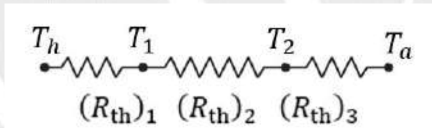

Fig 2.18 Thermal resistance diagram for hollow cylinder

- $(R_{\text{th}})_{\text{eq}} = (R_{\text{th}})_1 + (R_{\text{th}})_2 + (R_{\text{th}})_3$

Where

$$(R_{th})_1 = \frac{1}{h_1 \cdot A_1} = \frac{1}{h_1 \cdot 2\pi R_1 L}$$

$$(R_{\text{th}})_2 = \frac{\ln(R_2/R_1)}{2\pi k L}$$

$$(R_{\text{th}})_3 = \frac{1}{h_2 \cdot A_2} = \frac{1}{h_2 \cdot 2\pi R_2 L}$$

2.6.2 Equivalent Area for hollow cylinder

- In case of hollow cylinder equivalent area is equal to logarithmic mean area of A_1 *and* A_2

$$A = \frac{A_2 - A_1}{\ln(A_2/A_1)}$$

2.6.3 Solid Cylinder with Uniform Internal Heat Generation: -

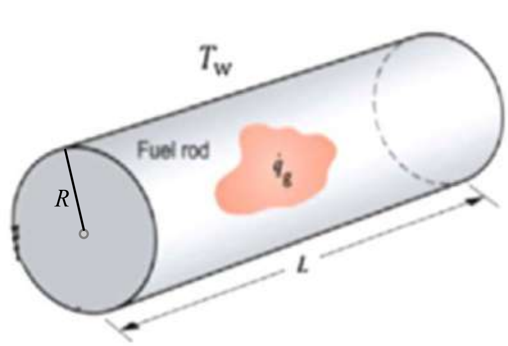

Fig 2.19 Solid cylinder with uniform heat generation

Assumption

- 1-D (Radial Direction)
- Steady State
- Uniform Heat generation
- k is constant

$$T(r) = T_w + \frac{\dot{q}_g}{4k}(R^2 - r^2) \text{ (Parabolic Variation)}$$

At r = 0, T = T_{Max}

$$T_{Max} = T_w + \frac{\dot{q}_g}{4k} R^2$$

2.7 General Heat Conduction Equation in Spherical Coordinate System

T (r, θ , ϕ, τ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(k_r r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k_\phi \frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k_\theta \sin\theta \frac{\partial T}{\partial \theta}\right) + \dot{q}_g = \frac{\partial}{\partial \tau}(\rho c T)$$

If material properties k, ρ & c are constant then

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{\dot{q}_g}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

Poisson Equation
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{\dot{q}_g}{k} = 0$$

Laplace Equation
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) = 0$$

Diffusion Equation
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

2.8 Conduction Heat Transfer through Hollow Sphere

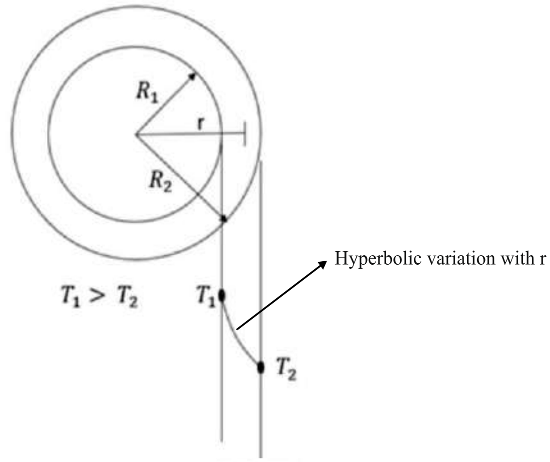

Fig 2.20 Hollow Sphere

Assumption:

- 1D (Radial Direction)
- Steady State
- No internal heat generation
- k is constant

$$\frac{T(r)-T_1}{T_2-T_1} = \frac{\frac{1}{r}-\frac{1}{R_1}}{\frac{1}{R_2}-\frac{1}{R_1}} \quad \text{(Hyperbolic variation with r)}$$

- Thermal Resistance in case of Hollow Sphere

$$(R_{th}) = \frac{(R_2 - R_1)}{4\pi\, k\, R_1 R_2}$$

- Conduction Heat Transfer through Hollow Sphere

$$\dot{Q} = \frac{(T_1 - T_2)}{\frac{R_2 - R_1}{4\pi\, k\, R_1 R_2}}.$$

2.8.1 Hollow Sphere with Convection boundary Condition

- Heat transfer rate $\quad \dot{Q} = \frac{T_h - T_1}{(R_{th})_1} = \frac{T_1 - T_2}{(R_{th})_2} = \frac{T_2 - T_a}{(R_{th})_3} = \frac{T_h - T_a}{(R_{th})_{eq}}$

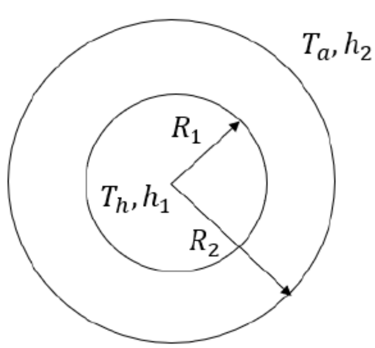

Fig 2.21 Hot fluid carrying hollow sphere exposed to the ambient

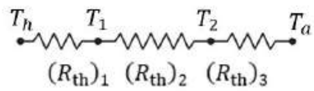

Fig 2.22 Thermal Resistance Diagram for Hollow Sphere

- $(R_{th})_{eq} = (R_{th})_1 + (R_{th})_2 + (R_{th})_3$

Where
$$(R_{th})_1 = \frac{1}{h_1 \cdot A_1} = \frac{1}{h_1 \cdot 4\pi R_1^2}$$
$$(R_{th})_2 = \frac{(R_2 - R_1)}{4\pi \, k \, R_1 R_2}$$
$$(R_{th})_3 = \frac{1}{h_2 \cdot 4\pi R_2^2}$$

2.8.2 Equivalent Area for Hollow Sphere

- In case of hollow sphere equivalent area is the geometric mean of A_1 *and* A_2 .

$$A = \sqrt{A_1 . A_2}$$

2.8.3 Solid Sphere with uniform internal heat Generation

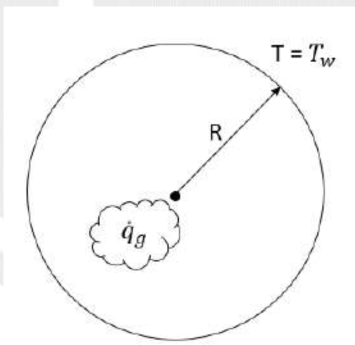

Fig 2.23 Solid sphere with uniform heat generation

Assumption

- 1D (Radial direction)
- Steady State
- Uniform Internal Heat Generation
- k is constant

$$\boxed{T(r) = T_w + \frac{\dot{q}_g}{6k}(R^2 - r^2)} \quad \text{(Parabolic Variation)}$$

At r = 0, T = T_{max}

$$T_{max} = T_w + \frac{\dot{q}_g}{6k} R^2$$

2.9 The Overall Heat Transfer Coefficient

- The heat transfer rate from hot fluid to cold fluid through plane wall is given by

$$\dot{Q}=\frac{T_h - T_c}{\left(\frac{1}{h_h A}\right)+\left(\frac{\delta}{kA}\right)+\left(\frac{1}{h_c A}\right)}$$

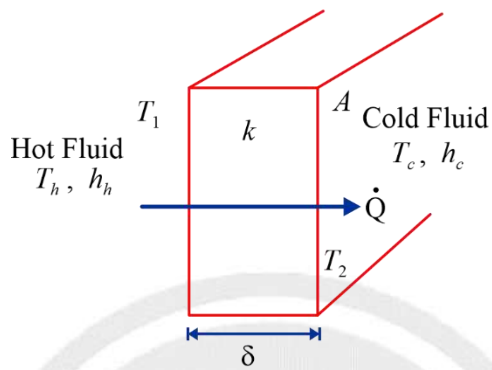

Fig. 2.24 Plane wall

- $\dot{Q}$ can be written in analogous manner to newton's law of cooling as

$$\dot{Q} = UA\,(T_h - T_c)$$

Where U → overall heat transfer coefficient $\left(\frac{W}{m^2K}\right)$

$$U=\frac{1}{\frac{1}{h_h}+\frac{\delta}{k}+\frac{1}{h_c}}$$

- Similarly, for hollow cylinder and sphere

$$\dot{Q} = U_i A_i (T_h - T_c) = U_o A_o (T_h - T_c)$$

Where U_i → Overall heat transfer coefficient on the basis of inner area.
U_o → Overall heat transfer coefficient on the basis of outer area.

2.10 Critical Radius of Insulation

- The critical radius of insulation is the radius at which heat transfer is maximum or thermal resistance minimum.

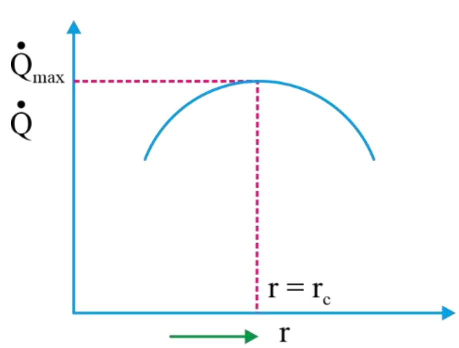

Fig. 2.25 Heat transfer variation with insulation radius

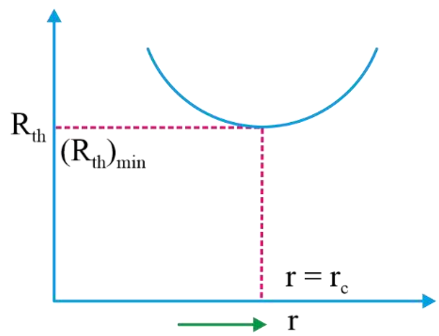

Fig. 2.26 Thermal resistance variation with insulation radius

- In case of cylinder

$$r_c = \frac{k_i}{h}$$

- In case of sphere

$$r_c = 2\frac{k_i}{h}$$

Where k_i = Thermal conductivity of insulating material.

h = Convective heat transfer coefficient

Note:

- Critical thickness: $t_c = r_c - r_o$
  (r_o = outer radius of cylinder/sphere)
- Critical diameter: $d_c = 2r_c$

- With increase in Insulation, total resistance first decreases, attain a minimum value and then increases OR with increase in insulation, rate of heat transfer first increases, attain a maximum value then start to decreases.
- If the outer radius is less than critical radius the heat transfer will be increased by adding more insulation and if outer radius is greater than the critical radius, an increase in insulation thickness will result in a decrease in heat transfer.
- No concept of critical thickness of insulation when insulation is provided inside the pipe or sphere, because providing insulation inside always decreases the heat transfer.
- Similarly, no concept of critical thickness of insulation when insulation is provided in a slab, because providing insulation always decrease the heat transfer.

❑❑❑

3 FINS

3.1 Introduction

- Fins are extended surfaces which are used to enhance the total heat transfer rate between the solid surface and the surrounding fluid

3.2 Fin Analysis

Assumptions

(i) Fin material is homogenous and isotropic

(ii) 1 D (x direction)

(iii) Steady state

(iv) No internal heat generation

(v) k = constant

(vi) Cross sectional area is uniform

(vii) Radiation heat transfer is neglected

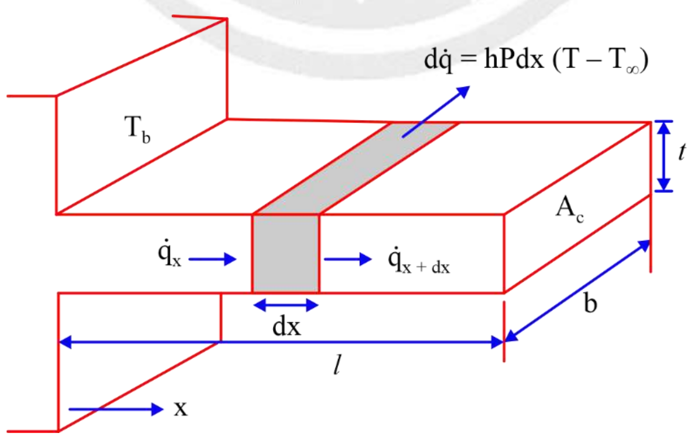

Fig 3.1 Rectangular fin of uniform cross section

Let,

k $\rightarrow$ Thermal conducting of fin material

h $\rightarrow$ Convective heat transfer coefficient

$A_c \rightarrow$ Cross sectional area

$A_s \rightarrow$ Surface area

$P \rightarrow$ Perimeter

$T_\infty \rightarrow$ Ambient temperature

$T_b \rightarrow$ Base temperature or root temperature.

$T \rightarrow$ Temperature at a distance x

$\theta = T - T_\infty,\ \theta_b = T_b - T_\infty$

Energy Balance,

$$E_{in} = E_{out}$$

$$\dot{q}_x = \dot{q}_{x+dx} + d\dot{q}_{convection}$$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$\theta = c_1 e^{mx} + c_2 e^{-mx}$$

Where $m = \sqrt{\dfrac{hP}{kA_c}}$ and $\theta = T - T_\infty$

3.3 Types of Fins

- Infinitely long fin
- Short Fin with insulated tip
- Short fin without insulated tip

3.3.1 Infinitely long fin (mL > 5)

- Temperature distribution $\dfrac{\theta}{\theta_b} = \dfrac{T - T_\infty}{T_b - T_\infty} = e^{-mx}$ (Temperature exponentially decreases with x)
- Heat transfer rate $\dot{Q}_{fin} = -k A_c \left.\dfrac{dT}{dx}\right|_{x=0}$

$$\dot{Q}_{fin} = \theta_b \sqrt{hPkA_c}$$

3.3.2 Adiabatic or Insulated tip at the fin end

- Temperature distribution

$$\frac{\theta}{\theta_b} = \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cos h\, m(L - x)}{\cos h(mL)}$$ (Temperature exponentially decreases with x)

- Heat transfer rate $\dot{Q}_{fin} = -k A_c \left.\dfrac{dT}{dx}\right|_{x=0}$

$$= \sqrt{hPkA_c}\, \theta_b \tan h(mL)$$

3.3.3 Convection at the fin end – Using concept of corrected length

- Corrected length $$L_c = L + \frac{A_c}{P}$$
- Temperature distribution $$\frac{\theta}{\theta_b} = \frac{\cosh m(L_c - x)}{\cosh(mL_c)}$$
- Heat transfer rate $$\dot{Q}_{fin} = -kA_c \left.\frac{dT}{dx}\right|_{x=o}$$
$$\dot{Q}_{fin} = \sqrt{hPkA_c}\,\theta_b \tanh(mL_c)$$

3.4 Fin Performance parameter

3.4.1 Fin efficiency (η_{fin})

- It is ratio of actual rate of heat transfer through the fin $(\dot{Q}_{act})$ to Maximum rate of heat transfer $(\dot{Q}_{max})$, if entire fin surface is maintained at the base temperature.
- Fin efficiency $$\eta_{fin} = \frac{\dot{Q}_{act}}{\dot{Q}_{max}}$$
$$= \frac{\dot{Q}_{fin}}{hA_s(T_b - T_\infty)}$$

  Where A_S is the surface area of the fin
- $$\eta_{fin_\infty} = \frac{1}{mL}$$
- $$\eta_{fin_{adia}} = \frac{\tanh(mL)}{mL}$$
- $$\eta_{fin_{non-adiabatic\ tip}} = \frac{\tanh(mLc)}{mLc}$$

3.4.2 Fin effectiveness (ε)

- It is ratio of rate heat transfer through the fin $(\dot{Q}_{fin})$ to rate of heat transfer without fin $(\dot{Q}_{without\ fin})$.

$$(\epsilon) = \frac{\dot{Q}_{fin}}{\dot{Q}_{without\ fin}}$$

- $$\epsilon_\infty = \sqrt{\frac{kP}{hA_c}}$$
- $$\epsilon_{adia} = \sqrt{\frac{kP}{hA_c}}\tanh(mL)$$

- $\epsilon_{\text{non-adiabatic tip}} = \sqrt{\dfrac{kP}{hA_c}} \tanh(mL_c)$

- The effectiveness of a fin is increased by

  (i) Using material having high thermal conductivity
  (ii) Low convective heat transfer coefficient
  (iii) Thin and closely packing of fins

3.4.5 Relation Between effectiveness (ε) and efficiency (η)

$$\frac{\text{Fin effectiveness}}{\text{Fin efficiency}} = \frac{A_s}{A_c}$$ (A_s – Surface area, A_c = cross sectional area)

Note:

1. If h = mk, $\dot{Q}_{fin} = \dot{Q}_{w/o\,fin}$ No use of fin
2. If h > mk, $\dot{Q}_{fin} < \dot{Q}_{w/o\,fin}$ Fin acts like insulator i.e., using fin decreases the heat transfer
3. If h < mk, $\dot{Q}_{fin} > \dot{Q}_{w/o\,fin}$ Using fin increases the heat transfer.

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4 TRANSIENT CONDUCTION OR UNSTEADY STATE HEAT CONDUCTION

4.1 Introduction

- To solve the problems of unsteady state conduction heat transfer 3 methods are used.

  (1) Lumped parameter analysis method

  (2) Heisler Chart Method

  (3) Error function Method

4.2 Biot Number

- It is a dimensionless number which is ratio of internal conduction resistance (ICR) within a solid body to external convection resistance (ECR) between the surface of solid and fluid media.

$$Bi = \frac{\text{Internal Conduction Resistance (ICR)}}{\text{External Convection Resistance (ECR)}} = \frac{\frac{L_c}{k}}{\frac{1}{h}} = \frac{hL_c}{k}$$

- L_c is the characteristic length of body

$$L_c = \frac{\text{Volume}}{\text{Surface area responsible for heat transfer}}$$

4.3 Lumped Parameter analysis method

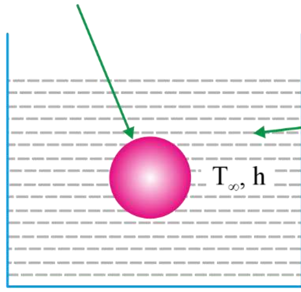
- Lumped parameter analysis method is used where ICR is very small as compare to ECR hence Bi < 0.1
- Lumped parameter analysis method is applicable where

  (i) h is very less

  (ii) L_c is very less

  (iii) k is very high (infinite thermal conductivity problems/ Negligible temperature gradient Problem)

Note:
- Since negligible temperature gradient exist therefore temperature is only function of time within the solid. T = *f*(time)



(Assume $T_i > T_\infty$)

Fig 4.1 Cooling of a hot solid body

Let,

m = mass of the solid body

ρ = density of solid body

c = specific heat of solid body

k = Thermal conductivity of solid

V = Volume of solid body

A = Surface area of solid body

T_∞ = Temperature of fluid

T_i = initial temperature of solid body

T = Temperature of solid body at any time τ

- (Neglect the effect of Radiation)
  From energy balance, at any time τ
  Rate of change of internal energy of solid body = Rate of convection heat transfer

$$-mc\frac{dT}{d\tau} = hA(T - T_\infty)$$

$$\boxed{\frac{T - T_\infty}{T_i - T_\infty} = e^{-\frac{hA}{\rho Vc}\tau} = e^{-B_i F_o} = e^{-\frac{\tau}{\tau^*}}}$$

(Applicable for both heating as well as cooling)

Where

Bi – Biot Number $\left(Bi = \frac{hL_c}{k}\right)$

F_o – Fourier number $\left(F_o = \frac{\alpha\tau}{L_c^2}\right)$,

α → Thermal diffusivity $\left(\frac{k}{\rho c}\right)$

τ^* – Thermal time constant $\left(\tau^* = \frac{\rho Vc}{hA}\right)$

- Temperature of solid body exponentially decreases with time or exponentially increases (when body is cool and medium is hot.

$$\frac{dT}{d\tau} = \text{Rate of heating or cooling}$$

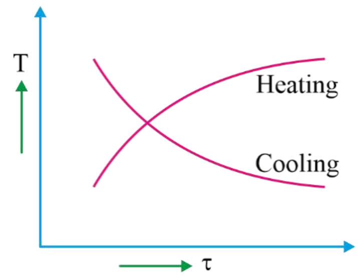


Fig 4.2 Transient temperature response of lumped capacitance solids for heating and cooling

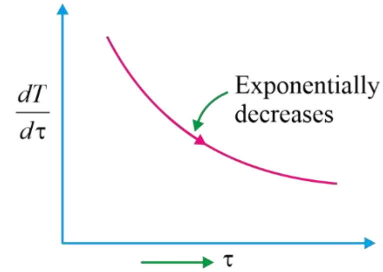


Fig 4.3 Rate of heating or cooling variation with time for lumped capacitance solids

4.3.1 The Characteristic Length (L_c)

$L_c = \dfrac{\text{Volume (V)}}{\text{Surface area exposed to surrounding (A)}}$	
Characteristic Length for sphere	$L_c = \dfrac{\frac{4}{3}\pi R^3}{4\pi R^2} = \dfrac{R}{3}$
Characteristic Length for solid cylinder	$L_c = \dfrac{\pi R^2 l}{2\pi R(l+R)}$ if $l >> R, L_c = \dfrac{R}{2}$
Characteristic Length for cube	$L_c = \dfrac{L^3}{6L^2} = \dfrac{L}{6}$
Characteristic Length for rectangular plate	$L_c = \dfrac{lbt}{2(lb+bt+lt)}$ if $l, b \gg t$ then, $L_c = \dfrac{t}{2}$
Characteristic Length for hollow cylinder	$L_c = \dfrac{\pi\left(r_0^2 - r_i^2\right)l}{2\pi r_0 l + 2\pi r_i l + 2\pi\left(r_0^2 - r_i^2\right)}$

4.3.2 Thermal Time Constant or Time Constant

- $\tau^* = \dfrac{\rho V c}{hA}$
- Shorter time constant means body respond to temperature change very fast.
- Time constant for thermocouple must be small.

Shorter Time Constant can be achieved by

- Decreasing V/A
- Using low density and low specific heat materials.
- Increasing the heat transfer coefficient

4.4 Heisler Chart Method

- Heisler Chart Method is used to determine the temperature variation and heat flow rate when both conduction and convection resistance are almost of equal importance ($0.1 < Bi < 100$)

4.5 Error Function Method

- Error Function Method is used to determine the temperature variation and heat flow rate when conduction resistance is very high as compare to convection resistance ($Bi > 100$).

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5 CONVECTION HEAT TRANSFER

5.1 Newton's Law of Cooling:

- Convection heat transfer between a surface and fluid is governed by Newton's law of cooling.

$$\dot{Q} = hA(T_s - T_\infty)$$

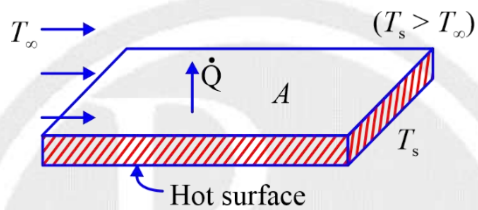

Fig 5.0 Convection heat transfer between hot surface and fluid

Where, $\dot{Q} =$ Rate of convection heat transfer

$A =$ Area exposed to heat transfer

$T_s =$ Surface temperature

$T_\infty =$ Fluid temperature

h = Coefficient of convective heat transfer

5.2 Coefficient of Convective Heat Transfer (h)

- It is defined as amount of heat transfer for a unit temperature difference between the fluid and unit area of surface in unit time.
- The unit of h

$$h = \frac{\dot{Q}}{A(T_s - T_\infty)} \quad W/m^2{}^\circ C \quad \text{or} \quad W/m^2K$$

- The value of ' $\boldsymbol{h}$ ' depends on the following factors
  (i) Thermodynamic and transport properties (e.g., viscosity. density, specific heat etc.)
  (ii) Nature of convection (Free/forced/mixed)
  (iii) Nature of fluid flow (Laminar/Turbulent)
  (iv) Type of fluid (Liquid/Gas)
  (v) Surface finish of plate (Smooth/Rough)
  (vi) Orientation of surface (Horizontal/Vertical/Inclined)

Type of convection	Range of ‘h’ (W/m^2-K)
Free convection in gases	3 – 25
Forced convection in gases	25 – 400
Free convection in liquids	50 – 350
Forced convection in liquids	350 – 3000
Condensation	3000 – 25000
Boiling	5000 – 50000

5.3 Criteria for Convection Heat Transfer

- Forced Convection ($Gr/Re^2 << 1$)

  Nu = f(Re, Pr)

- Free Convection or Natural Convection ($Gr/Re^2 >> 1$)

  Nu = f(Gr, Pr)

- Mixed Convection ($Gr/Re^2 = 1$)

  Nu = f(Re, Pr, Gr)

  Where

  Nu = Nusselt number

  Re = Reynolds number

  Pr = Prandtl number

  Gr = Grashof number

5.4 Important Dimensionless numbers used in Convection Heat Transfer

5.4.1 Nusselt Number (Nu)

1. It is ratio of conduction resistance to convection resistance offered by the fluid.

$$Nu = \frac{(R_{th})_{\text{conduction}}}{(R_{th})_{\text{convection}}}$$

2. Nusselt number also defined as ratio of convection heat transfer rate to the conduction heat transfer rate in fluid.

$$Nu = \frac{\dot{Q}_{\text{convection}}}{\dot{Q}_{\text{conduction}}}$$

$$Nu = \frac{hL_c}{k_f}$$

h = convective heat transfer coefficient
Lc = Characteristic length
k_f = thermal conductivity of fluid.

3. Larger Nusselt number means move effective convection heat transfer.

5.4.2 Reynold Number (Re)

- It is ratio of Inertia force to Viscous force.

  Mathematically,

$$Re = \frac{\rho V L_c}{\mu} = \frac{V L_c}{\nu}$$

where ρ = Density of fluid (kg/m^3)

V = Velocity of Fluid (m/s)

L_c = Characteristics length (m)

μ = Viscosity of fluid (Pa-s)

ν = kinematic viscosity (m^2/s)

5.4.3 Prandtl Number (Pr)

- It is ratio of kinematic viscosity (momentum diffusivity) to the thermal diffusivity.

  Mathematically

$$Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k_f}$$

Where μ = Viscosity of fluid (Pa-s)

c_p = Specific heat at constant pressure (J/kgK)

k_f = Thermal conductivity of fluid (W/mK)

5.4.4 Grashof Number (Gr)

- It is ratio of Buoyant force to viscous force.

  Mathematically

$$Gr = \frac{\text{Buoyancy force}}{\text{Viscous force}}$$

$$= \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

Where g = Gravitational acceleration (m/s^2)

β = Coefficient of volume expansion (K^{-1})

T_s = Temperature of surface (K)

T_∞ = Free stream temperature (K)

L_c = Characteristic length of the geometry (m)

ν = Kinematic viscosity of the fluid (m^2/s)

5.5 Forced Convection

1. External forced Convection
   (a) Constant Wall Temperature
   (b) Constant heat flux
2. Internal forced Convection
   (a) Constant Wall Temperature
   (b) Constant heat flux

5.5.1 External Forced Convection: Constant Wall Temperature: $T_s \neq f(x)$ = Constant

T_s = Plate temperature/wall temperature

T_∞ = Free stream temperature

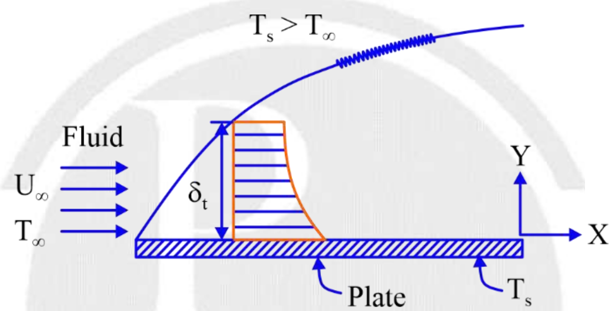

Fig 5.1 Thermal boundary layer for flow over a hot isothermal flat plate

Energy balance

At y = 0

Rate of conduction heat transfer in fluid = Rate of convection heat transfer

$$-k_f A \left.\frac{\partial T}{\partial y}\right|_{y=0} = hA(T_s - T_\infty)$$

$$\Rightarrow \qquad h = \frac{-k_f \left.\frac{\partial T}{\partial y}\right|_{y=0}}{T_s - T_\infty}$$

Local Nusselt number $\qquad Nu_x = \frac{h_x L_c}{k_f}$

$$\Rightarrow \qquad Nu_x = \frac{-\left.\frac{\partial T}{\partial y}\right|_{y=0} L_c}{T_s - T_\infty}$$

- Relation between Local convective heat transfer coefficient (h_x) and average convective heat transfer coefficient ($\bar{h}$)

$$\bar{h} = \frac{1}{L}\int_0^L h_x dx$$

5.5.2 Nusselt number in case of constant wall temperature

Laminar Flow	Turbulent Flow
($Re_x \leq 5 \times 10^5$)	($Re_x > 5 \times 10^5$)
• $Nu_x = 0.332\, Re_x^{0.5}\, Pr^{0.33}$ $\boxed{h_x \propto x^{-1/2}}$ • $\overline{Nu} = 0.664\, Re_L^{0.5}\, Pr^{0.33}$ $\boxed{\bar{h} = 2h_x}$	• $Nu_x = 0.029\, Re_x^{0.8}\, Pr^{0.33}$ $\boxed{h_x \propto x^{-1/5}}$ • $\overline{Nu} = 0.037\, Re_L^{0.8}\, Pr^{0.33}$ $\boxed{\bar{h} = \frac{5}{4} h_x}$

5.5.3 Nusselt Number in case of constant heat flux

Laminar Flow	Turbulent Flow
• $Nu_x = 0.453\, Re_x^{0.5}\, Pr^{0.33}$	• $Nu_x = 0.0308\, Re_x^{0.8}\, Pr^{0.33}$

Note:
- Nusselt No. & heat transfer coefficient in case of constant heat flux case is approximately 36% more as compare to constant wall temperature case. (In case of laminar flow)
- Nusselt No. & heat transfer coefficient in case of constant heat flux case is approximately 4% more as compare to constant wall temperature case (In case of turbulent flow)

5.5.4 Variation of h_x with nature of flow for a flat plate

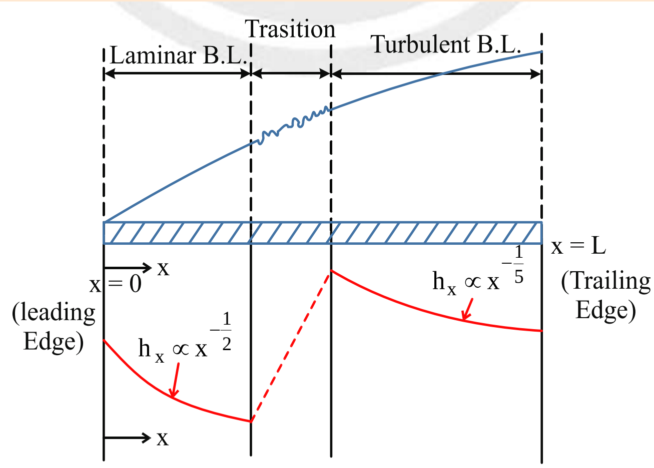

Fig 5.2 Convective heat transfer coefficient variation in flow over flat plate

Note:

Same variation occurs for local friction coefficient (C_{fx}) and for local wall shear stress (τ_{wx}).

5.5.5 Relation between Thermal Boundary Layer thickness and hydrodynamic boundary layer thickness

Laminar Flow	Turbulent Flow
$\frac{\delta_{hx}}{\delta_{tx}} = \Pr^{\frac{1}{3}}$	$\delta_{hx} \approx \delta_{tx}$

Note:

In case of laminar flow 3 cases are possible,

1. $P_r < 1$, $\delta_{hx} < \delta_{tx}$ Ex: Liquid metals.
2. $P_r > 1$, $\delta_{hx} > \delta_{tx}$ Ex: heavy oils
3. $P_r = 1$, $\delta_{hx} = \delta_{tx}$ Ex: gases

5.5.7 Reynold's Analogy

- It relates local convective heat transfer coefficient with local friction coefficient.

$$\frac{Nu_x}{\mathrm{Re}_x \Pr} = St_x = \frac{Cf_x}{2} (\text{when} \Pr = 1)$$

$$\left(St_x = \frac{h_x}{\rho U_\infty c_p} \right)$$

- Reynold's-Colburn Analogy

$$\mathrm{St}_x . \Pr^{2/3} = \frac{Cf_x}{2}$$

Where St_x = Local Stanton number

C_{fx} = Local skin friction coefficient

5.6 Internal Forced Convection

5.6.1 Hydrodynamic Boundary Layer Development

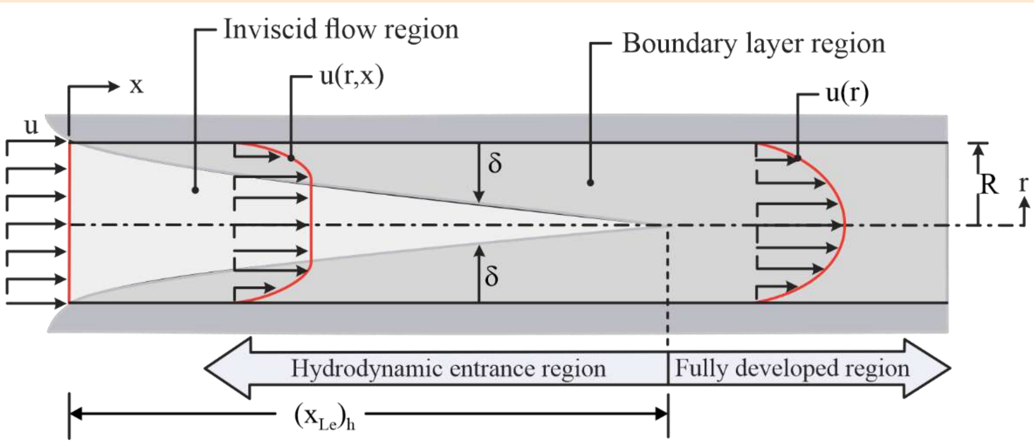

Fig 5.3 Hydrodynamic boundary layer development in internal flow

- In Hydrodynamic entrance region (Developing region) velocity of fluid layer is function of r as well as x, where as in developed region velocity is function of r only.
- In developed region $\left.\frac{du}{dx}\right|_r = 0$.

5.6.2 Thermal Boundary Layer Development

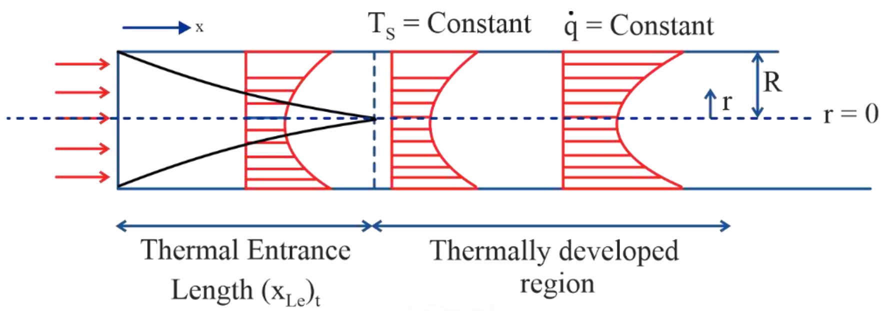

Fig 5.4 Thermal boundary layer development in internal flow

5.6.3 Thermal entrance length $(x_{Le})_t$

Laminar Flow	Turbulent flow
$(x_{Le})_t = 0.05 \text{ Re Pr D}$ $= (x_{Le})_h \text{ Pr}$	$(x_{Le})_t \approx 10 \text{ D}$

- For Thermally fully developed flow, note the following important points
  - (i) h/k is a constant, h will be a constant, if k = constant
  - (ii) Nu is a constant
  - (iii) θ is not a function of x, where $\theta = \frac{T(x,r)-T_s(x)}{T_m(x)-T_s(x)}$ $i.e \frac{d\theta}{dx} = 0$
- For thermal entry region, h is inversely proportional to x

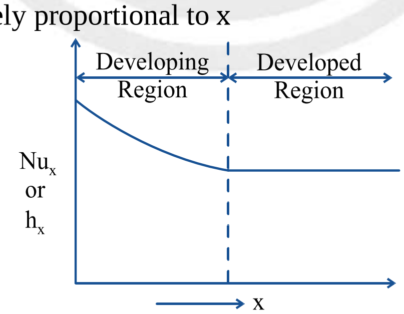

Fig 5.5 Convective heat transfer coefficient variation in internal flow

5.6.4 Average Velocity or Bulk mean velocity or Mean velocity (U_{avg})

- It is the hypothetical uniform velocity at a particular section that gives the same mass flow rate as actual velocity profile gives.

$$U_{avg} = \frac{2}{R^2}\int_0^R u(r).rdr$$

5.6.5 Bulk Mean Temperature /Average Temperature /Mixing cup temperature (T_m)

- It is hypothetical uniform temperature at a particular section that gives the same enthalpy rate as actual temperature profile gives.

$$T_m = \frac{\int_0^R u(r)T(r).rdr}{\int_0^R u(r).rdr}$$

5.6.6 Boundary Conditions for Internal flow Forced Convection

- In internal flow (L_c = D)

$$N_u = \frac{hL_c}{k_f} = \frac{h \times D}{k_f}$$

Laminar		Turbulent
Constant wall temperature $N_u = \frac{hD}{k_f} = 3.66$	Constant heat flux $N_u = \frac{hD}{k_f} = 4.364$	Ditties-Boelter Equation $Nu = \frac{hD}{k_f} = 0.023\,Re^{0.8}\,Pr^n$ $n = 0.4 \Rightarrow$ Heating $n = 0.3 \Rightarrow$ cooling

5.6.7 Mean Temperature Variation in Constant Wall Temperature Case

$T_s > T_{mi}$ (assume)

- T_s = constant, $T_s \neq f(x)$, $T_m = f(x)$
- Temperature distribution

$$\frac{T_m(x) - T_s}{T_{mi} - T_s} = e^{-\frac{hPx}{\dot{m}c_p}}$$

- Rate of heat transfer $(\dot{Q})$

$$\dot{Q} = \dot{m}c_p\left[T_{mi}(x) - T_{me}\right] = hA\left[\frac{\theta_1 - \theta_2}{\ln\frac{\theta_1}{\theta_2}}\right]$$

A = Area responsible for convection

$\theta_1 = T_s - T_{mi}$, $\theta_2 = T_s - T_{me}$

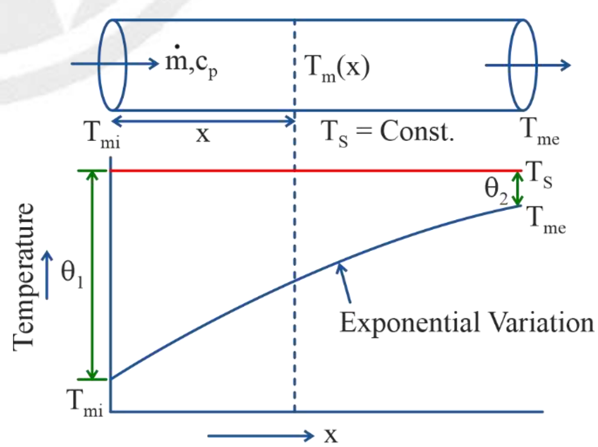

Fig 5.6 Mean temperature variation in internal flow with constant wall temperature

5.6.8 Mean Temperature Variation in Constant Heat Flux Case

$\dot{q}$ = constant, $\dot{q} \neq f(x)$

- $\frac{dT_m}{dx} = 0$ (T_m varies Linearly)
- In developing region

  $h\downarrow$, $(T_s - T_m)\uparrow$
- In developed region

  h = constant, $(T_s - T_m)$ = constant

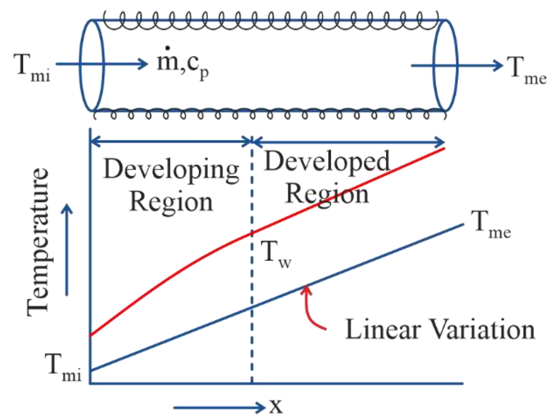

Fig 5.7 Mean temperature variation in internal flow with constant heat flux

5.7 Free or Natural Convection

- In free convection or natural convection motion of fluid particles occurs due to buoyancy which induced due to variation of density in the presence of temperature gradient.

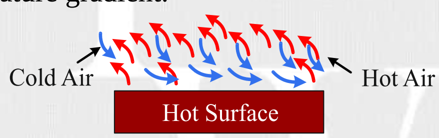

Fig 5.8 Natural convection

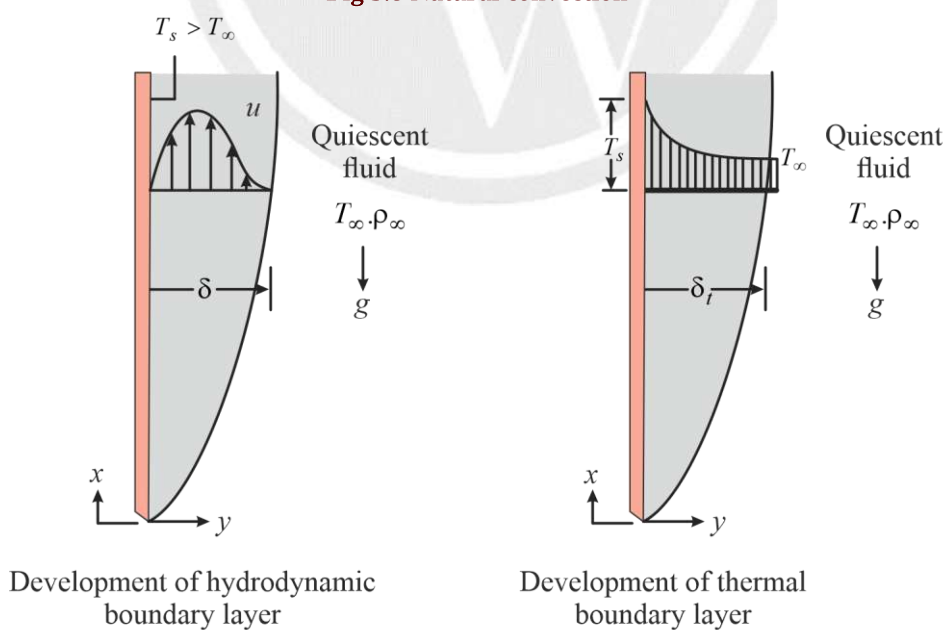

Fig 5.9 Hydrodynamic and thermal boundary layer development on a heated vertical plate.

Note:

If $\frac{Gr}{Re_L^2} \gg 1$, Free convection is predominant over forced convection.

5.7.1 Grashof Number (Gr)

- $$\text{Gr} = \frac{\text{buoyancy force}}{\text{viscous force}} = \frac{g\beta\Delta T L_c^3}{\nu^2} = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

Where

g = Gravitational acceleration (m/s^2)

T_s = Temperature of surface (K)

T_∞ = Temperature of fluid far from the surface (K)

L_c = Characteristic length of the geometry (m)

ν = Kinematic viscosity of the fluid (m^2/s)

β = Coefficient of volume expansion or isobaric expansivity (1/K)

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

$$\beta = \frac{1}{T_f}(\text{For ideal gas})$$

$$T_f = \frac{T_s + T_\infty}{2}$$

T_f = film temperature

5.7.2 Rayleigh Number (Ra)

- Rayleigh number is product of Grashof number (Gr) and Prandtl number (Pr)

$$\text{Ra} = \text{Gr} \times \text{Pr}$$

- $10^4 < \text{Ra} < 10^9$ – Laminar Boundry layer

  $\text{Ra} > 10^9$ – Turbulent boundary layer

5.7.3 Nusselt Number Expression for Free Convection Case

- Nu = f(Gr, Pr) or Nu = f(Ra)

  $\text{Nu} = \text{C Ra}^{1/4}$(Laminar)

  $\text{Nu} = \text{C Ra}^{1/3}$ (Turbulent)

  Where C is a Constant.

- For Vertical Plate or vertical Cylinder:

  $\text{Nu} = 0.59\ \text{Ra}^{1/4}$ (Laminar)

  $\text{Nu} = 0.13\ \text{Ra}^{1/3}$ (Turbulent)

- For Horizontal Plate/Horizontal Cylinder:

  $\text{Nu} = 0.54\ \text{Ra}^{1/4}$ (Laminar)

  $\text{Nu} = 0.14\ \text{Ra}^{1/3}$ (Turbulent)

Note:
In case of vertical cylinder characteristic length is length of cylinder where as in case of horizontal cylinder characteristic length is diameter of cylinder.

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6 HEAT EXCHANGER

6.1 Introduction

- A heat exchanger is a device which is used to transfer the heat energy from one fluid (hot fluid) to another fluid (cold fluid). The cold fluid is heated where as hot fluid gets cooled down.

6.2 Type of Heat Exchanger

6.2.1 On the basis of nature of heat exchange process:

(i) **Direct contact type heat exchanger:** In Direct contact type heat exchanger hot fluid and cold fluid are directly in contact with each other.
Ex. Cooling Tower, Open feed water heater, Jet Condenser

(ii) **Storage type or Regenerator type Heat exchanger:** Regenerator is the type of heat exchanger where heat energy from the hot fluid is intermittently stored in a thermal storage medium before it is transferred to the cold fluid. To accomplish this the hot fluid is brought into contact with the heat storage medium, then the hot fluid is replaced with the cold fluid, which absorbs the heat.
Ex. Heat exchanger used in gas turbines, Air Preheaters in steam power plant

(iii) **Transfer type or Recuperator Type:** Recuperator is a type of heat exchanger which has separate flow paths for each fluid and heat is transferred through the separating walls.
Ex. Concentric Tube type heat exchanger, Shell and tube type heat exchanger etc.

6.2.2 On the basis of relative direction of fluid motion:

1. Parallel flow/Co-current flow Heat Exchanger

- In parallel flow heat exchanger two fluids are moving in the same direction.

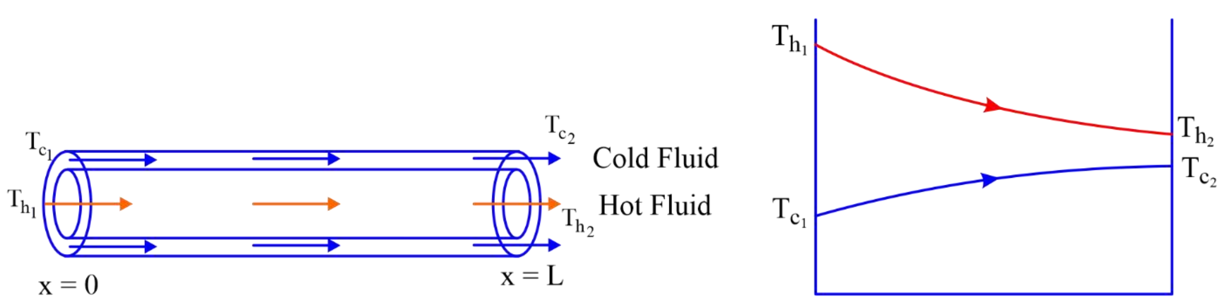

Fig. 6.1 Parallel flow heat exchanger

2. Counter-Flow Heat Exchanger

- In counter-flow heat exchanger two fluids are moving in the opposite direction.

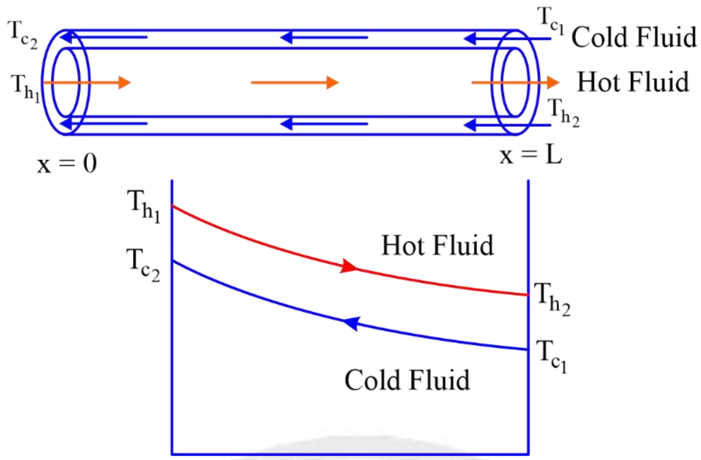

Fig. 6.2. Counter flow heat exchanger

3. Cross-Flow Heat Exchanger

- In cross flow heat exchanger two fluids are crossing at 90°

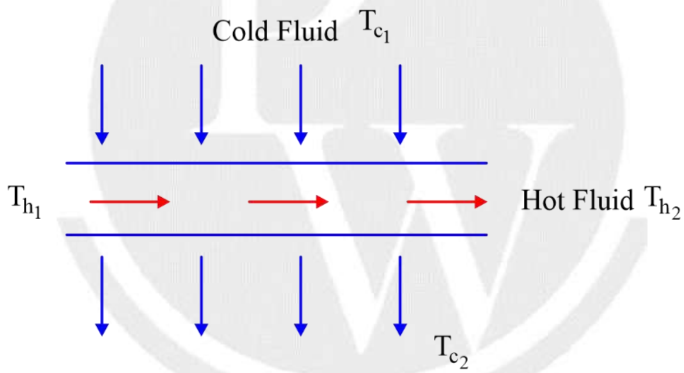

Fig. 6.3 Cross flow heat exchanger

6.2.3 On the basis of Phase change of the fluid

1. Evaporator type:

- Cold fluid gets evaporated by taking the heat energy from hot fluid.

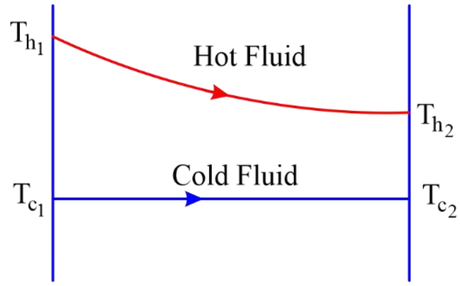

Fig. 6.4 Evaporator type Parallel Flow Heat Exchange

2. Condenser type:

- Hot fluid is condensed and releases latent heat of condensation which is absorb by the cold fluid and gets heated.

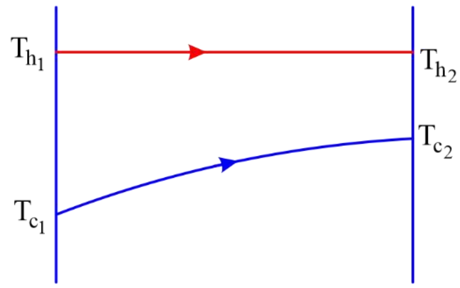

Fig. 6.5 Condenser type Parallel Flow Heat Exchange

6.3 Heat Exchanger Analysis

Let, $\dot{m}$ = Mass flow rate $\left(\frac{kg}{s}\right)$

c_p = Specific heat of fluid at constant pressure $\left(\frac{\text{J}}{\text{kg}\,°\text{C}}\right)$

T = Temperature of fluid

- Subscripts h and c refer to the hot and cold fluids respectively and 1 and 2 corresponds to the inlet and outlet conditions respectively.

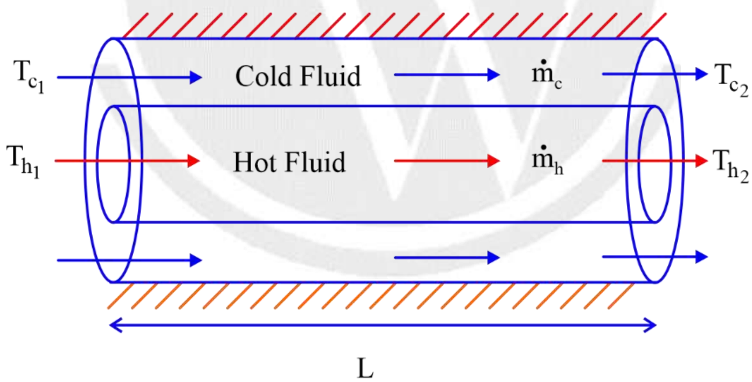

Fig. 6.6. Analysis of heat exchanger

Assumptions

- Flow is steady
- Heat Exchanger is adiabatic
- Change in Kinetic energy and potential energy are negligible.

Rate of heat energy lost by the hot fluid

$$\dot{Q} = \dot{m}_h c_{ph} (T_{h_1} - T_{h_2})$$

Rate of heat energy gained by the cold fluid

$$\dot{Q} = \dot{m}_c c_{pc} (T_{c_2} - T_{c_1})$$

Total heat transfer rate in the heat exchanger

$$\dot{Q} = UA\theta_m$$

By Energy Balance,

$$\dot{m}_c c_{pc}(T_{c_2} - T_{c_1}) = \dot{m}_h c_{ph}(T_{h_1} - T_{h_2}) = UA\theta_m$$

Where U = Overall heat transfer coefficient (W/m^{2}k)

A = Effective heat transfer area, and (m^2)

θ_m = Logarithmic Mean Temperature Difference (LMTD)

6.3.1 Logarithmic Mean Temperature Difference (LMTD)

- LMTD = $\dfrac{\theta_1 - \theta_2}{\ln \dfrac{\theta_1}{\theta_2}}$

  Where θ_1 = Temperature difference between hot and cold fluid at x = 0

  θ_2 = Temperature difference between hot and cold fluid at x = L

- For parallel flow heat exchanger

$$\theta_1 = T_{h_1} - T_{c_1}$$

$$\theta_2 = T_{h_2} - T_{c_2}$$

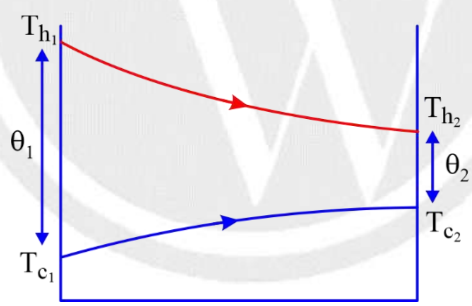

Fig. 6.7 Temperature profile for parallel flow heat exchanger

- LMTD for Counter flow heat exchanger

$$\theta_1 = T_{h_1} - T_{c_2}$$

$$\theta_2 = T_{h_2} - T_{c_1}$$

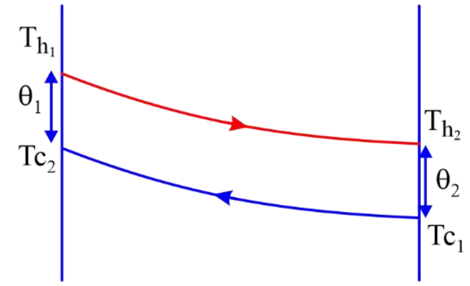

Fig. 6.8 Temperature profile for counter flow heat exchanger

- For Evaporator and condenser type heat exchanger, LMTD will be same either flow is parallel flow or counter flow.

Note:

Balance type heat exchanger

- Balanced type of heat exchanger is a counter flow heat exchanger with equal heat capacity rate.
- Temperature profile for hot fluid and cold fluid are straight line and parallel to each other.

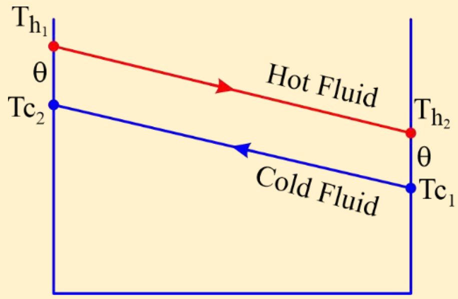

Fig. 6.8. (a) Temperature profile for balanced type heat exchanger

- For Balanced type heat exchanger.

$$\text{LMTD} = \theta_1 = \theta_2$$

6.4 Fouling Factor

- Fouling in relation to heat exchangers is the deposition and accumulation of unwanted material such as scale, suspended solids, insoluble salts and even algae on the internal surfaces of the heat exchanger.
- The fouling factor represents the theoretical resistance to heat flow due to the build-up of a fouling layer on the tube surfaces of the heat exchanger.
- Mathematically $R_f = \frac{1}{U_{foul}} - \frac{1}{U_{clean}}$

Where U_{foul}: Overall heat transfer coefficient for foul surface

U_{clean}: Overall heat transfer coefficient for clean surface

6.5 Effectiveness of Heat Exchanger

- The heat exchanger effectiveness (ε) is defined as the ratio of actual heat transfer to the maximum possible heat transfer. Thus

$$\varepsilon = \frac{\text{Actual heat transfer}}{\text{Maximum possible heat transfer}}$$

$$= \frac{\dot{Q}}{\dot{Q}_{\max}}$$

$$\varepsilon = \frac{\dot{m}_h c_{ph}\left(T_{h_1} - T_{h_2}\right)}{\left(\dot{m}c_p\right)_{\min}\left(T_{h_1} - T_{c_1}\right)} = \frac{\dot{m}_c c_{pc}\left(T_{c_2} - T_{c_1}\right)}{\left(\dot{m}c_p\right)_{\min}\left(T_{h_1} - T_{c_1}\right)}$$

6.6 Effectiveness-NTU method

6.6.1 Number of Transfer Unit (NTU)

$$\text{NTU} = \frac{\text{UA}}{(\dot{m}c_p)_{\min}}$$

Where U = Overall heat transfer coefficient (W/m^{2}K)

$\dot{m}$ = Mass flow rate (kg/s)

$(\dot{m}c_p)_{\min}$ = Minimum heat capacity rate (W/K)

6.6.2 Effectiveness (ε) of parallel/counter flow heat exchanger

$$\epsilon_{\text{parallel}} = \frac{1-e^{-NTU(1+C)}}{1+C}$$

$$\epsilon_{\text{counter}} = \frac{1-e^{-NTU(1-C)}}{1-C\cdot e^{-NTU(1-C)}}$$

Where C = Heat capacity ratio

$$= \frac{(\dot{m}c_p)_{\min}}{(\dot{m}c_p)_{\max}}$$

6.6.3 Special cases

- Heat exchanger of equal heat capacity rate (C = 1)

$$\epsilon_{\text{parallel}} = \frac{1-\exp^{(-2NTU)}}{2}$$

$$\epsilon_{\text{counter}} = \frac{\text{NTU}}{1+\text{NTU}}$$

- For evaporator and condenser type heat exchanger (C = 0)

$$\epsilon_{\text{parallel}} = \epsilon_{\text{counter}} = 1-\exp^{(-NTU)}$$

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7 RADIATION HEAT TRANSFER

7.1 Introduction

- Radiation is the energy emitted by matter by virtue of their own temperature due to thermal excitation of the molecules.
- Radiation is assumed to propagate in the form of electromagnetic waves. In heat transfer study, we are interested in thermal radiation.
- Thermal Radiation wavelength ranges from 0.1 μm to 100 μm. Thermal Radiation consists

  (a) Some part of UV rays – (0.1 μm – 0.4 μm)

  (b) Complete Visible lights – (0.4 μm – 0.7 μm)

  (c) Infrared rays – (0.7 μm – 100 μm)
- Thermal radiation exhibit characteristics similar to visible light.
- Thermal radiation can be reflected, refracted and are subject to scattering and absorption when they passes through medium.

7.2 Absorptivity (α), Reflectivity (ρ), Transmittivity (τ)

- Absorptivity (α) : Fraction of incident radiation absorbed.
- Reflectivity (ρ) : Fraction of incident radiation reflected.
- Transmittivity (τ) : Fraction of incident radiation transmitted.

Energy Balance

$$\dot{Q}_a + \dot{Q}_t + \dot{Q}_r = \dot{Q}_i$$

$$\frac{\dot{Q}_a}{\dot{Q}_i} + \frac{\dot{Q}_t}{\dot{Q}_i} + \frac{\dot{Q}_r}{\dot{Q}_i} = 1$$

$$\boxed{\alpha + \tau + \rho = 1}$$

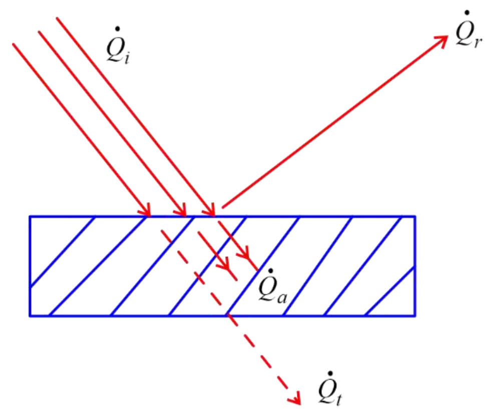

Fig 7.1 Absorption, reflection, and transmission processes associated with a semi-transparent medium.

- For black body

$$\alpha = 1, \rho = 0, \tau = 0$$

- For opaque body

$$\tau = 0, \alpha + \rho = 1$$

- For white body or perfect reflector

$$\rho = 1, \alpha = 0, \tau = 0$$

7.3 Black Body

$$\alpha = 1, \rho = 0, \tau = 0$$

- A body is said to be black if it absorbs all incident radiation.
- The term black is used, since most black coloured surfaces normally shows high value of absorptivity.
- There are some surfaces which absorb nearly all incident radiation but do not appear black.
- Therefore, we can say, black surfaces are black body but black body need not be black in colour.

  Ex. ice, snow etc.

- Black body is a perfect emitter as well as a perfect absorber.
- At a given temperature and wavelength, no surface can emit more radiation energy than a black body.
- A black body is a diffuse emitter which means it emits radiation uniformly in all direction. Also, a black body absorbs all incident radiation regardless of wavelength and direction.
- A large cavity with a small opening closely resembles to a black body.

7.4 Emissivity

7. 4. 1 Total Emissivity or Emissivity

- Emissivity of a surface is defined as the ratio of radiation emitted by the surface to the radiation emitted by a black body at the same temperature.

  Total Emissivity $(\varepsilon) = \dfrac{E(T)}{E_b(T)}$

7. 4. 2 Spectral Emissivity or Monochromatic Emissivity

- Spectral Emissivity of a surface is defined as the ratio of spectral emissive power of the surface to the spectral emissive power a black body at the same temperature.

$$\text{Spectral Emissivity } (\varepsilon_\lambda) = \frac{E_\lambda(T)}{E_{b\lambda}(T)}$$

7.5 Radiation Intensity (I_e)

- It is defined as the amount of energy emitted per unit solid angle by per unit area of the radiating surface.

$$I_e = \frac{E}{\pi}$$

7.6 Irradiation (G) & Radiosity(J)

7.6.1 Irradiation (G)

- Total incident radiation on a surface from all directions per unit time and per unit area of surface.

7.6.2 Radiosity (J)

- It refers to all of the radiant energy leaving a surface per unit time, per unit area of surface.

$$J = E + \rho G$$

Where

E = Total emissive power (W/m^2)

ρ = Reflectivity

G = Irradiation (W/m^2)

J = Radiosity (W/m^2)

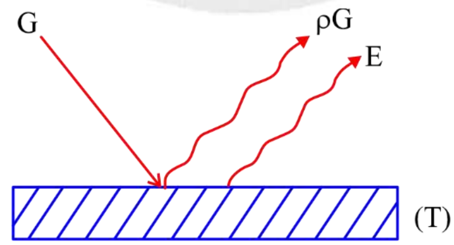

Fig 7.2 Irradiation and radiosity associated with an opaque, non-black surface

7.7 Gray Body

- A body for which emissivity is constant, is known as Gray body

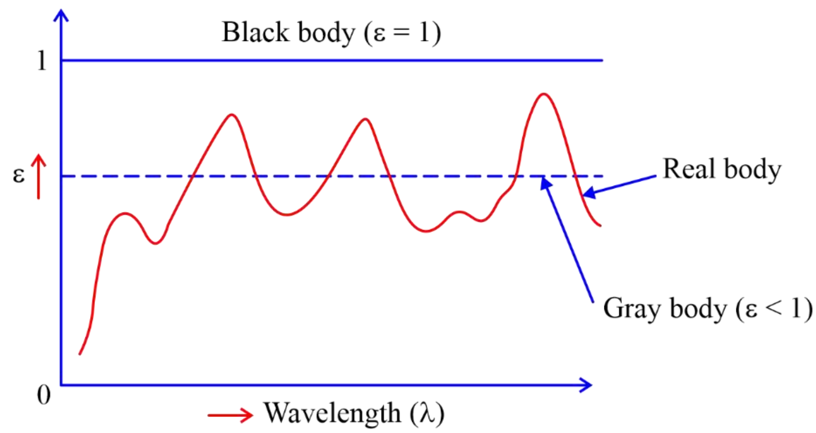

Fig 7.3 Emissivity Distribution for Black, Real and Grey body

7. 8 Various Laws used in Radiation

7. 8. 1 Planck's Law

- Planck's law is the basic law of radiation, other laws are derived from Planck's law.
- As per Planck's law for a given black body, at a given temperature monochromatic emissive power strongly depends on wavelength.

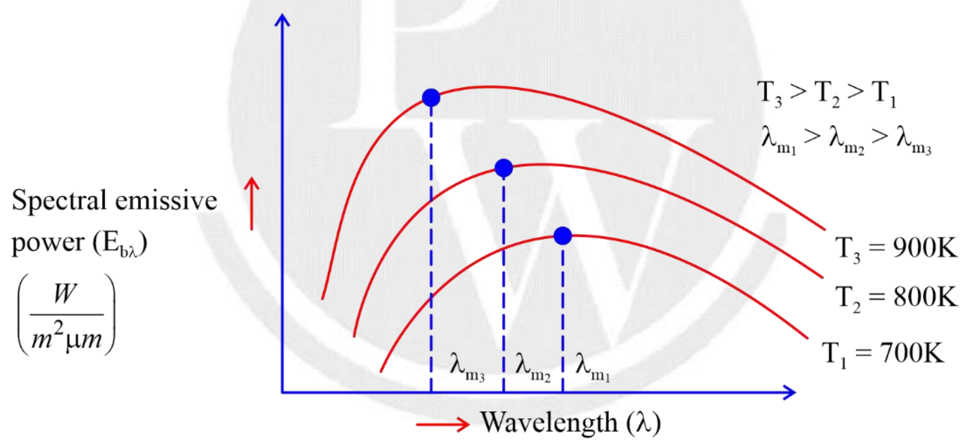

Fig 7.4 Spectral Emissive Power Distributing for a Black Body

According to this,

$$E_{b\lambda} = \frac{2\pi c^2 h\lambda^{-5}}{\exp\left(\frac{ch}{\lambda kT}\right)-1} = \frac{C_1\lambda^{-5}}{\exp\left(\frac{C_2}{\lambda T}\right)-1}$$

where,

$E_{b\lambda}$ = Monochromatic (single wavelength) emissive power of a black body,

c = Velocity of light in vacuum

h = Planck's constant

λ = Wavelength, (in μm)

k = Boltzmann constant

C_1 = 3.74×10^{-16} W-m^2, $C_2 = 1.438 \times 10^{-2}$ mK

T = Absolute temperature, K

Observation:

(i) Corresponding to particular temperature, monochromatic emissive power of a black body first increases, reaches maximum and then start to decrease.

(ii) As temperature increases, wavelength at which monochromatic emissive power is maximum shifted towards the shorter wavelength side.

(iii) At a given temperature, area under the graph ($E_{b\lambda}$ vs λ) gives the total emissive power of a black body at that temperature.

7. 8. 2 Stefan-Boltzmann law

- According to Stefan-Boltzmann law "At a particular temperature, total radiation energy emitted by a black body per unit time per unit area of all possible wavelength in all possible direction is directly proportional to fourth power of absolute temperature.

$$E_{b\lambda} = \int_0^\infty E_{b\lambda} \cdot d\lambda \text{ W/m}^2$$

$$E_b = \sigma_b T^4 \text{ (W/m}^2\text{)}$$

$$E_b = \sigma_b AT^4 \text{ } (W)$$

where, σ_b = Stefan – Boltzmann constant = 5.67×10^{-8} W/m^2 K^4

7. 8. 3 Wien's Displacement Law

- It gives a relationship between the temperature of a black body and the wavelength at which the maximum value of monochromatic emissive power occurs.
- A peak monochromatic emissive power occurs at a particular wavelength.
- As temperature increases wavelength at which monochromatic emissive power is maximum, shifted towards the shorter wavelength side.

Mathematically,

$$\lambda_{max} T = \text{Constant}$$

$$\lambda_{max} T = 2900\mu\text{m-K}$$

$$(E_{b\lambda})_{max} = 1.285 \times 10^{-5} \text{ T}^5 \frac{W}{m^3}$$

$$(E_{b\lambda})_{max} \propto T^5$$

- Maximum monochromatic emissive power of a black body is proportional to the 5th power of absolute temperature.

7.8.4 Kirchhoff's Law

- As per Kirchhoff's law, the emissivity of a body is equal to its absorptivity when the body remains in thermal equilibrium with its surroundings.

$$\varepsilon = \alpha$$

7.8.5 Lambert's Cosine Law

- The law states that the intensity of radiation I_θ from a radiating plane surface in any direction is directly proportional to the cosine of the angle between the radiation emission and the normal surface vector.

$$I_\theta = I_n \cos\theta$$

Where I_n = Intensity of radiation in the direction of its normal

θ = Angle subtended by normal to the radiating surface and direction vectors of emission of the receiving surface.

- Equation is only true for a radiation surface whose radiation intensity is constant.

7. 9 Shape Factor

- The shape factor may be defined as "The fraction of radiative energy that is diffused from one surface and strikes the other surface directly with no intervening reflections.

$$F_{1-2} = \frac{\dot{Q}_{1-2}}{\dot{Q}_1}$$

where, $\dot{Q}_1$ = Rate of total energy emitted by surface 1

$\dot{Q}_{1-2}$ = Fraction of rate of energy leaving surface ' 1 ' and reaching to surface '2'.

7. 9. 1 Characteristics of shape factor

- Shape factor only depends only on geometry and orientation of the surface. Shape factor is also called view factor or geometry factor.
- Reciprocity theorem

$$A_1F_{1-2} = A_2F_{2-1}$$

- Shape factor algebra:

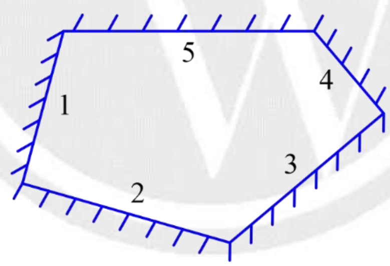

Fig 7.5 Radiation exchange in an enclosure.

$F_{11} + F_{12} + F_{13} + \ldots\ldots\ldots\ldots\ldots\ldots = 1$

$F_{21} + F_{22} + F_{23} + \ldots\ldots\ldots\ldots\ldots\ldots = 1$

$F_{31} + F_{32} + F_{33} + \ldots\ldots\ldots\ldots\ldots\ldots = 1$

& so on.

- Shape factor for some important surface

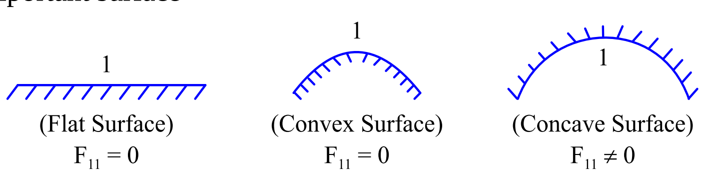

Fig 7.6 Shape factor for various surfaces

- If Emitting surface or radiating surface (1) is divided into two parts 3 and 4,

$$A_1F_{12} = A_3F_{32} + A_4F_{42}$$

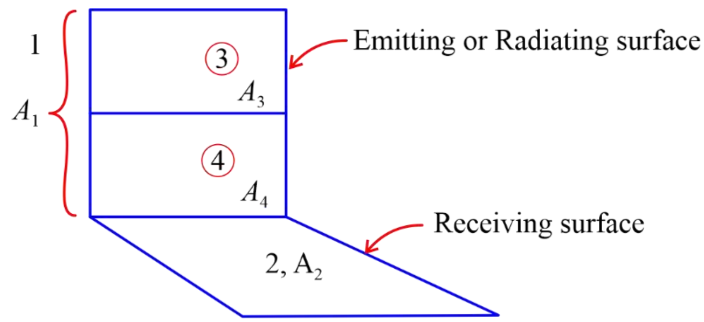

Fig 7.7 View factors for Perpendicular Rectangles with a Common Edge

- If Receiving surface (2) is divided into two parts 3 and 4,

$$F_{12} = F_{13} + F_{14}$$

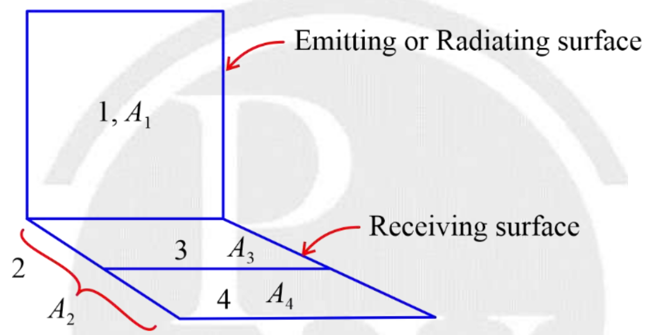

Fig 7.8 View factors for Perpendicular Rectangles with a Common Edge

7.10 Net Radiation Heat Transfer between bodies

7.10.1 Radiation Heat Exchange between two black bodies

$$\dot{Q}_{net} = A_1F_{12}\,\sigma_b\left(T_1^4 - T_2^4\right)$$
$$= A_2F_{21}\,\sigma_b\left(T_1^4 - T_2^4\right)$$

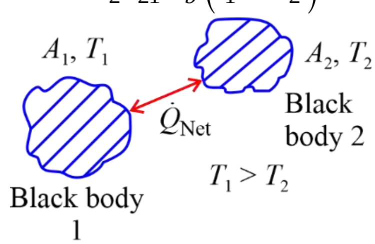

Fig 7.9 Radiation heat transfer between two black body

7.10.2 Radiation Heat exchange between two Non-black body

$$\dot{Q}_{net} = \frac{\sigma_b(T_1^4 - T_2^4)}{\frac{1-\varepsilon_1}{\varepsilon_1 A_1} + \frac{1}{A_1F_{12}} + \frac{1-\varepsilon_1}{\varepsilon_2 A_2}}$$

7.10.3 Radiation Heat exchange between Two Infinite Parallel Surfaces

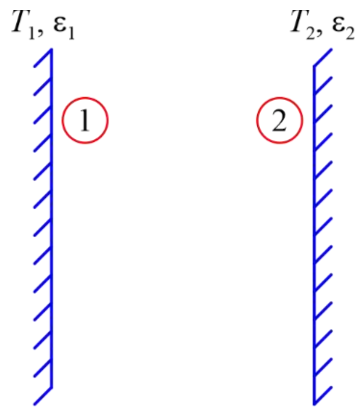

Fig 7.10 Radiation between two infinite parallel surfaces

$$\dot{Q} = A_1 (F_g)_{12} \sigma_b (T_1^4 - T_2^4)$$

$$(F_g)_{12} = \frac{1}{\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1}{F_{12}} + \frac{1-\varepsilon_2}{\varepsilon_2} \cdot \frac{A_1}{A_2}}$$

$$F_{12} = 1, \frac{A_1}{A_2} = 1$$

$$(F_g)_{12} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}$$

7.10.4 Radiation Heat exchange between two Concentric Cylinders

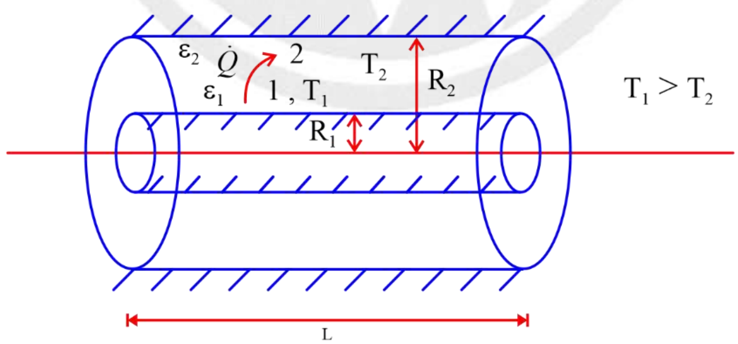

Fig 7.11 Radiation between two concentric cylinders

$$\dot{Q} = A_1 (F_g)_{12} \sigma_b (T_1^4 - T_2^4)$$

$$(F_g)_{12} = \frac{1}{\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1}{F_{12}} + \left(\frac{1-\varepsilon_2}{\varepsilon_2}\right) \frac{A_1}{A_2}}$$

$$F_{12} = 1, \frac{A_1}{A_2} = \frac{R_1}{R_2}$$

7.10.5 Radiation Heat exchange between two Concentric Spheres

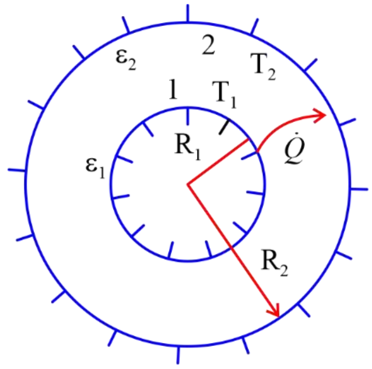

Fig 7.12 Radiation between two concentric sphere

$$\dot{Q} = A_1 (F_g)_{12} \sigma_b (T_1^4 - T_2^4)$$

$$(F_g)_{12} = \frac{1}{\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1}{F_{12}} + \frac{1-\varepsilon_2}{\varepsilon_2} \cdot \frac{A_1}{A_2}}$$

And $\quad F_{12} = 1, \; \frac{A_1}{A_2} = \frac{R_1^2}{R_2^2}$

7.10.6 Radiation Heat exchange between A Small Convex Object Within A Large Enclosure

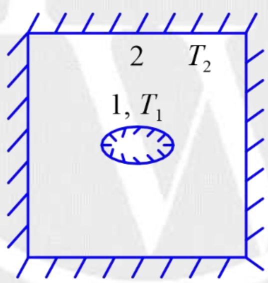

Fig 7.13 Radiation between a small body enclosed by a large body

$$F_{12} = 1 \text{ and } A_1/A_2 = 0 \; (A_2 >> A_1)$$

And $\quad \dot{Q} = A_1 \varepsilon_1 \sigma_b (T_1^4 - T_2^4)$

7.11 Radiation Shield Concept

- Radiation shield is used to reduce the heat transfer by increasing the resistance. Plate which is used for radiation shield should have high reflectivity and lower emissivity.

$$\left(\dot{Q}_{net}\right) \text{ with 1R.S.} = \frac{\sigma_b (T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{2}{\varepsilon_3} - 2} \left(\frac{\text{w}}{\text{m}^2}\right)$$

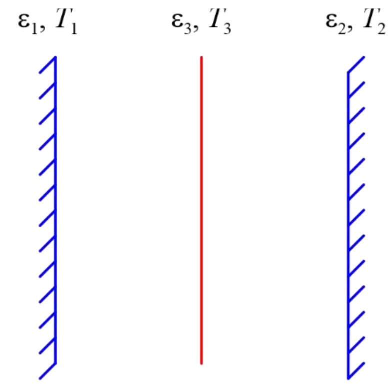

Fig 7.14 Radiation between parallel plate with radiation shield

- If N, radiation shield of different emissivity $\varepsilon_{S_1}, \varepsilon_{S_2} \cdots \varepsilon_{SN}$ are used

$$(\dot{Q}_{\text{Net}})_{N\text{ R.S.}} = \frac{\sigma_b(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + 2\sum_{i=1}^{N}\frac{1}{\varepsilon_{S_i}} - (N+1)}\left(\frac{\text{w}}{\text{m}^2}\right)$$

- If N radiation shield of same emissivity are used

$$\varepsilon_{S_1} = \varepsilon_{S_2} = \varepsilon_{S_N} = \varepsilon$$

$$\dot{Q}_{\text{Net}} = \frac{\sigma_b(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{2}{\varepsilon}N - (N+1)}$$

- If shields and plate have same emissivity then

$$(\dot{Q}_{\text{Net}})_{\text{with } N\text{ R.S.}} = \frac{1}{N+1}(\dot{Q}_{\text{Net}})_{\text{without R.S.}}$$

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Engineering Mechanics

Engineering Mechanics

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1 BASIC CONCEPTS

1.1 Force

- Force is the interaction between two bodies.
- It is the action of one body on another body that tries to change the motion of the other body.

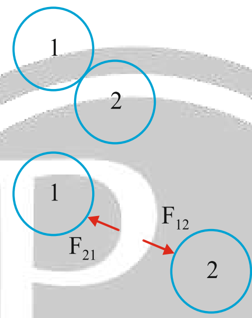


Fig. 1.1 Interaction of force between two bodies

Force is a vector quantity. It is characterized by –

(1) Point of application

(2) Line of action (direction)

(3) Magnitude

1.1.1 Principle of transmissibility of force

The point of application of force can be moved anywhere along the line of action of force without changing the external effects of the force on the body.

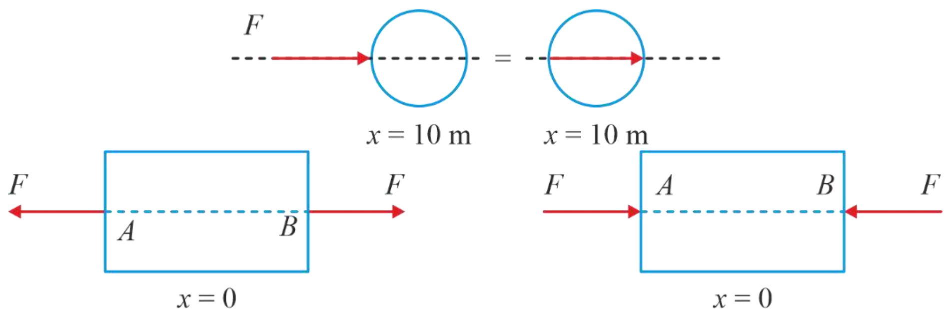


Fig. 1.2 Transmission of force along its line of action

1.2 Moment of a force

Moment of a force is the measure of tendency of the force to rotate a body about a particular point or axis.

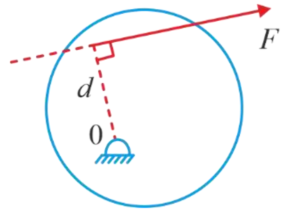


Fig. 1.3 Force at a distance from center of rotation

Moment of F about O

$$M_0 = F \times d \text{ (cw)}$$
$$d = \text{Perpendicular distance of F from O}$$

1.3 Moment of a couple

Couple is caused by two non-concurrent parallel forces of same magnitude and having opposite directions. Couple is the product of Force and the perpendicular distance between them.

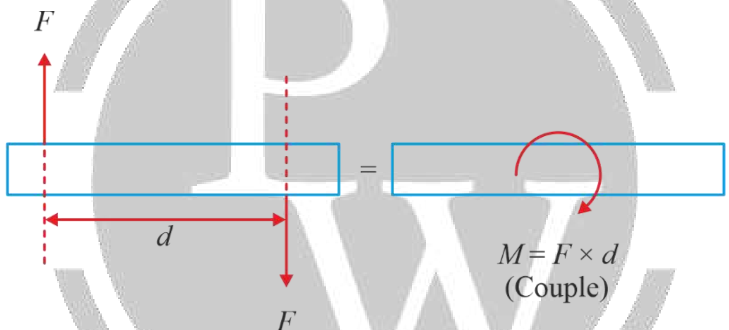


Fig. 1.4 Moment of couple

1.4 Resolution of a Force

1.4. 1 Resolution of a force into two components

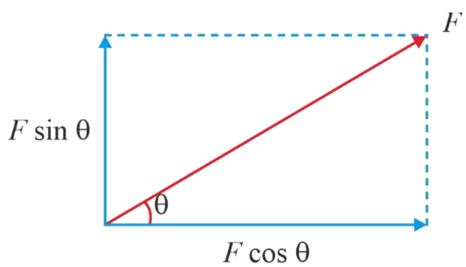


Fig. 1.5 Resolution of force into two components

1.4. 2 Resolution of a force into a force and a couple

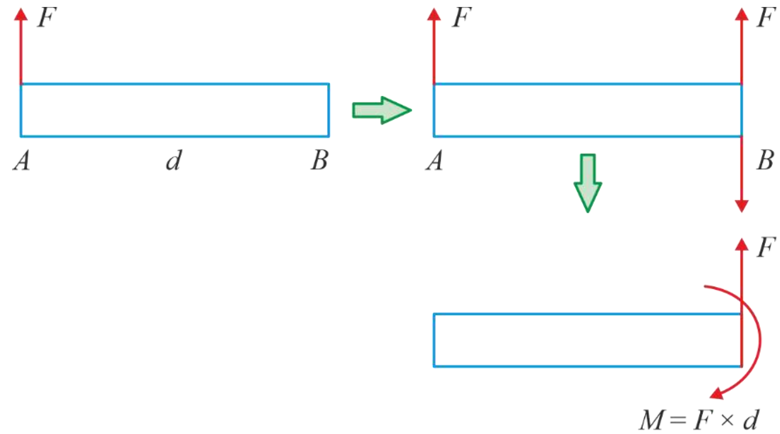


Fig. 1.6 Resolution of force into a force and a couple

1.5 Varignon's theorem

Moment of a force about any point is equal to the sum of the moments of the components of that force about the same point.

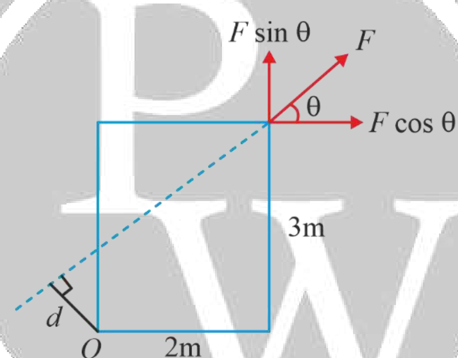


Fig. 1.7 Moment of Force about a point

$$M_0 = F \times d = F \cos\theta \times 3 - F \sin\theta \times 2$$

1.6 Newton's Three Laws of Motion

1.6.1 First Law

If the resultant force acting on a body is zero, the body will remain at rest (if originally at rest) or will move with constant speed in a straight line (if originally in motion)

1.6.2 Second Law

If the resultant force acting on a body is not zero, the body will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

$$F \propto a$$
$$F = ma$$

1.6.3 Third Law

The forces of action and reaction between bodies in contact have the same magnitude, same line of action, and opposite sense.

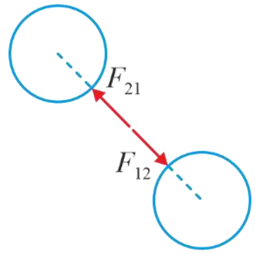


Fig. 1.8 The forces of action and reaction

$$F_{12} = -F_{21}$$

1.7 Resultant of a force system

1.7.1 Resultant of a Parallel force system:

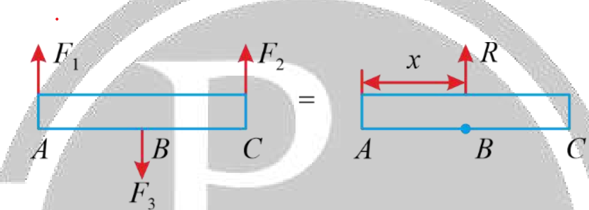


Fig. 1.9 Equivalent force of a parallel force system

$$R = F_1 + F_2 - F_3$$

Note:

We can find the distance x by equating the moment of the resultant R with the moment of all the forces at any point.

1.7.2 Resultant of a concurrent force system:

Two forces: (Parallelogram law)

If two forces acting a point, are represented in magnitude and direction by the two adjacent sides of a parallelogram, their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn from the same point.

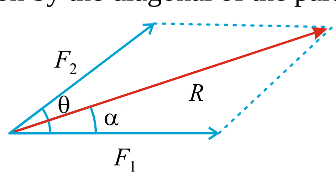


Fig. 1.10 Resultant of two forces and its direction

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$$\alpha = \tan^{-1}\left(\frac{F_2\sin\theta}{F_1 + F_2\cos\theta}\right)$$

Several forces: (Polygon law)

If more than two forces acting at a point are represented by consecutive sides of a polygon taken in order then their resultant will be represented by closing side of polygon taken in opposite order.

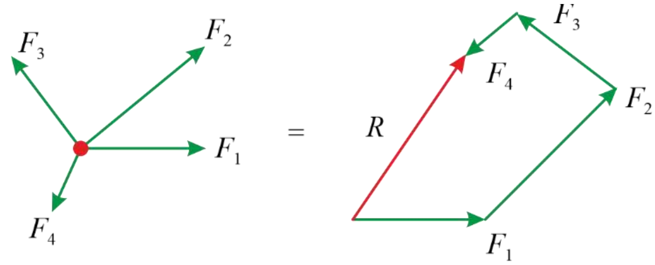


Fig. 1.11 Representation of forces in a polygon

Several forces: (Method of Projections)

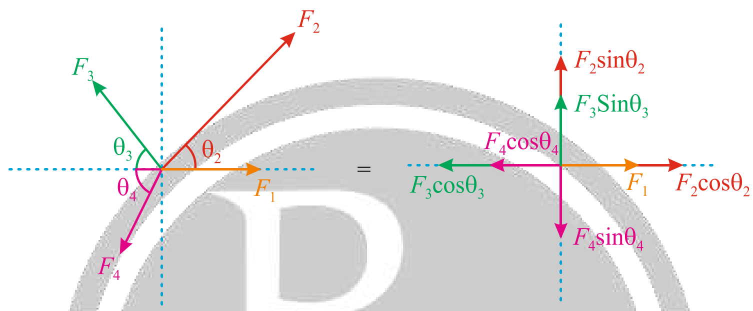


Fig. 1.12 Projection of forces on two perpendicular axes

$$\Sigma F_x = F_1 + F_2\cos\theta_2 - F_3\cos\theta_3 - F_4\cos\theta_4$$

$$\Sigma F_y = F_2\sin\theta_2 + F_3\sin\theta_3 - F_4\sin\theta_4$$

$$R = \sqrt{\Sigma F_x^2 + \Sigma F_y^2}$$

$$\alpha = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$

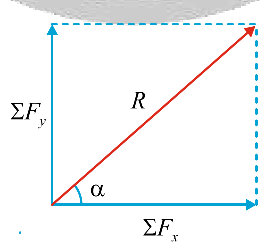


Fig. 1.13 Resultant of two perpendicular forces and its direction

1.7.3 Resultant of a non-concurrent force system

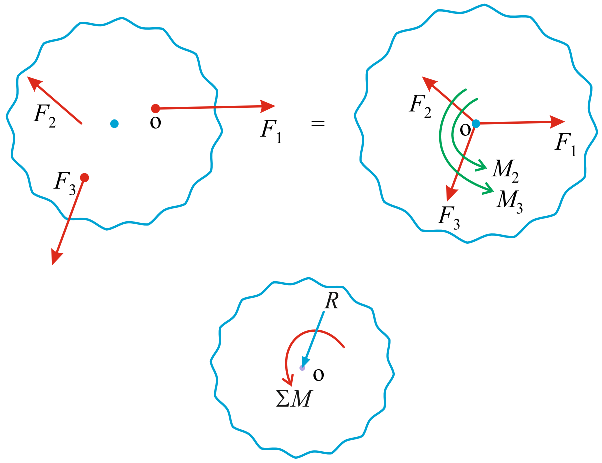


Fig. 1.14 A non-concurrent force system and its resultant

R is the resultant of forces F_1, F_2 and F_3 and $\sum M$ is the net moment about O.

1.8 Free Body Diagram

An FBD is a sketch of the selected system consisting of a body, part of body or a collection of interconnected bodies completely isolated from other bodies showing the interaction of all other bodies by force.

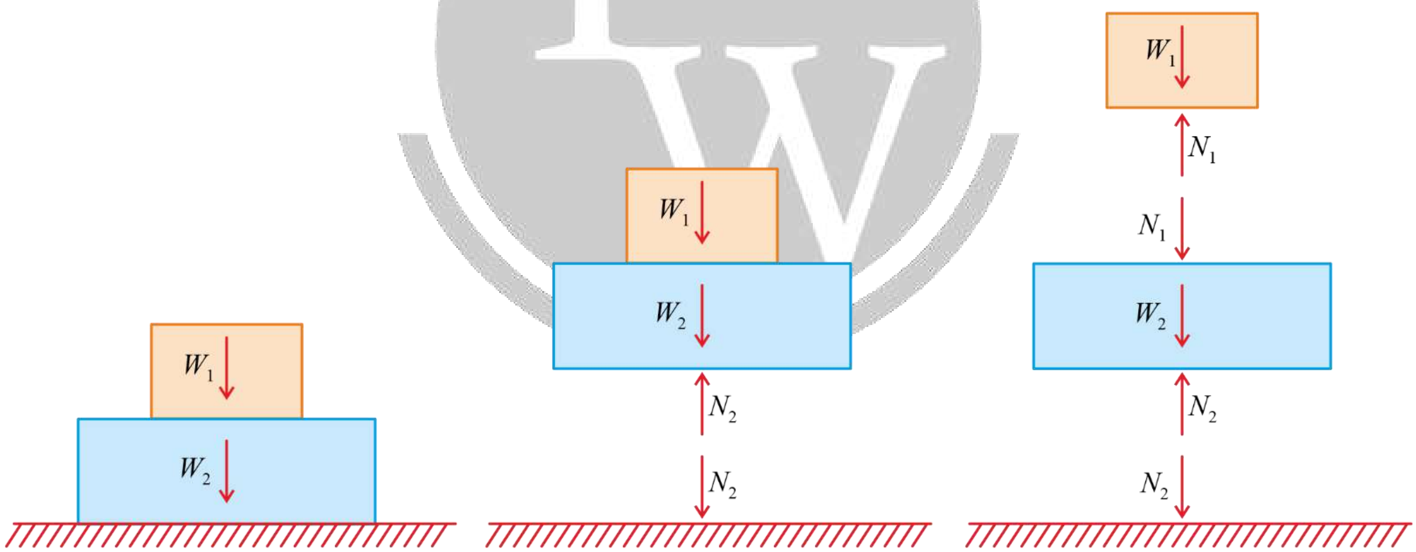


Fig. 1.15 Free body diagrams

1.9 Types of supports and their reactions

1.9.1 Roller/Frictionless surface

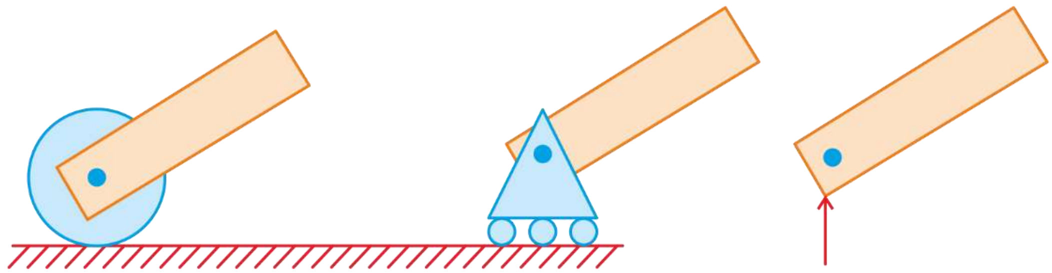


Fig. 1.16 Roller support and reaction force

Note:

We can assume any direction of reaction force (Upward or downward). If the calculated value of reaction is (+ve), the assumed direction is right otherwise wrong.

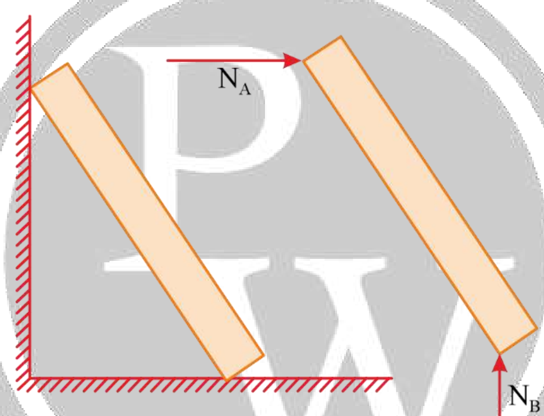
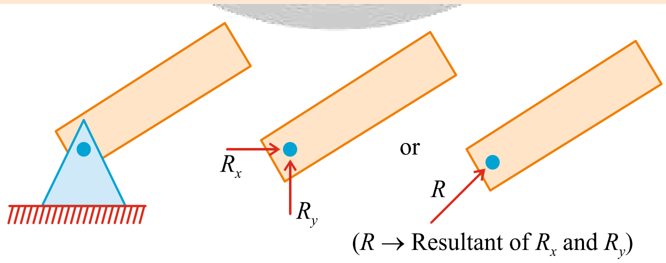


Fig. 1.17 A block leaning against wall and its FBD

1.9.2 Hinge/Friction surface



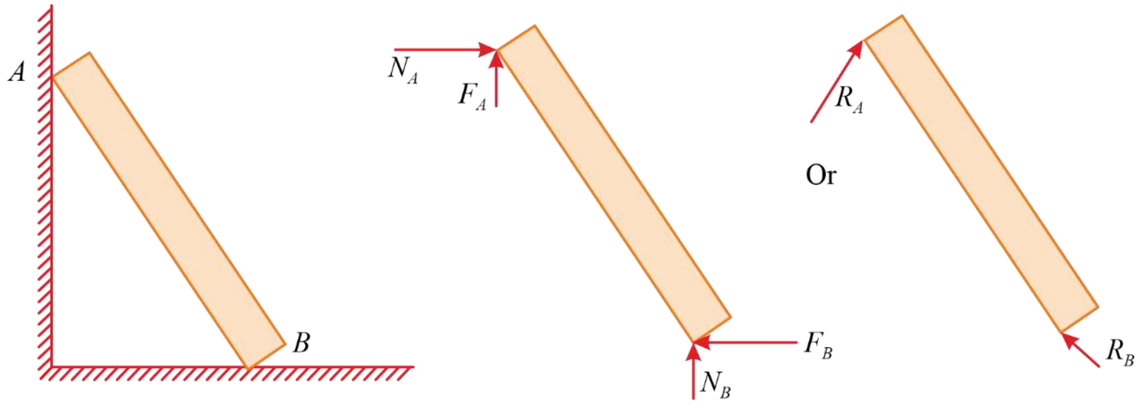


Fig. 1.18 Hinge support and FBD of block leaning against rough wall

1.9.3 Fixed support

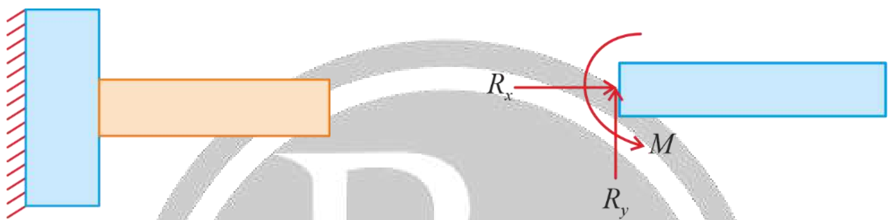


Fig. 1.19 Fixed support and its reactions

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2 EQUILIBRIUM OF RIGID BODIES

2.1 Equilibrium of a rigid body

When a body is subjected to forces and the body is either at rest or moving with constant velocity in the same direction, the body is said to be in equilibrium.

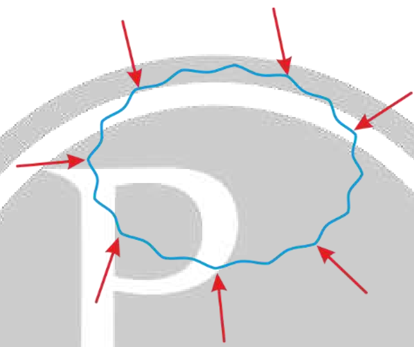

Fig. 2.1 A body in equilibrium under the application of loads

For the rigid body to be in equilibrium, necessary conditions are -

$$\begin{Bmatrix} \Sigma F_x = 0, \Sigma F_y = 0, \Sigma F_z = 0 \\ \Sigma M_x = 0, \Sigma M_y = 0, \Sigma M_z = 0 \end{Bmatrix}$$

For coplanar force system

$$\Sigma F_x = 0, \Sigma F_y = 0$$

$$\Sigma M_z = 0$$

The above equations are known as equilibrium equations.

2.1.1 Statically Determinate System

If number of unknowns = number of equilibrium equation, then system is statically determinate.

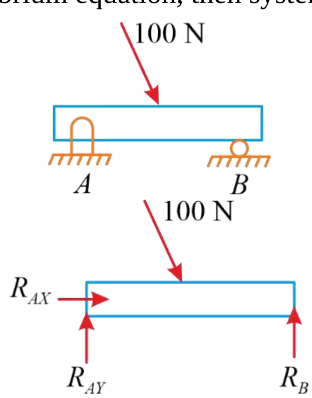

Fig. 2.2 Statically determinate system

Number of unknowns (R_{AX}, R_{AY}, R_B) = 3

Number of equilibrium equations ($\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_z = 0$) = 3

Note:
All the problems of engineering mechanics are statically determinate.

2.1.2 Statically Indeterminate System

If number of unknowns > number of equilibrium equations, then system is statically indeterminate.

Note:
Statically indeterminate problems are solved in strength of materials.

2.1.3 Equilibrium of a two-force system:

If a body, subjected to two forces, is in equilibrium, the two forces must have the same magnitude, the same line of action, and opposite sense.

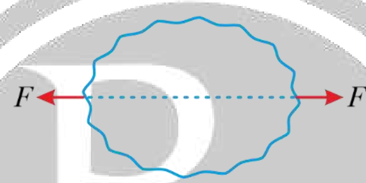

Fig. 2.3 A body in equilibrium under the application of two forces

2.1.4 Equilibrium of a three-force system:

If a body subjected to three forces is in equilibrium, the lines of action of the three forces must be either parallel or concurrent.

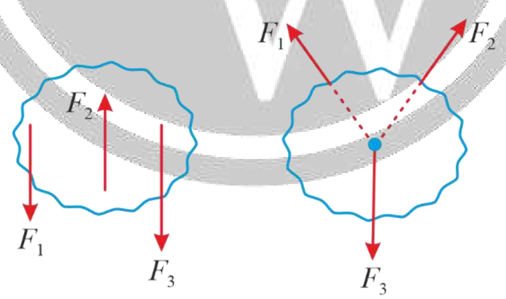

Fig. 2.4 A body in equilibrium under the application of three forces

Lami's Theorem

It states that, when a body under the action of three concurrent and coplanar forces, is in equilibrium, then each force is proportional to the sine of the angle between the other two forces.

$$\frac{F_1}{\text{Sin}\,\theta_1} = \frac{F_2}{\text{Sin}\,\theta_2} = \frac{F_3}{\text{Sin}\,\theta_3}$$

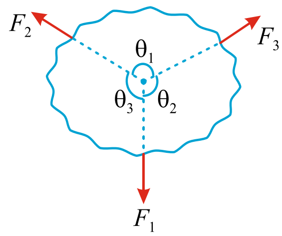

Fig. 2.5 Three concurrent forces on a stationary body

Note:

For calculating angles between forces, draw all the forces either diverging from a point or converging to a point.

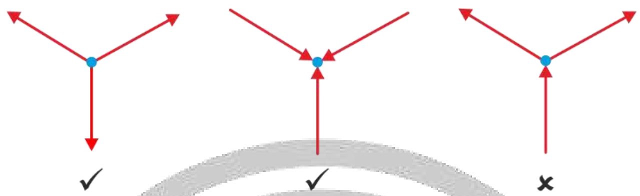

Fig. 2.6 Method to calculate angles between forces in Lami's Theorem

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3 FRICTION

3.1 Friction

Friction is a force that resists the relative motion between two contacting surfaces.

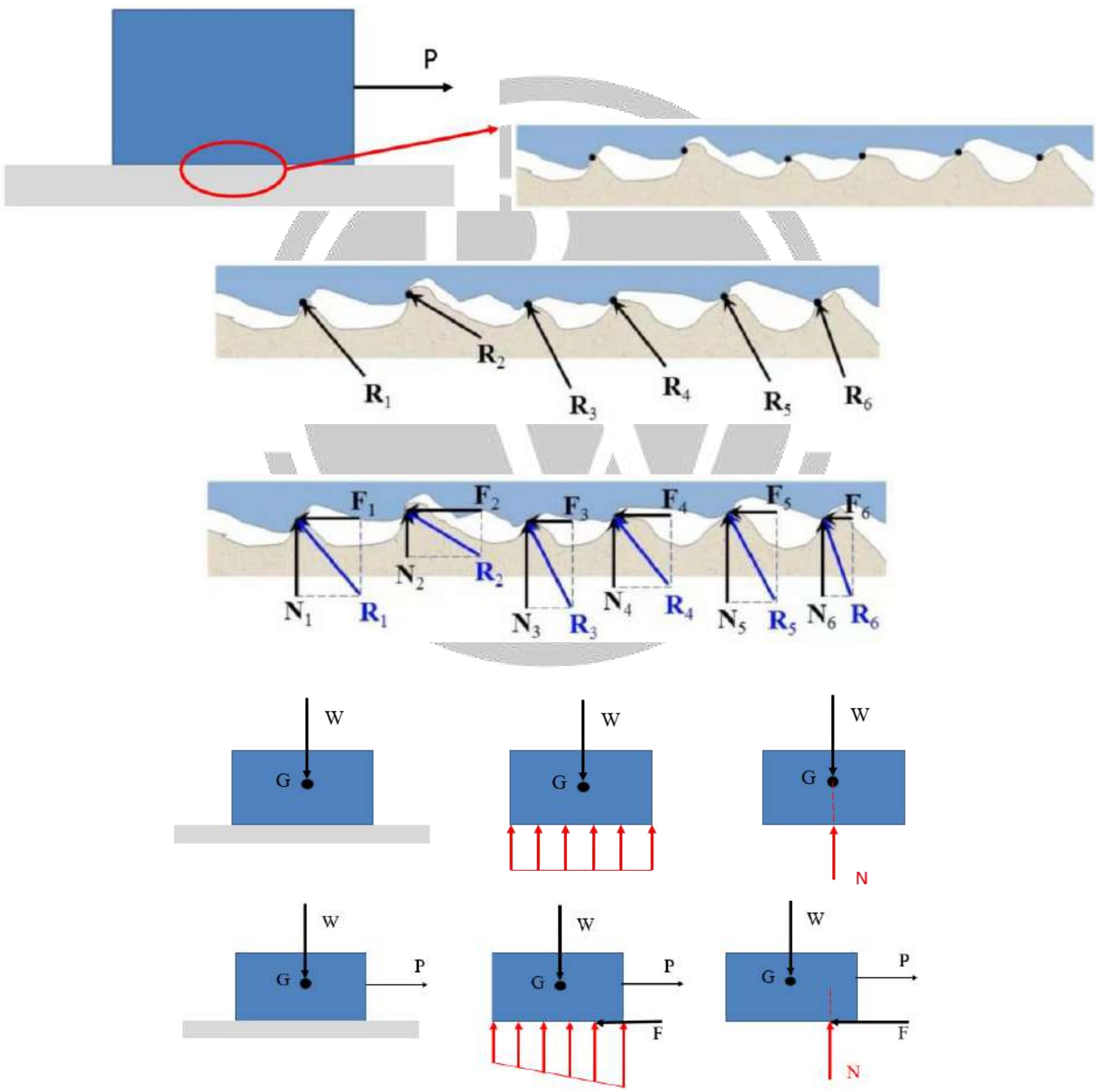


Fig. 3.1 Understanding of friction on microscopic level

> **Note:**
> The normal reaction N does not pass through the center of gravity G.

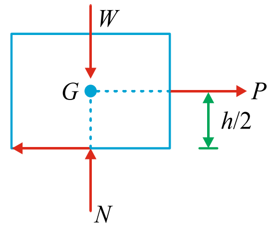


Fig. 3.2 FBD of a block moving on a rough surface

$$\Sigma M_G = F \times \frac{b}{2} \neq 0$$

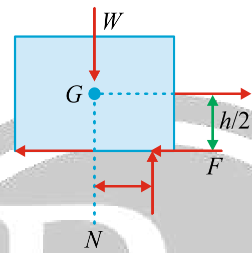
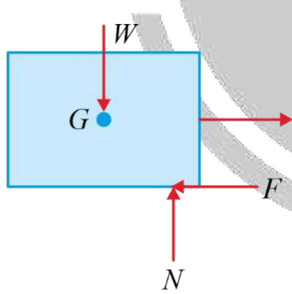


Fig. 3.3 FBD of a block moving on a rough surface

$$\Sigma M_G = F \times \frac{h}{2} - N \times x = 0$$

3.1.1 Limiting Friction:

It is the maximum static friction between two dry surfaces.



$N = W$
$F = P$

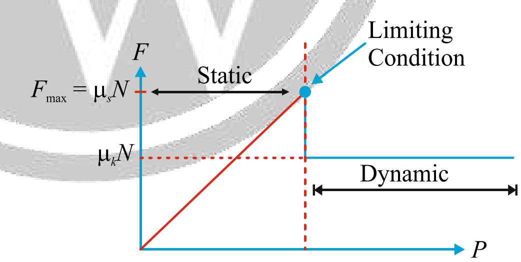


Fig. 3.4 Load and friction force diagram

$\mu_s \rightarrow$ static coefficient of friction

$\mu_k \rightarrow$ kinetic coefficient of friction

$\mu_k < \mu_s$

$F_{max} = \mu_s N$ (Maximum static friction force)

- If $P < \mu_s N$ — Body remains static.
- If $P = \mu_s N$ — Body is on the range of moving. (static)
- If $P > \mu s N$ — Body starts moving.

3.1.2 Laws of dry friction

- Static friction force is directly proportional to applied load.
- Friction force is independent of area of contact.
- Kinetic friction force is independent of velocity.

3.2 Sliding vs Tipping

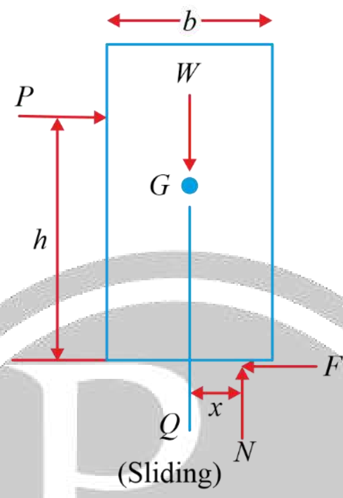


Fig. 3.5 Block sliding on a rough surface

$$\sum M_Q = 0$$

$$P \times h - N \times x = 0$$

$$P \times h = N \times x$$

$$x = \frac{Ph}{W}$$

As the distance x increases, normal force N moves towards the edge. At the time of tipping, $x = b/2$ and N is at the edge.

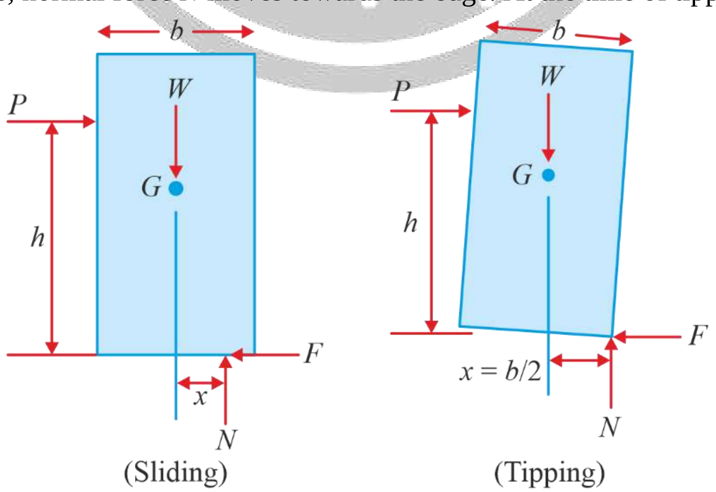


Fig. 3.6 Block sliding and tipping on a rough surface

3.3 Friction Angle (ϕ)

It is the angle between **normal force** and **resultant of normal and friction force** when the body is on the verge of moving.

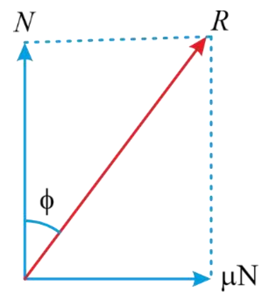


Fig. 3.7 Resultant of normal and friction force

$$\tan\phi = \frac{\mu N}{N}$$

$$\phi = \tan^{-1}\mu$$

3.4 Angle of Repose

If a body is resting on an inclined surface, then the angle of repose is the maximum angle at which the body can be at rest without slipping down.

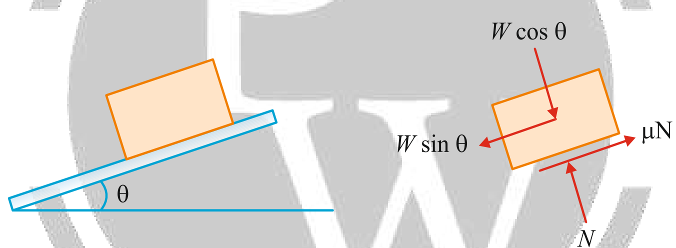


Fig. 3.8 Block sliding on an inclined surface

$$W\sin\theta = \mu N$$

$$W\cos\theta = N$$

$$\tan\theta = \mu$$

$$\tan\theta = \tan\phi$$

$$\theta = \phi$$

3.5 Belt Friction

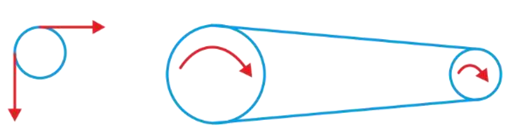


Fig. 3.9 Tensions in the belt of a belt-pulley system

$\theta \rightarrow$ angle of contract

$T_1 \rightarrow$ maximum tension in belt

$T_2 \rightarrow$ minimum tension in belt

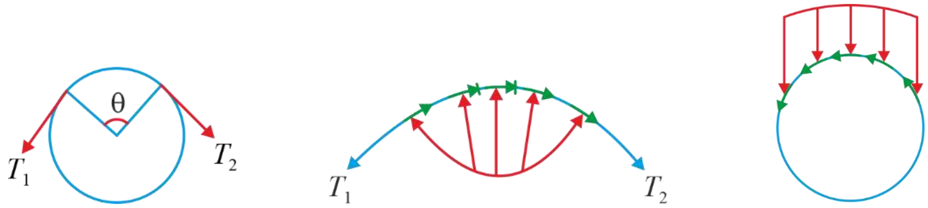


Fig. 3.10 Pressure on the pulley due to tension in belt

$$\boxed{\frac{T_1}{T_2} = e^{\mu\theta}}$$ (θ must be in radians)

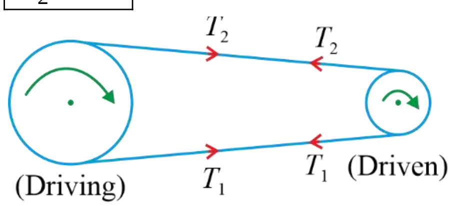


Fig. 3.11 Belt pulley system and tension in the belt

3.6 Rolling Friction


Fig. 3.12 Objects including rolling friction

Roller is at rest,

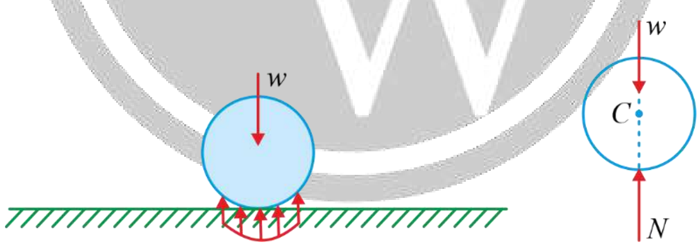


Fig. 3.13 FBD of roller at rest

A force P is applied to the roller to move the roller at constant velocity,

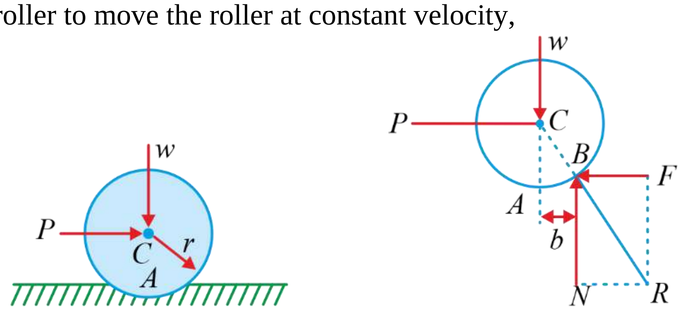


Fig. 3.14 FBD of roller under rolling friction and applied load

R = Resultant of normal force N and friction force F

b = Coefficient of rolling resistance (in mm)

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4 TRUSS

4.1 Truss

Truss is a structure which consists of several members connected together to support load.

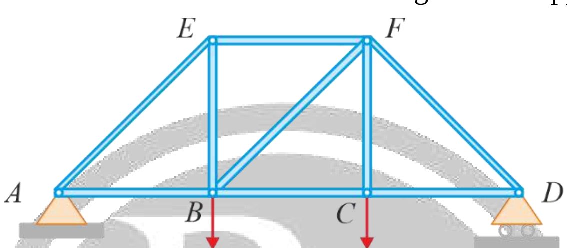

Fig. 4.1 A truss

4.2 Types of Trusses

Truss

- Plane Truss (all members are in a single plane)
- Space Truss (members are in different planes)

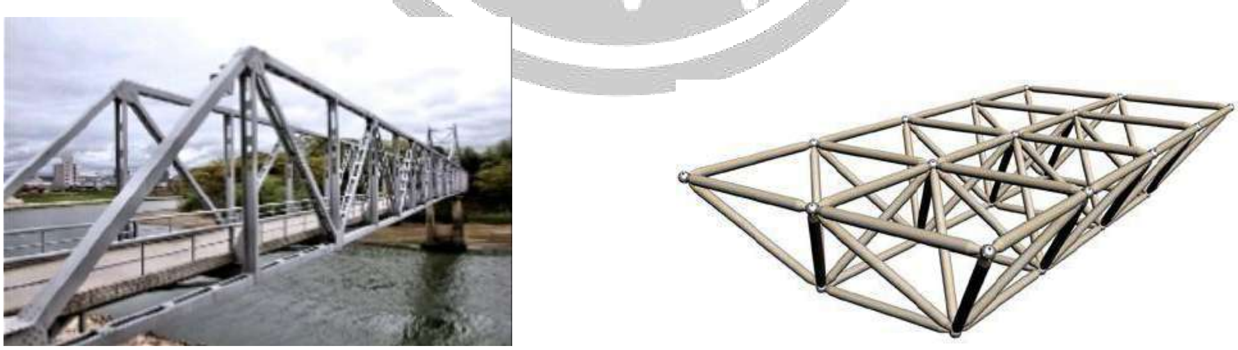

Fig. 4.2 Plane truss and space truss

4.3 Analysis of Plane Truss

- All members are connected only at the ends.
- All joints are frictionless pin joints.
- Loads are only applied at the joints
- All the members are subjected to only axial loads or in other words all the members are two force members.

- Weight of the members is assumed negligible.

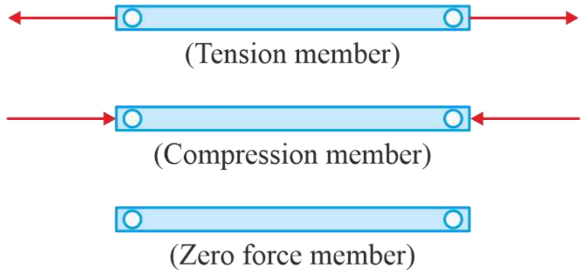

Fig. 4.3 Representation of forces on a member in truss

4.4 Perfect Truss

- A perfect truss is a truss which has just enough number of members to keep the truss in equilibrium.
- A perfect truss is statically determinate.

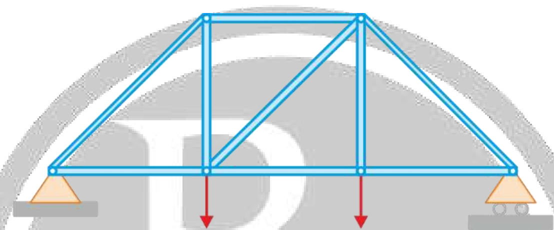

Fig. 4.4 Perfect truss

$m \rightarrow$ number of members

$j \rightarrow$ number of joints

Perfect trust must be statically determinate, so

Number of unknowns = Number of equilibrium equations

$$m + 3 = 2j$$

$$m = 2j - 3$$

- If $m = 2j - 3$ – Perfect truss (stable)
- If $m < 2j - 3$ – unstable truss
- If $m > 2j - 3$ – statically Indeterminate/ over rigid (stable)

4.5 Method of Joints

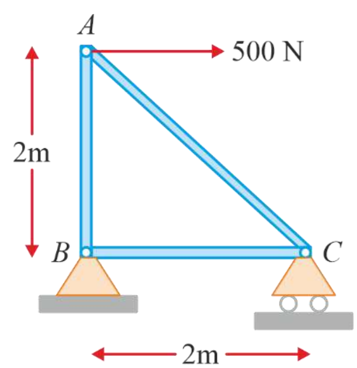

Fig. 4.5 A simple truss

FBD of joint A

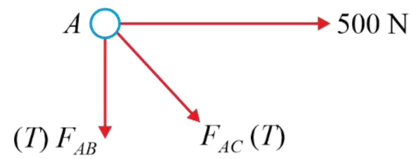

Fig. 4.6 Forces at point A in the truss

$$\begin{aligned} \Sigma F_x &= 0 \\ F_{AC} \cos 45 + 500 &= 0 \\ F_{AC} &= -707.1 \text{ (C)} \\ \Sigma F_y &= 0 \\ F_{AB} &- F_{AC} \sin 45° = \text{O} \\ F_{AB} &= 500 \text{ N } (T) \end{aligned}$$

FBD of joint C

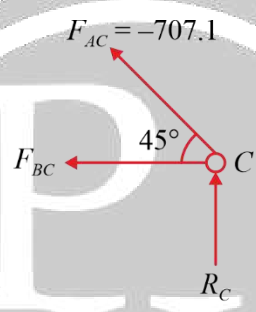

Fig. 4.7 Forces at point C in the truss

$$\begin{aligned} \Sigma F_x &= 0 \\ F_{BC} - 707.1 \cos 45° &= 0 \\ F_{BC} &= 500 \text{ N } (T) \end{aligned}$$

4.6 Method of Sections

Find the axial force in member BD.

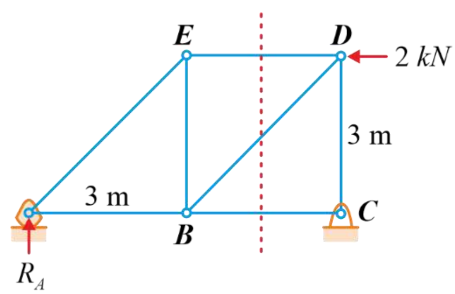

Fig. 4.8 A section taken in the truss

$$\begin{aligned} \Sigma M_c &= 0 \\ R_A \times 6 &= 2 \times 3 \\ R_A &= 1 \text{ kN} \end{aligned}$$

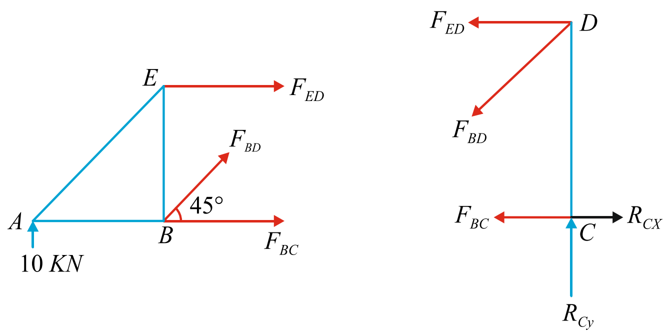

Fig. 4.9 Forces in the two parts of truss after taking section

$$\Sigma F y = 0$$
$$F_{BD} \sin 45° + 1 = 0$$
$$F_{BD} = -1.414 \text{ kN}$$

4.7 Truss vs Frames

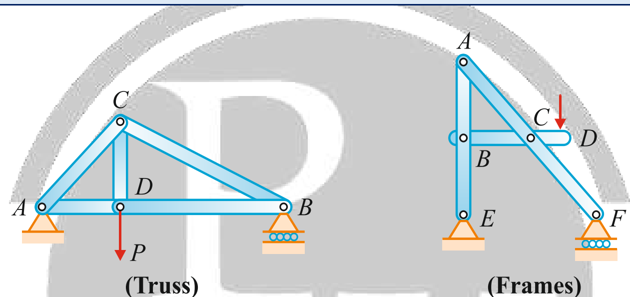

Fig. 4.10 Truss and frame

- In case of truss, all members are two force members, whereas there is at least one multi force member in frames.

4.8 Analysis of Plane Frames

- Frames are structures containing at least one multi force member.
- The members are connected at the ends or in between the ends.
- Loads act at the joints as well as on the members.

Two force members

Straight member
No force between the ends

Multi force members

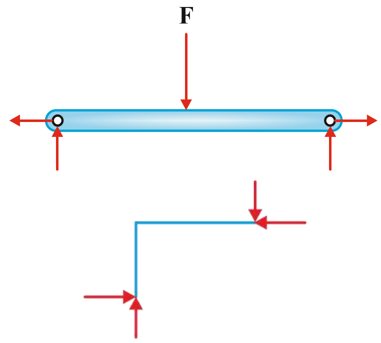

Fig. 4.11 Forces in the members

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5 CENTROID, COG, COM AND MOI

5.1 Centroid

It is the geometric centre of an object.

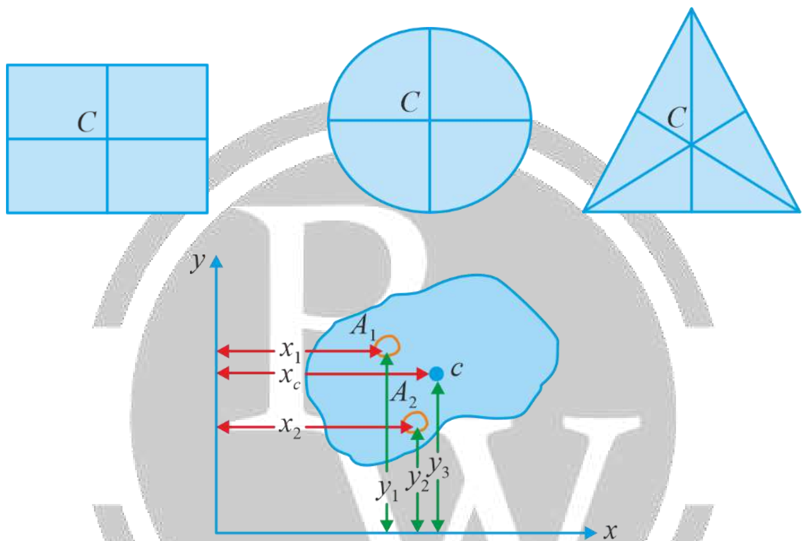


Fig. 5.1 Centroid of various shapes

$$x_c = \frac{A_1x_1 + A_2x_2 \ldots\ldots + A_nx_n}{A_1 + A_2 \ldots\ldots + A_n}$$

$$y_c = \frac{A_1y_1 + A_2y_2 \ldots\ldots + A_ny_n}{A_1 + A_2 \ldots\ldots + A_n}$$

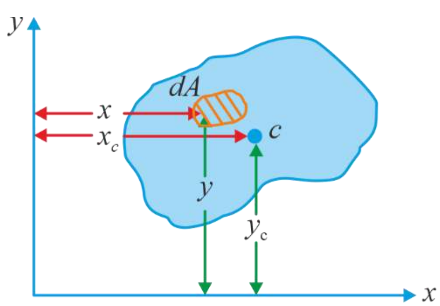


Fig. 5.2 Centroid of a lamina

$$x_c = \frac{\int x\,dA}{\int dA}$$

$$y_c = \frac{\int y dA}{\int dA}$$

5.2 Center of Mass

It is an imaginary point where the mass of the body can be assumed to be concentrated.

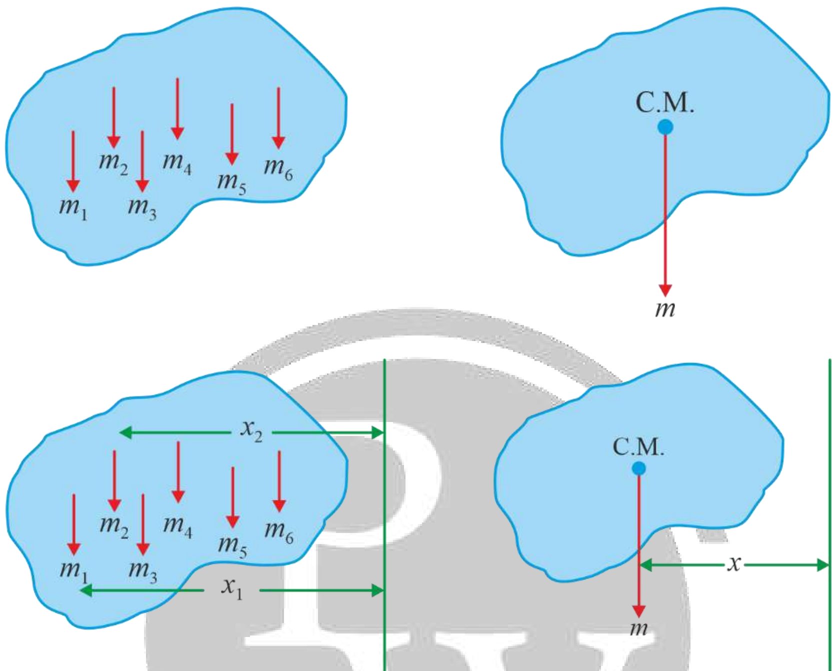


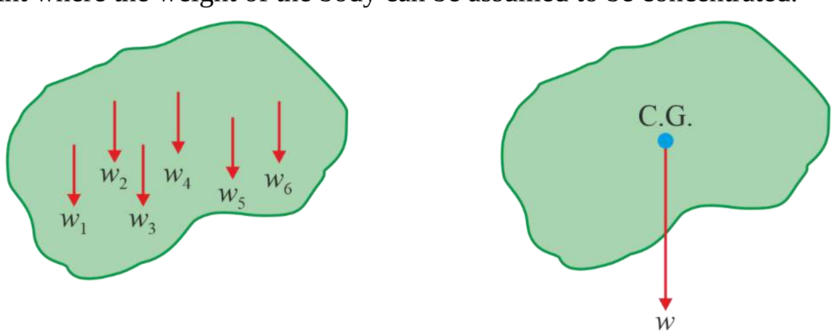
Fig. 5.3 Center of mass of an object

$$\boxed{x = \frac{m_1x_1 + m_2x_2 .... + m_nx_n}{m_1 + x_2 .... + m_n}}$$

> **Note:**
> If density of the body is same at every point, centroid and centre of mass will be same.

5.3 Center of Gravity

It is an imaginary point where the weight of the body can be assumed to be concentrated.



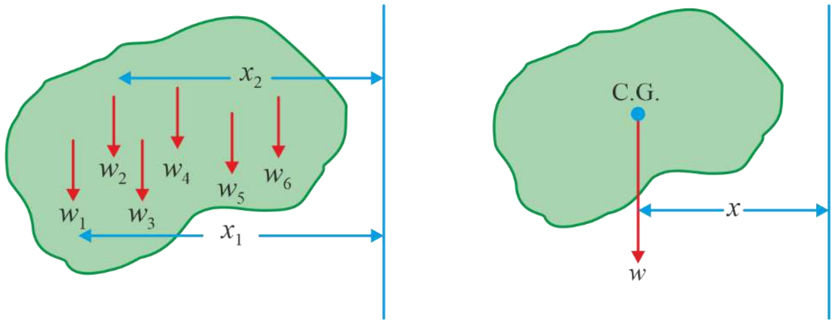


Fig. 5.4 Center of gravity of an object

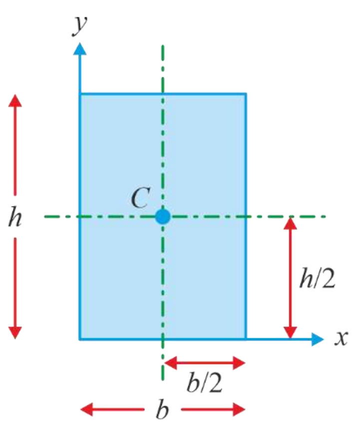
$$\boxed{x = \frac{W_1 x_1 + W_2 x_2 .... + W_n x_n}{W_1 + W_2 .... + W_n}}$$

Note:

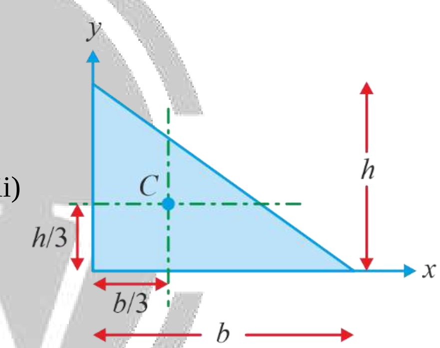
For small bodies g is constant, hence center of mass and center of gravity is same.

5.4 Centroid of some common areas

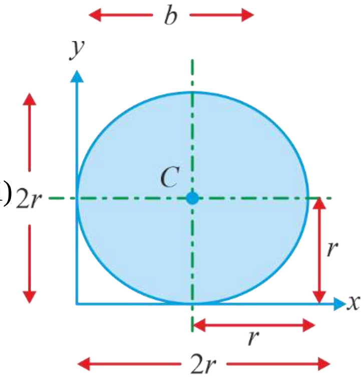
(i)



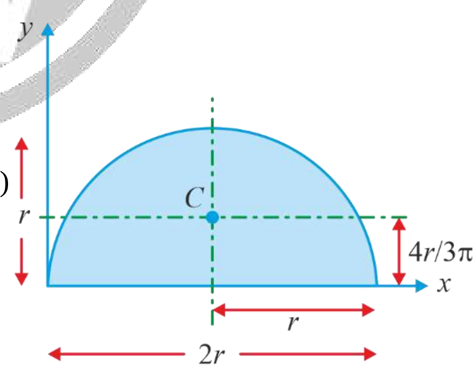
(ii)



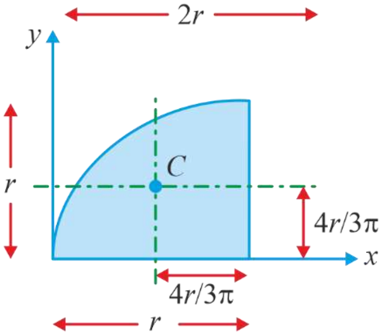
(iii)



(iv)



(v)



5.5 Moment of Inertia of Area (Second Moment of Area)

- It is a geometrical property of an area which indicates how its points are distributed about an axis.
- It signifies the resistance of an area against the applied moment (bending moment or twisting moment) about an axis.

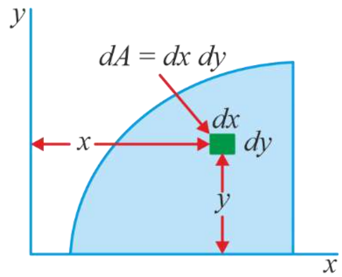


Fig. 5.5 MOI of an area

$$\boxed{I_x = \int y^2 dA}$$

$$\boxed{I_y = \int x^2 dA}$$

5.5.1 Polar moment of inertia

It is the moment of inertia of an area about the normal axis (z axis).

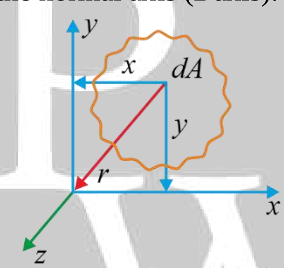


Fig. 5.6 Polar MOI of a section

$$r^2 = x^2 + y^2$$

$$I_z = \int r^2 dA = \int (x^2 + y^2) dA$$

$$I_z = \int x^2 dA + \int y^2 dA$$

$$\boxed{I_z = I_x + I_y = J}$$

5.5.2 Parallel Axis Theorem

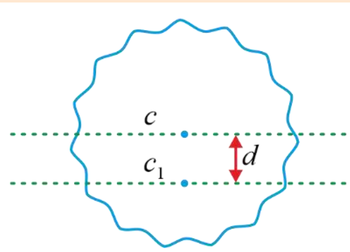


Fig. 5.7 An axis parallel to centroidal axis

$$I_{c_1} = I_c + Ad^2$$

5.5.3 Radius of Gyration

Consider an area A whose moment of inertia about an axis is I. Suppose the area is concentrated into a thin strip parallel of to that axis as shown in figure, such that its moment of inertia about the axis is same as I. Then the distance at which this strip is to be placed from the axis is called radius of gyration of the area about that axis.

$$I = Ak^2$$

$$k = \sqrt{\frac{I}{A}}$$

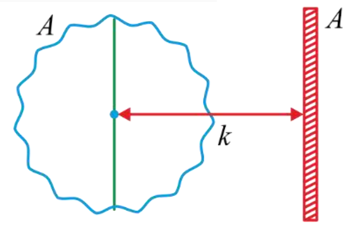


Fig. 5.8 Radius of gyration of a section

5.5.4 Moment of inertia of some common areas

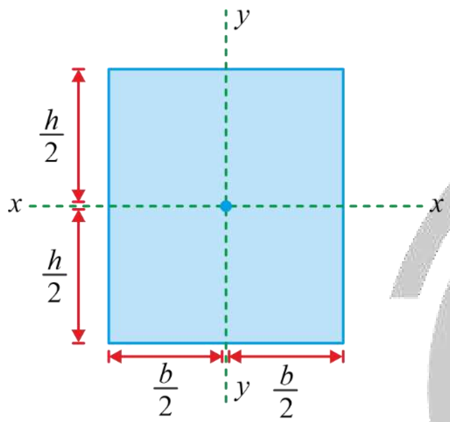
(i)


Fig. 5.9 Rectangular section

$$I_x = \frac{bh^3}{12}$$

$$I_y = \frac{hb^3}{12}$$

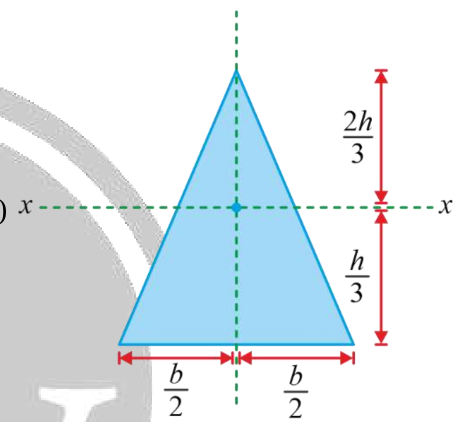
(ii)


Fig. 5.10 Triangular section

$$\boxed{I_x = \frac{bh^3}{36}}$$

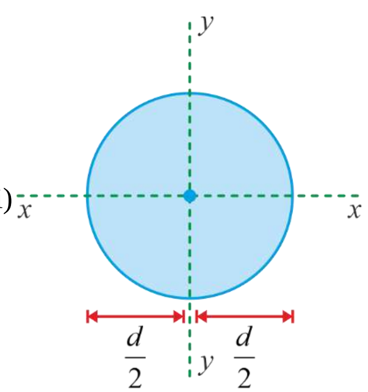
(iii)


Fig. 5.11 Circular section

$$\boxed{I_x = I_y = \frac{\pi}{64}d^4}$$

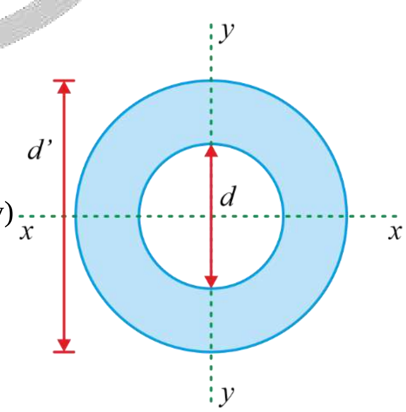
(iv)


Fig. 5.12 Tube section

$$\boxed{I_x = I_y = \frac{\pi}{64}\left(d_0^4 - d_i^4\right)}$$

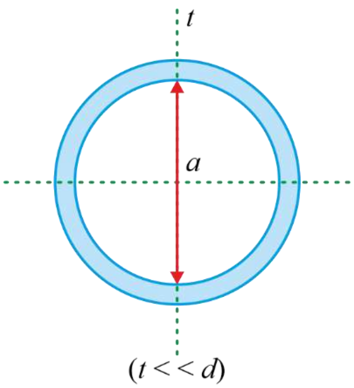
(v) 

Fig. 5.13 Ring section

$$\boxed{I_x = I_y = \frac{\pi d^3 t}{8}}$$

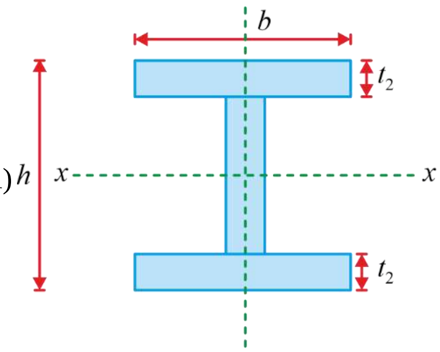
(vi) 

Fig. 5.14 I section

$$\boxed{I_x = \frac{BH^3}{12} - \frac{bh^3}{12}}$$

$b = B - t_1$

$h = H - 2t_2$

5.6 Moment of Inertia of Mass

It indicates how the mass of the body is distributed about an axis.

It signifies the resistance of a body against the applied torque about an axis.

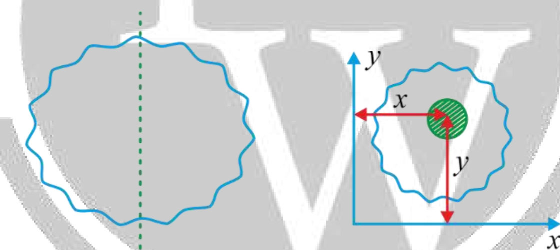


Fig. 5.15 Moment of Inertia of mass

$$\boxed{I_x = \int y^2 . dm} \text{ kg-m}^2$$

$$\boxed{I_y = \int x^2 dm} \text{ kg-m}^2$$

5.6.1 Parallel Axes Theorem

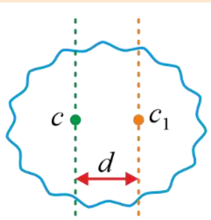


Fig. 5.16 An axis parallel to axis at COM of an object

$$\boxed{I_{C_1} = I_C + md^2}$$

5.6.2 Radius of Gyration

If the total mass of the body were concentrated to a point then the distance of that point at which it would have a moment of inertia the same as the body's actual distribution of mass, is called radius of gyration.

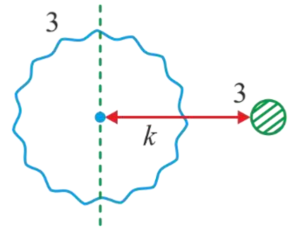


Fig. 5.17 Radius of gyration of an object

$$I = mk^2$$

$$\boxed{k = \sqrt{\frac{I}{m}}}$$

5.6.3 Moment of inertia of some common bodies

(i) Slender rod

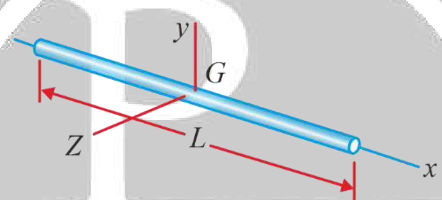


Fig. 5.18 Slender Rod

$$\boxed{I_y = I_z = \frac{mL^2}{12}}$$

$$I_z = I_z + m \times \left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{4}$$

$$\boxed{I_z = \frac{mL^2}{3}}$$

(ii) Thin Disc

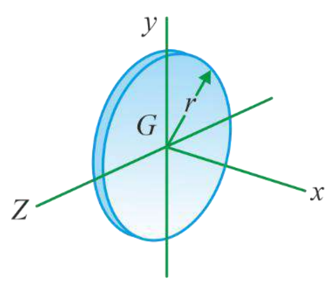


Fig. 5.19 Thin Disc

$$\boxed{I_x = \frac{mr^2}{2}}$$

(iii) Cylinder

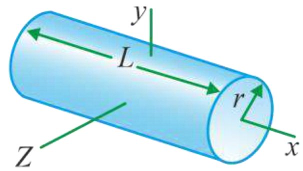


Fig. 5.20 Cylinder

$$I_x = \frac{mr^2}{2}$$

(iv) Sphere

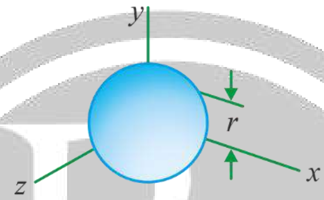


Fig. 5.21 Sphere

$$I_x = I_y = I_z = \frac{2}{5}mr^2$$

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6 KINEMATICS OF PARTICLES

6.1 Types of Motion in a plane

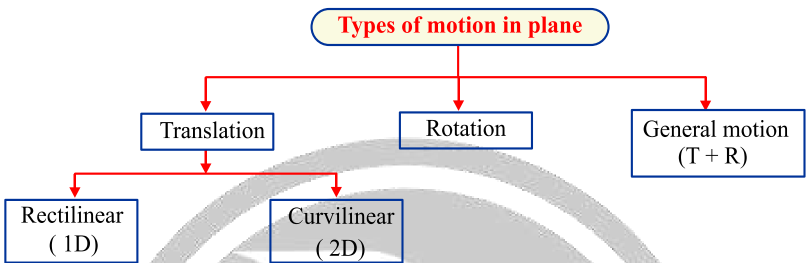


Fig. 6.1 Types of motion

6.1.1 Translational motion

- All the particles of the body move along identical parallel path.
- All the particles of the body have same displacement.

(i) Rectilinear Translational Motion: (1D)

- All the particles of the body move along **straight lines.**

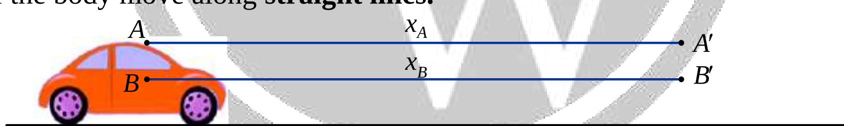
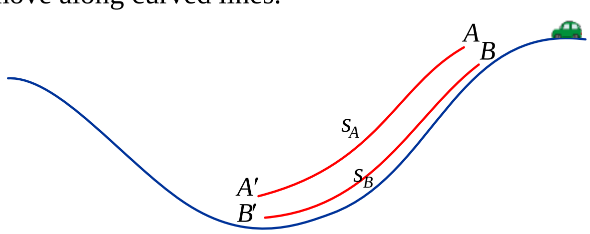


Fig. 6.2 A car moving in a straight line

$$x_A = x_B$$
$$v_A = v_B$$
$$a_A = a_B$$

(ii) Curvilinear Translational Motion: (2D)

- All the particles of the body move along curved lines.



$$s_A = s_B$$
$$v_A = v_B$$
$$a_A = a_B$$

Fig. 6.3 A car moving along a curved path

6.1.2 Rotational motion

- All the particles of the body move in circular paths.
- All the particles of the body have different displacement.

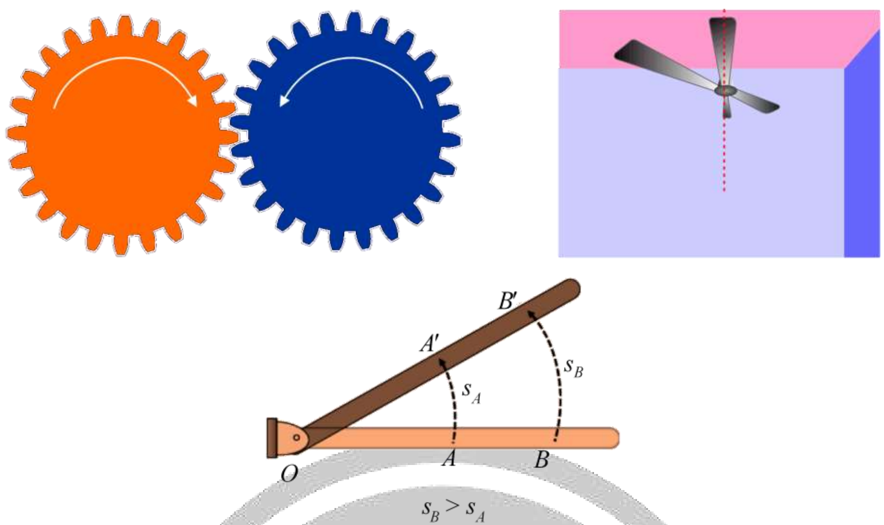


Fig. 6.4 Some examples of rotational motion

6.1.3 General Motion

- The body translates and rotates simultaneously.
- All the particles of the body move in different paths.
- All the particles of the body have different displacement.

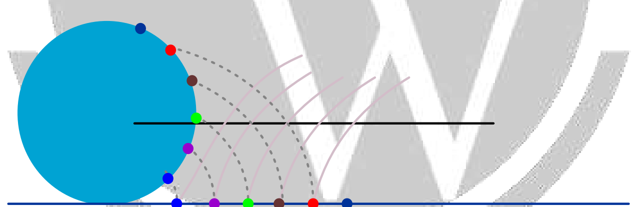


Fig. 6.5 A roller undergoing general motion

6.1.4 Dynamics of particles vs Rigid Bodies

- If shape and size of the body is not affecting the motion, we can consider the body to be a particle.
- By saying that the bodies are analysed as particles, we mean that only their motion as an entire unit will be considered; any rotation about their own mass center will be neglected.
- There are cases, however, when such a rotation is not negligible; the bodies cannot then be considered as particles. Such motions will be analysed in later chapters, dealing with the dynamics of rigid bodies

6.2 Translational Motion

- All the particles of the body move along identical parallel path.
- All the particles of the body have same displacement.

Rectilinear Translational Motion: (1D)

- All the particles of the body move along straight lines.

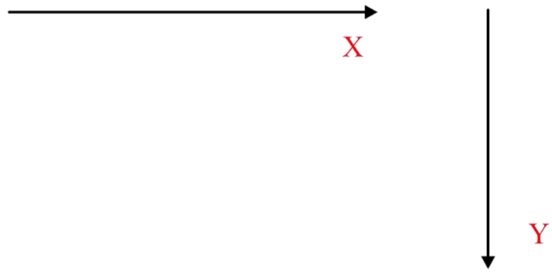


Fig. 6.6 Two perpendicular straight lines

6.2.1 Motion Variables

(i) Position

(ii) Displacement

(iii) Velocity

(iv) Acceleration

These are the variables which define the motion of the body.

(i) Position

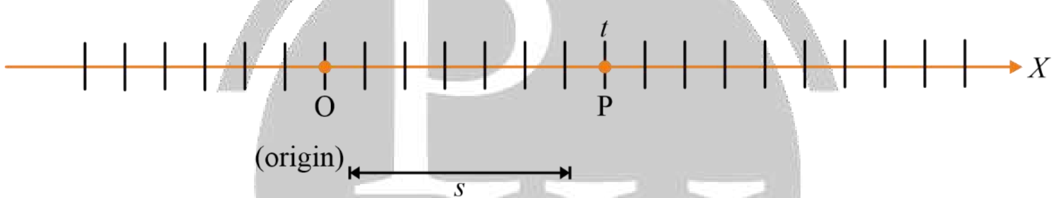


Fig. 6.7 Position of a particle at time t

(ii) Displacement

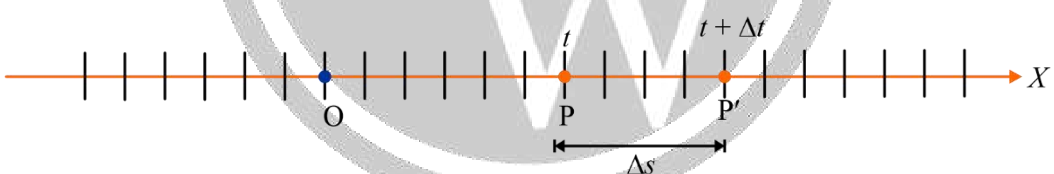


Fig. 6.8 Displacement of a particle at time t

$\Delta s \rightarrow$ displacement within time Δt.

(iii) Velocity

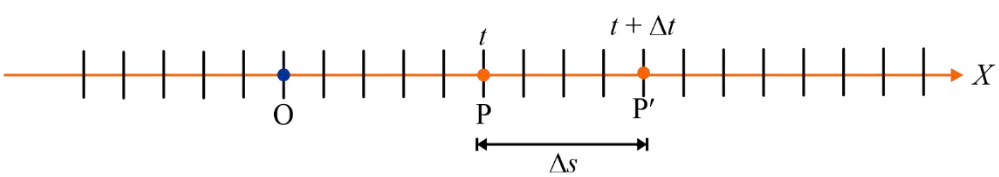


Fig. 6.9 Velocity of point at time t

$$\boxed{V_{avg} = \frac{\Delta s}{\Delta t}} \frac{\text{m}}{\text{s}}$$

Instantaneous velocity, $$\boxed{V = \frac{ds}{dt}}$$

(iv) Acceleration

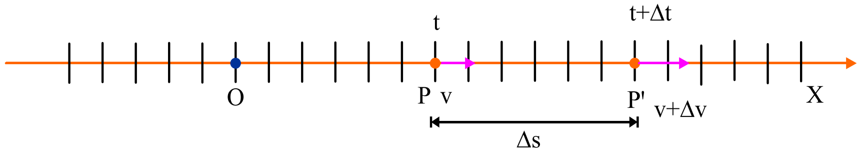


Fig. 6.10 Acceleration of a particle at time t

$$\boxed{a_{avg} = \frac{\Delta v}{\Delta t}} \ \frac{m}{s^2}$$

Instantaneous acceleration

$$\boxed{a = \frac{dv}{dt}}$$

$$a = \frac{dv}{ds} \cdot \frac{ds}{dt} \quad \Rightarrow \quad \boxed{v \cdot \frac{dv}{ds}}$$

Note:
Negative acceleration (deceleration) means velocity is decreasing.

6.2.2 Graphical Interpretation

(1) Displacement – Time graph

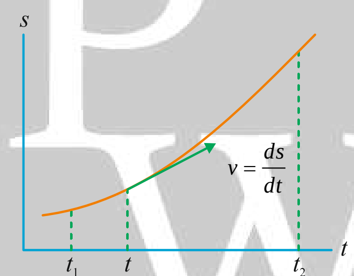


Fig. 6.11 Displacement-time graph

The slope of the displacement – time graph at any time is equal to the velocity at that time.

(2) Velocity – Time graph

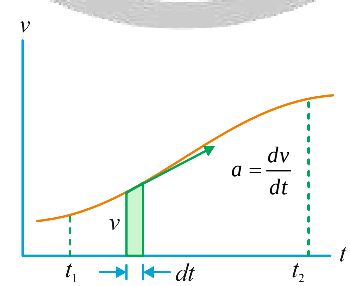


Fig. 6.12 Velocity-time graph

- The slope of the velocity – time graph at any time t is equal to the acceleration at that time.
- The net displacement of the particle during the interval from t_1 to t_2 is the corresponding area under the velocity – time graph, which is

$$\int_{s_1}^{s_2} ds = \int_{t_1}^{t_2} v\,dt \quad or \quad s_2 - s_1 = \text{(area under } \upsilon\text{-t curve)}$$

(3) Acceleration – Time graph

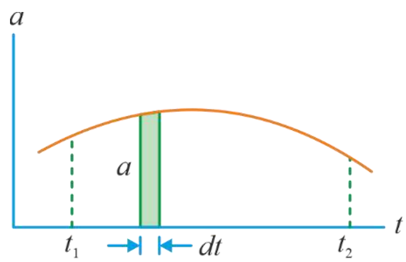


Fig. 6.13 Acceleration-time graph

- The net change in velocity of the particle during the interval from t_1 to t_2 is the corresponding area under the acceleration – time graph, which is

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \;\; or \;\; v_2 - v_1 = \text{(area under a-t curve)}$$

6.2.3 Uniformly Accelerated Motion (a = constant)

(i) $a = \dfrac{dv}{dt}$

$$\int_u^v dv = \int_o^t a\,dt$$

$$v - u = at$$

$$\boxed{v = u + at}$$

(ii) $v = \dfrac{ds}{dt}$

$$\int_o^s ds = \int_o^t v\,dt$$

$$s = \int_o^t (u + at)\,dt$$

$$\boxed{s = ut + \frac{1}{2}at^2}$$

(iii) $a = \dfrac{vdv}{ds}$

$$\int_u^v v\,dv = \int_o^s a\,ds$$

$$\frac{v^2 - u^2}{2} = as$$

$$\boxed{v^2 - u^2 = 2as}$$

6.3 Curvilinear Translational Motion

- All the particles of the body move along curved path.

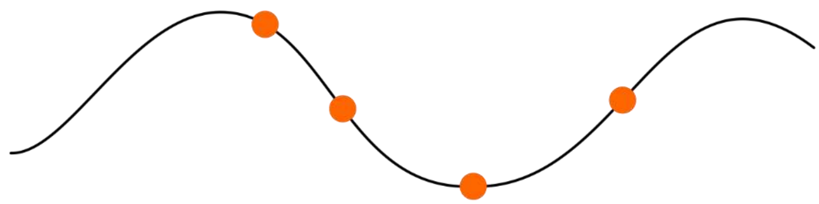


Fig. 6.14 A particle undergoing curvilinear motion

6.3.1 Motion Variables

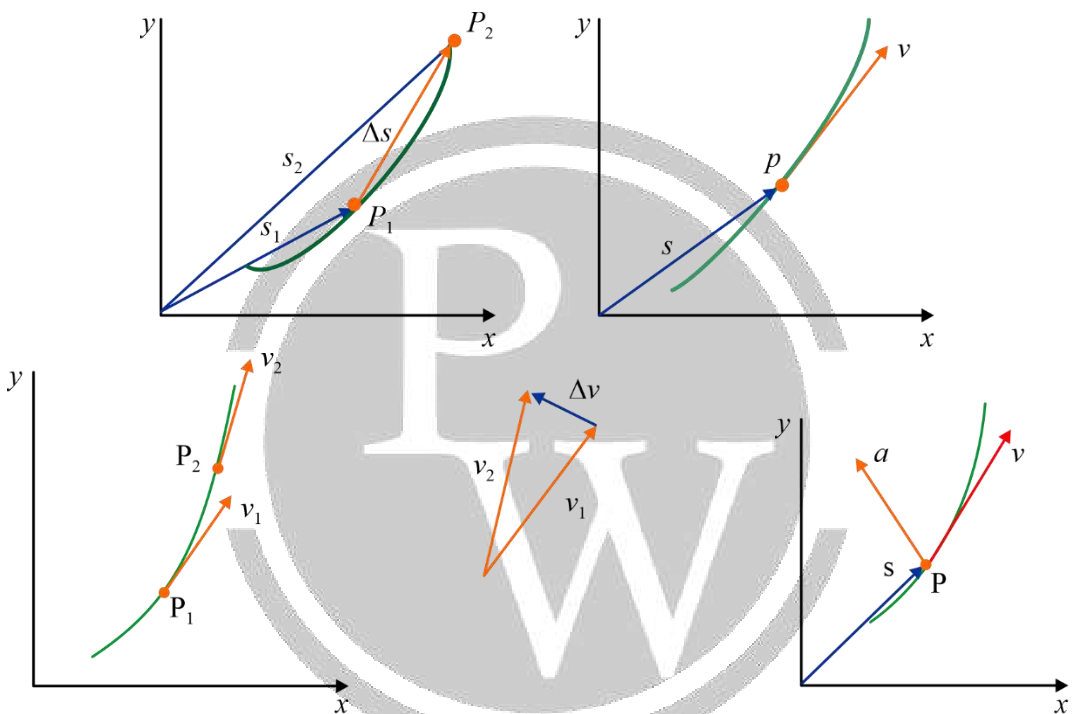


Fig. 6.15 Motion variables

$$\vec{s} = x\hat{i} + y\hat{j}$$

$$|s| = \sqrt{x^2 + y^2}$$

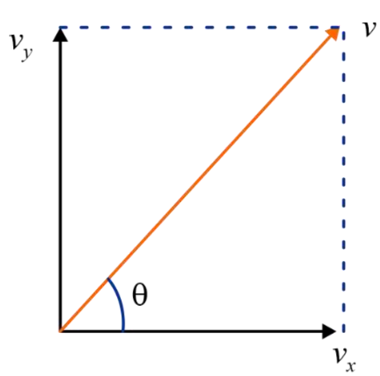


Fig. 6.16 Resultant velocity and its direction

$$\vec{v} = \frac{d\vec{s}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j}\right)$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$|v| = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}\frac{v_y}{v_x}$$

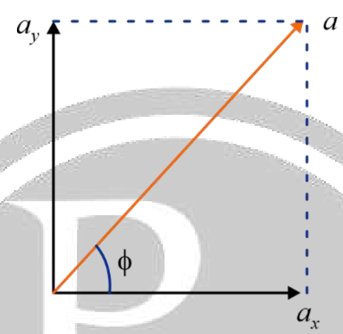


Fig. 6.17 Resultant acceleration and its direction

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(v_x\hat{i} + v_y\hat{j}\right)$$

$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

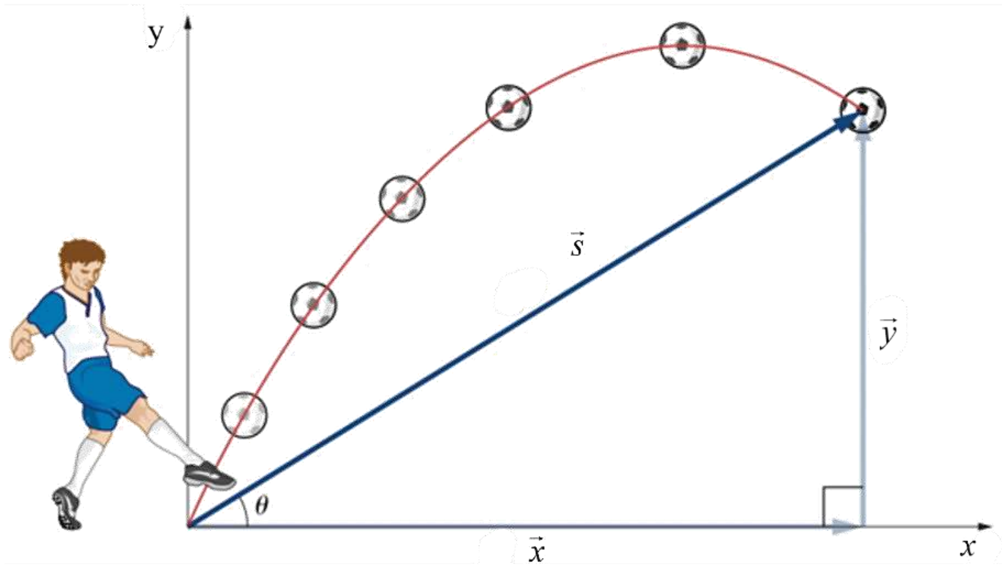
$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

$$|a| = \sqrt{a_x^{\,2} + a_y^{\,2}}$$

$$\phi = \tan^{-1}\frac{a_y}{a_x}$$

6.4 Projectile Motion

Projectile motion is experienced by an object or particle that is thrown near the Earth's surface and moves along a curved path under the action of gravity only.



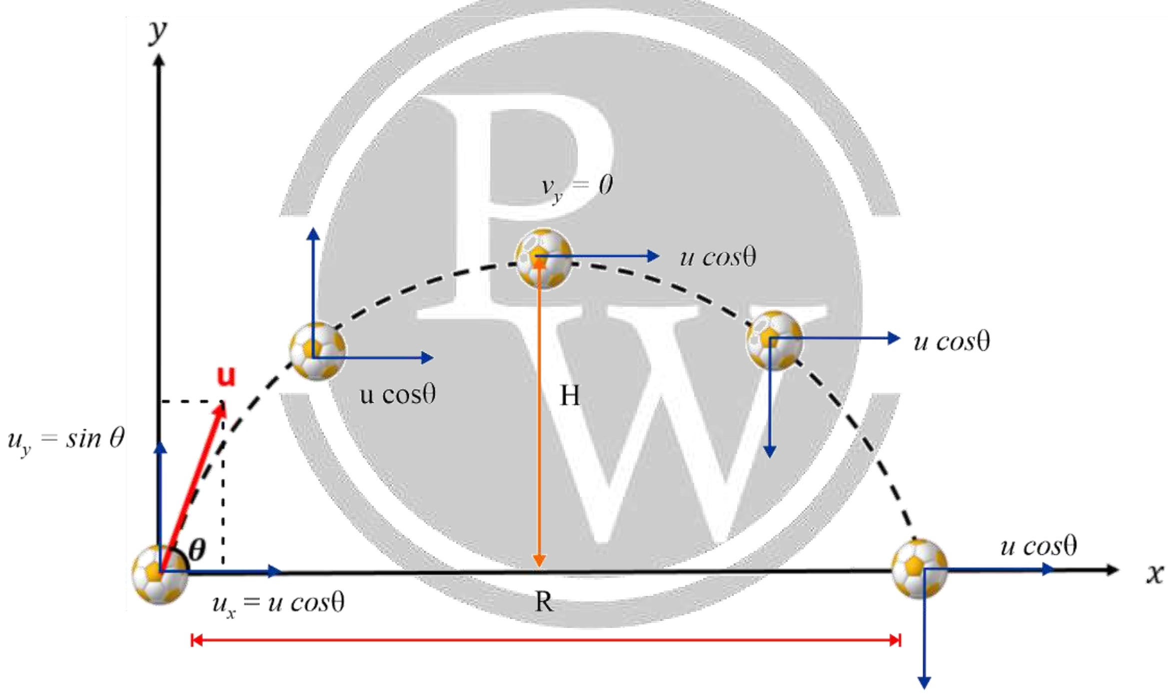


Fig. 6.18 Projectile motion

x	y
$a_x = 0$	$a_y = \pm g$
$v_x = u_x$	$v_y = u_y + a_y.t$
$s_x = u_x \times t$	$s_y = u_y t + 1/2 a_y t^2$

Important parameters to calculate:

1. Time of Flight (T) – total time of motion
2. Maximum Height (H) – maximum displacement in y
3. Rang (R) – maximum displacement in x

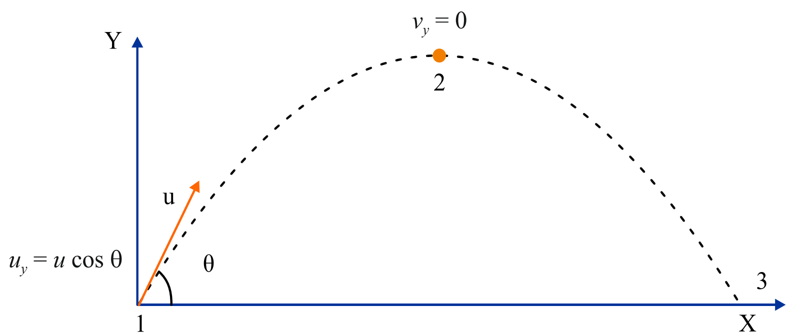


Fig. 6.19 Velocities of a particle in projectile motion

6.4.1 Time of flight (T)

Motion from 1 to 2 in y direction

$$v_y = u_y + a_y.\,t$$

$$0 = u\sin\theta - g.t$$

$$t = \frac{u\sin\theta}{g}$$

∵ Total time of flight, $T = 2.t$

$$\boxed{T = \frac{2u\sin\theta}{g}}$$

Alternatively:

Motion in y direction from 1 to 3

$$s_y = u_y.T + ½\, a_y.\, T^2$$

$$0 = u\sin\theta.\, T - ½\, g\, T^2$$

$$\boxed{T = \frac{2u\sin\theta}{g}}$$

6.4.2 Maximum Height (H)

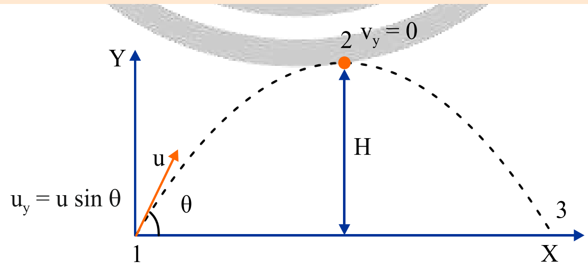


Fig. 6.20 Vertical velocities of a particle in projectile motion

Motion in Y from 1 to 2

$$v_y^2 - u_y^2 = 2a_y\, s_y$$

$$0 - (u\sin\theta)^2 = -2g.H$$

$$\boxed{H = \frac{u^2\sin^2\theta}{2g}}$$

6.4.3 Range (R)

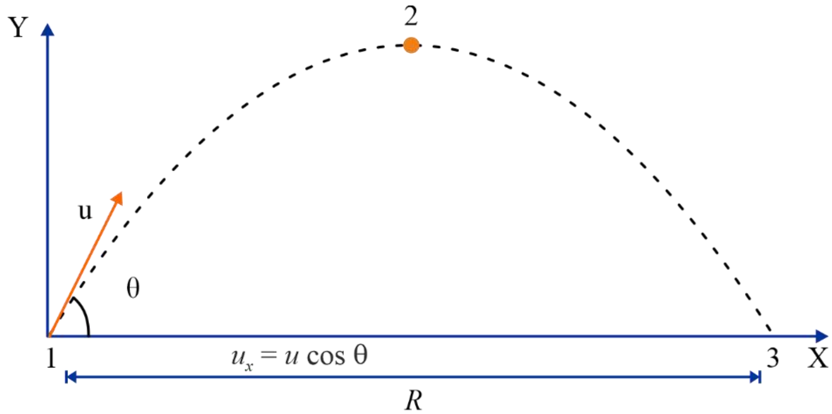


Fig. 6.21 Horizontal velocity of a particle in projectile motion

Motion in x from 1 to 3

$$s_x = u_x \cdot T + 1/2\, a_x \cdot T^2$$

$$R = u\cos\theta \times \frac{2u\sin\theta}{g} + 0$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}}$$

Maximum Range

$$R = \frac{u^2 \sin 2\theta}{g}$$

$$\sin 2\theta = 1 = \sin 90°$$

$$\theta = 45°$$

$$\boxed{R_{max} = \frac{u^2}{g}}$$

Range vs θ

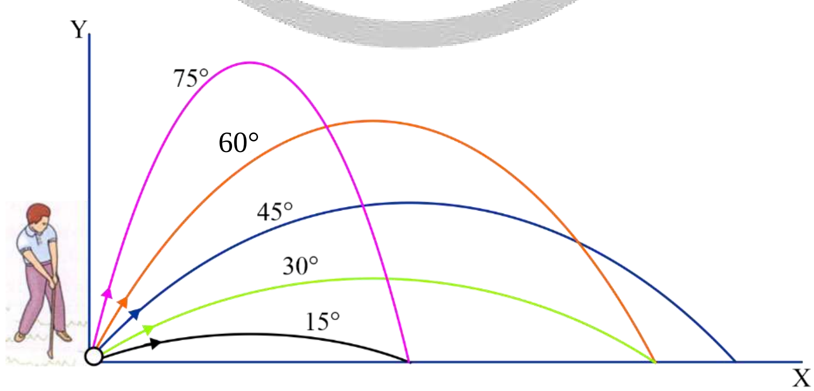


Fig. 6.22 Projectile motion at different angles

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7 KINEMATICS OF RIGID BODIES

7.1 Rotational Motion

- All the particles of the body move in circular paths.
- All the particles of the body have different displacement.

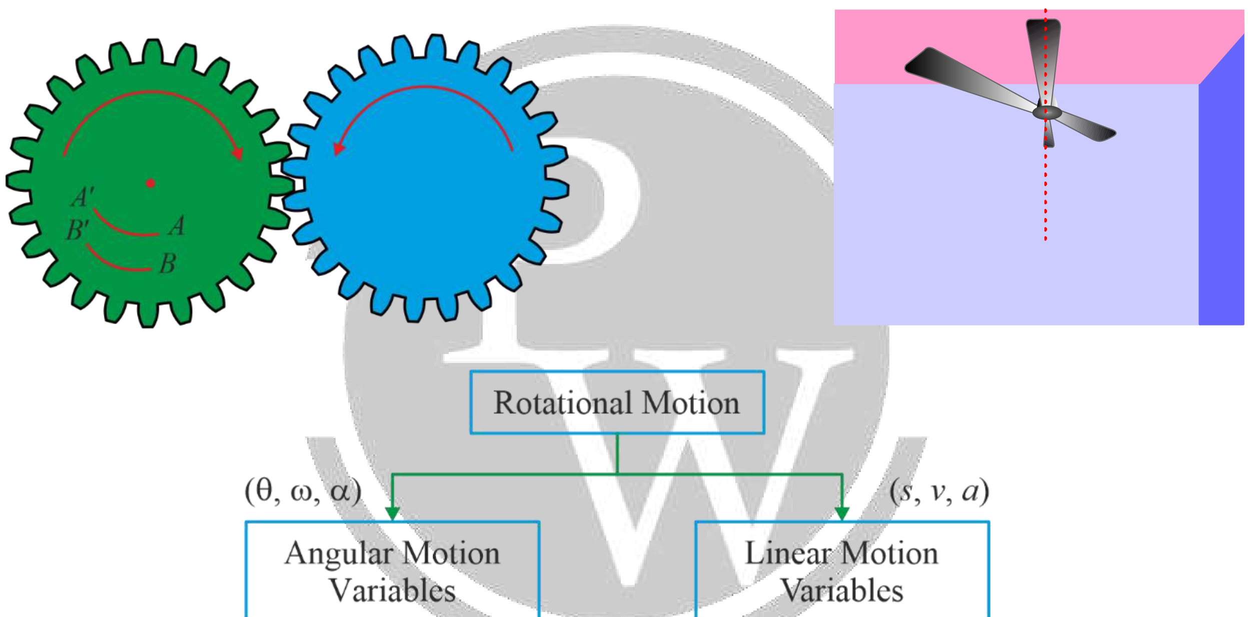

Fig. 7.1 Rotational motion and motion variables

7.1.1 Angular Motion Variables

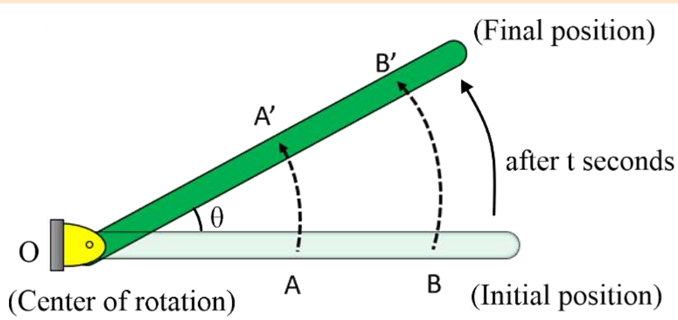

Fig. 7.2 A link rotating about a point

(1) Angular Displacement

$$\theta_A = \theta_B \qquad \text{(radians)}$$

- Angular displacement of every particle is same (θ).

(2) Angular velocity

$$\omega = \frac{dv}{dt} \qquad \text{(rad/s)}$$

ω of every particle is same.

$$\omega_A = \omega_B$$

(3) Angular acceleration

$$\alpha = \frac{d\omega}{dt} \qquad (\text{rad/s}^2)$$

$$\alpha = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\alpha = \omega\cdot\frac{d\omega}{d\theta}$$

$$\alpha_A = \alpha_B$$

$$\theta \rightarrow S$$

$$\omega \rightarrow v$$

$$\alpha \rightarrow a$$

Constant acceleration	Continuous Motion
1. $\omega = \omega_0 + \alpha t$	1. $\omega = \frac{d\theta}{dt}$
2. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	2. $\alpha = \frac{d\omega}{dt}$
3. $\omega^2 - \omega_0^2 = 2\alpha\theta$	3. $\alpha = \omega\cdot\frac{d\omega}{d\theta}$

7.1.2 Linear Motion Variables

(1) Linear displacement

$$s = r\,\theta$$

where r is the distance of the point from the center of rotation

$$s_A \neq s_B$$

In rotational motion, as r changes linear displacement also changes.

(2) Linear velocity

$$v = r.\omega$$

$$v_A \neq v_B$$

(3) Linear acceleration

$$a = r.\alpha$$

$$a_A \neq a_B$$

7.2 General Motion (Translation + Rotation)

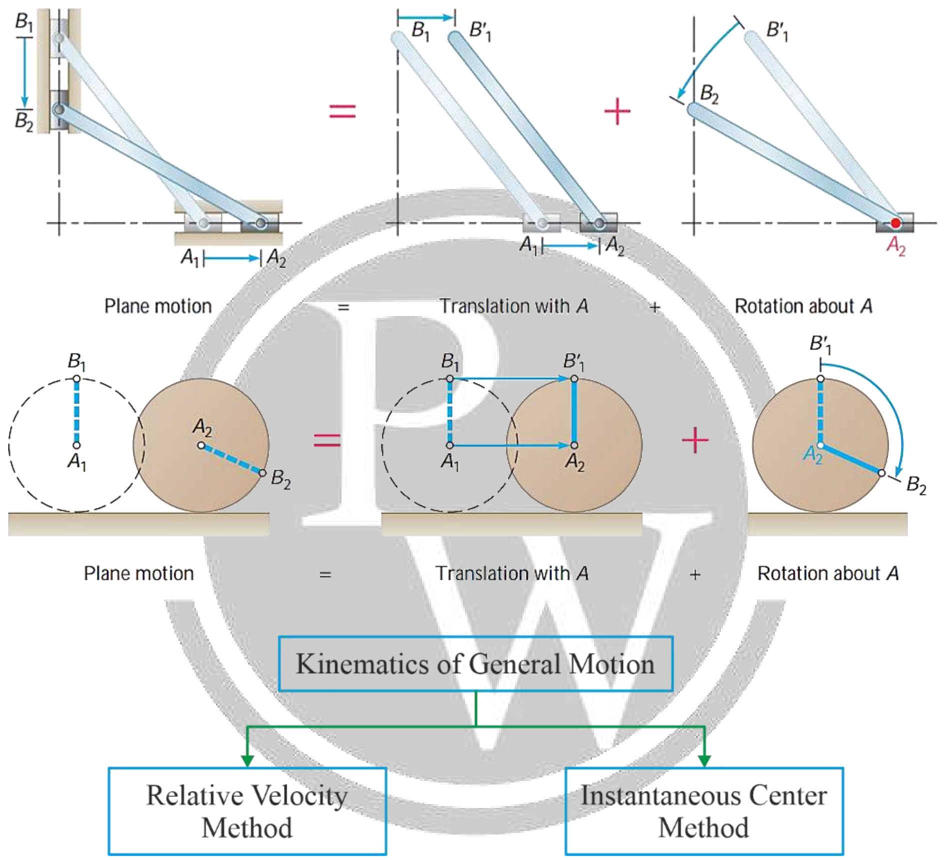

Fig. 7.3 General motion and methods to find motion variables

7.2.1 Relative Velocity Method

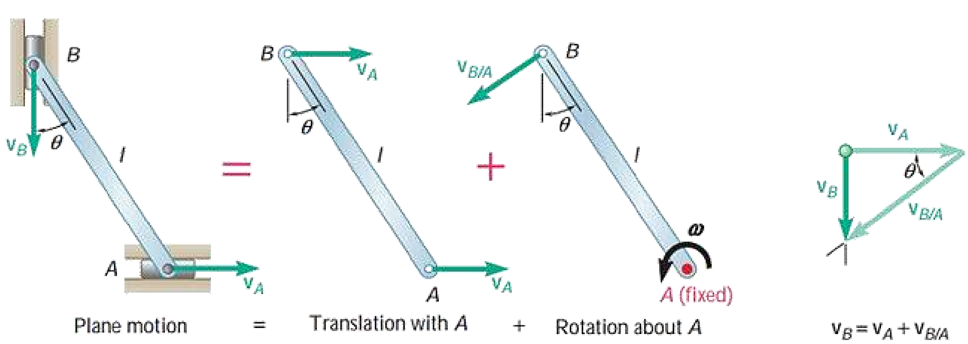

Fig. 7.4 A double slider mechanism and its velocity diagram

With A chosen as the reference point, the velocity of B is the vector sum of the translational portion V_A, plus the rotational portion $V_{B/A}$, which has the magnitude $V_{B/A} = \omega r$

Q. A rod of length 2 m is sliding in a corner as shown. At an instant when the rod makes an angle of 30 degrees with the horizontal plane, the velocity of point A on the rod is 8 m/s. Find

(i) Velocity of the end B (in m/s)

(ii) Velocity of the mid point (in m/s)

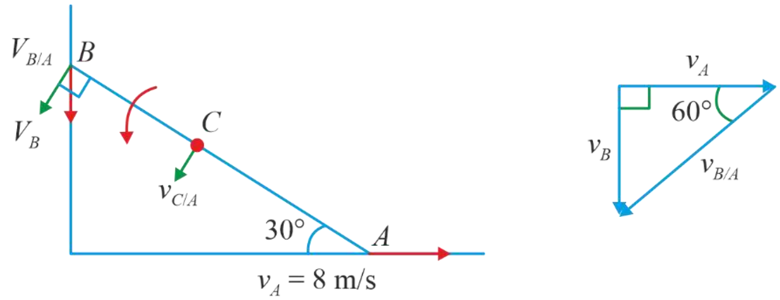

Fig. 7.5 Velocities of points in link AB and its velocity diagram

$$\vec{v}_B = v_A + \vec{v}_{B/A}$$

$$\tan 60° = \frac{v_B}{v_A}$$

$$v_B = 13.85 \text{ m/s}$$

$$v_{B/A} = \sqrt{v_A^2 + v_B^2}$$

$$\omega \times BA = \sqrt{8^2 + 13.85^2}$$

$$\omega \times 2 = 16$$

$$\omega = 8 \text{ rad/s}$$

$$\vec{v}_C = \vec{v}_A + \vec{v}_{C/A}$$

$$v_{C/A} = \omega \times CA = 8 \times 1 = 8 \text{ m/s}$$

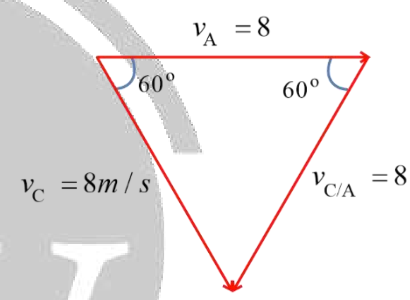

Fig. 7.6 Velocity diagram

From equilateral triangle

$$v_c = 8 \text{ m/s}$$

Alternatively:

Relative velocity of B w.r.t. A along AB = 0

$\therefore$ $\vec{v}_B$ along AB = $\vec{v}_A$ along AB

$$v_B \cos 60° = v_A \cos 30°$$

$$v_B = 8 \times \frac{\cos 30°}{\cos 60°}$$

$$v_B = 13.85 \text{ m/s}$$

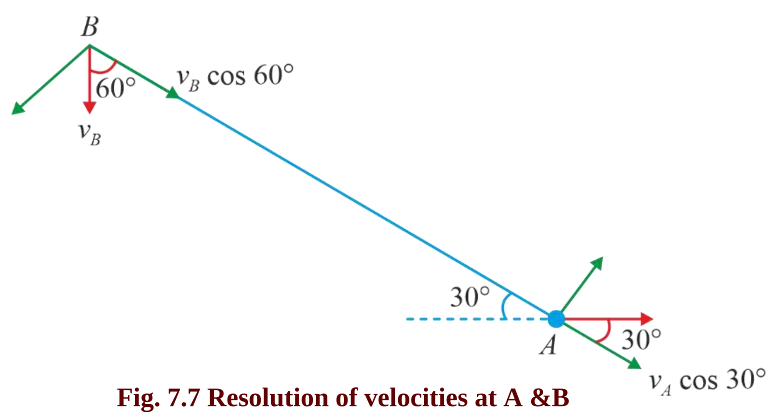

Fig. 7.7 Resolution of velocities at A &B

7.2.2 Instantaneous Center Method

- Combined motion of rotation and translation, may be assumed to be a motion of pure rotation about some imaginary center known as instantaneous center.
- As the position of body goes on changing, therefore the instantaneous center also goes on changing.

$$V_A = \omega \times IA$$

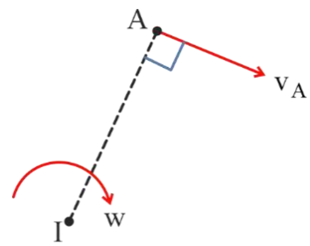

Fig. 7.8 Instantaneous center of A

Location of I Center

Case 1: When the direction of velocities of two particles A and B are known and V_A is not parallel to V_B.
I center lie at the intersection of two lines drawn at A and B, perpendicular to V_A and V_B respectively.

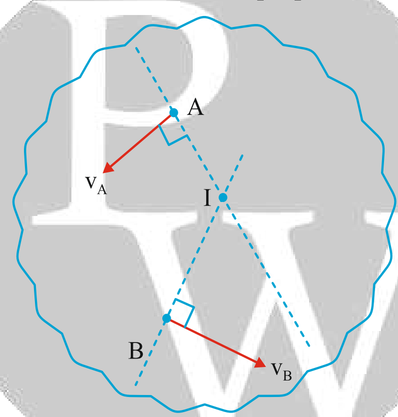

Fig. 7.9 I center when V_A is not parallel to V_B

Case 2: When the direction and magnitude of velocities of two particles A and B are known and V_A is parallel to V_B.
I center lie at the line connecting A and B.

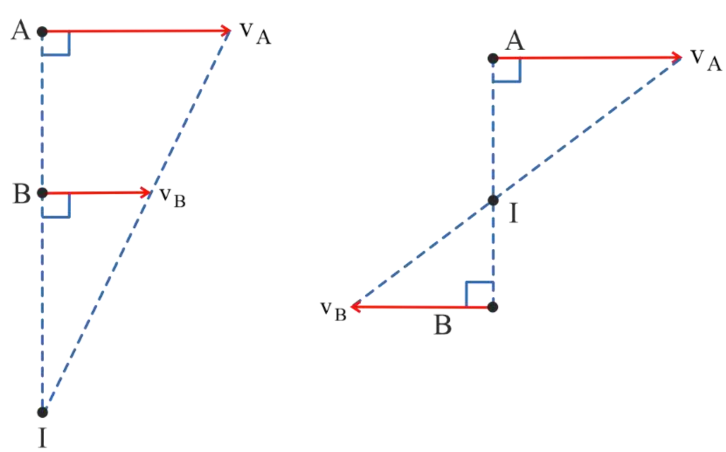

Fig. 7.10 I center when V_A is parallel to V_B

Case 3: Rolling without slipping on a fixed surface.

I center lie at the point of contact at that instant.

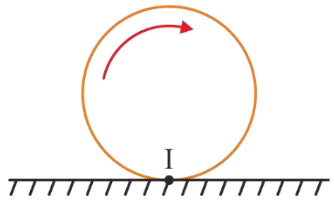

Fig. 7.11 I center of rolling object

Q. A rod of length 2 m is sliding in a corner as shown. At an instant when the rod makes an angle of 30 degrees with the horizontal plane, the velocity of point A on the rod is 8 m/s. Find

(i) Velocity of the end B (in m/s)

(ii) Velocity of the mid point (in m/s)

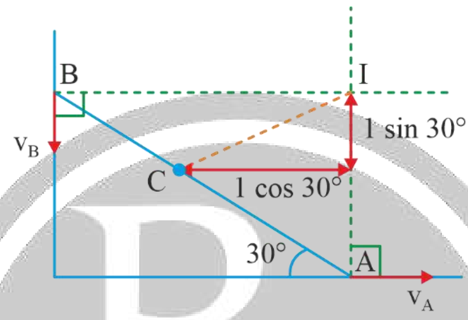

Fig. 7.12 I center of link AB in double slider 4 bar mechanism

$$v_A = \omega \times IA$$
$$8 = \omega \times 2 \sin 30°$$
$$\omega = 8 \text{ rad/s}$$
$$v_B = \omega \times IB$$
$$= 8 \times 2 \cos 30°$$
$$v_B = 13.85 \text{ m/s}$$
$$IC = \sqrt{(1\cos 30°)^2 + (1.\sin 30°)^2} = 1\,\text{m}$$
$$v_c = \omega \times IC$$
$$= 8 \times 1$$
$$v_c = 8 \text{ m/s}$$

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8 KINETICS OF PARTICLES

8.1 Kinetics of Particles

Relation between forces and motion variables (s, v, a)

- D'Alembert's method
- Work - Energy method
- Impulse - momentum method

8.1.1 D'Alembert's Principle

D'Alembert's Principle introduces a new force- Inertia Force. D'Alembert's principle is another form of Newton's second law of motion.

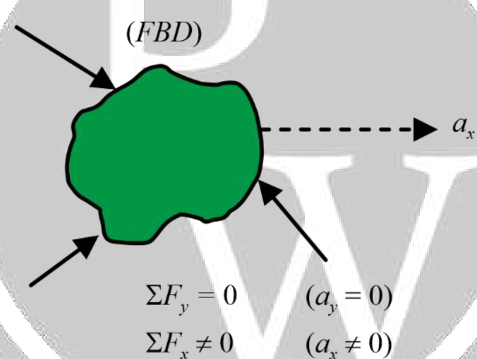


Fig. 8.1 An object with an acceleration in x-direction

Newton's law of motion:

$$\Sigma F_x = m \cdot a_x$$

$$\Sigma F_x - ma_x = 0$$

Total force in x

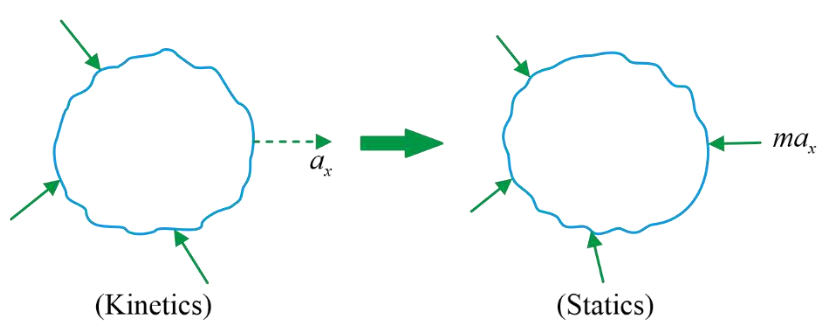


Fig. 8.2 Conversion of an object from kinetics to statics by introducing inertia force

1. An elevator of mass 3000 kg is moving vertically downwards with a constant acceleration. Starting from rest, it travels a distance of 40 m during an interval of 10 seconds. Find the cable tension (in N) during this time. Assume g = 10 m/s^2.

Sol.

$$\Sigma F_y = ma_y$$

$$30000 - T = 3000\text{ a}$$

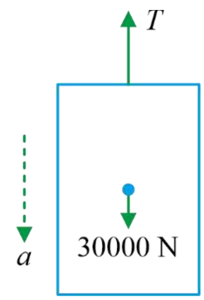
$$s = ut + \frac{1}{2}at^2$$

$$40 = 0 + \frac{1}{2}a \times 10^2$$

$$a = 0.8\text{ m/s}^2$$

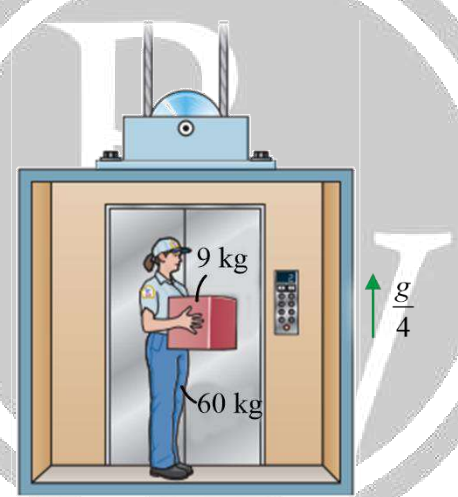
$$30000 - T = 3000 \times 0.8$$

$$T = 27{,}600\text{ N}$$

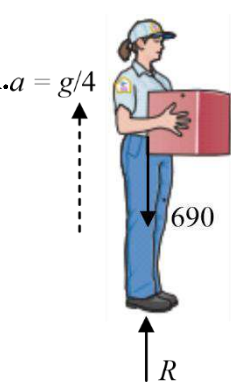


(FBD)

2. A 60-kg woman holds a 9-kg package as she stands within an elevator which briefly accelerates upward at a rate of $g/4$. Determine the force R which the elevator floor exerts on her feet during the acceleration interval. Assume g = 10 m/s^2.



Sol.

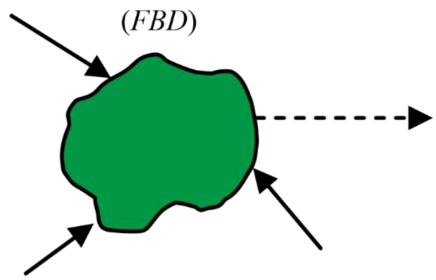


$$\Sigma F_y = ma_y$$

$$R - 690 = 69 \times \frac{g}{4} = 69 \times \frac{10}{4}$$

$$R = 862.5\text{ N}$$

8.1.2 Work – Energy Method



ΣF = Net force in direction of motion

Fig. 8.3 FBD of an object

Work done on body = Charge in K.E. of body

$$\Sigma F \times s = \frac{1}{2} m (v^2 - u^2)$$

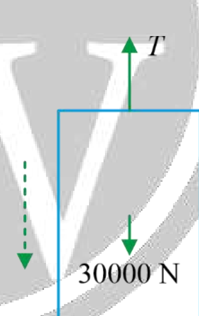
$$\Sigma F = \text{Net force in direction of motion.}$$

3. An elevator of mass 3000 kg is moving vertically downwards with a constant acceleration. Starting from rest, it travels a distance of 40 m during an interval of 10 seconds. Find the cable tension (in N) during this time. Assume g = 10 m/s^2.

Sol.

$$\Sigma F \times s = \frac{1}{2} m (v^2 - u^2)$$

$$(30000 - T) \times 40 = \frac{1}{2} \times 3000(v^2 - 0)$$



$$s = ut + \frac{1}{2} at^2$$

$$40 = 0 + \frac{1}{2} a \times 10^2$$

$$a = 0.8 \text{ m/s}^2$$

$$v = u + at$$

$$= 0 + 0.8 \times 10$$

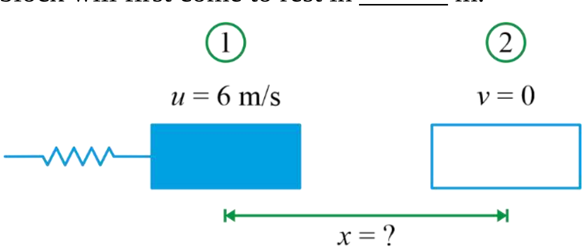
$$v = 8 \text{ m/s}$$

$$(30000 - T) \times 40 = \frac{1}{2} \times 3000 (8^2 - 0)$$

$$T = 27{,}600 \text{ N}$$

4. The 10 kg block is moving to the right at 6 m/s when the 360 N/m spring is unstretched. If the horizontal plane is frictionless, the block will first come to rest in _______ m.

Sol.



Conservation of Energy

$$\underset{(1)}{\text{(Total energy)}} = \underset{(2)}{\text{(Total energy)}}$$

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2$$

$$10 \times 6^2 = 360 \times x^2$$

$$x = 1 \text{ m}$$

Work – Energy

$$\text{Work done} = \text{Change in K.E.}$$

$$0 = \frac{1}{2}m\,(0 - u^2) + \frac{1}{2}kx^2$$

$$\frac{1}{2}mu^2 = \frac{1}{2}kx^2$$

- Conservation of energy is only applicable if there is no friction.
- If there is friction, apply work-energy.

8.1.3 Conservation of Momentum

When the resultant of the external forces acting on a system of particles is zero, the linear momentum of the system is conserved.

Fig. 8.4 A system of particles with net external force zero

$$\Sigma F \cdot t = m.(v - u)$$

$$0 = m.(v - u)$$

$$u = v$$

8.2 Collision (Impact) of Elastic Bodies

Impact occurs when two bodies collide with each other during a very short period of time.

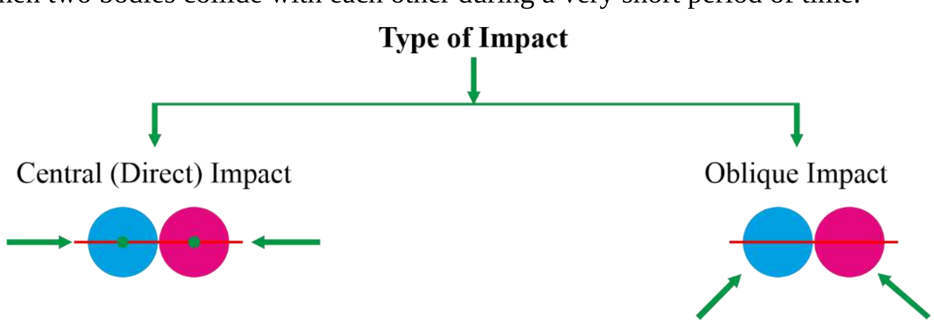


Fig. 8.5 Types of impact

8.2.1 Central (Direct) Impact

Fig. 8.6 Before and after velocities in Direct impact

Let $(u_1 > u_2)$ $v_1 = ?$ Let $(v_2 > v_1)$

$v_2 = ?$

Principle of Conservation of Momentum:

Initial momentum of system = Final momentum of system

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \quad \text{. . . (i)}$$

$$\begin{pmatrix} \vec{v} = +ve \\ \overleftarrow{v} = -ve \end{pmatrix}$$

Newton's Law of Restitution

- For a given pair of bodies in collision, the ratio of relative velocity after collision and relative velocity before collision is always same.
- This ratio is known as coefficient of restitution (e). Its value depends on the material.

$$e = 0 \text{ to } 1$$

$$e = 1 \text{ for perfectly elastic bodies}$$

$$e = 0 \text{ for perfectly plastic bodies}$$

$$e = \frac{v_2 - v_1}{u_1 - u_2} \quad \text{. . . (ii)}$$

- For perfectly elastic collision, kinetic energy of the system is conserved.

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

- If $e < 1$, some kinetic energy is lost.

$$\text{Loss in K.E.} = \left(\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2\right) - \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right)$$

Special Cases

Case 1: If there is perfectly plastic collision between two bodies, the bodies stick together and move as single body after collision.

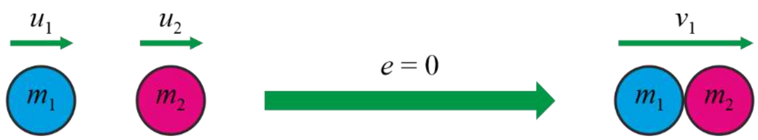


Fig. 8.7 Direct impact of perfectly plastic bodies

$$e = \frac{v_2 - v_1}{u_1 - u_2} = 0$$

$$v_2 = v_1$$

Case 2: If there is perfectly elastic collision between two bodies, of equal masses, the velocities of the bodies interchange after collision.

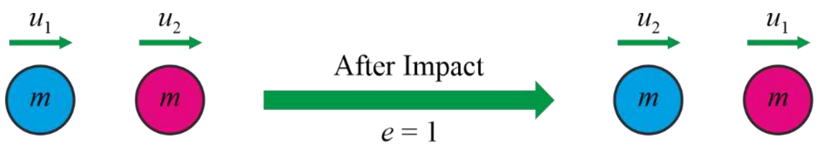


Fig. 8.8 Direct impact of perfectly elastic bodies

$$e = 1 = \frac{v_2 - v_1}{u_1 - u_2}$$

$$v_2 - v_1 = u_1 - u_2 \quad \text{. . . (i)}$$

$$mu_1 + mu_2 = mv_1 + mv_2$$

$$u_1 + u_2 = v_1 + v_2 \quad \text{. . .(ii)}$$

$$v_1 = u_2$$

$$v_2 = u_1$$

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9 KINETICS OF RIGID BODIES

9.1 Kinetics of Rigid Bodies

Relation between forces and motion variables. (s, v, a)

- D'Alembert's method
- Work - Energy method
- Impulse - momentum method

Rotation + General motion

9.1.1 D'Alembert's Method

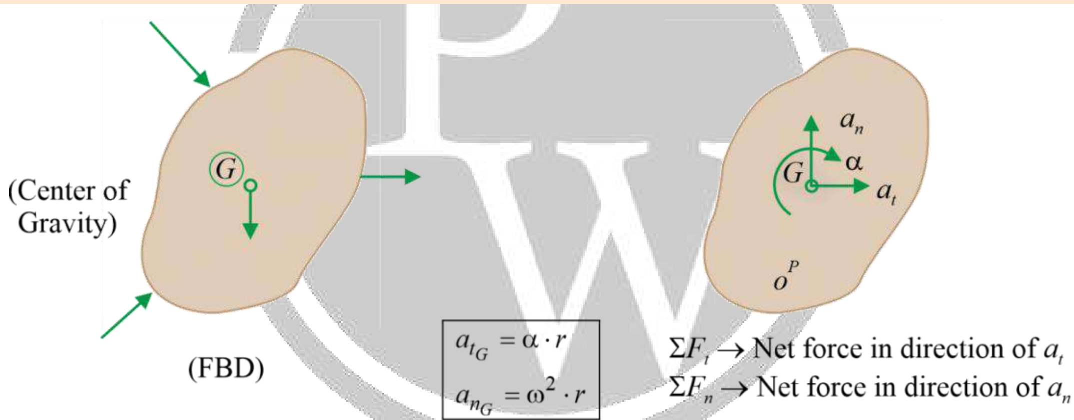


Fig. 9.1 An object under application of external loads and its accelerations

$$\Sigma F_t = m \cdot a_t$$

$$\Sigma F_n = m \cdot a_n$$

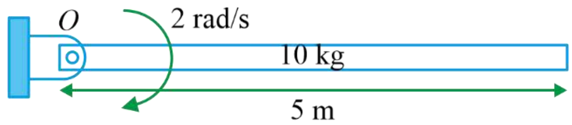
$$\Sigma M_G = I_G \cdot \alpha$$

OR

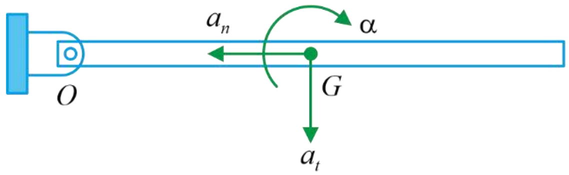
$$\Sigma M_P = I_P \cdot \alpha$$

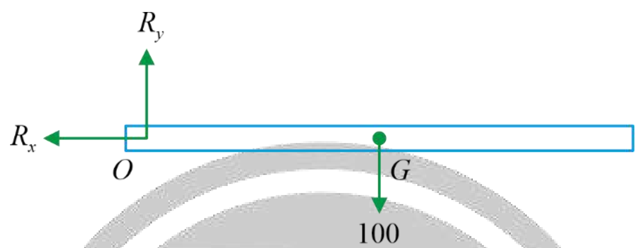
I_G = Mass MOI of body about G

1. A slender rod of mass 10 kg and length 5 m is rotating with an angular velocity of 2 rad/s at the instant as shown in figure. Find the angular acceleration of the rod and the horizontal and vertical reaction at the hinge.



Sol.





$$\Sigma F_n = m \cdot a_n = m \cdot \omega^2 r = m\omega^2(\text{OG})$$

$$R_X = 10 \times 2^2 \times 2.5$$

$$R_X = 100 \text{ N}$$

$$\Sigma F_t = m \cdot a_t = m \cdot \alpha \cdot r = m \cdot \alpha\ (\text{OG})$$

$$(100 - R_y) = 10 \times \alpha \times 2.5 \quad \ldots \text{(i)}$$

$$\Sigma M_G = I_G \cdot \alpha = \left(\frac{ml^2}{12}\right) \cdot \alpha$$

$$R_y \times 2.5 = \left(\frac{10 \times 5^2}{12}\right) \cdot \alpha \quad \ldots \text{(ii)}$$

By solving equation (i) and (ii)

$$R_y = 25 \text{ N}$$

$$\alpha = 3 \text{ rad/s}^2$$

Alternatively:

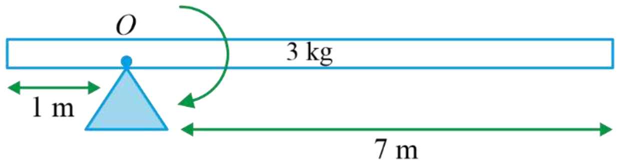
$$\Sigma M_0 = I_0 \cdot \alpha = \left(\frac{ml^2}{3}\right) \cdot \alpha$$

$$100 \times 2.5 = \left(\frac{10 \times 5^2}{3}\right) \times \alpha$$

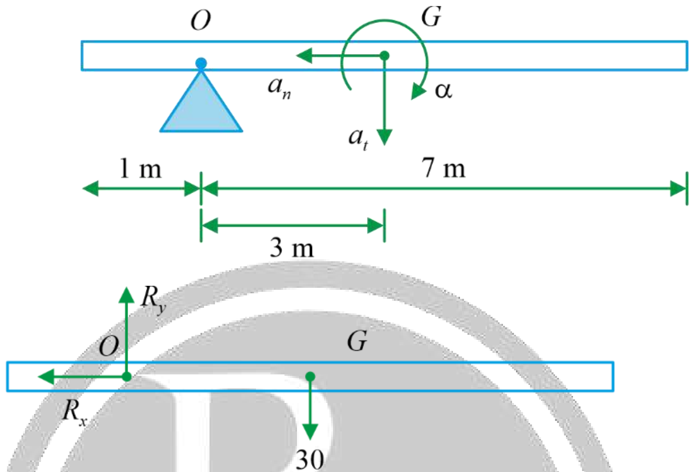
$$\alpha = 3 \text{ rad/s}^2$$

2. A uniform slender rod (8 m length and 3 kg mass) rotates in a vertical plane about a horizontal axis 1 m from its end as shown in the figure. The magnitude of the angular acceleration (in rad/s^2) of the rod at the position shown is _______.

(g = 10 m/s^2) **[GATE-ME-14:2M]**



Sol.



$$I_0 = I_G + m \cdot (OG)^2$$

$$= \frac{3\times 8^2}{12} + 3\times 3^2$$

$$I_0 = 43\ kg - m^2$$

$$\Sigma M_0 = I_0\,\alpha$$

$$30 \times 3 = 43 \cdot \alpha$$

$$\alpha = 2.093 \text{ rad/s}^2$$

9.1.2 Work – Energy Method

Work done on the body = Change in K.E. of the body.

Kinetic Energy

(1) Pure Rotation:

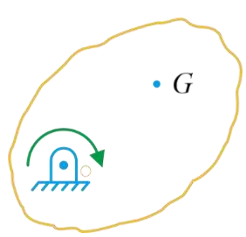


Fig. 9.2 An object under pure rotation

$$\text{K.E.} = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

OR

$$= \frac{1}{2}[I_G + m \cdot r^2]\,\omega^2$$

$$\text{K.E.} = \frac{1}{2}I_o\omega^2$$

(2) General Motion (Rotation + Translation):

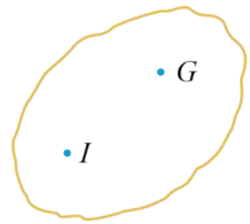


Fig. 9.3 Center of gravity and I center of an object

$$\text{K.E.} = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

OR

$$\text{K.E.} = \frac{1}{2}I_I \cdot \omega^2$$

Work Done

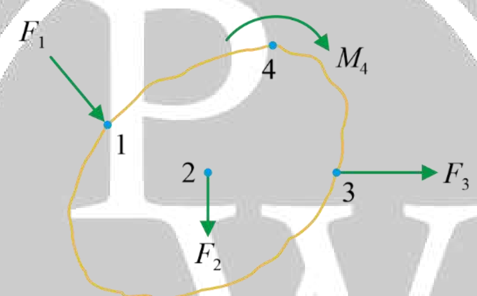
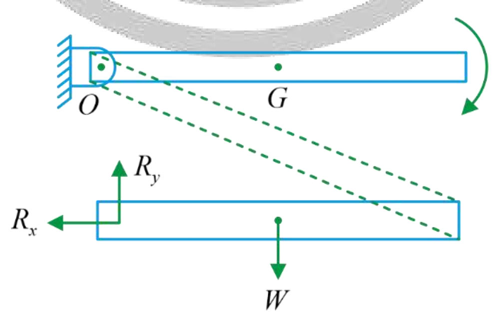


Fig. 9.4 FBD of an object under application of loads and moment

$$\text{W.D.} = \pm F_1s_1 \pm F_2s_2 \pm F_3s_3 \pm M_4\theta_4$$

Forces with zero work done

(1)



W.D. by R_x and R_y = 0

as displacement of O = 0

(2) Rolling without Slipping:

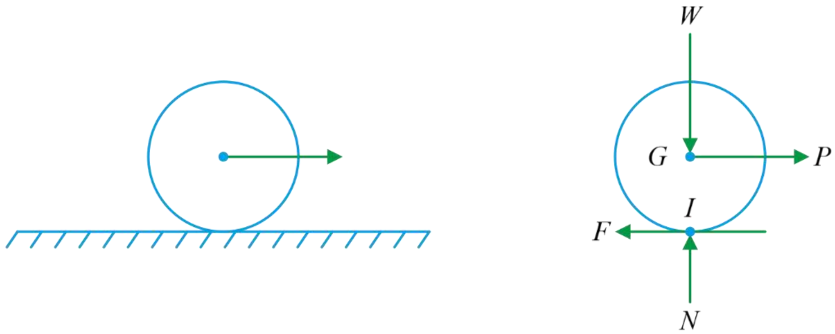
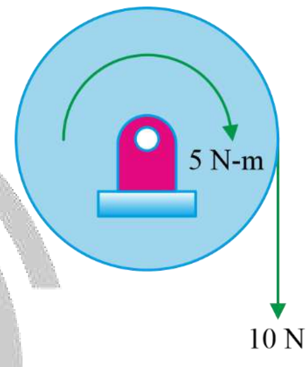


Fig. 9.5 A roller rolling without slipping and its FBD

W.D. of $F = 0$

as $V_I = 0$

3. The 30-kg disk of radius 0.2 m shown in figure is pin supported at its center. Determine the number of revolutions it must make to attain an angular velocity of 20 rad/s starting from rest. It is acted upon by a constant force of 10 N, which is applied to a cord wrapped around its periphery, and a constant couple moment of 5 N-m. Neglect the mass of the cord in the calculation.



Sol.

$$S_P = \theta \times r$$

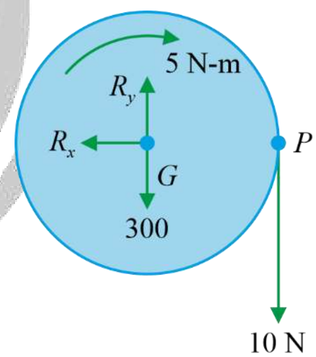
$$S_P = \theta \times 0.2$$

$$10 \times S_P + 5 \times \theta = \frac{1}{2}m(v_G^2 - u_G^2) + \frac{1}{2}I_G(\omega_2 - \omega_0^2)$$

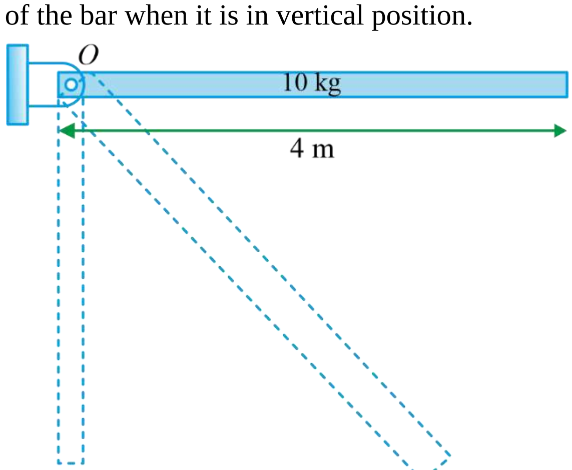
$$10 \times 0.2 \times \theta + 5 \times \theta = \frac{1}{2}m(0-0) + \frac{1}{2} \times \frac{30 \times 0.2^2}{2}(20^2 - 0)$$

$$\theta = 17.14 \text{ radius}$$

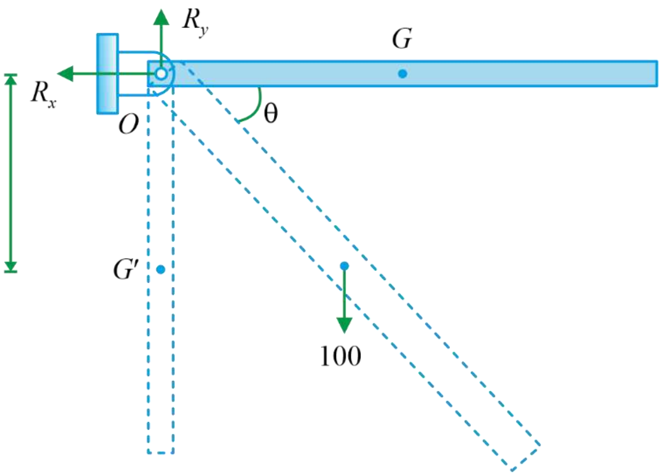
$$\text{No. of revelations} = \frac{\theta}{2\pi} = \frac{17.14}{2\pi} = 2.73$$



4. A uniform bar of mass 10 kg and length 4 m hangs from a frictionless hinge. It is released from rest in the horizontal position. Find the angular velocity of the bar when it is in vertical position.



Sol. $S_G = \frac{4}{2} = 2\text{m}$



$$100 \times S_G = \frac{1}{2}m(v_G^2 - u_G^2) + \frac{1}{2}I_G(\omega^2 - \omega_0^2)$$

OR

$$100 \times S_G = \frac{1}{2}I_0(\omega^2 - \omega_0^2)$$

$$100 \times 2 = \frac{1}{2}\left(\frac{10 \times 4^2}{3}\right)(\omega^2 - 0)$$

$$\boxed{\omega = 2.74} \text{ rad/s}$$

9.1.3 Impulse – Momentum Method

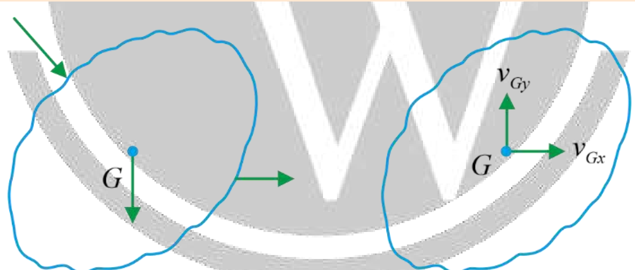


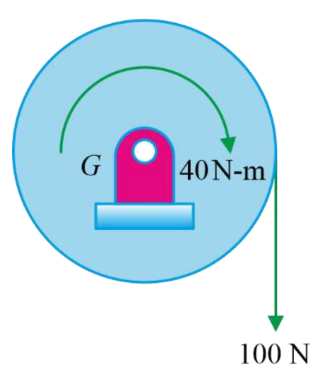
Fig. 9.6 An object under the application of load and its velocities

$$\Sigma F_x \times t = m(v_{Gx} - u_{Gx})$$

$$\Sigma F_y \times t = m(v_{Gy} - u_{Gy})$$

$$\Sigma M_G \times t = I_G(\omega - \omega_0)$$

5. The 20 kg disk of radius 0.75 m shown in figure is pin supported at its center. It is acted upon by a constant force of 100 N, which is applied to a cord wrapped around its periphery, and a constant couple moment of 40 N-m. Find the angular velocity of the disc and horizontal and vertical reactions at the hinge, 2 sec after starting from ret.



Sol.

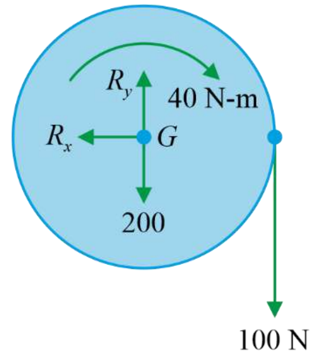
$$v_{Gx} = 0$$

$$u_{Gx} = 0$$

$$v_{Gy} = 0$$

$$u_{Gy} = 0$$

$$\Sigma F_x \times t = m(v_{Gx} - u_{Gx})$$
$$R_x \times 2 = 0$$
$$R_x = 0$$
$$\Sigma F_y \times t = m(v_{Gy} - u_{Gy})$$
$$(R_y - 200 - 100) \times 2 = 0$$
$$R_y = 300 \text{ N}$$
$$\Sigma M_G \times t = I_G(\omega - \omega_0)$$
$$(100 \times 0.75 + 40) \times 2 = \frac{20 \times 0.75^2}{2}(\omega - 0)$$
$$\omega = 40.88 \text{ rad/s.}$$



9.2 Lagrange's Equation of Motion

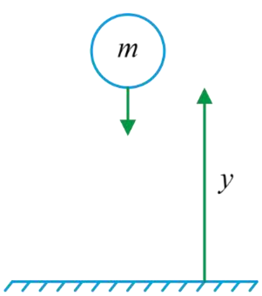
$$\boxed{L = T - U}$$
(Lagrangian) (K.E.) (P.E.)

q = Generalised coordinate.

Equation of Motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$
$$\left(\dot{q} = \frac{dq}{dt}\right)$$

6.



$$q = y, \; \dot{q} = \dot{y}$$
$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2$$
$$U = mgh = mgy$$
$$L = T - U$$
$$L = \frac{1}{2}m\dot{y}^2 - mgy$$
$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial y} = -mg$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = m\ddot{y}$$

Equation of motion:

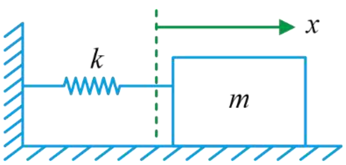
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

$$m\ddot{y} - (-mg) = 0$$

$$\ddot{y} + g = 0$$

$$\ddot{y} = -g \text{ (–ve sign indicates } a \text{ is } \downarrow)$$

7.



$$q = x,\ \dot{q} = \dot{x}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = \frac{1}{2}kx^2$$

$$L = T - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial x} = -kx$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

Equation of motion:

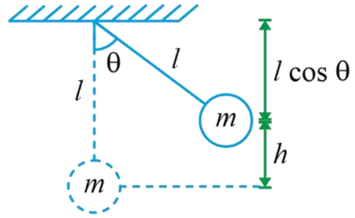
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m\ddot{x} - (-kx) = 0$$

$$m\ddot{x} + kx = 0$$

$$\boxed{\ddot{x} + \left(\frac{k}{m}\right)x = 0} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

8.



(String is massless)

$$q = \theta,\ \dot{q} = \dot{\theta}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega l)^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

$$U = mgh = mg\,(l - l\cos\theta)$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl + mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

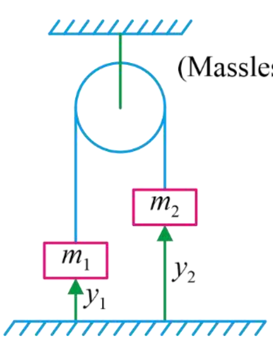
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2\ddot{\theta} - (-mgl\sin\theta) = 0$$

$$l^2\ddot{\theta} + gl\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \frac{g}{l}\theta = 0} \quad (\sin\theta \approx \theta)\ \ \omega_n = \sqrt{\frac{g}{l}}$$

9.



(Massless and frictionless pulley)

$$y_1 + y_2 = c$$

$$y_2 = c - y_1$$

$$\dot{y}_2 = -\dot{y}_1$$

$$T = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2$$

$$U = m_1gy_1 + m_2gy_2 = m_1gy_1 + m_2g\,(c - y_1)$$

$$L = T - U = \frac{1}{2}(m_1 + m_2)\dot{y}_1^2 - m_1gy_1 - m_2gc + m_2gy_1$$

$$\frac{\partial L}{\partial y_1} = -m_1g + m_2g = (m_2 - m_1)g$$

$$\frac{\partial L}{\partial \dot{y}_1} = (m_1 + m_2)\dot{y}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) = (m_1 + m_2)\ddot{y}_1$$

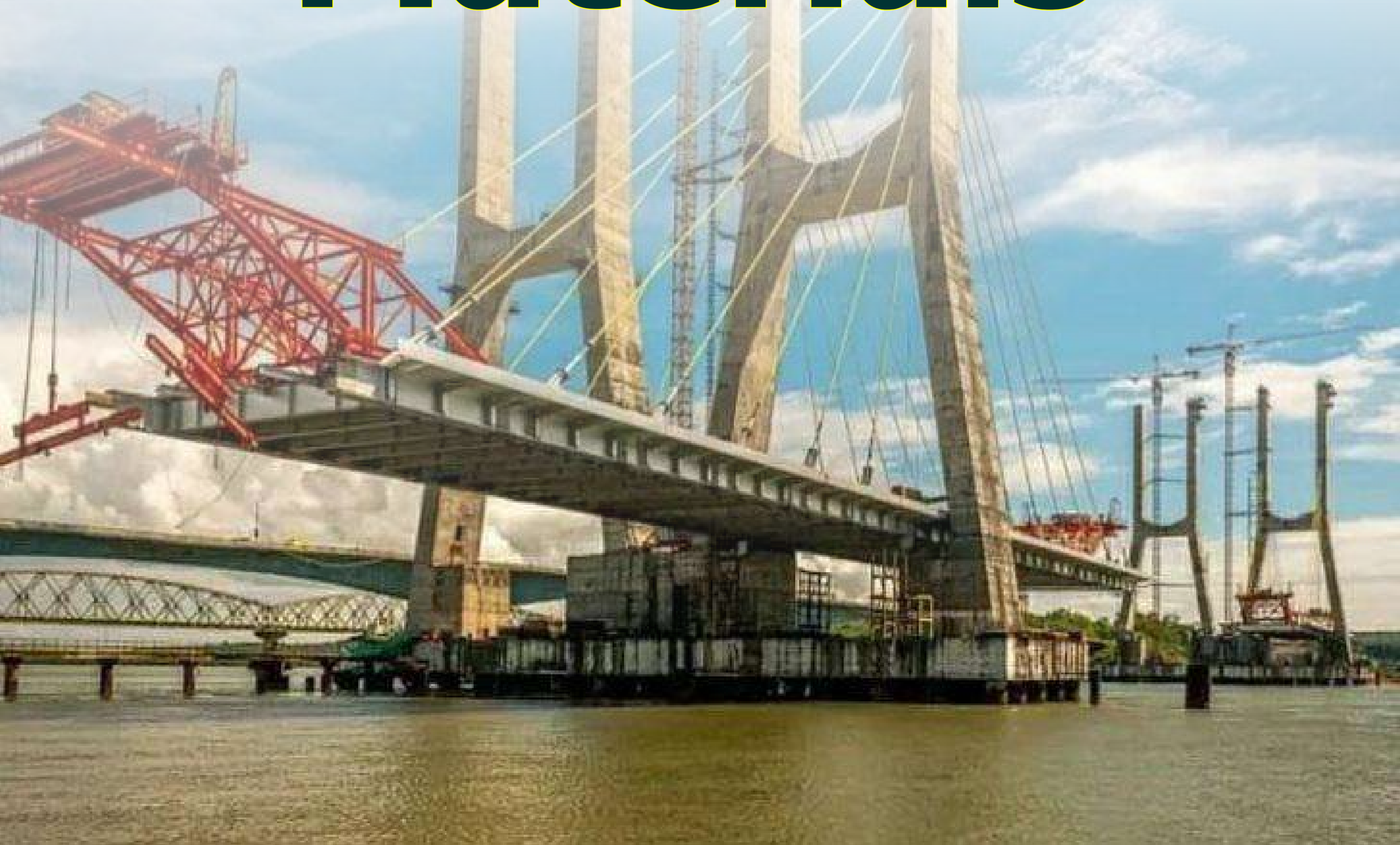
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0$$

$$(m_1 + m_2)\ddot{y}_1 - (m_2 - m_1)g = 0$$

$$\ddot{y}_1 = \frac{(m_2 - m_1)g}{(m_1 + m_2)}$$

❑❑❑

Strength of Materials



Strength of Materials

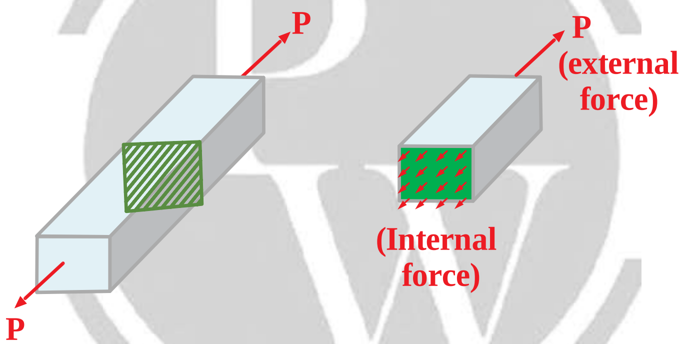
INDEX

1 INTRODUCTION & PROPERTIES OF MATERIAL

1.1 Introduction to stress

STRESS

- It is the measure of internal resistance of the body against external load.
- It is defined as internal force per unit area at a given point on any plane.
- SI unit is Pa (N/m^2).



$$\sum F_x = 0$$

Fig. 1.1 Stress in a bar subjected to axial force

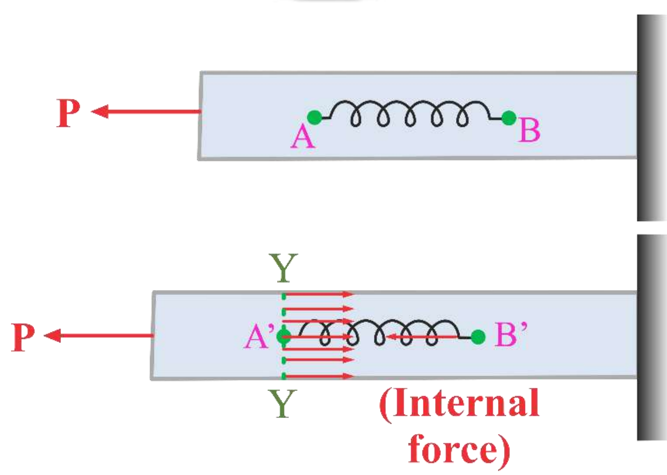


Fig. 1.2 Internal resisting force due to external load

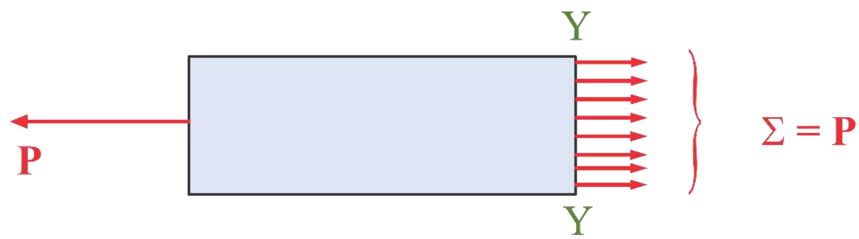


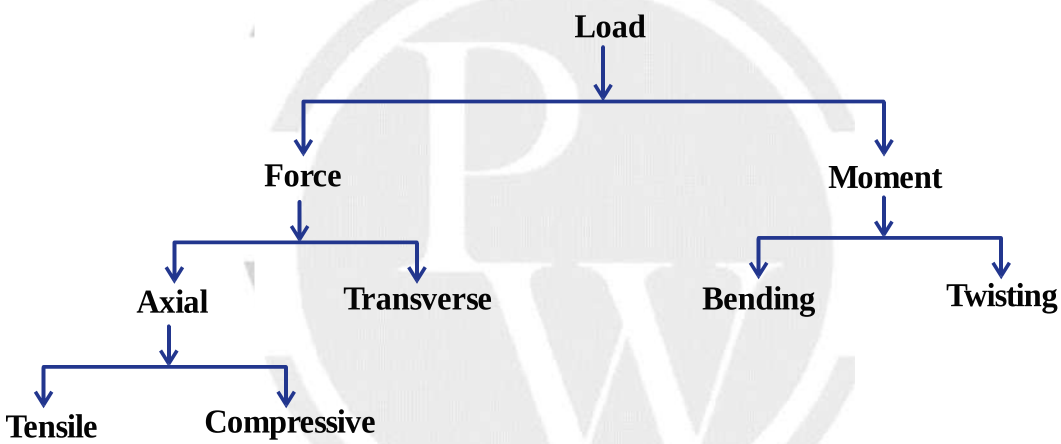
Fig. 1.3 Cut section of the bar to represent internal resisting force

$$\sum F_x = 0$$

$$Stress = \frac{P}{A} \qquad \frac{N}{m^2} = Pa$$

$$\frac{N}{mm^2} = \frac{N}{10^{-6} m^2} = 10^6 \frac{N}{m^2} = MPa$$

1.2 Types of Loads



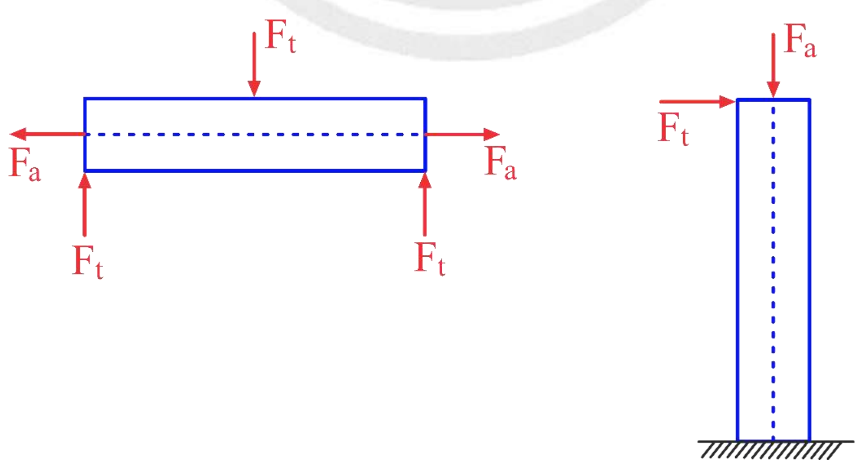


Fig. 1.4 Axial and Transverse force representation

- $F_a \rightarrow$ Axial force
- $F_t \rightarrow$ Transverse force

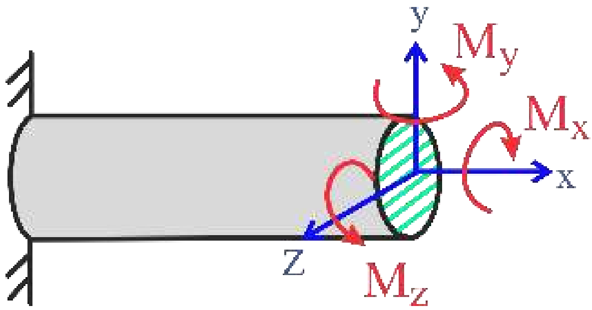


Fig. 1.5 Bending and Twisting Couples

- x ⟶ Normal to Plane
- y/z ⟶ Parallel to Plane
- M_x ⟶ Twisting/Torsional moment/Torque
- M_y/M_z ⟶ Bending moment

1.3 Types of stresses

- **Stress**
  - **Direct (Force)**
    - **Axial (σ)**
      - **Tensile**
      - **Compressive**
    - **Shear (τ)**
  - **Indirect (Moment)**
    - **Bending (σ)**
      - **Tensile**
      - **Compressive**
    - **Torsional (τ)**

1.3.1 Direct stress

- Direct stresses are developed due to external force directly acting on the plane.

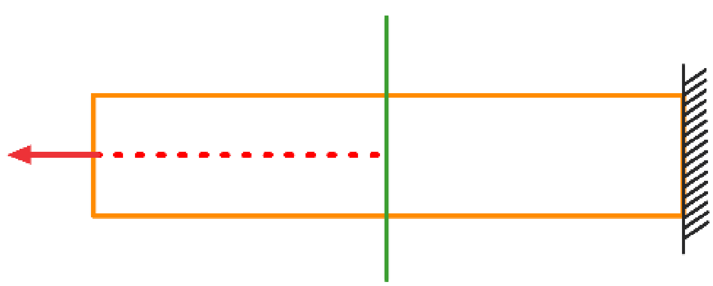


Fig. 1.6 Bar subjected to direct axial stress

1.3.2 Indirect stress

- Indirect stresses are developed due to moments, when external force is not passing through the centroid of the plane.

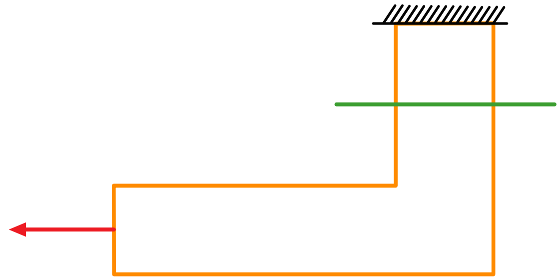


Fig. 1.7 Bar subjected to indirect stress

1.3.3 Normal & Shear Stress

- Normal stress (σ) is developed when the internal forces are acting normal (perpendicular) to the plane and shear stress (τ) is developed when the internal forces are acting parallel to the plane.
- If the internal force is acting at some angle to the plane, both normal stress and shear stress are developed on the plane

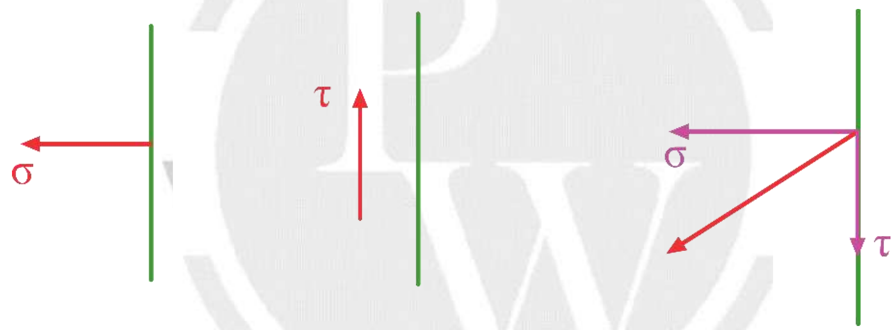
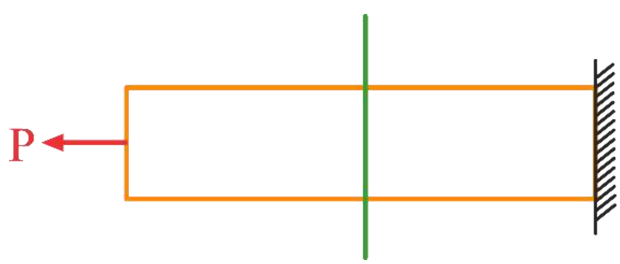


Fig. 1.8 Normal and shear stress representation

1.3.4 Direct Axial Stress



$$\sigma = \frac{P}{A}$$

Fig. 1.9 Direct axial stress

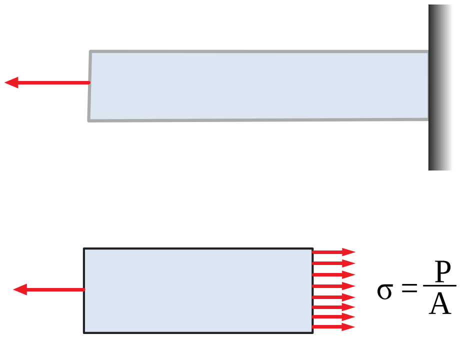


Fig. 1.10 Direct axial tensile stress

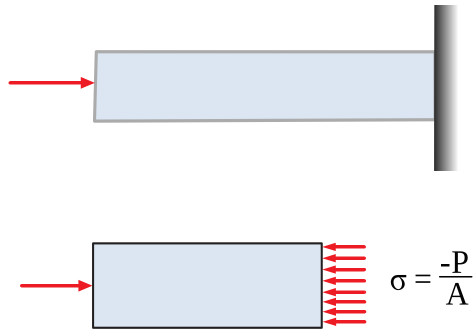


Fig. 1.11 Direct axial compressive stress

1.3.5 Direct Shear Stress

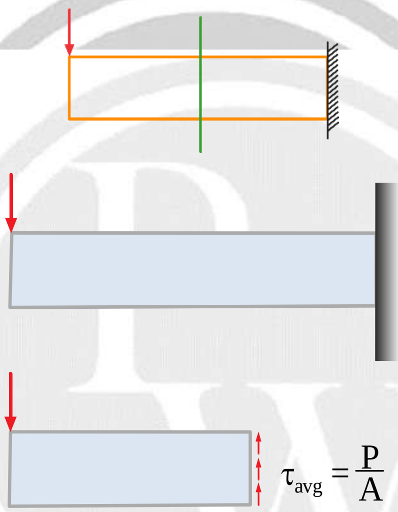


Fig. 1.12 Direct Shear stress

1.3.6 Bending Stress

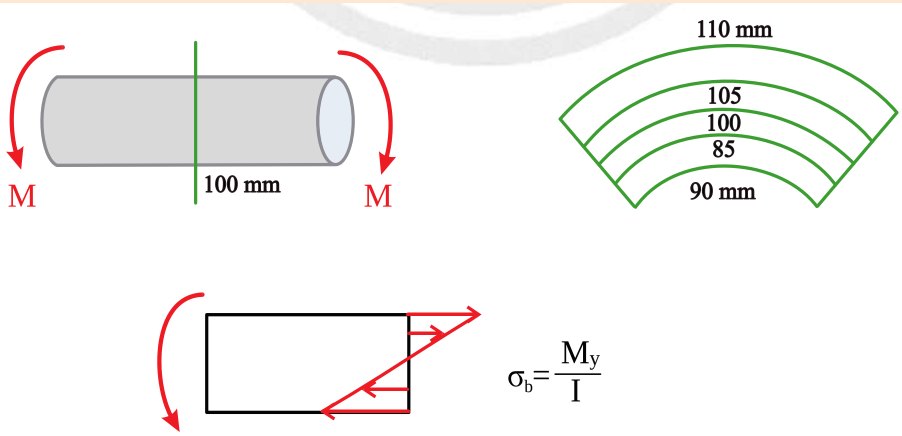
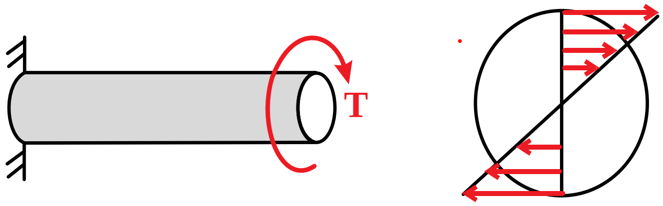


Fig. 1.13 Bending stress

1.3.7 Torsional Stress



$$\tau = \frac{Tr}{J}$$

Fig. 1.14 Torsional stress

1.4 Stress analysis under general loading

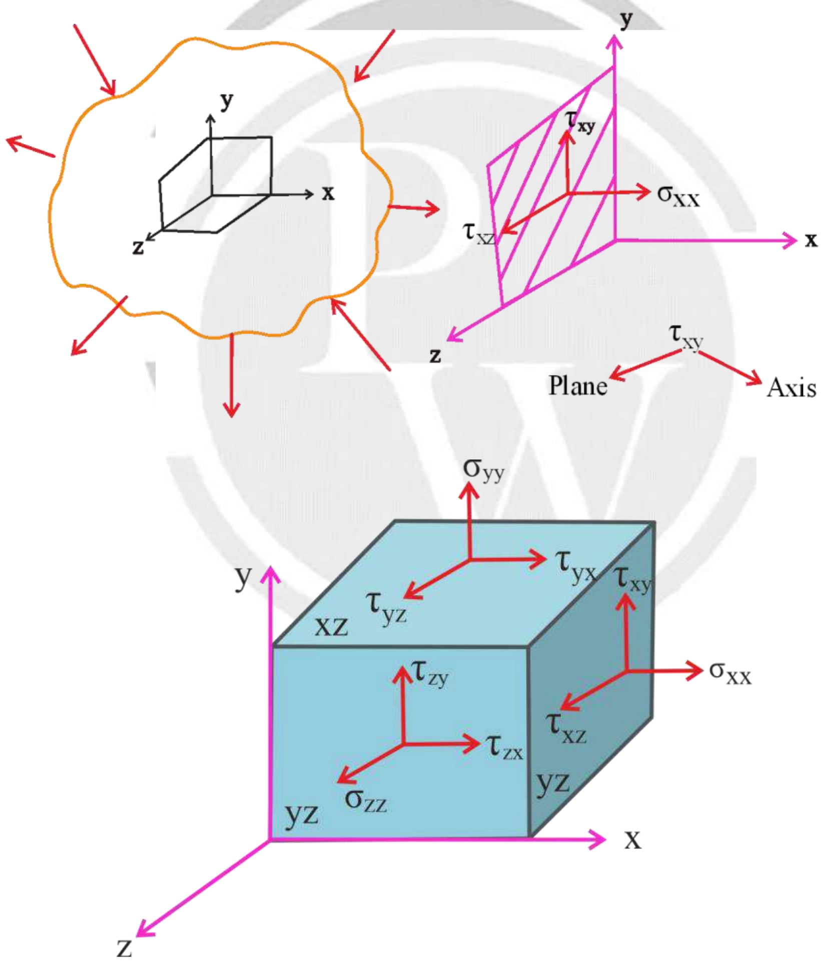
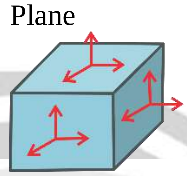


Fig. 1.15 Triaxial state of stress at a point

$$\begin{array}{ccc} & & \begin{array}{ccc} x & y & z \end{array} \\ (yz) & \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \end{array} \qquad \left.\begin{array}{l} \tau_{xy} = \tau_{yx} \\ \tau_{xz} = \tau_{zx} \\ \tau_{yz} = \tau_{zy} \end{array}\right\} \text{Complimentary Shear Stress}$$

(yz), (xz), (xy) label the rows x, y, z respectively.

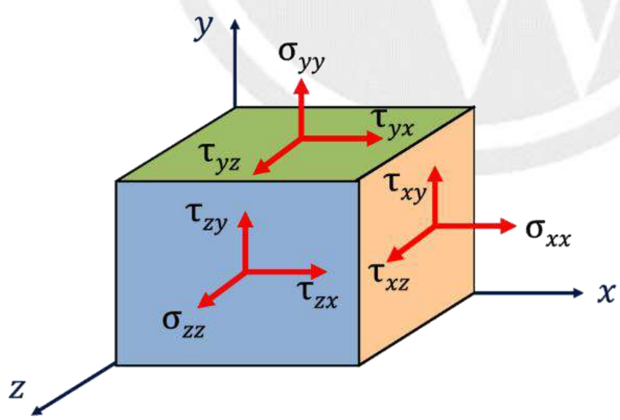
(Symmetric matrix)

Scalar	Vector	Tensor
• Magnitude	• Magnitude • Direction $\rightarrow \begin{Bmatrix} 100 \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} i \\ j \\ k \end{matrix}$	• Magnitude • Direction • Plane 

1.5 Tensor

- Tensors are quantities which are characterized by magnitude, direction and plane.
- Examples – Stress, Strain and Moment of Inertia

1.5.1 Stress Tensor



$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \end{array}$$

Fig. 1.16 Stress Tensor at a point

- 6 Independent Stress components
- 3 Dependent Stress components

1.6 Complimentary shear stress

- If there is a shear stress on one plane, there must be an equal and opposite shear stress on the perpendicular plane known as complimentary shear stress.

$\Sigma Fy = 0$

$\Sigma Fx = 0$

$\Sigma M = 0$

$(\tau_{xy} \times bc) \times a = (\tau_{yx} \times ab) \times c$

$(\tau_{xy} \times abc) = \tau_{yx} \times abc$

$\tau_{xy} = \tau_{yx}$

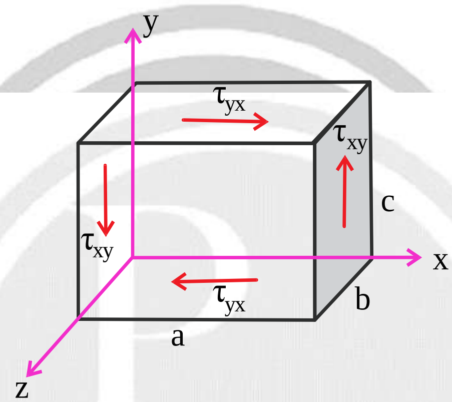
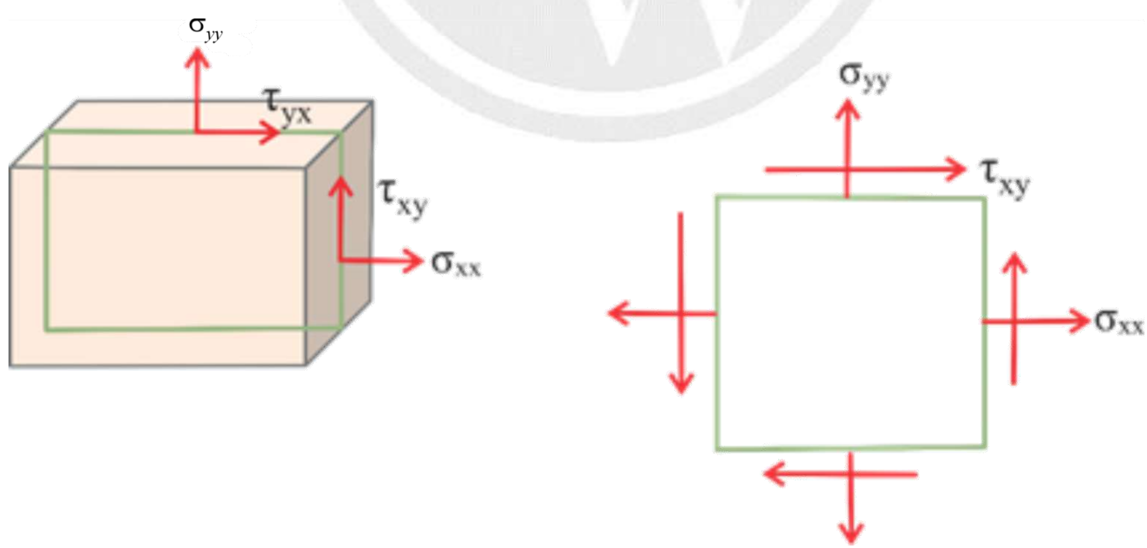


Fig. 1.17 Cross shear or complimentary shear

1.7 Bi-axial stress (Plane Stress)



$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

Fig. 1.18 Bi-axial state of stress or Plane stress condition

- Examples of Plane Stress

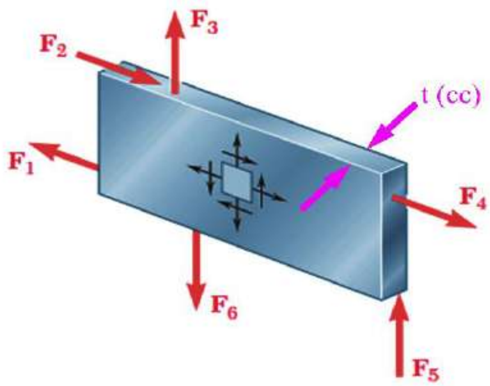


Fig. 1.19. Thin plate subjected to forces acting in the midplane of the plate

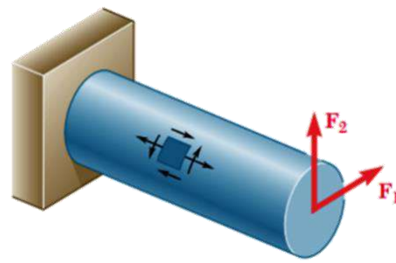
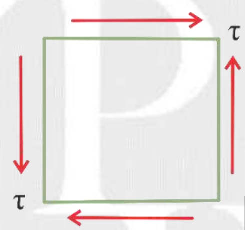


Fig. 1.20. On the free surface of a structural element or machine

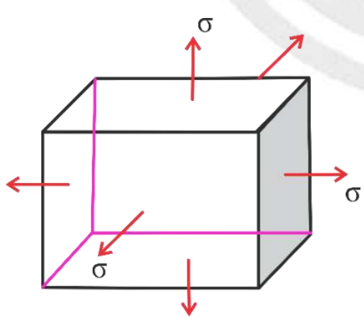
1.8 Pure Shear Stress



$$\begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}$$

Fig. 1.21 Pure shear state of stress

1.9 Hydrostatic Stress

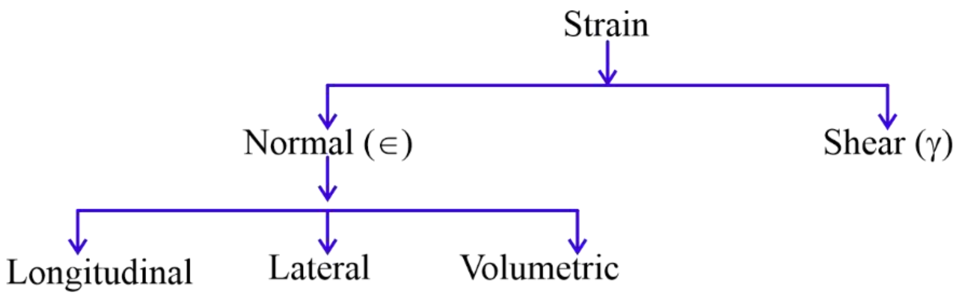


$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

$$\sigma_x = \sigma_y = \sigma_z = \sigma$$

Fig. 1.22 Hydrostatic state of stress

1.10 Types of strain



1. 11 Normal Strain (ϵ)

- It is the measure of change in size.
- It is defined as change in a dimension per unit original dimension.

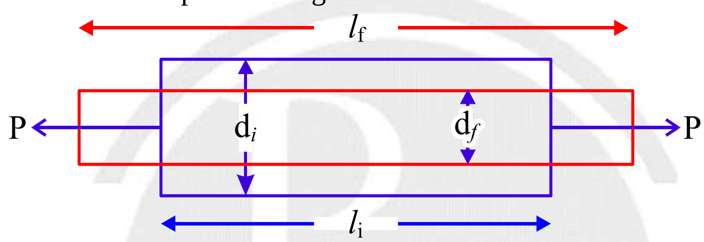


Fig.1.23 Bar Subjected to pure axial loading

$\varepsilon_{long} = \varepsilon_x = \frac{\Delta l}{l}$

$\varepsilon_{lateral} = \varepsilon_y = \varepsilon_z = \frac{\Delta d}{d}$

$\varepsilon_{vol} = \frac{\Delta v}{v} = \varepsilon_x + \varepsilon_y + \varepsilon_z$

1.12 Shear Strain (γ)

- It is the measure of change in shape.
- It is a defined as change in angle between two mutually perpendicular planes.

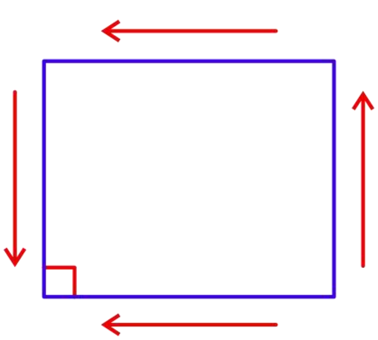


Fig. 1.24 Pure shear state of stress

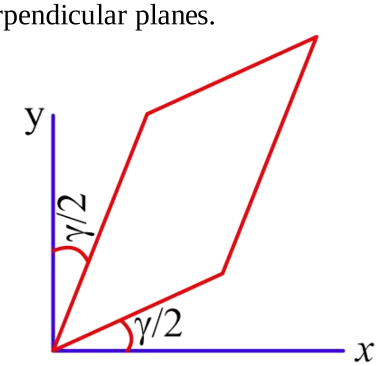


Fig. 1.25 Distorted member due to shear load

1.12.1 Strain Tensor:

$\sigma \rightarrow \varepsilon$

$\tau \rightarrow \frac{\gamma}{2}$

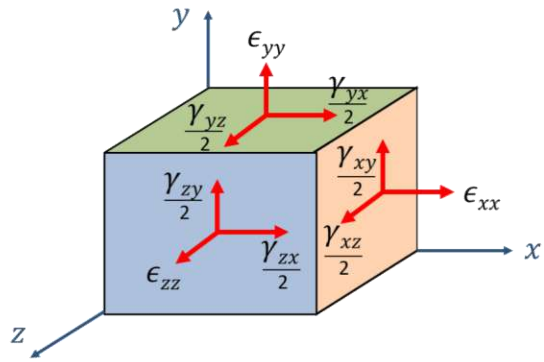


Fig.1.26 Triaxial state of strain at a point

1.13 Types of Material

Types of Material

- Homogeneous
- Non homogenous

- Isotropic
- Anisotropic
- Orthotropic

1.13.1 Homogenous Material:

Material properties are same at all points in the same direction.

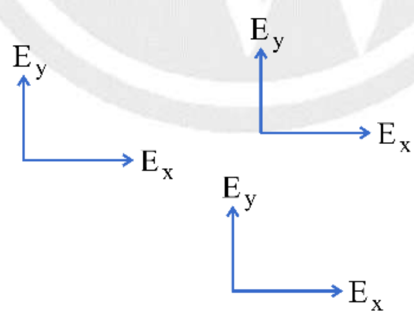


Fig.1.27 Homogeneous Material

1.13.2 Non - Homogenous Material:

Material properties are different at all points in the same direction.

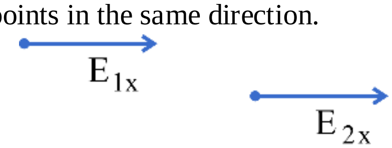


Fig.1.28 Non-Homogeneous Material

1.13.3 Isotropic Material:

Material properties are same in every direction at a point.

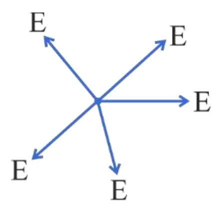


Fig.1.29 Isotropic Material

1.13.4 Anisotropic Material:

Material properties are different in every direction at a point.

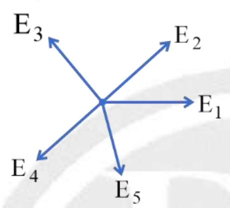


Fig.1.30 Anisotropic Material

1.13.5 Orthotropic Material:

Material properties are different in mutually perpendicular directions at a point.

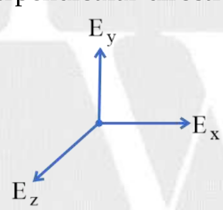


Fig.1.31 Orthotropic Material

1.14 Elastic Constants

- Elastic constants are material properties.
- They are the relation between the stress and strain.
- The magnitude of strain under external load, depends on elastic constants of the material.
  Stress = Elastic Constant × Strain

Types of Elastic Constants

- Modulus of elasticity (E)
- Modulus of rigidity (G or C)
- Bulk modulus (K)
- Poisson's Ratio $\left(\mu \text{ or } \frac{1}{M}\right)$

1.14.1 Modulus of Elasticity/Young's Modulus (E):

Ratio of normal stress and normal (longitudinal) strain.

$$E = \frac{\sigma}{\varepsilon} \qquad 1\ GP_a = 10^9 P_a = 10^3 MP_a$$

$$E_{steel} = 200GP_a$$

$$E_{Cu} = 100GP_a$$

$$E_{Al} = 70GP_a$$

1.14.2 Modulus of Rigidity/Shear Modulus (C or G):

Ratio of shear stress and shear strain.

$$G = \frac{\tau}{\gamma}$$

1.14.3 Bulk Modulus (K):

Ratio of hydrostatic stress and volumetric strain.

$$K = \frac{\sigma}{\varepsilon_v}$$

1.14.4 Poisson's Ratio (1/m):

Ratio of magnitude of lateral strain and longitudinal strain.

$\mu = 0$ to 0.5

$\mu = 0 \rightarrow$ Cork

$\mu = 0.5 \rightarrow$ Rubber

$\mu = 0.25$ to $0.33 \rightarrow$ Metals

$$\mu = -\frac{\varepsilon_{lateral}}{\varepsilon_{long}}$$

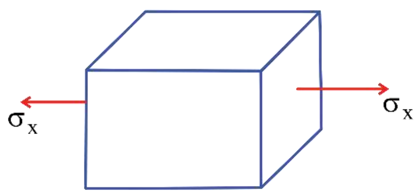
1.15 Relation between Elastic Constants

(a) $E = 2G(1 + \mu)$

(b) $E = 3K(1 - 2\mu)$

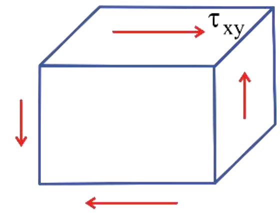
1.16 Hooke's Law

- Stress is directly proportional to corresponding strain within proportional limit.
- Constants of proportionality are the elastic constants.



$$\sigma_x \propto \varepsilon_x$$
$$\sigma_x = E.\varepsilon_x$$

Fig. 1.32 Member subjected to axial load



$$\tau_{xy} \propto \gamma_{xy}$$
$$\tau_{xy} = G.\gamma_{xy}$$

Fig. 1.33 Member subjected to shear load

Generalised Hooke's Law

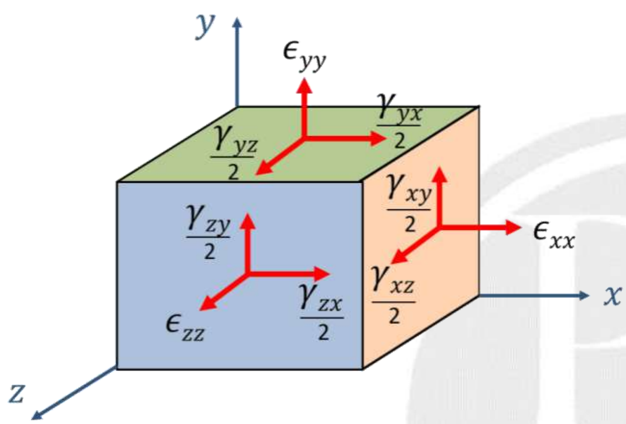


Fig. 1.34 Triaxial state of strain at a point

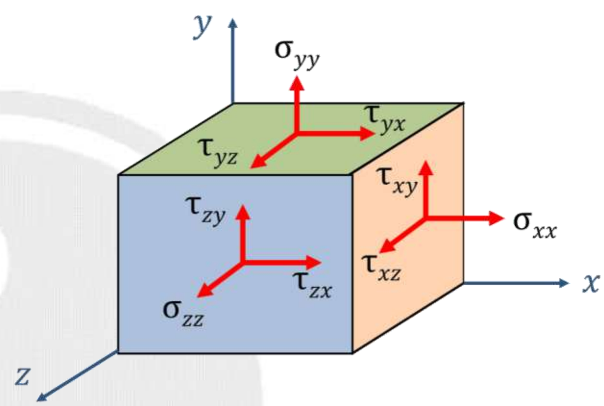


Fig. 1.35 Triaxial state of stress at a point

$$\varepsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\varepsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G}$$
$$\varepsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

Minimum number of Independent Elastic Constants

Material	Triaxial Stress	Biaxial Stress
Isotropic	2	2
Orthotropic	9	4
Anisotropic	21	6

1.17 Stress – Strain Curve for Ductile Materials

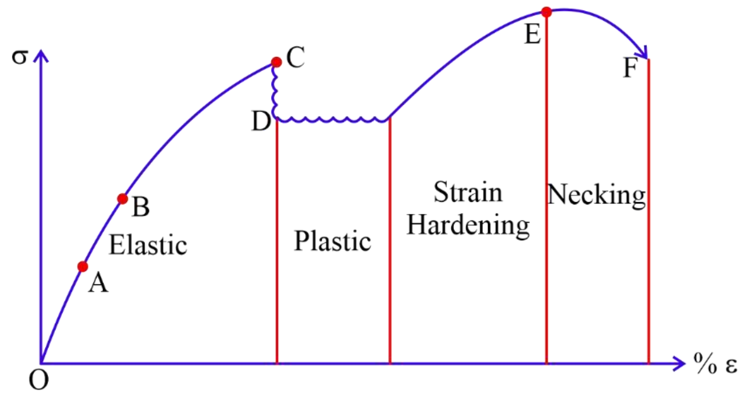


Fig. 1.36 Engineering stress vs Engineering strain diagram for ductile material

1.18 Stress – Strain Curve for Brittle Materials

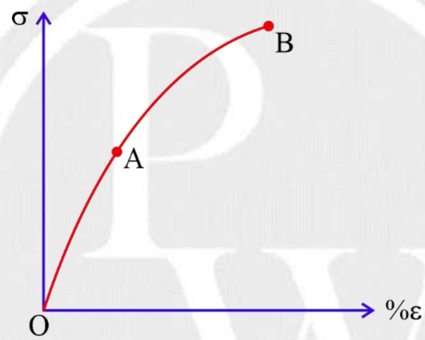


Fig. 1.37 Stress vs Strain diagram for Brittle Material

1.18.1 Strength:

Maximum magnitude of stress that the material can sustain without failure.

(i) Yield Strength: Maximum stress that the material can sustain without yielding.

(ii) Ultimate Strength: Maximum stress that the material can sustain without fracture.

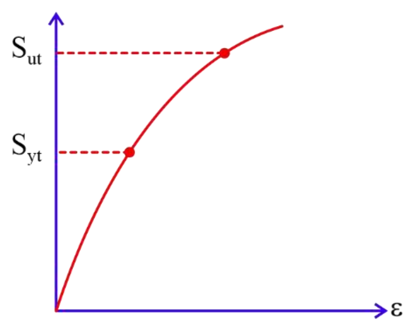


Fig. 1.38 Engineering stress vs Engineering strain diagram

Ductile: $S_{yt} = S_{yc} > S_{ys}$

Brittle: $S_{uc} > S_{us} > S_{ut}$

Ductility:

Ability of a material to deform plastically.

% change in length $= \frac{\Delta l}{l} \times 100\%$

% change in area $= \frac{\Delta A}{A} \times 100\%$

Resilience:

Ability of a material to absorb strain energy without permanent deformation.

Modulus of resilience = S.E./Volume

$= \frac{1}{2}\sigma \times \varepsilon = \frac{1}{2} \times \sigma \times \frac{\sigma}{E}$

$= \frac{\sigma^2}{2E}$

$= \frac{{S_{yt}}^2}{2E}$

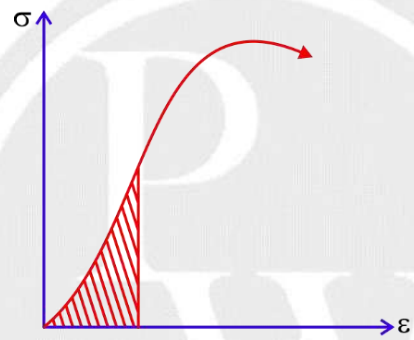


Fig. 1.39 Modulus of Resilience

Toughness:

Ability of a material to absorb strain energy without fracture.

Modulus of toughness = Toughness / Volume

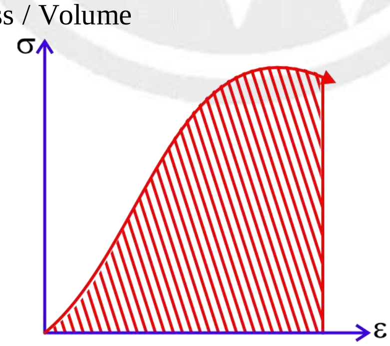


Fig. 1.40 Modulus of Toughness

1.19 True Stress - Strain

$\sigma_E = \frac{P}{A_0}$ $\sigma_T = \frac{P}{A_f}$

$\varepsilon_E = \frac{\Delta l}{l_0}$ $\varepsilon_T = l_n \frac{l_f}{l_0} = l_n \frac{A_0}{A_f}$

- $\varepsilon_T = \ell_n(1 + \varepsilon_E)$
- $\sigma_T = \sigma_E(1 + \varepsilon_E)$

Here σ_E = Engineering stress, σ_T = True stress, ε_E = Engineering strain, ε_T = True strain

1.20 Power Law

$\sigma_T = k{\varepsilon_T}^n$

k ⟶ strength coefficient

n ⟶ Strain hardening exponent (0 to 1)

At ultimate point

$$\varepsilon_T = n$$

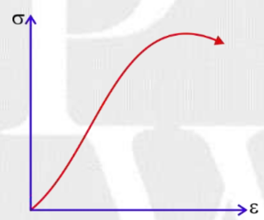


Fig. 1.41 Stress vs strain diagram

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2 AXIALLY LOADED MEMBERS

2.1 Axially Loaded Members

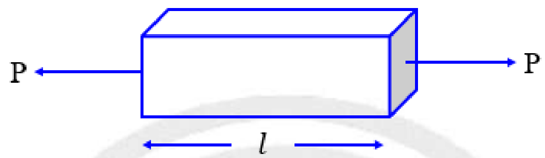
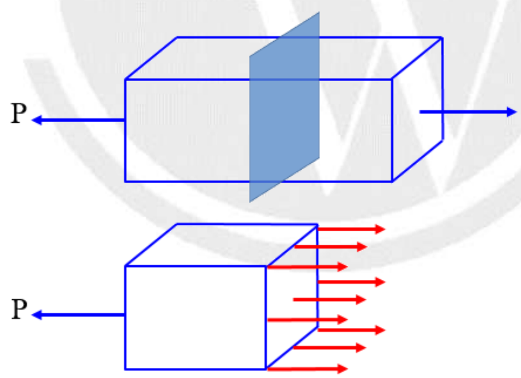


Fig.2.1 Bar subjected to axial load

Assumptions

- Material of the bar is homogeneous and isotropic.
- Bar is of constant cross-sectional area.
- Axial load passes through the centroid of the cross section.
- Stresses are within proportional limit.



$$\sigma = \frac{P}{A}$$

Fig.2.2 Stress representation on the plane of cross section

Elongation of bar

$$\Delta l = \frac{Pl}{AE} \qquad \Delta l = \int_0^l \frac{Pdx}{AE}$$

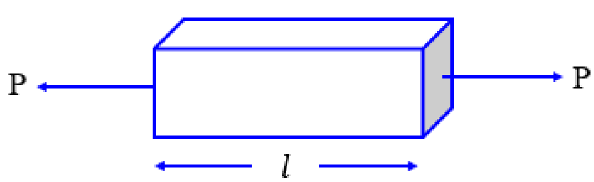


Fig.2.3 Bar subjected to Pure Axial load

Calculation of Internal Force P

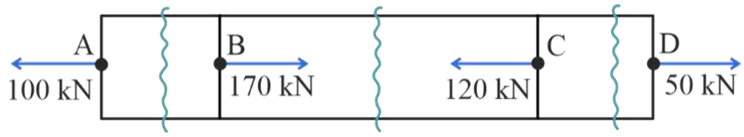


Fig.2.4 Bar subjected to variable axial loads

$$P_{AB} = 100\,kN$$
$$P_{BC} = 100 - 170 = -70kN$$
$$P_{CD} = 50kN$$

2.2 Elongation of Prismatic Bar due to self-weight

$$\Delta l = \frac{\gamma l^2}{2E}$$

or

$$\Delta l = \frac{Wl}{2AE}$$

Here,

W = self-weight

= γAL

γ = weight/volume 'or' weight density

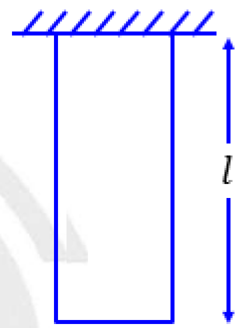


Fig.2.5 Prismatic Bar under self-weight

2.3 Elongation of Conical Bar due to self-weight

$$\Delta l = \frac{\gamma l^2}{6E}$$

or

$$\Delta l = \frac{Wl}{2AE}$$

Here W = self-weight $= \dfrac{\gamma Al}{3}$

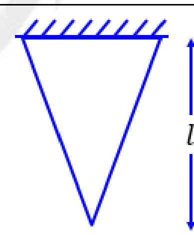


Fig.2.6 Conical Bar under self-weight

2.4 Elongation of Circular Tapered Bar

$$\Delta l = \frac{Pl}{\frac{\pi}{4} d_1 . d_2 . E}$$

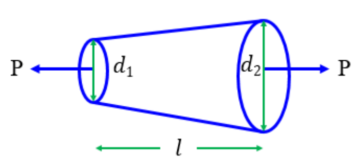


Fig.2.7 Elongation of circular tapered bar under axial load

2.5 Elongation of Rectangular Tapered Bar

$$\Delta l = \frac{Pl}{\dfrac{(b_2 - b_1)}{\ln\left(\dfrac{b_2}{b_1}\right)}.t.E}$$

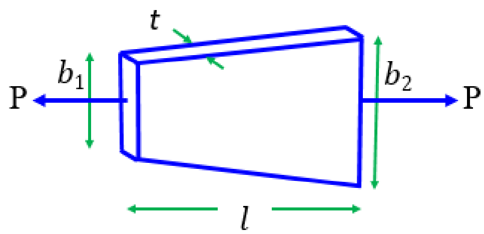


Fig.2.8 Elongation of rectangular tapered bar under axial load

2.6 Statically Indeterminate Bars

$$\Sigma F_x = 0$$

$$R_A = 100 + R_C$$

Compatibility Equation:

$$\Delta l = 0$$

$$\Delta l_{AB} + \Delta l_{BC} = 0$$

$$\frac{R_A \times l_{AB}}{A_{AB} \times E_{AB}} + \frac{R_C \times l_{BC}}{A_{BC} \times E_{BC}}$$

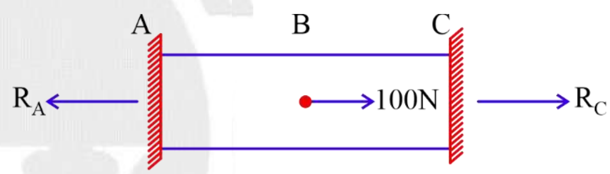


Fig.2.9 Bars fixed at both the ends, under axial load

2.7 Thermal Stress

$$\Delta l_{\text{thermal}} = l \propto \Delta T$$

$\propto \rightarrow$ coefficient of thermal expansion $(/°C)$

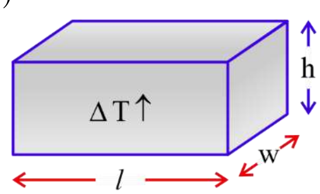


Fig.2.10 Free expansion of rectangular block

2.7.1 Thermal Stress in Bars fixed in one direction:

$$\Delta l_{total} = 0$$

$$\Delta l_{thermal} + \Delta l_{mech} = 0$$

$$(l \propto \Delta T) - \left(\frac{Pl}{AE}\right) = 0$$

$$\sigma = \frac{P}{A} = E \propto \Delta T$$

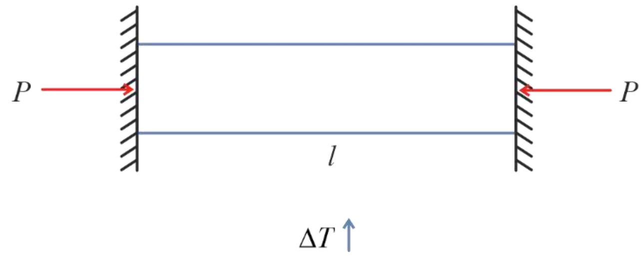


Fig.2.11 Thermal Stress in Bars fixed in one direction

2.7.2 Thermal Stress in Bars fixed in two directions:

$$\sigma_x = \sigma_y = \frac{E \propto \Delta T}{(1-v)}$$

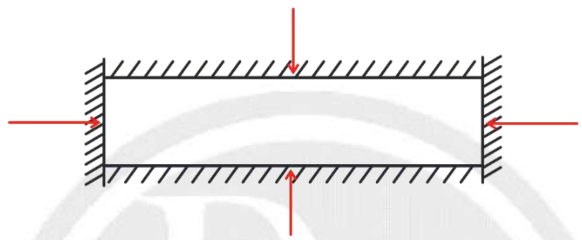


Fig.2.12 Thermal Stress in Bars fixed in two directions

2.7.3 Thermal Stress in Bars fixed in all directions:

$$\sigma_x = \sigma_y = \sigma_z = \frac{E \propto \Delta T}{(1-2v)}$$

Fig.2.13 Thermal Stress in Bars fixed in all directions

2.7.4 Thermal Stress in a Bars when there is a gap/yielding of supports

Fig.2.14 Thermal Stress in a Bars when there is a gap/yielding of supports

$$\Delta l_{total} = \delta$$

$$\Delta l_{thermal} + \Delta_{mech} = \delta$$

$$(l \propto \Delta T) - \left(\frac{Pl}{AE}\right) = \delta$$

2.7.5 Thermal Stress in a tapered bar fixed in one direction

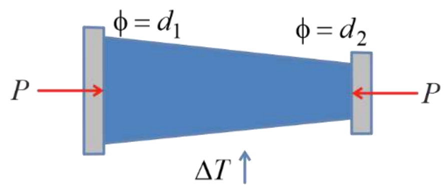


Fig.2.15 Thermal Stress in tapered bar fixed in one direction

$$\Delta l_{total} = 0$$

$$\Delta l_{thermal} + \Delta l_{mech} = 0$$

$$(l \propto \Delta T) - \left(\frac{Pl}{\frac{\pi}{4} d_1 d_2 E} \right) = 0$$

2.7.6 Thermal Stress in a compound bar

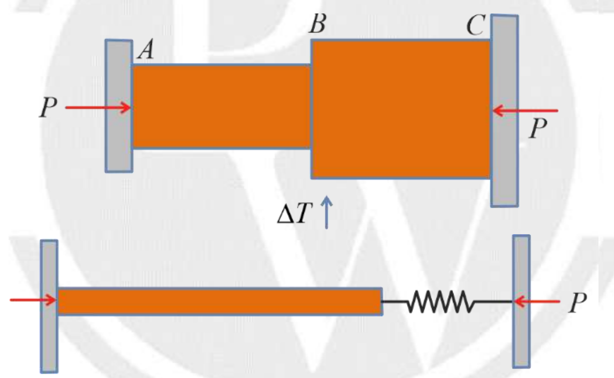


Fig.2.16 Thermal Stress in a compound bar

$$\Delta l_{mech} = \frac{Pl}{AE} + \frac{P}{k}$$

2.7.7 Thermal Stress in Composite Bar

$\propto_{Al} > \propto_S$

$(\Delta l_{total})_S = (\Delta l_{total})_{Al}$

$(\Delta \propto \Delta T)_S + \left(\frac{Pl}{AE} \right)_S = (l \propto \Delta T)_{Al} - \left(\frac{Pl}{AE} \right)_{Al}$

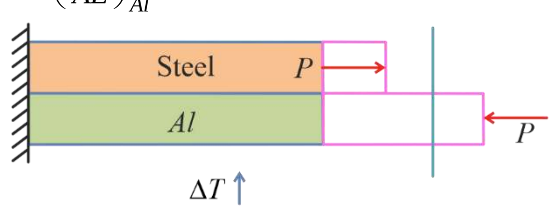


Fig.2.17 Thermal Stress in a composite bar

2.8 Strain Energy due to Axial Load

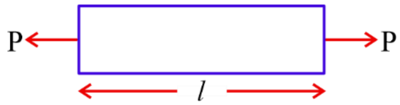


Fig.2.18 Strain Energy due to Axial Load

$$U = \frac{P^2 l}{2AE}$$

2.9 Axial Impact Load

$$\sigma_s = \frac{W}{A} \qquad \Delta l_s = \frac{Wl}{AE}$$

$$\sigma_I = \sigma_s \times IF$$

$$\Delta l_I = \Delta l_s \times IF$$

$$IF = 1 + \sqrt{1 + \frac{2h}{\Delta l_s}}$$

For suddenly applied load ($h \to 0$)

IF = 2

Here,

IF = Impact factor

σ_s = Stress due to static load

Δl_s = Elongation due to static load

σ_I = Stress due to impact load

Δl_I = Elongation due to impact load

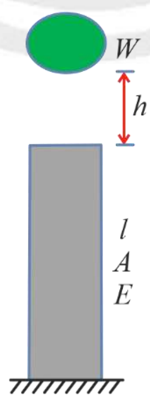


Fig.2.19 Member subjected to Impact axial load

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3 TORSION IN CIRCULAR SHAFTS

3.1 Torsion Equation

Assumptions

- Material of the shaft is homogeneous and isotropic.
- Stresses are within proportional limit.
- All the transverse sections remain plane and undistorted after twisting. In other words, the diameter of the shaft remains straight after twisting.

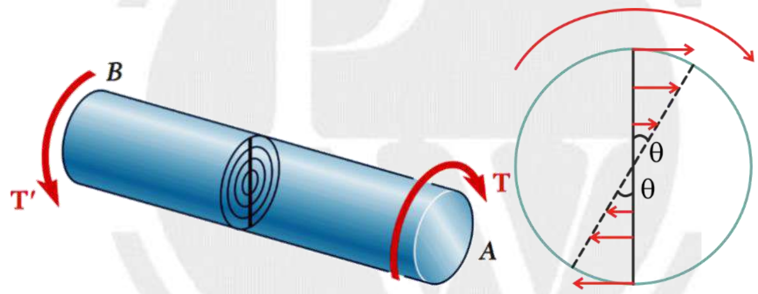

Fig.3.1 Shaft Subjected to pure torsion

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$$

J = Polar moment of inertia

r = radial distance from axis

θ = angle of twist (radians)

3.1.1 Maximum Shear Stress

$$\frac{T}{J} = \frac{\tau_{max}}{r_{max}}$$

$$\tau_{max} = \frac{T.r_{max}}{J} = \frac{T \times d/2}{\pi/32\, d^4}$$

$$\tau_{max} = \frac{16T}{\pi d^3} \quad \text{[For solid shaft]}$$

$$\tau_{max} = \frac{16T}{\pi d^3(1-K^4)} \quad \left(K = \frac{d_1}{d_0}\right) \rightarrow \text{[For hollow shaft]}$$

3.1.2 Angle of Twist

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\theta = \frac{Tl}{GJ} \quad \text{(in radians)}$$

3.1.3 Polar Section Modulus

- It is the measure of the strength (maximum applicable torque) of shaft.
- It depends on the shape and size of the cross section.

$$\frac{T}{J} = \frac{\tau_{max}}{r_{max}}$$

$$T \propto \frac{J}{r_{max}}$$

$$\frac{J}{r_{max}} = Z_P$$

$Zp \uparrow \rightarrow T_R \uparrow \rightarrow \tau_{max} \downarrow \rightarrow$ chances of failure $\downarrow$

For same area of cross-section

$$(z_p)_H > (z_p)_S$$

$$T_H > T_S$$

3.1.4 Torsional rigidity (GJ)

It is the measure of the resistance to deformation under twisting moment.

3.1.5 Torsional stiffness (*q*)

It is the magnitude of torque required for unit angle of twist.

$$q = \frac{T}{\theta} = \frac{GJ}{l}$$

3.1.6 Internal Torque

$$T_{AB} = 100\,Nm$$

$$T_{BC} = 100 - 240 = -140\,Nm$$

$$T_{CD} = 220\,Nm$$

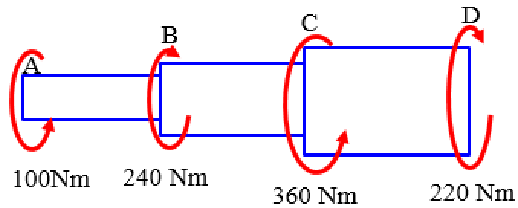

Fig.3.2 Shaft Subjected to variable twisting moments

3.2 Statically Indeterminate Shaft

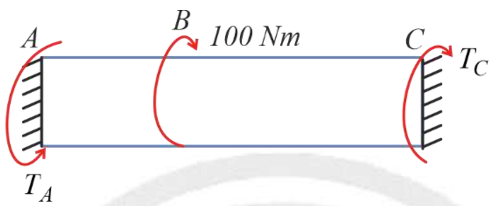

Fig.3.3 Shaft fixed at both the ends and subjected to twisting moments

$\Sigma M = 0$

$T_A = 100 + T_C$

Compatibility eq^n

$\theta_{AC} = 0$

$\theta_{AB} + \theta_{BC} = 0$

$$\frac{T_A \times l_{AB}}{G_{AB} \times J_{AB}} + \frac{T_C \times l_{BC}}{G_{BC} \times J_{BC}} = 0$$

3.3 Composite Shaft

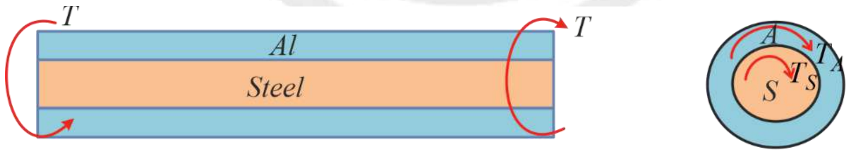

Fig.3.4 Composite shaft subjected to twisting moment

$\Sigma M = 0$

$T_S + T_A = T$

Compatibility equation:

$\theta_S = \theta_A$

$$\frac{T_S \times l}{G_S \times J_S} = \frac{T_A \times l}{G_A \times J_A}$$

3.4 Strain Energy due to Torsion

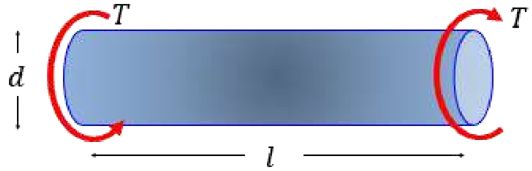

Fig.3.5 Strain Energy due to Torsion

$$U = \frac{T^2 l}{2GJ}$$

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4 SHEAR FORCE & BENDING MOMENT

4.1 Beams

Beams are Structural member used to support transverse loads.

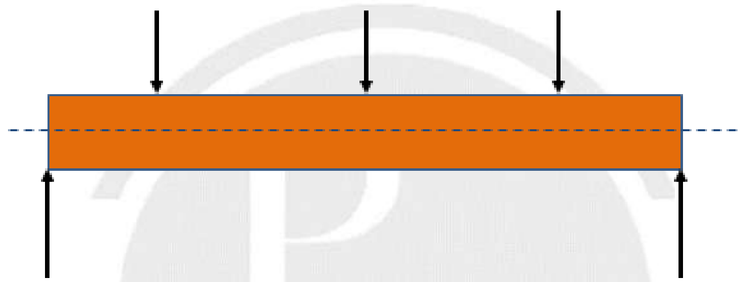


Fig.4.1 Beam subjected to transverse shear load

4.1.1 Types of Beams

Types of Beams

Statically Determinate

1. Cantilever Beam
2. Simply Supported Beam
3. Overhang Beam

Statically Indeterminate

1. Propped Cantilever Beam
2. Continuous Beam
3. Fixed (Built-in) Beam

(A) Cantilever Beam

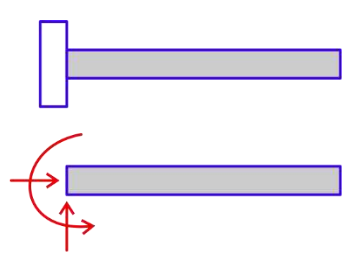


Fig.4.2 Cantilever Beam

$\sum F_x = 0$

$\sum F_y = 0$

$\sum M = 0$

(B) Simply Supported Beam

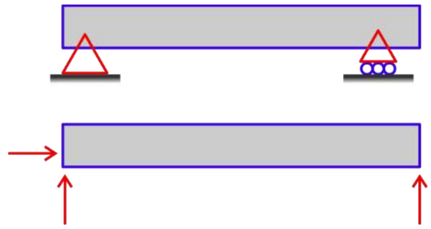


Fig.4.3 Simply Supported Beam

(C) Overhanging Beam

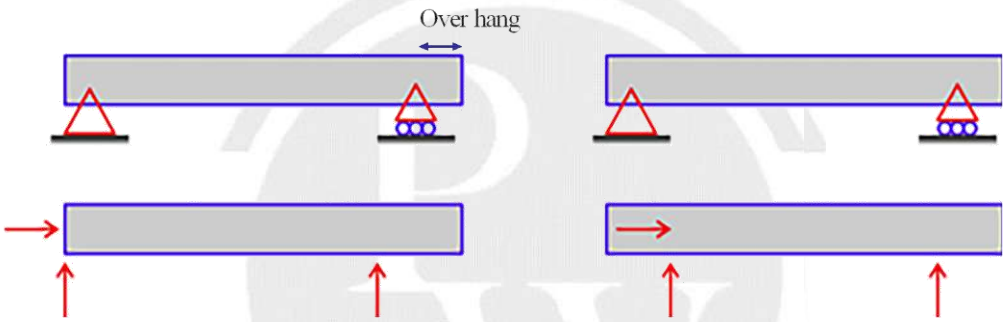


Fig.4.4 Overhanging Beam

(D) Propped Cantilever Beam

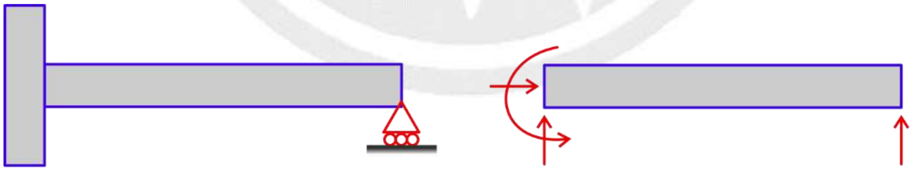


Fig.4.5 Propped Cantilever Beam

(E) Continuous Beam

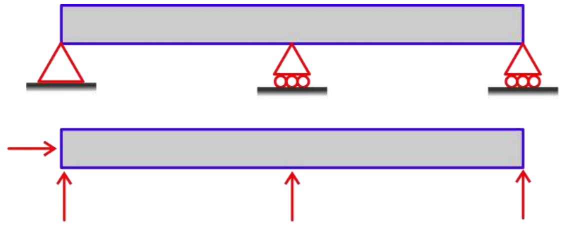


Fig.4.6 Continuous Beam

(F) Fixed Beam

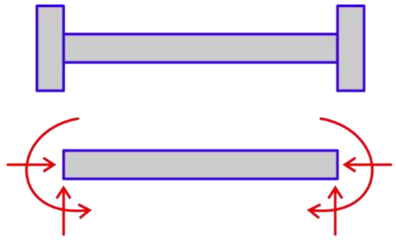


Fig.4.7 Fixed Beam

(G) Beam with Internal Hinge

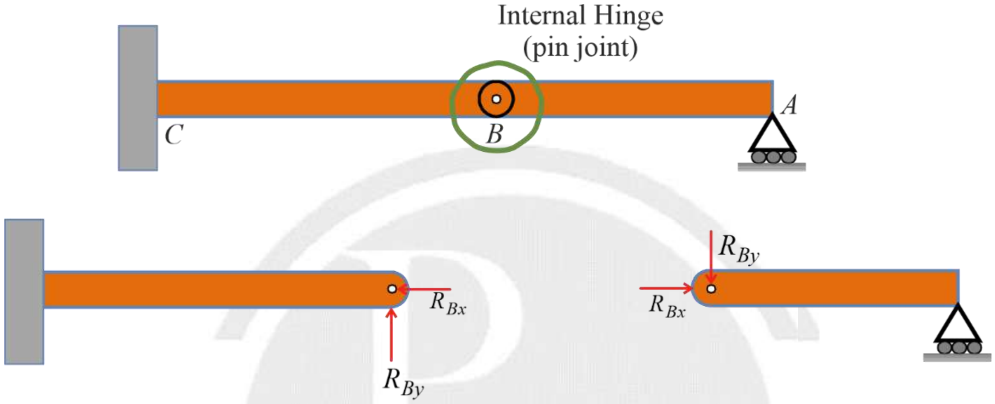
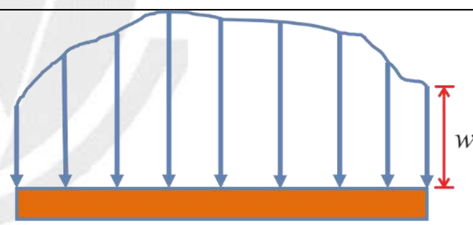


Fig.4.8 Beam with Internal Hinge

(H) Distributed loads

Load intensity = w (KN/m) Total load = Area under the loading diagram	 **Fig.4.9 Beam subjected to distributed loads**

(I) Uniformly Distributed load

Total load = wl

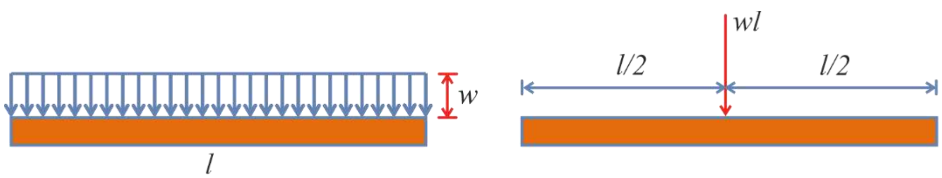


Fig.4.10 Beam subjected to uniformly distributed load

(J) Uniformly Varying Load

Total load $= \frac{1}{2}.w.l$

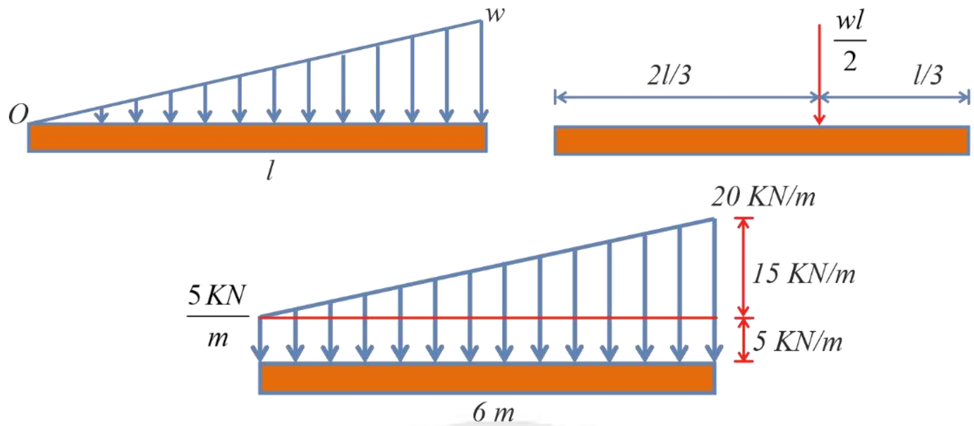


Fig.4.11 Beam subjected to uniformly varying load

Total load $= 6\times5+\frac{1}{2}\times6\times15$

= 30 + 45

= 75 KN

4.2 Shear Force

- Shear force is the transverse internal force at a section.
- It is equal to the sum of total transverse force either on the left or right side of the section.

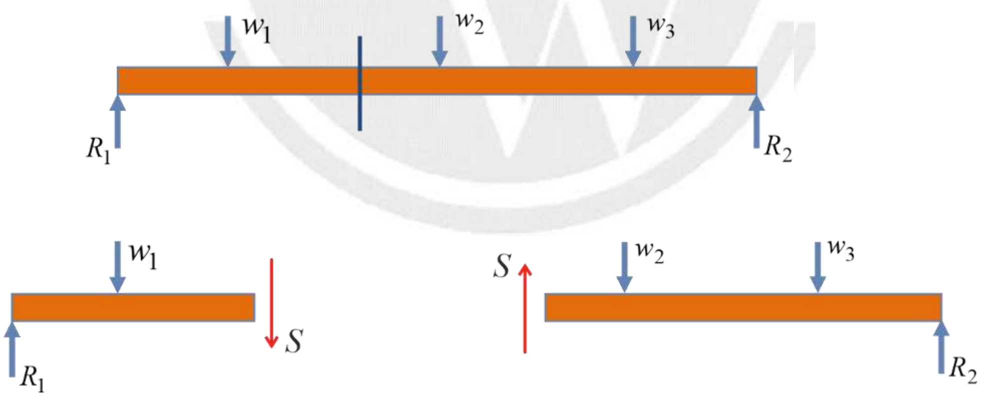


Fig.4.12 Beam subjected to transverse shear loads

Sign Convention

Fig.4.13 Sign convention of shear force

4.3 Bending Moment

- Bending moment is the internal moment at a section.
- It is equal to the sum of moment of all the forces either on the left or right side of the section.

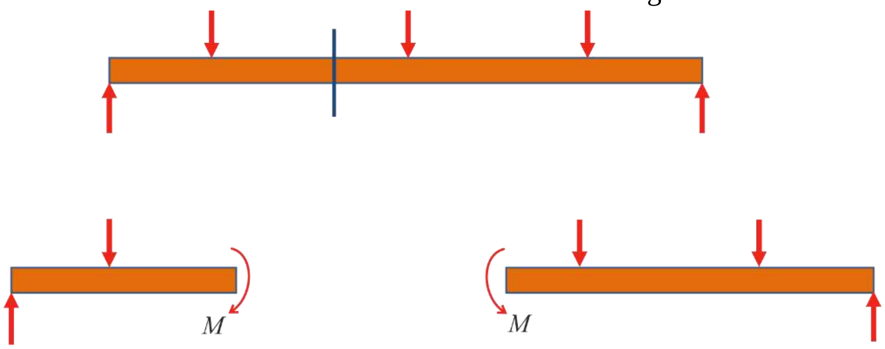


Fig.4.14 Bending moment due to transverse shear loads

Sign Convention:

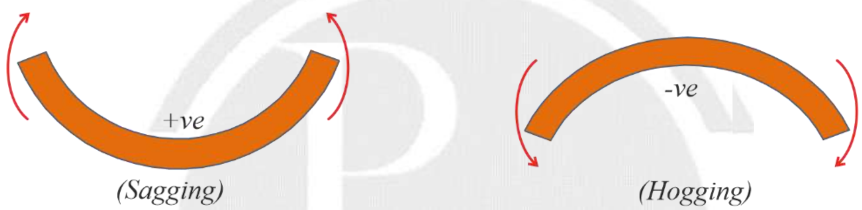


Fig.4.15 Bending moment sign convention

4.4 Relation Between Load Intensity(w), Shear Force (F) and Bending Moment (M)

(a) $\frac{ds}{dx} = w$ $\quad\quad$ $\int_A^B dM = \int_A^B S dx$

(b) $\frac{dM}{dx} = S$ $\quad\quad$ $M_B - M_A =$ Area of SFD between A and B

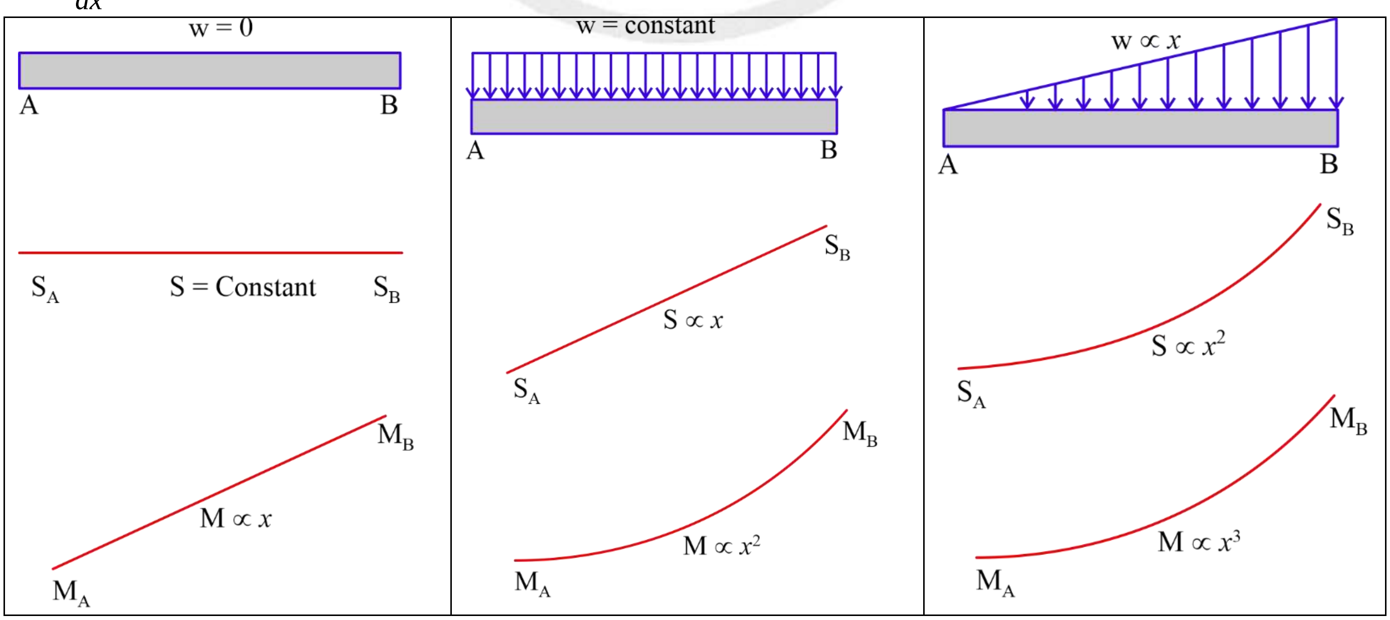


Fig.4.16 Relation Between Load Intensity, Shear Force and Bending Moment

4.4.1 Sudden Change in Shear Force

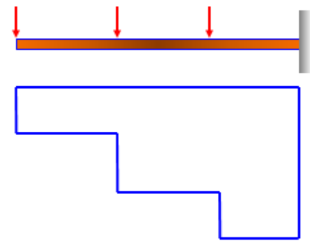


Fig.4.17 Shear force diagram

4.4.2 Sudden Change in Bending Moment

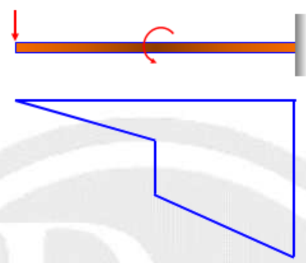


Fig.4.18 Bending Moment Diagram

4.4.3 Point of Maximum Bending Moment

Bending moment is maximum at a section if

- The sign of shear force changes at the section

  Or

- There is a couple at that section

4.5 Point of Contra flexure

It is the point at which sign of bending moment changes and the curvature of beam changes from sagging to hogging or hogging to sagging.

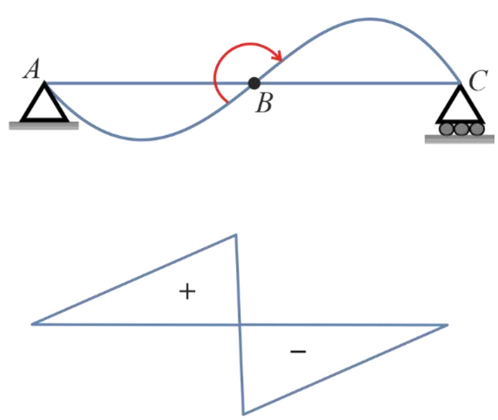


Fig.4.19 BMD Representing Point of Contra Flexure

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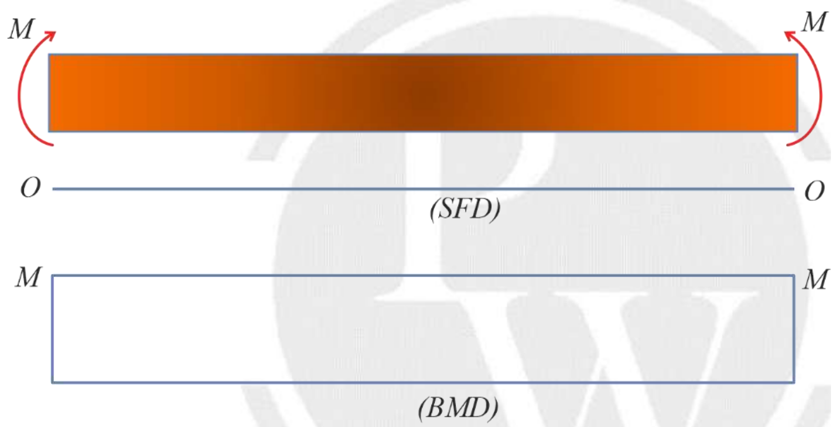
5 BENDING STRESS IN BEAMS

5.1 Bending Stress in Beams

5.1.1 Pure bending

- A beam is under pure bending when it is subjected to constant bending moment.
- Pure bending occurs when shear force is zero.



$$\frac{dM}{dx} = S = O$$

Fig.5.1 Beam subjected to pure bending

5.1.2 Euler – Bernoulli's Beam Theory

(A) Assumptions

- Material of the beam is homogeneous and isotropic.
- Young`s modulus in tension and compression is same.
- Stresses are within proportional limit.
- The beam is under pure bending.
- All the transverse sections remain plane after bending.
- Beam is initially straight and bends into a circular arc.
- Cross section of the beam is symmetric about the plane of loading.
- All the transverse sections remain plane after bending.

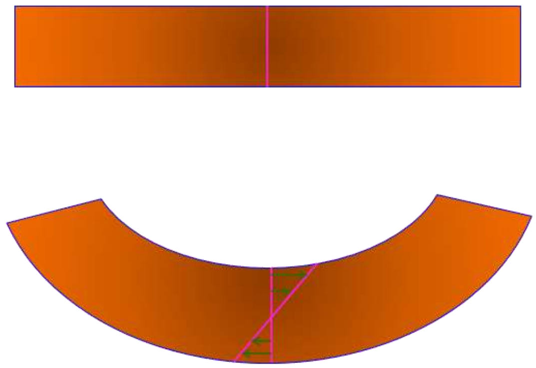


Fig.5.2 Beam subjected to sagging bending moment

- Cross section of the beam is symmetric about the plane of loading.

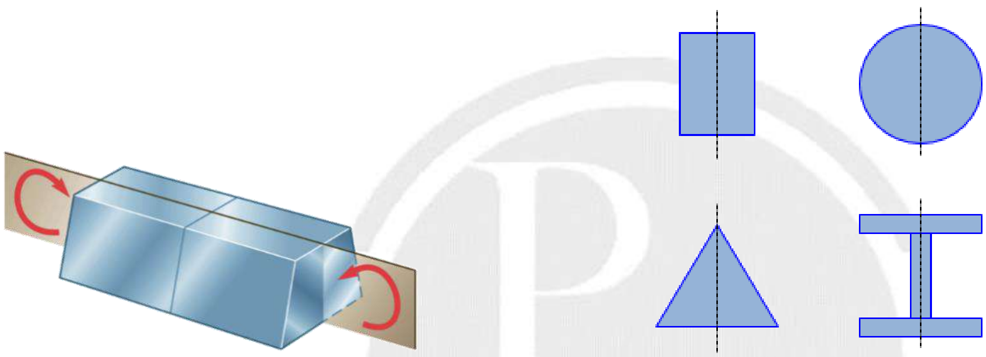


Fig.5.3 Cross section of beam under bending

(B) Neutral Layer

Undeformed longitudinal layer.

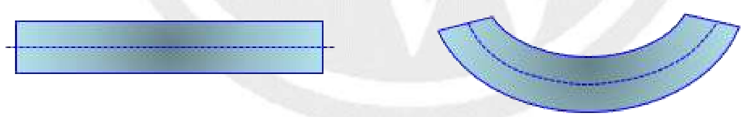


Fig.5.4 Undeformed neutral axis during bending

(C) Neutral axis

- Axis about which beam bends.
- Intersection of neutral layer with the cross section.

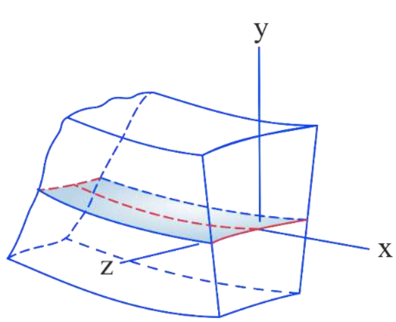


Fig.5.5 Neutral Surface and Neutral axis

Neutral axis passes through the centroid of the section, if

- The material is homogenous.
- There is no plastic deformation.

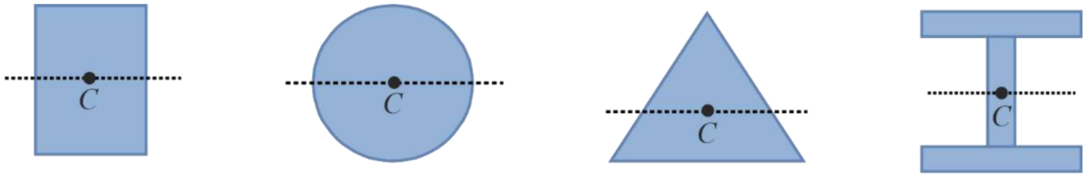


Fig.5.6 Various cross sections of beams

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

y ⟶ vertical distance from N.A.

I ⟶ MOI about NA

R ⟶ Radius of curvature of Neutral fiber

$\sigma \propto y$

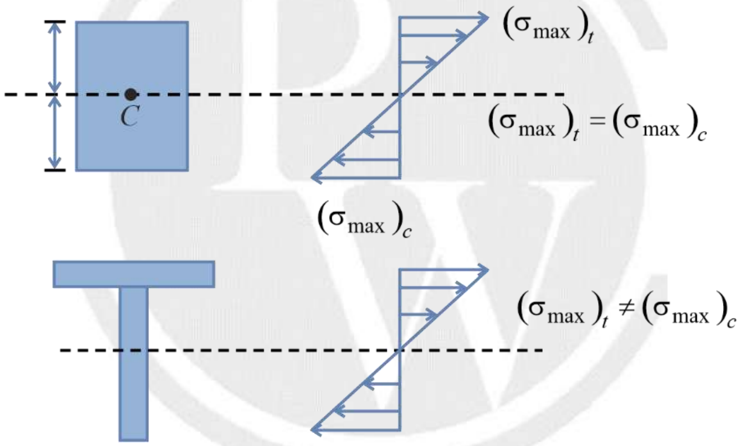


Fig.5.7 Bending stress distribution

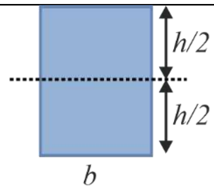
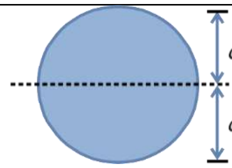
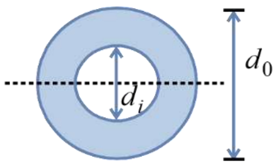
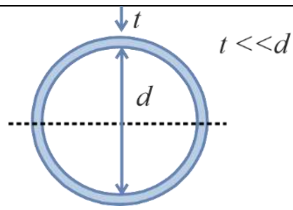
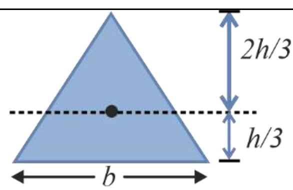
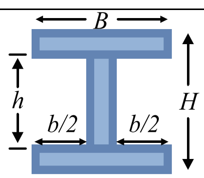
(D) Section Modulus

- It is the measure of the strength of beam (maximum applicable bending moment).

$$Z = \frac{I}{y_{max}}$$

For same cross-sectional area

$Z_I > Z_L > Z_0$

Cross Section	Y_{max}	I_{NA}	Z
	$\frac{h}{2}$	$\frac{bh^3}{12}$	$\frac{bh^2}{6}$
	$\frac{d}{2}$	$\frac{\pi}{64}d^4$	$\frac{\pi}{32}d^3$
	$\frac{d_0}{2}$	$\frac{\pi}{64}\left(d_0^4 - d_i^4\right)$	$\frac{\pi}{32}d_0^3\left(1-K^4\right)$ $K = \frac{d_i}{d_0}$
	$\frac{d}{2}$	$\frac{\pi d^3 t}{8}$	$\frac{\pi d^2 t}{4}$
	$\frac{2h}{3}$	$\frac{bh^3}{36}$	$\frac{bh^2}{24}$
	$\frac{H}{2}$	$\frac{BH^3}{12} - \frac{bh^3}{12}$	

- **Flexural rigidity (EI)**

  It is used in the design of beam based on rigidity criteria

- **Flexural stiffness**

  $\frac{EI}{l}$

5.2 Combined Axial load and Bending Moment

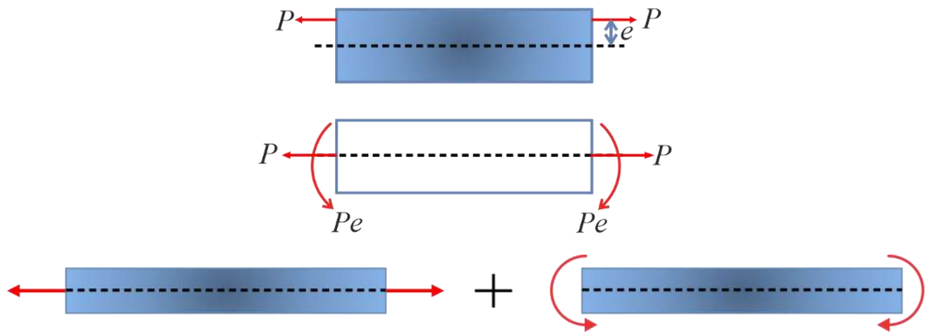


Fig.5.8 Beam subjected to combined axial load and bending moment

$$(\sigma_{max})_t = \frac{P}{A} + \frac{My_{max}}{I}$$

$$(\sigma_{max})_C = \frac{P}{A} - \frac{My_{max}}{I}$$

5.3 Beam of Uniform Strength

- Beam of uniform strength is a beam subjected to same maximum bending stress throughout the length.
- Beam of uniform strength has varying cross section.

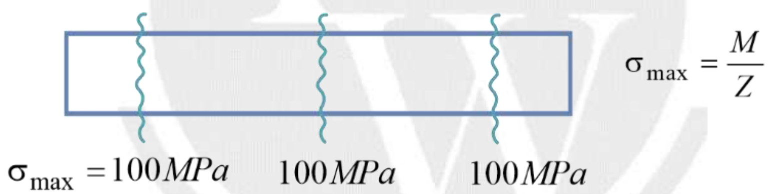


Fig.5.9 Beam of Uniform Strength

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6 SHEAR STRESS IN BEAMS

6.1 Shear Stress in Beams

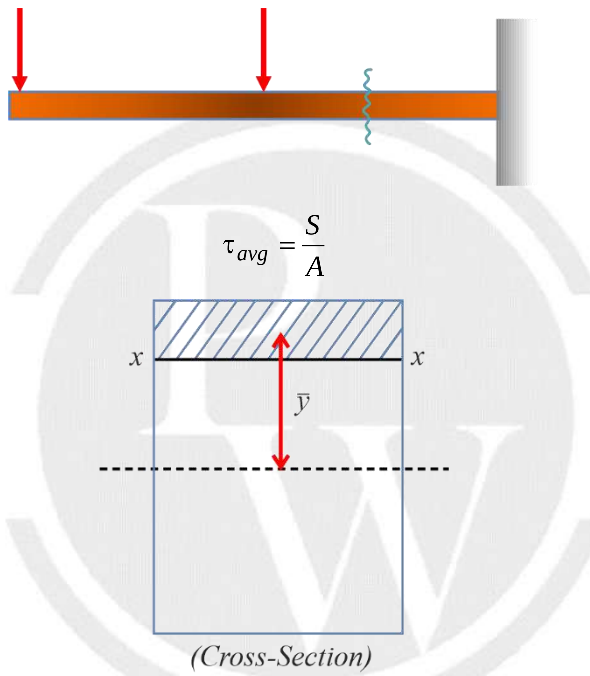


Fig.6.1 Shear stress in Beams

$$\tau = \frac{s(A\bar{y})}{Ib}$$

b → width of layer X-X

$(A\bar{y})$ → First moment of area above / below x-x about NA.

6.1.1 Rectangular Section

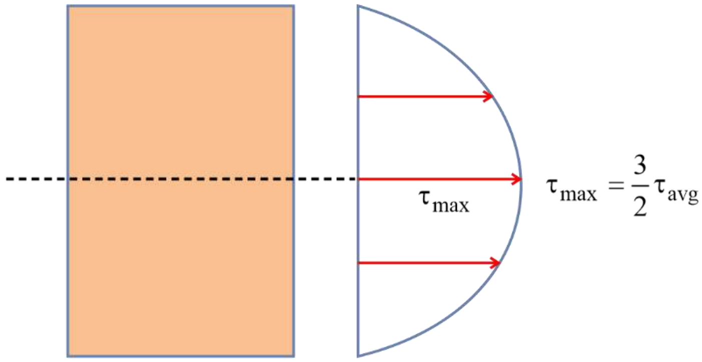


Fig.6.2 Shear stress distribution for rectangular cross section

6.1.2 Circular Section

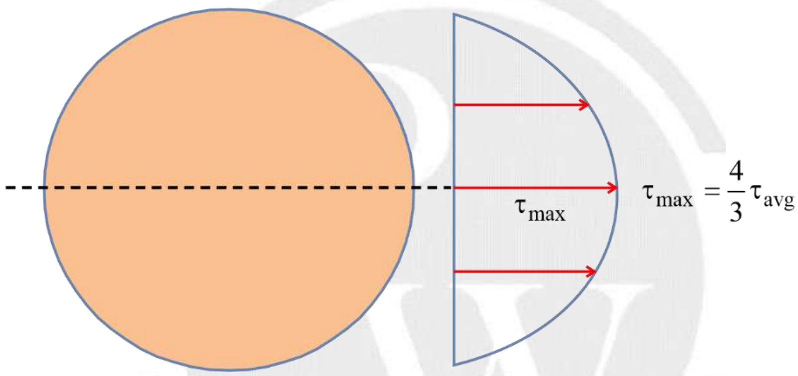


Fig.6.3 Shear stress distribution for circular cross section

6.1.3 Triangular Section

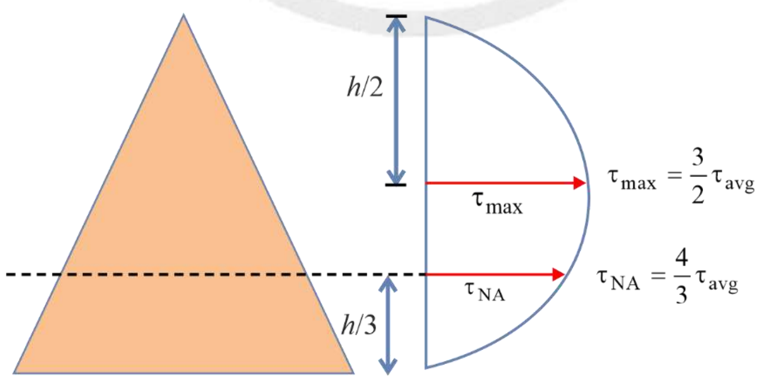


Fig.6.4 Shear stress distribution for triangular cross section

6.1.4 Diamond Section

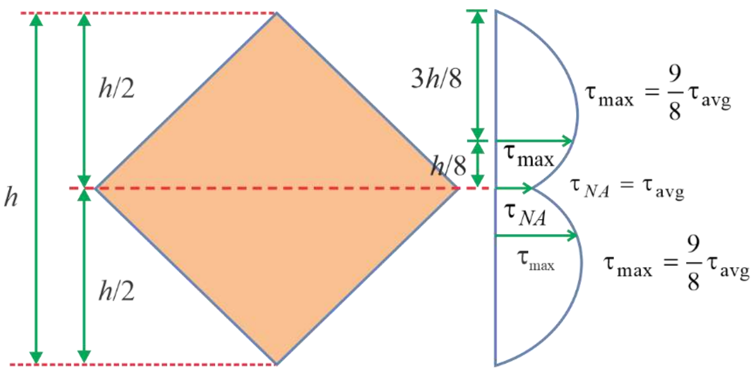


Fig.6.5 Shear stress distribution for diamond cross section

6.1.5 I Section

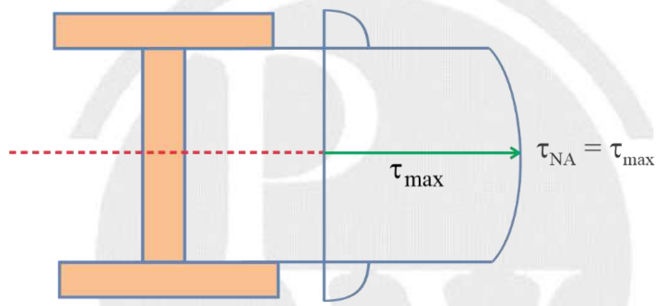


Fig.6.6 Shear stress distribution for I section beam

6.1.6 T Section

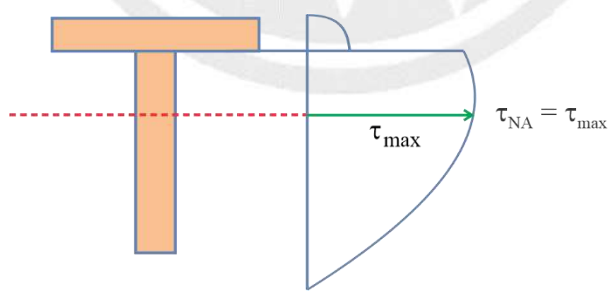
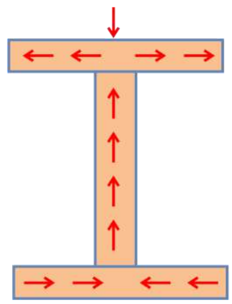


Fig.6.7 Shear stress distribution for T section beam

6.2 Shear Flow

In thin walled members (I section, T section) shear flow is the shear force per unit length.



$$q = \frac{SA\bar{y}}{I}$$

Fig.6.8 Shear flow in I section

6.3 Shear Centre

It is the point on the beam section at which the transverse load can be applied without causing twisting.

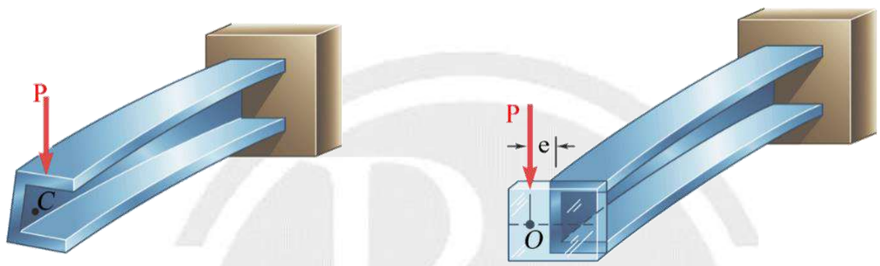
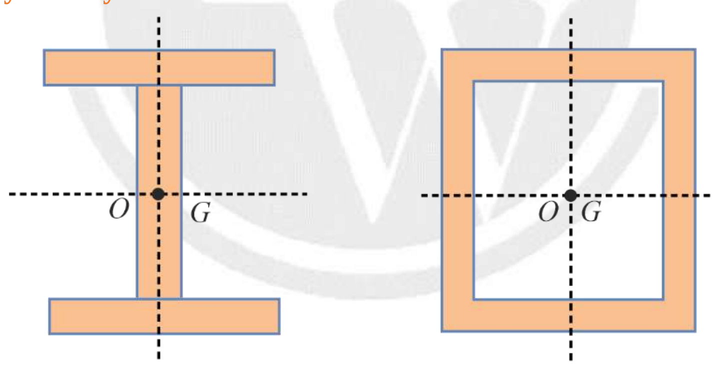


Fig.6.9 Shear center for thin-walled sections

(A) Sections with two axes of symmetry



$G \rightarrow$ *centroid*

$O \rightarrow$ *shear center*

Fig.6.10 Shear center when there are two axes of symmetry

(B) Sections with one axis of symmetry

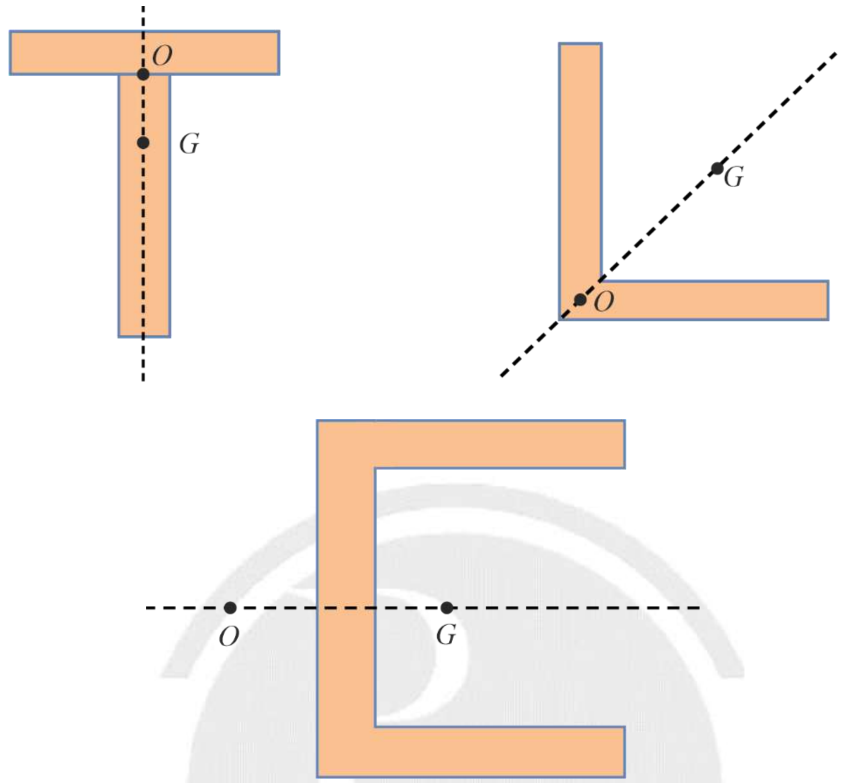


Fig.6.11 Shear center when there is one axes of symmetry

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7 DEFLECTION OF BEAMS

7.1 Deflection of Beams

Deflection represents linear deviation of a point and the slope represents the angular deviation of the point on the longitudinal axis of the beam

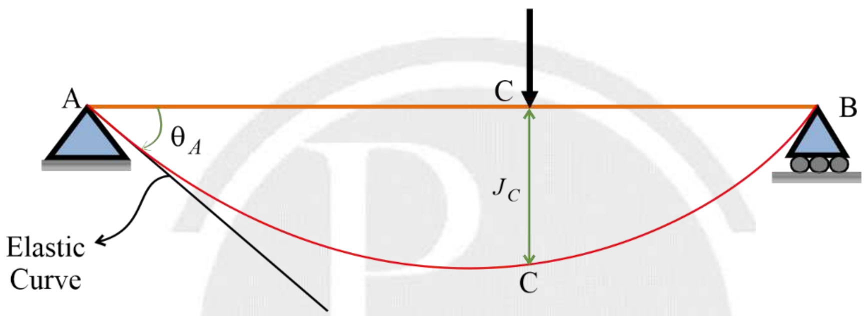


Fig.7.1 Deflection of Beam

Methods to find slope and deflection of beams

- Double Integration and Macaulay's method
- Moment – Area method
- Strain Energy method

7.2 Double Integration Method

$$EI\frac{d^2y}{dx^2} = M_x$$

$$EI\frac{dy}{dx} = \int M_x + C_1 \qquad \ldots\ldots(1)$$

$$EIy = \int\int M_x + C_1x + C_2 \qquad \ldots\ldots.(2)$$

From equation 1 and 2 slope and deflection can be determined at any location of the beam.

7.3 Macaulay's Method or Modified double integration method

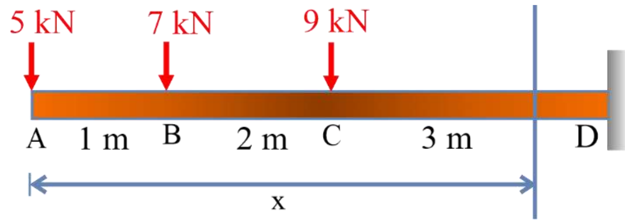


Fig.7.2 Macaulay's method for beam subjected to multiple loads

$$EI\frac{d^2y}{dx^2} = -5x - 7(x-1) - 9(x-3)$$

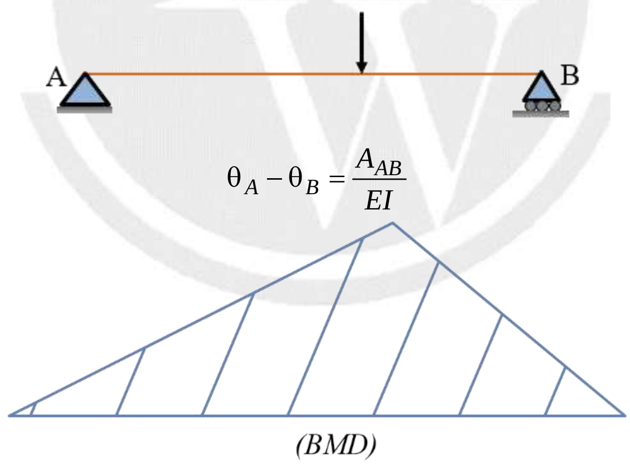
$$EI\frac{dy}{dx} = \frac{-5x^2}{2} - \frac{7(x-1)^2}{2} - \frac{9(x-3)^2}{2} + C_1$$

This method is preferred for simply supported beam unsymmetric loading and cantilever beam subjected to multiple loads. Here the terms within the bracket are known as Macaulay function and they are integrated as whole.

7.4 Moment – Area Method

7.4.1 Mohr's 1st Theorem

The change in slope between any two points A and B on the elastic curve is equal to the area of the bending moment diagram between A and B divided by EI.



$$\theta_A - \theta_B = \frac{A_{AB}}{EI}$$

Fig.7.3 Slope calculation from Mohr's 1st Theorem

7.4.2 Mohr's 2nd Theorem

The vertical deviation of any point A on the elastic curve from the tangent of a point B on the elastic curve is equal to the first moment of area of bending moment diagram between A and B about point A divided by EI.

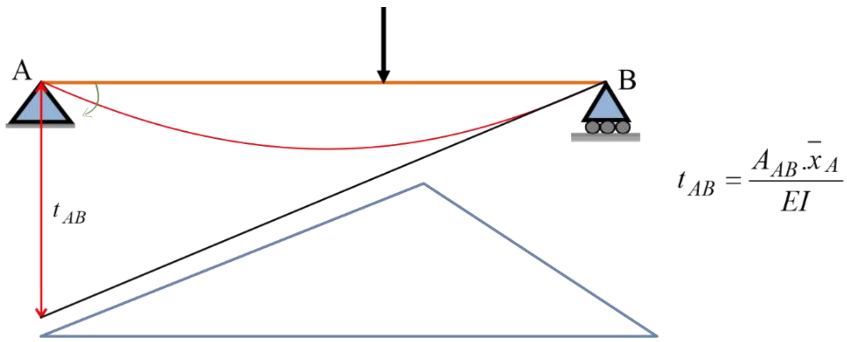


Fig.7.4 Deflection calculation from Mohr's 2nd Theorem

7.5 Strain Energy due to Bending

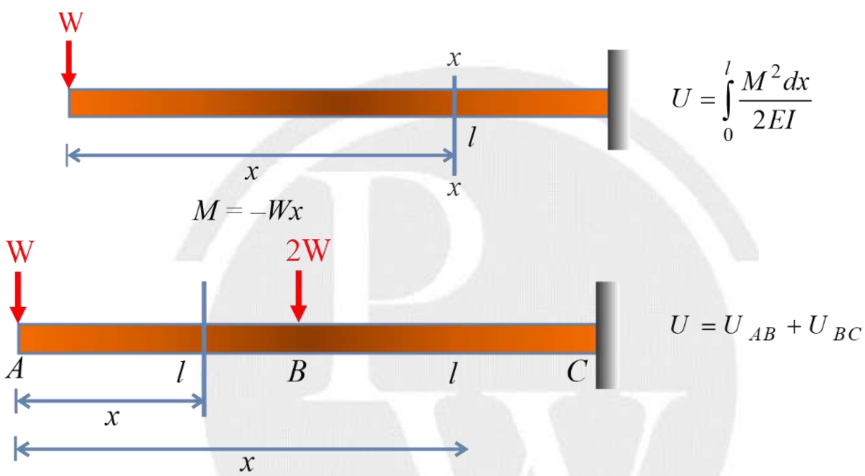


Fig.7.5 Strain energy due to bending

$M_{AB} = -Wx$ ($x = 0$ to l)

$M_{BC} = -Wx - 2W(x - l)$ ($x = l$ to $2l$)

7.6 Castigliano's Theorem

The partial derivative of the total strain energy in a structure with respect to any force at a point is equal to the deflection at that point in the direction of the force.

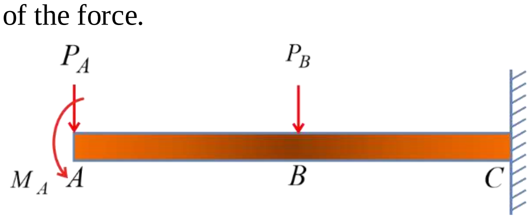


Fig.7.6 Castigliano's Theorem

$$U = \text{Total S.E.}$$

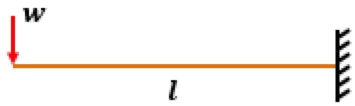
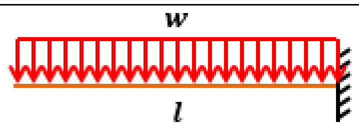
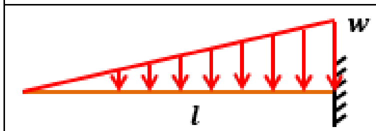
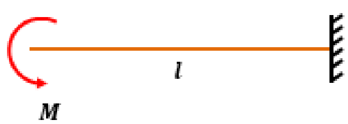
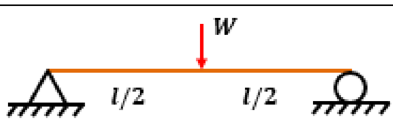
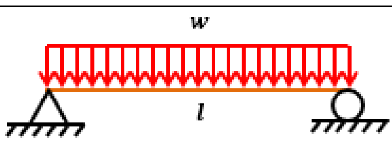
$$\frac{\partial U}{\partial P_A} = y_A$$

$$\frac{\partial U}{\partial P_B} = y_B$$

The partial derivative of the total strain energy in a structure with respect to a moment at a point is equal to the slope at that point.

$$\frac{\partial U}{\partial M_A} = \theta_A$$

7.7 Slope and Deflection of standard case

Loading	θ_{max}	y_{max}
	$\frac{Wl^2}{2EI}$	$\frac{Wl^2}{3EI}$
	$\frac{Wl^3}{6EI}$	$\frac{Wl^4}{8EI}$
	$\frac{Wl^3}{24EI}$	$\frac{Wl^4}{30EI}$
	$\frac{Ml}{EI}$	$\frac{Ml^2}{2EI}$
	$\frac{Wl^2}{16EI}$	$\frac{Wl^3}{48EI}$
	$\frac{Wl^3}{24EI}$	$\frac{5Wl^4}{384EI}$

7.8 Principle of Superposition

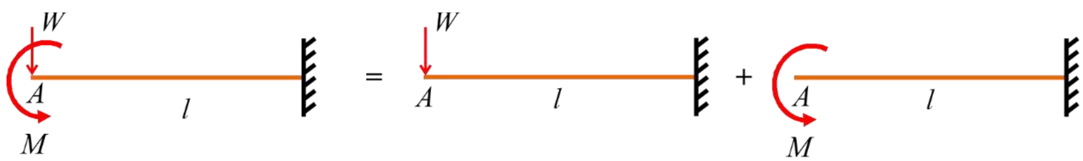


Fig.7.7 Principle of Superposition

$$\theta_A = \frac{Wl^2}{2EI} + \frac{Ml}{EI}$$

$$y_A = \frac{Wl^3}{3EI} + \frac{Ml^2}{2EI}$$

7.9 Maxwell's Reciprocal Theorem

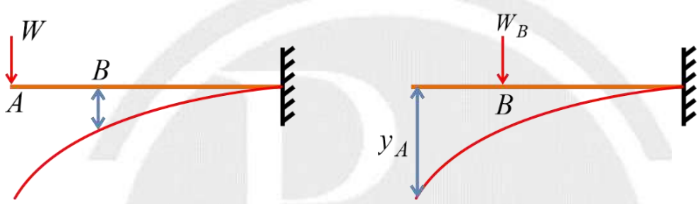


Fig.7.8 Maxwell's Reciprocal Theorem

$W_A . y_A = W_B . y_B$

Special Case in Cantilever Beams

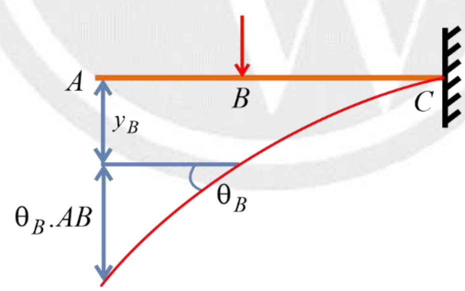
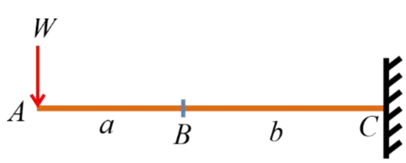
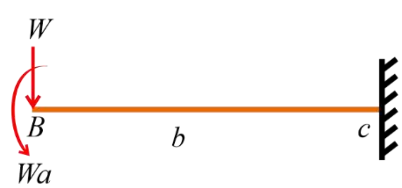


Fig.7.9 Elastic curve becomes straight line (AB)

$\theta_A = \theta_B$ (Since elastic curve becomes straight line)

$y_A = y_B + \theta_B . AB$ (This equation is valid only when elastic curve becomes straight line)





$\theta_B = ?$

$y_B = ?$

7.10 Statically Indeterminate Beams

(1) In this case, deflection at A is zero.

Compatibility equation

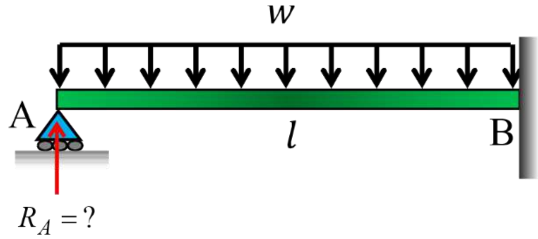


Fig.7.10 Propped cantilever beam reaction calculation

$y_A = 0$

$\downarrow y_A$ due to w + $\uparrow y_A$ due to R_A = 0

$$\frac{wl^4}{8EI} - \frac{R_A l^3}{3EI} = 0$$

$$R_A = \frac{3wl}{8}$$

(2) In this case deflection at point A in the beam is equal to the deflection in the spring

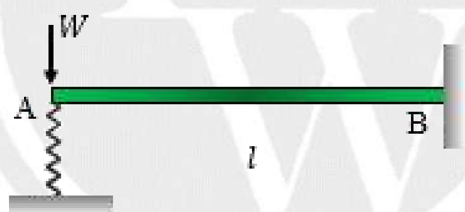


Fig.7.11 Spring support at one end of cantilever beam

$y_A = y_{spring}$

$\downarrow y_A$ due to w + $\uparrow y_A$ due to R_S = $\downarrow y_{spring}$

$$\frac{wl^3}{3EI} - \frac{R_S l^3}{3EI} = \frac{R_S}{K}$$

$R_S = ?$

FBD

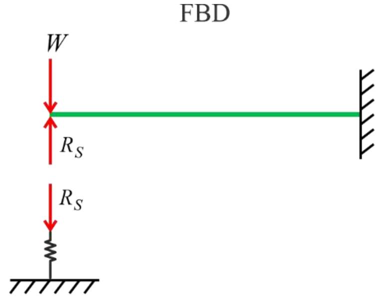


Fig.7.12 Free body diagram

(3) In this case, deflection at point B and C will be same

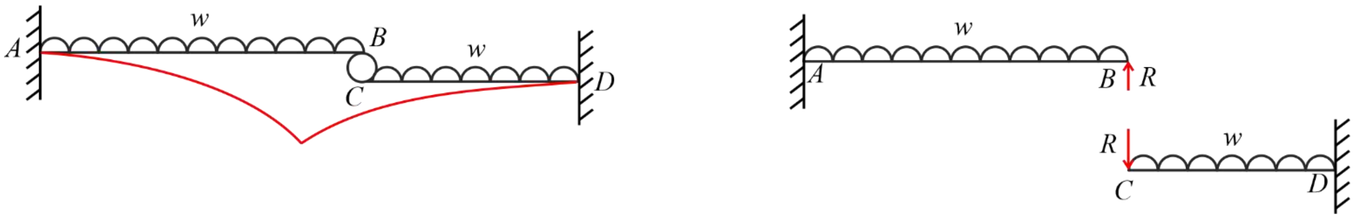


Fig.7.13 Two cantilever beam attached at their free ends

$y_B = y_C$

$\theta_B \neq \theta_C$

$y_B = y_C$

$\downarrow y_b$ due to w + $\uparrow y_B$ due to R

$= \downarrow y_c$ to w + $\downarrow y_c$ due to R

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8 COMPLEX STRESS

8.1 Complex Stress

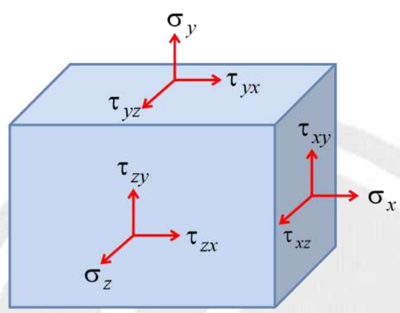
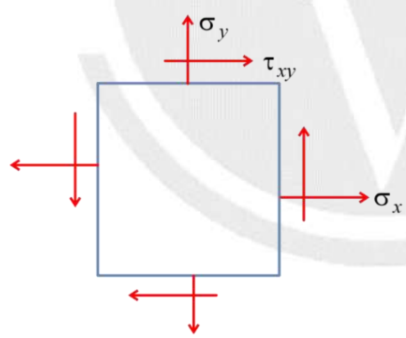


Fig.8.1 Point is subjected to triaxial state of stress

8.2 Plane Stress



$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

Fig.8.2 Point is subjected to biaxial state of stress

8.3 Stresses on Oblique Planes

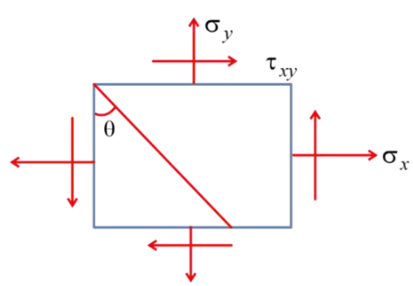


Fig.8.3 Stresses on oblique planes

$$\sigma_\theta = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_\theta = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

Sign Convention

(a) σ:

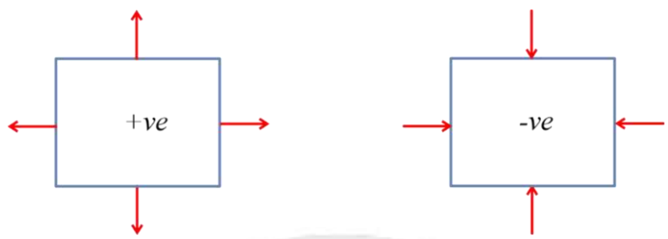


Fig.8.4 Normal stress sign convention

(b) τ:

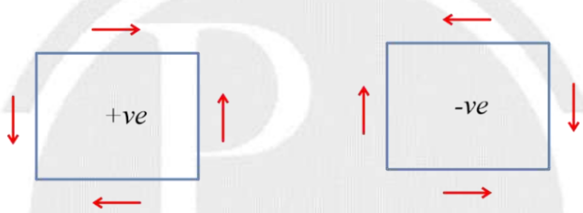


Fig.8.5 Shear stress sign convention

(c) θ:

Fig.8.6 Sign convention for θ (location of oblique plane)

8.4 Mohr's Circle

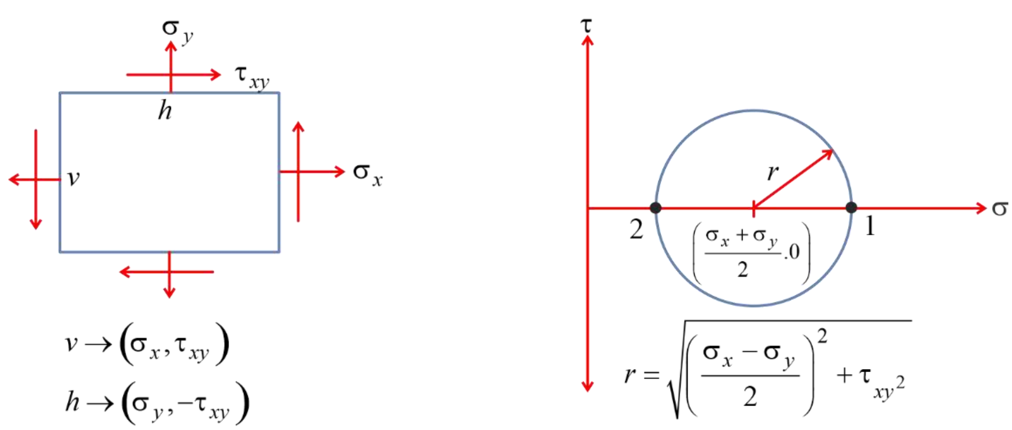


Fig.8.7 Mohr's circle for biaxial state of stress

8.5 Principal Planes

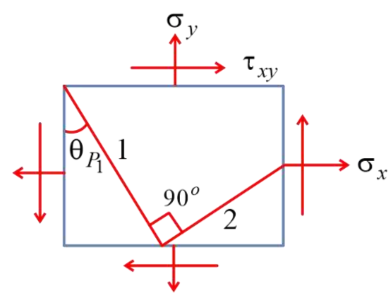


Fig.8.8 Location of Principal Planes

$$\theta_{P_1} = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)$$

$$\theta_{P_2} = \theta_{P_1} + 90^o$$

8.6 Principal Stresses

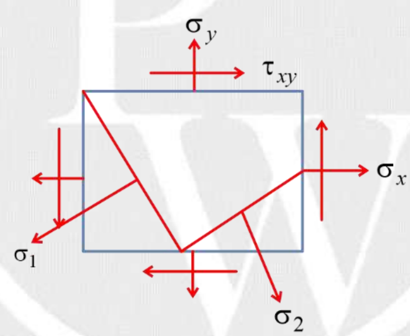


Fig.8.9 Principal stresses in complex state of stress

$$\sigma_{1,2} = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}$$

$$\sigma_1 + \sigma_2 = \sigma_x + \sigma_y$$

$$\sigma_1.\sigma_2 = \sigma_x.\sigma_y - \tau_{xy^2}$$

8.7 Maximum Shear Stress

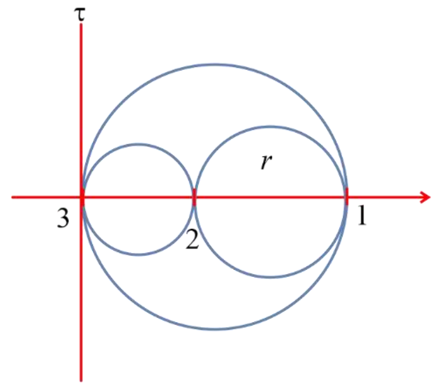


Fig.8.10 Mohr's circle for triaxial state of stress

$$(\tau_{max})_{\text{in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2}$$

or

$$\left(\frac{\sigma_1 - \sigma_2}{2}\right)$$

$$\tau_{max} = \max{}^m \text{ of } \left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|$$

8.8 Combined Bending & Twisting

$$\sigma_{max} = \frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$$

$$\tau_{max} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}$$

$$M_{eq} = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

$$T_{eq} = \sqrt{M^2 + T^2}$$

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9 COMPLEX STRAIN

9.1 Complex Strain

Strain analysis is similar to the stress analysis, just replace normal stress by normal strain and shear stress by half of the shear strain

$$\sigma \rightarrow \varepsilon$$

$$\tau \rightarrow \frac{\gamma}{2}$$

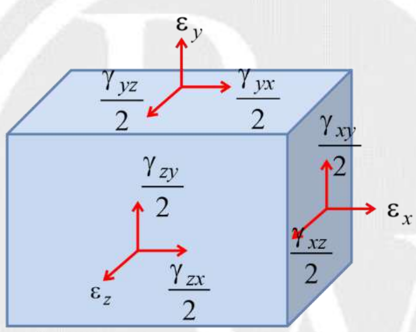
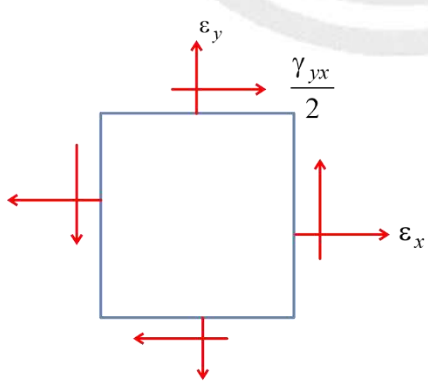


Fig.9.1 Point is subjected to triaxial state of strain

9.2 Plane Strain



$$\begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_y \end{bmatrix}$$

Fig.9.2 Point is subjected to biaxial state of strain

9.3 Strains on Oblique Planes

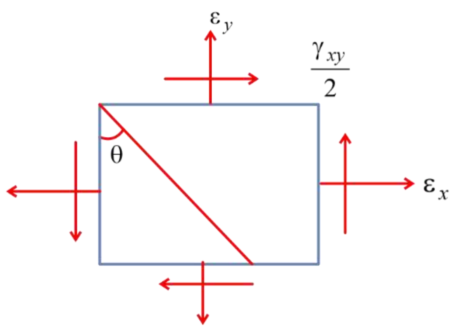


Fig.9.3 Strains on oblique planes

$$\varepsilon_\theta = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_\theta}{2} = -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

9.4 Mohr's Circle

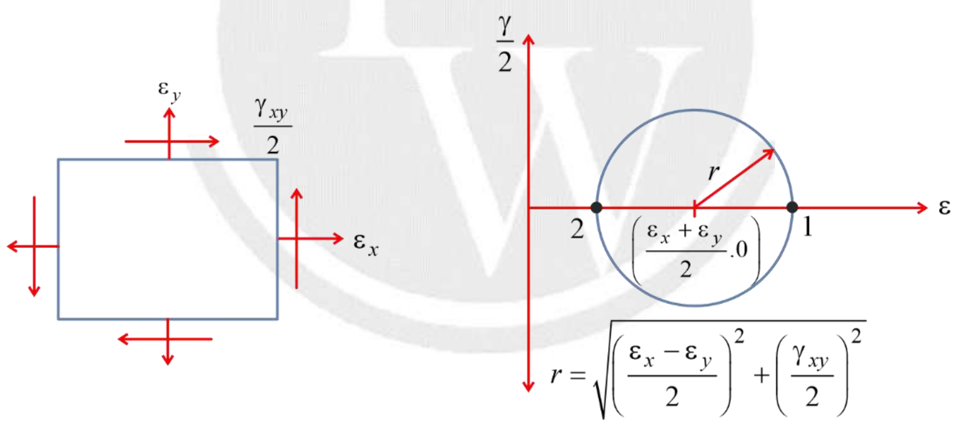


Fig.9.4 Mohr's circle for biaxial state of strain

9.5 Principal Planes

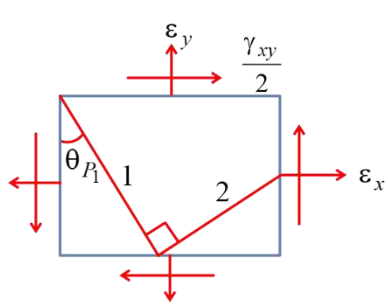


Fig.9.5 Location of Principal Planes

$$\theta_{P_1} = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right)$$

$$\theta_{P_2} = \theta_{P_1} + 90^o$$

9.6 Principal Strains

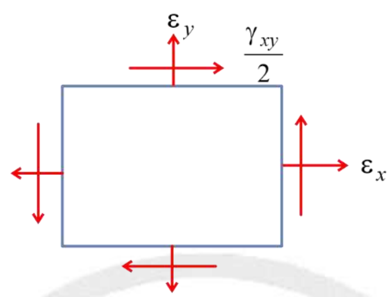


Fig.9.6 Principal strain for biaxial state of strain

$$\varepsilon_{1,2} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right) + \left(\frac{\gamma_{xy}}{2}\right)}$$

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_x + \varepsilon_y$$

$$\varepsilon_1 . \varepsilon_2 = \varepsilon_x . \varepsilon_y - \left(\frac{\gamma_{xy}}{2}\right)^2$$

9.7 Maximum Shear Strain

$$(\gamma_{\max})_{in-plane} = (\varepsilon_1 - \varepsilon_2)$$

$$\gamma_{\max} = \max^m \, of \, |\varepsilon_1 - \varepsilon_2|, |\varepsilon_2 - \varepsilon_3|, |\varepsilon_3 - \varepsilon_1|$$

9.8 Strain Rosette

Combination of three strain gauges arranged in three different directions.

Fig.9.7 Strain rosette

$$\varepsilon_A = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_A + \frac{\gamma_{xy}}{2}\sin 2\theta_A$$

$$\varepsilon_b = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_b + \frac{\gamma_{xy}}{2}\sin 2\theta_B$$

$$\varepsilon_C = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_C + \frac{\gamma_{xy}}{2}\sin 2\theta_C$$

9.8.1 Rectangular Strain Rosette

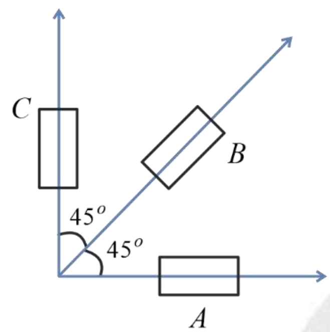


Fig.9.8 Rectangular strain rosette

$$\theta_A = 0^o$$

$$\theta_B = 45^o$$

$$\theta_C = 90^o$$

$$\varepsilon_x = \varepsilon_A$$

$$\varepsilon_y = \varepsilon_C$$

$$\gamma_{xy} = 2\varepsilon_B - (\varepsilon_A + \varepsilon_C)$$

9.8.2 Delta Strain Rosette

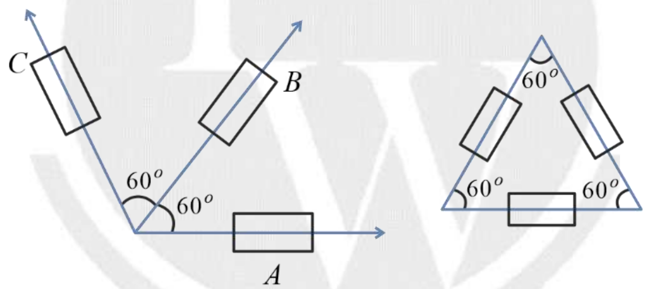


Fig.9.9 Delta strain rosette

9.8.3 Star Strain Rosette

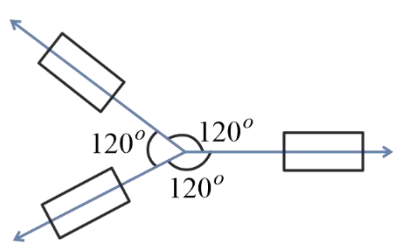


Fig.9.10 Star strain rosette

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10 PRESSURE VESSELS

10.1 Pressure Vessels

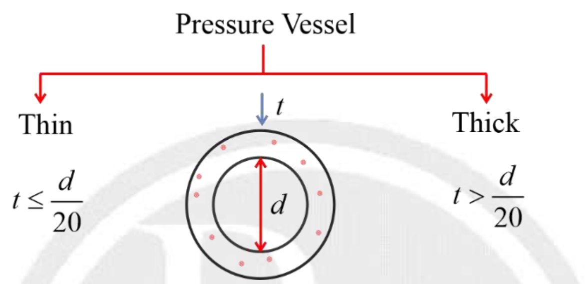


Fig.10.1 Pressure vessel

10.2 Thin Cylinder

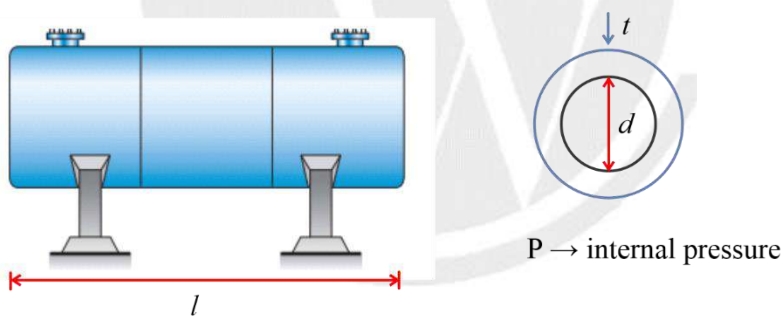


Fig.10.2 Thin Cylindrical Pressure Vessel

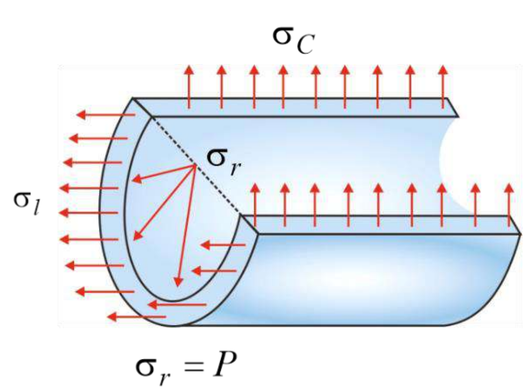


Fig.10.3 Various stresses on thin cylindrical pressure vessel

10.2.1 Longitudinal Stress

$$\sigma_l = \frac{Pd}{4t}$$

10.2.2 Circumferential/Hoop Stress

$$\sigma_C = \frac{Pd}{2t}$$

$$\sigma_1 = \sigma_c, \sigma_2 = \sigma_l$$

$$\sigma_{\max} = \frac{Pd}{2t}$$

$$(\tau_{\max})_{in\ plane} = \frac{Pd}{8t}$$

$$\tau_{\max} = \frac{Pd}{4t}$$

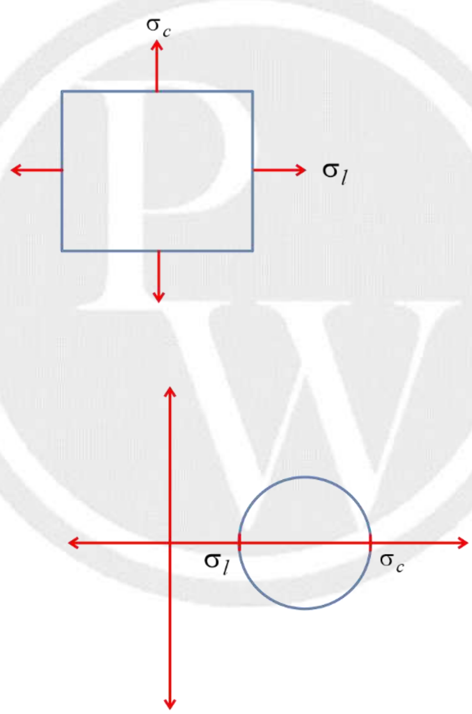


Fig.10.4 Mohr's circle for biaxial state of stress of thin cylindrical pressure vessel subjected to internal pressure

10.2.3 Longitudinal Strain

$$\varepsilon_l = \frac{\sigma_l}{E} - \nu\frac{\sigma_c}{E}$$

$$\varepsilon_l = \frac{Pd}{4tE}(1-2v) = \frac{\Delta l}{l}$$

10.2.4 Circumferential/Hoop Strain

$$\varepsilon_c = \frac{\sigma_c}{E} - \nu\frac{\sigma_l}{E}$$

$$\varepsilon_c = \frac{Pd}{4tE}(2-\nu) = \frac{\Delta d}{d}$$

10.2.5 Volumetric Strain

$$\varepsilon_v = \varepsilon_l + 2\varepsilon_c$$

$$\varepsilon_v = \frac{Pd}{4tE}(5-4\nu) = \frac{\Delta v}{v}$$

10.3 Thin Sphere

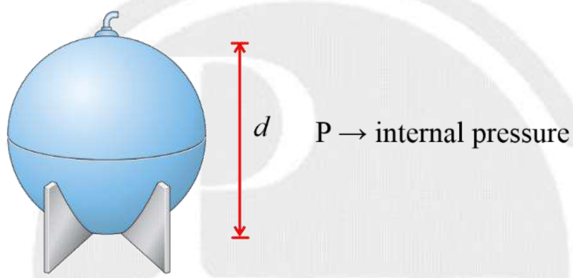


Fig.10.5 Thin spherical pressure vessel

10.3.1 Circumferential/Hoop Stress

$$\sigma_c = \frac{Pd}{4t}$$

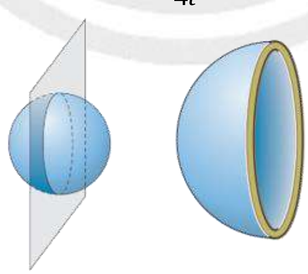
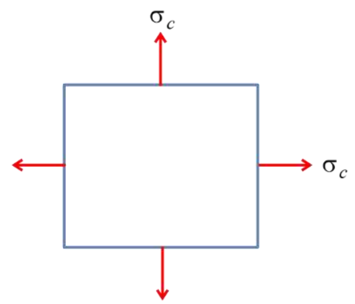


Fig.10.6 Cross sectional view of thin spherical pressure vessel



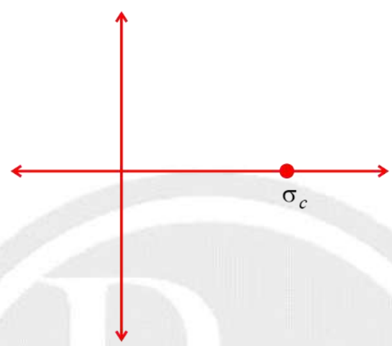


Fig.10.7 Mohr's circle for biaxial state of stress of thin spherical pressure vessel subjected to internal pressure

$$\sigma_1 = \sigma_2 = \sigma_c$$

$$\sigma_{\max} = \frac{Pd}{4t}$$

$$(\tau_{\max})_{in\ plane} = 0$$

$$\tau_{\max} = \frac{Pd}{8t}$$

10.3.2 Circumferential/Hoop Strain

$$\varepsilon_c = \frac{\sigma_c}{E} - v\frac{\sigma_c}{E}$$

$$\varepsilon_c = \frac{Pd}{4tE}(1-v) = \frac{\Delta d}{d}$$

10.3.2 Volumetric Strain

$$\varepsilon_v = 3\varepsilon_c$$

$$\varepsilon_v = \frac{3Pd}{4tE}(1-v) = \frac{\Delta v}{v}$$

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11 COLUMNS

11.1 Columns

Column is a structural member used to support axial compressive loads.

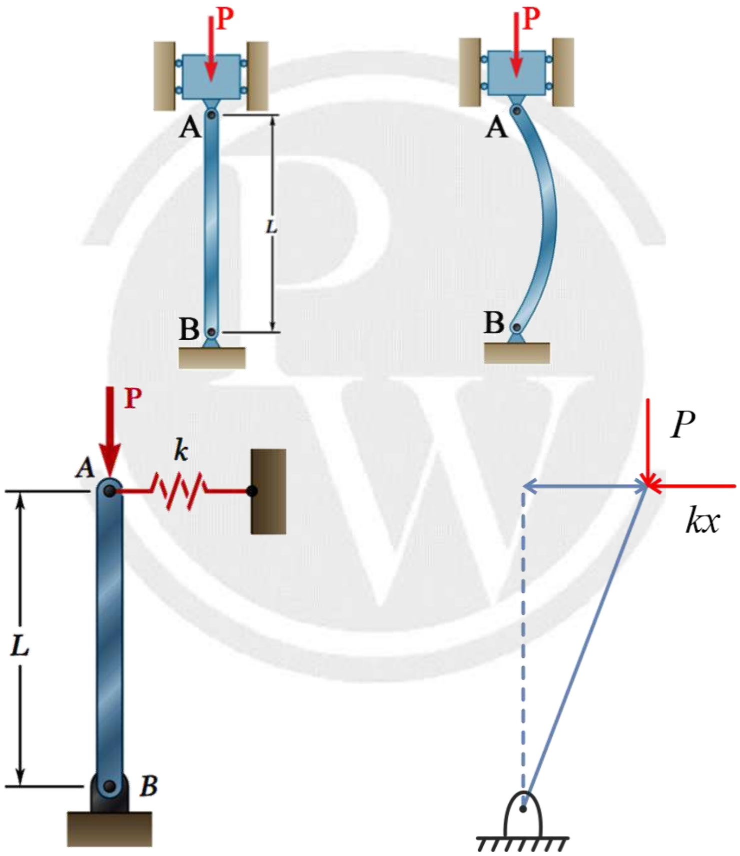

Fig.11.1 Columns

If $Px < kx.l$ – Stable
If $Px > kx.l$ – Unstable
If $Px = kx.l$ – Critical
$P_{cr} = k.l$

11.2 Euler's theory of Buckling

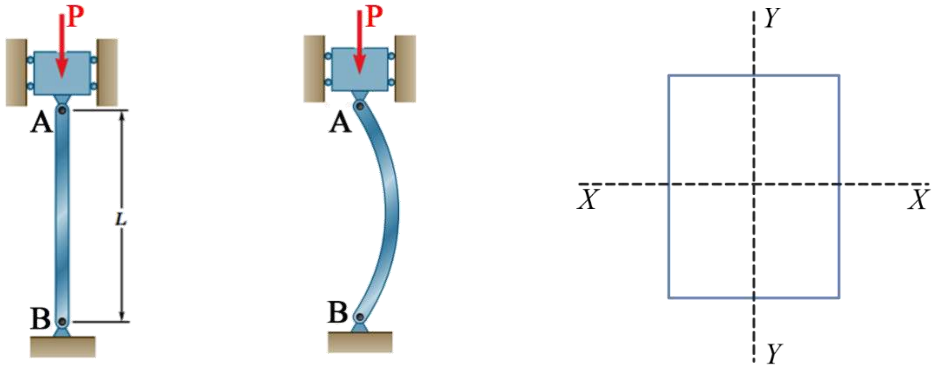

Fig.11.2 Euler's theory of Buckling for column

$$P_{cr} = \frac{\pi^2 EI_{min}}{l_e^2}$$

$I_{min} = min^m$ of I_X, I_Y

$l_e \rightarrow$ effective length

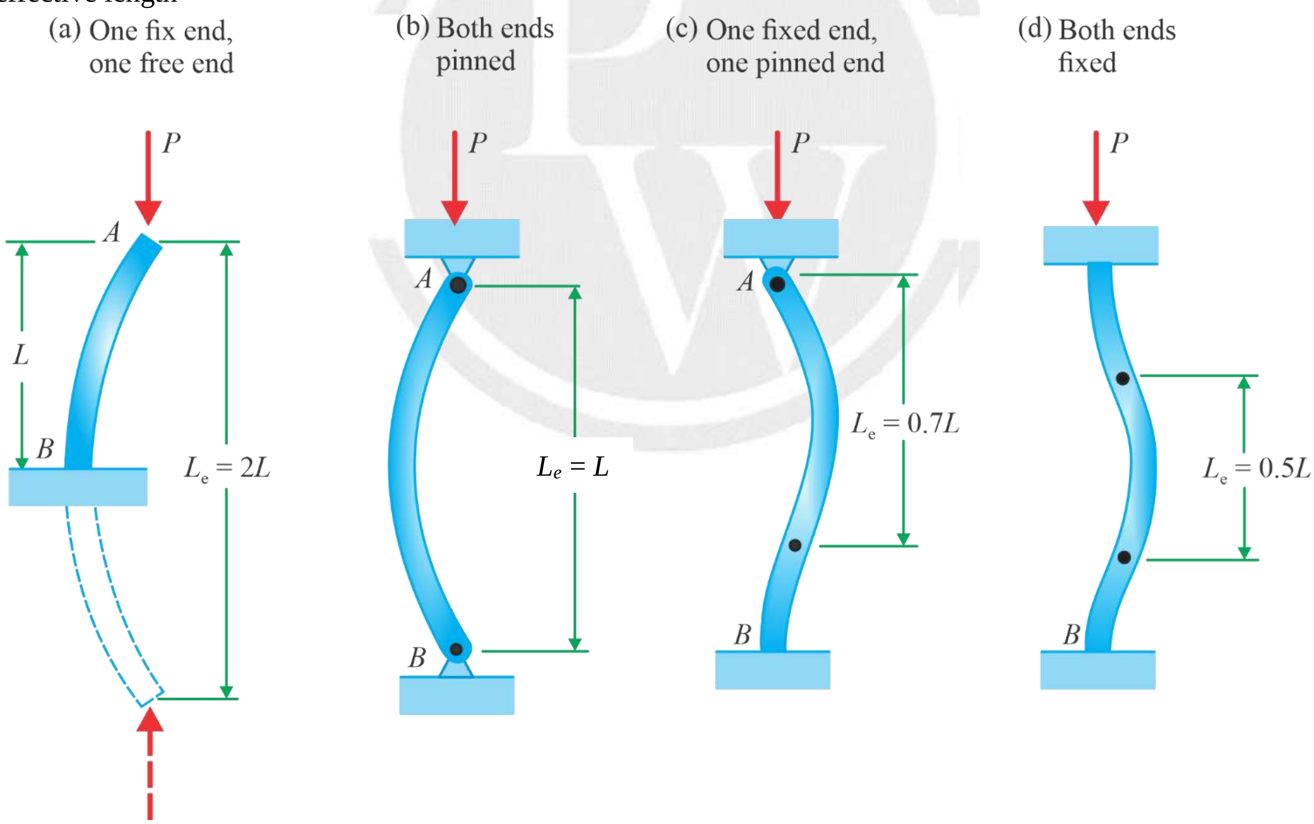

Fig.11.3 Various end conditions for column

$$P_{cr} \propto \frac{1}{\ell_c^2}$$

End Conditions		Effective length l_e
(1)	One end fixed, other free	2L
(2)	Both ends hinged	L
(3)	One end fixed, other hinged	$\frac{l}{\sqrt{2}}$
(4)	Both ends fixed	$\frac{l}{2}$

11.3 Limitation of Euler's theory of Buckling

$$\lambda \geq \pi\sqrt{\frac{E}{\sigma_c}}$$

$$\lambda = \frac{l_e}{k_{\min}} = \text{Slenderness ratio}$$

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Machine Design

Machine Design

INDEX

1 Design Against Static Load

1.1 Introduction

1.1.1 Type of Loads

Static Load:

Magnitude, direction & point of Application of load does not change with time or remains constant with time (t)
E.g., Load acting on the beam and columns of the room.

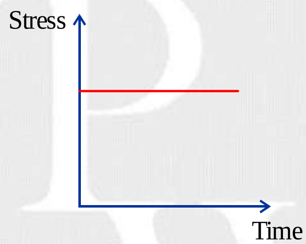


Fig.1.1: Static stress

Fluctuating Load:

Magnitude, Direction, 'or' Point of application of load changes with time.
E.g., Load on any bridge; load on the piston of an IC engine

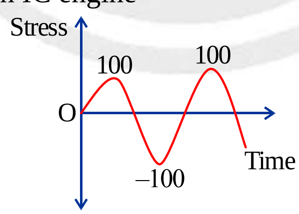


Fig.1.2: Fluctuating stress

1.1.2 Strength of Material

The magnitude of stress corresponding to which if the value of induced stress exceeds; the material fails. That particular value of stress is known as strength of material.
It is a property of material & depends on the nature of loading i.e., load is fixed.

$$\text{Induced Stress} > \text{Strength} \leftarrow \text{fail}$$

$$\text{Induced Stress} < \text{Strength} \leftarrow \text{safe}$$

For ductile material: Failure criteria is Yield strength.
For brittle material: Failure criteria is ultimate strength.

1.1.3 Design under 1-D Tensile Stress Condition

Fig.1.3: Axial tensile stress

Tensile Strength: (Strength of material under 1D tensile stress condition):

$$\sigma_{ft}\begin{cases} = S_{yt}\,(\text{for Ductile}) \\ = S_{ut}\,(\text{for Brittile}) \end{cases}$$

For safe design: $\sigma \le \sigma_{ft}$

For more safer design (we use factor of safety)

Factor of safety = $\dfrac{\text{Failure stress}}{\text{Allowable stress}} = \dfrac{\text{Strength of material}}{\text{Working stress}}\{\text{FOS}>1\}$

$$\sigma \le \frac{\sigma_{ft}}{\text{Fos}}$$

1.1.4 Design under 1-D Compressive Stress Condition

Fig.1.4: Axial compressive stress

Compressive Strength

Compressive Strength (Strength of material under 1D tensile stress condition):

$$\sigma_{fc}\begin{cases} = S_{yc}\,(\text{for Ductile}) \\ = S_{uc}\,(\text{for Brittile}) \end{cases}$$

For safe design;

$$\sigma \le \sigma_{fc}$$

Design must be safer:

$$\sigma \le \frac{\sigma_{fc}}{FOS}$$

> **Note:**
> 1. Margin of safety = Factor of safety –1

1.1.5 Even Material & Uneven Material

Even Material

Material whose strength in compression & strength in tension are equal to each other; then the material is an even Material. i.e., For even Material;

$$\sigma_{ft} = \sigma_{fc}$$

Generally ductile Material are even material.

Uneven Material

Material whose strength in compression & strength in tension are not same; then the material is known as uneven material.
i.e., for Uneven Material;

$$\sigma_{fc} \neq \sigma_{fc}$$

Generally, brittle material is uneven material.

$$S_{uc} > S_{ut}$$

1.2 Theory of Failure (TOF)

There is no need of theory of failure if there is uniaxial 1-D loading condition since we can estimate the strength by tensile test. But in complex stress condition as shown in figure 1.5 testing is not practically possible and for such cases, we require theory of failure.

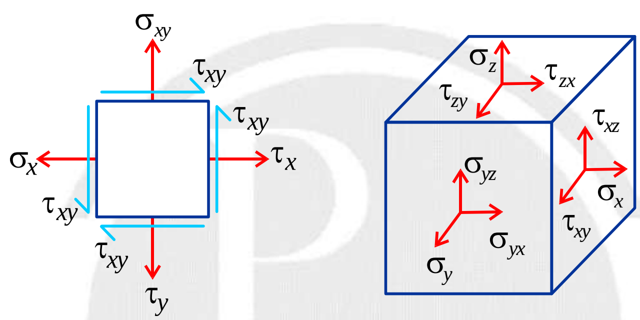
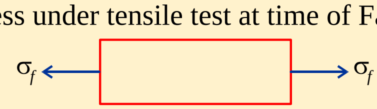


Fig.1.5: Complex Stress Condition

Theory of failure compares the actual complex stress condition of critical point with the tensile test by using some criteria of failure. Various theory of failures (TOF) is:

(1) Max. Principal stress or Max-Normal Stress TOF or Rankine TOF
(2) Max Shear stress TOF or Tresca and J.J. Guest TOF
(3) Strain Energy TOF or Haigh's TOF
(4) Distortion Energy TOF or VON Misses TOF
(5) Principal Strain TOF or St. Venant TOF.

Note:
Principal Stress under tensile test at time of Failure



{$\sigma_f \rightarrow$ failure stress}

$$\sigma_f = \begin{cases} S_{yt} \text{ (For ductile material)} \\ S_{ut} \text{ (For brittle material)} \end{cases}$$

$$\text{Principle stress} = \begin{bmatrix} \sigma_1 = \sigma_f \\ \sigma_2 = 0 \\ \sigma_3 = 0 \end{bmatrix}$$

1.2.1 Rankine 'or' Maximum Principal Stress 'or' Maximum Normal Stress TOF

Criteria of failure: Maximum principal stress or Maximum normal stress

If Maximum normal stress of the machine element crosses the maximum normal stress under tensile test at the time of failure, then; machine element will fail according to this TOF.

or

According to Rankine TOF, Machine element will be safe if the maximum normal stress induced in the Machine element is less than the maximum normal stress under tensile test at the time of failure.

For safe design:

$$(\sigma_{\max})_{\text{actual}} \le (\sigma_{\max})_{\text{f}}$$

$$(\sigma_{\max})_{\text{actual}} \le \sigma_f$$

For design

$$(\sigma_{\max})_{\text{actual}} \le \frac{\sigma_f}{fos}$$

$$(\sigma_{\max})_{\text{actual}} \le \sigma_{\text{perm.}}$$

> **Note:**
> This theory of failure is suitable for brittle materials as brittle materials are weak in tension, hence they fail due to maximum principal stress.

1.2.2 Tresca & J.J. Guest TOF or Maximum Shear stress TOF

Criteria of failure:

Maximum Shear Stress

Maximum shear stress:

$$\tau_{\max} = \max\left[\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right]$$

Maximum shear stress under tensile test at the time of failure:

Under tensile test:

$$\sigma_1 = \sigma_f, \sigma_2 = 0, \sigma_3 = 0$$

$$(\tau_{\max})_f = \max\left[\left|\frac{\sigma_f - 0}{2}\right|, \left|\frac{0 - 0}{2}\right|, \left|\frac{0 - \sigma_f}{2}\right|\right] = 0.5\sigma_f$$

Statement:

If Maximum shear stress of the machine element crosses the maximum shear stress under tensile test at the time of failure, then; machine element will fail according to this TOF.

or

Machine element will be safe if the maximum shear stress induced in the Machine element is less than the maximum shear stress under tensile test at the time of failure.

For safe design

$$(\tau_{max})_{actual} \le (\tau_{max})_f$$

At time of design, for safer design

$$(\tau_{max})_{actual} \le \frac{0.5\sigma_f}{FOS}$$

1.2.3 Haigh's 'or' Strain Energy TOF

Criteria of Failure:

Strain energy per unit volume.

Strain Energy per unit Volume at any point:

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \text{ J/m}^3$$

Strain Energy per unit Volume at any point under tensile test at the time of failure:

Under tensile test: $\sigma_1 = \sigma_f, \sigma_2 = 0, \sigma_3 = 0$

$$u_f = \frac{1}{2E}\left[\sigma_f^2\right] \text{ J/m}^3$$

Statement:

According to Haigh's TOF. if Maximum strain energy per unit volume at the critical point of the machine element crosses the strain energy per unit volume under tensile test at the time of failure, then machine element will fail according to this TOF.

or

Machine element will be safe if the strain energy per unit volume at the critical point induced in the machine element is less than the strain energy per unit volume under tensile test at the time of failure.

For safe design:

$$u \le u_f$$

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \le \frac{1}{2E}\left[\sigma_f^2\right]$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \le \sigma_f^2$$

$$\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)} \le \sigma_f$$

For safer design:

$$\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)} \le \left(\frac{\sigma_f}{\text{Fos}}\right)$$

For 2D stress condition: $\sigma_3 = 0$

$$\sqrt{\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2} \le \left(\frac{\sigma_f}{\text{Fos}}\right)$$

1.2.4 Distortion Energy 'or' Von-mises TOF

Criteria of Failure:

Distortion energy per unit volume.

Distortion Energy per unit Volume at any point:

$$u_d = \frac{1}{6G}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \text{ J/m}^3$$

Distortion Energy per unit Volume at any point under tensile test at the time of failure:

Under tensile test: $\sigma_1 = \sigma_f, \sigma_2 = 0, \sigma_3 = 0$

$$u_{df} = \frac{1}{2E}\left[\sigma_f^2\right] \text{ J/m}^3$$

Statement:

According to Von-mises TOF. if Maximum distortion energy per unit volume at the critical point of the machine element crosses the distortion energy per unit volume under tensile test at the time of failure, then machine element will fail according to this TOF.

or

Machine element will be safe if the distortion energy per unit volume at the critical point induced in the Machine element is less than the distortion energy per unit volume under tensile test at the time of failure.

For safe design:

$$u_d \le u_{df}$$

$$\frac{1}{6G}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \le \frac{1}{6G}\sigma_f^2$$

$$\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \le \sigma_f^2$$

$$\sqrt{\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)} \le \sigma_f$$

For safer design:

$$\sqrt{\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right]} \le \left(\frac{\sigma_f}{\text{Fos}}\right)$$

For 2D stress condition: $\sigma_3 = 0$

$$\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \le \left(\frac{\sigma_f}{\text{Fos}}\right)$$

1.2.5 Maximum Principal Strain 'or 'Maximum normal Strain or St. Venant TOF

Criteria of Failure:

Max. principal strain or Max normal strain

Maximum normal strain at any point:

$$\varepsilon_{\max} = \max\begin{cases} \frac{1}{E}(\sigma_1 - \mu\sigma_2 - \mu\sigma_3) \\ \frac{1}{E}(\sigma_2 - \mu\sigma_1 - \mu\sigma_3) \\ \frac{1}{E}(\sigma_3 - \mu\sigma_1 - \mu\sigma_2) \end{cases}$$

Maximum normal strain at any point under tensile test at the time of failure:

Under tensile test: $\sigma_1 = \sigma_f, \sigma_2 = 0, \sigma_3 = 0$

$$(\varepsilon_{\max})_f = \max\begin{cases} \frac{1}{E}(\sigma_f) \\ \frac{1}{E}(\mu\sigma_f) = \frac{\sigma_f}{E} \\ \frac{1}{E}(\mu\sigma_f) \end{cases}$$

Statement:

According to St. Venant's TOF. if maximum normal strain at the critical point of the machine element crosses the maximum normal strain under tensile test at the time of failure, then machine element will fail according to this TOF.

or

Machine element will be safe if the maximum normal strain at the critical point induced in the Machine element is less than the maximum normal strain under tensile test at the time of failure.

For safe design:

$$(\varepsilon_{\max})_a \le (\varepsilon_{\max})_f$$

$$(\varepsilon_{\max})_a \le \frac{\sigma_f}{E}$$

For safer design:

$$(E_{max})_a \le \frac{\sigma_f}{E \times Fos}$$

> **Note:**
> **Steps of Design**
> (a) Identify critical point.
> (b) Find principal stress at critical point.
> (c) Select and use suitable theory of failure.

1.3 Combined Bending & Torsion

Solid Circular shaft	Hollow Circular Shaft
$\sigma_{max} = \sigma_1 = \frac{16}{\pi D^3}[M + \sqrt{M^2 + T^2}]$	$\sigma_{max} = \sigma_1 = \frac{16}{\pi D_0^3(1-k^4)}\left[M + \sqrt{M^2 + T^2}\right]$
$\sigma_2 = \frac{16}{\pi D^3}\left[M - \sqrt{M^2 + T^2}\right]$	$\sigma_2 = \frac{16}{\pi D_0^3(1-k^4)}\left[M - \sqrt{M^2 + T^2}\right]$
$\sigma_3 = 0$	$\sigma_3 = 0$
$\tau_{max} = \frac{16}{\pi D^3}\left[\sqrt{M^2 + T^2}\right] = \left[\frac{\sigma_1 - \sigma_2}{2}\right]$	$\tau_{max} = \left\lvert\frac{\sigma_1 - \sigma_2}{2}\right\rvert = \frac{16}{\pi D_0^3(1-k^4)}\left[\sqrt{M^2 + T^2}\right]$

1.3.1 Equivalent Bending Moment (M_e)

$$M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2}\right]$$

1.3.2 Equivalent Torsional Moment (T_e)

$$T_e = \sqrt{M^2 + T^2}$$

1.4 Region of Safety for 2D stress condition

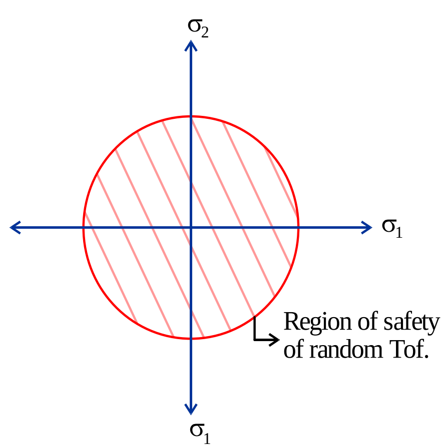
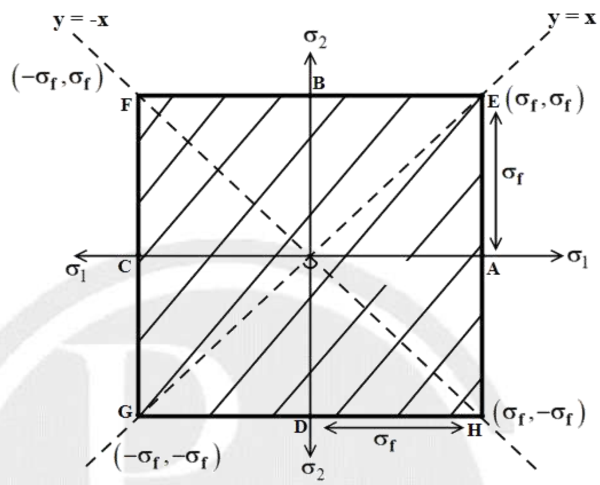
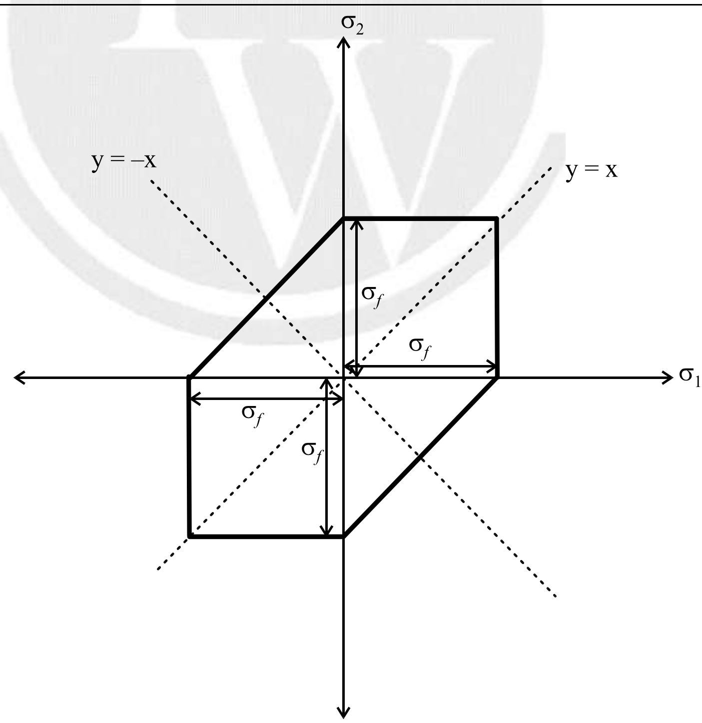
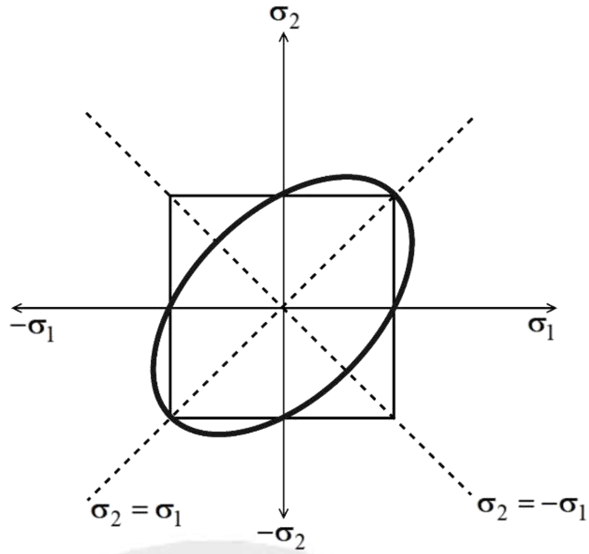
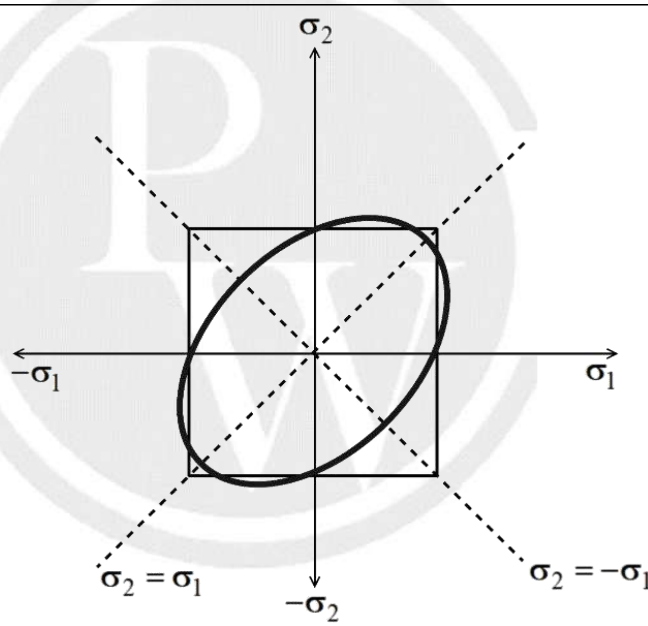


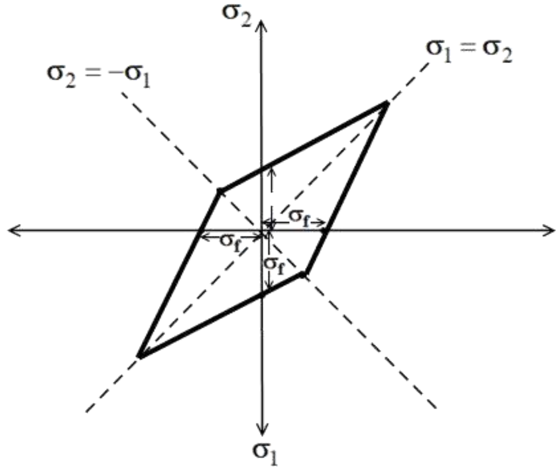
Fig.1.6: Region of safety

- If the actual loading lies under region of safety; Material is safe with FOS greater than 1.
- If the actual loading lies just on the boundary of region of safety; Material is just safe with FOS equal to 1.
- If actual loading lies outside region of safety; Material is unsafe.

1.4.1 Region of safety of Various TOF under 2 D Stress condition

TOF	Region of safety
Rankine TOF	 Fig.1.7: Safety region for Rankine TOF
Tresca TOF	 Fig.1.8: Safety region for Tresca TOF

Vonmises TOF	 **Fig.1.8: Safety region for Vonmises TOF**
Strain energy TOF	 **Fig.1.9: Safety region for Strain energy TOF**

Maximum normal strain TOF	 **Fig.1.10: Safety region for Maximum normal strain TOF**

1.4.2 Comparison of Tresca's, Rankine & Von-Mises TOF for 2-D Stress Condition

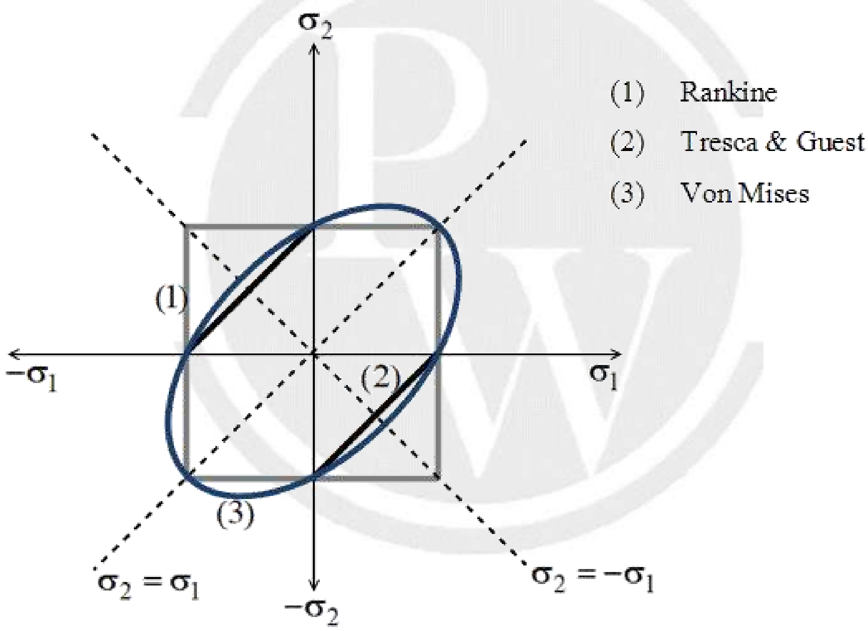


Fig.1.11: Safety region for Tresca, Rankine & Von Mises TOF

1. The theory of failure whose region of safety is smaller it will be safer.

(a) Safer TOF in II & IV Quadrant: $\xrightarrow[\text{Safety increases}]{\text{Rankine < Von Mises < Tresca \& Guest}}$

(b) Safer TOF in I & III Quadrant (Except $\sigma_1 = \sigma_2$) $\xrightarrow[\text{Safety increases}]{\text{Von Mises < Rankine = Tresca \& Guest}}$

(c) Safer TOF in I & III Quadrant (when $\sigma_1 = \sigma_2$): $\xrightarrow[\text{Safety increases}]{\text{Von Mises = Rankine = Tresca \& Guest}}$

2. For any given stress condition FOS will be less for safer theory of failure**.**

(a) II & IV Quadrant:

$$\xrightarrow[\text{FOS}]{\text{Rankine} > \text{Von Mises} > \text{Tresca \& Guest}}$$

(b) I & III Quadrant (Except $\sigma_1 = \sigma_2$):

$$\xrightarrow[\text{FOS}]{\text{Von Mises} > \text{Rankine} = \text{Tresca \& Guest}}$$

(c) I & III Quadrant ($\sigma_1 = \sigma_2$):

$$\xrightarrow[\text{FOS}]{\text{Von Mises} = \text{Rankine} = \text{Tresca \& Guest}}$$

6. For any given FOS, design will be strong for safer theory of failure.

7. Safer theory of failure is less Economical.

1.5 Shear Strength under pure Shear for Various TOF

Theory of failure	Shear strength
Rankine TOF	$\tau_f = \sigma_f$
Tresca & Guest TOF	$\tau_f = 0.5\sigma_f$
Von-Mises TOF	$\tau_f = 0.577\sigma_f$
Haigh TOF	$\tau_f = \sigma_f / \sqrt{2(1+\mu)}$
St. Venant TOF	$\tau_f = \sigma_f / \sqrt{(1+\mu)}$

1.6 Equivalent Normal Stress using Various theory of Failure

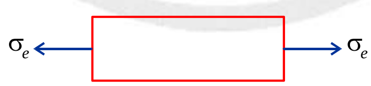


Fig.1.12: Equivalent Normal Stress

(1) Rankine TOF

$$\sigma_e = (\sigma_{\max})_{actual}$$

(2) Tresca& J.J. Guest TOF

$$\frac{\sigma_e}{2} = (\tau_{\max})_{actual} \Rightarrow \sigma_e = 2(\tau_{\max})_{actual}$$

(3) Von-Mises TOF

$$(u_d)_{eq.} = (u_d)_{actual} \Rightarrow \frac{1}{6G} \times \sigma_e^2 = (u_d)_{actual}$$

$$\Rightarrow \frac{1}{6G}[\sigma_e^2] = \frac{1}{6G}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$\Rightarrow \sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}$$

(4) Haigh TOF

$$u_{eq} = u_{actual}$$

$$\Rightarrow \frac{1}{2E}\sigma_e^2 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

$$\Rightarrow \sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}$$

(5) St. Venant TOF

$$(\varepsilon_{\max})_e = (\varepsilon_{\max})_{actual}$$

$$\frac{\sigma_e}{E} = (\varepsilon_{\max})_{actual}$$

1.7 Suitable TOF for various material

(1) For Brittle Material, Rankine TOF is more suitable.

(2) For ductile material, Tresca and Von mises TOF are more suitable. Out of which Tresca TOF is safer and Von-mises. TOF is more economical.

(3) Under Hydrostatic stress condition ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma$), we cannot use Tresca and Von-mises TOF for any type of material, because under hydrostatic stress condition value of maximum shear stress and distortion energy per unit volume is zero.

❑❑❑

2 DESIGN AGAINST FLUCTUATING LOAD

2.1 Introduction

Fluctuating Stress: It Stress at any point continuously fluctuates with respect to time.

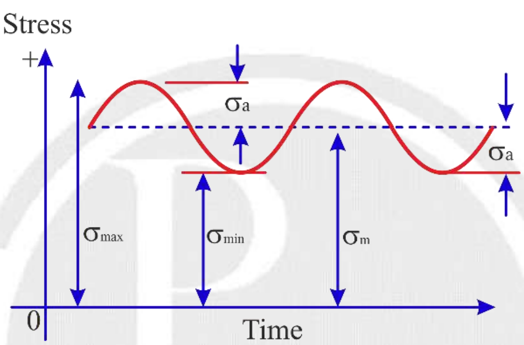


Fig.2.1: Fluctuating stress

2.1.1 Various terminology of fluctuating stress condition

1. **Average or Mean Stress** (σ_m)**:** $\sigma_m = \dfrac{\sigma_{max} + \sigma_{min}}{2}$

2. **Amplitude Stress (σa):** $\sigma_a = \dfrac{\sigma_{max} - \sigma_{min}}{2}$

3. **Stress Ratio *(R)***: $R = \dfrac{\sigma_{min}}{\sigma_{max}}$ $\{0 \le R \le 1\}$

4. **Amplitude Ratio *(A)*:** $A = \dfrac{\sigma_a}{\sigma_m} = \dfrac{\sigma_{max} - \sigma_{min}}{\sigma_{max} + \sigma_{min}} = \dfrac{1-R}{1+R}$

Note:

Put σ_{min} & σ_{max} values with appropriate sign as per the nature of stress. These are positive for tensile stress and negative for compressive stress.

2.1.2 Special types of fluctuating stress condition

Repeating Stress/Repeated Stress condition

Special type of fluctuating stress where stress at the critical point varies from 0 to σ. (i.e., $\sigma_{min} = 0$ & $\sigma_{max} = \sigma$)

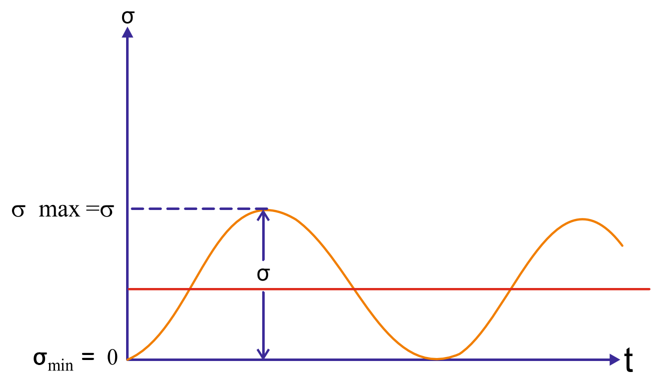


Fig.2.2: Repeating stress

For repeating stress condition: $\sigma_m = \frac{\sigma_{max}}{2}, \sigma_a = \frac{\sigma_{max}}{2}, R = 0, A = 1$

Reverse Stress Condition

Special type of fluctuating Stress condition where stress at critical point varies from ($-\sigma_1$) to ($+\sigma_2$) i.e., out of σ_{min} & σ_{max} one is compressive & other is tensile ($\sigma_{min} = -\sigma_1; \sigma_{max} = \sigma_2$).

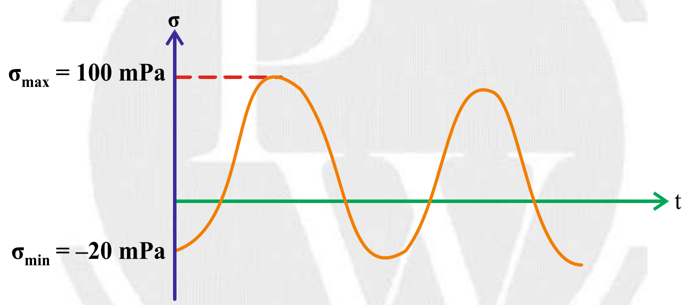


Fig.2.3: Reverse stress

Completely Reverse Stress condition:

If stress at critical points varies from $-\sigma$ to σ i.e. both σ_{min} & σ_{max} are of equal magnitude & opposite in nature, then that stress condition is known as completely reverse stress condition.

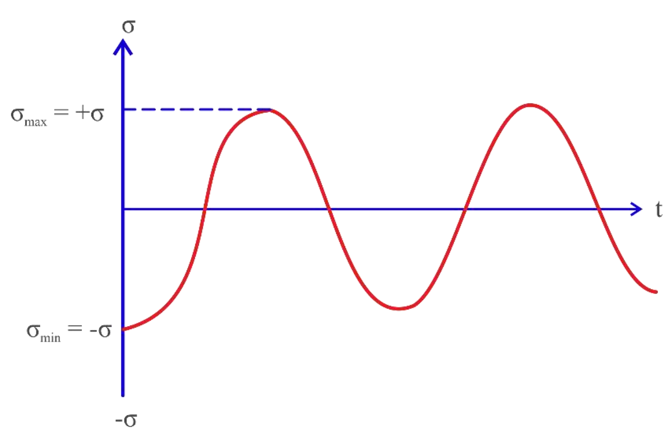


Fig.2.4: Completely Reverse stress

For repeating stress condition: $\sigma_m = 0, \sigma_a = \sigma, R = -1, A = \infty$

Note:
The bending stress at any point on a rotating shaft is state of completely reversed stress condition.

2.1.3 Fatigue Failure

- It is the failure of a machine element due to fluctuating stress at a stress level lower than the yield or ultimate strength of the material.
- Microcracks are developed in the material due to stresses. As the load is repeated, the cracks increase in size and during the final load cycle, when the material is not able to support the load, failure occurs.
- Microcracks are developed near the discontinuity where the stress is maximum due to stress concentration.

2.2 Stress concentration

Stress concentration is defined as the localization of high stresses due to the irregularities present in the component and abrupt changes of the cross section. Due to effect of stress concentration stress near irregularity will be more than nominal stress σ_0 (stress calculated by using elementary equation studied in subject strength of material with assumptions no irregularity is present is known as nominal stress)

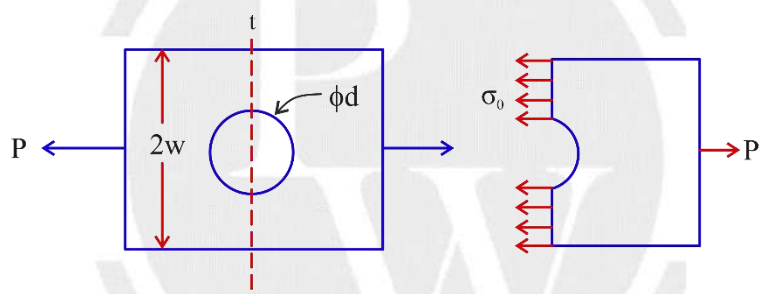


Fig.2.5: Nominal stress

Nominal stress near irregularity: $\sigma_0 = \dfrac{P}{(2w - d)t}$

Since irregularity is present in the above system therefore stress near irregularity will be more than nominal stress due to stress concentration.

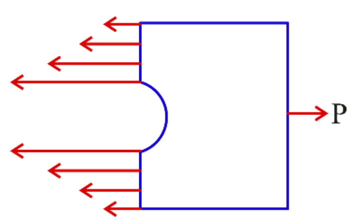


Fig.2.6: Effect of stress concentration with irregularities

Stress concentration occurs near irregularity due to disturbance in the path of stress flow due to which stress flow lines comes near and stress increases. (It is just like, when in flowing fluid stream lines comes nearer due to disturbance and velocity of flow increases.)

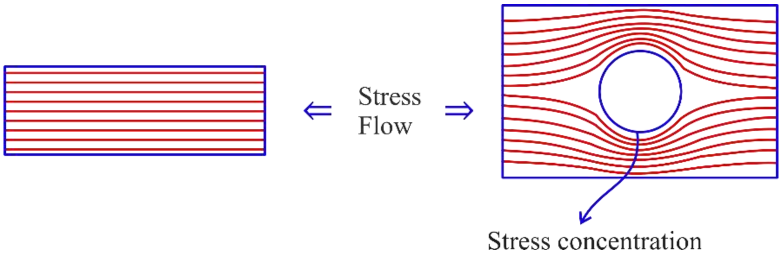
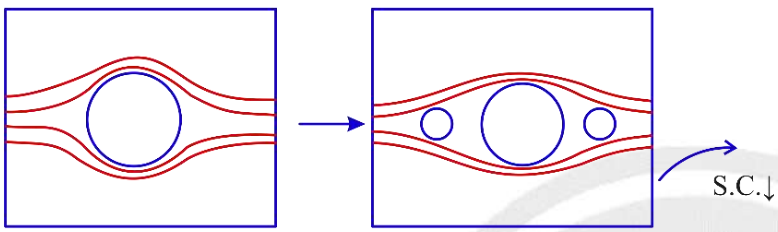


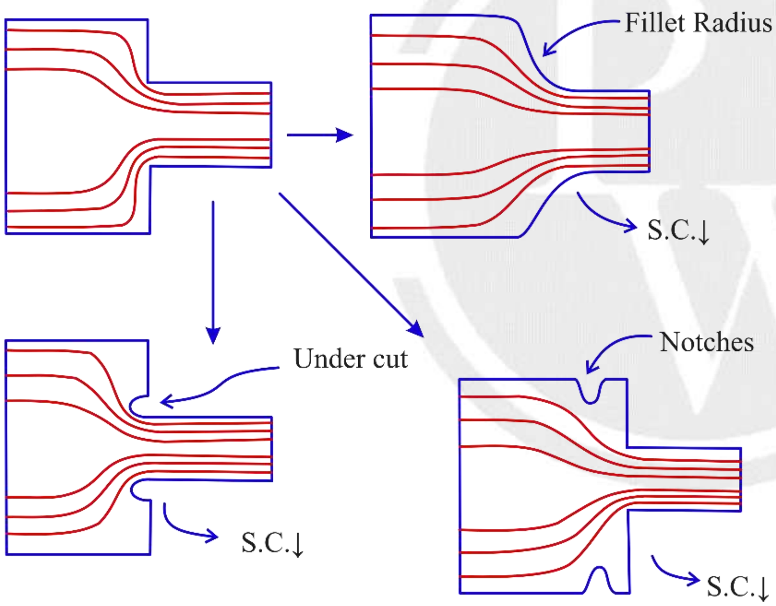
Fig.2.7: Stress flow

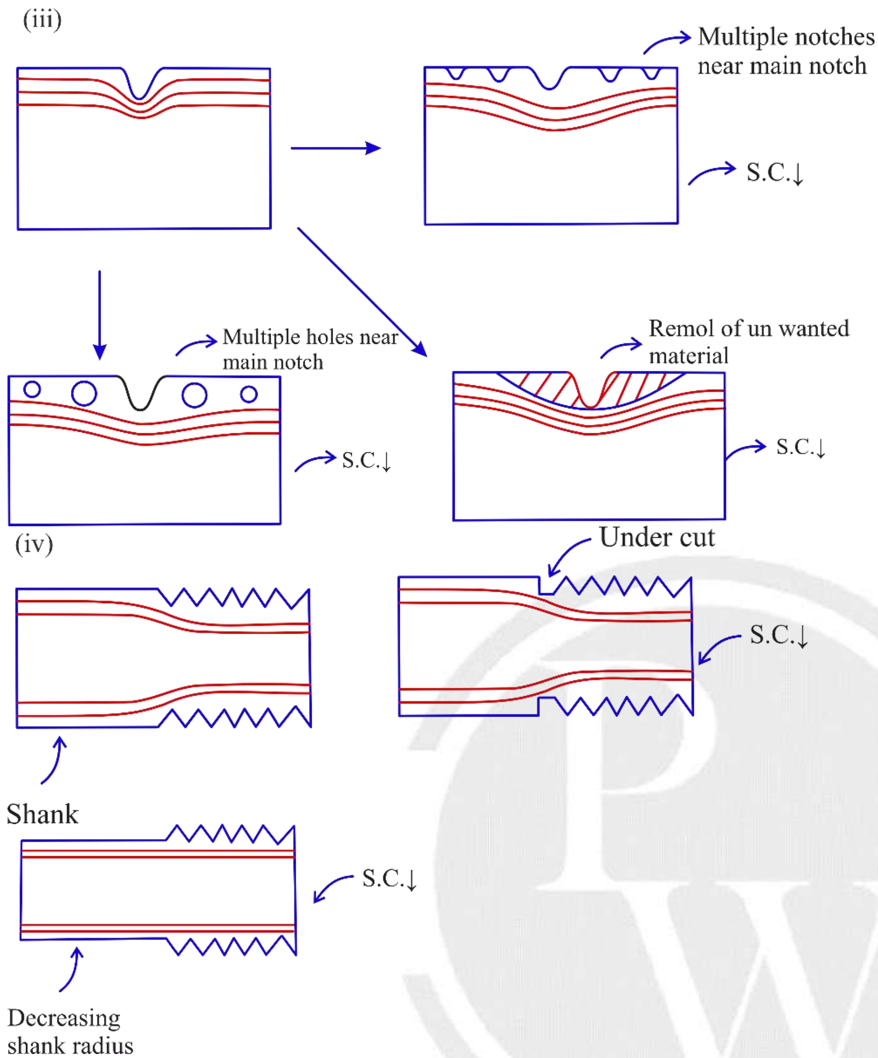
Methods to Reduce Stress Concentration (S.C.) when irregularity present (try to smoothen the stress flow):

(i)



(ii)





2.2.1 Stress Concentration factor

Stress Concentration factor:

(a) Theoretical stress concentration factor (K_t)

(b) Actual stress concentration factor (K_f)

Theoretical Stress Concentration Factor (K_t):

- To consider the effect of stress concentration and find out localized stresses, a factor called theoretical stress concentration factor is used. It is denoted by K_t. It is also called Geometry Stress Concentration Factor.
- It depends on the geometry of irregularity corresponding to type of loading and it will not depend on the material.
- The theoretical stress concentration factor and charts are based on photo-elastic analysis of the epoxy model.
- Since theoretical stress concentration factor is calculated for epoxy material (which is assumed 100% sensitive for stress concentration), hence it will relate maximum possible stress near irregularity due to stress concentration with nominal stress, because actual stress near irregularity will also depend upon material.
- Theoretical stress concentration factor is defined as:

$$K_t = \frac{\text{Maximum possible value of stress near irregularity}}{\text{Nominal stress obtained by elementary equations}} = \frac{\sigma_{max}}{\sigma_0} \Rightarrow \sigma_{max} = K_t \sigma_0 .$$

Where, $\sigma_0 \rightarrow$ nominal stress

$\sigma_{max} \rightarrow$ Maximum possible stress near irregularity due to stress concentration

- **Empirical relation:** Theoretical stress concentration factor for elliptical hole under axial load by assuming width of plate >>> size of hole:

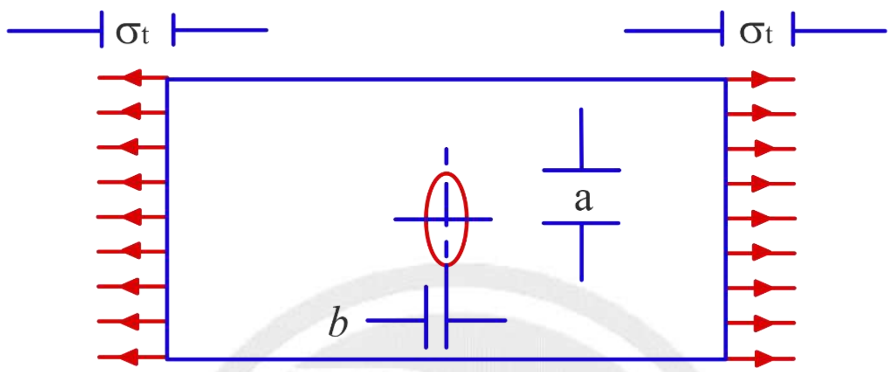


Fig.2.8: Stress concentration with elliptical hole

$$K_t = 1 + 2\left(\frac{a}{b}\right)$$

b $\rightarrow$ length of semi axis along the applied load.

a $\rightarrow$ length of semi axis perpendicular to applied load.

For circular hole: a = b: $K_t = 3$

Fatigue/Actual Stress Concentration Factor

- Experiments have shown that the actual stress concentration factor (K_f) is less than (K_t).
- It depends on the size of the stress concentration and the material.
- Fatigue stress concentration factor is defined as:

$$K_f = \frac{\text{Actual stress near irregularity due to stress concentration}}{\text{Nominal stress obtained by elementary equations}} = \frac{\sigma_a}{\sigma_0} \Rightarrow \sigma_a = K_f \sigma_0$$

Notch Sensitivity Factor

- The notch sensitivity of a material is a measure of how sensitive a material is towords notches or geometric discontinuities.
- It is defined as the ratio of increase of actual stress over nominal stress and increase of theoretical stress (or maximum possible stress due to stress concentration) over nominal stress.

$$q = \frac{K_f \sigma_0 - \sigma_0}{K_t \sigma_0 - \sigma_0} \Rightarrow q = \frac{K_f - 1}{K_t - 1} \qquad (0 < q < 1)$$

$$\Rightarrow \qquad K_f = 1 + q(K_t - 1)$$

For 100% sensitivity: $K_f = K_t$	For 0% Sensitivity: $K_f = 1$
$1 \le K_f \le K_t$	

Effect of Stress concentration on Ductile and Brittle material

(a) Effect of Stress Concentration on Ductile Material under Static Load.

Stress constriction effect can be neglected in ductile material under static load due to redistribution of stress near irregularity which occurs due to plastic deformation.

(b) Effect of Stress Concentration on brittle Material under Static Load.

Stress constriction effect cannot be neglected in brittle material under static load, because brittle material does not fail due to plastic deformation. In brittle material if we will neglect the stress concentration, then crack may develop near irregularity which propagates further and material can fail, hence we cannot neglect the effect of stress concentration in brittle material under static load.

(c) Effect of Stress Concentration on ductile and brittle material under fluctuating Load

We cannot neglect the effect of stress concentration in both ductile & brittle material under fluctuating load, because material under fluctuating Stress condition fails due to fatigue.

2.3 Design under completely reverse stress condition

2.3.1 Fatigue strength, life and endurance limit

Fatigue Strength (S_f):

The maximum possible magnitude of amplitude of completely reverse stress condition for life N number of stress cycles without fatigue failure is known as fatigue strength corresponding to N-cycles life.

Endurance Strength or Endurance Limit (S_e)

- Endurance strength of a material defined as the maximum amplitude of completely reversed stress that the material can sustain for an unlimited number of cycles without fatigue failure.
- It is determined either by rotating beam experiment or by using approximate relation between endurance strength and ultimate strength.

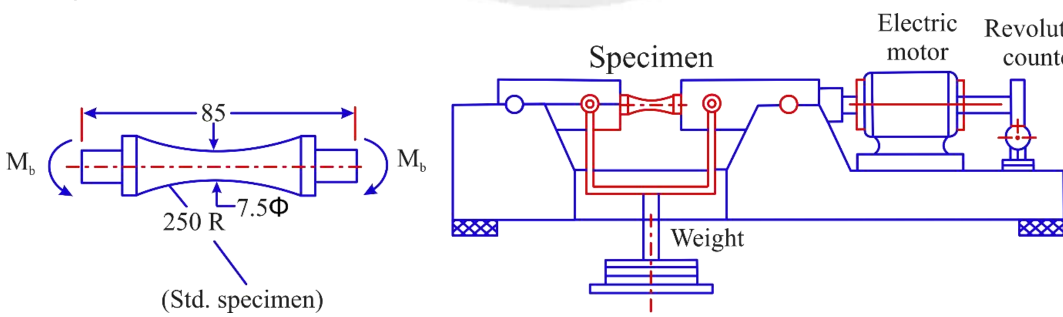


Fig.2.9: Rotating beam experiment

S-N Curve

- Through rotating beam experiment, S-N curve is drawn on log paper of base 10. S-N curve for ferrous Material is shown in figure:

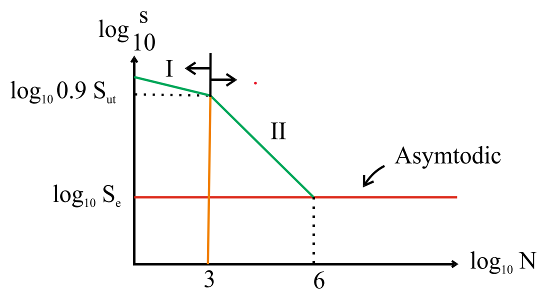


Fig.2.10: S-N Curve for ferrous material

I ⇒ Low Cycle Fatigue

II⇒ High Cycle Fatigue

I ⇒ Low Cycle Fatigue: Any fatigue failure when the number of stress cycles are less than 1000. Components are designed on the basis of ultimate tensile strength or yield strength with a suitable factor of safety.

II ⇒ High Cycle Fatigue: Any fatigue failure when the number of stress cycles are more than 1000. Components designed on the basis of fatigue strength for finite life and endurance strength for infinite life.

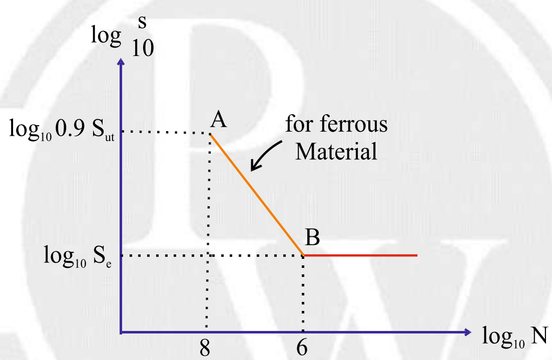


Fig.2.11: High cycle fatigue region of S-N curve for ferrous material

For ferrous material S-N curve becomes asymptotic corresponds to life 10^6 cycle (point B in figure 2.11), hence fatigue strength corresponds life 10^6 will be endurance strength for ferrous material.

S-N curve for Al:

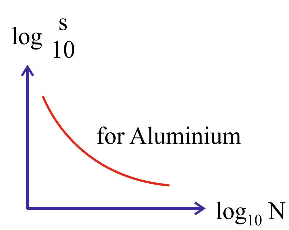


Fig.2.12: S-N curve for Al

Corrected Endurance Strength

- The endurance limit of actual component is different from the endurance limit of a rotating beam specimen due to a number of factors.
- The difference arises due to the fact that there are standard specifications and working conditions for the rotating beam specimen, while the actual components have different specifications and work under different conditions.
- Different modifying factors known as derating factor are used in practice to account this difference.

Corrected endurance strength, $S_e = k_a\ k_b\ k_c\ k_d\ k_e\ (S_e)_{std}$

k_a = Surface finish factor

k_b = Size factor

k_c = Reliability factor

k_d = Factor to consider stress concentration

k_e = Temp. Factor

$K_L \Rightarrow$ Load factor

(i) Surface finish factor (k_a): This factor depends upon the surface finish of the machine element. For mirror polished surface like specimen (k_a =1) and if the surface finish is poor (k_a < 1)

(ii) Size factor (k_b): This factor depends upon the size of the body if the size increases chances of irregularities increase and endurance strength decrease.

If $\phi \leq 7.5$mm, $k_b = 1$ and if $\phi \geq 7.5$mm, $k_b < 1$

(iii) Reliability factor (k_c)

- The endurance strength determined by rotating beam experiment or the approximated relation is only 50% reliable, means 50% components will survive with this endurance strength.
- To ensure that more than 50% components survive, the component must be designed with lower endurance strength.
- As the reliability (% of components that survive) increases, reliability factor (k_c) decreases.

(iv) Factor to consider stress concentration. (k_d)

$$k_d = \frac{1}{k_f}$$

$k_f \Rightarrow$ actual stress concentration factor

$k_f = 1 + q(k_t - 1)$

If q not given; then take $q = 1$

(v) Load factor (K_L)

From testing endurance strength is calculated for completely reversed bending stress condition

For completely reversed axial stressed condition we consider load factor which is generally 0.8. In exam load factor will be given and if not given then take load factor =1.

Design Process:

Completely reversed Stress
$(\sigma_m = 0, \sigma_a = \sigma)$

Infinite life
$(\geq 10^6$ cycles)

Finite life
$(< 10^6$ cycles)

2.3.2 Design under completely Reversed stress for infinite life

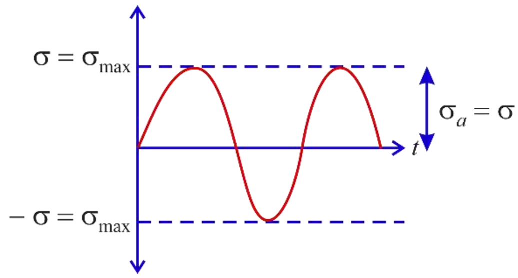


Fig.2.13: Completely Reversed stress

For infinite life: Criteria of failure is corrected endurance strength (*Se*)

Safe condition:

Amplitude stress of critical point under completely reverse stress condition should be less than corrected endurance strength (S_e).

$$\sigma_a \le S_e$$

For design:

$$\sigma_a \le \frac{S_e}{FOS}$$

2.3.3 Design under completely Reversed stress for finite life (say N)

For finite life: Criteria of failure is fatigue strength corresponds to required life (S_f).

Safe condition:

Amplitude stress of critical point under completely reverse stress condition should be less than fatigue strength corresponds to required life (S_f).

$$\sigma_a \le S_f$$

For design:

$$\sigma_a = \frac{S_f}{FOS}$$

2.3.4 Miner's Equation (Cumulative Damage in Fatigue)

Stress	Fraction	Individual Life	Actual Life for that ± σ acting
$\pm \sigma_1$	α_1	N_1	$n_1 = \alpha_1 N$
$\pm \sigma_2$	α_2	N_2	$n_2 = \alpha_2 N$
$\pm \sigma_3$	α_3	N_3	$n_3 = \alpha_3 N$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \Rightarrow \sum \alpha_i = 1$$

$$\frac{\alpha_1}{N_1} + \frac{\alpha_2}{N_2} + \frac{\alpha_3}{N_3} = \frac{1}{N} \Rightarrow \sum \frac{\alpha_i}{N_i} = \frac{1}{N}$$

2.4 Design under fluctuating stress condition other than completely reverse stress condition

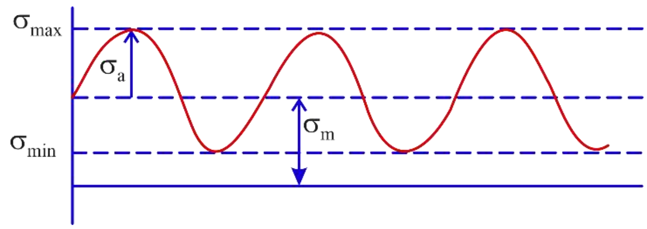


Fig.2.14: Fluctuating stress condition other than completely reverse stress condition

Mean and amplitude stress at critical point

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}, \; \sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}$$

For fluctuating stress condition other than completely reverse stress condition, there are infinite combinations are possible and for all combinations testing is not a good idea. So, scientists have given their criteria to design under fluctuating load condition like Soderberg criteria, Goodmen criteria, Gerber criteria etc.

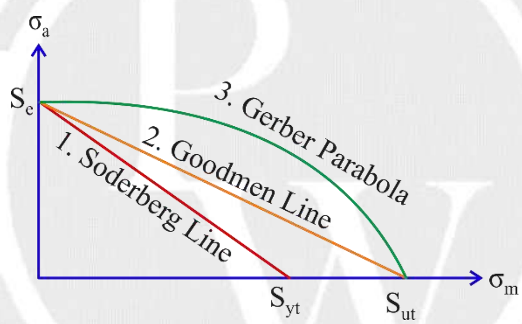


Fig.2.15: Soderberg, Goodmen and Gerber parabola

2.4.1 Soderberg criteria/Line

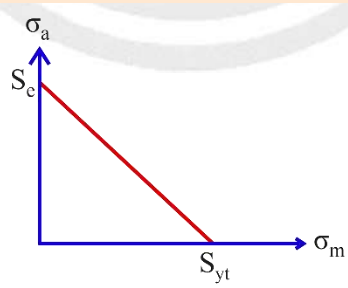


Fig.2.16: Soderberg line

Equation of Soderberg line:

$$\frac{\sigma_m}{S_{yt}} + \frac{\sigma_a}{S_e} = 1$$

For design: use FOS = n and replace S_{yt} by $\frac{S_{yt}}{n}$ and S_e by $\frac{S_e}{n}$.

$$\frac{\sigma_m}{\frac{S_{yt}}{n}} + \frac{\sigma_a}{\frac{Se}{n}} = 1$$

$$\Rightarrow \qquad \frac{\sigma_m}{S_{yt}} + \frac{\sigma_a}{S_e} = \frac{1}{n} \qquad \text{(Soderberg criteria for design)}$$

2.4.2 Goodmen criteria/Line

Goodmen Criteria

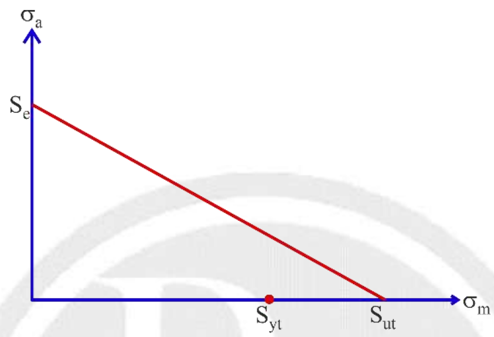


Fig.2.17: Goodmen line

Equation of Goodmen Line:

$$\frac{\sigma_m}{S_{ut}} + \frac{\sigma_a}{S_e} = 1$$

For design: use FOS = n and replace S_{ut} by $\frac{S_{ut}}{n}$ and S_e by $\frac{S_e}{n}$.

$$\frac{\sigma_m}{\frac{S_{ut}}{n}} + \frac{\sigma_a}{\frac{Se}{n}} = 1$$

$$\Rightarrow \qquad \frac{\sigma_m}{S_{ut}} + \frac{\sigma_a}{S_e} = \frac{1}{n} \qquad \text{(Goodman criteria of design)}$$

2.4.3 Gerber criteria/Parabola

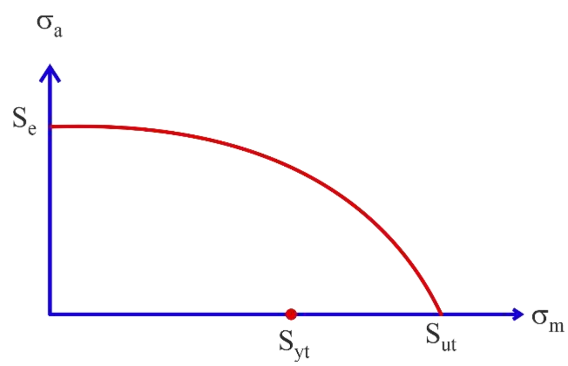


Fig.2.18: Gerber Parabola

Equation of Gerber Parabola

$$\left(\frac{\sigma_m}{S_{ut}}\right)^2 + \frac{\sigma_a}{S_e} = 1$$

For design: use FOS = n and replace S_{ut} by $\frac{S_{ut}}{n}$ and S_e by $\frac{S_e}{n}$.

$$\left(\frac{n.\sigma_m}{S_{ut}}\right)^2 + \frac{n.\sigma_a}{S_e} = 1 \qquad \textbf{(Gerber criteria of design)}$$

2.4.4 Comparison of Soderberg, Goodmen & Gerber Criteria

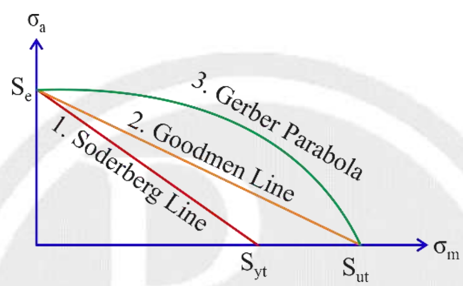


Fig.2.19: Comparison of Soderberg, Goodmen, Gerber Parabola

Region of Safety

Soderberg < Goodmen < Gerber

Key points:

- Criteria whose reason of safety is smaller is safer $\xrightarrow[\text{Safety}\downarrow]{\text{Soderberg > Goodmen > Gerber}}$.
- For any given stress condition FOS will be less for safer criteria $\xrightarrow[\text{FOS}]{\text{Gerber > Goodmen > Soderberg}}$.
- For any given FOS, design will be strong for more safer criteria.
- More safer criteria are less Economical

2.4.5 Langer criteria/Line

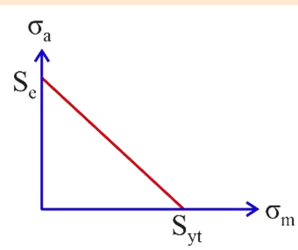


Fig.2.20: Langer criteria

Assumption:

Material fail only due to yielding. If σ_m is more then it gives good results but if σ_a is more than it does not give accurate result.

Equation of Langer Line.

$$\frac{\sigma_m}{S_{yt}} + \frac{\sigma_a}{S_{yt}} = 1$$

For design: use FOS = n and replace S_{yt} by $\frac{S_{yt}}{n}$.

$$\frac{\sigma_m}{S_{yt}} + \frac{\sigma_a}{S_{yt}} = \frac{1}{n} \qquad \textbf{(Langer's criteria for design)}$$

2.4.6 Modified Goodmen Criteria

Safer criteria out of Goodmen and Langer's criteria is modified goodmen criteria.

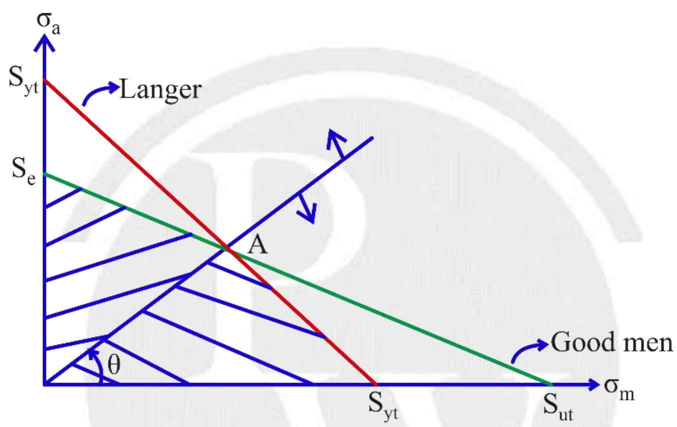


Fig.2.21: Modified Goodmen Criteria

To find FOS: Lesser FOS out of Langer criteria and Goodmen criteria will be FOS by Modified Goodmen criteria.

To design: Stronger design out of Langer criteria and Goodmen criteria will be design by Modified Goodmen criteria.

2.4.7 ASME Criteria/Ellipse

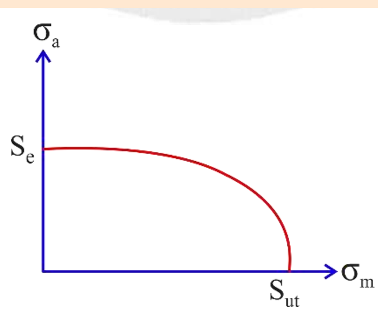


Fig.2.22: ASME criteria

Equation of ASME Ellipse;

$$\left(\frac{\sigma_m}{S_{ut}}\right)^2 + \left(\frac{\sigma_a}{S_e}\right)^2 = 1$$

For design: use FOS = n and replace S_{ut} by $\frac{S_{ut}}{n}$ and S_e by $\frac{S_e}{n}$.

$$\left(\frac{\sigma_m}{S_{ut}}\right)^2 + \left(\frac{\sigma_a}{S_{ut}}\right)^2 = \frac{1}{n^2}$$ $\Rightarrow$ **ASME criteria for design.**

2.4.8 Design Against Fluctuating Stress condition under combined loading condition

- We have studied Soderberg; Goodman etc. criteria for the 1–D fluctuating loading condition, but here 2-D & 3-D stress are present.
- In order to use the above criteria, we need to convert 2-D & 3-D Stress condition into equivalent 1-D stress condition using theory of failure for equivalent stress studied in static loading one by one for both amplitude stress condition & Mean Stress conditions.
- After that we can easily design using the suitable criteria for fluctuating stress condition.

2.5 Significance of mean compressive stress under fluctuating stress condition

$\sigma_a \Rightarrow$ always +ve

$\sigma_m \Rightarrow$ can be +ve; 0, –ve

If σ_m is +ve; it will lead to propagation of crack & will result in fracture.

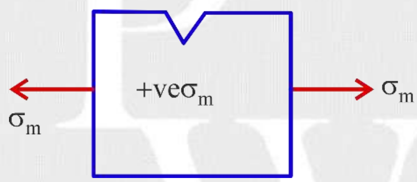


Fig.2.23: Propagation of crack under tensile stress

And if σ_m is –ve; it will not try to propagate the crack, hence results in less chance of failure.

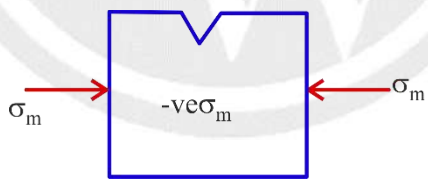


Fig.2.24: Under compressive stress

In order to increase the fatigue strength under fluctuating load we can induce the residual compressive stress by process like cold working; burnishing; shot pinning; case hardening; Cloning etc.

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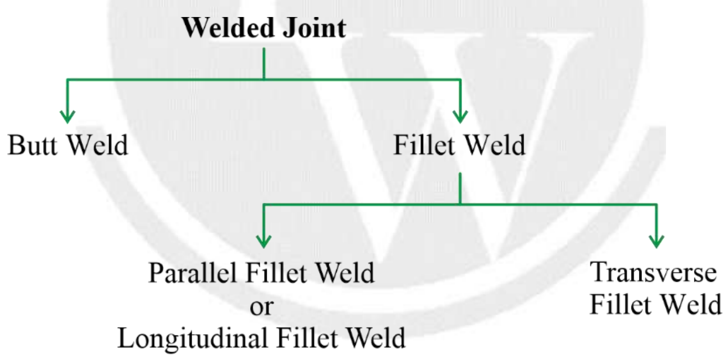
3 DESIGN OF WELDED JOINT

3.1 Introduction of Welded Joints

A welded joint is a permanent joint which is obtained by the fusion of the edges of the two parts to be joined together, with or without the application of pressure and a filler material.

- It is stronger than riveted joint.
- Weight addition is less compared to riveted joints
- Cheaper than riveted joints
- Due to application of heat during welding, material properties changes
- Stress concentration occurs due to uneven heating.

3.2 Classification of Welded Joints



3.2.1 Butt weld

When two plates which are in same plane are welded together, then weld is known as Butt weld.

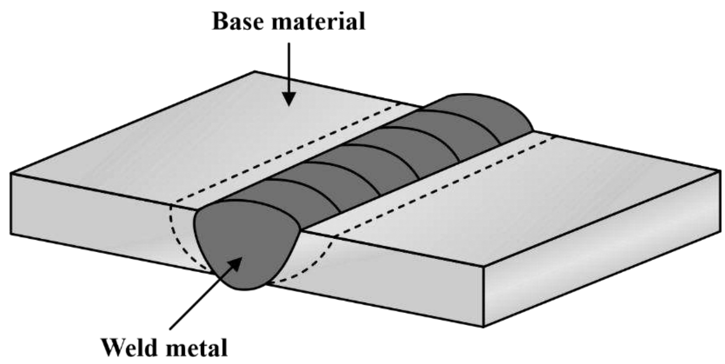


Fig.3.1: Butt weld

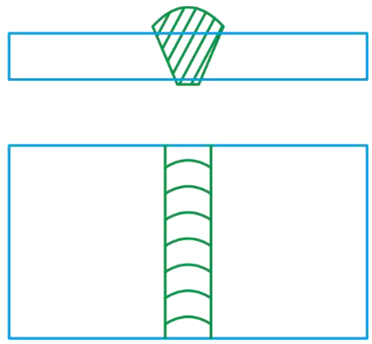


Fig. 3.2: Front and top view

3.2.2 Fillet Weld

Two plates kept in overlapping planes.

Filled weld is further classified in two types:

1. **Parallel fillet weld**

   When direction of the weld length is parallel to the direction of the force acting on the joint, then it is called a parallel joint.

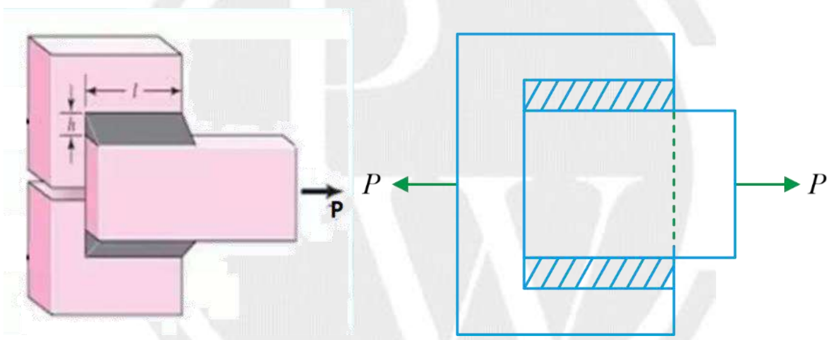


Fig. 3.3: Parallel fillet weld

2. **Transverse fillet weld**

   When direction of the weld length is perpendicular to the direction of the force acting on the joint, then it is called a transverse joint.

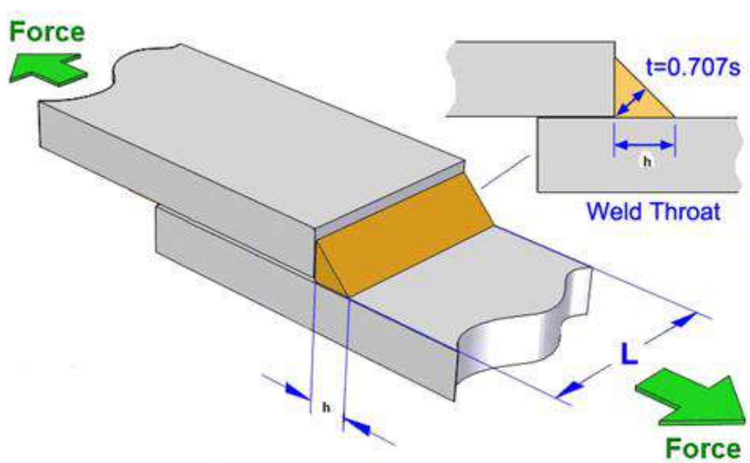


Fig. 3.4: Transverse fillet weld

Terminologies of Fillet weld:

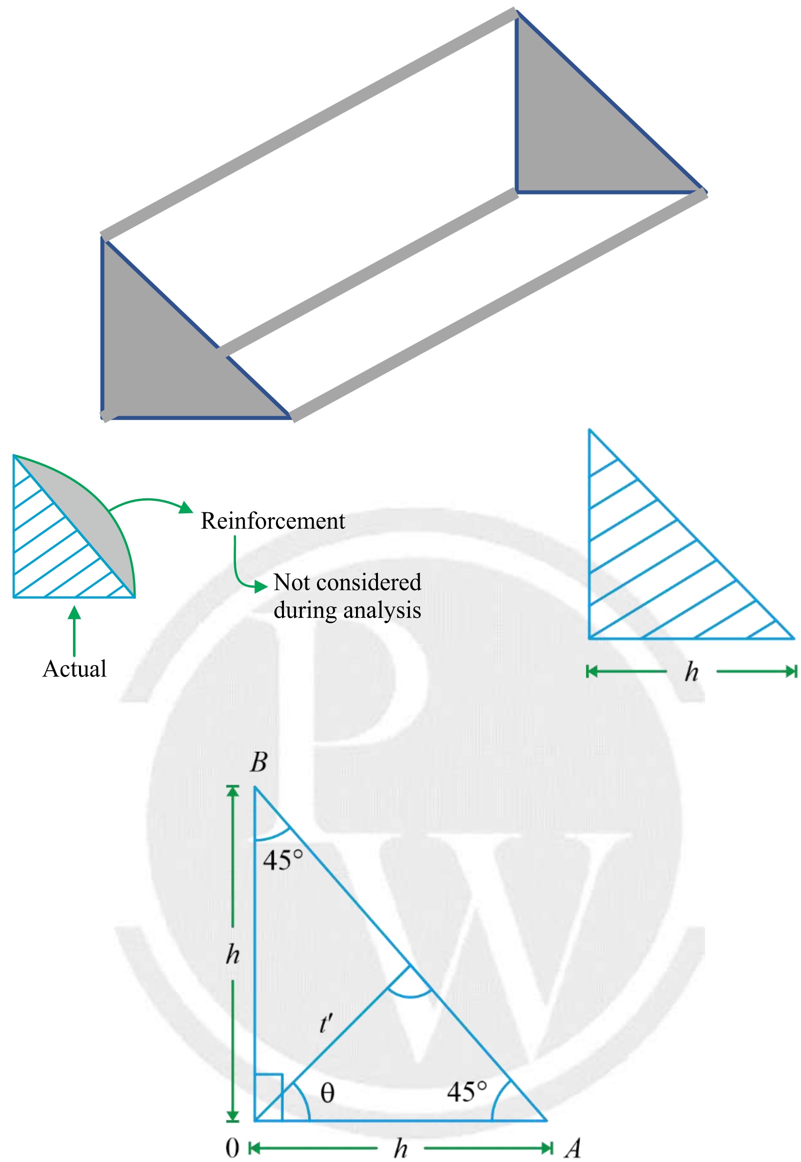


Fig. 3.5: Terminology of fillet weld

L = Length of weld

h = size or leg of fillet weld

$t' = \dfrac{h}{\sin\theta + \cos\theta}$ = thickness at θ angle from leg

$t = \dfrac{h}{\sqrt{2}}$ = throat of the fillet weld (or minimum thickness of fillet weld) = thickness at $\theta = 45^0$

A_t = Area of throat = tL

3.3 Analysis of Butt weld

Fig.3.6: Butt weld under force P

Minimum thickness of the weld is called throat. Which is thickness t of the weld as shown in figure. In throat reinforcement part is not consider because it is provided to compensate the flaws in the weld.

$\therefore$ Tensile stress in the weld is,

$$\sigma_t = \frac{P}{tL} \Rightarrow \boxed{P = \sigma_t \, t \, L}$$

Where,

σ_t = tensile stress in the weld

L = length of the weld

t = throat thickness

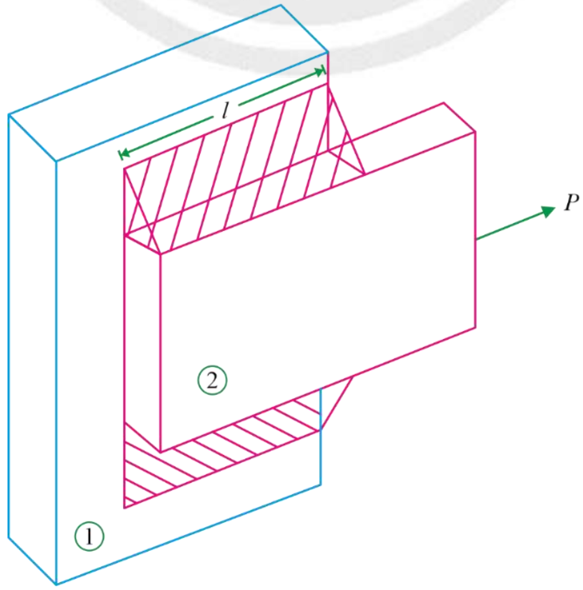
P = axial force in weld

If efficiency (η) of the joint is given, then,

$$\boxed{P = \sigma_t \, t L \eta}$$

3.4 Analysis of Fillet weld

3.4.1 Symmetrically loaded parallel fillet weld



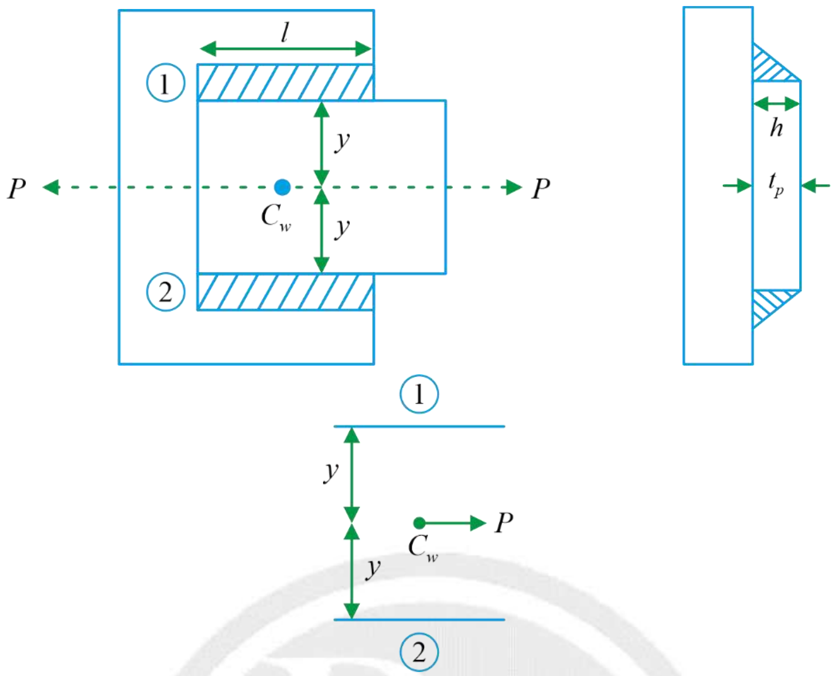


Fig.3.7: Symmetrically loaded Parallel fillet weld

l = Length of weld

h = size or leg of fillet weld = t_p

t $= \dfrac{h}{\sqrt{2}}$ = throat of the fillet weld (or minimum thickness or fillet weld) = thickness at $\theta = 45^0$

A_t = Total area of throat = 2 *tl*

- Since line of action of force *P* is passing through combine centroid of the weld, hence only primary stress (or direct stress) will induce in the weld.
- Whenever force *P* is parallel to the weld, shear stress will be induced in the weld which will be maximum at throat and expression to calculate shear stress at throat is.

$$\tau_{max} = \frac{P}{\text{total area of throat}}$$

$$\Rightarrow \qquad \tau_{max} = \frac{P}{2\,t\,l}$$

For design; $\tau_{max} \le \tau_P$ (Permissible shear stress)

$$\Rightarrow \qquad \frac{P}{2\,t\,l} \le \tau_P$$

$$\Rightarrow \qquad \boxed{P \le \tau_P\,(2tl)}$$

Note:
In welded joint:
- Stress due to force = **Primary stress**
- Stress due to moment = **Secondary stress**

3.4.2 Unsymmetrical loaded parallel fillet weld

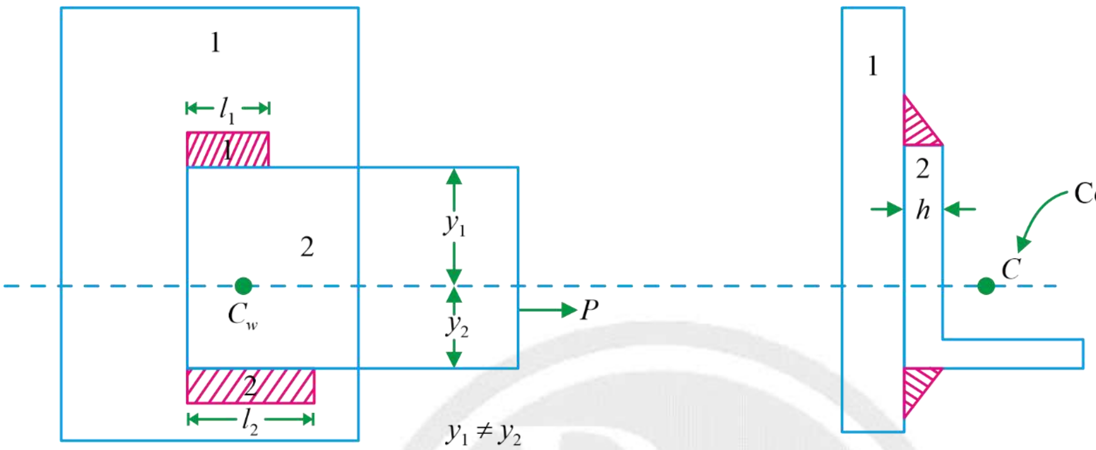


Fig.3.8: Unsymmetrical loaded parallel fillet weld.

- To ensure that no secondary stress (stress due to moment) should induce in the weld, line of action of force P pass through the combine centroid of the weld and for that following condition must be satisfied:

$$\boxed{l_1 y_1 = l_2 y_2}$$

- Since force P is passing through combine centroid of weld, only primary shear stress will induce in the weld, whose expression is

$$\Rightarrow \quad \tau_{max} = \frac{P}{\text{total area of throat}}$$

$$\Rightarrow \quad \tau_{max} = \frac{P}{(l_1 + l_2)\, t}$$

$$\Rightarrow \quad \boxed{P = \tau_{max} \times (l_1 + l_2)\, t}$$

3.4.3 Transverse fillet weld

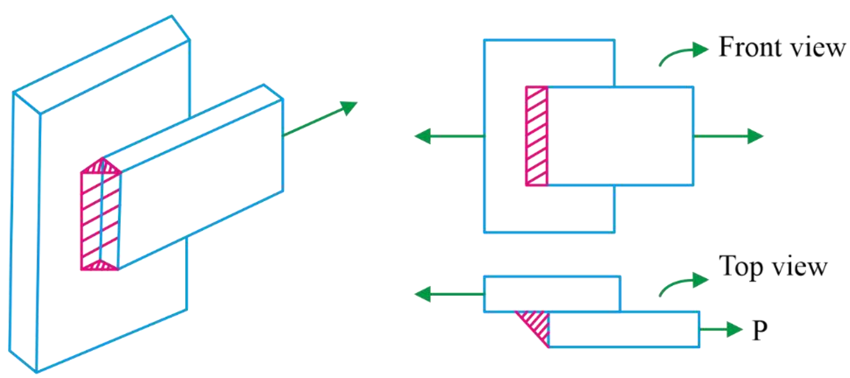
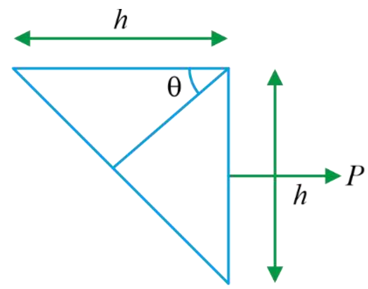


Fig.3.9: Analysis of Transverse fillet weld



- Both normal and shear stress induces in the weld.
- Actual maximum shear stress in the weld: $\tau_{max} = \frac{1.207\,P}{hL}$ [at θ = 22.5°]
- Actual maximum tensile stress in the weld: $\sigma_{t\,max} = \frac{1.207\,P}{hL}$ [at θ = 67.5 °]
- To be on safer side, instead of calculating actual maximum shear and tensile stress due to force P in transverse fillet weld, we calculate primary shear stress and tensile stress in the weld by dividing throat area in the force P as given below during analysis.

Maximum shear stress, $$\tau_{max} = \frac{P}{A_t}$$

⇒ $$\tau_{max} = \frac{P}{tL} = \frac{\sqrt{2}P}{hL}$$

⇒ $$\boxed{P = \tau_{max} \times \left(\frac{h}{\sqrt{2}} L\right)}$$

Maximum tensile stress,

$$\sigma_{t\,max} = \frac{P}{A_t} = \frac{P}{tL}$$

$$P = \sigma_{t\,max}\, tL$$

⇒ $$\boxed{P = \sigma_{t\,max} \frac{h}{\sqrt{2}} L}$$

3.5 Eccentric loaded weld joint

3.5.1 Eccentric loaded weld joint type I

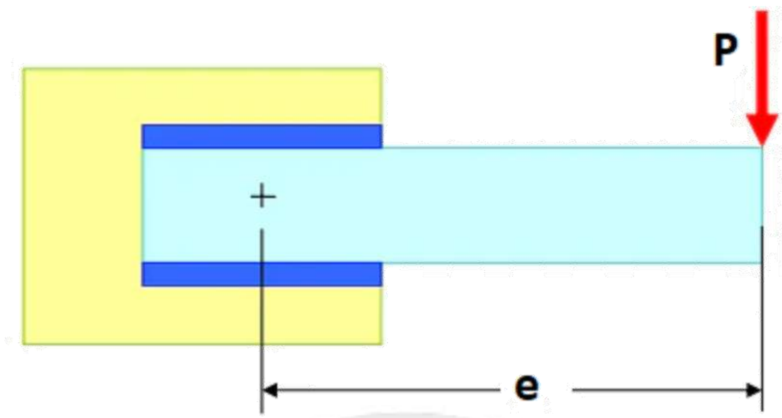


Fig. 3.10: Eccentric loaded weld joint – Type I

Any eccentric force can be replaced by an equal and same direction force (P) acting through the Centre of Gravity (COG) + a couple (M = P · e) lying in the same plane.

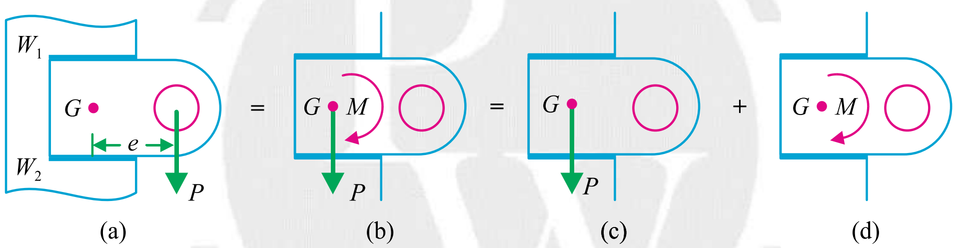


Fig.3.11: Shifting of eccentric load at center of gravity

The total stress felt by this weldment will be:

(1) Due to force P

(2) Due to moment M = P·e

Effect of Force P

- Primary shear stress (τ′ will induce)
- **Magnitude**: Primary shear stress will be same at every point as given below,

$$\boxed{\tau' = \frac{P}{\text{total area of throat}} = \frac{P}{A_t}}$$

- **Direction:** Direction of primary shear will be same at each point and will be opposite to force *P*.

  In figure 3.12 direction of primary shear stress of point *A* is drawn.

Effect of Moment M = P × e

- Due to moment $M = P \times e$, secondary shear stress (τ'') will induce in the weld.
- **Magnitude:**

$$\tau'' = \frac{M}{J} r$$

Where, J = Polar moment of inertia of the weld about COG.

τ'' = Secondary shear stress at a distance r from COG.

- **Direction:**

Step I: Draw a line which is perpendicular to the line joining the point whose secondary shear stress we are drawing and COG.

Step II: In this line take the sense of secondary shear stress such that it should try to rotate the weldment about COG opposite to the sense in which moment $M = P \times e$ it trying to rotate the weldment.

In figure 3.12 direction of secondary shear stress of point A is drawn.

Resultant Shear Stress

Find the vector resultant of primary and secondary shear stress.

Resultant shear stress of point A,

$$\boxed{\tau_A = \sqrt{(\tau'_A)^2 + (\tau''_A)^2 + 2\tau'_A\,\tau''_A \cos\theta_A}}$$

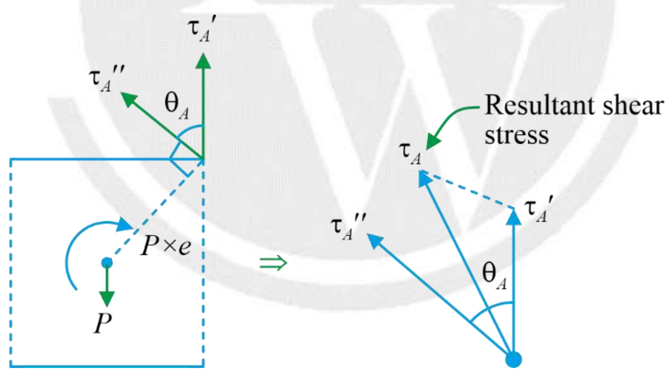


Fig.3.12: Shear stress at point A

Critical Point

During design we need to find resultant shear stress on critical point. Critical point is the point, where resultant shear stress is maximum.

Steps to find critical point:

Step I: Consider all the farthest points from COG.

Step II: Draw the direction of primary and secondary shear stress in these points.

Step III: In the selected points mark the points whose distance from COG (i.e. r) is maximum.

Step IV: In the selected points mark the points whose vector angle between primary and secondary shear stress (i.e. θ) is minimum.

Step V: Points where r is maximum and θ is minimum (i.e., common points of step III and IV) are critical points.

Note:

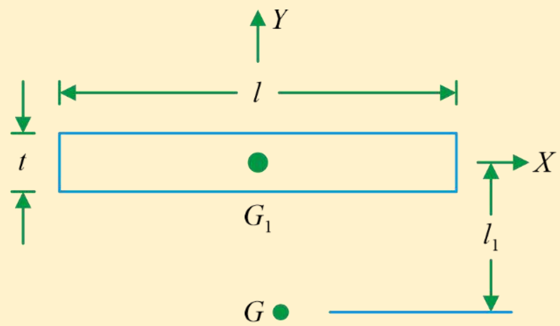
Method of find polar moment of inertia

Here, l = length of weld

t = throat of weld

G_1 = centre of gravity of weld

G = centre of gravity of group of welds



$$\therefore \quad I_{xx} = \frac{lt^3}{12}$$

$$I_{yy} = \frac{tl^3}{12}$$

$$J_{G1} = I_{xx} + I_{yy} = I_{yy} \qquad [I_{xx} \text{ is very small}]$$

$$\Rightarrow \quad J_{G1} = \frac{tl^3}{12}$$

$\therefore$ By Parallel axis theorem,

$$J_G = J_{G1} + Ar^2 = \frac{tl^3}{12} + ltr_1^2 = lt\left[\frac{l^2}{12} + r_1^2\right]$$

If there are no of weld with polar moment of inertia, $J_1, J_2, J_3 \ldots\ldots J_n$, then

$$J_{\text{resultant}} = J_1 + J_2 + J_3 + \ldots\ldots + J_n$$

3.5.2 Eccentric loaded weld joint type II

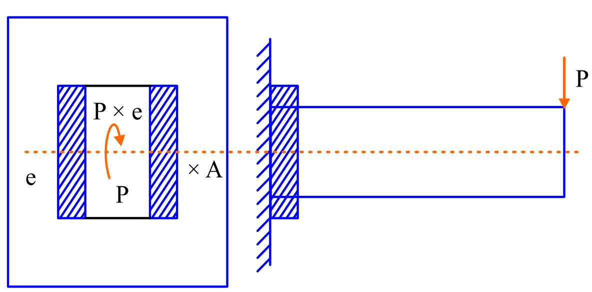


Fig.3.13: Eccentric loaded weld joint – Type II

Effect of Force P

Due to force P primary shear stress (τ') will induce in the weldment which is calculated by,

$$\tau' = \frac{P}{\text{Area of throat}} = \frac{P}{A_t}$$

Effect of bending moment ($M_b = P \times e$):

Due to moment ($M_b = P \times e$) bending stress or secondary normal stress will induce in the weldment which is calculated by

$$\sigma_b = \frac{M_b}{I} y$$

where,

I=MOI of combine weld about neutral axis (which is centroidal x axis in this case)

y=Perpendicular distance from neutral axis.

For design:

- Calculate primary shear stress (τ') and maximum bending stress ($\sigma_{b,m}$) i.e. bending stress at the point where y is maximum.

$$\tau' = \frac{P}{A_t} \;\&\; \sigma_{b,m} = \frac{M_b}{I} y_{\max}$$

- Find principal stresses and maximum shear stresses at the critical point and use appropriate Theory of failure for design.

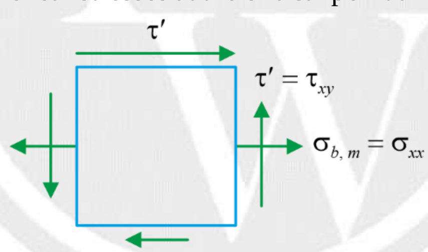


Fig.3.14 Stress condition at critical point

$$\sigma_{xx} = \sigma_{b,m}$$

$$\sigma_{yy} = 0$$

$$\tau_{xy} = \tau'$$

Principle Stresses

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$\Rightarrow$

$$\boxed{\sigma_{1,2} = \frac{\sigma_{b,m}}{2} \pm \sqrt{\left(\frac{\sigma_{b,m}}{2}\right)^2 + (\tau')^2}}$$

and $\sigma_3 = 0$

Maximum shear stress,

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\Rightarrow \quad \tau_{max} = \sqrt{\left(\frac{\sigma_{b,m}}{2}\right)^2 + (\tau')^2}$$

3.5.3 Important relation for circumferential weld

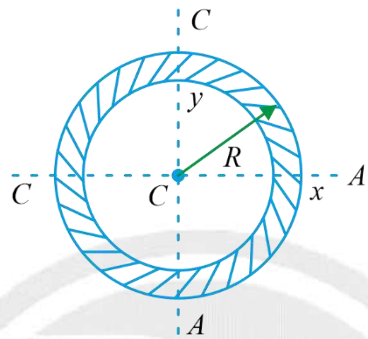


Fig.3.15: Circumferential weld

t = Throat of weld

A_t = Area of throat = $2\pi Rt$

$I_{CXA} = I_{CYA} = \pi + \pi t R^3$

J = Polar MOI about COG

$= 2\pi\, t\, R^3$

Circumferential fillet weld under torsion:

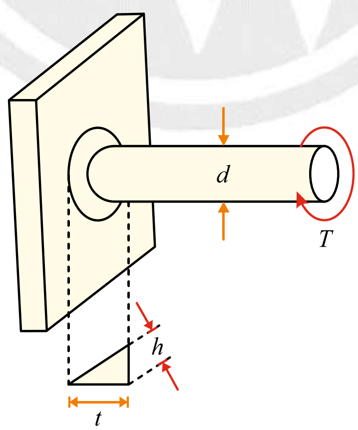


Fig.3.16: Circumferential weld under torsion

Maximum shear stress induces in the weld,

$$\tau_{max} = \frac{T}{J} r_{max}$$

$$\Rightarrow \quad \tau_{max} = \frac{T}{2\pi t R^3} \times R = \frac{T}{2\pi t R^2}$$

$$\Rightarrow \quad \tau_{max} = \frac{T}{2\pi\left(\frac{h}{\sqrt{2}}\right)\left(\frac{d}{2}\right)^2}$$

$$\Rightarrow \quad \boxed{\tau_{max} = \frac{2\sqrt{2}\,T}{\pi h d^2}}$$

Circumferential fillet weld under bending:

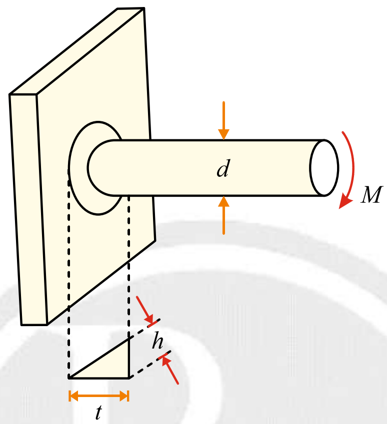


Fig.3.17: Circumferential weld under bending

Maximum bending stress induce in the weld,

$$\sigma_{b,max} = \frac{M}{I}\,y_{max}$$

$$\Rightarrow \quad \sigma_{b,max} = \frac{M}{\pi t R^3} \times R = \frac{M}{\pi t R^2}$$

$$\Rightarrow \quad \sigma_{b,max} = \frac{M}{\pi\left(\frac{h}{\sqrt{2}}\right)\left(\frac{d}{2}\right)^2}$$

$$\Rightarrow \quad \boxed{\sigma_{b,max} = \frac{4\sqrt{2}\,M}{\pi h d^2}}$$

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4 DESIGN OF BOLTED & RIVETED JOINT

4.1 Introduction of bolted joint

Threaded Joints has numerous applications; eg lead screw in Lathe; In Screw Jack; Bolted Joints etc. Here we would restrict our study to Bolted Joints Only.

Bolted / Threaded Joints are TEMPORARY JOINT because both plates and joint are reusable.

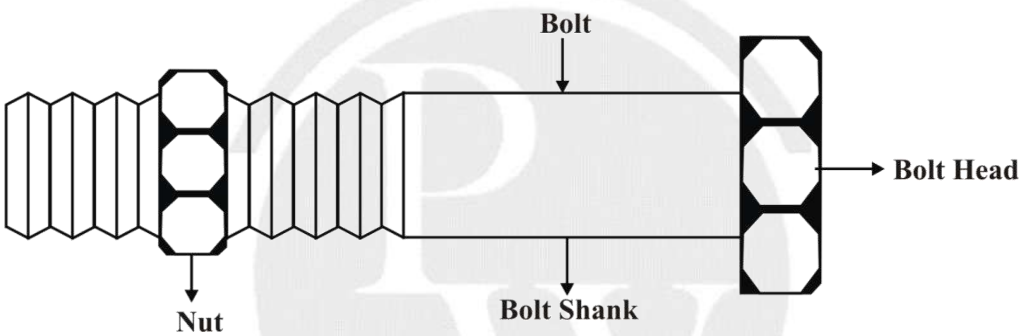


Fig.4.1: Bolted joint

4.1.1 Various terminology of bolted joint

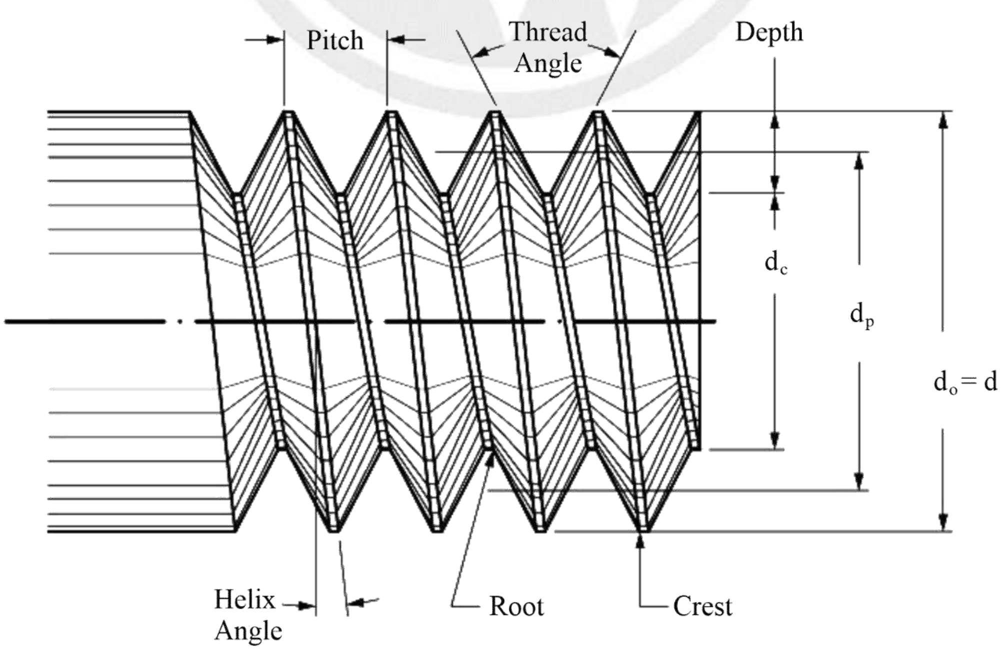


Fig.4.2: Terminologies of bolted joint

Terminologies of Screw Threads:

Crest: Peak point of thread profile
Root: Bottom-most point of thread profile
Major Diameter $(d_0 = d)$**:** Major diameter is the diameter of an imaginary cylinder that bounds the crest of an external thread. It is also called as nominal diameter

Minor/Core Diameter $(d_i = d_c)$**:** Minor diameter is the diameter of an imaginary cylinder that bounds the roots of an external thread. It is also called as core diameter.
Pitch (p)**:** Distance between two similar points on adjacent threads measured parallel to the axis of the thread.

Pitch diameter (d_p)**:** It is diameter of an imaginary cylinder the surface of which would pass through the threads at such points as to make the width of the threads (AB) equal to the width of the spaces cut by the surface of cylinder between two threads (BC). i.e. AB = BC.

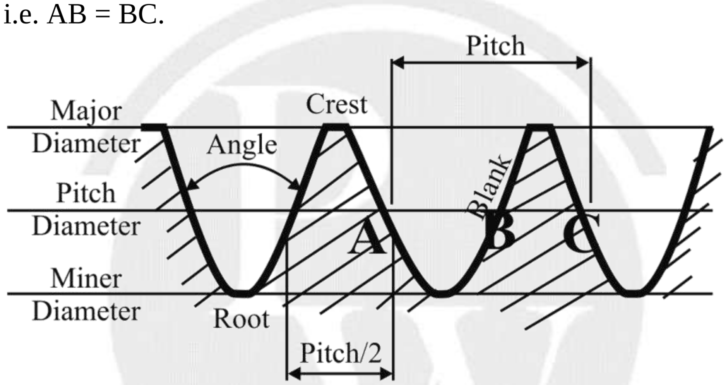


Fig.4.3: Threads of bolt

4.1.2 Various types of threads

(i) Square thread:

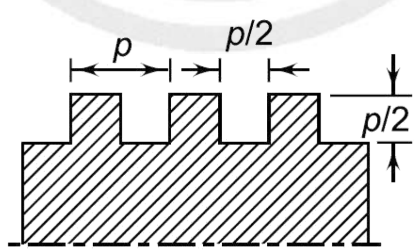


Fig.4.4: Square thread

- Used for transmission of power.
- Used in machine tool spindle, screw jack etc.
- Less wear
- Difficult to manufacture.
- Low strength.

(ii) Trapezoidal thread:

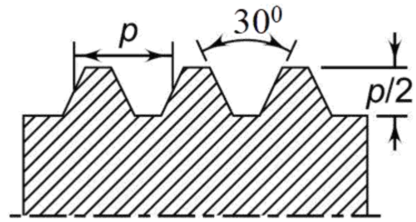


Fig.4.5: Trapezoidal thread

- Modification of square thread.
- Stronger than square thread due to more thickness at core.
- Easier to manufacture.
- More wear

(iii) Acme thread:

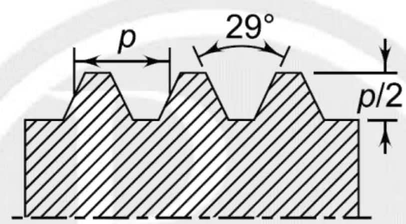


Fig.4.6: Acme thread

- Special type of trapezoidal thread called Acme Thread
- In an acme thread, the thread angle is 29° instead of 30°

(iv) Buttress thread:

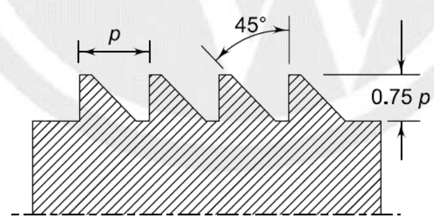


Fig.4.7: Buttress thread

- Combined advantages of square and ACME threads.

4.1.3 Lead (L)

Lead is the distance that the nut moves parallel to the axis of the screw, when the nut is given one turn. Or lead is axial movement due to 1 complete rotation of nut.

Lead = n * pitch

Where, n = number of starts in thread

n = 1 for single start thread

n = 2 for double start thread

n = 3 for triple start thread and so on

Axial movement $\left(x\right)$ due to θ_C angular rotation of nut or k turns of nut:

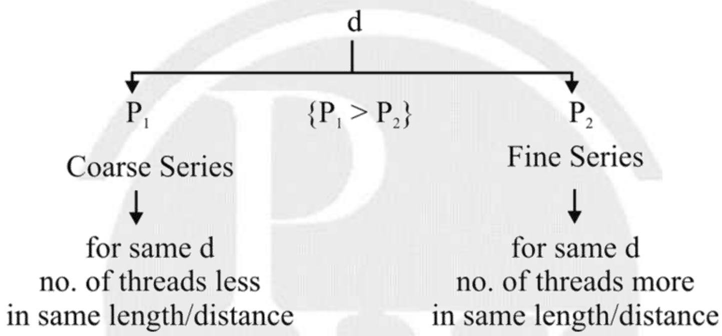
Angular rotation	Number of turns	Axial movement
2π	1	L
1 rad.	1	$\frac{L}{2\pi}$
θ_C	kL	$\frac{L}{2\pi}\theta_C$

$$x = \frac{L}{2\pi}\theta_c$$

4.1.4 Coarse and fine series of thread

There are two types of threads on the basis of their application:

1. **Coarse series-**recommended for general industrial applications
2. **Fine series-**recommended for sophisticated and delicate parts



Both these thread types have different designations:

1. **Coarse series-** Designated by the letter 'M' followed by the value of nominal diameter in mm. (Do not forget units)
   So, M12 means the nominal diameter of the thread is 12 mm.
2. **Fines Series-** Designated by the letter 'M' followed by the value of nominal diameter and then pitch in mm followed by 'X' symbol.
   So, M12 × 1.25 means nominal diameter = 12 mm and pitch = 1.25 mm.

Note:
The letter 'M' stands for ISO-Metric thread.

4.2 Preload in Nut & Bolt Assembly due to tightening

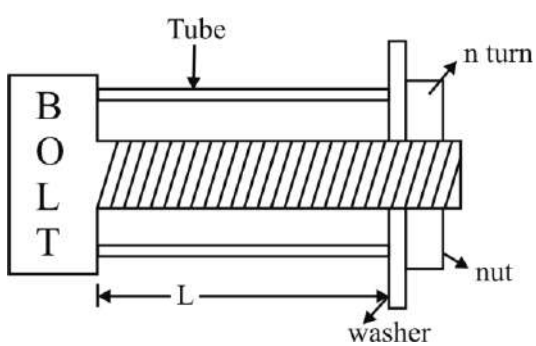


Fig.4.8: Preload in Nut and bolt

L_b = $L_t = L$

A_t = Cross sectional area of connecting member under compression due to tightening

A_b = Cross sectional Area of bolt or core area of bolt $\frac{\pi}{4}d_c^2$

E_{cm} = Young's Modulus of Elasticity of connecting member

E_b = Young's Modulus of Elasticity of bolt

Axial stiffness of bolt; $\boxed{k_b = \frac{A_b E_b}{L_b}}$

Axial stiffness of connecting member under compression due to tightening; $\boxed{k_{cm} = \frac{A_{cm}E_{cm}}{L_{cm}}}$

Axial movement of nut during tightening $= x$

- Initially the tube is at its free length i.e., washer is just touching the tube; No force is applied by nut & washer against bolt head to the tube.
- Due to tightening connecting member will be in compression and bolt will be in tension. Magnitude of force induced on the bolt is equal but opposite in nature to that of force induced in the connecting member due to tightening of the nut.

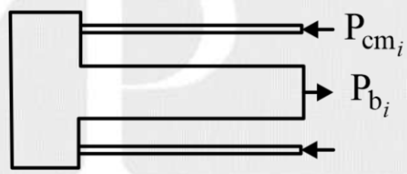


Fig.4.9: Axial force induced due to tightening

$$|P_{t_i}| = |P_{b_i}| = P_i$$

Compression Nature   Tensile Nature

Axial movement of nut during tightening:

$$x = |\text{Compression in conneting member}| + |\text{Elongation of bolt}|$$

$$\boxed{x = \left|\frac{P_{t_i}}{K_t}\right| + \left|\frac{P_{b_i}}{K_b}\right|}$$

$$\boxed{x = \frac{P_i}{K_t} + \frac{P_i}{K_b}}$$

Tensile stress on bolt due to preload = $(\sigma_{b_i})_{preload} = \frac{P_i}{A_b} = \frac{P_i}{\frac{\pi}{4}d_c^2}$

Compressive stress on connecting member due to preload = $(\sigma_{cm})_{preload} = \frac{P_i}{A_{cm}}$

Stresses in Bolt due to Bolt Preload:

1. Tensile stress in the cross-section: $\sigma_t = \dfrac{P_i}{\frac{\pi}{4} d_c^2}$

2. Torsional shear stress developed in the cross-section: $\tau = \dfrac{16T}{\pi d_c^3}$

3. Direct shear stress across the threads: $\tau = \dfrac{P_i}{\pi d_c t n}$

   Where, t = thickness of threads, n= no. of threads in contact with threads

4.3 Bolted Joint under External Load without Preload

4.3.1 Bolted Joint under External Tensile Load without Preload

Connecting members are connected by one bolted joint:

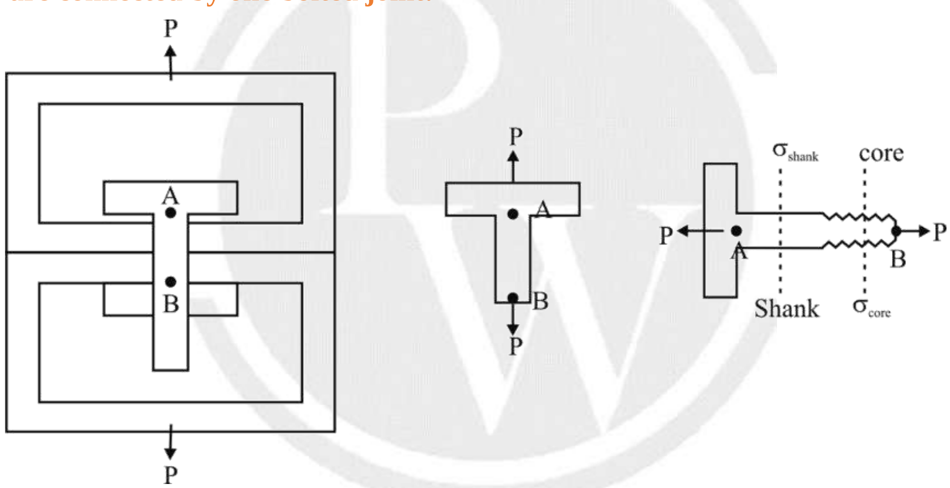


Fig.4.10: Bolted joint under tensile load without preload

- Without preload all the external tensile force will transfer in the bolts.
- Tensile force induced on bolt = P
- Maximum tensile stress induced in the bolt = $\dfrac{P}{A_c = A_b}$

  Where, $A_c = A_b$ = Core area of bolt

Connecting members are connected by n identical bolts:

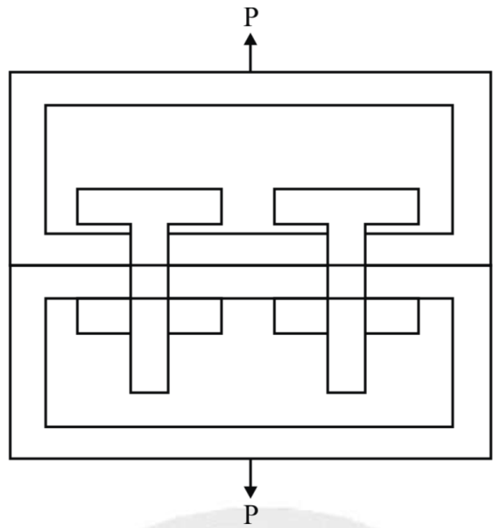


Fig.4.11: Connecting member with *n* identical bolts under external tensile load without preload

- Tensile force induced on each bolt = $\frac{P}{n}$
- Maximum tensile stress induced in the bolt = $\frac{P/n}{A_c = A_b}$

Where, $A_c = A_b$ = Core area of bolt

4.3.2 Bolted Joint under External Shear Load without Preload

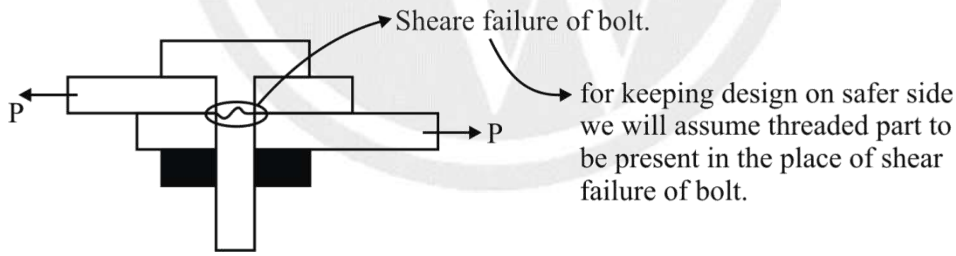


Fig.4.12: Bolted joint with external shear load without preload

$$\tau = \frac{P}{A_C}$$

For design $\tau \le \tau_P$

If more than one bolt is present;

$$\tau = \frac{P}{nA_C} \quad \{n \Rightarrow \text{no. of bolt}\}$$

$$\frac{P}{nA_C} \le \tau_P$$

4.4 Bolted Joint under External Tensile Load with Preload

4.4.1 Case I: Connecting members are connected by one bolted joint

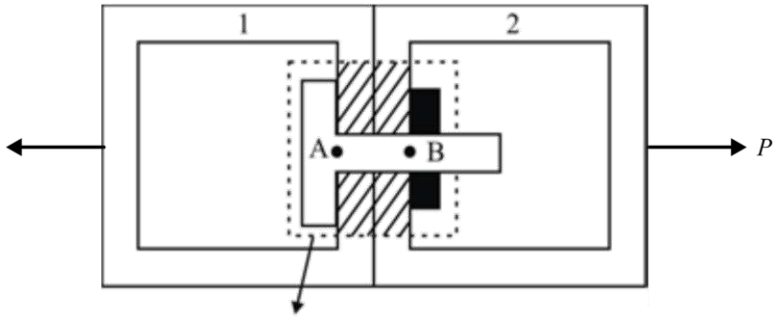


Fig.4.13: Bolted joint under external tensile load with preload

P_i = Preload on bolt

Effect of Preload on bolt = P_i (tensile)

Effect of preload on connecting member = – P_i (compressive)

K_b = Stiffness of bolt {only that part which elongates)

K_{CM} = Equivalent stiffness of that part of connecting members which is under contraction due to preload.

$C = \frac{K_b}{K_{CM} + K_b}$ = Stiffness correction factor

P_b = Effect of external load on bolt

P_{cm} = Effect of external load on part of connecting members which is under contraction due to preload

To find the effect of external tensile load with preload, part of connecting members which is under contraction due to preload and bolt will be treated as composite bar in parallel.

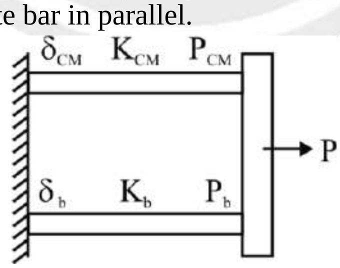


Fig.4.14: Effect of external tensile load with preload

Due to external load P,

$$\delta_{CM} = \delta_b = \delta$$

$$\frac{P_{CM}}{K_{CM}} = \frac{P_b}{K_b} = \frac{P}{K_{eq}} \quad \{K_{eq} = K_b + K_{CM})$$

$$\boxed{P_b = \left(\frac{K_b}{K_{CM} + K_b}\right) P = CP}$$

And,

$$P_{CM} = \left(\frac{K_{CM}}{K_{CM} + K_b}\right)P$$

$$P_{CM} = (1 - C)P$$

Total force on bolt: $(P_b)_{total} = P_i + P_b = P_i + CP$

Total load on connecting member: $(P_{cm})_{total} = -P_i + P_{cm} = -P_i + (1 - C)P$

Stress in bolt = $\sigma_b = \frac{(P_b)_{total}}{A_b} = \frac{P_i + CP}{A_b}$

Stress in connecting member = $\sigma_{CM} = \frac{(P_{cm})_{total}}{A_{CM}} = \frac{-P_i + (1 - C)P}{A_{CM}}$

For leak proof joint: $\sigma_{CM} < 0 \Rightarrow -P_i + (1 - C)P < 0$

4.4.2 Case II: Connecting members are connected by n identical bolts

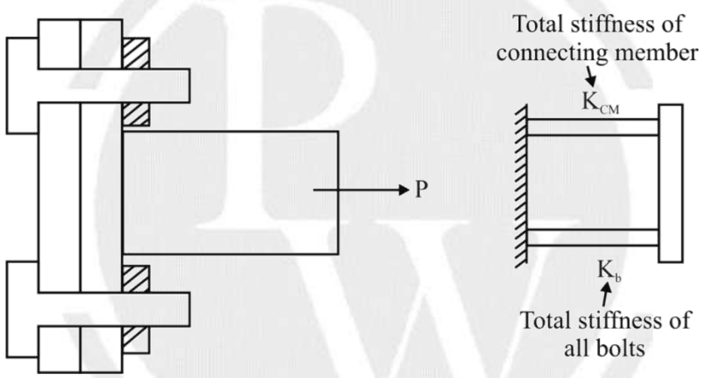


Fig.4.15: Connecting members with n identical bolts and under external tensile load with preload

Consider
n = Number of bolts
$K_{b'}$ = Stiffness of each bolt

$K_b = nK_{b'}$ = Equivalent stiffness of all bolts= Combine stiffness of all the bolts

$A_b = A_C = \frac{\pi}{4}d_C^2$ = Core area of each bolt.

P_i = Preload on each bolt

Effect of Preload on bolt = + P_i (tensile)

Effect of preload on connecting member = $-n P_i$ (compressive)

Due to external force P ($\because$ preload present) combine stiffness of all the bolt and the connecting member will be treated as composite bar in parallel.

Effect of external force on each bolt, $P_b = \frac{CP}{n}$

Effect of external force on connecting member, $P_{CM} = (1 - C)P$

Total force on each bolt: $(P_b)_{total} = P_i + P_b = P_i + \frac{CP}{n}$

Total load on connecting member: $(P_{cm})_{total} = -nP_i + P_{cm} = -nP_i + (1-C)P$

Stress in each bolt = $\sigma_b = \frac{(P_b)_{total}}{A_b} = \frac{P_i + \frac{CP}{n}}{A_b}$

Stress in connecting member = $\sigma_{CM} = \frac{(P_{cm})_{total}}{A_{CM}} = \frac{-nP_i + (1-C)P}{A_{CM}}$

For leak proof joint: $\sigma_{CM} < 0 \Rightarrow -nP_i + (1-C)P < 0$

4.5 Effect of preload under fluctuating external load condition

Preload on each bolt $= P_i$

Consider, External load fluctuating from 0 to P.

Then, in bolt, external load will fluctuate from 0 to CP (Assuming connecting members are connected with one bolted joint)

Total load fluctuation in bolt $= P_i$ to $(P_i + CP)$

$\sigma_{min} = \frac{P_i}{A_b}$ and $\sigma_{max} = \frac{P_i + CP}{A_b}$

Amplitude stress, $\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} = \frac{CP}{2A_b}$

Mean stress, $\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} = \frac{P_i}{A_b} + \frac{CP}{2A_b}$

If $P_i >> CP$, then $\sigma_a << \sigma_m$ and if $\sigma_a << \sigma_m$ then fluctuation of stress will be less with respect to mean stress. Hence preload reduces the effect of fluctuation.

Note:

Advantage of Preload

- Leak Proof Joint { $F_{CM} < 0 \Rightarrow [-nP_i + (1-c)P] < 0$ }
- Reduces fluctuation in Load
- Increases fatigue Life
- Reduces effect of External Load = $[(P_b)_{total} = P_i + CP]$

4.6 Bolt of uniform strength

Bolts of uniform strength:

If you look at any regular bolt, if the diameter of shank will be more than the effective diameter of the threaded region. The reason for this is the diameter of threaded part has somewhat reduced due to thread making. Hence, when a load P acts through the bolt, threaded part is stressed more, so the strain energy stored in threaded part will be more than the strain energy stored in unthreaded part. To increase the amount of strain energy that this bolt can store, we have to increase the stress in unthreaded (shank) part as well.

So, we make some changes in the bolt so that the value of stress is same at every cross section, be it threaded part or unthreaded part. Such an ideal bolt is known as **bolt of uniform strength.**

Option 1: Either by reducing the unthreaded cross-sectional area (to increase stress) by turning operation as you can see in image(A) of figure 4.16.

Option 2: Either by reducing the unthreaded cross-sectional area by drilling a hole through it as you can see in image (B) of figure 4.16.

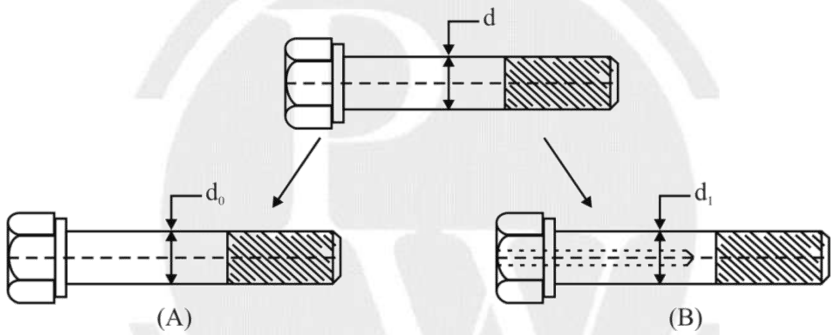


Fig.4.16: Bolts of uniform strength

Method in image (A) is preferred because:

Reason 1: Drills come in standard sizes so we cannot drill any hole of desired diameter directly. We might have to go for reaming after that. No such problem in turning.

Reason 2: Drilling a hole will cause unwanted stress concentration.

4.7 Introduction of riveted joint

A rivet has a cylindrical shank with a head at one end. It is used to produce permanent joints between two plates.

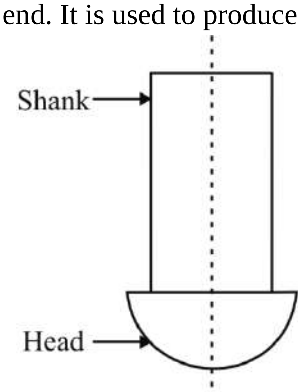


Fig.4.17: Rivet

Rivet is inserted into the holes of plates (which are to be joined) and then its protruding part is upset by a hammer.

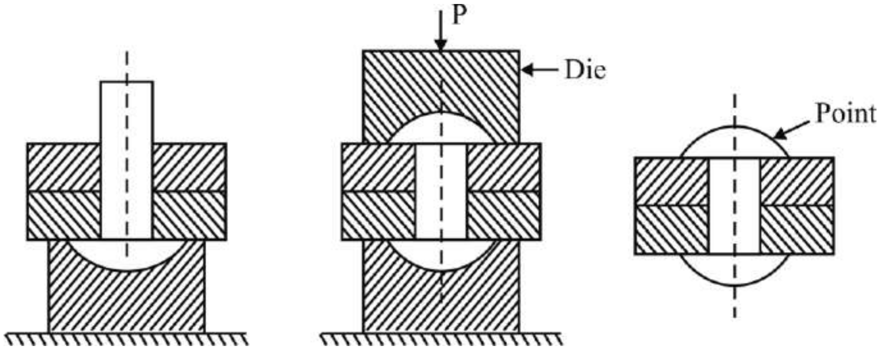


Fig.4.18: Riveting process

Joints made by rivets are called permanent because the parts can be only the dismantled by damaging rivets.

Note:
(i) Diameter of shank < Diameter of rivet hole
(ii) A rivet is specified by shank diameter of the rivet. A 20 mm rivet means a rivet having 20 mm as the shank diameter

4.7.1 Classification of riveted joint

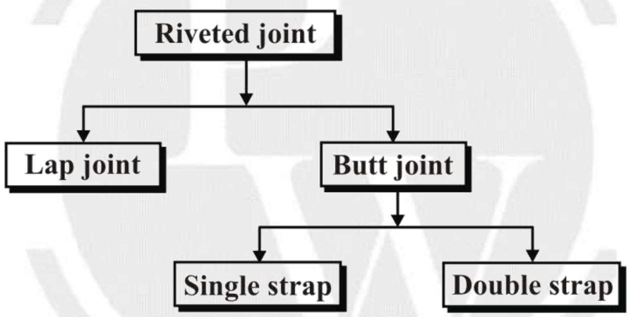


Fig.4.19: Classification of riveted joint

I. Lap Joint: Riveted joint between two overlapping plates.

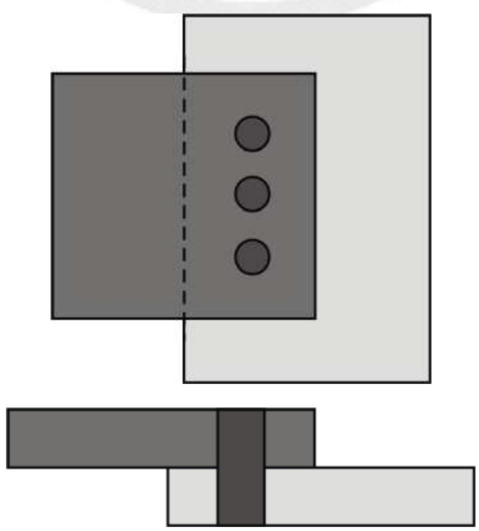


Fig.4.20: Lap Joint

II. Butt Joint: In this type of riveting, the plates to be joined are kept in the same plane, without forming an overlap. Another plate known as **cover or strap plate** is placed over either one side or on both side of the main plates, then it is riveted with main plates.

If only one cover plate is placed on the main plate, then that butt joint is known as **Single strap butt joint**.

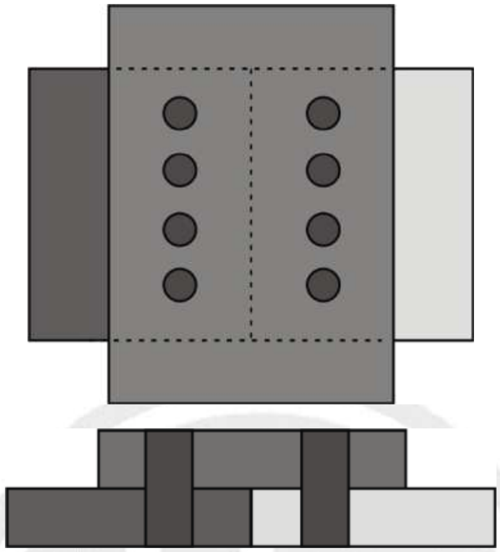


Fig.4.21: Single strap Butt joint

If cover or strap plate is placed on the both sides of main plate, then that butt joint is known as **Double strap butt joint**.

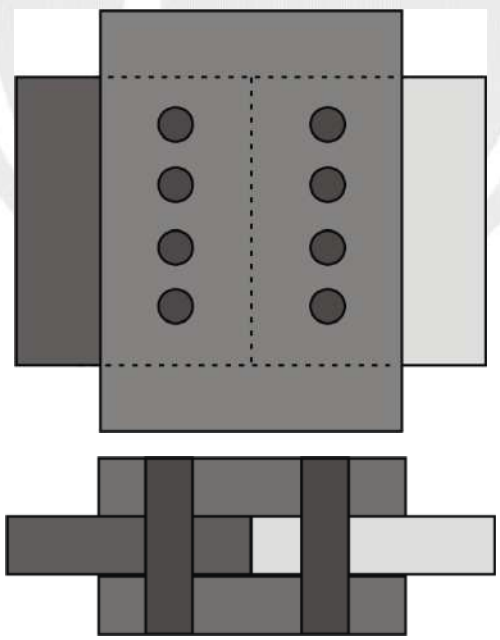


Fig.4.22: Double strap butt joint

Classification of riveted joint Based on how many rows of rivets in joint:

Single riveted joint: This riveted joint has one row of the rivet in a lap joint or only one row of the rivet on each plate of butt joint. **(Refer figure 4.23)**

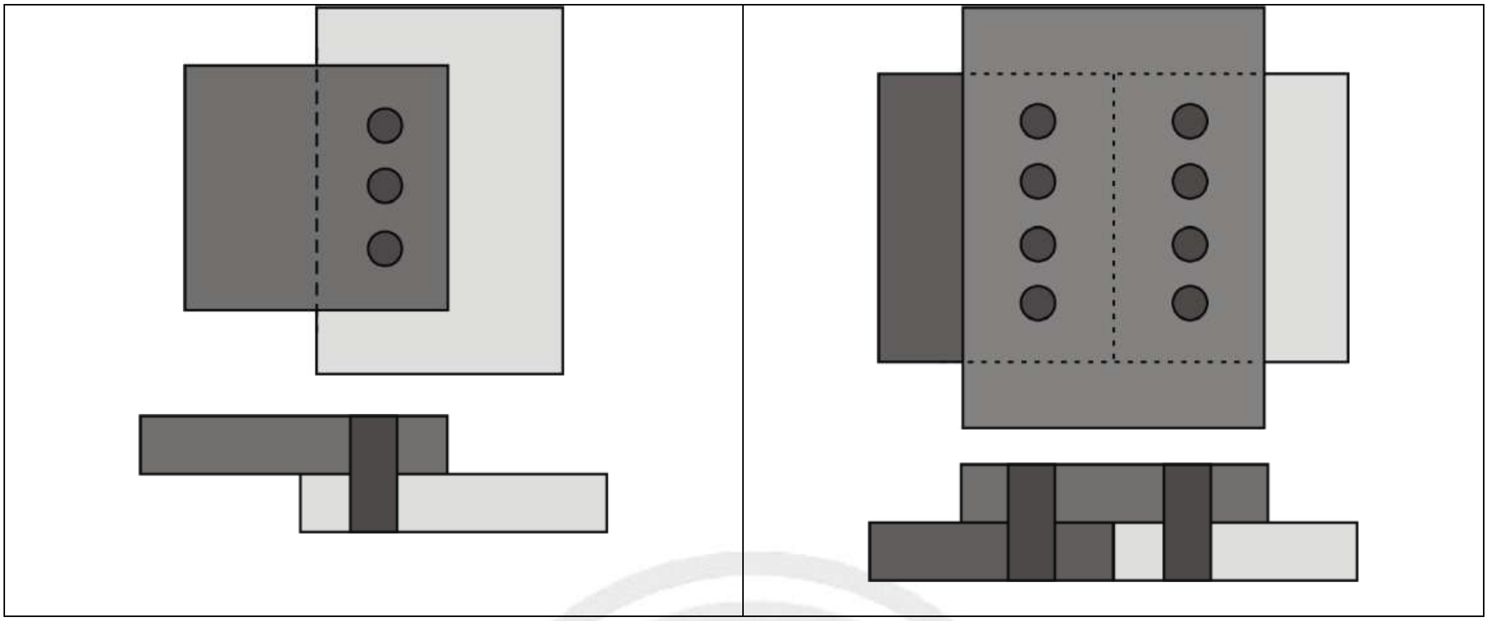


Fig.4.23: Single riveted joint

Double riveted joint: Two rows of rivets are used in a lap joint or two rows of rivet are used in each main plate of butt joint. Similarly, there are triple riveted, quadruple riveted joints are there.

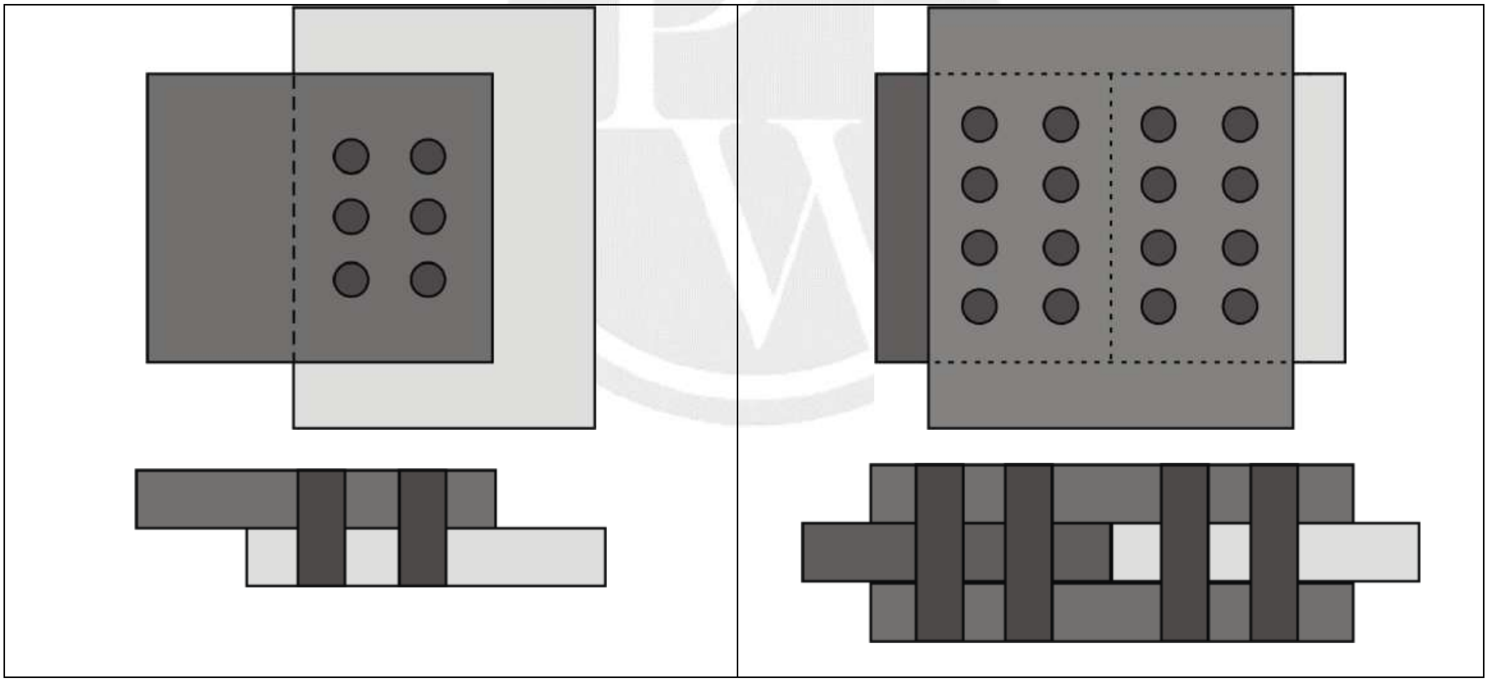
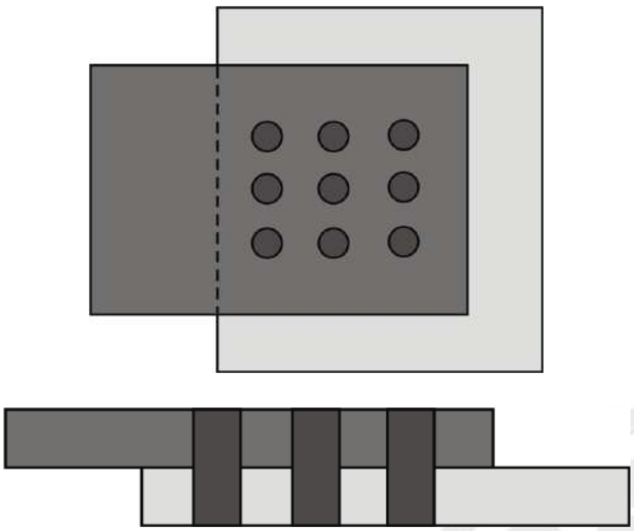
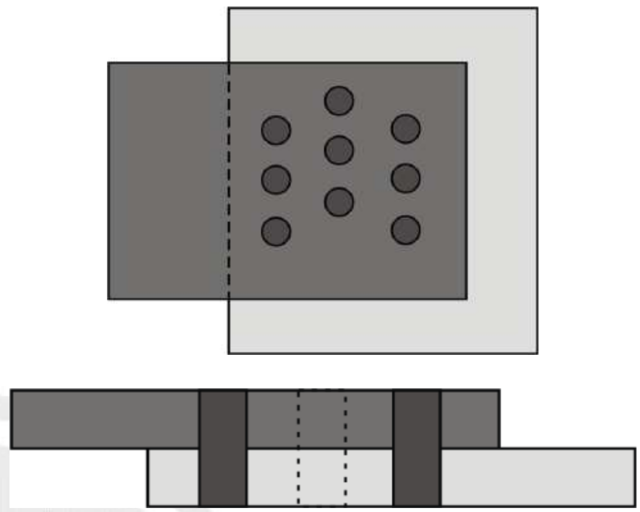


Fig.4.24: Double riveted joint

Classification of riveted joint based on the arrangement of rivets in adjacent rows of rivet

Chain riveted joints: The rivets in the adjacent rows are opposite to each other (in same transverse line).	**Zigzag riveted joints:** When the rivets in adjacent rows are not in chain arrangement.
 Fig.4.25: Chain riveted join	 Fig.4.26: Zigzag riveted joint

4.7.2 Terminology of riveted joint

Pitch(p): The pitch of the rivet is defined as the distance between the center of one rivet to the center of adjacent rivet in the same row.

Margin / Edge distance (m):

The margin is the distance between the edge of the plate to the center line of the rivets in the nearest row.

Transverse pitch (p_b):

Transverse pitch also called back pitch or row pitch. It is the distance between two consecutive rows of the rivet on the same plate.

Diagonal pitch (p_d):

Diagonal pitch is the distance between the center of one rivet to the center of adjacent rivet located in the adjacent row.

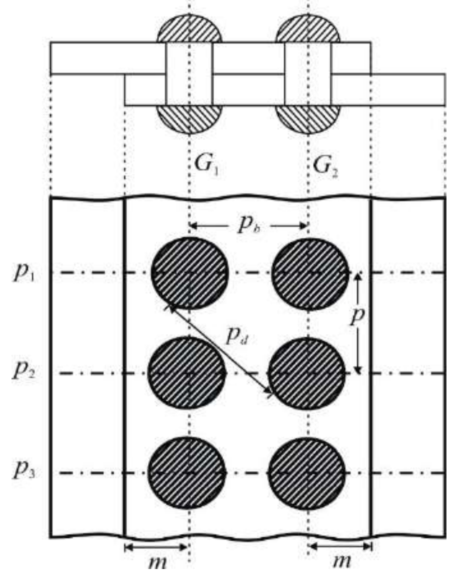


Fig.4.27: Terminologies of riveted joint

4.8 Load Carrying capacity of riveted joint

4.8.1 When number of rivets in each row is same

Consider,

w = width of each plate

t = thickness of each plate

d = diameter of rivet

d_h = diameter of rivet hole

m = Number of rows of rivet

n = Number of rivets in each row

N = mn = total number of rivets

Note:

(i) For Butt riveted joint, find value of M, n and N only for one plate

(ii) For figure 4.28 m is 2, n is 3 and N is 6.

σ_{tp} = Permissible tensile stress of plate

τ_p = Permissible shear stress of rivet

σ_{cp} = Permissible crushing stress of rivet.

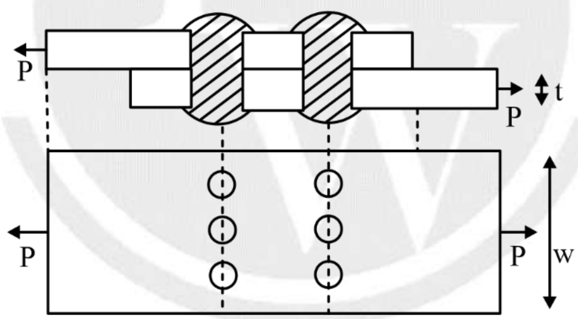


Fig.4.28: Riveted joint with tensile load

1. **Tearing load carrying capacity (or tearing strength) of the plate (P_t):**
   When number of rivets in each row is same then, tearing failures occurs from the row which is nearest to the load as shown in figure.

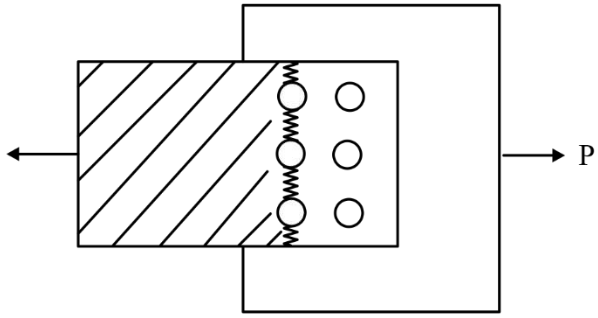


Fig.4.29: Tearing failure

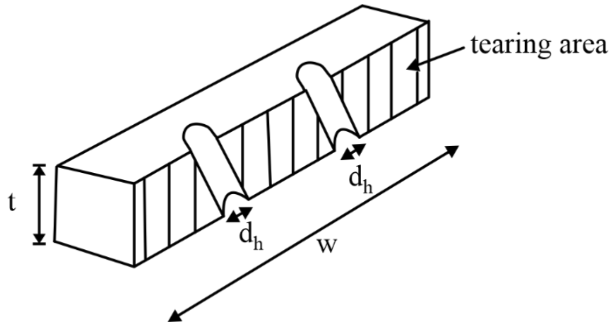


Fig.4.30: Tearing area view

Tensile stress in the plate of nearest row

$$\sigma_t = \frac{P}{\text{tearing area}} = \frac{P}{(w - nd_h)t} \le \sigma_{tp}$$

$$\Rightarrow P \le \sigma_{tp}(w - nd_h)t$$

Hence load carrying capacity to avoid tearing failure,

$$P_t = \sigma_{tp}(w - nd_h)t$$

2. **Shearing load carrying capacity or shearing strength (P_s):**

Shearing failures occur when shearing of all rivets occurs.

Shear stress in rivet,

$$\tau = \frac{P}{\text{total shear area}}$$

Note:

Total shear area of rivets

$$= k \times \frac{\pi}{4} d^2 \times N$$

Where,

k = Number of shear area in one rivet.

For single shear, k = 1(refer figure 4.31)

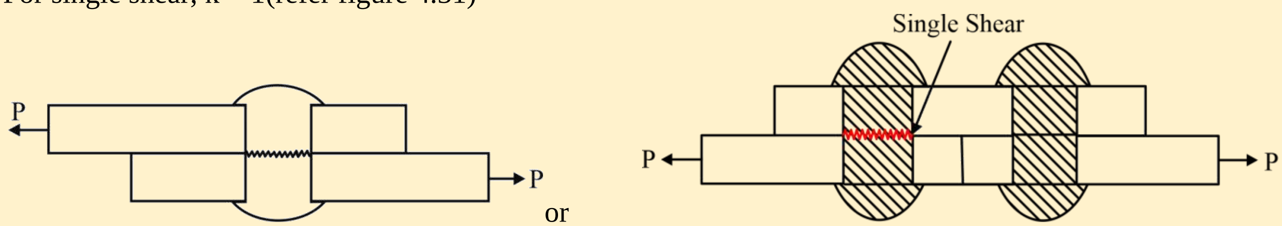


Fig.4.31: Single Shearing Failure

For double shear, k = 2

(Refer figure 4.32)

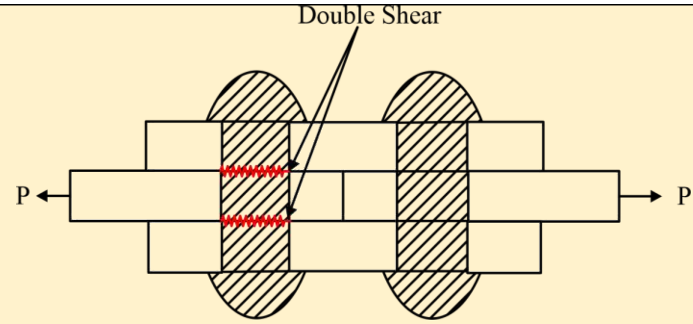


Fig.4.32: Double Shear Failure

Hence, shear stress in rivet,

$$\tau = \frac{P}{k \times \frac{\pi}{3} d^2 \times N} \le \tau_p$$

$$P \le \tau_p \times \left(k \times \frac{\pi}{4} d^2\right) \times N$$

Hence load carrying capacity to avoid shearing failure is

$$P_S = \tau_p \left(k \times \frac{\pi}{4} d^2\right) \times N$$

3. Crushing load carrying capacity or crushing strength (P_C):

In crushing failure rivets fails due to compressive stress and crushing failure occur when crushing of all rivets occurs.

Crushing stress in rivet:

$$\sigma_C = \frac{P}{\text{Total crushing area}}$$

Note:

Crushing area of one rivet = dt

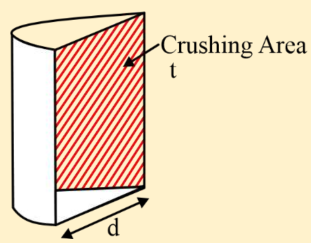


Fig.4.33: Crushing area

Total crushing area = dt × N

Hence,

$$\sigma_c = \frac{P}{dtN} \le \sigma_{cp}$$

$$\Rightarrow P \le \sigma_{cp} dtN$$

Hence load carrying capacity to avoid crushing failure is

$$\boxed{P_C = \sigma_{cp} dtN}$$

4. Actual load carrying capacity (P_{LCC})

$P_{LCC} =$ Minimum of load carrying capacity by considering all type of failure.

$$\boxed{P_{LCC} = \text{Min}(P_t, P_s, P_c)}$$

5. Efficiency (η)

It is the ratio of actual load carrying capacity (P_{LCC}) to the load carrying capacity of solid plate without rivet and hole $(P_{\text{Solid Plate}})$

$$\eta = \frac{P_{LCC}}{P_{\text{Solid Plate}}}$$

$P_{\text{Solid Plate}}$:

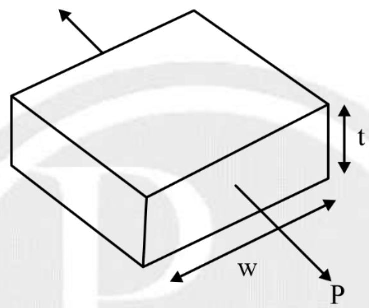


Fig.4.34: Solid plate without rivet and hole

Tensile stress, $\sigma_t = \frac{P}{wt} \le \sigma_{tP}$

$\Rightarrow P \le \sigma_{tP} wt$

Hence, $P_{\text{Solid Plate}} = \sigma_{tP} wt$

Efficiency of joint:

$$\boxed{\eta = \frac{P_{LCC}}{\sigma_{tP} wt}}$$

Note:

If width of the plate is not given, then to find efficiency of the joint calculate load carrying capacity per pitch length. To calculate load carrying capacity per pitch length replace width w by pitch p and number of rivets in each row i.e. n by 1.

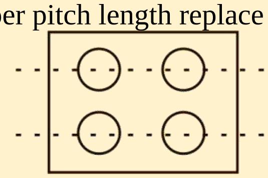


Fig.4.35

Tearing strength / Pitch length: $P_{t/PL} = \sigma_{tp}(p - d_h)t$

Shearing strength / Pitch length: $P_{S/PL} = \tau_p \times k \times \frac{\pi}{4} d^2 \times m$

Crushing strength / Pitch length: $P_{C/PL} = \sigma_{cp} \times dt \times m$

Actual load carrying capacity: $P_{LCC/PL} = \text{Min}(P_{t/PL}, P_{S/PL}, P_{C/P_L})$

Efficiency: $\eta = \frac{P_{LCC/PL}}{\sigma_{tP} \times pt}$

4.8.2 When number of rivets in each rows are different (Diamond Pattern)

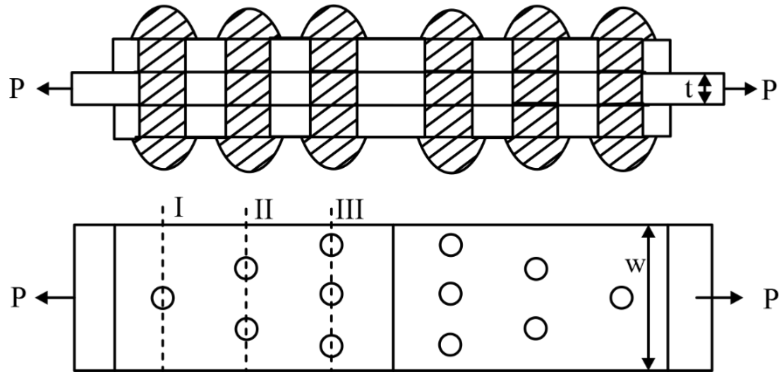


Fig.4.36: Diamond pattern rivet joint

1. **Shearing load carrying capacity (P_s)**

$P_s = \tau_P \times \text{Total shear area}$

[In figure 4.36 k = 2 (double shear), total number of rivets = 6]

$$P_S = \tau_p \times \left(2 \times \frac{\pi}{4} d^2\right) \times 6$$

2. **Crushing load carrying capacity** (P_C)

$P_C = \sigma_{cp} \times (\text{Total crushing area})$

[For Figure 4.36 total number of rivets = 6]

$$\boxed{P_C = \sigma_{cp} \times dt \times 6}$$

3. **Load carry capacity to avoid tearing from first row** $\left(P_{t_I}\right)$:

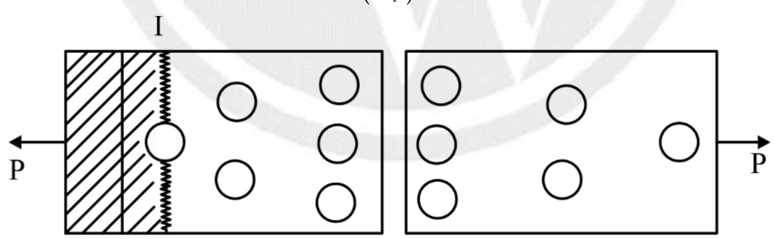


Fig.4.37: Tearing of the first row

$P_{t_I} = \sigma_{tp} \times (w - d_h)t$

4. **Load carrying capacity to avoid tearing of second row** $(P_{t_{II}})$

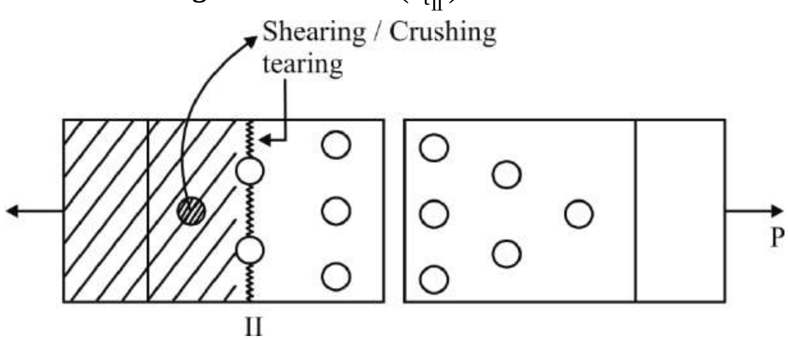


Fig.4.38: Tearing of the second row

$$P_{t_{II}} = \sigma_{tp}(w - 2d_h)t + \min\left(\tau_P \times k \times \frac{\pi}{4}d^2, \sigma_{cp} \times dt\right)$$

5. **Load carrying capacity to avoid tearing of 3rd row ($P_{t_{III}}$)**

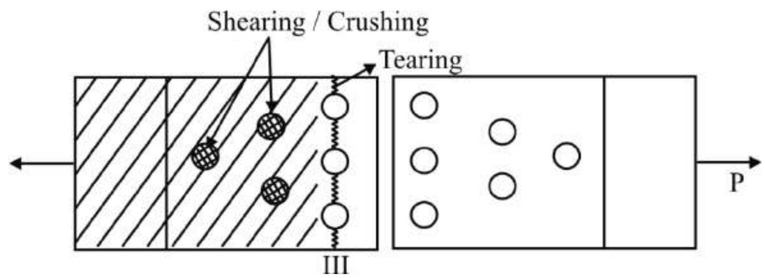


Fig.4.39: Tearing of the 3rd row

$$P_{t_{III}} = \sigma_{tp}(w - 3d_h)t + \min\left(\tau_P \times k \times \frac{\pi}{4}d^2 \times 3, \sigma_{cp} \times dt \times 3\right)$$

6. **Actual load carrying capacity** (P_{LCC}):

$$P_{LCC} = \min(P_S, P_C, P_{t_I}, P_{t_{II}}, P_{t_{III}})$$

7. **Efficiency** (η): $\eta = \dfrac{P_{LCC}}{\sigma_{tp} wt}$

4.9 Eccentric loading in bolted and riveted joint

4.9.1 Type I: Eccentricity in the plane of cross-section of bolt or rivet

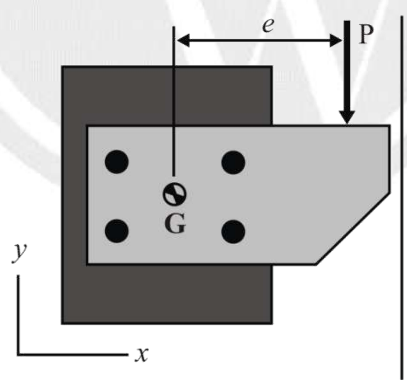


Fig.4.40: Eccentric load on the rivet

Shift the force P at the COG:

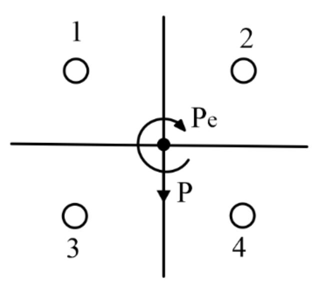


Fig.4.41: Shifting of load

(i) Direct force P will act at the COG which causes primary shear in each rivet/bolt.
(ii) Moment M = Pe will act at COG which causes secondary shear in each rivet/bolt.

Effect of force P:

- Primary shear will induce.
- **Magnitude of primary shear force (P_S):** Primary shear force in each rivet/bolt will be same & is calculated by:

$$P_S' = \frac{P}{\text{No.of rivet / bolt}}$$

- **Magnitude of primary shear stress (τ′):** Primary shear stress will be same in each bolt/rivet & is calculated by

$$\tau'_1 = \tau'_2 = \tau'_3 = \tau'_4 = \frac{\text{Primary shear force on each bolt/rivet}}{A}$$

- **Direction:** Direction of primary shear stress will be same at each point of bolt/rivet & will be opposite to force P. As shown in figure 4.42.

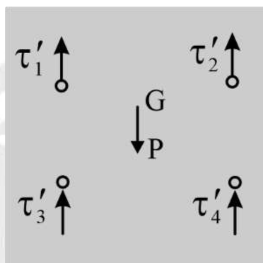


Fig.4.42: Direction of primary shear stress

Effect of moment $M = P \times e$:

- Due to moment $M = P \times e$, secondary shear will induce in each rivet/bolt.
- Magnitude of secondary shear force (P''_{s_i}) and secondary shear stress (τ''_i)

$$P''_{s_i} = Cr_i$$

Where P''_{s_i} = Secondary shear force on i^{th} bolt/rivet.

r_i = distance of center of i^{th} rivet/ bolt from center of gravity,

C is proportional constant & is calculated by, $C = \frac{Pe}{(r_1^2 + r_2^2 + r_3^2 + ...r_n^2)}$ N/m

Bolt/Rivet	Secondary shear force	Secondary shear Stress
1.	$P_{s_1}'' = Cr_1$	$\tau_1'' = \frac{P_{s_1}''}{A}$
2.	$P_{s_2}'' = Cr_2$	$\tau_2'' = \frac{P_{s_2}''}{A}$
3.	$P_{s_3}'' = Cr_3$	$\tau_3'' = \frac{P_{s_3}''}{A}$
4.	$P_{s_4}'' = Cr_4$	$\tau_4'' = \frac{P_{s_4}''}{A}$

Direction

Step I: Draw a line which is perpendicular to the line joining the center of bolt/rivet whose secondary shear stress we are drawing & COG.

Step II: In this line take the sense of secondary shear stress such that it should try to rotate the rivet/bolt opposite to the sense in which moment $M = P \times e$ if trying to rotate the rive/bolt.

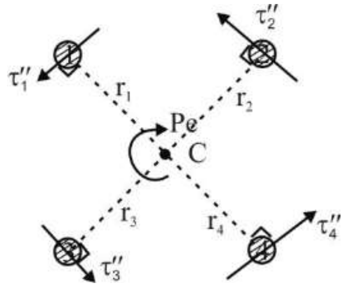


Fig.4.43: Direction of secondary shear stress

Resultant Shear Stress:

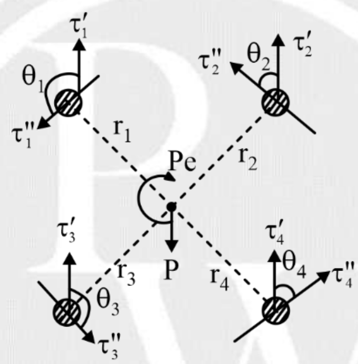


Fig.4.44: Combined primary and secondary shear stress

Find the vector resultant of primary & secondary shear stress.

Resultant shear stress of bolt/rivet 1,

$$\tau_1 = \sqrt{(\tau_1')^2 + (\tau_2'')^2 + 2\tau_1'\tau_2''\cos\theta_1}$$

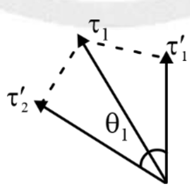


Fig.4.45: Direction and magnitude of resultant shear stress

Critical Rivet/bolt

During design we need to find resultant shear stress on critical Rivet/bolt. Critical Rivet/bolt is the Rivet/bolt where resultant shear stress is maximum.

Steps to find critical Rivet/bolt:

Step I: Draw the direction of primary and secondary shear stress of each rivet/bolt.

Step II: Select the bolt/rivet whose distance from COG (i.e. r) is maximum.

Step III: Select the bolt/rivet whose vector angle between primary & secondary shear stress (i.e. θ) is minimum.

Step IV: Bolt/rivet where r is maximum and θ is minimum (i.e. common rivets/bolts of step III & IV) will be critical bolt/rivet).

4.9.2 Type II: Eccentric load perpendicular to the axis of bolts

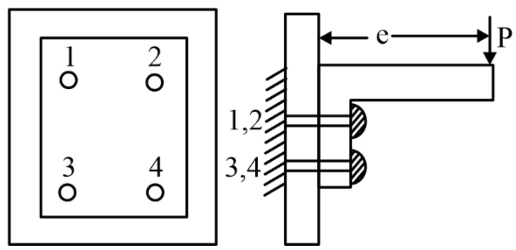


Fig.4.46: Eccentric load perpendicular to axis of the bolts

In this case, moment due to force P will try to till the bracket about tilting point C as shown in figure 4.47.

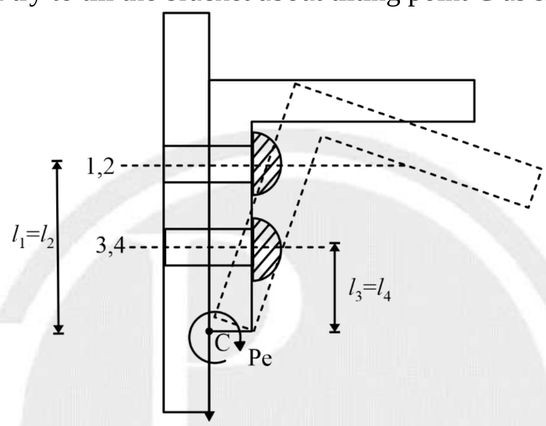


Fig.4.47: Tilting of rivet

Hence, in this case we will transfer the force P at tilting point C.
After transferring at tilting point C, point C will be subject to
(i) Direct fore P which will cause primary shear in each bolt/rivet
(ii) Moment $M_t = P \times e$ which causes secondary tensile in each bolt/rivet.

Effect of force P:

- Primary shear will induce.
- Magnitude of primary shear force (P_S'): Primary shear force in each rivet/bolt will be same & is calculated by:

$$P_S' = \frac{P}{\text{No.of rivet / bolt}}$$

- **Magnitude of primary shear stress (τ′):** Primary shear stress will be same in each bolt/rivet & is calculated by:

$$\tau'_1 = \tau'_2 = \tau'_3 = \tau'_4 = \frac{\text{Primany shear force on each bolt/rivet}}{A}$$

Effect of moment $M_t = P \times e$:

- Due to moment $M_t = P \times e$, secondary tensile fore (P_t'') will induce in each bolt/rivet which is calculated by: $P_{t_i}'' = k \times \ell_i$

Where, P_{t_i}'' = secondary tensile force of i^{th} bolt/rivet

ℓ_i = Distance of i^{th} bolt/rivet from tilting point C.

k is proportional constant & is calculated by $k = \dfrac{P \times e}{\ell_1^2 + \ell_2^2 + ... + \ell_n^2}$

Bolt/Rivet	Secondary tensile force	Secondary tensile stress
1	$P_{t_1}'' = k\ell_1$	$\sigma_{t_1}'' = \frac{P_{t_1}''}{A}$
2	$P_{t_2}'' = k\ell_2$	$\sigma_{t_2}'' = \frac{P_{t_2}''}{A}$
3	$P_{t_3}'' = k\ell_3$	$\sigma_{t_3}'' = \frac{P_{t_3}''}{A}$
4	$P_{t_4}'' = k\ell_4$	$\sigma_{t_4}'' = \frac{P_{t_4}''}{A}$

For design:

- Calculate stresses on critical bolt/rivet. Bolt/rivet whose secondary tensile stress is maximum (say $\sigma_{t,m}$) will be critical bolt/rivet because primary shear stress is same for all bolt/rivet (say τ_s'). For secondary tensile stress to be maximum, ℓ should be maximum. Hence Bolt/rivet whose ℓ is maximum will be critical bolt/rivet.
- Find principal stress & maximum shear stress at the critical point & use appropriate theory of failure.

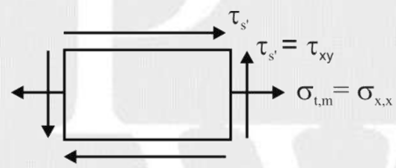


Fig.4.48: Combined tensile and shear stress

$$\sigma_{xx} = \sigma_{t,m},\ \sigma_{yy} = 0,\ \tau_{xy} = \tau_s'$$

Principle stresses:

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_{1,2} = \frac{\sigma_{t,m}}{2} \pm \sqrt{\left(\frac{\sigma_{t,m}}{2}\right)^2 + (\tau_s')^2}$$

and

$$\sigma_3 = 0$$

Maximum shear stress:

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{t,m}}{2}\right)^2 + (\tau_s')^2}$$

Note:

Steps for designing bolt or rivet for Type II Eccentric Loading

(a) Find ℓ for each bolt & decide critical bolt or rivet

(b) Find primary and secondary shear stress for critical bolt or rivet.

(c) Find principal stresses and maximum shear stress for critical bolt or rivet.

(d) Design the bolt or rivet according to given TOF

4.9.3 Type III: Eccentric load parallel to the axis of bolts

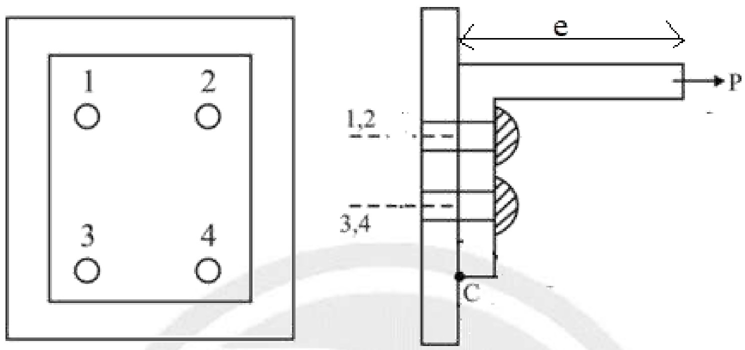


Fig.4.49: Eccentric load parallel to the axis of the bolt

Transfer the force P at tilting point C:

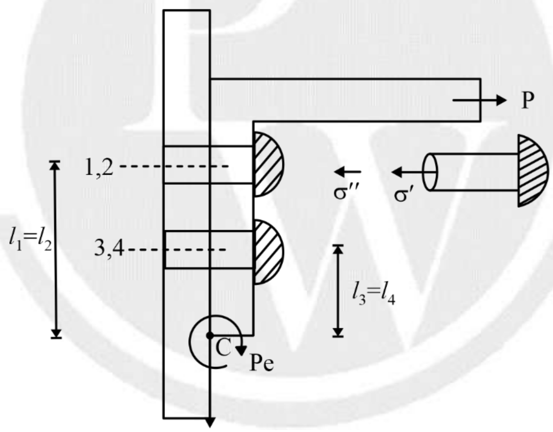


Fig.4.50: Transfer of load

After transferring, the point C will be subjected to

(i) Direct force P which causes primary tensile in each bolt/rivet.

(ii) Moment $M_t = P \times e$ which causes secondary tensile in each bolt/rivet

Effect of force P:

- Due to force P each bolt/rivet will be subjected to primary tensile.
- Magnitude of primary tensile force:
  It will be same in each bolt/rivet & is calculated by

$$P_{t_1}' = P_{t_2}' = P_{t_3}' = P_{t_4}' = P_t' = \frac{P}{\text{no.of bolt / rivet}}$$

Primary tensile stress in each bolt/rivet

$$\sigma_{t_1}' = \sigma_{t_2}' = \sigma_{t_3}' = \sigma_{t_4}' = \sigma_t' = \frac{P_t'}{A}$$

Effect of moment $M_t = P \times e$:

- Due to moment $M_t = P \times e$, secondary tensile fore (P_t'') will induce in each bolt/rivet which is calculated by: $P_{t_i}'' = k \times \ell_i$

  Where, P_{t_i}'' = secondary tensile force of i^{th} bolt/rivet

  ℓ_i = Distance of i^{th} bolt/rivet from tilting point C.

  k is proportional constant & is calculated by $k = \dfrac{P \times e}{\ell_1^2 + \ell_2^2 + ... + \ell_n^2}$

Bolt/Rivet	Secondary tensile force	Secondary tensile stress
1	$P_{t_1}'' = k\ell_1$	$\sigma_{t_1}'' = \dfrac{P_{t_1}''}{A}$
2	$P_{t_2}'' = k\ell_2$	$\sigma_{t_2}'' = \dfrac{P_{t_2}''}{A}$
3	$P_{t_3}'' = k\ell_3$	$\sigma_{t_3}'' = \dfrac{P_{t_3}''}{A}$
4	$P_{t_4}'' = k\ell_4$	$\sigma_{t_4}'' = \dfrac{P_{t_4}''}{A}$

Resultant tensile stress of any i^{th} bolt/rivet (σ_{t_i}):

$$\sigma_{t_i} = \sigma_{t_i}' + \sigma_{t_i}''$$

For design: Find the resultant tensile stress in the critical bolt/rivet. Bolt/rivet whose distance from tilting point i.e. ℓ is maximum will be critical bolt/rivet.

❑❑❑

5 DESIGN OF CLUTCH

5.1 Introduction

It is used to **connect or disconnect** the source of power from the remaining parts of the power transmission system at the will of the operator.

It should be capable to transmit required torque or power from input shaft to output shaft in engaged position.

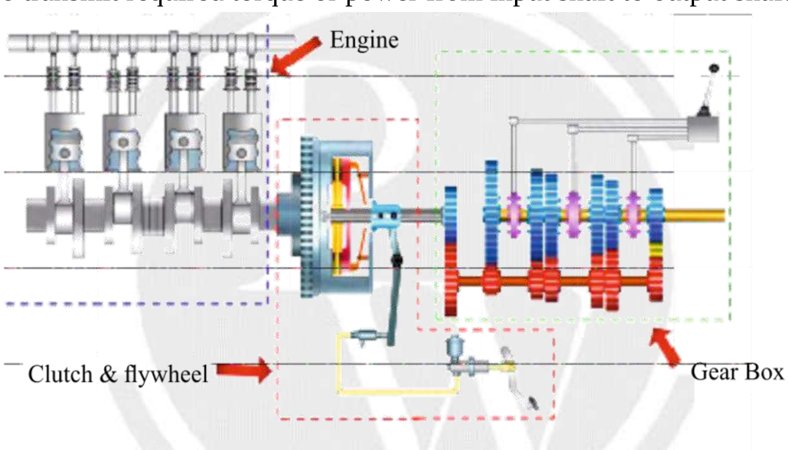


Fig.5.1: Power transmission system

5.1.1 Types of Clutch

Types of Clutch
- Positive Clutch
- Friction Clutch

(I) Positive Clutch:

- Positive engagement (no slip) due to presence of teeth. (Refer figure 5.2 and 5.3)
- High torque transmitting capacity.
- Sudden engagement.
- Cannot be engaged at high speed to avoid jerk.
- Used in power press, rolling mills etc.
- It is further classified as:

(a) Square jaw clutch: It can transmit torque in both directions.

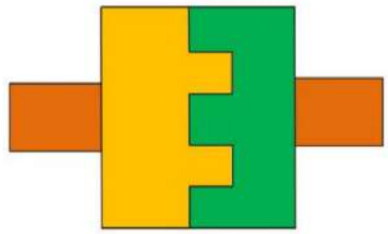


Fig.5.2: Square jaw clutch

(b) Spiral jaw clutch: It can transmit torque in one direction only.

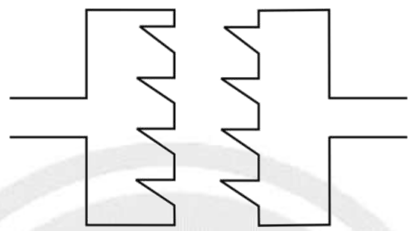


Fig.5.3: Spiral jaw clutch

(II) Friction Clutch:

- The power is transmitted by the friction between surfaces attached to the driving and driven shafts.
- Slip during engagement
- Smooth engagement
- Torque transmitting capacity depends upon axial force applied in engages position.
- Can engaged at high speed without jerk.
- Used in automobiles.

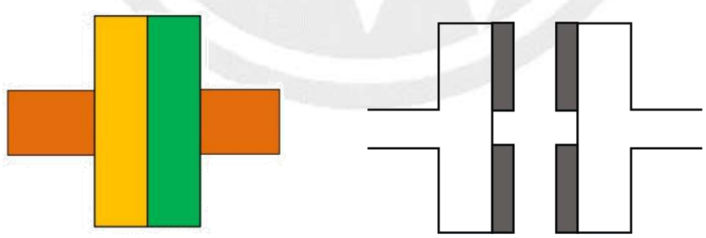


Fig.5.4: Friction clutch

Types of Friction Clutches

- Plate Clutch
  - Single Plate
  - Multi Plate Clutch
- Cone Clutch
- Centrifugal Clutch

5. 2 Plate Clutch

5. 2. 1 Introduction of Plate Clutch

Plate Clutch:

- The shape of the clutch is like a circular flat plate.
- It is an axial clutch.
- Depending upon the no. of plates on the shaft plate clutches are of two types:

  (a) Single plate clutch: Number of plates on driven shaft is 1.

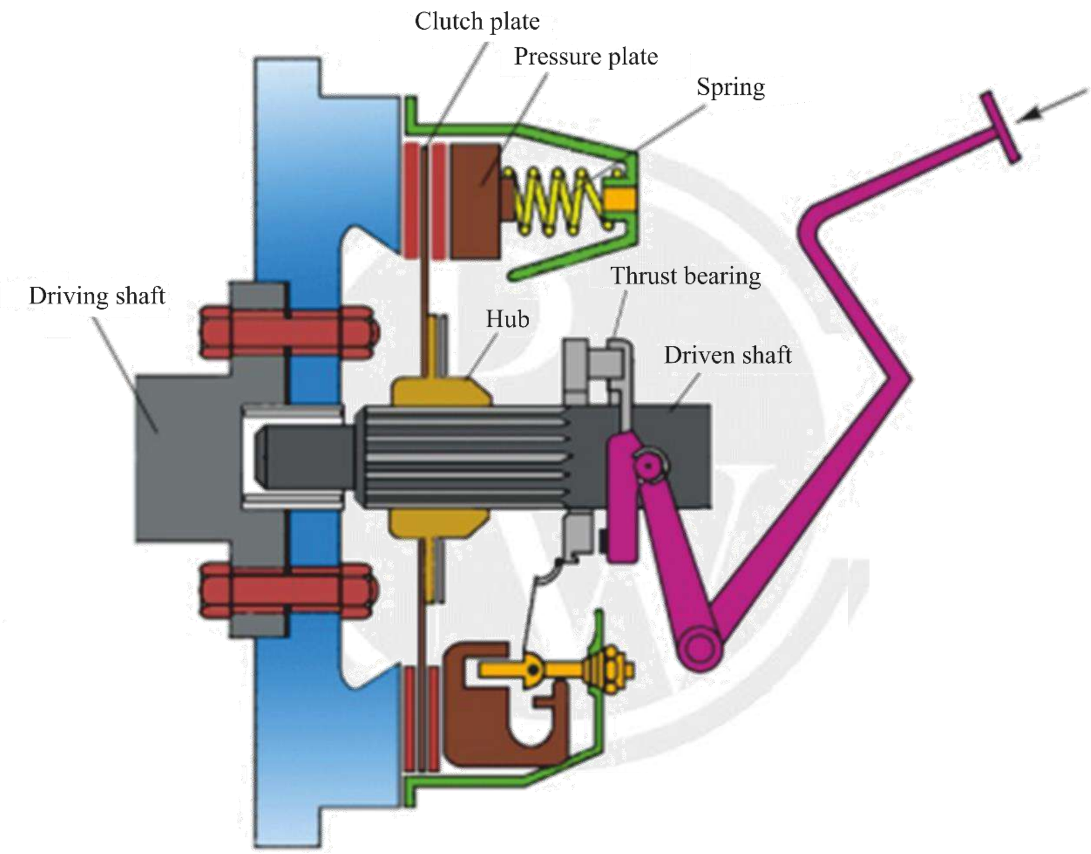


Fig.5.5: Single plate clutch

  (b) Multiplate Clutch: Multiple plates on the driving & driven shaft.

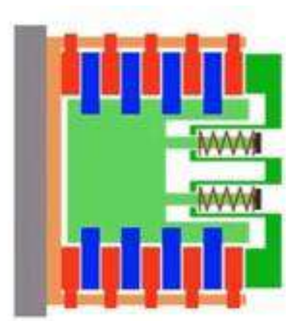


Fig.5.6: Multiple clutch

5. 2. 2 Analysis of Plate Clutch

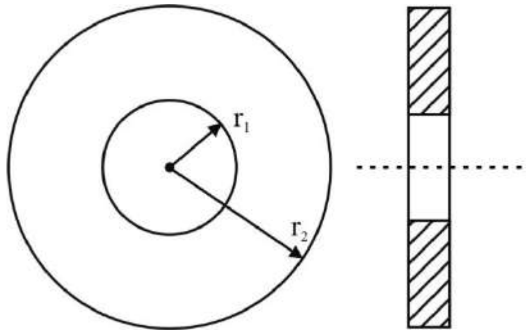


Fig.5.7: Cross-section view of the clutch

r_i = Inner radius of friction lining.

r_0 = Outer radius of friction lining.

W = Axial force on clutch at engaged position

T = Maximum torque transferred without slip.

p = Pressure on clutch plate at a radial distance r

p_{max} = Maximum pressure

μ = Coefficient of friction.

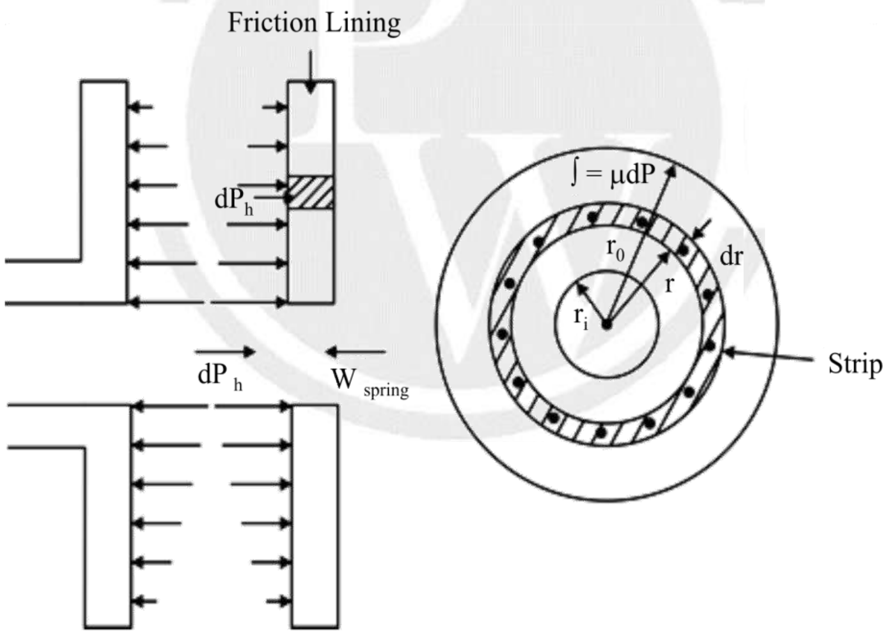


Fig.5.8: Analysis of plate clutch

(i) Total pressure (or normal) force on lining (P_n) and spring force (W)

$$W = P_n = \int_{r_i}^{r_o} p \times 2\pi r\, dr$$

(ii) Torque transmitted by clutch (T)

$$T = n\int_{r_i}^{r_o} \mu p \times 2\pi r^2 dr$$

In order to find reaction between *p* and r we have two theories:

1. Uniform Pressure theory
2. Uniform wear theory

Uniform Pressure Theory (UPT):

According to this theory if the clutch is new and there is no significant wear on the surface of the clutch, we can assume that the pressure is uniform over the area.

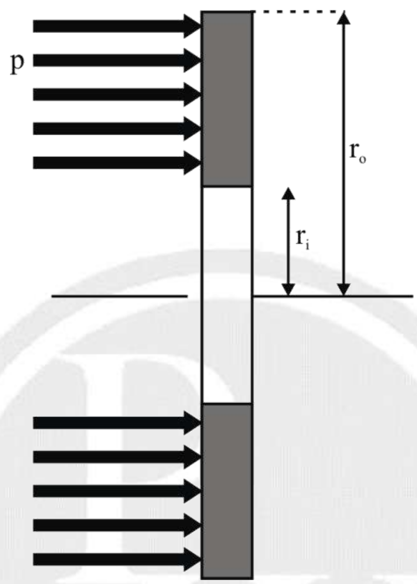


Fig.5.9: Uniform pressure

For UPT:

(i) $W = P_n = p\pi(r_o^2 - r_i^2)$

(ii) $T = n\mu W\left(\dfrac{2}{3}\dfrac{r_o^3 - r_i^3}{r_o^2 - r_i^2}\right)$

$\Rightarrow$ $T = n\mu W R_m$

Where, $R_m = \dfrac{2}{3}\dfrac{r_o^3 - r_i^3}{r_o^2 - r_o^2}$ = mean radius according to **UPT**.

Uniform Wear Theory (UWT):

According to this theory when the clutch becomes old and there is wear on the surface, then rate of wear can be assumed to be uniform.

Rate of wear = constant

Rate of wear $\propto pr$

Hence in UWT:

$pr = \text{constant} = C = p_{max}\, r_i$

$\Rightarrow p = \dfrac{C}{r}$

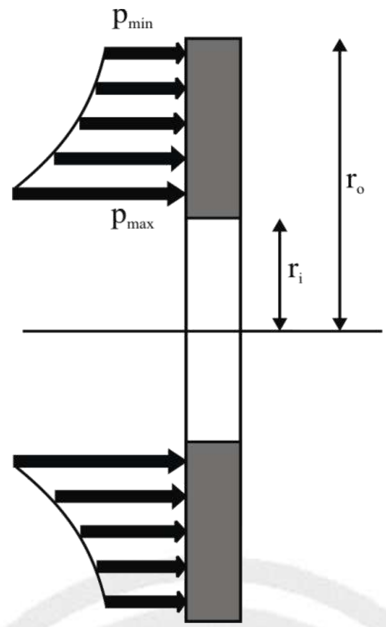


Fig.5.10: Uniform wear

(i) $W = P_n = 2\pi C(r_o - r_i)$

where, $C = pr = p_{mar} r_i$

(ii) $T = n \times \mu W\left(\frac{r_o + r_i}{2}\right) = n \times \mu W R_m$

Where, $R_m = \frac{r_o + r_i}{2}$ = mean radius according to UWT.

Note:

(i) **Number of pairs of contacting surfaces:**

Plate Clutch	n
Single plate clutch with one side effective	1
Single plate clutch with both side effective	2
Multiple Clutch	$n_1 + n_2 - 1$ where, n_1 = Number of plates on driving shaft n_2 = Number of plates on driven shaft

(ii) In plate clutch, if n (number of pairs of contacting surfaces), coefficient of friction (μ), permissible pressure and outer radius (r_o) due to space limitation are fixed, then to have maximum torque transmitting capacity inner radius (r_i) of the friction lining should be: $r_i = \frac{r_o}{\sqrt{3}}$

(iii) The only important thing to understand in friction clutches is the knowledge when to apply which theory. Generally, question will mention which theory to use but if it does not, use this to determine which theory to use:

a. If question mentions that clutch is **new**, use uniform pressure theory.
b. If question mentions that clutch is **old**, use uniform wear theory.
c. If question mentions that clutch is **worn-out**, use uniform pressure theory.
d. In questions when it is mentioned that clutch is being used for **power transmission**, use uniform wear theory.
e. If question mentions **nothing** then also use uniform wear theory.

5.2.3 Friction radius (r_f)

Hypothetical or Imaginary radius where total frictional force is assumed to be acting such that torque corresponding to this condition is same as actual torque.

Friction Radius	
UPT	**UWT**
$r_f = \frac{2}{3}\frac{r_o^3 - r_i^3}{r_o^2 - r_i^2}$	$r_f = \frac{r_o + r_i}{2}$

5.2.4 Single Plate vs Multi Plate Clutch

S. No.	Single Plate Clutch	Multi Plate Clutch	
1	Less torque transmitting capacity for same size	1.	More torque transmitting capacity for same size
2.	Less heat generation, hence no coolant is required. **(Dry clutch)**	2.	High heat generation, hence we need coolant. **(Wet clutch)**
3.	Used in big automobiles like buses, cares etc.	3.	Used in small automobiles like bikes due to limitation of size

5. 3 Cone Clutches

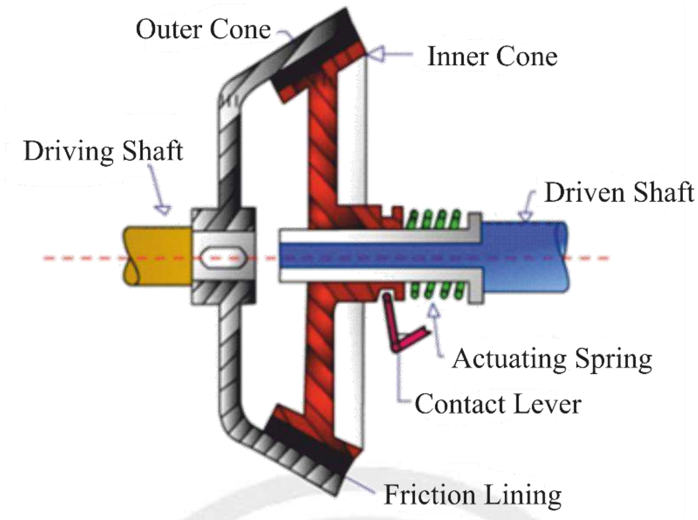


Fig.5.11: Cone clutch

5.3.1 Analysis of Cone Clutch

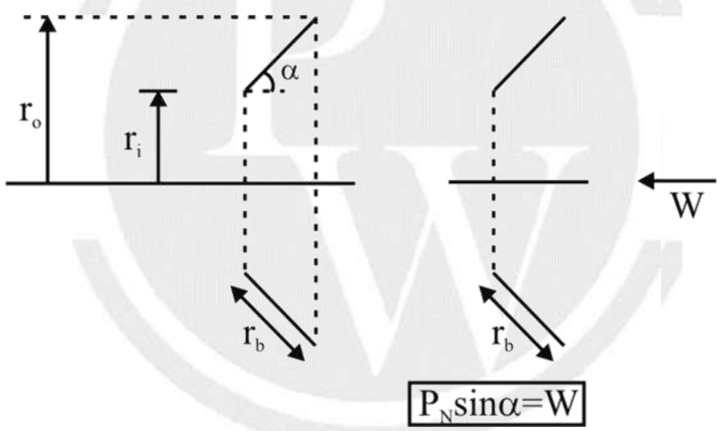


Fig.5.12: Analysis of Cone clutch

(i) Face width, $\boxed{b = \dfrac{r_o - r_i}{\sin\alpha}}$

(ii) Total pressure or normal force on friction lining (P_n)

$$P_n = \int_{r_i}^{r_o} p_x 2\pi r \frac{dr}{\sin\alpha} = \frac{W}{\sin\alpha}$$

(iii) Spring or axial force at engaged position (W)

$$\boxed{W = \int_{r_i}^{r_o} p \times 2\pi r\, dr}$$

(iv) Torque transmitted by clutch (T),

$$\boxed{T = \int_{r_i}^{r_o} \mu p \times 2\pi r^2 \frac{dr}{\sin\alpha}}$$

For UPT:

(i) $W = p \times \pi(r_o^2 - r_i^2)$

(ii) $P_n = \dfrac{W}{\sin\alpha}$

(iii) $\boxed{T = \mu W \times \left(\dfrac{2}{3}\dfrac{r_o^3 - r_i^3}{r_o^2 - r_i^2}\right) \times \dfrac{1}{\sin\alpha}}$

For UWT:

(i) $W = 2\pi C(r_o - r_i)$

where, $C = pr = P_{mar}\ r_i$

(ii) $P_n = \dfrac{W}{\sin\alpha}$

(iii) $T = \mu W\left(\dfrac{r_o + r_i}{2}\right) \times \dfrac{1}{\sin\alpha}$

5.3.2 Advantage and disadvantage of cone clutch

Advantage of Cone Clutch:

For same size (r_O & r_i), friction lining (μ); permissible pressure and effort (W_{spring}) torque transmitting capacity increases for cone clutch by factor $\left(\dfrac{1}{\sin\alpha}\right)$ in comparison to single plate clutch.

Disadvantages of Cone Clutch:

- Multi plate cone clutch is not possible
- Engaging & disengaging is not so smooth & chances of self-locking.
  $\alpha \downarrow$ changes of self-locking $\uparrow$

Note:

For smooth engagement and disengagement: $\alpha > \tan^{-1}(\mu)$

5.4 Centrifugal clutch

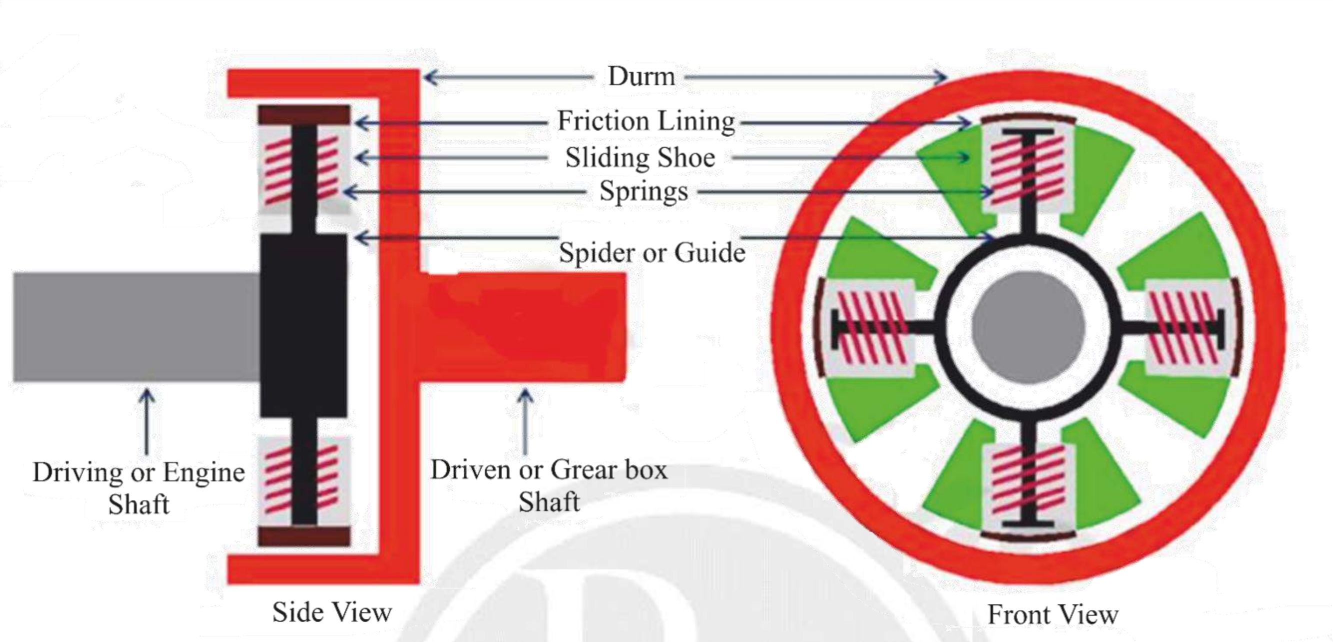


Fig.5.13: Centrifugal clutch

5.4.1 Analysis of centrifugal clutch

Key points:

- Centrifugal clutch is an automatic clutch where the clutch is engaged or disengaged automatically based on the speed of the driving shaft.
- At two speeds centrifugal force & spring force are equal, net force between drum and shoe will be zero and clutch will be in disengaged position.
- As the speed increases both centrifugal force & spring force F_c & F_s increases, at a certain speed ω_1, spring force reaches its maximum value (F_S). If the speed is increased beyond ω_1 only centrifugal force increases (F_C) & spring force remains constant at its max value (F_S). Hence there is a net force $F_C - F_S$ on each shoe which engages the clutch.
- It is a radial clutch.

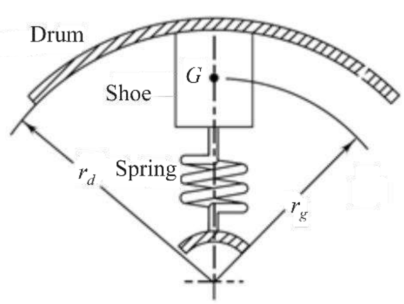


Fig.5.14: Analysis of centrifugal clutch

m = mass of each shoe (kg)

r_g = radius of the centre of gravity of the shoe in engaged position

r_d = inner radius of drum.

z = number of shoes

ω_1 = Speed at which just engagement starts

ω = Actual speed

F_S = Maximum spring force or spring force at all engaged position = centrifugal force at just engaged position $(F_{c_1} = m r_g \omega_1^2)$

$\Rightarrow F_S = F_{c_1} = m r_g \omega_1^2$

If $\omega < \omega_1 \leftarrow$ Disengaged position, and if $\omega > \omega_1 \leftarrow$ engaged position

At engaged position $(\omega > \omega_1)$:

Centrifugal force at each shoe, $F_C = m r_g \omega^2$

Spring force at each shoe, $F_S = F_{C_1} = m r_g w_1^2$

Net normal force between shoe and drum,

$F_C - F_S = m r_g (\omega^2 - \omega_1^2)$

Friction force = $\mu(F_C - F_S) = \mu m r_g (\omega^2 - \omega_1^2)$

Torque transmitted by each shoe,

$= \mu(F_C - F_S) r_d = \mu m r_g (\omega^2 - \omega_1^2) r_d$

Total torque transmitted (T)

$\boxed{T = \mu(F_C - F_S) r_d \times z}$

$\Rightarrow T = \mu m r_g (\omega^2 - \omega_1^2) r_d \times z$

5. 4. 2 Application of centrifugal clutch

Application:

1. Centrifugal clutches are used in light weight vehicles such as Elalf carts, move pads, lawn movers etc.
2. In centrifugal clutch the engagement is very smooth.
3. Centrifugal clutches are also used in heavy duties applications, cranes, cement mills etc. where the engine has to be started at no load condition.

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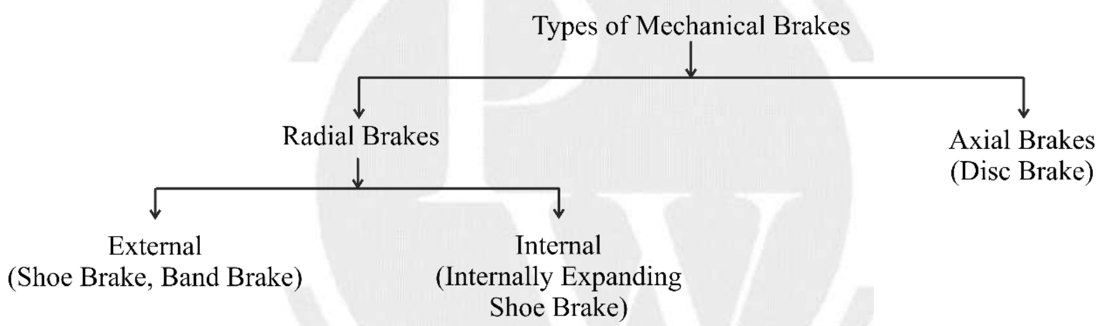
6 DESIGN OF BRAKE

6.1 Introduction of Brake

- A brake is defined as a device, which is used to absorb the energy possessed by a moving system or mechanism by means of friction to slow down or completely stop the motion of a moving system, such as a rotating drum, machine or vehicle
- Types of Brakes – Mechanical Brakes, Hydraulic Brakes, Electrical Brakes etc.
- Our study is limited to only mechanical brakes.



6.2 Shoe Brake

6.2.1 Introduction of Shoe Brake

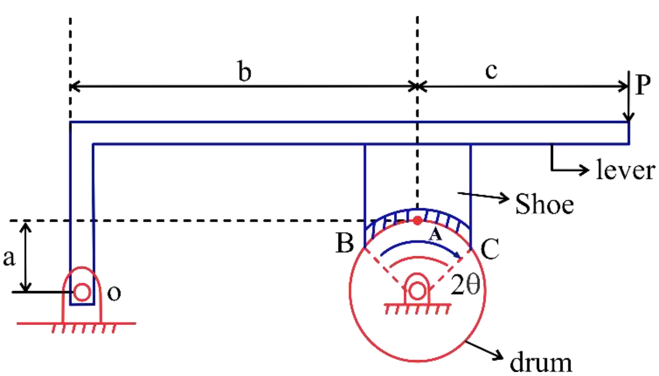


Fig. 6.1: Shoe Brake

- **Fulcrum (O):** Point at which lever is hinged.
- **Braking effort (P):** Effort required in lever to get braking action in drum.
- FBD of lever and drum.

  Shoe of the lever and drum in region BAC are contacting each other. For simplicity we can consider only point A (midpoint of region BAC) is contacting each other to make the FBD of lever and drum.
- If FBD is drawn by considering single contact point A, then use equivalent coefficient of friction (μ_e) to calculate friction force which is calculated as discussed below:

For short shoe Brake $\left(2\theta < \frac{\pi}{4}\right)$	**For long shoe brake** $\left(2\theta > \frac{\pi}{4}\right)$
$\mu_e = \mu$	$\mu_e = \frac{4\mu \sin\theta}{2\theta + \sin 2\theta}$

Note:
(i) 2θ: Block angle in rad.
(ii) If block angle (2θ) is not given, then consider brake as short shoe brake.

6.2.2 Analysis of Shoe Brake

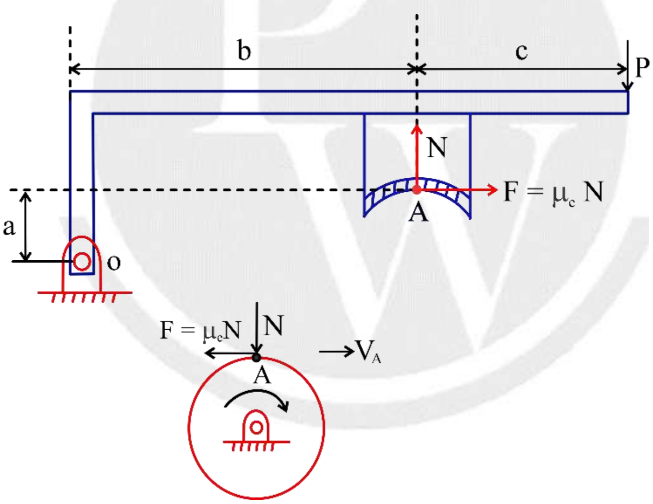


Fig. 6.2: FBD of shoe brake

Step I: First draw the FBD of drum:

Normal force (N): Normal force N in the drum will be towards centre of drum

Friction force ($F = \mu_e N$): In the FBD of drum friction force ($F = \mu_e N$) will be opposite to direction of velocity of point A.

Step II: Draw the FBD of lever:

In the FBD of lever contact force will be equal and opposite to the contact forces drawn on the FBD of drum.

- Braking torque (T_B)

  $T_B = F \times r = \mu_e N \times r$

- Relation between P & N:

  In the FBD of lever,

$$\sum M_{fulcrum,O} = 0$$

$$\Rightarrow P(b+c) - N(b) + \mu_e Na = 0$$

$$\Rightarrow P = \frac{Nb - \mu_e Na}{b+c}$$

Note:

Always take the $\sum M_{fulcrum,O} = 0$ to find relation between P & N. Don't try to memorise above relation

Other important relation:

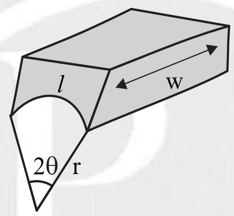


Fig. 6.3

Maximum pressure,

$$p_{max} = \frac{N}{w \times r \times 2\theta} \text{ (For long shoe brake)}$$

$$p_{max} = \frac{N}{wl} \text{ (For short shoe brake)}$$

6.2.3 Self-energising brake and Self-locking Condition

Self-energising brake:

When friction reduces the braking effort, then brake is known as self-energising brake.

Self-locking Condition:

If braking effort P become zero or negative, then brake automatically get locked with the drum and this condition is called self-locking condition.

To avoid self-locking condition, P > 0

Now we will discuss various cases of Shoe brake.

6.2.4 Line of action of friction force passes through fulcrum and drum is rotating CW

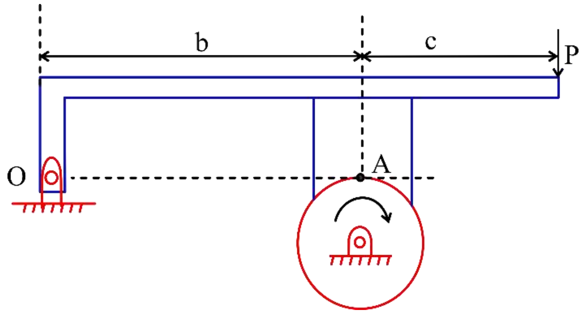


Fig. 6.4: Line of action of friction force passes through fulcrum O

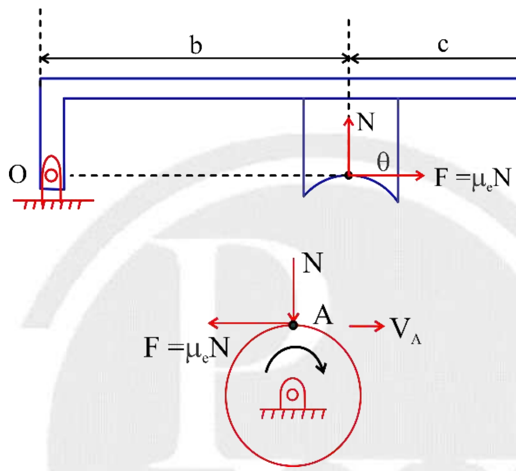


Fig. 6.5: FBD of lever and drum

In FBD of lever,

$$\sum M_O = 0 \Rightarrow P(b+c) - Nb = 0$$

$$\Rightarrow P = \frac{Nb}{b+c}$$

6.2.5 Line of action of friction is above fulcrum O and drum is rotating CW

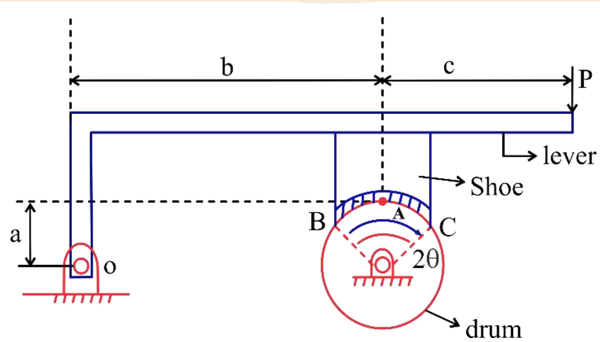


Fig. 6.6: Line of action of friction force is above fulcrum and CW drum rotation

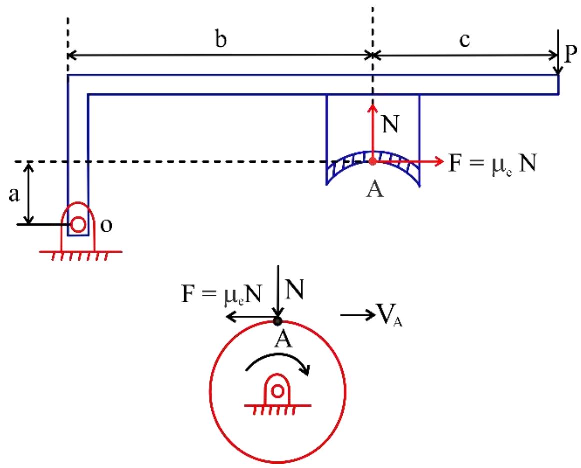


Fig. 6.7: FBD lever and drum

In the FBD of lever,

$$\sum M_{\text{fulcrum},O} = 0$$

$$\Rightarrow P(b + c) - N(b) + \mu_e Na = 0$$

$$\Rightarrow P = \frac{Nb - \mu_e Na}{b + c}$$

Important Points:

- Since friction force is reducing braking effort, hence above brake is self-energising brake.
- If P becomes 0 or negative, then brake will be in self-locking condition (SIC),

To avoid SLC:

$$P > 0 \Rightarrow b - \mu_e a > 0$$

$$\Rightarrow b > \mu_e a$$

6.2.6 Line of action of friction is above fulcrum O and drum is rotating ACW

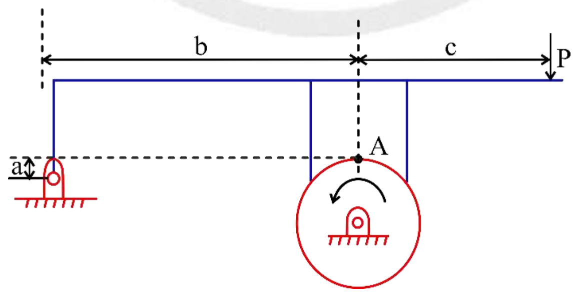


Fig. 6.8: Line of action of friction force is above fulcrum O and ACW drum rotation

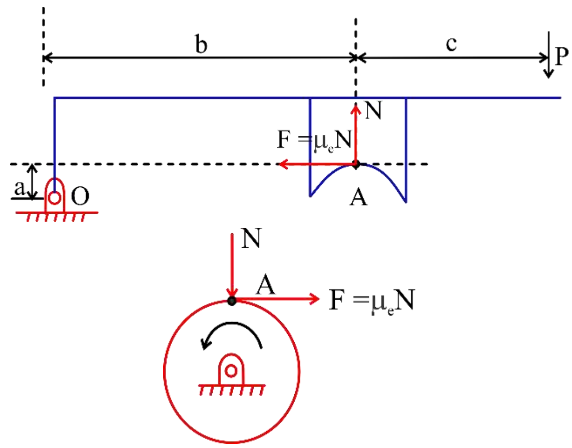


Fig. 6.9: FBD of lever and drum

In FBD of lever

$$\sum M_o = 0 \Rightarrow P(b+c) - Nb - \mu_e Na = 0$$

$$\Rightarrow P = \frac{N(b+\mu_e a)}{b+c}$$

Since friction is not reducing braking effort, hence above brake is not a self-energizing brake.

6.2.7 Line of action of friction is below fulcrum O and drum is rotating CW

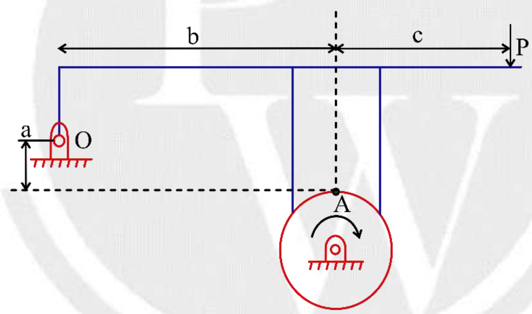


Fig. 6.10: Line of action of friction force below fulcrum and CW drum rotation

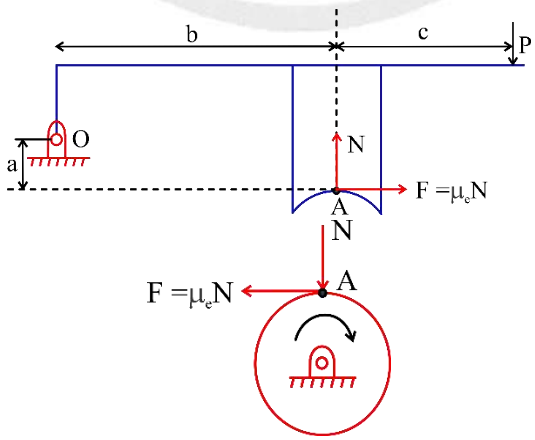


Fig. 6.11: FBD of lever and drum

$$\sum M_o = 0 \Rightarrow P(b+c) - Nb - \mu_e Na = 0$$

$$\Rightarrow P = \frac{Nb + \mu_e Na}{b+c}$$

Since friction is not reducing braking effort, hence above brake is not a self-energizing brake.

6.2.8 Line of action of friction is below fulcrum O and drum is rotating ACW

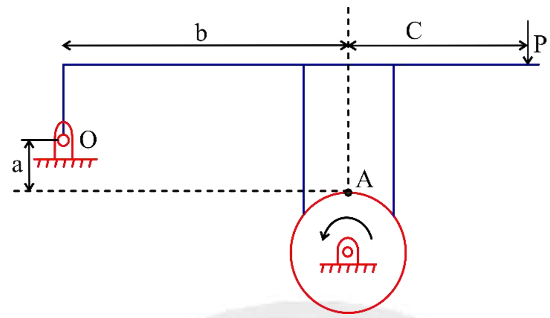


Fig. 6.12: Line of action of friction force is below fulcrum and ACW drum rotation

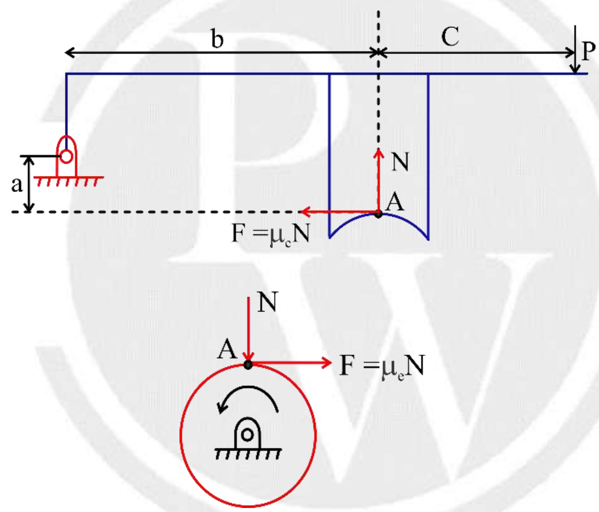


Fig. 6.13: FBD of below offset ACW drum rotation

In FBD of lever,

$$\sum M_o = 0$$

$$\Rightarrow P(b+c) - Nb + \mu_e Na = 0$$

$$\Rightarrow P = \frac{Nb - \mu_e Na}{b+c}$$

Important Points:

- Since friction force is reducing braking effort, hence above brake is self-energising brake.
- If P becomes 0 or negative, then brake will be in self-locking condition, (SIC)

To avoid SLC:

$$P > 0 \Rightarrow b - \mu_e a > 0$$
$$\Rightarrow b > \mu_e a$$

Note:

(i) If direction of moment of friction force & braking effort (P) about fulcrum O in the FBD of lever is same, then brake will act as a self-energizing brake.

(ii) If brake is not a self-energizing brake, then brake can never be in self-locking condition.

(iii) If brake is self-energizing brake then, brake may or may not be in self-locking condition.

If $P \leq 0$	If $P > 0$
(Brake will be in self locking condition)	(Brake will not be in self locking condition)

6.3 Band Brake

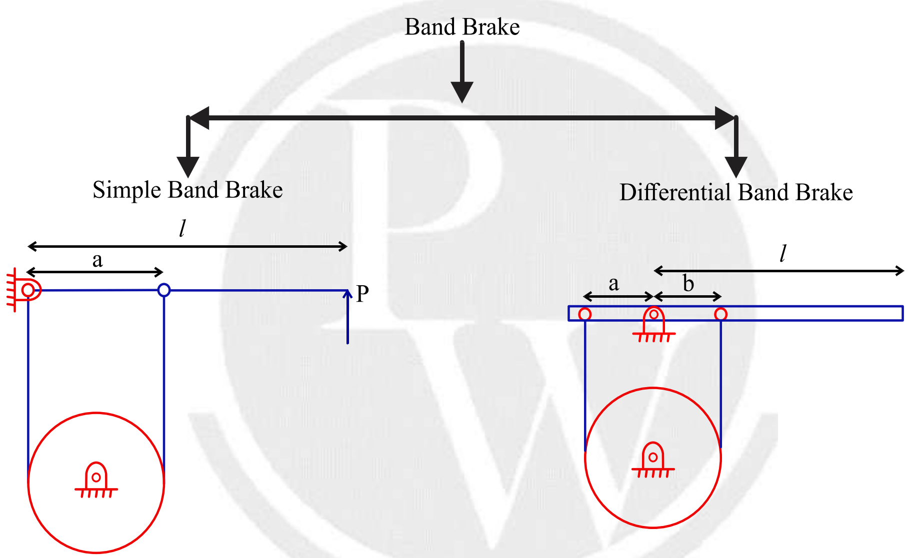


Fig. 6.14: Band brake

Identifying tight side and slack side of the band: Side opposite to the direction of drum rotation will be tight side and other side will be slack side in the FBD of drum and band **(Refer figure 6.15)**

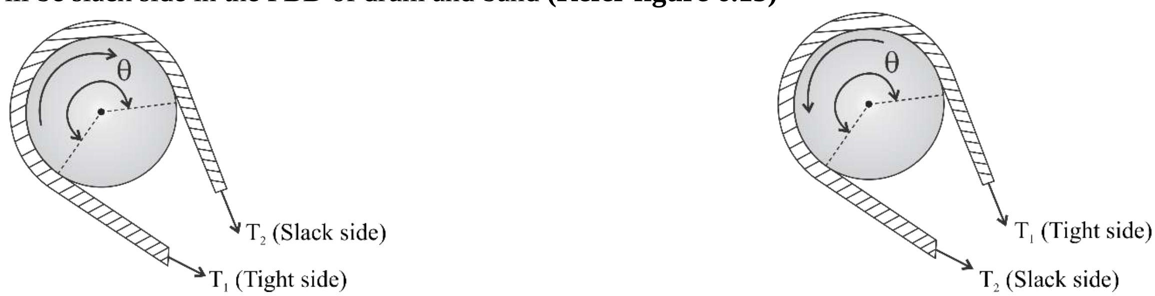


Fig. 6.15: Identifying Tight and slack side

Important Relations:

T_1 = Tension in tight side of band

T_2 = Tension in slack side of band

r = radius of drum

t = thickness of band

$r_e = r + \frac{t}{2} \approx r$ (If t is not given)

b = width of band

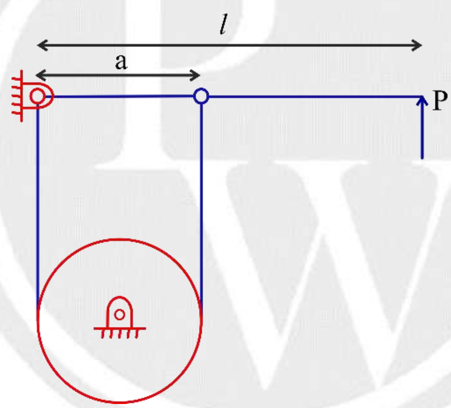
θ = angle of wrap

Braking torque (T_B): $T_B = (T_1 - T_2)r_e$

Relation between T_1 and T_2: $\frac{T_1}{T_2} = e^{\mu\theta}$

Maximum tensile stress in band: $\sigma_{t,max} = \frac{T_1}{bt}$

6.3.1 Analysis of simple band brake



For CW rotation of drum

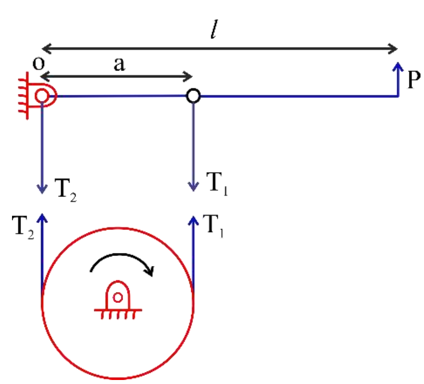


Fig. 6.16: Analysis of simple Band brake for CW rotation of drum

In FBD of lever,

$$\sum M_O = 0 \Rightarrow Pl - T_1 a = 0$$

$$\Rightarrow P = \frac{T_1 a}{l}$$

For ACW rotation of drum

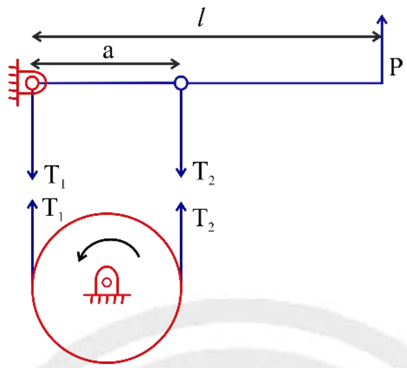


Fig. 6.17: Analysis of Simple Band brake for ACW rotation

In FBD of lever,

$$\sum M_O = 0 \Rightarrow Pl - T_2 a = 0$$

$$\Rightarrow P = \frac{T_2 a}{l}$$

6.3.2 Analysis of differential band brake

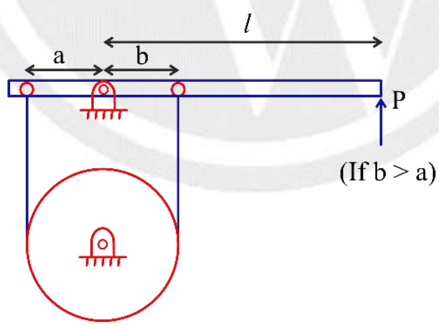


Fig. 6.18: Differential band brake (b > a)

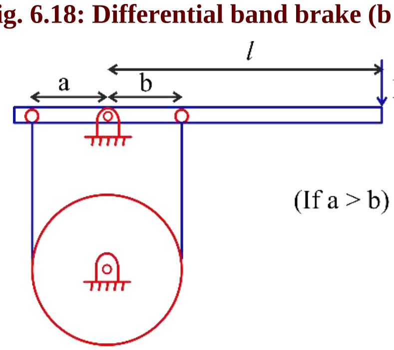


Fig. 6.19: Differential band brake (a > b)

Analysis for b > a (figure 6.18) and for CW rotation of drum:

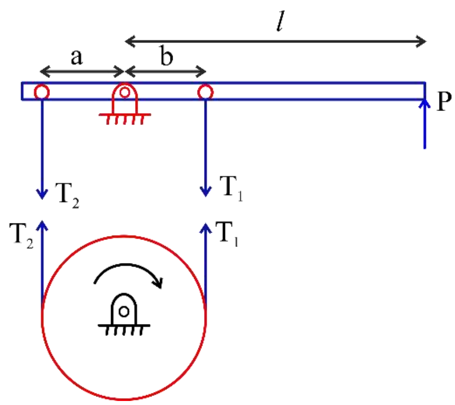


Fig. 6.20: FBD of differential band brake for CW rotation

In FBD of lever,

$$\sum M_O = 0 \Rightarrow Pl - T_1 b + T_2 a = 0$$

$$\Rightarrow P = \frac{T_1 b - T_2 a}{l}$$

Key points:

- Since $T_2 b$ part is reducing braking effort, hence above brake is self-energizing brake.
- To avoid SLC

$P > 0 \Rightarrow T_1 b > T_2 a$

$$\Rightarrow \frac{T_1}{T_2} > \frac{a}{b}$$

$$\Rightarrow e^{\mu\theta} > \frac{a}{b}$$

LHS = $e^{\mu\theta} > 1$ and since b > a therefore RHS $= \frac{a}{b} < 1$, therefore LHS (i.e. $e^{\mu\theta}$) will always be greater than RHS (i.e. $\frac{a}{b}$).

Hence, above condition will always satisfy & brake will never be in self-locking condition.

Analysis for b > a (figure 6.18) and for ACW rotation of drum

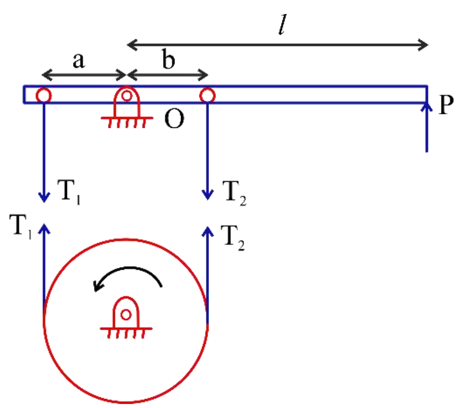


Fig. 6.21: FBD of differential band brake for ACW rotation

In FBD of lever,

$$\sum M_O = 0 \Rightarrow Pl - T_2 b + T_1 a = 0$$

$$\Rightarrow P = \frac{T_2 b - T_1 a}{l}$$

Key Points

- Since $T_1 a$ part is reducing braking effort therefore above brake is self-energizing brake.
- To avoid SLC,

  $P > 0 \Rightarrow T_2 b - T_1 a > 0$

  $$\Rightarrow \frac{T_2}{T_1} > \frac{a}{b} \Rightarrow \frac{T_1}{T_2} < \frac{b}{a}$$

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7 DESIGN OF ROLLING & SLIDING CONTACT BEARING

7.1 Introduction of Bearing

Bearings are machine elements that support a moving element (shaft or axle) with minimum friction.

(a) Function of bearings-

- Support the load actiing on shaft.
- Permit rotation of shaft with minimum friction.

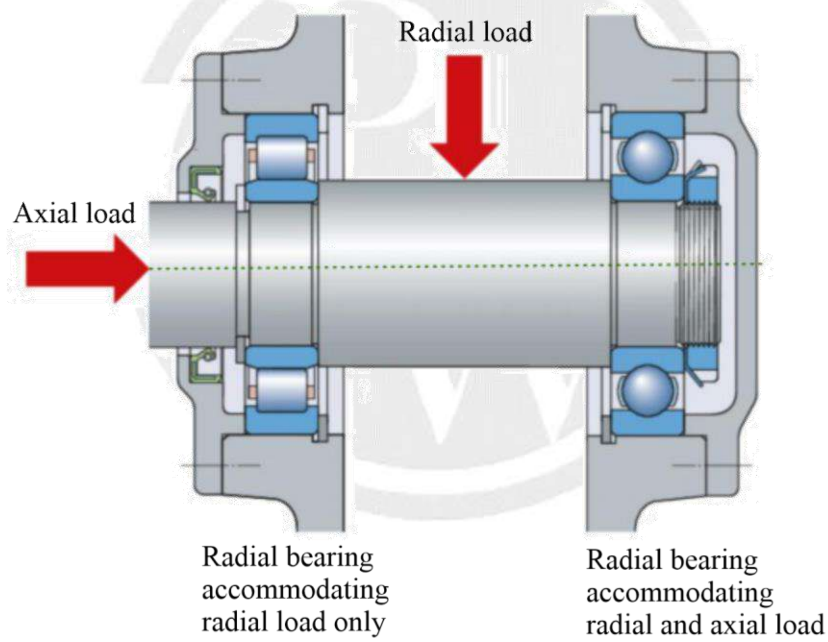


Fig. 7.1

Types of Bearings

- **Rolling Contact Bearings**
- **Sliding Contact Bearings**

7.2 Rolling Contact Bearings

- Rolling motion between the fixed and moving surfaces.
- Low starting and running friction, hence they are also known as Antifriction Bearings.
- More noise at very high speeds.
- More initial cost.
- Used in machine tool spindles, automobile front, and rear axles, gearbox, and small-size electric motors.
- In rolling contact bearings, the shafts are supported on the bearing surface through rolling elements such as balls and rollers.
- The relative motion between the shaft and bearing surface is rolling motion hence the friction is very low
- At low speeds the friction is negligible hence these bearings are also called as anti-friction bearings.

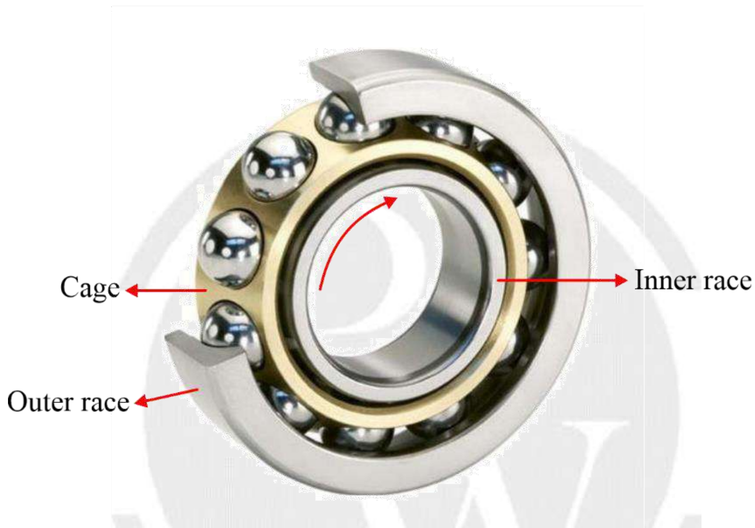


Fig. 7.2: Rolling contact bearing

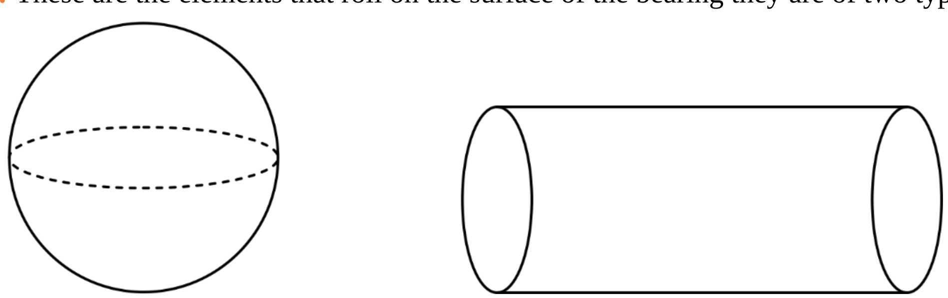
7.2.1 Construction of rolling contact bearing

(1) Inner Race: It is a circular ring connected to the shaft through interference fit it rotates with the shaft.

(2) Outer Race: It is a circular ring connected to the fix support through interference if it remains stationary.

(3) Cage: It is used to hold the rolling elements together so that the distance between them is maintained.

(4) Rolling Elements: These are the elements that roll on the surface of the bearing they are of two types:



(a) Ball **(b) Rollers**

Fig. 7.3: Rolling elements of rolling contact bearing bearing

- In case of balls less contact area, less friction & less load and it is point contact and bearing is known as ball bearing.
- In case of rollers more contact area, more load carrying capacity and more friction and it is line contact and bearings is known as roller bearing.

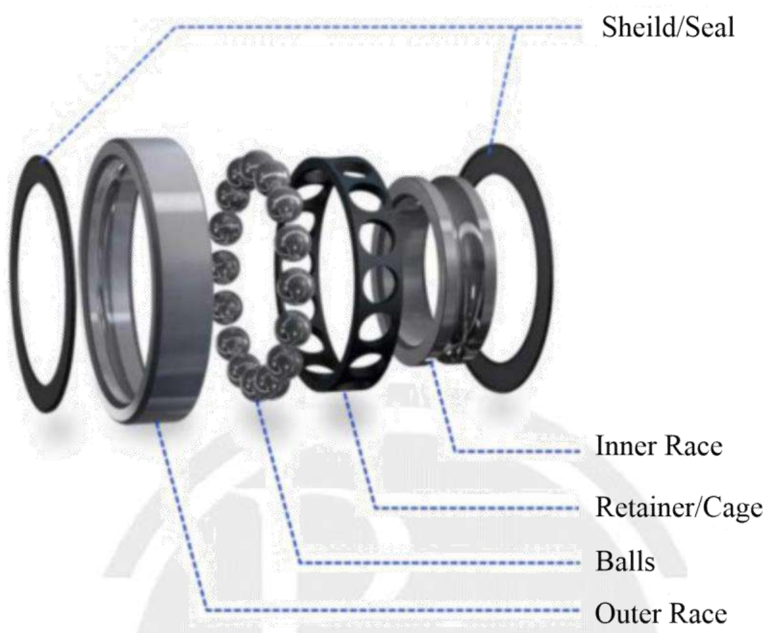


Fig. 7.4: Parts of rolling contact bearing

7.2.2 Types of Rolling Contact Bearings:

(1) Ball Bearing:

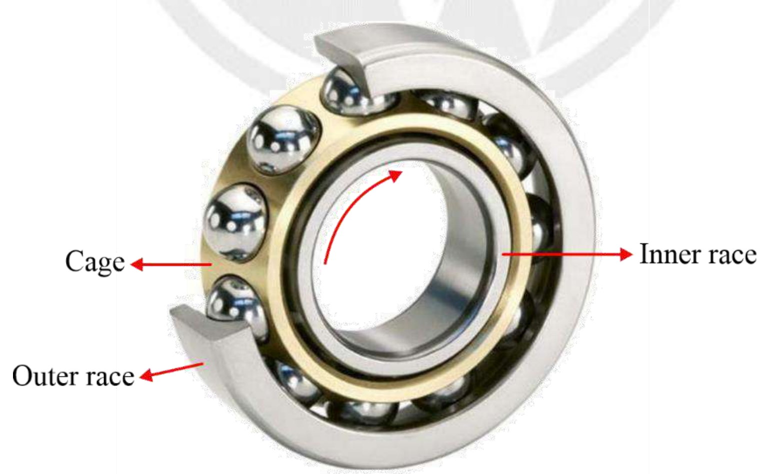


Fig. 7.5: Ball bearing

- Point contact
- Less load carrying capacity
- Less friction

(2) Roller Bearing:

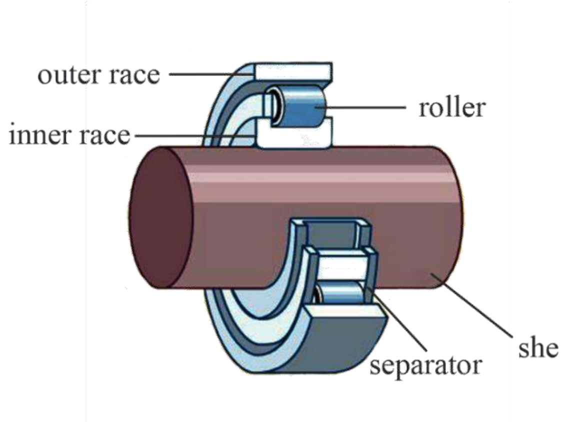


Fig. 7.6: Roller bearing

- Line contact
- High load carrying capacity
- High friction

7.2.3 Types of Ball Bearings:

(1) Deep Groove Ball Bearing:

- These are the most frequently used ball bearings

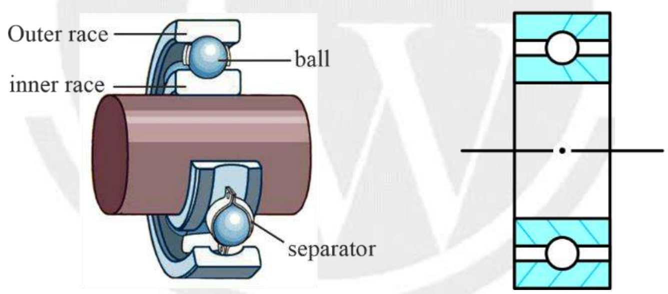


Fig. 7.7: Deep groove ball bearing

- Due to the depth of grooves balls are constrained in axial direction hence they can support radial as well as axial loads.
- Compared to axial loads they can support large radial loads.
- They have poor rigidity compared to roller bearing.

(2) Angular contact Ball Bearing:

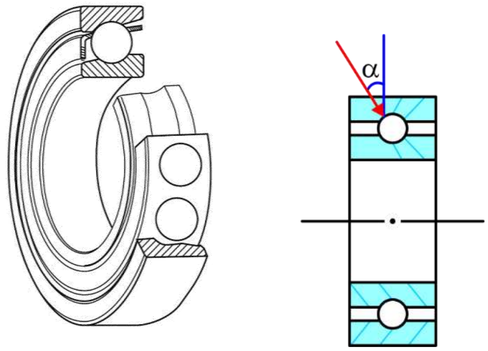


Fig. 7.8: Angular contact Ball Bearing

- In angular contact bearings the grooves in inner and outer race are so shaped that the line of reaction at the contact between the balls and races makes an angle with axis of bearing.
- This reaction has a radial and axial component therefore these bearings can support radial as well as axial roads.
- These bearings can only support axial load in one direction. Hence, they are often used in pair so that they can support axial load in both directions.

(3) Self-Aligning Ball Bearing:

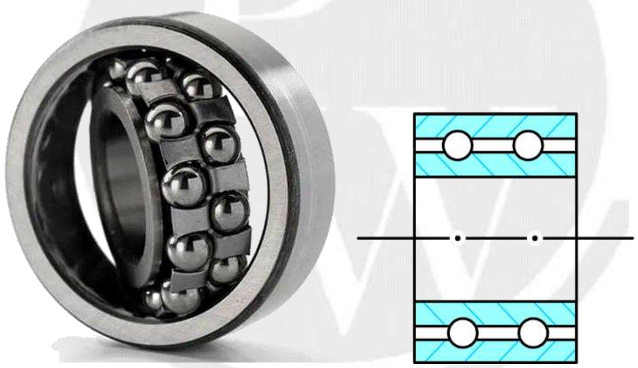


Fig. 7.9: Self Aligning Ball Bearing

- Self-aligning bearings are used where the axis of shaft and bearing can be misaligned due to tolerance or excessive deformation.

(4) Thrust Ball Bearing:

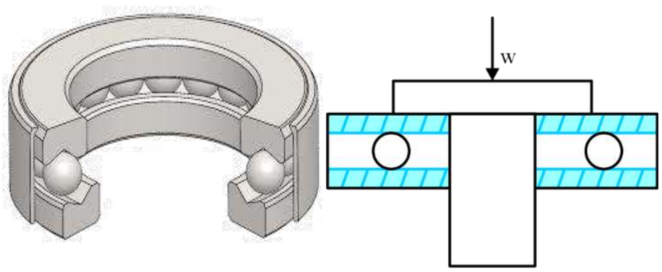


Fig. 7.10: Thrust Ball Bearing

7.2.4 Types of Roller Bearings

(1) Cylindrical Roller Bearings:

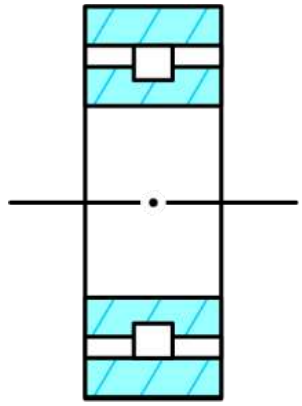



Fig. 7.11: Cylindrical roller bearing

- Cylindrical roller bearings can only support radial loads
- Due to the line contact between the rollers and races they have high radial load carrying capacity.
- These bearings are more rigid than ball bearings

(2) Needle Bearing:

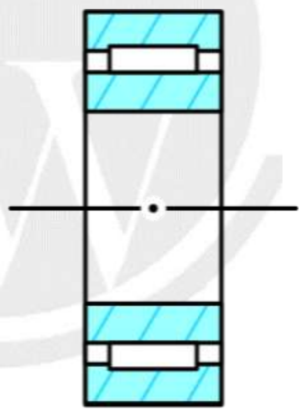



Fig. 7.12: Needle bearing

- If length of rolling element (l) is >>> than diameter of roller (d) $\left[\text{Generally if } \frac{l}{d} > 4\right]$, then that rolling contact bearing is known as Needle Bearing.
- Used where radial space is less
- Can support oscillating loads.

(3) Tapered roller bearing:

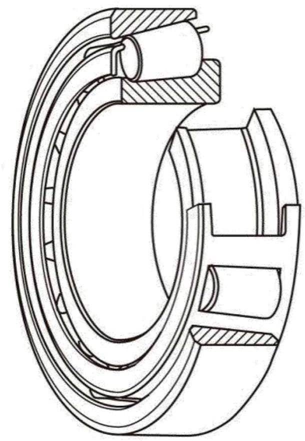


Fig. 7.13: Tapered roller bearing

- The line of reaction between the rollers and the races make an angle with the axes of bearings hence these bearings can support radial as well as axial loads.

(4) Self aligning roller bearing:

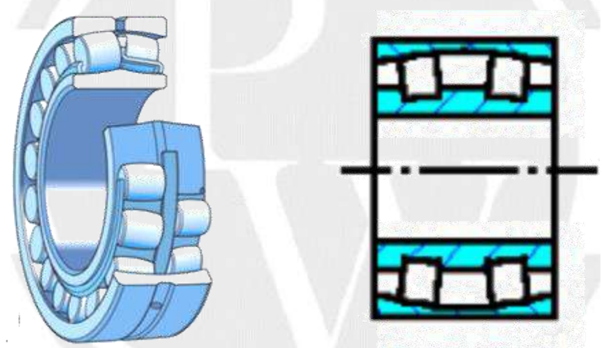


Fig. 7.14: Self aligning roller bearing

- Self- aligning roller bearings are used where the axis of shaft and bearing can be misaligned due to tolerance or excessive deformation.

7. 3. Design of rolling contact bearing

7. 3.1 Static Load Carrying Capacity

- Static load is the load acting on the bearing when the shaft is stationary.
- It produces permanent deformation in balls and races, which increases with increasing load.
- It is defined as the static load which corresponds to a total permanent deformation of balls and races, at the most heavily stressed point of contact, equal to 0.0001 of the ball diameters.

$$C_0 \frac{k.d^2n}{5}$$

k → constant which depends on material
d → diameter of ball
n → number of balls
C_0 → static load carrying capacity

7.3.2 Life of Bearing

- Life of a bearing is defined as the number of revolutions that the bearing can make before it fails. It is denoted in million revolutions.

Life at reliability R(L_R):

Life at reliability R(L_R) at any load P means at least R percent of bearing will sustain stated life under load P.

(a) Rated life ($L_{90\%}$ or L): Life for 90% Reliability

(b) Average life ($L_{50\%}$ or L_{av}): Life for 50% reliability.

Relationship between life at different reliablity under any load

$$\frac{L_{R_2}}{L_{R_1}} = \left[\frac{\ln\left(\frac{1}{R_2}\right)}{\ln\left(\frac{1}{R_1}\right)}\right]^{1/1.17}$$

Relation between average life ($L_{50\%}$) and rated life ($L_{90\%}$): $L_{50\%} = 5L_{90\%}$

Relation between life in hours and life in mR:

$$\text{Life in } mR = \frac{\text{Life in } hr \times 60 \times N}{10^6}$$

where N = speed of the shaft in rpm.

7.3.3 Equivalent Load (P)

- The equivalent dynamic load is defined as the constant radial load in radial bearings (or thrust load in thrust bearings), which if applied to the bearing would give same life as that which the bearing will attain under actual condition of forces.

  $P = XVF_r + YF_a$

  Where,

F_r = radial load

F_a = axial load

X = radial load factor

Y = Axial load factor

V = Race rotation factor

= 1 (when inner race rotates)

= 1.2 (when outer race rotates)

7.3.4 Relation between equivalent load and rated life

$$\frac{L_2}{L_1} = \left(\frac{P_1}{P_2}\right)^k$$

Where, L_2 = rated life under load P_2

L_1 = rated life under load P_1

k = 3 for ball bearing and k = $\frac{10}{3}$ for roller bearing.

7.3.5 Dynamic Load Carrying Capacity:

- It is the equivalent load at which the rated life of bearing is one million revolutions.

Relation between life (L) corresponds to load P and dynamic load carrying capacity (C):

$$L = \left(\frac{C}{P}\right)^k$$

Where, L = rated life corresponds to equivalent load P

C = dynamic load carrying capacity

7.3.6 Cyclic loading in rolling contact bearing

Loads	Rpm	%time	Revolutions in 1 min.
P_1	N_1	α_1	$n_1 = \alpha_1 N_1$
P_2	N_2	α_2	$n_2 = \alpha_2 N_2$
P_3	N_3	α_3	$n_3 = \alpha_3 N_3$
			$N_e = \sum n$

Equivalent speed in rpm = $N_e = \sum n$.

Equivalent load $P = \left[\frac{P_1^k n_1 + P_2^k n_2 + ...}{n_1 + n_2 + ...}\right]^{1/k}$

Where k = 3 for ball bearing and $\frac{10}{3}$ for roller bearing

7.4 Designation of Bearings:

P – Q – R – S

Here,

- P = Type of bearing:

  0, 1, 6, 8 – ball bearing

  2, 3, 4, 5, 7, 9 – roller bearing

- Q = Indicate series:
  1 – Extra light 2 – Light
  3 – Medium 4 – Heavy
- RS = when multiply by 5, it gives shaft diameter in mm.

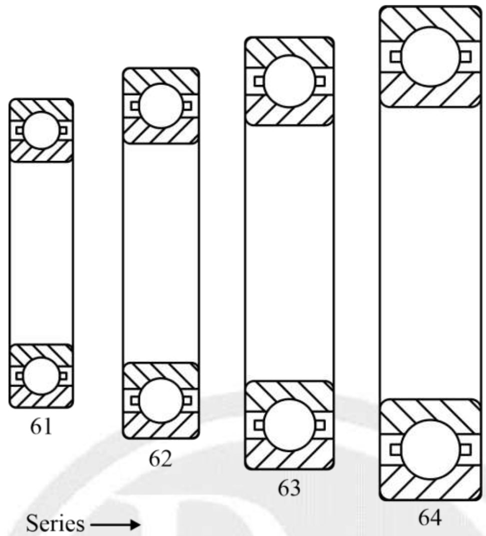


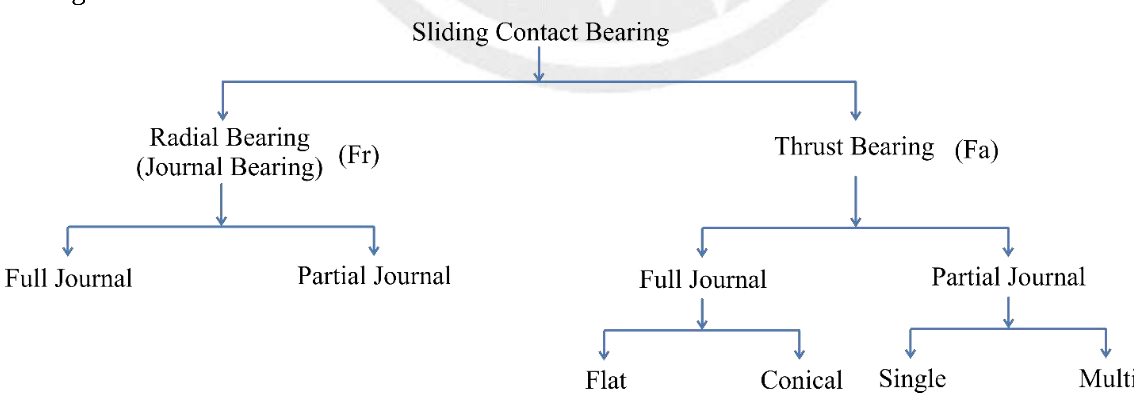
Fig. 7.15: Designation of bearing

- Due to large size balls, load carrying capacity is high.

7. 5 Introduction of sliding contact bearings

- Relative motion between the shaft and the support is sliding motion.
- Due to sliding motion friction, wear and heat generation is very high, hence lubrication is required.

  Applications: Crankshaft bearings in petrol and diesel engines, centrifugal pumps, large size electric motors steam and gas turbines.



7.6 Types of Sliding contact bearing:

7.6.1 Journal Bearings

(1) Full Journal Bearing

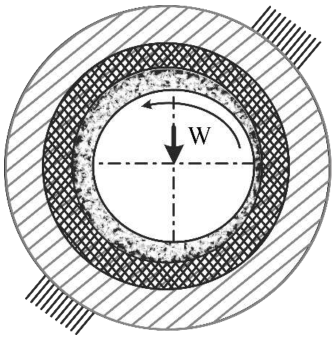


Fig. 7.16: Full journal bearing

- These bearings can support load in any direction
- This journal bearing is fully covered.

(2) Partial Journal Bearing

- These bearings can support radial load in one direction.
  Ex: Axles of Railway wagons

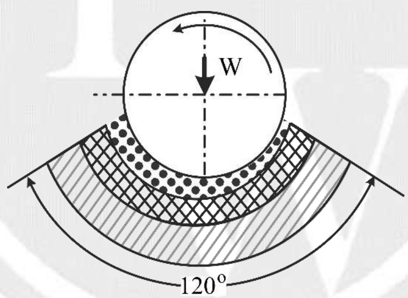


Fig. 7.17: Partial journal bearing

7.6.2 Thrust Bearings

(1) Pivot Bearings

The end of the shaft is in contact with bearing surface.

(a) Flat pivot Bearing:

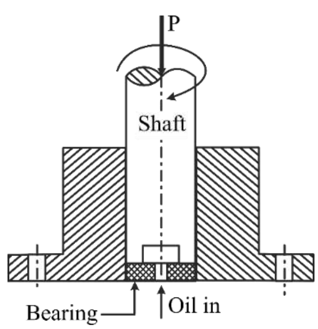


Fig. 7.18: Flat pivot bearing

(b) Conical pivot Bearing: It has more contact area so more friction and has less pressure.

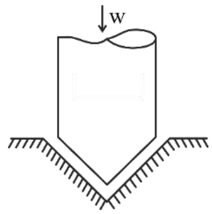


Fig. 7.19: Conical pivot Bearing

(2) Collar Bearings

The shaft is supported by the collars which are in contact with bearings

(a) **Single Collar:**

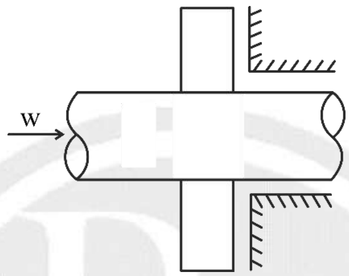


Fig. 7.20: Collar bearing

(b) **Multi Collar:**

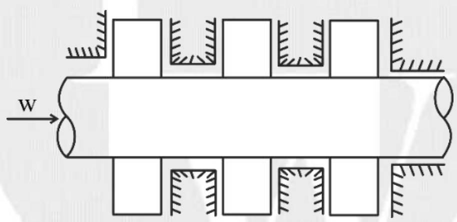
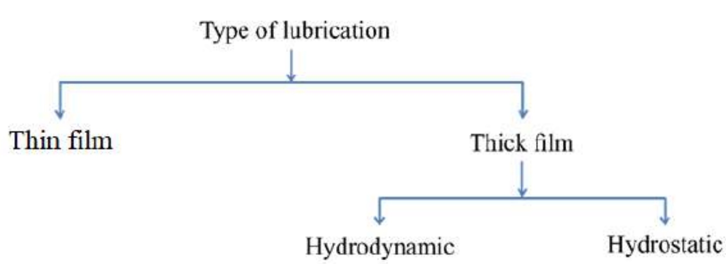


Fig. 7.21: Multi collar bearing

7.7 Lubrication in sliding contact bearing

7.7.1 Types of Lubrication



(1) Thin Film Lubrication:

- There is partial contact between journal bearing surface as the thickness of oil is very less
- It is used where the load & the speed of the journal is low.

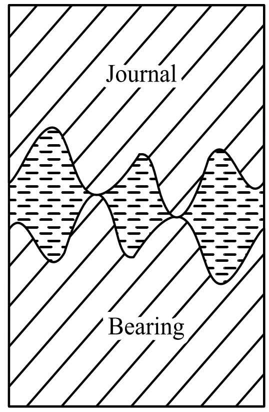


Fig. 7.22: Thin film lubrication

(2) Thick Film lubrication

- Due to thick film of lubricating oil there is no contact between journal & bearing surface
- It is used where the load & the speed of the journal is high

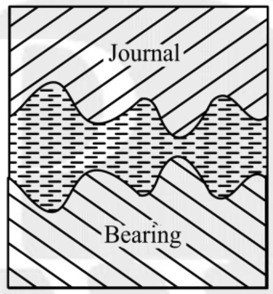


Fig. 7.23: Thick Film lubrication

(a) Hydrodynamic Lubrication:

In Hydro dynamic lubrication, the lubrication is done by the speed of journal. Initially at low speeds of journal there is a partial contact between the surface as the journal starts to rotate a concurring film is developed in the direction of motion where the pressure starts to build and at certain speed the journal is completely separated from the surface.

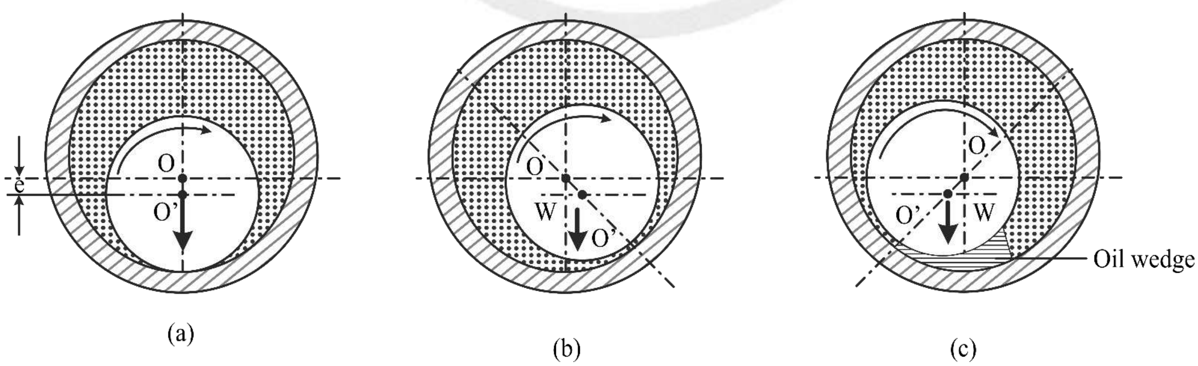


Fig. 7.24: Hydrodynamic Lubrication

Fig. (a): Journal is at rest: Full contact between journal and bearing.
Fig. (b): Journal starts to rotate: Partial contact between journal and bearing.
Fig. (c): Journal rotating at high speed: No contact between journal and bearing.

(b) Hydrostatic Lubrication

In hydrostatic lubrication, the lubricating oil is pressurised by external source as the speed of the journal is not enough.

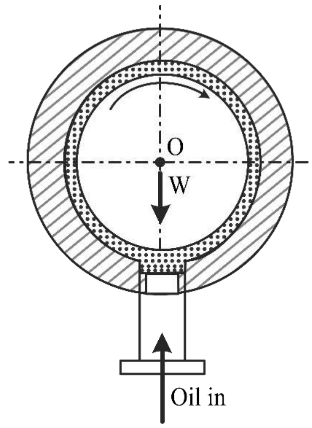


Fig. 7.25: Hydrostatic Lubrication

7.8 Analysis of Hydrodynamic Journal Bearing

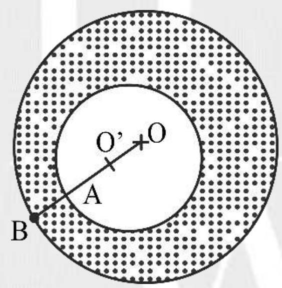


Fig. 7.26: Hydrodynamic journal bearing

R = Radius of Bearing = OB

D = Diameter of Bearing = 2R

r = radius of shaft / journal = O'A

d = diameter of shaft / journal = 2r

L = Length of Bearing

ω = speed of shaft in rad/s $= \dfrac{2\pi N}{60}$

N = speed of shaft in rpm

n_s = speed of shaft in rps $= \dfrac{N}{60}$

W = Radial load in bearing

p = Bearing pressure

z = dynamic viscosity of lubricant in Pa – s

μ_e = Equivalent coefficient of friction

e = eccentricity = OO'

- **Radial Clearance (C_R):** $C_R = R - r$
- **Diametral Clearance:** $C_D = D - d = 2C_R$
- **Minimum Oil thickness (h_0):**

  OB = OO' + O'A + AB

  $\Rightarrow R = e + r + h_0$

  $\Rightarrow h_0 = R - r - e$

  $\Rightarrow h_0 = C_R - e$
- **Eccentricity ratio (ε)**

  $$\varepsilon = \frac{e}{C_R} = \frac{C_R - h_0}{C_R} = 1 - \frac{h_0}{C_R}$$
- **Bearing pressure (p):** $p = \frac{W}{Ld}$
- **Equivalent coefficient of friction (μ_e) as per Petroff's equation:**

  $$\mu_e = 2\pi^2\left(\frac{zn_s}{p}\right)\left(\frac{r}{C_R}\right)$$

  Where, μ_e = equivalent coefficient of friction
- **Bearing characteristic number (BCN):** $BCN = \frac{zn_s}{p}$
- **Frictional torque loss due to friction:** $T_f = \mu_e w_r$
- **Power Loss:** $\omega T_f = 2\pi n_s T_f = \frac{2\pi N}{60} T_f$
- **Heat Generation rate (Hg):** H_g = Power loss
- **Some field number (S):** $S = \left(\frac{zn_s}{p}\right)\left(\frac{r}{C_R}\right)^2$

7.9 McKee's Investigation Curve

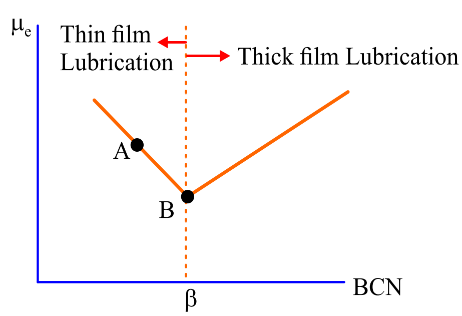


Fig. 7.27: McKee's Curve

- Bearing modulus **(β):** BCN for least coefficient of friction is known as bearing modulus.
- If we keep BCN at B, any reduction in velocity will shift the point toward A. This will increase μ_e, which will increase Temperature. Increase in temperature will reduce viscosity z, which will reduce BCN. Reduction in BCN will shift point more towards left side of A. This will keep happening until shaft will come in contact with the bearing. So BCN is not kept near B.
- Generally, BCN is kept between 3 β to 15 β.

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8 DESIGN OF GEAR

8.1 Introduction of spur gear

Spur Gear

- Used to connect parallel shafts.
- Teeth are parallel to axis.
- Due to sudden engagement, noise and vibration at high speed.
- Only radial load, no axial load.

Fig.8.1: Spur gear

8.2 Force analysis in spur gear

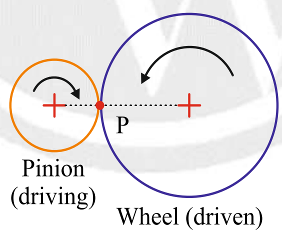

Fig.8.2: Spur gear line diagram

Consider,

m = module of gear

T_p = Number of teeth on pinion

T_w = Number of teeth of wheel

R_p = pitch circle radius of pinion $= \dfrac{mT_p}{2}$

R_w = pitch circle radius of wheel $= \dfrac{mT_w}{2}$

Φ = pressure angle

G = Gear ratio $= \dfrac{T_w}{T_p}$

N_p = Speed of pinion in rpm

ω_p = Speed of pinion in rad/s $= \dfrac{2\pi N_p}{60}$

N_w = Speed of wheel in rpm

ω_w = Speed of wheel in rad/s $= \dfrac{2\pi N_w}{60}$

F_t = tangential component of force

F_r = radial component of force

F_n = Normal force

M_{tp} = Torque in pinion

M_{tw} = Torque transmitted to wheel

- Since the driving gear rotates due to some external force its tangential component of force is opposite in direction of rotation
- In driven gear tangential component is in the same direction of rotation.
- F_r is drawn from pitch point towards centre of gear

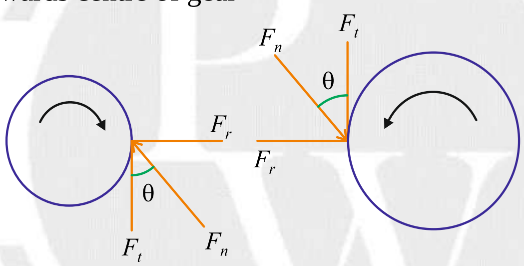

Fig.8.3: Force components on spur gear

- $F_t = F_n \cos\phi$
- $F_r = F_n \sin\phi = F_t \tan\phi$
- $M_{tp} = F_t R_p$
- $M_{tw} = F_t R_w$
- $\dfrac{N_p}{N_w} = \dfrac{T_w}{T_p}$
- $(\text{Power})_p = (\text{Power})_w$

$$\Rightarrow \omega_p M_{tp} = \omega_w M_{tw}$$

$$\Rightarrow \frac{2\pi N_p}{60} M_{tp} = \frac{2\pi N_w}{60} M_{tw}$$

$$\Rightarrow N_p M_{tp} = N_w M_{tw}$$

8.3 Design of spur gear

8. 3. 1 Beam strength of gear

Beam Strength:

Beam strength of a Spur gear is maximum value of tangential force that the teeth of Gear can sustain without failure.

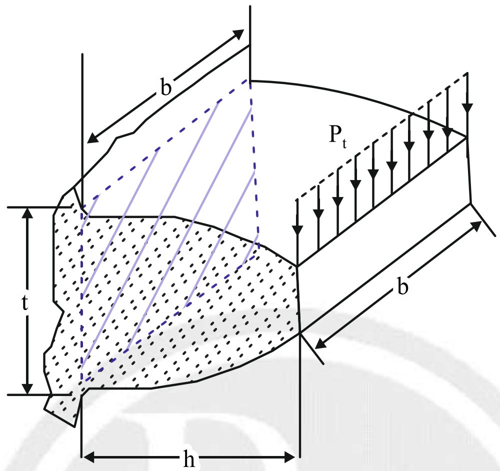

Fig.8.3: Spur gear tooth

Assumptions:

(a) Axial compressive stress due to F_N which is negligible

(b) F_T is uniformly distributed over the width

(c) Only one pair of teeth are in contact at a time

(d) F_T is static

Lewis Formula:

$$P_t = mb\sigma_b Y \frac{K_1}{K_2}$$

Where,

m = module
b = face width of gear
σ_b = bending stress in gear tooth
Y = Lewis form factor
K_1 = product of factors which are less than 1
K_2 = product of factor which are greater than 1
If no factors are given then take K_1 & K_2 = 1

For Design

- Out of pinion and gear, design by using the gear which is weaker.
- If material of both wheel and pinion is same, then pinion will always be weaker.

- If material of wheel and pinion is different, calculate product of permissible bending stress and Lewis form factor for pinion $\left[\left(\sigma_{b,p}Y\right)_p\right]$ and wheel $\left[\left(\sigma_{b,p}Y\right)_w\right]$

  If $\left(\sigma_{b,p}Y\right)_p < \left(\sigma_{b,p}Y\right)_w$, then pinion will be weaker.

  If $\left(\sigma_{b,p}Y\right)_p > \left(\sigma_{b,p}Y\right)_w$, then wheel will be weaker.

- For weaker gear during design, put σ_b = permissible bending stress of that gear $\left(\sigma_{b,p}\right)$, then we will get the expression of beam strength:

  $$P_t = mb\sigma_{b,p}Y\frac{K_1}{K_2}$$

8. 3. 2 Wear strength of spur gear (S_w)

Consider,

D_p = pitch circle diameter of pinion,
D_w = pitch circle diameter of wheel,
b = face width
E_p = Young Modulus of pinion,
E_w = Young Modulus of wheel,
σ_{es} = surface endurance limit
BHN = Brinell hardness Number

Wear strength of spur gear (S_w)

$$S_w = D_p bQK$$

Where,
D_p = pitch circle diameter of pinion
b = face width

Q = ratio factor $= \frac{2G}{G \pm 1} = \frac{2D_w}{D_w \pm D_p}$ [+ sign is used for external gearing and – sign is used for internal gearing]

$$K = \text{Load Stress Factor} = \frac{\sigma_{es}^2 \sin\phi\left[\frac{1}{E_p} + \frac{1}{E_w}\right]}{1.4} = 0.16\left(\frac{BHN}{100}\right)^2$$

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9 DESIGN OF SPRING

9.1 Introduction

A spring is defined as an elastic body, whose function is to distort when loaded and to recover its original shape when the load is removed.

The important functions and applications of springs are:

- to absorb shocks and vibrations e.g. vehicle suspension to store energy, e.g. clocks, toys
- to control motion by maintaining contact between two elements e.g. cams and followers
- to measure force, weighing balances and scales.

9. 2 Helical Spring

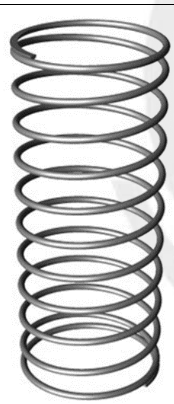

Fig.9.1: Open coil Helical spring

Fig.9.2: Close Coil Helical spring

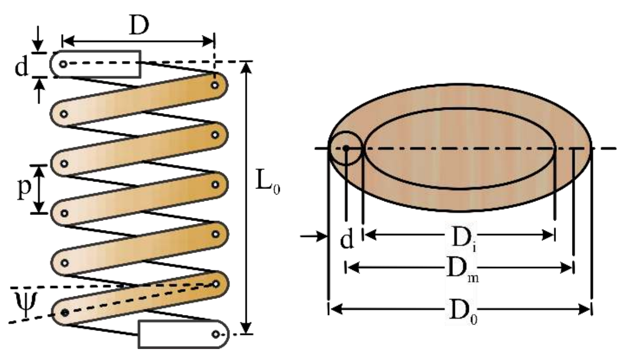

Fig.9.3: Helical spring

In the helical spring shown in figure 9.1.

$p =$ Pitch,

$L_o =$ Free length of spring,

$\Psi =$ Coil angle or helix angle,

$n =$ Number of actives coils,

$d =$ Wire diameter,

$D_0 =$ Outer diameter of coil,

$D_i =$ Inner diameter of coil and

$D_m = D =$ Mean diameter of coil $= 2R$,

$C =$ Spring index $= \dfrac{D}{d}$

Note:
- If $\Psi < 10^\circ \rightarrow$ Close coil helical spring
- If $\Psi > 10^\circ \rightarrow$ Open coil helical spring

9. 2. 1 Analysis of close coil Helical spring

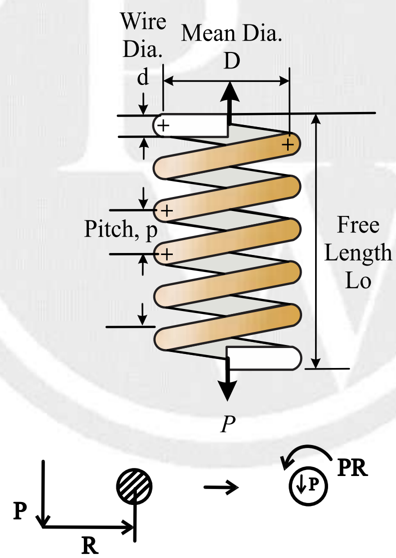

Fig.9.4: Analysis of close coil Helical spring

(i) Maximum Shear Stress (τ_{max})

- $\tau_{max} = \dfrac{16PR}{\pi d^3} + \dfrac{4P}{\pi d^2}$

  [Neglecting Curvature effect]

- $\tau_{max} = \dfrac{16PR}{\pi d^3}$

  [Neglecting curvature effect and effect of direct shear]

- $\tau_{max} = k_w \dfrac{16PR}{\pi d^3}$

  [Considering Curvature effect and effect of primary shear] where, k_w is wharl's correction factor.

  $$k_w = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

(ii) Stiffness of spring (k)

$$k = \frac{Gd^4}{64R^3n}$$

(iii) Deflection of spring (δ)

$$\delta = \frac{P}{k} = \frac{64PR^3n}{Gd^4}$$

(iv) Strain energy (U)

$$U = \frac{1}{2}P\delta = \frac{1}{2}k\delta^2 = \frac{32P^2R^3n}{Gd^4}$$

9.3 Spring in series and parallel

Case 1: Spring are in series:

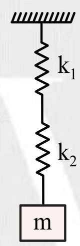

Fig.9.5: Spring in series

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}$$

Case 2: Springs are in parallel:

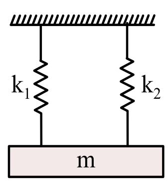

Fig.9.6: Spring in parallel

$$k_e = k_1 + k_2$$

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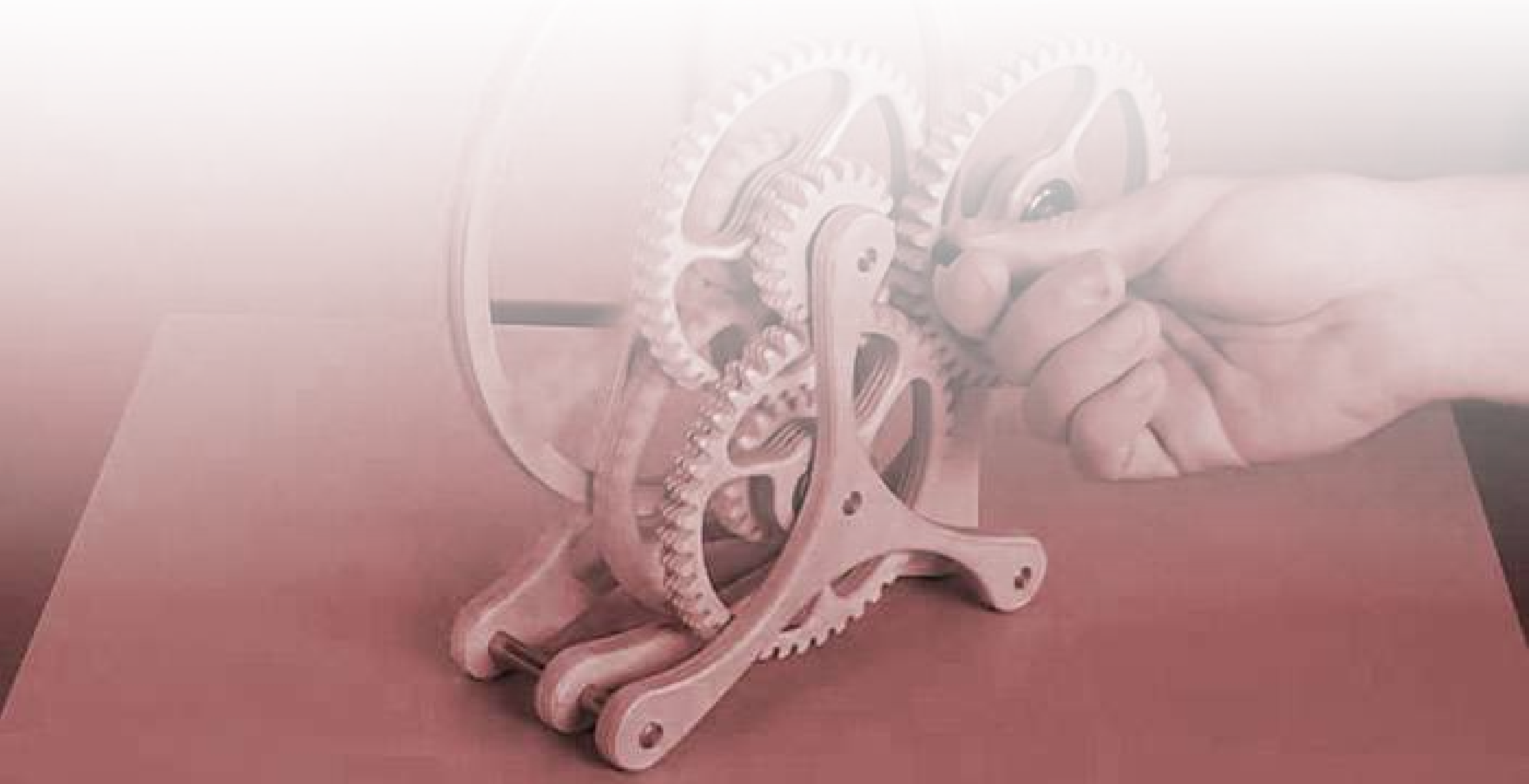
Theory of Machines



Theory of Machines

INDEX

1 SIMPLE MECHANISMS

1.1 Types of Constrained Motion

- **Completely Constrained:** Motion in definite direction of force applied.
- **Successfully Constrained:** Motion is possible in more than one direction but it is made in only one direction with the help of third link.
- **Incompletely (unconstrained) Motion:** Motion is possible in more than one direction depending on direction of force.

1.2 Resistant Body

- A body/link which can transfer motion/power at least in one direction

1.3 Kinematic Link

- **Rigid link:** Crank, Connecting rod, Piston, etc.
- **Flexible link:** Belt, Rope, Chain drives, etc.
- **Fluid link:** Hydraulic brakes, Ram, Jack, etc.

Links may also be classified as

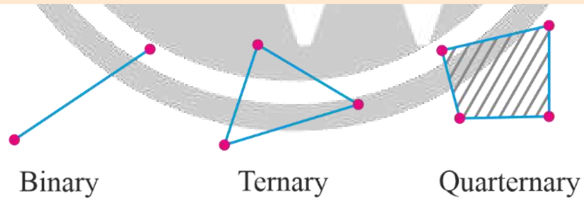


Fig. 1.1 Links

1.4 Kinematic Pair

- Joint of two links having constrained relative motion between them.

(a) According to nature of Relative Motion:

- Turning pair (Revolute/Pin Joint)
- Sliding Pair (Prismatic Pair)
- Screw Pair (Helical Pair)
- Rolling Pair
- Spherical Pair

(b) According to nature of contact:

- **Lower Pair:** Surface/Area contact
- **Higher Pair:** Line/Point contact.

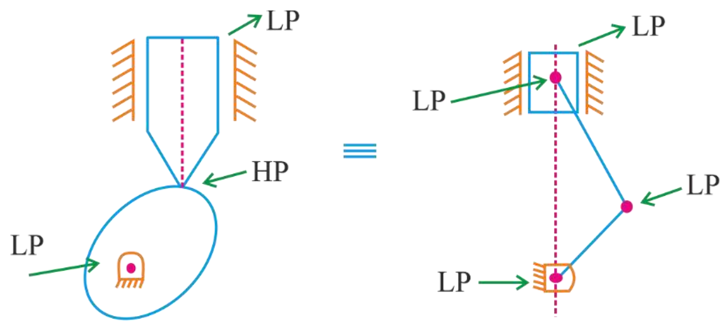


Fig. 1.2 Cam and Follower

$$\boxed{1\,\text{Higher Pair} \cong 2\,\text{Lower Pairs}}$$

(c) According to nature of mechanical constraint:

- **Self-closed pair:** Permanent contact between links.
- **Forced closed pair:** Links are in contact either by gravity action or spring action.

1.5 Types of Joint

- Binary Joints →
- Ternary Joint →
- Quaternary Joint →

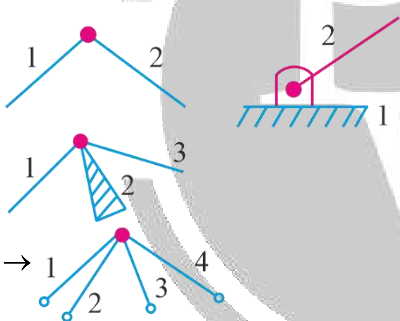


Fig. 1.3 Joints

> **Note:**
>
> If n links are connected at joint ≡ (n-1) Binary Joints

1.6 Kinematic Chain

- Closed chain having constrained relative motion.
- If one link of kinematic chain is fixed it becomes mechanism.

1.7 Degree of Freedom (Mobility)

- Minimum number of independent variables required to define position/motion of the system.
- For rigid body in space:

$$\boxed{\text{DOF} = 6 - \text{Number of restraints}}$$

- Restraint represents those motions which are not possible.
- For 2-D planar mechanisms:

$$\boxed{F = 3(l-1) - 2j - h}$$

l = number of links

h = number of higher pairs.

j = number of binary joints.

1.8 Case of Redundant Link

- Link which don't produce any extra constraint. Such link should not be counted while finding DOF.

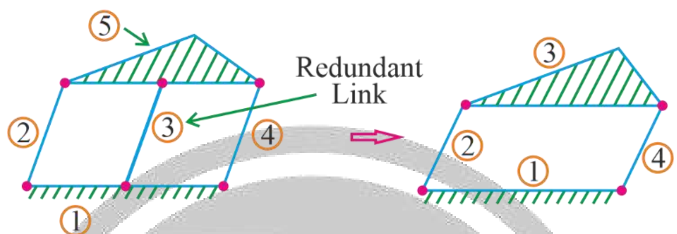


Fig. 1.4 Mechanism with Redundant link

1.9 Redundant Degree of Freedom

- Sometimes one or more links of a mechanism can be moved without causing any motion to rest of links of mechanism. such link is said to have redundant degree of freedom (F_r)

$$\boxed{F = 3(l-1) - 2j - h - F_r}$$

F_r = number of dummy motion (number of redundant DOF)

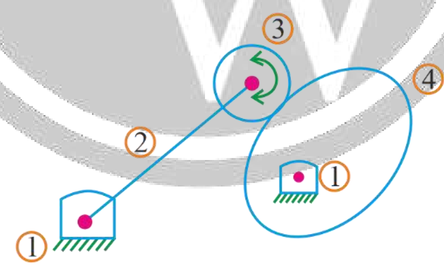


Fig. 1.5 Mechanism with Redundant DOF

Link 3 $\rightarrow$ have redundant DOF

Physical significance of DOF of Mechanism

F = 3 (4 –1) – 2 (3) – 1–1 = 1

- If F = 1 $\rightarrow$ Kinematic Chain
- If F = 0 $\rightarrow$ Frame /structure
- If F < 1 $\rightarrow$ Super structure/Indeterminate Structure.
- If F > 1 $\rightarrow$ Unconstrained Chain

- DOF of open chain:

$\boxed{F = 3l - 2j - h} \rightarrow$ No link is fixed.

1.10 Grubler's Criteria

- Origin of first kinematic chain,

$$F = 1, h = 0$$
$$F = 3(l-1) - 2j - h$$
$$1 = 3l - 3 - 2j - 0 \quad \Rightarrow \quad \boxed{3l - 2j - 4 = 0}$$

$l_{min} = 4$ to make a kinematic chain with all lower pairs.

- Four bar mechanism
- Single slider crank mechanism
- Double slider crank mechanism

1.11 Four bar Mechanism

- Four links with four turning pairs
- Input and output links $\rightarrow$ rotates only
- Complete rotation $\rightarrow$ Crank
- Partial rotation $\rightarrow$ Rocker

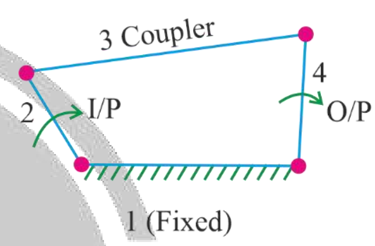


Fig. 1.6 Four bar chain mechanism

1.12 Grashoff's Law

- For continuous relative motion:

$\boxed{(s+l) \le (p+q)} \rightarrow$ Class-I mechanisms

s = length of shortest link
l = length of longest link
p, q = length of other two links.

Case: I $\quad$ **($s + l$) < ($p + q$): Law satisfies [class I]**

- Shortest link fixed: Double crank mechanism.
- Shortest link is coupler: Double rocker mechanism
- Link adjacent to shortest link is fixed: Crank –Rocker Mechanism

Case: II $\quad$ **($s + l$) = ($p + q$): Law satisfies [class I]**

E.g.: $s = 20cm$, $l = 50cm$, $p = 40cm$, $q = 30cm$

Parallelogram Linkage

l – Fixed: Double crank
s – Fixed: Double crank

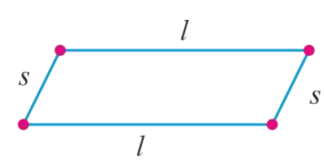


Fig. 1.7 Parallelogram Linkage

Deltoid Linkage

s – Fixed: Double crank
l – Fixed: Crank-rocker mechanism

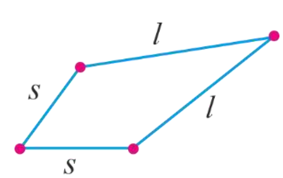


Fig. 1.8 Deltoid Linkage

Case: III (*s* + *l*) > (*p* + *q*): Law does not satisfy

- Class-II mechanism
- Double rocker mechanism

1.13 Transmission Angle for 4-bar Mechanism

Angle between coupler link and output link

= transmission angle.

$$\mu\begin{cases}\text{minimum;} & \theta = 0^\circ \\ \text{maximum;} & \theta = 180^\circ\end{cases}$$

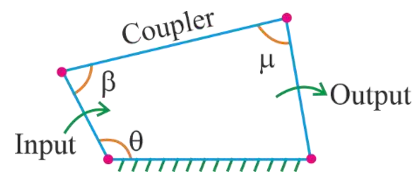


Fig. 1.9 Transmission angle

1.14 Single Slider Crank Chain

- In 4-bar chain, one turning pair replaced by sliding pair.

(a) First Inversion of single slider-crank mechanism:

Cylinder is fixed

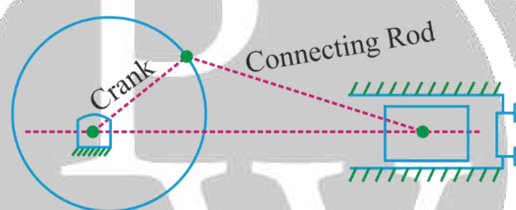


Fig. 1.10 Single slider Crank Chain

Rotary motion $\longleftrightarrow$ Reciprocating motion

$$\left\{\begin{array}{l}\text{Application:} \\ \bullet\ \text{Reciprocating Engine} \\ \bullet\ \text{Reciprocating compressor}\end{array}\right\}$$

(b) Second inversion of slider crank mechanism:

Crank Fixed

Application:

- Whitworth quick return mechanism
- Rotary engine (Gnome engine)

Whitworth Quick Return Mechanism:

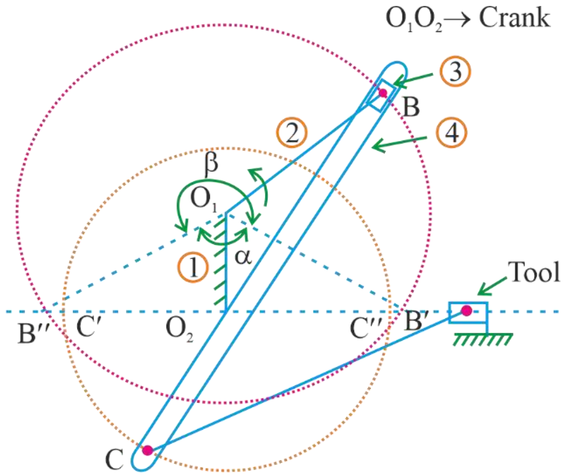


Fig. 1.11 Whitworth Quick Return Mechanism

Forward stroke: (B' B B")

$$\boxed{\text{Quick Return Ratio} = \frac{\text{time of cutting}}{\text{time of return}} = \frac{\beta}{\alpha} > 1}$$

(c) Third inversion of slider Crank Mechanism:

Connecting Rod is fixed.

Application:

- Oscillating cylinder Engine.
- Crank & slotted lever mechanism.

  Used in shaping and Slotting machine

  Velocity is max at the mid stroke.

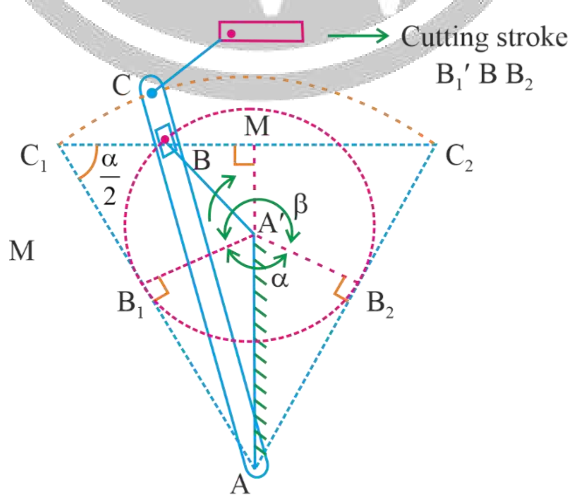


Fig. 1.12 Crank and slotted lever mechanism.

CA = length of slotted Bar.

A′B = length of crank

$$\boxed{QRR = \frac{(\text{time})_{\text{cutting}}}{(\text{time})_{\text{return}}} = \frac{\beta}{\alpha} > 1}$$

AA′ = length of connecting rod.

Stroke length = $2(C_1A) \times \frac{A'B_1}{AA'}$

(d) Fourth inversion of slider crank mechanism:

Slider is fixed

Application:

- Hand Pump / Bull Engine.

1.15 Double Slider Crank Chain

- Four bar chain with two turning pairs; two sliding pairs.
  - **(a) First Inversion:** Slotted bar is fixed.
    Application: Elliptical Trammel used to draw ellipse
  - **(b) Second Inversion:** Any one slider is fixed.
    Application: Scotch Yoke Mechanism [To convert reciprocating motion into rotation and vice versa]
  - **(c) Third Inversion:** Link connecting sliders is fixed.
    Application: Oldham coupling [used to connect shafts having lateral misalignment.]

$$\boxed{\omega_{driver} : \omega_{driven} = 1.1}$$

1.16 Mechanical Advantage

- Ratio of output (Force/torque) to the input (force/torque), at any instant.

$$\boxed{MA = \frac{F_o}{F_I} = \frac{T_o}{T_I} = \frac{\text{Load}}{\text{Effort}}}$$

- Relation between *MA* and efficiency

$$\eta = \frac{p_o}{p_i} = \frac{F_o \cdot V_0}{F_i V_i} = \frac{T_o \cdot \omega_o}{T_i \cdot \omega_i}$$

$$\boxed{MA = \eta \cdot \frac{V_i}{V_o} = \eta \frac{\omega_i}{\omega_o}}$$

1.17 Toggle Positions in 4-bar Chain

It represents extreme position of output link as a rocker in 4-bar chain.

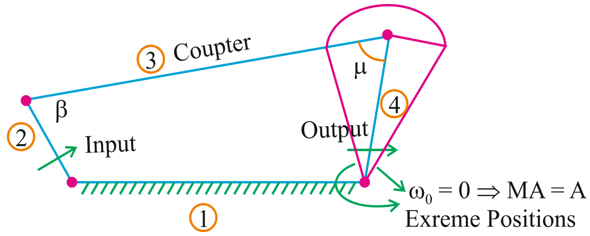


Fig. 1.13 Toggle Positions

1.18 Intermittent Motion Mechanism

- Motion is repeated after specific time internal while input is given continuously.
- Geneva mechanism $\rightarrow$ Indexing
- Ratchet Mechanism $\rightarrow$ Clocks

1.19 Straight Line Motion Mechanism

Exact Straight-line motion mechanism

- Hart's mechanism
- Scott Russell Mechanism
- Modified Scott Russell mechanism.

Approx. straight-line motion mechanism

- Grass Hooper's Mechanism
- Robert's mechanism
- Tchebichoff's mechanism
- Watt indicator mechanisms
- Pantograph

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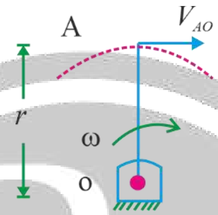
2 VELOCITY AND ACCELERATION ANALYSIS

2.1 Velocity of a Point on Link

- Relative velocity of one point of a link w.r.t. other point is always perpendicular to the link.
- For a link undergoing pure translation, relative velocity between points is zero.



$\boxed{V_{AO} = r.\omega}$ perpendicular to line

Fig. 2.1 Vector

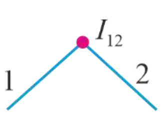
2.2 Instantaneous Centre (I-centre)

- Any link at any instant can be assumed in pure rotation about imaginary point in space known as I-centre.
- I-centre may keep on changing as links move.
- For two bodies having velocity motion between them, there is an I-centre.
- In a mechanism, if l = number of links. Then

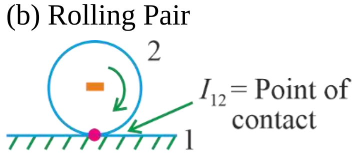
  Number of I-Centres = $\dfrac{l(l-1)}{2}$

- Basic I-centres:

  (a) Turning Pair



  (b) Rolling Pair



  (c) Sliding Pair →

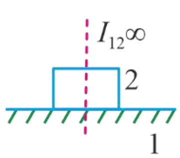


Fig. 2.2 I-centre

- **Centrode:** Locus of I-centres for relative motion between links.
- **Axode:** The line passing through 1-centre and perpendicular to the plane of motion is known as instantaneous axis. The locus of instantaneous axis for a link during the whole motion is known as Axode. It is surface

Motion	Centrode	Axode
General	Curve	Curved surface
Rectilinear	Straight Line	Plane surface
Pure Rotation	Point	Straight Line

2.3 Kennedy's Theorem

- For example. For link 2, 3, 5 I_{23}, I_{35} I_{25} will be in a line, for any three links having relative motion among them, their three I-centres must lie in a straight line.

2.4 Angular Velocity Ratio Theorem

$$\boxed{\omega_m\left(I_{mn}I_{1m}\right)=\omega_n\left(I_{mn}I_{1n}\right)}$$

If I_{1m} and I_{1n} lies at same side of I_{mn} then sense of and ω_m & ω_n will be same

2.5 Maximum Velocity During Cutting Stroke and Return Stroke in Crank-Slotted Bar

- During cutting stroke:

$\omega \rightarrow$ Uniform angular velocity of crank

$r \rightarrow$ crank radius

$l_1 \rightarrow$ length of slotted bar

$l_e \rightarrow$ length of connecting rod (fixed link)

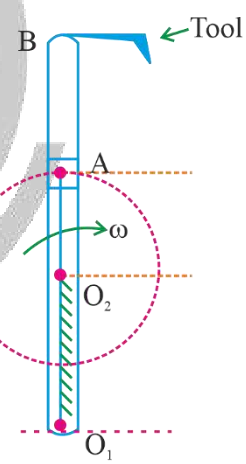


Fig. 2.3 Crank and Slotted lever mechanism

$$\omega_{S.L.\text{max cutting stroke}}=\frac{r\omega}{l+r}$$

$$\omega_{S.L.\text{max return stroke}}=\frac{r\omega}{l-r}$$

> **Note:**
>
> $\omega_{S.L.}$ & Speed of cutter will be max at mid stroke when slotted lever becomes vertical

- For any rotating link, we have two components of accelerations

(i) Radial/centripetal acceleration

(ii) Tangential acceleration

2.5.1 Acceleration Analysis

$$\boxed{\left(a_{BA}\right)_{\text{radial}}=\frac{V_{BA}^2}{(BA)}=(AB)\cdot\omega^2}$$

$$\boxed{\left(a_{BA}\right)_{\tan g.}=(AB)\cdot\alpha}\rightarrow\perp^r \text{ perpendicular to link}$$

$$\left|a_{BA}\right|=\sqrt{\left(a_{BA}\right)_{\text{radial}}^2+\left(a_{BA}\right)_{\text{tang.}}^2}$$

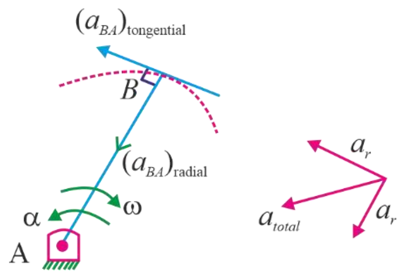


Fig. 2.4 Acceleration Components

2.6 Coriolis Acceleration

- This acceleration is associated with the slider when it is sliding on a rotating body.

$$a_c = 2\times V\times\omega$$

V = sliding velocity of slider

ω = angular velocity of body on which slider is sliding.

Direction of:

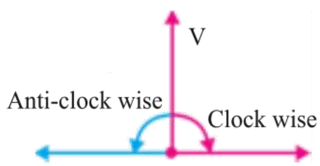
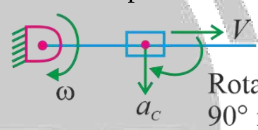
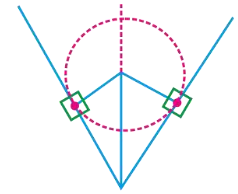
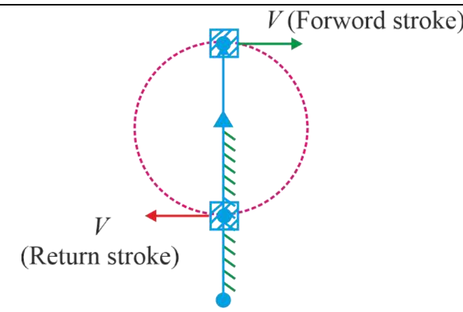
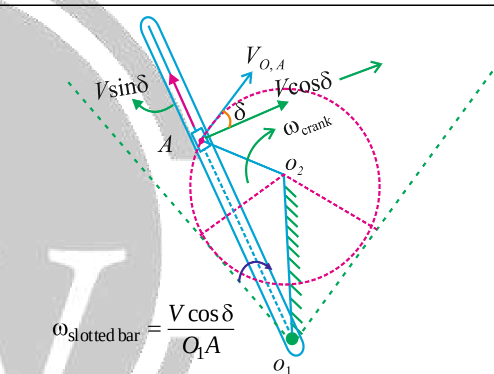
1. Take the sense of ω 2. Rotate V in that sense by 90°  	For Example: 

Fig. 2.5 Coriolis Acceleration

Note:
- Coriolis Acceleration exists in Crank and slotted bar QRMM.
- At four positions, a^c becomes zero.

At the extreme positions. At extreme ends, $\omega_{\text{slotted bar}} = 0 \Rightarrow a_C = 0$	 **Fig. 2.6 Coriolis Acceleration**
At the mid-stroke At mid-stroke, slider isn't sliding ($V_s = 0$) $\Rightarrow\ a_C = 0$.	 **Fig. 2.7 Coriolis Acceleration**
• Between extreme position a mid-stroke. $a^C = 2 \times V_{\text{sliding}} \times \omega_{S.L.}$ $\Rightarrow \left[a^C = 2(V\sin\delta)\right] \cdot \left(\frac{V\cos\delta}{O_1 A}\right)$	 **Fig. 2.8 Crank and Slotted Lever Mechanism**

2.7 Rubbing Velocity

- If r = radius of pin at joint O,

  V_r = (Rubbing velocity) = $r(\omega_1 \pm \omega_2)$

  (+) = when links move in opposite direction

  (–) = when links move in same direction.

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3 CAM AND FOLLOWERS

3.1 Introduction

- A cam is mechanical member used to impart desired motion to a follower by direct contact.
- The cam may be rotating or reciprocating whereas the follower may be rotating, reciprocating or oscillating.
- It is used in automatic machine, IC engine, machine tools, printing control mechanism.

Element of Cam & Follower

- A driver member known as the cam.
- A driven member called the follower.
- A frame which supports the cam and guides the follower.

Point to remember:

A cam and follower combination belong to the category of higher pair.

3.2 Types of Cam

3.2.1 According to the shape:

- **Wedge and flat cams**

A wedge cam has a translational motion. The follower can either translate or oscillate.

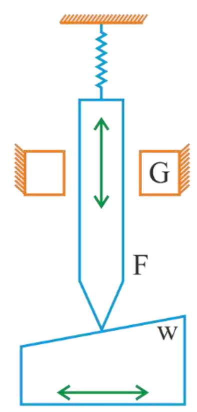


Fig. 3.1 Wedge and flat cams

- **Radial or Disc Cams**

A cam in which the follower moves radially from the centre of rotation of the cam is known as a radial or disc cam.

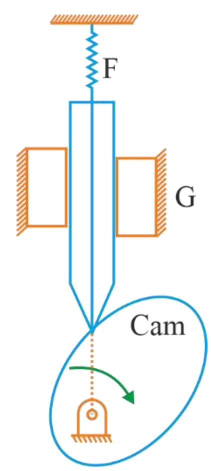


Fig. 3.2 Radial or Disc Cams

- **Spiral cams**
  A spiral cam is a face cam in which a groove is cut in the form of a spiral. It is used in computer.

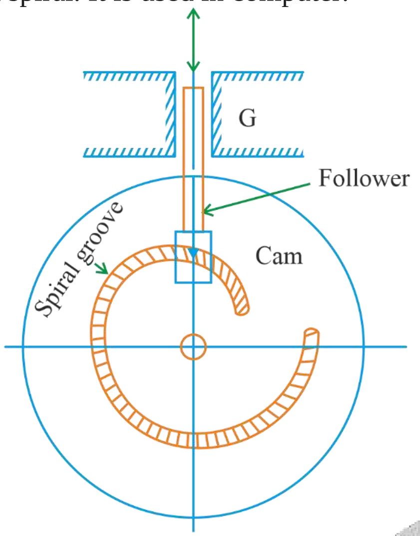


Fig. 3.3 Spiral cams

- **Cylindrical cams**
  In a cylindrical cam, a cylinder which has a circumferential contour cut in surface, rotates about its axis. It is also known as barrel or drum cams.

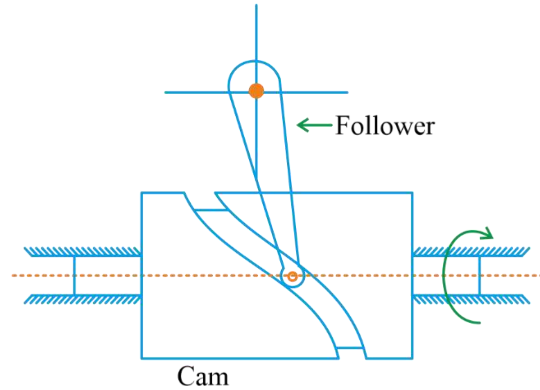


Fig. 3.4 Cylindrical cams

- **Conjugate cams**
  It is a double disc cam and preferred when the requirement is low wear, low noise, better control of the follower, high speed, high dynamic loads etc.

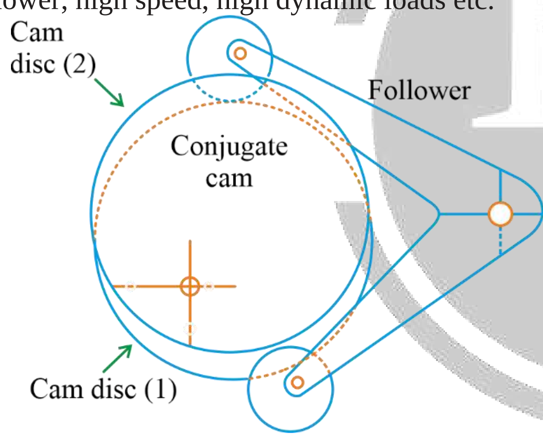


Fig. 3.5 Conjugate cams

- **Globoidal cams**
  It has two types of surfaces i.e. convex or concave. It is used when moderate speed and the angle of oscillation of the follower is large.

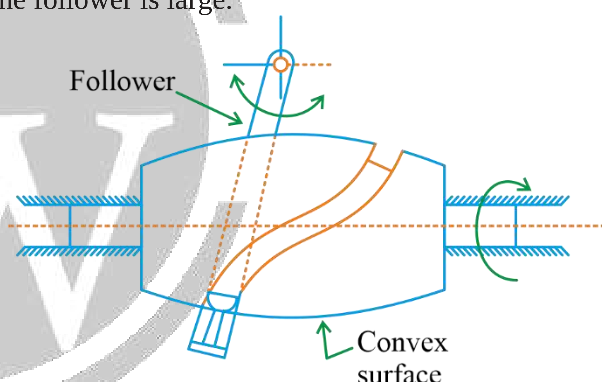


Fig. 3.6 Globoidal cams

3.2.2 According to the follower movement:

- **Rise-Return-Rise (RRR)**
  In this, there is alternate rise and return of the follower with no period of dwells. The follower has a linear or an angular displacement.

- **Dwell-Rise-Return Dwell (D-R-R-D)**
  In this cam, there is rise and return of the follower after a dwell.

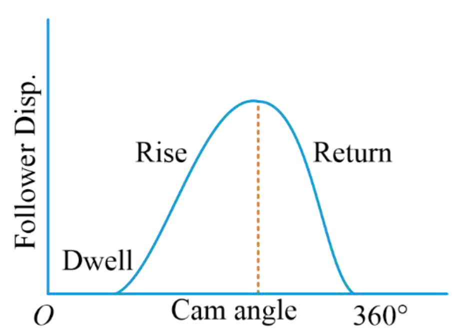


Fig. 3.8

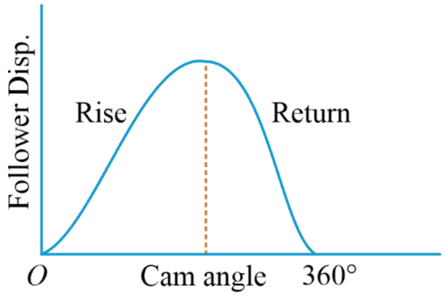
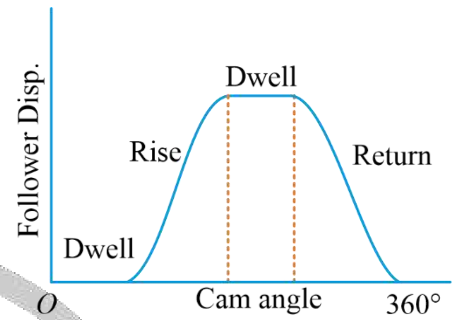


Fig. 3.7

- **Dwell-Rise-Dwell-Return-Dwell (D-R-D-R-D)**

  The dwelling of the cam is followed by rise and dwell and subsequently by return and dwell. In case the return of the follower is by a fall, the motion may be known as Dwell-Rise-Dwell (DRD).



D-R-D-R-D
Fig. 3.9

3.3 Type of followers

3.3.1 According to the Shape

- **Knife-edge follower:**
  Its use is limited as it produces a great wear of the surface at the point of contact

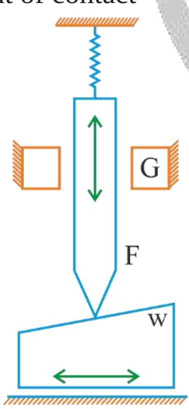


Fig. 3.10 Knife-edge follower

- **Roller follower:**
  At low speeds, the follower has a pure rolling action, but at high speeds, some sliding also occurs.

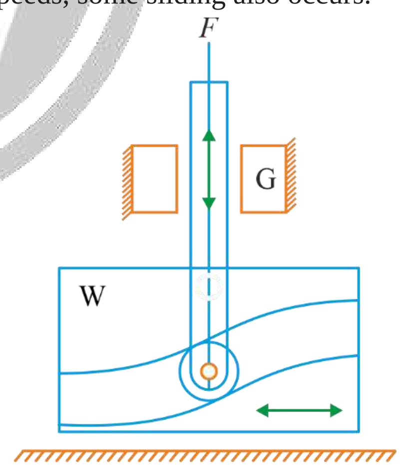


Fig. 3.11 Roller follower

- **Mushroom follower:**
  It does not pose the problem of jamming the cam

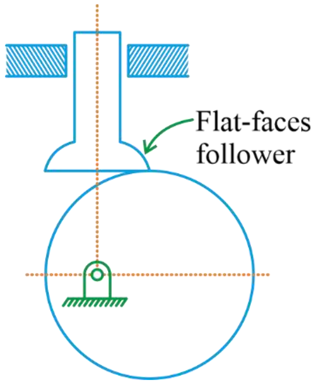


Fig. 3.12 Mushroom follower

3.3.2 According to the Movement

- **Reciprocating follower:**
  In this type, as the cam rotates the follower reciprocates or translates in the guides.

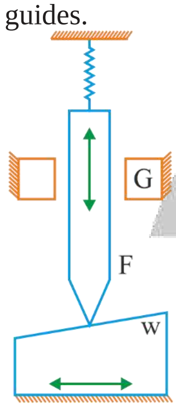


Fig. 3.13 Reciprocating follower

- **Oscillating follower:**
  The follower is pivoted at a suitable point on the frame and oscillates as the cam makes the rotary motion.

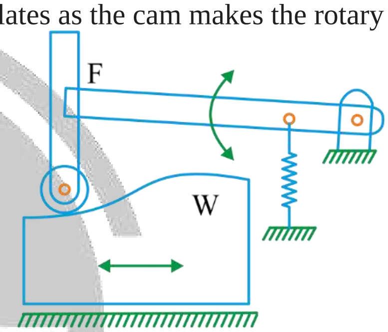


Fig. 3.14 Oscillating follower

3.3.3 According to the Location

- **Radial follower:**
  The follower is known as a radial follower if the line of movement of the follower passes through the centre of rotation of the cam

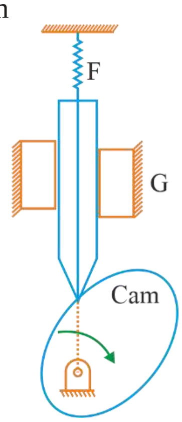


Fig. 3.15 Radial follower

- **Offset follower:**
  If the line of movement of the roller follower is offset from the centre of rotation of the cam, the follower is known as an offset follower.

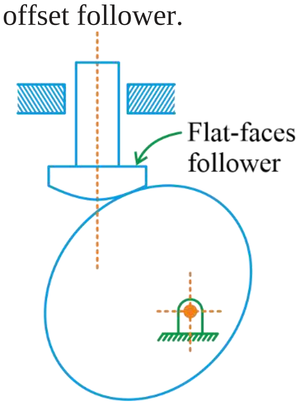


Fig. 3.16 Offset follower

3.4 Terminology and Force Exerted by Cam

3.4.1 Terminology of Cam

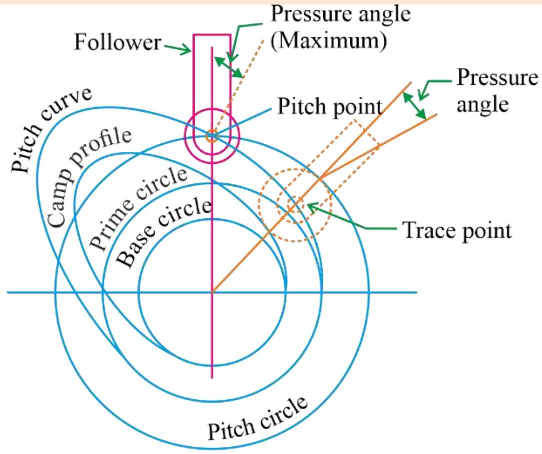


Fig. 3.17 Terminology of cam

- **Base circle**
  It is the **smallest** circle **tangent** to the cam profile drawn from the centre of rotation of a radial cam.
- **Pressure angle:** The pressure angle, representing the steepness of the cam profile, is the angle between the normal to the pitch curve at a point and the direction of the follower motion. It varies in magnitude at all instants of the follower motion.
- **Pitch Point:** It is the point on pitch curve at which the pressure angle is **maximum**.
- **Prime circle:** The **smallest** circle drown **tangent** to the pitch curve is known as prime circle.
- **Angle of Ascent (ϕ_a):** It is the angle through which the cam turns during the time the follower rise.
- **Angle of dwell (δ):** Angle of dwell is the angle through which the cam turns while the follower **remains stationary** at the highest or **lowest position.**
- **Angle of Decent (ϕ_d):** It is the angle through which the cam turns during the time the follower returns to the initial position.
- **Angle of Action:** It is the total angle moved by cam during the time, between the beginning of rise and the end of return of the follower.
- **Point to remember:**
- The dynamic effects of acceleration (jerks) usual speed of the cams.

3.4.2 Force Exerted by Cam

The force exerted by a cam on the follower is always normal to the surface of the cam at the point of contact. The vertical component (F cosα) lifts the follower whereas the horizontal component (F sin α) exerts lateral pressure on the bearing.

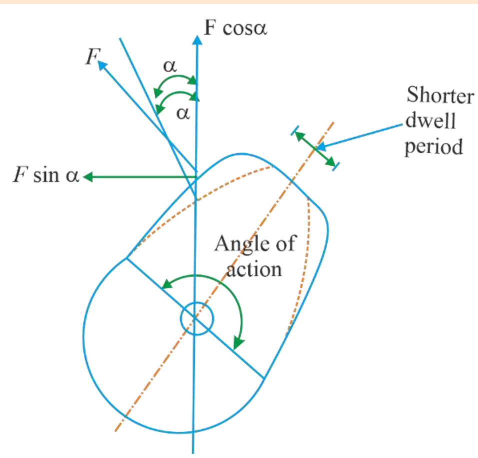


Fig. 3.18 Force Component on cam

3.5 Standard follower Motion Analysis

s = follower displacement (instantaneous)

h = maximum follower displacement

V = velocity of the follower

f = acceleration of the follower

θ = cam rotation angle (instantaneous)

φ = cam rotation angle for the maximum follower displacement

β = angle on the harmonic circle

3.5.1 Simple Harmonic Motion (SHM)

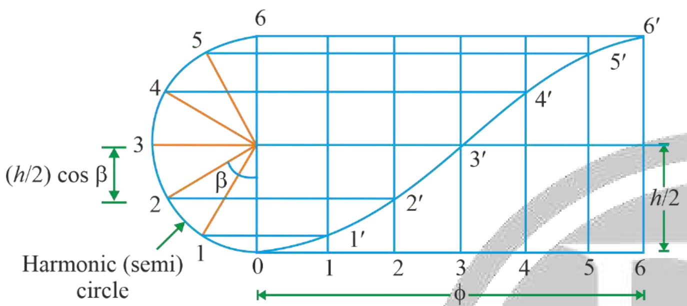


Fig. 3.19 Simple Harmonic Motion (SHM)

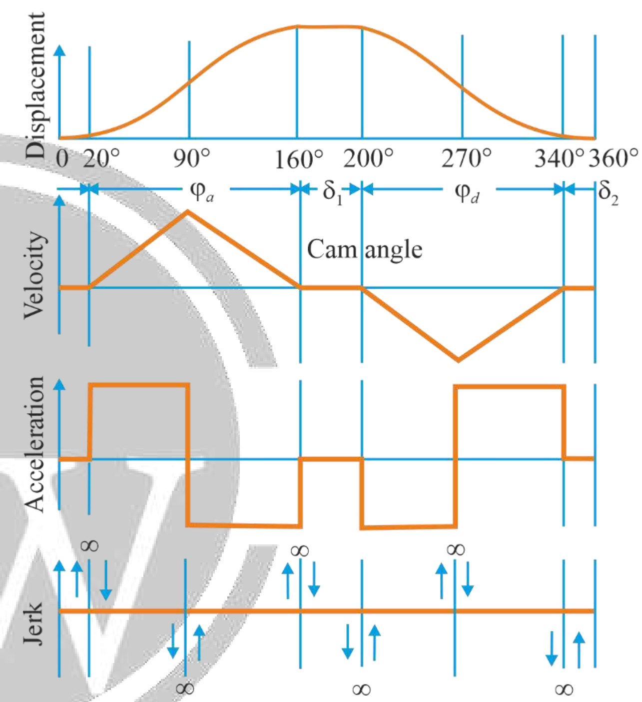


Fig. 3.20 Simple Harmonic Motion (SHM)

- **Displacement of follower**

$$s = \frac{h}{2}\left(1 - \cos\frac{\pi\omega t}{\varphi}\right)$$

- **Maximum velocity**

$$v_{\max} = \frac{h}{2}\frac{\pi\omega}{\varphi} \text{ at } \theta = \frac{\varphi}{2}$$

- **Maximum acceleration**

$$f_{\max} = \frac{h}{2}\left(\frac{\pi\omega}{\varphi}\right) \text{at } \theta = 0$$

3.5.2 Constant Acceleration and Deacceleration (Parabolic)

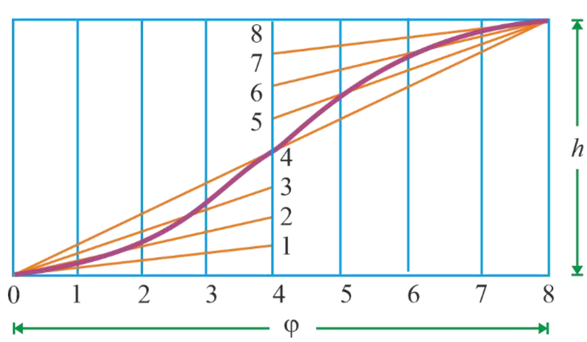


Fig. 3.21 Constant Acceleration and Deacceleration (Parabolic)

- **Displacement of follower**

$$s = \frac{h}{2} \text{ and } t = \frac{\varphi/2}{\omega}$$

- **Maximum velocity of follower**

$$v_{max} = \frac{4h\omega}{\varphi^2}\frac{\varphi}{2} = \frac{2h\omega}{\varphi}$$

at $\theta = \frac{\varphi}{2}$.

- **Acceleration of follower**

$$f = \frac{2h/2}{\varphi^2/4\omega^2} = \frac{4h\omega^2}{\varphi^2}$$

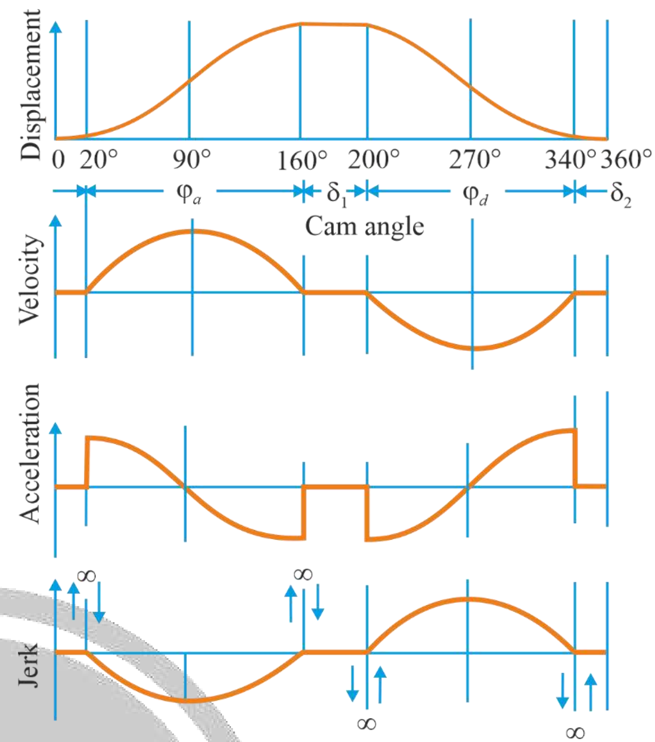


Fig. 3.22 Constant Acceleration and Deacceleration (Parabolic)

3.5.3 Constant Velocity

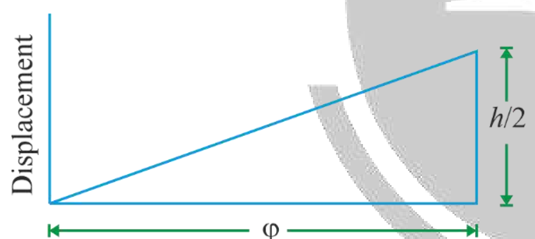


Fig. 3.23 Constant Velocity

- **Displacement of followers**

$$s = h\frac{\theta}{\varphi} = h\frac{\omega t}{\varphi}$$

- **Velocity of followers**

$$v = \frac{ds}{dt} = \frac{h\omega}{\varphi} \text{ constant}$$

- **Acceleration of follower**

$$f = \frac{dv}{dt} = 0$$

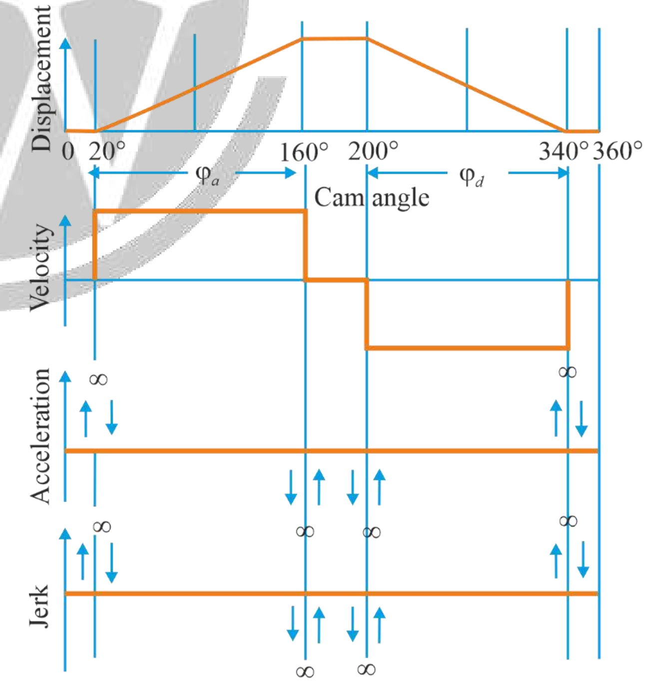


Fig. 3.24 Constant Velocity

3.5.4 Cycloidal

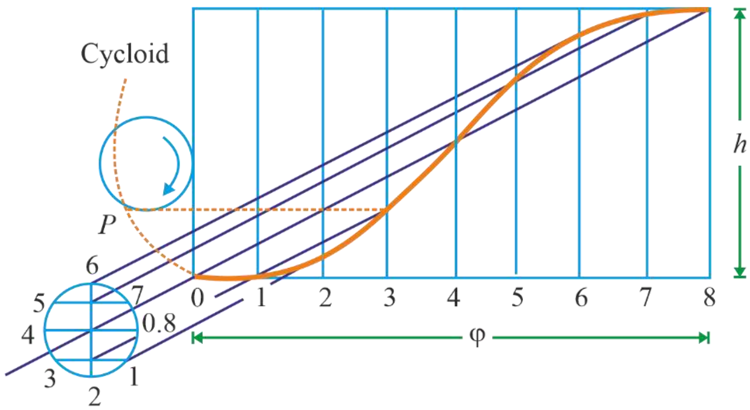


Fig. 3.25 Cycloidal

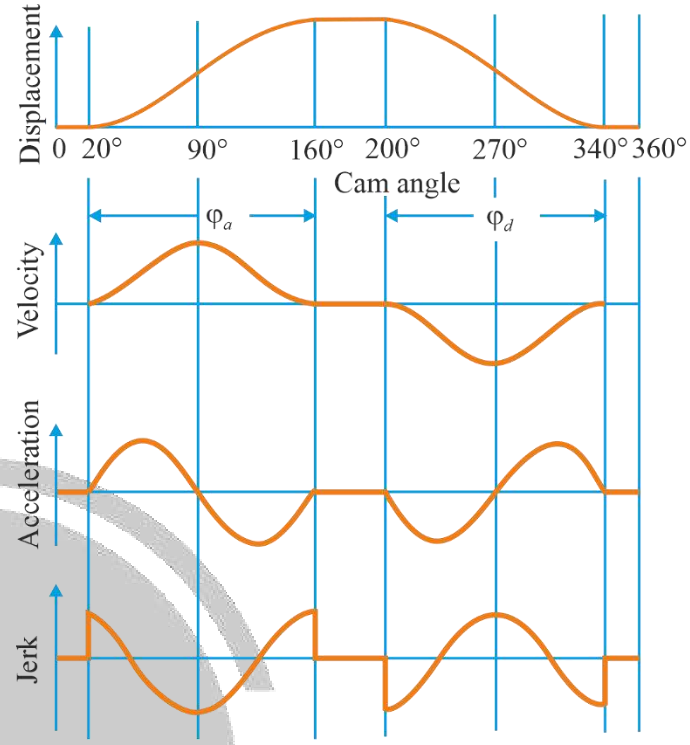


Fig. 3.26 Cycloidal

- **Displacement of follower**

$$s = \frac{h}{\pi}\left(\frac{\pi\theta}{\varphi} - \frac{1}{2}\sin\frac{2\pi\theta}{\varphi}\right)$$

- **Maximum velocity of follower**

$$v_{\max} = \frac{2h\omega}{\varphi} \text{ and } \theta = \frac{\varphi}{2}$$

- **Maximum acceleration of followers**

$$f_{\max} = \frac{2h\pi\omega^2}{\varphi^2} \text{ and } \theta = \frac{\varphi}{4}$$

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4 GEARS AND GEAR TRAIN

4.1 Gears & Its Classification of Gears

- Gears are used to transfer power/motion from one shaft to anther in such away than ratio $\frac{\omega_1}{\omega_2}$ remains constant during entire motion.
- According to relative position of their shaft axis:

4.1.1 Parallel Shaft

1. **Spur gear:**
   - Straight tooth parallel to axis of gear.
   - No axial thrust
   - High impact stress on tooth
2. **Helical Gear:**
   - Straight tooth inclined to axis of rotation.
   - Low impact stress due to gradual load application.
   - Problem of axial thrust.
3. **Double Helical (Herring bone) Gear:**
   - Axial thrust problem eliminated.

4.1.2 Intersecting Shaft

1. **Straight Bevel Gear:**
   - Connects shaft at an angle running at low speed.
   - If size of both gear is same: Mitre gears
   - Subjected to impact stress
2. **Spiral Bevel/Helical Bevel:**
   - Low impact stress
   - Problem of axial thrust

4.1.3 Skew shaft (non-parallel and non-intersecting)

- Pure rolling motion isn't possible.

1. **Crossed Helical Gears:**
   - Two shafts can be set at any angle by choosing suitable helix angle.
2. **Worm Gear:**
   - Very large speed reduction ratio

   Worm (Driver) $\rightarrow$ Very less dia. & very large spiral angle.

   Worm (Wheel) $\rightarrow$ Very large dia. & very less spiral angle.

Note:
- In power transmission, smaller bodies are made drivers.
- Hence generally gears are also known as speed reduction device.

4.2 Terminology of Gear:

(a) Circular Pitch (P_c):

It is distance from one point of tooth to the same point of adjacent tooth measured along pitch circle.

$$p_c = \frac{\pi D}{T}$$

(b) Diametral Pitch (P_d):

It is defined as ratio of number of teeth to diameter of pitch circle

$$p_d = \frac{T}{D}$$

(c) Module (m):

It is ratio of diameter of pitch circle to number of teeth

$$m = \frac{D}{T}$$

(d) Gear Ratio:

It is the ratio of number of teeth on gear to the number of teeth on pinion

$$G = \frac{T_G}{T_p}$$

(e) Velocity Ratio:

It is ratio of ω_{driver} to ω_{driven} gear

D $\rightarrow$ Pitch circle diameter; T $\rightarrow$ tooth.

- For two mating gears, circular pitch **and** module must be same.

4.3 Meshing of Involute Gears

- Normal drawn at any point on involute curve will become tangent to its base circle.

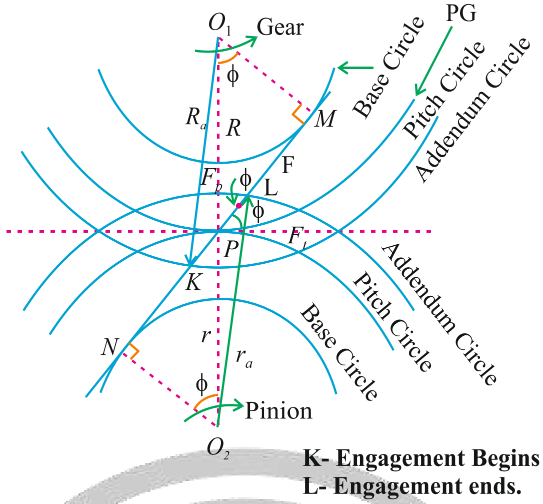

Fig. 4.1 Involute Teeth

4.3.1 Line of Action [NM]

- Driving gear tooth, exerts force on driven gear tooth at contact point Q along line of action MN.
- Pitch point (P) and Point of contact (Q) lies along a line of action.

4.3.2 Pressure Angle (ϕ)

- Angle between line of action and common tangent to pitch circle
- Line of action remains fixed (in case of involute gear) while point of contact changes continuously along line of action between K and L as gear tooth engage and disengage
- Time interval in which point of contact (Q) is travelling from start of engagement to end of engagement is termed as one engagement period.

4.3.3 Path of Contact/Contact Length

- Contact length = Approach length + Recess length

i.e., $KL = KP + PL$

Path of approach – (KP):

$$KP = \sqrt{R_a^2 - R^2\cos^2\phi} - R\sin\phi$$

Path of Recess – (PL):

$$PL = \sqrt{r_a^2 - r^2\cos^2\phi} - r\sin\phi$$

So, contact length = $KP + PL$ = $\sqrt{R_a^2 - R^2\cos^2\phi} + \sqrt{r_a^2 - r^2\cos^2\phi} - (R+r)\sin\phi$

4.3.4 Arc of Contact

- Length travelled by pinion/gear along their pitch circle in one engagement period.

Arc of contact = Arc of approach + Arc of Recess

$$\boxed{\text{Arc of Contact} = \frac{\text{Path of contact}}{\cos\phi}}$$

Note:

In one engagement period:

$$\left.\begin{aligned}(\text{Angle Turned})_{\text{By pinion}} &= \frac{\text{Arc of contact}}{r}\\(\text{Angle Turned})_{\text{By gear}} &= \frac{\text{Arc of contact}}{R}\end{aligned}\right\}$$

4.3.5 Contact Ratio

- It gives number of pairs of teeth engaged in one engagement period.

$$\boxed{\text{Contact Ratio} = \frac{\text{Arc of contact}}{\text{Circular Pitch}}}$$

4.3.6 Path of Contact in Rack and Pinion Arrangement

Path of contact = $KP + PL$

$$PL = \sqrt{r_a^2 - r^2\cos^2\phi} - r\sin\phi$$

$$\boxed{KP = \frac{\text{Addendum of Rack}}{\sin\phi}}$$

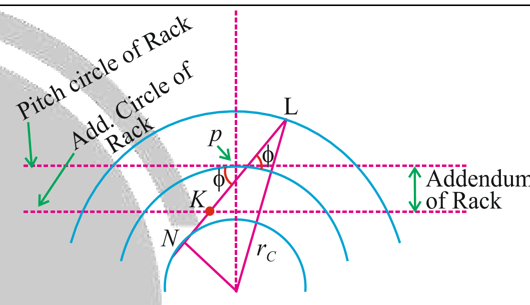

Fig. 4.2 Rack and Pinion in Mesh

4.4 Power Transmission in Gears

- F = normal thrust between teeth

$\Rightarrow$ $\boxed{F_t = F\cos\phi}$ $\rightarrow$ Tangential component (Power Transmission)

$\Rightarrow$ $\boxed{F_R = F\sin\phi}$ $\rightarrow$ Radial component (Thrust action on bearing.)

$\Rightarrow$ Torque = F_t × pitch circle radius.

$\Rightarrow$ Power Transmitted, $P = F_a \cdot \omega \times \text{Torque}$

$$p = \frac{2\pi NT}{60} \rightarrow watt$$

N $\rightarrow$ rpm

T $\rightarrow$ N-m

$\Rightarrow$ Efficiency of gear drive:

$$\boxed{\eta = \frac{p_o}{p_i} = \frac{T_o \cdot \omega_o}{T_i \cdot \omega_i} = \frac{F_o \cdot V_o}{F_i \cdot V_i}}$$

4.5 Velocity of Sliding

$V_{\text{sliding}} = (\omega_p + \omega_g)$ × distance between pitch point and point of contact

$V_{\text{sliding}} = (0)$ motion is pure rolling at pitch point = Zero

4.6 Interference in Involute Gears

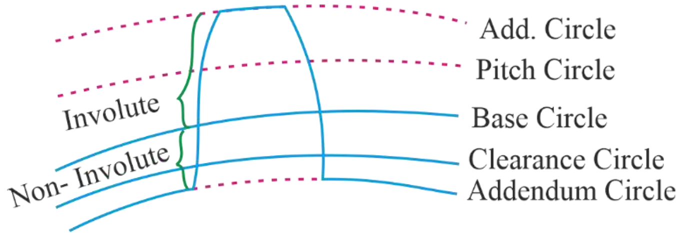

Fig. 4.3 Gear tooth

- The profile of gear is involute outside the base circle and inside base circle it is non-involute.
- Mating of involute profile of one gear with non-involute profile of other results into interference.

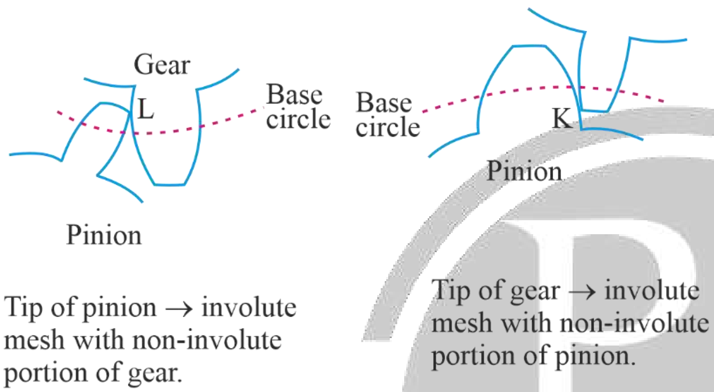

Fig. 4.4 Path of Contact

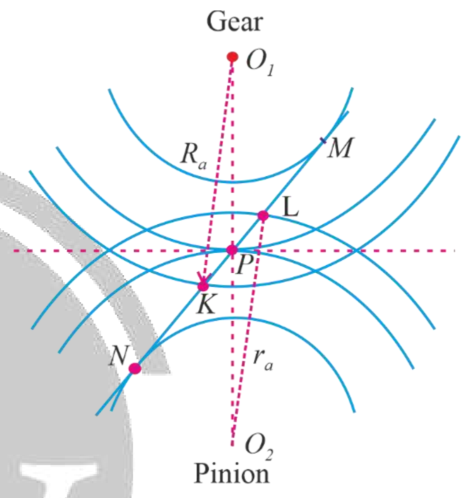

Fig. 4.5 Path of Contact

Interference occurs if:

$= \quad R_a > O_1N$ or $r_a > O_2M$

So, Last safety point of K is N.

Last safety point of L is M.

4.7 Methods to Prevent Interference

1. **Undercutting of Gear:** Removal of material of non-involute portion below base circle.

   Limitation: Strength of tooth decreases at the base, so used only in low power transmission.

2. **Increase Pressure Angle:**

   ϕ can be increased by reducing base circle radius.

   $R_B \downarrow \;\Rightarrow$ non-involute portion $\downarrow \;\Rightarrow$ interference $\downarrow$

   Limitation: $\phi_{max} \approx 20^{o}$ to 25^{o}

3. **Stubbing the tooth:** Portion of tip of tooth of gear is removed, thus preventing that portion to contact non-involute portion.

4. **Limitation:** Contact ratio decreases

5. Increasing Number of teeth

By increasing number of teeth on same pitch circle, tooth size decreases.

$R_A \downarrow$

Interference $\downarrow$

4.8 Minimum Tooth Required to Prevent Interference

- **For pinion:**

$$\boxed{t_{\min} = \frac{2A_p}{\sqrt{1+G(G+2)\sin^2\phi}-1}}$$

- **For gear:**

$$\boxed{T_{\min} = \frac{2A_g}{\sqrt{1+\frac{1}{G}\left(\frac{1}{G}+2\right)\sin^2\phi}-1}}$$

Where, A_p, A_g = Fractional addendum (Addendum of pinion and gear for one module (m))

$$\left[G = \frac{T}{t}\right]$$

- Full depth involute: Addendum = module
- Stub involute: Addendum < module

  Minimum tooth required on pinion to prevent interference in rack and pinion Arrangement:

  $$t_{\min} = \frac{2A_r}{\sin^2\phi}$$

  $A_r \rightarrow$ Fractional addendum on rack

Note:

(a) In case, if fractional addendum on gear and pinion is same i.e., $A_p = A_g$, then gear will do interference first. Make gear safe by providing $T_{\min}$.

(b) Say addendum of on gear $A_g = 0.8 \Rightarrow$ 20% portion of gear is stubbed.

4.9 Cycloidal Tooth Profile

- Phenomena of interference is absent.
- Tooth have spreading flank and thus stronger.

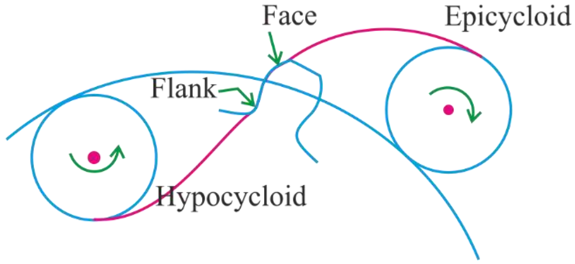

Fig. 4.6 Cycloidal Tooth Profile

- $\left.\begin{array}{l}\text{At engagement}: \phi = \phi_{max} \\ \text{At pitch point}: \phi = 0^{o} \\ \text{At engagement}: \phi = \phi_{max}\end{array}\right\} \rightarrow \text{variable } \phi, \text{less smooth running}$
- $\left.\begin{array}{l}\text{Flank} \rightarrow \text{Hypocycloid} \\ \text{Face} \rightarrow \text{Epicycloid}\end{array}\right\} \rightarrow \text{complicated design, difficult to menufecture}$

4.10 Gear Trains

When more than two gears are made to mesh with each other to transmit power from one shaft to another, such a combination is called gear train

$$\text{Speed Ratio} = \frac{\omega_{\text{driver}}}{\omega_{\text{driven}}} = \frac{1}{\text{Train Value}}$$

4.10.1 Simple Gear Train

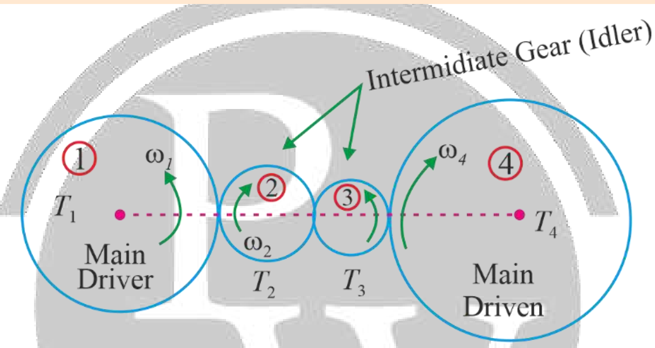

Fig. 4.7 Gear Train

$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}; \quad \frac{\omega_2}{\omega_3} = \frac{T_3}{T_2}; \quad \frac{\omega_3}{\omega_4} = \frac{T_4}{T_3}$$

$$\left[\frac{\omega_1}{\omega_4} = \frac{T_4}{T_1} = \text{Speed Ratio}\right]$$

- Intermediate gears don't affect speed ratio (Idler gears)
- If number of idlers are odd than direction of driver and driven are same
- If number of idlers are even than direction of driver and driven are same are opposite.

4.10.2 Compound Gear Train

- At least one of the intermediate shafts have more than one gear in use.

4.10.3 Reverted Gear Train

- Form of compound gear train which connects two co-axial shafts.

$$\frac{\omega_1}{\omega_2} = \frac{T_2}{T_1}; \quad \frac{\omega_3}{\omega_4} = \frac{T_4}{T_3} \qquad (\therefore\ \omega_2 = \omega_3)$$

$$\Rightarrow \left[\frac{\omega_1}{\omega_4} = \frac{T_2 \cdot T_4}{T_1 \cdot T_3} = \text{Speed Ratio}\right]$$

Also, $r_1 + r_2 = r_3 + r_4$

$$\Rightarrow m(t_1 + t_2) = m'(t_3 + t_4)$$

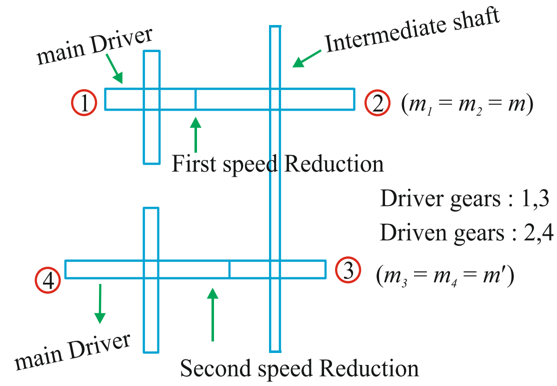

Fig. 4.8 Reverted Gear Train

4.10.4 Epicyclic Gear Train or Planetary

- If apart from motion of gear, any of the gear axis is also rotating about same axis, it is known as epicyclic gear train.
- DOF of epicyclic gear train is 2
- To rotate axis of gear, a link is used known as arm/carrier.
- One gear is fixed either sun or ring.

4.10.5 Fixing Torque/Holding Torque:

$$\Rightarrow \boxed{(T_{O/P} \cdot \omega_{O/P}) + \eta(-T_{I/P} \cdot \omega_{I/P}) = 0} \qquad \text{...(I)}$$

η = efficiency of gear train.

For equilibrium, net torque on gear train = 0

$$\boxed{T_{O/P'} + T_{I/P} + T_{\text{fixing}} = 0} \qquad \text{... (II)}$$

- For given input torque $T_{I/P}$ we can find output torque from equations (I) and then fixing torque from equation (II).

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5 ANALYSIS OF SINGLE SLIDER CRANK MECHANISM

5.1. Analysis of Single Slider Crank Mechanism

If crank is rotating with constant angular speed ω in clockwise direction.

r = crank radius

l = length of connecting rod

θ = Angle turned by crank from line of stroke or from TDC/BDC.

$n = \frac{l}{r}$ = obliquity ratio

ω = crank speed of rotation.

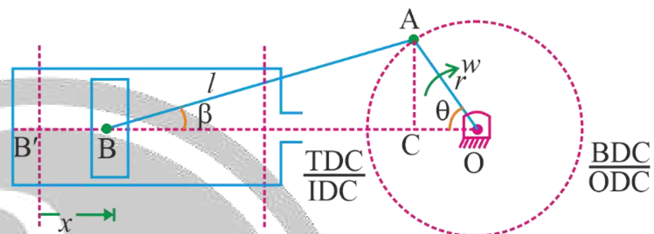


Fig. 5.1 Single Slider Crank Mechanism

5.1.1 Piston Displacement

$x = OB' - OB = (l + r) - (l\cos\beta + r\cos\theta)$

From figure $AC = l\sin\beta = r\sin\theta$

$$\sin\beta = \frac{\sin\theta}{n} \Rightarrow \cos\beta\frac{\sqrt{n^2 - \sin^2\theta}}{n}$$

$$x = (nr + r) - \left[nr \cdot \frac{\sqrt{n^2 - \sin^2\theta}}{n} + r\cos\theta\right]$$

$$x = r(1 - \cos\theta) + r\left(n - \sqrt{n^2 - \sin^2\theta}\right)$$

$$\boxed{x = r\left[(1 - \cos\theta) + \left(n - \sqrt{n^2 - \sin^2\theta}\right)\right]}$$

5.1.2 Piston Velocity

$$V = \frac{dx}{dt} = r\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right] \cdot \frac{d\theta}{dt}$$

$$V = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}}\right] \rightarrow \text{exact.}$$

If n is large, $n^2 - \sin^2\theta \approx n^2$

$$\boxed{V = r\omega\left[\sin\theta + \frac{\sin 2\theta}{2n}\right]} \rightarrow \text{Approximate.}$$

5.1.3 Piston Acceleration

$$a = \frac{dV}{dt} = \frac{dV}{d\theta}\cdot\frac{d\theta}{dt}$$

$$a = r\omega^2\left[\cos\theta + \frac{\cos 2\theta}{n}\right] \rightarrow \text{Approx.}$$

5.1.4 Angular Velocity of Connecting Rod

$$\sin\beta = \frac{\sin\theta}{n} \Rightarrow \cos\beta\cdot\frac{d\beta}{dt} = \frac{\cos\theta}{n}\cdot\frac{d\theta}{dt}$$

$$\frac{\sqrt{n^2 - \sin^2\theta}}{n}\cdot(\omega_{Cr}) = \frac{\cos\theta}{n}\times\omega_{\text{crank}}$$

$$\boxed{\omega_{CR} = \omega_{\text{crank}}\cdot\frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}}$$

If $\quad n >>> \sin\theta$

$\Rightarrow \quad \boxed{\omega_{CR} = (\omega_{\text{crank}})\cdot\frac{\cos\theta}{n}}$

5.1.5 Angular Acceleration of Connecting Rod

$$\alpha_{CR} = \frac{\omega_{CR}}{dt} \quad \Rightarrow \boxed{\alpha_{CR} = -\omega_{\text{crank}}^2\cdot\frac{\sin\theta}{n}}$$

5.2 Force Analysis of Slider – Crank Mechanism

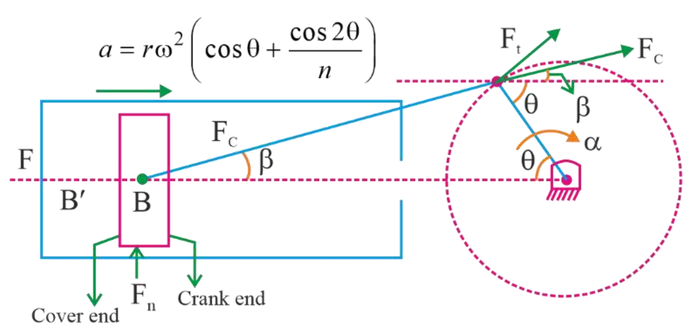
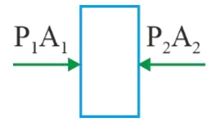


Fig. 5.2 Force component on Single Slider Crank Mechanism

m = mass of reciprocating part

Assumption – mass/Inertia Effect of connecting rod is neglected.

5.2.1 Fluid Pressure Force



P_1 = Fluid pressure at cover end.

P_2 = Fluid pressure at crank end.

A_1 = Area of piston exposed to fluid pressure at cover end.

A_2 = Area of piston exposed to fluid pressure at crank end.

$A_1 = \frac{\pi}{4}D^2$; D → Bore size

$A_2 = \frac{\pi}{4}\left(D^2 - d^2\right)$; d → Piston rod diameter

$$\boxed{F_P = P_1A_1 - P_2A_2}$$

5.2.2 Piston Effort/ Effective Driving Force on Piston

$$\boxed{F = F_P - m \times a - F \pm mg}$$

F → Friction

mg → In case of vertical engines

ma → Inertia force (F_i)

5.2.3 Force Transmitted along Connecting Rod

$$F_C \cos\beta = F \Rightarrow \boxed{F_C = \frac{F}{\cos\beta}}$$

5.2.4 Normal Thrust against Cylinder Walls

$$F_n = F_C \sin\beta = F \tan\beta$$

$$\Rightarrow \boxed{F_n = F \tan\beta}$$

5.2.5 Radial Thrust on Crankshaft Bearing

$$F_R = F_C \cos(\theta + \beta)$$

$$\Rightarrow \boxed{F_R = \frac{F}{\text{Cos}\beta} \cdot \cos(\theta + \beta)}$$

5.2.6 Crank Effort (F_t)

$$F_t = F_C \sin(\theta + \beta)$$

$$\Rightarrow \boxed{F_t = \frac{F}{\cos\beta} \cdot \sin(\theta + \beta)}$$

5.2.7 Torque/Turning Moment on Crankshaft

$$T = F_t \cdot r$$

$$\Rightarrow \boxed{T = F_r \cdot \frac{\sin(\theta+\beta)}{\cos\beta}}$$

Or, $\boxed{T = F_r\left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2-\sin^2\theta}}\right)}$ $\rightarrow$ from this expression we say that Torque, $\tau = f(\theta)$

Therefore, T $\neq$ Constant

Note:

If it is required to find speed at which gudgeon pin load reverse its direction. Then F_C will become first zero than reverse its direction i.e., $F_C = \frac{F}{\cos\beta} = 0 \quad \Rightarrow \boxed{F = 0}$

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6 FLYWHEEL

6.1 Introduction

- A flywheel is a rotating disc or rim that is used to store rotational kinetic energy $\left(\frac{1}{2}I\omega^2\right)$.
- Flywheels have a significant mass moment of inertia and thus resist changes in rotational speed.
- Energy is transferred to a flywheel by applying torque to it, thereby increasing its rotational speed, and hence it stores the energy.
- Conversely, a flywheel releases stored energy by applying torque to a mechanical load, thereby its rotational speed decreases.
- For example, a flywheel is used to maintain constant angular velocity of the crankshaft in a reciprocating engine. In this case, the flywheel which is mounted on the crankshaft stores energy when torque is exerted on it by a firing piston and it releases its energy to mechanical loads when piston is not exerting torque on it.

6.1.1 Turning Moments Diagrams

- A plot of T vs. θ is known as the turning moment diagram. The inertia effect of the connecting rod is usually ignored while drawing these diagrams. The turning moment diagrams for different types of engines are being given below:

(1) Single cylinder - Double acting steam engine -

Fig shows the turning - moment diagram for a single cylinder double – acting engine.

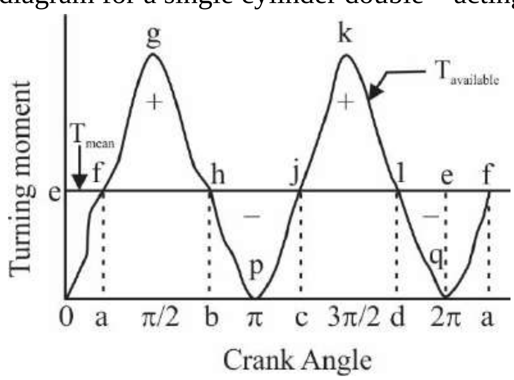

Fig. 6.1 Turning moment diagram

> **Note:**
> Generally, engines are connected against a constant load. In such case $T_{resisting}$ or T_{load} will be equal to T_{mean}.

- It can be observed that during the outstroke (ogp) the turning moment is maximum when the crank angle is little less than 90° and zero when the crank angle is zero and 180°. A somewhat similar turning moment diagram is obtained during the instroke (pkq).

- The area of the turning-moment diagram is proportional to the work done per revolution as the work is the product of the turning moment & the angle turned.
- The mean torque against which the engine works is given by

$$T_{mean} = \frac{\text{work done in one cycle}}{\text{Time period}}$$

$$oe = \frac{\text{Area ogpkq}}{2\pi}$$

- Where oe is the mean torque and is mean height of the turning moment diagram.

- The maximum engine speed is at b & d and minimum speed is at c and a. the greatest speed is the greater of the two maximum speeds and the least speed is the lesser of the two minimum speeds.
- The difference between the greatest and the least speeds of the engine over one revolution is known as the fluctuation of speed.

(2) Multi – Cylinder Engines –

- Fig. 6.2 shows the turning – moment diagram for a multi–cylinder engine

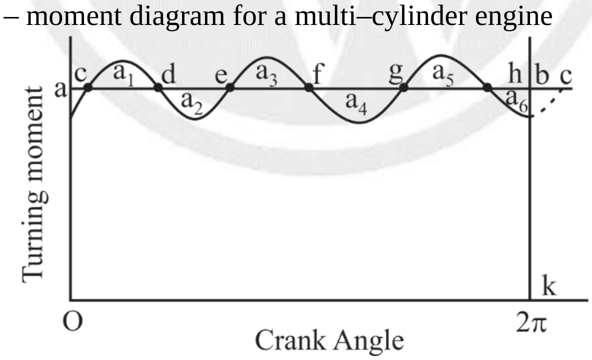

Fig. 6.2 Turning moment diagram

- The mean torque line ab intersects the turning moment curve at c, d, e, f, g and h. The area under the wavy curve is equal to the area oabk.
- The speed of the engine will be maximum when the crank positions correspond to d, f and h, and minimum corresponding to c, e and g.

6.2 FLUCTUATION OF ENERGY

- Let a_1, a_3 and a_5 be the areas in work units of the portions above the mean torque ab of the turning – moment diag. (fig (c)). These areas represent quantities of energies added to the flywheel. Similarly, areas a_2, a_4 and a_6 ab represent quantities of energies taken from the flywheel.
- The energies of the flywheel corresponding to position of the crank are as follows:

Crank position	Flywheel energy
c	E
d	$E + a_1$
e	$E + a_1 - a_2$
f	$E + a_1 - a_2 + a_3$
g	$E + a_1 - a_2 + a_3 - a_4$
h	$E + a_1 - a_2 + a_3 - a_4 + a_5$
c	$E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

- The greatest of these energies is the maximum kinematic energy of the flywheel and for the corresponding crank position, the speed is maximum.

- The least of these energies is the least kinetic energy of the flywheel and for the corresponding crank position, the speed is minimum.

(1) Coefficient of fluctuation of energy:

- It is the ratio of maximum fluctuation of energy to the total work done in a cycle.

$$\boxed{C_E = \frac{\Delta KE_{max}}{\text{WD per cycle}}}$$

- Maximum fluctuation of speed

$$\boxed{\Delta\omega_{max} = (\omega_1 - \omega_2)}$$

(2) Coefficient of fluctuation of speed:

- The difference between the greatest speed and the least speed is known as the maximum fluctuation of speed & the ratio of the maximum fluctuation of speed to the mean speed is the coefficient of fluctuation of speed.

$$C_s = \frac{\omega_1 - \omega_1}{\omega_{mean}}$$

Where

$$\omega_{mean} = \frac{\omega_1 + \omega_2}{2}$$

- Maximum fluctuation of energy

$$(\Delta KE)_{max} = \Delta E = \frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_2^2$$

$$= I\left(\frac{\omega_1 + \omega_2}{2}\right)(\omega_1 - \omega_2)$$

$$\Delta E = I\omega \frac{(\omega_1 - \omega_2)}{\omega} \times \omega$$

$$\boxed{\Delta KE_{max} = I\omega^2 C_s}$$

also known as fundamental equation of flywheel

Where I = moment of inertia of the fly wheel

ω_1 = maximum speed

ω_2 = minimum speed

ω = mean speed = $\frac{\omega_1 + \omega_2}{2}$

C_s = coefficient of fluctuation of speed

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7 BALANCING

7.1 Types of Balancing

- Balancing is the process of designing and modify machinery so that the unbalance is reduced to an acceptable level and if possible is eliminated entire.
- Generally, it is done by redistributing the mass which may be accomplished by addition or removal of mass from various mess members.
- There are two basic types of unbalances - rotating unbalance and reciprocating unbalance - which may occur separately or in combination.

Balancing
- Rotating unbalance
  - Fun = $me\omega^2$
    - Magnitude = constant
    - Direction = changes
- Reciprocating unbalance

7.1.1 Static Balancing

- A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation.
- Single transverse plane
- The net dynamic force acting on the shaft is equal to zero.

$$\Sigma\vec{F} = 0$$

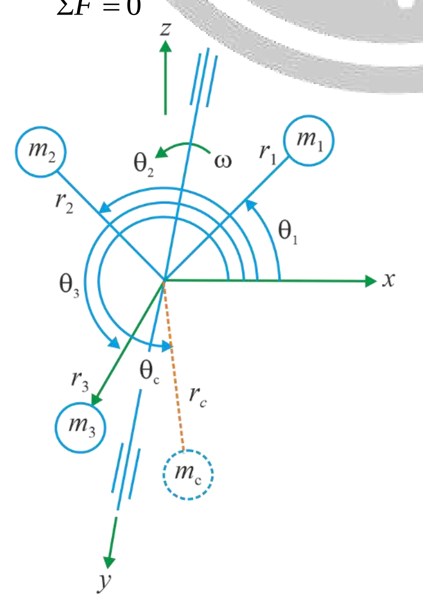

Fig. 7.1 Single Plane Unbalance System

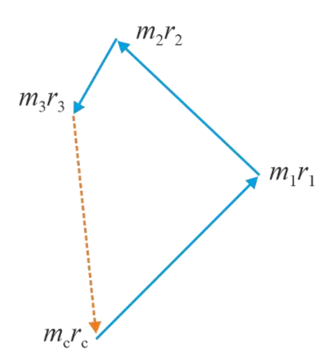

Fig. 7.2 Force Polygon

7.1.2 Dynamic Balancing

- When several masses rotate in different plane, the centrifugal forces, in addition to being out of balance, also form couples.
- A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.
- More than one transverse plane

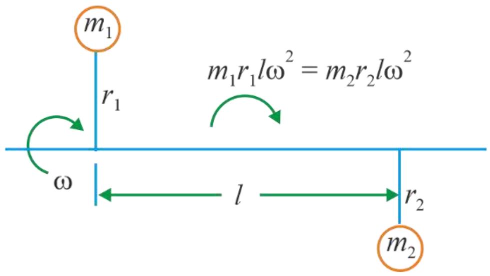

Fig. 7.3 Multi transverse Plane Unbalance System

- The net couple due to the dynamic forces acting on the shaft is equal to zero. In other words, the algebraic sum of the moments about any point in the plane must be zero.

  $\Sigma\vec{F} = 0$
- $\Sigma\vec{M} = 0$

7.2 Balancing of Reciprocating Mass

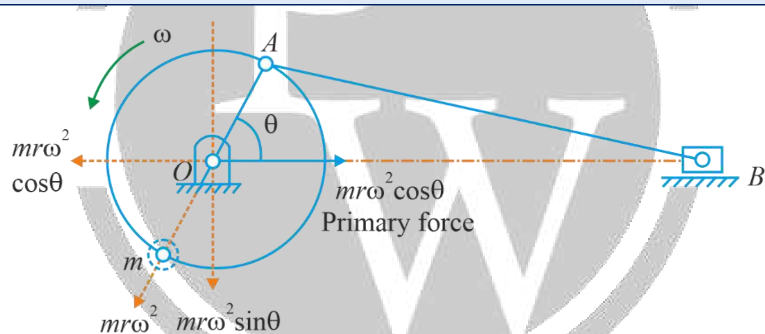

Fig. 7.4 Single Slider Crank Mechanism

- Total unbalance force due to reciprocating mass (m):

$$F = mr\omega^2\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= mr\omega^2\cos\theta + mr\omega^2\frac{\cos 2\theta}{n}$$

- Primary force = $mr\omega^2$ cosθ. The secondary force = $mr\omega^2\left(\frac{\cos 2\theta}{n}\right)$
- Maximum value of the primary force = $mr\omega^2$
- Maximum value of the secondary force = $\frac{mr\omega^2}{n}$

7.2.1 By Balancing Fraction (c) of the Reciprocating Mass

- Primary force balanced by the mass = $cmr\omega^2 \cos\theta$
- Primary force unbalanced by the mass = $(1 - c)\, mr\omega^2 \cos\theta$
- Vertical component of centrifugal force which remains unbalanced
- $= cmr\omega^2 \sin\theta$
- Resultant unbalanced force at any instant

$$= \sqrt{[(1-c)mr\omega^2 \cos\theta]^2 + [cmr\omega^2 \sin\theta]^2}$$

- The resultant unbalanced force is minimum when $c = \frac{1}{2}$
- If m_P is the mass at the crankpin and c is the fraction of the reciprocating mass m to be balanced, the mass the crankpin may be considered as $(cm + m_P)$ which is to be completely balanced.

7.3 Parameters Related to the Partial Balancing in Locomotives

7.3.1 Hammer Blow

- Hammer-blow is the maximum vertical unbalanced force caused by the mass provided to balance the reciprocating masses.
- Its value is $mr\omega^2$. Thus, it varies as a square of the speed.
- At high speeds, the force of the hammer-blow could exceed the static load on the wheels and the wheels can be lifted off the rail when the direction of the hammer-blow will be vertically upwards.

7.3.2 Variation of Tractive Force

- Total unbalanced primary force or the variation in the tractive force

$$= -(1-c)mr\omega^2(\cos\theta - \sin\theta)$$

- For its maximum value

$$\theta = 135° \text{ or } 315°$$
$$\theta = 135°$$

- Maximum variation of Tractive force:

$$= \pm\sqrt{2}(1-c)mr\omega^2$$

7.3.3 Swaying Couple

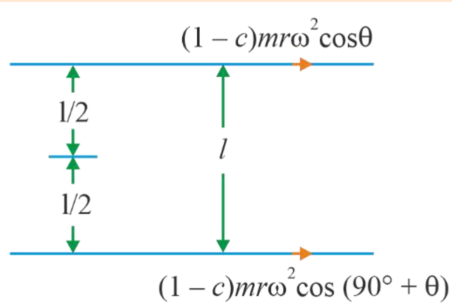

Fig. 7.5 Swaying Couple

- Swaying couple = moments of forces about the engine centre line

$$= [(1-c)mr\omega^2 \cos\theta]\frac{1}{2} - [(1-c)mr\omega^2 \cos(90° + \theta)]\frac{1}{2}$$

$$= (1-c)mr\omega^2(\cos\theta + \sin\theta)\frac{1}{2}$$

- For its maximum value:

θ = 45° or 225°

- Maximum swaying couple = $\pm\frac{1}{\sqrt{2}}(1-c)mr\omega^2 l$

7.3.4 Secondary Balancing

- Secondary force = $mr\omega^2 \frac{\cos 2\theta}{n}$
- Its frequency is twice that of the primary force and the magnitude $\frac{1}{n}$ times the magnitude of the primary force.

Parameter	Actual	Imaginary
Angular velocity	ω	2ω
Length of crank	r	$\frac{r}{4n}$
Mass at the crank pin	m	m

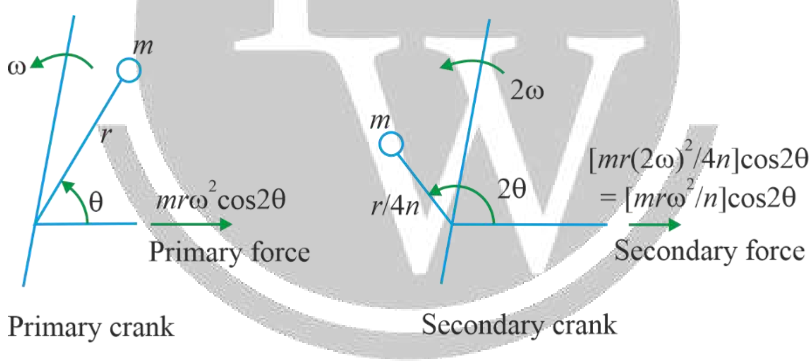

Fig. 7.6 Secondary Balancing

- Centrifugal force induced in the imaginary crank = $\frac{mr(2\omega)^2}{4n}$
- Component of this force along line stroke = $\frac{mr(2\omega)^2}{4n}\cos 2\theta$

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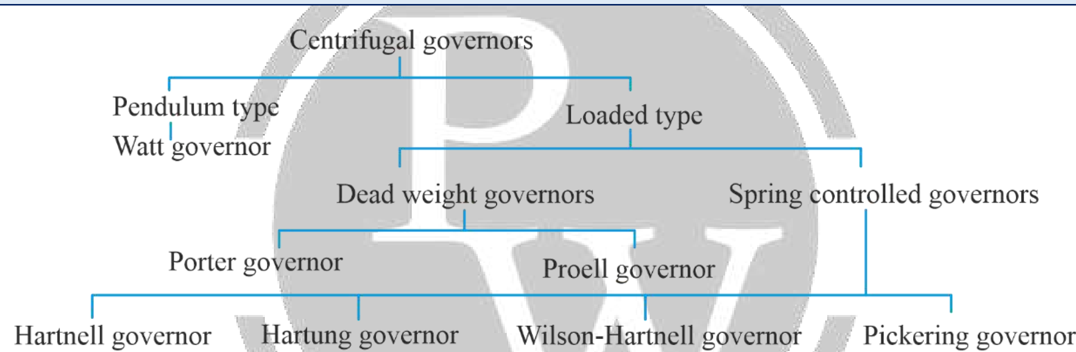
8 GOVERNOR

8.1 Introduction

- Governor is to regulate the mean speed of an engine.
- Automatically controls the supply of working fluid (fuel) to the engine with the varying load conditions and keeps the mean speed within certain limits.

8.2 Types of Governor



8.3 Terminology

- **Height of a governor:** It is vertical distance from the centre of the ball to a point where the axes of the arms (or arms produced) intersect on the spindle axis.
- **Equilibrium speed:** It is speed at which the governor balls, arms etc., are in complete equilibrium and the sleeve do not tend to move upwards or downwards.
- **Mean equilibrium speed.** It is speed at the mean position of the balls or the sleeve.
- **Maximum and minimum equilibrium speeds:** The speeds at the maximum and minimum radius of rotation of the balls, without tending to move either way is known as maximum and minimum equilibrium speeds respectively
- **Sleeve lift.** It is the vertical distance which the sleeve travels due to change in equilibrium speed.

8.4 Watt Governor

m = Mass of the ball in kg,

w = Weight of the ball in newtons = $m.g$,

T = Tension in the arm in newtons,

ω = Angular velocity of the arm and ball about the spindle axis in rad/s,

r = Radius of the path of rotation of the ball i.e., horizontal distance from the centre of the ball to the spindle axis in metres,

F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$, and

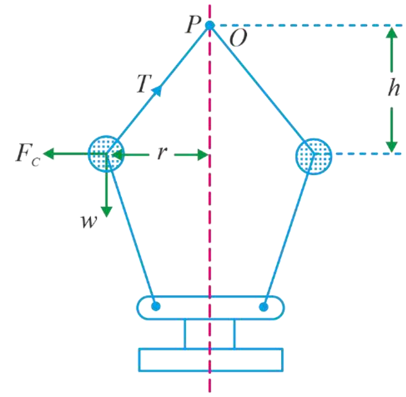


Fig. 8.1 Watt governor

8.4.1 Height of the Governor

$$h = \frac{9.81}{(2\pi N/60)^2} \quad = \frac{895}{N^2} \text{ metres}$$

Note:

The height of a governor h, is inversely proportional to N^2. Therefore, at high speeds, the value of h is small. This governor only works satisfactorily at relatively low speeds *i.e.*, from 60 to 80 r.p.m.

8.4 Porter Governor

m = Mass of each ball in kg,

w = Weight of each ball in newtons = $m.g$,

M = Mass of the central load in kg,

W = Weight of the central load in newtons = $M.g$,

r = Radius of rotation in metres,

α = Angle of inclination of the arm (or upper link) to the vertical, and

β = Angle of inclination of the line (or lower link) tot the vertical.

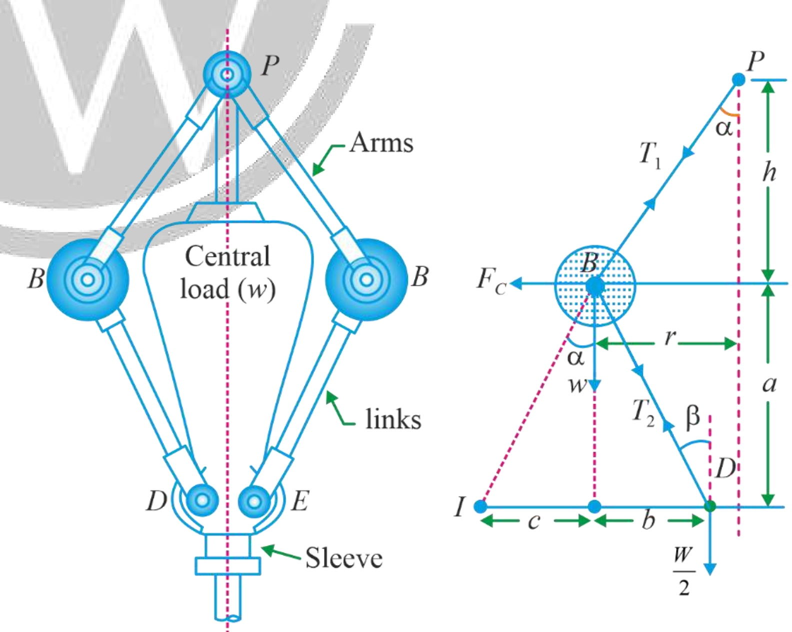


Fig. 8.2 Porter governor

8.4.1 Equilibrium Equation About I-centre

$$mr\omega^2 \times a = mg \times c + \frac{Mg \pm f}{2}(c+b)$$

8.4.2 Height of the Governor

If length of arms is equal to the length of links and the points P and D lie on the same vertical line, then

$$\tan\alpha = \tan\beta \text{ or } k = \tan\alpha/\tan\beta = 1$$

If f = Frictional force acting on the sleeve

$$h = \frac{m.g + \left(\frac{M.g \pm f}{2}\right)(1+k)}{m.g} \times \frac{895}{N^2}$$

When $\tan\alpha = \tan\beta$ or $k = 1$, then

$$h = \frac{m+M}{m} \times \frac{g}{\omega^2}$$

8.5 Proell Governor

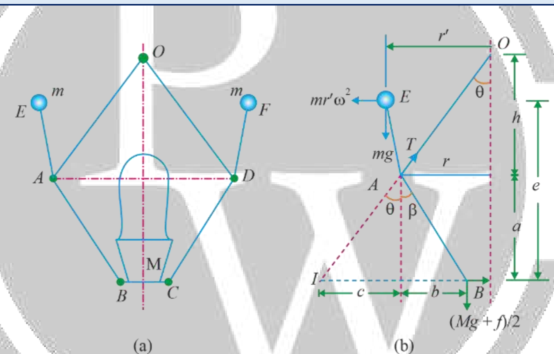


Fig. 8.3 Proell Governor

8.5.1 Equilibrium equation about I centre

$$mr'\omega^2 e = mg(c + r - r') + \frac{Mg \pm f}{2}(c+b)$$

$$N^2 = \frac{895}{h}\frac{a}{e}\left(\frac{2mg + (mg \pm f)(1+k)}{2mg}\right)$$

8.5.2 Height of governor

If $k = 1$, $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{mg + (Mg \pm f)}{mg}\right)$$

If $f = 0$, $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{2m + M(1+k)}{2m}\right)$$

If $k = 1$, $f = 0$ $$h = \frac{895}{N^2}\frac{a}{e}\left(\frac{m+M}{m}\right)$$

8.6 Hartnell Governor

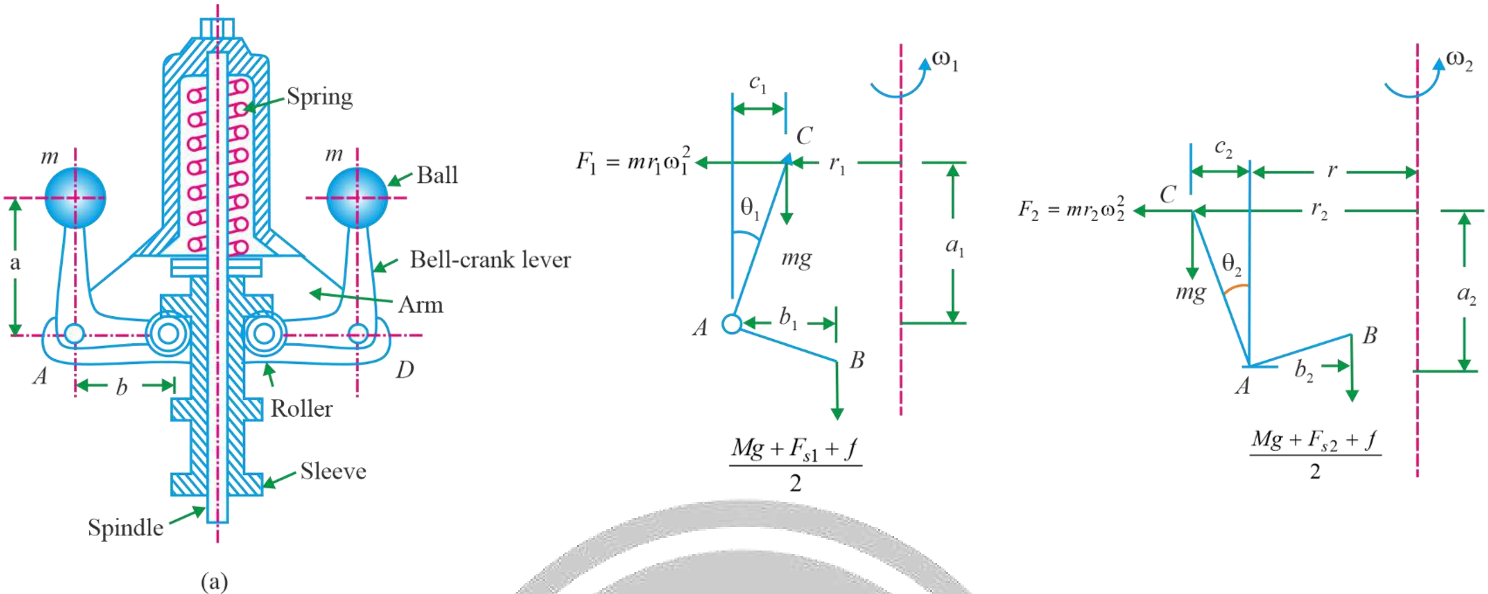


Fig. 8.4 Hartnell governor

8.6.1 Stiffness of Spring

$$s = \frac{2}{r_2 - r_1} \cdot \left(\frac{a}{b}\right)^2 \cdot (F_2 - F_1) \quad = \quad 2\left(\frac{a}{b}\right)^2 \left(\frac{F_2 - F_1}{r_2 - r_1}\right)$$

8.6.2 Lift of Sleeve

$$h_1 \quad = \quad \theta.b = \frac{r_2 - r_1}{a} \cdot b$$

8.7 Parameters related to Governor

8.7.1 Sensitiveness of Governor

It is ratio of the mean equilibrium speed to the difference between the maximum and minimum equilibrium speeds.

N_1 = Minimum equilibrium speed,

N_2 = Maximum equilibrium speed, and

N = Mean equilibrium speed = $\frac{N_1 + N_2}{2}$

Sensitiveness of governor $= \frac{N}{N_1 - N_2}$ $\qquad = \frac{(N_1 + N_2)}{2(N_2 - N_1)} = \frac{\omega_1 + \omega_2}{2(\omega_2 - \omega_1)}$

8.7.2 Stability of Governors

For a stable governor, if the equilibrium speed increases, the radius of governor balls must also increase.

Note:
A governor is said to be unstable if the radius of rotation decreases as the speed increases.

8.7.3 Isochronous Governors

- when the equilibrium speed is constant (*i.e.*, range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronism is the stage of infinite sensitivity.
- **Condition of isochronous in Hartnell Governor:**

$$\frac{M \cdot g + S_1}{M \cdot g + S_2} = \frac{r_1}{r_2}$$

8.7.4 Hunting

- if the speed of the engine fluctuates continuously above and below the mean speed. And results in hitting stopper repeatedly. This is caused by a too sensitive governor which changes the fuel supply by a large amount when a small change in the speed of rotation takes place.

8.7.5 Effort of a Governor

- it is the mean force exerted at the sleeve for a given percentage change of speed (or lift of the sleeve).
- **For porter governor:** $\frac{E}{2} = \frac{cg}{1+k}[2m + M(1+k)]$
- **For watt governor:** $\frac{E}{2} = cmg$
- **For Hartnell governor:** $\frac{E}{2} = c(Mg + F_S)$

Where "c" fraction of change in speed

8.7.6 Power of a Governor

- it is the work done at the sleeve for a given percentage change of speed. It is the product of the mean value of the effort and the distance through which the sleeve moves.

$$\text{Power} = \text{Mean effort} \times \text{lift of sleeve}$$

- **For porter governor power:**

$$= \left[m + \frac{M}{2}(1+k)\right] gh \left(\frac{4c^2}{1+2c}\right)$$

- **For porter governor power (k = 1):**

$$= (m+M)gh\left(\frac{4c^2}{1+2c}\right)$$

- **Coefficient of Insensitiveness** = $\left(\frac{N_1 - N_2}{N}\right)$

8.8 Controlling Force (Fc)

The centrifugal force on each ball of a governor is balanced by an equal and opposite force acting radially inwards known as controlling force.

8.8.1 For Porter Governor Controlling Force

$$= \tan\theta\left[mg + \frac{Mg \pm f}{2}(1+k)\right]$$

8.8.2 For Hartnell Governor Controlling Force

$$= \frac{1}{2}(Mg + F_s \pm f)\frac{b}{a}$$

- The intersection of the speed curves with the controlling force curve provides the speeds of the governor corresponding to the radii. (Assume f = 0)

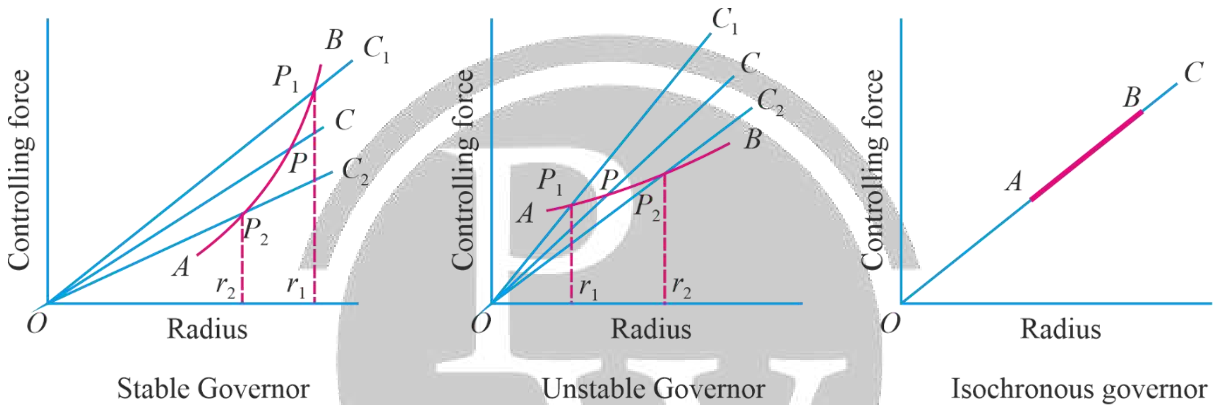


Fig. 8.5 Controlling force (without friction)

Note:

If friction is considered, two more curves of the controlling force are obtained. Thus, in all, three curves of the controlling:

(i) for steady run (neglecting friction)

(ii) While the sleeve moves up (f positive)

(iii) While the sleeve moves down (f negative)

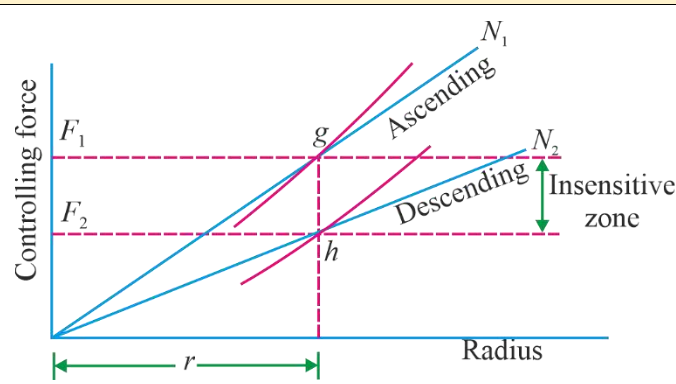


Fig. 8.6 Controlling force with friction

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9 GYROSCOPE

9.1 Introduction

- Gyroscopic effect comes into action when axis of rotation of a rotating body (propeller, wheel etc.) is turned round about an axis perpendicular to axis of spin.
- Axis of spin, axis of precession and axis of gyroscopic couple are perpendicular to each other.

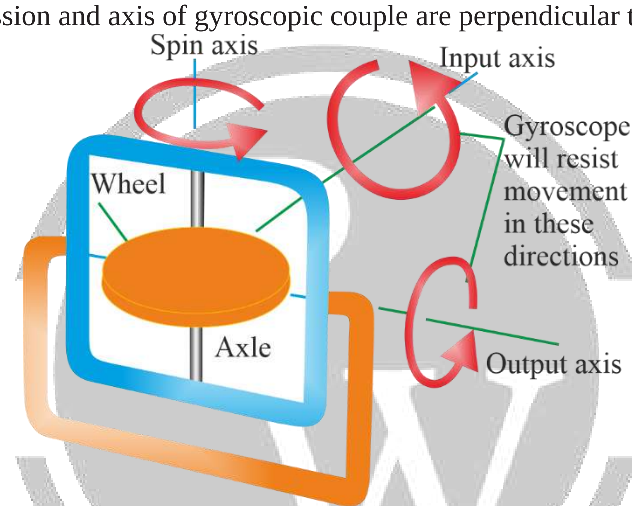


Fig. 9.1 Gyroscope

9.1.1 Analysis

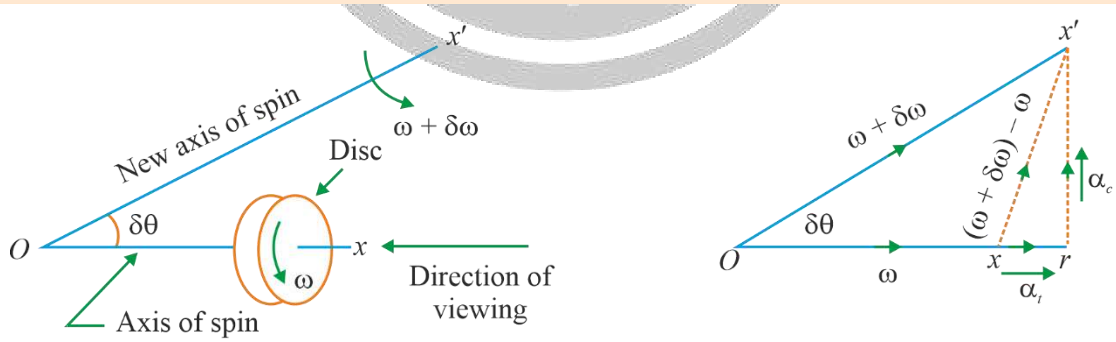


Fig. 9.2 Gyroscope Couple axis

α_c = angular acceleration responsible for changing direction of ω

α_t = Angular acceleration responsible for change in magnitude of ω

$$\alpha_t = \frac{\delta\omega}{\delta t}$$

$$\alpha_c = \omega \times \frac{d\theta}{dt} = \omega \cdot \omega_P$$

- Total angular acceleration of the disc:

$$= \text{vector } xx' = \text{vector sum of } \alpha_t \text{ and } \alpha_c$$

$$= \frac{d\omega}{dt} + \omega \cdot \omega_P$$

- **Angular velocity of precession:** Angular velocity of the axis of spin $\omega_p = \frac{d\theta}{dt}$.
- **Axis of precession:** The axis, about which the axis of spin is to turn
- **Precessional angular motion:** The angular motion of the axis of spin about the axis of precession.

Note:

- The axis of precession is perpendicular to the plane in which the axis of spin is going to rotate.
- If the angular velocity of the disc remains constant at all positions of the axis of spin, then $\frac{d\omega}{dt}$ is zero; and thus, α_t is zero.
- If the angular velocity of the disc changes the direction, but remains constant in magnitude, then angular acceleration of the disc is given by

$$\alpha_c = \omega.d\theta/dt = \omega.\omega_p$$

- The angular acceleration α_c is known as gyroscopic acceleration.

9.2 Gyroscopic Couple

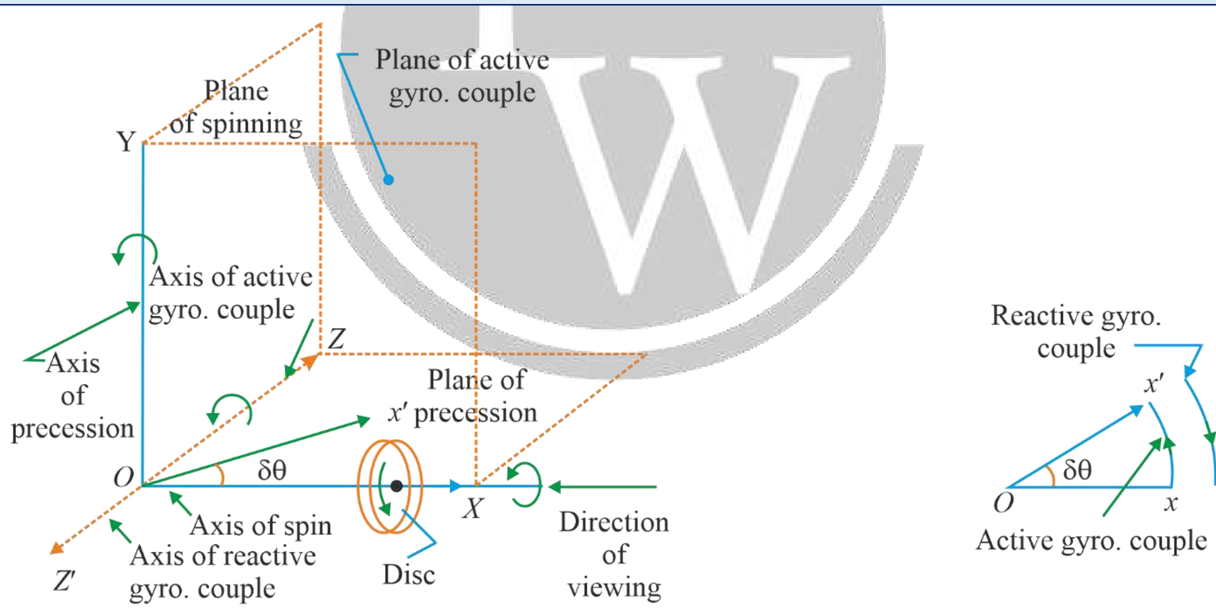


Fig. 9.3 Gyroscope Couple axis

- The rate of change of angular momentum:

$$C = \underset{\delta t \to 0}{Lt}\ I.\omega \times \frac{\delta\theta}{\delta t} = I.\omega \times \frac{d\theta}{dt} = I.\omega.\omega_P \qquad \left(\because \frac{d\theta}{dt} = \omega_P\right)$$

9.3 Effect of the Gyroscopic Couple on an Aeroplane

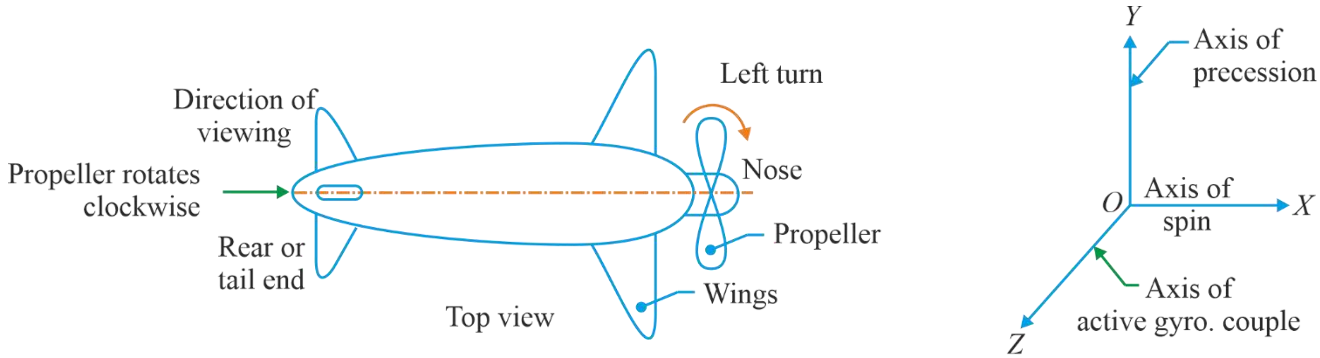


Fig. 9.4 Aeroplan Spin Axis

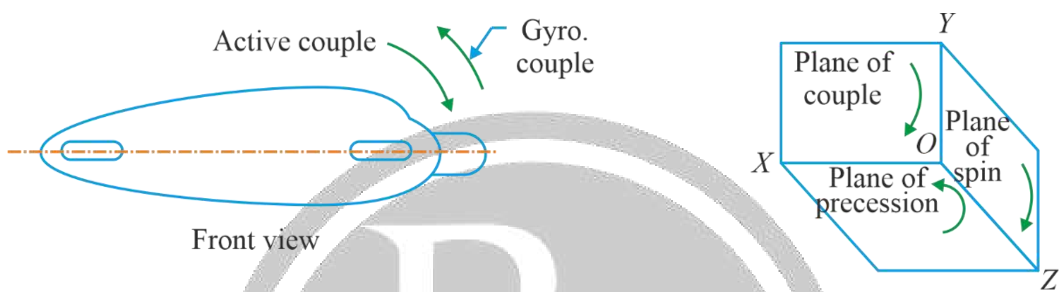


Fig. 9.5 Gyroscope Couple

Let ω = Angular velocity of the engine in rad/s,

m = Mass of the engine and the propeller in kg,

k = Its radius of gyration in metres,

I = Mass moment of inertia of the engine and the propeller in kg-m^2

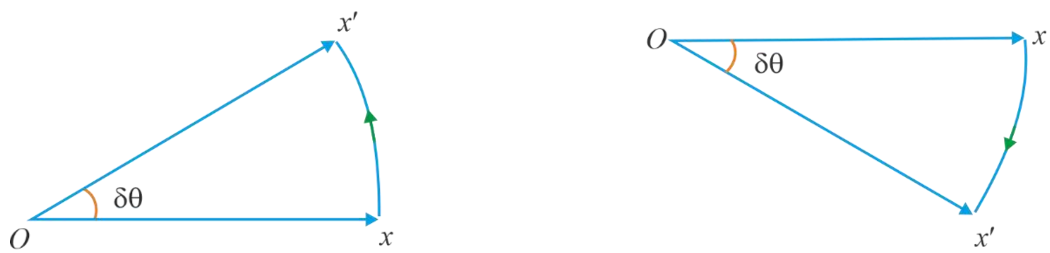
$= m.k^2$,

V = Linear velocity of the aeroplane in m/s,

R = Radius of curvature in metres, and

ω_p = Angular velocity of precession = $\dfrac{V}{R}$ rad/s

$\therefore$ Gyroscopic couple acting on the aeroplane,



(a) Aeroplane taking left turn (b) Aeroplane taking right turn

9.3.1 Conclusion

- When the aeroplane takes a **right turn** under similar conditions as discussed above (Propeller rotating clockwise when viewed from back side of aeroplane), the effect of the reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.
- When the engine or propeller rotates in **anticlockwise direction** when viewed from the rear or tail end and the aeroplane takes a **left turn**, then the effect of reactive gyroscopic couple will be to **dip the nose** and **raise the tail** of the aeroplane.
- When the aeroplane takes a **right turn** under similar conditions as mentioned in note 2 above, the effect of reactive gyroscope couple will be to **raise the nose** and **dip the tail** of the aeroplane.
- When the engine or propeller rotates in **clockwise direction** when viewed from the front and the aeroplane takes a left turn, then the effect of reactive gyroscopic couple will be to **raise the tail** and **dip the nose** of the aeroplane.
- When the aeroplane takes a **right turn** under similar conditions as mentioned in note 4 above, the effect of reactive gyroscopic couple will be to **raise the nose** and **dip the tail** of the aeroplane.

9.4 Effect of the Gyroscopic Couple on a Naval ship

- The fore end of the ship is called bow and the rear end is known as **stern** or **aft**. The left hand and right-hand sides of the ship, when viewed from the stern are called **port** and **star-board** respectively.

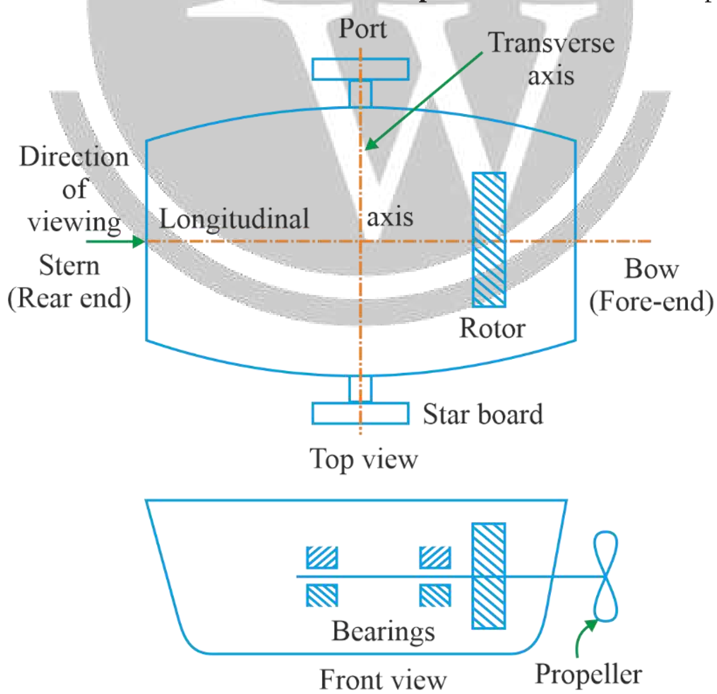


Fig. 9.6 Naval Ship

9.4.1 Naval Ship during Steering (left turn)

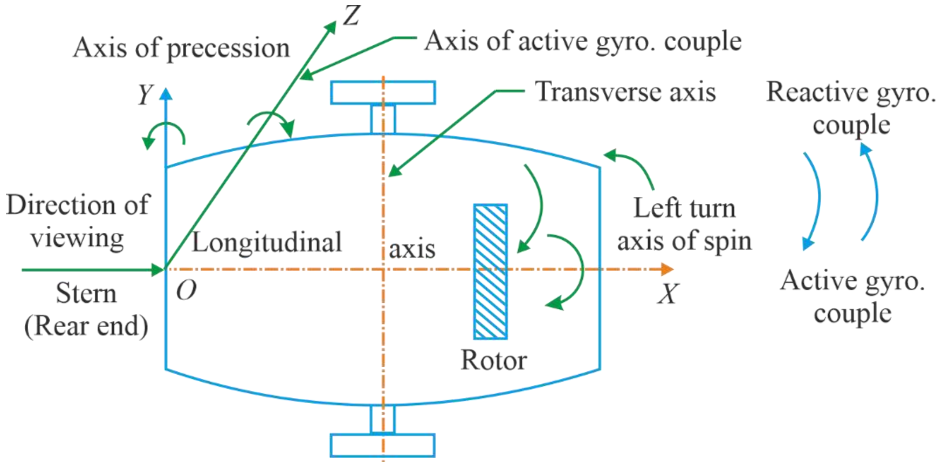
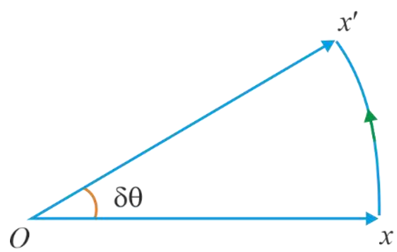


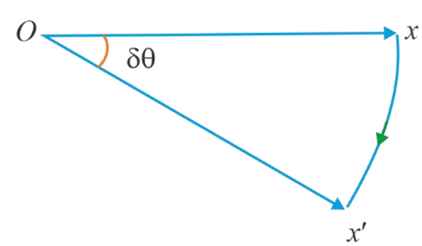
Fig. 9.7 Left turn of Naval Ship

The effect of this reactive gyroscopic couple is to raise the bow and lower the stern.

- When the ship steers to the right under similar conditions as discussed above, the effect of the reactive gyroscopic couple, it will be to **raise the stern** and **lower the bow**.
- When the rotor rates in the anticlockwise direction. When viewed from the stern and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **lower the bow** and **raise the stern**.
- When the ship is steering to the right under similar conditions, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern**.
- When the rotor rotates in the clockwise direction when viewed from the bow or fore end and the ship is steering to the left, then the effect of reactive gyroscopic couple will be to **raise the stern** and **lower the bow**.
- When the ship is steering to the right under similar conditions as discussed in note 4 above, then the effect of reactive gyroscopic couple will be to **raise the bow** and **lower the stern**.



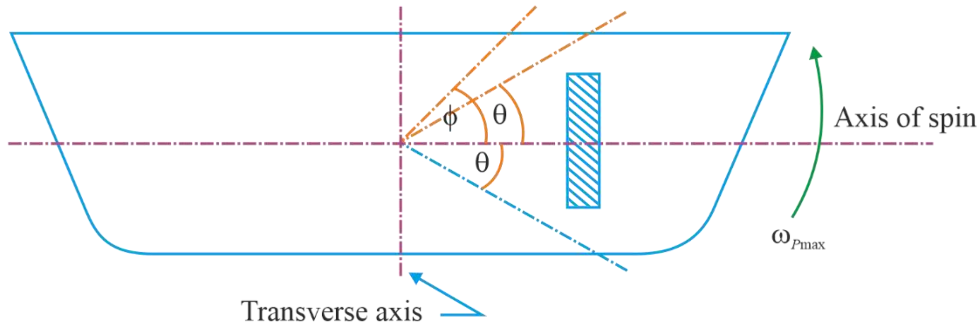
(a) Streeing to the left



(b) Streeing to the right

9.4.2 Naval Ship during Pitching

Pitching is the movement of a complete ship up and down in a vertical plane about transverse axis.



(a) Pitching of a naval ship

(b) Piching upward (c) Pitching downward

Fig. 9.8 Naval Ship with Pitching

- Angular velocity of S.H.M (ω_1)

$$= \frac{2\pi}{\text{Time period of S.H.M. in seconds}} = \frac{2\pi}{t_p}\text{rad/s}$$

- Angular velocity of precession (ω_p)

$$\omega_p = \frac{d\theta}{dt} = \frac{d}{dt}(\phi \sin \omega_1 . t) = \phi\omega_1 \cos \omega_1 t \quad [\phi = \text{amplitude of pitching}]$$

- Maximum angular velocity of precession ($\omega_{P\max}$)

$$\omega_{P\max} = \phi \cdot \omega_1 = \phi \times 2\pi / t_p$$

- Assume I = Moment of inertia of the rotor in kg-m^2

  ω = angular velocity of the rotor in rad/s.

- Minimum gyroscopic couple,

$$C_{\max} = I.\omega.\omega_{P\max}$$

- The angular acceleration during pitching,

$$\alpha = \frac{d^2\theta}{dt^2} = -\phi(\omega_1)^2 \sin \omega_1 t$$

- Maximum angular acceleration during pitching,

$$\alpha_{\max} = (\omega_1)^2$$

9.4.3 Naval Ship during Rolling

- In case of rolling of a ship, the axis of precession (*i.e.*, longitudinal axis) is always parallel to the axis of spin for all positions. Hence, there is no effect of the gyroscopic couple acting on the body of a ship.

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10 MECHANICAL VIBRATIONS

10.1 Natural Vibrations for Systems having Single Degree of Freedom

Natural Vibration

Vibration in which there is no kinetic friction at all as well as there is no external force after initial release of system.

10.2 Force Method

D' Alembert's Principle:

- System of forces/system of torques acting on a moving body is in dynamic equilibrium with the inertial force/inertia torque of the body.

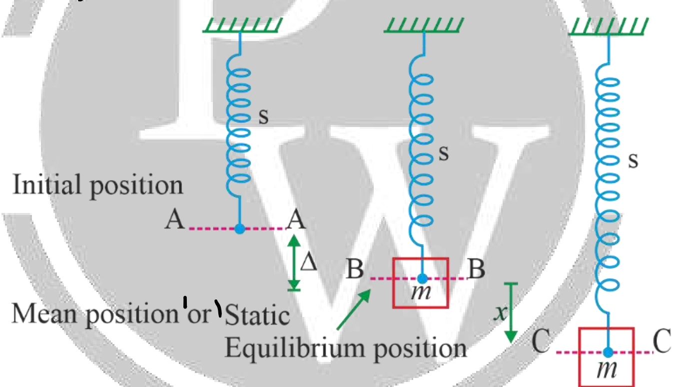
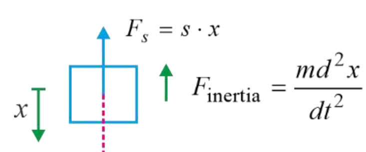


Fig. 10.1 Spring and Mass System

- As mass attached to spring, static deflection generated in spring is Δ. At this position system attains equilibrium position. (B-B).
- At static equilibrium, mg = sΔ
- If system is disturbed by small displacement x from equilibrium position.



- $\frac{md^2x}{dt^2} + sx = 0$ [D'Alembert Principle]

  On comparison with standard SHM equation

$$\boxed{\omega_n = \sqrt{\frac{s}{m}}}, \quad \boxed{F = \frac{1}{2\pi}\sqrt{\frac{s}{m}}}$$

$\downarrow$ (rad/ s) $\qquad \downarrow$ s^{-1}

The solution of equations (I) is of the form –

$$\boxed{x = A\sin\left(\sqrt{\frac{s}{m}}t + \phi\right)}$$

A = amplitude (constant)

10.3 Energy Method

- In natural vibrations, the kinetic friction assumed as zero.
- Total energy of system during vibration = Constant

  i.e., $\frac{dE}{dt} = 0$

In this case,

$$E = \frac{1}{2}mv^2 + \frac{1}{2}s\,x^2$$

$$\Rightarrow \quad \frac{dE}{dt} = \frac{1}{2}\left[2mv\frac{dv}{dt} + 2s\,x\frac{dx}{dt}\right]$$

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \left(\frac{2md^2x}{dt^2} + 2s\,x\right)v = 0$$

$$v \neq 0 \qquad \rightarrow 2m\ddot{x} + 2sx = 0$$

$$\Rightarrow \quad \ddot{x} + \left(\frac{s}{m}\right)x = 0$$

$$\Rightarrow \quad \omega = \sqrt{\frac{s}{m}}$$

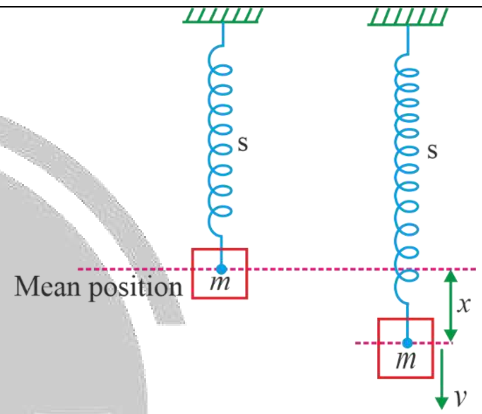


Fig. 10.2 Spring and Mass System

10.4Torque Method

- This method is very useful for the cases in which body is rotating about a point

  Ex. (i) Oscillation of pendulum

  (ii) System in pure rolling motion – Pure rotation observed from point of contact.

$$\boxed{I\ddot{\theta} + s\theta = 0}$$

10.5 Combination of Spring

10.5.1 Parallel Combination

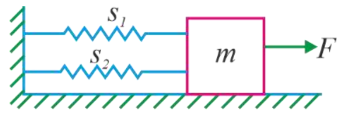


Fig. 10.3 Spring in Parallel

same elongation but different force in each spring.

$$F = F_1 + F_2$$

$$\Rightarrow s_{eq} \cdot x = s_1 x + s_2 x \quad \Rightarrow s_{eq} = s_1 + s_2$$

10.5.2 Series Combination

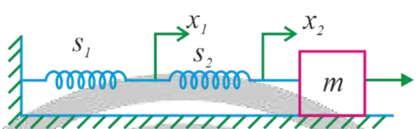


Fig. 10.4 Spring in Series

- same force but different deflection

$$\Rightarrow x = x_1 + x_2$$

$$\frac{F}{s_{eq}} = \frac{F}{s_1} + \frac{F}{s_2} \Rightarrow \left[\frac{1}{s_{eq}} = \frac{1}{s_1} + \frac{1}{s_2}\right]$$

10.5.3 Cutting of Spring in ratio m: n ratio

$$s_A = \left(\frac{m+n}{m}\right)s$$

$$s_B = \left(\frac{m+n}{n}\right)s$$

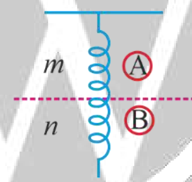


Fig. 10.5 Spring

10.5.4 Inertia Effect of Spring Mass

m_s = mass of spring

- If a simple spring mass system is undergoing natural vibration. Then equivalent mass of system is, $m_{eq} = m + \frac{m_s}{3}$

$$\Rightarrow \quad \omega_n = \sqrt{\frac{s}{m_{eq}}} \qquad \Rightarrow \omega_n = \sqrt{\frac{s}{m + \frac{m_s}{3}}}$$

10.6 Rayleigh Method

Conditions

- Only applied in natural vibration
- Only applied in spring-mass system 'or' equivalent
- There should be only one mass in the system that must be point mass.

If Δ = static deflection under suspended
Mass = 'm'
from static equilibrium

$$mg = s\Delta \quad \Rightarrow \quad \frac{s}{m} = \frac{g}{\Delta}$$

$$\Rightarrow \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

10.7 Free-Damped Vibrations

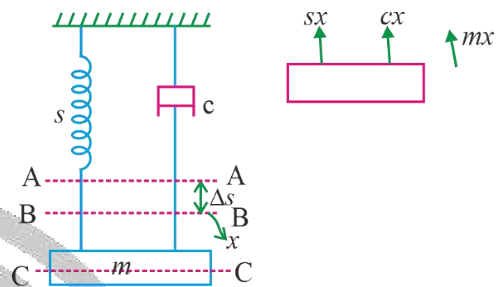


Fig. 10.6 Spring damped Mass System

$(x = 0, t = 0) \rightarrow$ static equilibrium
At $(t = t, x = x)$

$$\boxed{\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{s}{m}x = 0} \qquad \text{...(I)}$$

$$\left(D^2 + \frac{c}{m}D + \frac{s}{m}\right)x = 0$$

$$x \neq 0, \quad \Rightarrow D^2 + \frac{c}{m}D + \frac{s}{m} = 0$$

Auxiliary equation, $\alpha^2 + \frac{c}{m}\alpha + \frac{s}{m} = 0$

$$\alpha_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{s}{m}\right)}$$

Solution of equation (I) is –

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$$

10.7.1 Damping Factor or Damping Ratio

$$\zeta = \sqrt{\frac{\left(\frac{c}{2m}\right)^2}{\frac{s}{m}}} = \frac{\frac{c}{2m}}{\sqrt{\frac{s}{m}}} = \frac{c}{2m\omega_n}$$

$$\Rightarrow \quad \zeta = \frac{c}{2\sqrt{ms}}$$

$$\boxed{\zeta = \frac{\text{Actual damping coeff.}}{\text{Critical damping coeff.}} = \frac{c}{c_c}}$$

- If $\zeta < 1 \rightarrow$ Under damped
  $\zeta = 1 \rightarrow$ Critical damped
  $\zeta > 1 \rightarrow$ Over damped
- For $\zeta = 1, \quad c_c = 2\sqrt{ms}$
- In terms of damping factor –

$$\boxed{\frac{d^2x}{dt^2}+(2\zeta\omega_n)\frac{dx}{dt}+\omega_n^2 x=0}$$

$$x = Ae^{\alpha_1 t}+Be^{\alpha_2 t}$$

Where,

$$\alpha_1,\alpha_2 = -\zeta\omega_n \pm\sqrt{\zeta^2\omega_n^2-\omega_n^2}$$

$$\alpha_1,\alpha_2 = \omega_n\left(-\zeta\pm\sqrt{\zeta^2-1}\right)$$

- **Case-I: Overdamped System (ζ > 1)**

  Roots of auxiliary equations are real and unequal.

  $$x = Ae^{\alpha_1 t}+Be^{\alpha_2 t}$$

  For, $\alpha_1,\alpha_2 = \omega_n\left(-\zeta\pm\sqrt{\zeta^2-1}\right)$

  System can't vibrate due to overdamping. (Aperiodic motion) and x = 0 with time.

- **Case-2: Critically Damped (ζ = 1)**

  Roots of auxiliary equations are equal, $\alpha_{1,2} = -\omega_n$

  $\Rightarrow\ x=(A+Bt)e^{-\omega_n t}$

  As $t\to\infty,\ x\to 0$

  Aperiodic motion. $x\to 0$ in shortest possible time without oscillation.

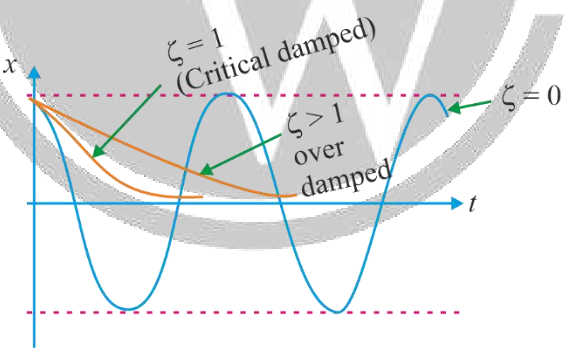


Fig. 10.7 Damping Factor

- **Case-3: Underdamped (ζ = 1)**

  Roots of auxiliary equations are imaginary

  $$\alpha_{1,2} = \left(-\zeta\pm i\sqrt{1-\zeta^2}\right)\omega_n$$

  Now, $x = Xe^{-\zeta\omega_n t}\cdot\sin(\omega_d t+\phi)$

  X, ϕ $\rightarrow$ Constant determined by initial conditions.

Note:

In above solution, part $Xe^{-\zeta\omega_n t}$ represents exponentially decreasing of amplitude with time and $\sin(\omega_d t+\phi)$ represents repetition of motion.

10.7.2 Oscillating Frequency of Underdamped Vibration

$$\boxed{\omega_d = \sqrt{1-\zeta^2}\cdot\omega_n} \qquad \{\zeta<1 \Rightarrow \omega_d<\omega_n\}$$

- Resultant motion is oscillatory with frequency ω_d and decreasing magnitude with time.

$$\boxed{\text{Time period, } T_d = \frac{2\pi}{\omega_d}}$$

X_0 = displacement at start of first cycle

X_1 = displacement at end of cycle

X_n = displacement at end of n^th^ cycle

$$x = Xe^{-\zeta\omega_n t}\cdot\sin\left[\frac{2\pi}{T_d}\cdot t+\phi\right]$$

At $t = T_d$

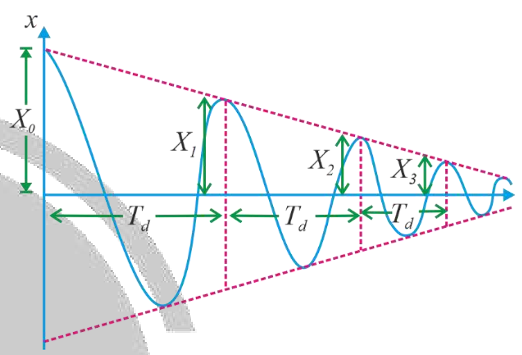


Fig. 10.8 Frequency under damped condition

$$\Rightarrow \quad X_o = X\sin\phi$$

$$X_1 = Xe^{-\zeta\omega_n T_d}\cdot\sin(2\pi+\phi)$$

$$X_1 = X_e^{-\zeta\omega_n T_d}\cdot\sin\phi$$

$$X_2 = X_e^{-2\zeta\omega_n T_d}\cdot\sin\phi$$

$$\vdots$$

$$X_n = Xe^{-n.\zeta\omega_n T_d}\cdot\sin\phi$$

$$X_{n+1} = Xe^{-(n+1)\zeta\omega_n T_d}\cdot\sin\phi$$

$$\Rightarrow \quad \boxed{\frac{X_0}{X_1} = \frac{X_1}{X_2} = \frac{X_2}{X_3} = ... = \frac{X_n}{X_{n+1}} = e^{\zeta\omega_n T_d}}$$

$\Rightarrow$ Ratio of amplitude of two successive oscillations is constant.

10.7.3 Logarithmic Decrement (δ)

$$\delta = \ln\left(\frac{X_n}{X_{n+1}}\right) = \ln\left(e^{\zeta\omega_n T_d}\right)$$

$$\Rightarrow \quad \delta = \zeta\omega_n T_d = \zeta\omega_n\left(\frac{2\pi}{\omega_d}\right)$$

$$\delta = \frac{2\pi\zeta\omega_n}{\left(\sqrt{1-\zeta^2}\right)\omega_n} \Rightarrow \boxed{\left(\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

10.8 Forced Damped Vibrations

$$m\ddot{x} + c\dot{x} + sx = F_o \sin\omega t$$

$$\Rightarrow \frac{d^2x}{dt^2} + \left(\frac{c}{m}\right)\frac{dx}{dt} + \left(\frac{s}{m}\right)x = \left(\frac{F_o}{m}\right)\sin\omega t$$

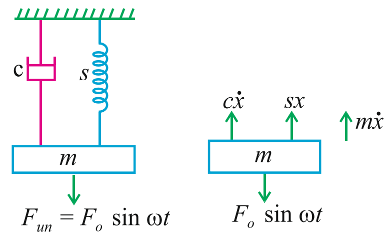


Fig. 10.9 Spring Damped Mass System

10.8.1 Amplitude of Steady State Response

$$\boxed{A = \frac{\frac{F_o}{s}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

$\frac{F_o}{s} \rightarrow$ static deflection of spring under F_o.

$\phi \rightarrow$ phase l_{ag} of displacement relative to velocity vector.

$$\boxed{\tan\phi = \frac{\left(2\zeta\frac{\omega}{\omega_n}\right)}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}}$$

10.8.2 Magnification Factor

- Ratio of amplitude of steady state response to static deflection under the action of F_o.

$$\boxed{MF = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

Note:

Irrespective of amount of damping maximum amplitude of vibration occurs before resonance $\left(\frac{\omega}{\omega_n} = 1\right)$

10.8.3 Vibration Isolation and Transmissibility

- Transmissibility (ε) is defined as force transmitted (to foundation) to force applied. It is the measure of effectiveness of vibration is dating material.

$$\boxed{\varepsilon = \frac{F_t}{F_o} = \sqrt{\frac{1+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}}$$

- At resonance, $\omega=\omega_n$

$$\Rightarrow \ \varepsilon = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta}$$

- If no damper is used – i.e., $\zeta = 0$

$$\Rightarrow \ \varepsilon = \frac{1}{\pm\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]}$$

Note:

For all possible values of g, transmissibility

$\varepsilon = 1$ when $\frac{\omega}{\omega_n}=0$ and $\frac{\omega}{\omega_n}=\sqrt{2}$

$$\varepsilon = \begin{cases} \text{greater than 1;} & 0<\frac{\omega}{\omega_n}<\sqrt{2} \\ \text{less than 1;} & \frac{\omega}{\omega_n}>\sqrt{2} \end{cases}$$

Rotary Unbalance force

m_{ro} = mass of rotary part of machine.

e = distance between centre of mass of rotary part and axis of rotation.

ω = forced frequency

$$\Rightarrow \ \boxed{F_{un} = \left(m_{ro}\cdot e\cdot \omega^2\right)\sin\omega t}$$

Reciprocating Unbalance force

m_{rec} = mass of reciprocating part of machine

r = crank radius = $\frac{\text{stroke}}{2}$

ω = Forced frequency

$$\Rightarrow \quad \boxed{F_{un} = \left(m_{rec} \cdot r \cdot \omega^2\right) \sin \omega t}$$

10.9 Whirling Speed of Shaft

- When a rotor is mounted on a shaft, its centre of mass does not usually coincide with the centre of the shaft. Therefore, upon rotation, shaft bend in the direction of eccentricity of the centre of mass.
- Critical or whirling or whipping speed is the speed at which the shaft tends to vibrate violent in the transverse direction.
- Synchronous whirl –

  Spinning speed = Rotation speed

- Critical/Whirl speed:

  At which shaft tend to vibrate violently in transverse direction.

  s = stiffness of shaft

  e = Initial eccentricity of centre of mass of rotor

  m = mass of rotor.

  y = additional deflection of rotor due to centrifugal force

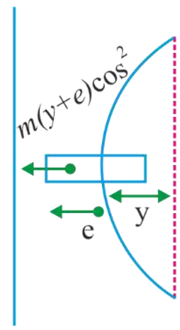


Fig. 10.10 Disk with Shaft

$$\Rightarrow \quad sy = my\omega^2 + me\omega^2$$

$$\Rightarrow \quad y = \frac{me\omega^2}{s - m\omega^2}$$

$$\boxed{y = \frac{e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1}}$$

If $\omega_n = \omega$ (critical speed), resonance occurs, $y \to \infty$

$$\boxed{\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\Delta}}}$$

10.10 Torsional Vibration

I = mass MOI of disc

G = modulus of rigidity of shaft

J = polar MOI (area) of shaft

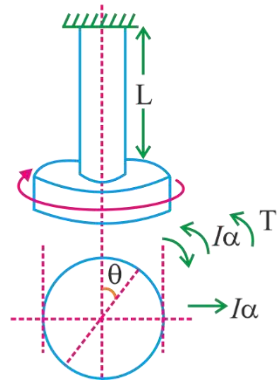


Fig. 10.11 Torsional Vibration System

$$T = \left(\frac{GJ}{L}\right)\theta$$

$$T = q\theta$$

$$\Rightarrow \quad I\alpha + T = 0$$

$$\Rightarrow \quad I\frac{d^2\theta}{dt^2} + \left(\frac{GJ}{L}\right)\theta = 0$$

$$\Rightarrow \quad \boxed{\omega = \sqrt{\frac{q}{I}}}$$

If I′ = mass MOI of shaft, then - $\left[I_{eq} = I + \frac{I'}{3}\right]$

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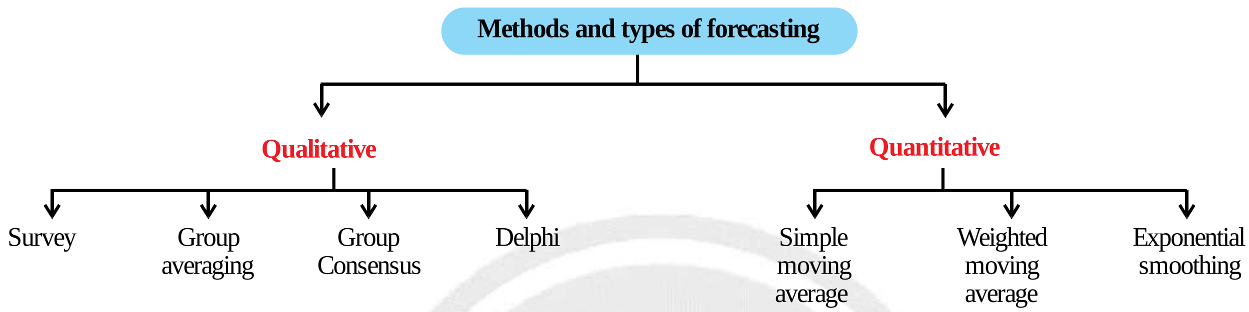
Industrial Engineering

Industrial

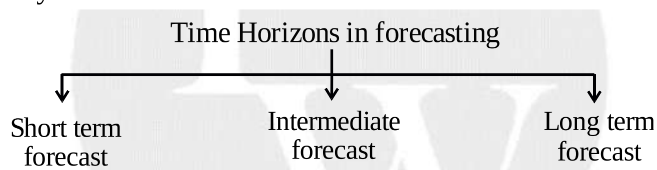
INDEX

1 FORECASTING



1.1 Forecasting

- DEMAND forecast is basically concerned with the estimation of DEMAND.



Note:

- To make the proper arrangement for training the personnel is not a purpose of long-term forecasting
- Direct survey method can be used for forecasting the sales potential of a new product.
- For sales forecasting, pooling of expert opinions is made use of in Delphi technique.
- Rolling horizon in forecast is used for easy updating of changes and maintaining same length of forecast horizon by adding a new period when one period is over.

1.2 Quantitative Forecasting

- This method is generally used for **short-** and medium-term forecasting.
- **Time series model**: In this method the variable that is being forecast is decomposed into its components (N)
- **Associative model**: Used when changes in one or more independent variables can be used to predict the changes in the dependent variable

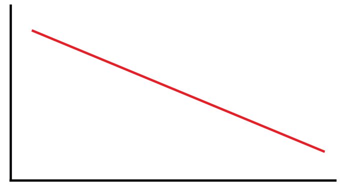


Fig. 1.1 Trend Component

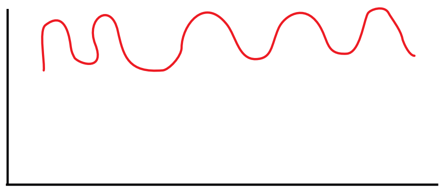


Fig. 1.2 Seasonal Component

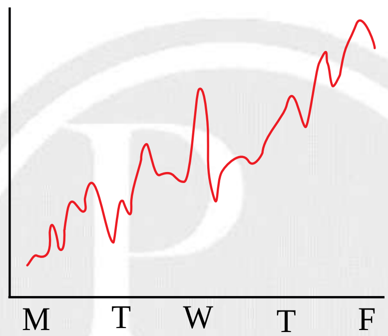


Fig. 1.3 Random Component

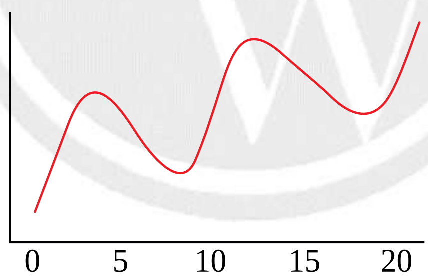


Fig. 1.4 Cyclical Component

Note:

- ***Time series*** analysis technique of sales-forecasting can be applied to only **medium** and **short-range** forecasting.
- ***Qualitative*** information about the market is necessary for **long-range** forecasting.

1.3 Simple Moving Average

Moving Average obtained by adding and averaging the value from a given number of periods repeatedly, each time deleting the oldest value and adding a new value.

1.4 Weighted Moving Average

In this method the weights are given to the values vary, the highest weight is given to the latest value and the weight decreases as the values become old.

1.5 Exponential Smoothing

- Assumes the most recent observations have the highest predictive value
- The weightage of the data diminishes exponentially as the data become older
- **Gives more weight to recent time periods**

$$F_{t+1} = F_t + \underbrace{\alpha\left(D_t - F_t\right)}_{e_t}$$

F_{t+1} = Forecast value for time t + 1
D_t = Actual value at time t
α = Smoothing constant

OR

$F_{t+1} = \alpha D_t + \alpha(1-\alpha) D_{t-1} + \alpha(1-\alpha)^2 D_{t-2} + \ldots$
$F_{t+1} = F_t + \alpha (D_t - F_t)$
If $\alpha = 0$ $\quad F_{t+1} = F_t$ ---------------- STABLE
If $\alpha = 1$ $\quad F_{t+1} = D_t$ ---------------- RESPONSIVE

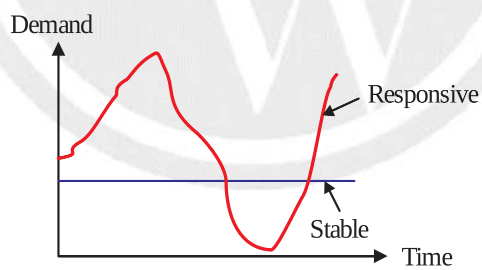


Fig. 1.5 Demand v/s time

Note:

- ***Higher the value of α***, more responsive the forecast will be and this is desirable for forecasting of ***new products***.
- Whereas ***lower value of α*** makes the forecast more stable and this desirable for ***old and stable products***.

1.6 Error Analysis

It is assumed that the forecasting model should over estimate and under estimate with equal magnitude so that the errors produces by forecasting model will fit into a Normal distribution curve.

1.7 Measuring Forecasting Accuracy

• Mean Absolute Deviation (MAD) • *measures the total error in a forecast without regard to sign*	$\text{MAD} = \dfrac{\sum_{i=1}^{N} \lvert \text{Demand} - \text{forecast} \rvert}{n}$
• Cumulative Forecast Error (CFE) • *Measures any bias in the forecast*	$\text{CEF} = \sum_{i=1}^{N} (\text{Demand} - \text{forecast})$
• Mean Square Error (MSE) • *Penalizes larger errors*	$\text{MSE} = \sum_{i=1}^{N} \dfrac{(\text{Demand} - \text{forecast})^2}{n}$
• Tracking Signal • Measures if your model is working • **Good tracking signal has low values**	$\text{TS} = \dfrac{\text{CFE}}{\text{MAD}}$
• Mean Error of BIAS	$\dfrac{\sum_{i=1}^{N} (D_f - F_t)}{N}$

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2 Break Even Analysis

2.1 Total Cost of a Product is the Sum of *Fixed Cost* & *Variable Cost*.

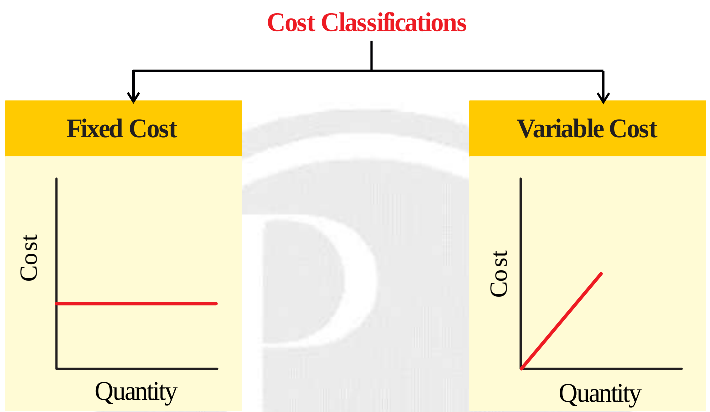

Fig. 2.1 Fig. 2.2

- Fixed cost = F
- Variable cost/unit = V
- Selling price/unit = S
- number of units being produce = Q
- total cost (TC) = F + VQ
- total revenue (TR) = SQ
- profit (P) = SQ – (F + VQ)
- At break-even point profit = 0
- TR = TC

$SQ^x = F + VQ^x$

$Q^x = F/(S - V)$

$SQ > VQ$

$S > V$

$Q^x = Q_{BE}$ = Break even quantity

Break-Even Chart

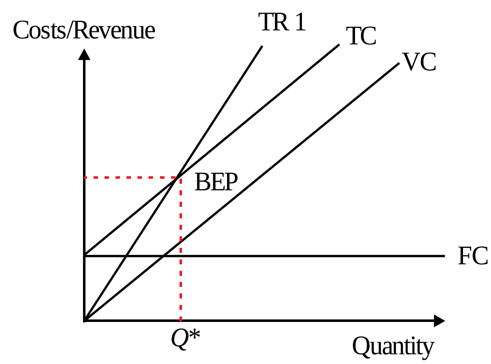

Fig. 2.3 Cost vs quantity

2. 2 Another Use of BEP to Compare two Machines/Processes

Where, F_A = Machine A fixed cost

F_B = Machine B fixed cost

V_A = Machine B variable cost

Q = Quantity

At break-even point

Total cost of machine A = Total cost of machine B

$$F_A + QV_A = F_B + QV_B$$

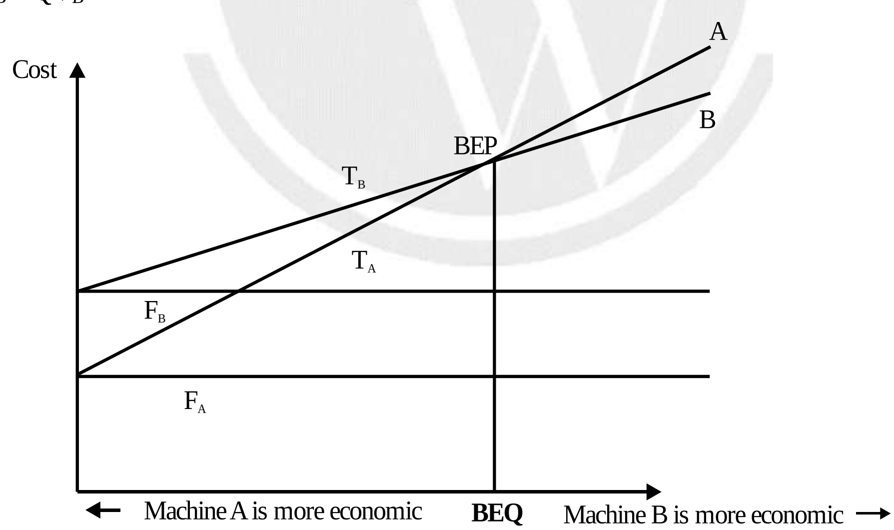

Fig. 2.4

2.3 Contribution

Contribution is the measure of economic value that tells how much the sale of one unit of the product will ***contribute to cover fixed cost, with the remainder going to profit***

Contribution = Sales – Total Variable Cost (Q.V)

Contribution = F + P

2.4 Profit Volume Ratio or Margin of Safety (%)

The margin of safety is simple the excess of actual output over the break-even output. In terms this is simply the excess of sales revenue over the break-even sales revenue.

$$\text{MoS (\%)} = \left(\frac{\text{Actual sales} - \text{Break even sales}}{\text{Actual sales}}\right) \times 100$$

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3 PRODUCTION PLANNING & CONTROL

3.1 Production Planning and Control

- PPC acts as the nerve center for planning the activity and sending the continuous stream of directions to all manufacturing departments.

ORGANISATION CHART FOR P.P.C. DEPARTMENT

P.P.C.

Production Planning

- Order Booking
- Production budget
- Material Records
- Methods
- Machines
- Tools and Jigs
- Operation Layout
- Time estimating
- Scheduling

Production Control

- Despatching
- Expediting

Inventory Control

- Stores management
- Quality Control
- Handling
- Receiving
- Simplification
- Standardization

3. 2 PPC Includes the following Function and Activities

3.2. 1 Routing

Routing consists of deciding the path the item would take in its transformation from raw material to final product. It involves the creation of a rout sheet which contains the following information.

Note:

- **Routing in production planning and control refers to the** Sequence of operations to be performed.
- The routing function in a production system design is concerned with optimizing material flow through the plant.
- The correct sequence of these activities is
  - (a) Analysis of the product and breaking it down into components
  - (b) Taking makes or buys decision
  - (c) Determination of operations and processing time requirement
  - (d) Determination of the lot size

3. 2. 2 Loading

The work of transformation of the raw material has to be converted into workloads on individual machine. This process is known as loading.

3. 2. 3 Balancing

It is the process of insuring that the individual work load on each of the machine is more or less and the same.

3. 2. 4 Scheduling

- Firstly, think about the same points,
  - (a) When the work will be done
  - (b) Within, what time it will be completed
  - (c) Component design

  GANT chart, PERT & CPM are the tools of scheduling

Note:

- Scheduling–
  - (a) is a general time table of manufacturing
  - (b) is the time phase of loading
  - (c) is loading all the work in process on
- Preparation of master production schedule is an iterative process
- Schedule charts are made with respect to jobs while load charts are made with respect to machines.
- Sales forecast, Component design, & Time standards are to be considered for production scheduling.
- Machine load chart gives simultaneously, information about the progress of work and machine loading.

3.2. 5 Dispatch

It deals with the smooth introduction of work on to the shop floor. It includes release of work orders, release of tools, drawing etc. On the shop floor in there required sequence. ***Authorizing a production work order to be launched*.**

> **Note:**
> In a low volume production, the dispatching function is not concerned with *Requisition of raw materials, parts and components*

3. 2. 6 Follow Up

This function of PPC deals with the control or the feedback part to ensure smooth running of all the previous operation

3.3 Sequencing and Scheduling

In this topic we *determine order (sequence) for a series of job* to be done on a finite number of service facilities in some pre-assigned order so as to **optimize the total cost(time) involved.**

3.4 Sequencing of N Jobs on 1 M/C

- There are certain parameters which are relevant in case of scheduling of N Jobs on 1 M/C:
  - **(a) Job flow time:** the flow time for a job is the time from some starting point until that job is completed.
  - **(b) Make span time:** it is the time when processing begins on the first job in the set until the last one is completed.
  - **(c) Mean flow time:** total flow time divided by number of jobs
  - **(d) Average tardiness:** the tardiness of the job is the amount of time after its due date that the JOB is completed. If the JOB is completed before due date then the tardiness is ZERO

3.5 Sequencing of N Jobs on 1 M/C

- **Shortest Processing Time (SPT**). Jobs with the shortest processing time are scheduled first

  NOTE: It minimizes average flow time and total cost
- **Earliest Due Date (EDD).** Jobs are sequenced according to their due dates.
  NOTE: It minimizes tardiness
- Critical Ratio (CR)

  CR = Due Date/Processing Time
  Schedule the job with increasing order of CR value.
- **Slake time remaining (STR).** This rule sequencing the job in increasing order of their slack time remaining.

  STR = Due Date – Processing Time

3.6 Algorithm of Johnson's Rule

(1) Identify the job with the smallest processing time (on either machine)

(2) If the smallest processing time involves:

(a) Machine 1, schedule the job at the beginning of the schedule

(b) Machine 2, schedule the job toward the end of the schedule

(3) If there is some unscheduled job, go to step 1 otherwise stop.

(4) If $M1_k = M2_r$ then in this case process the k^{th} job first and r^{th} job last.

If both equal values occur on machine M1 then select the one which has lowest value on M2 first. If both equal values occur on machine M2 then select the one with the lowest value on M1 first.

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4 LINEAR PROGRAMMING

Mathematical programming is used to find the best or optimal solution to a problem that requires a decision or set of decisions about how best to use a set of limited resources to achieve a state goal of objectives.

- $C_1, C_2 \ldots\ldots C_n$= are known as profit coefficient.
- $a_1, a_2 \ldots\ldots a_{mn}$ = are known as technological coefficient.
- $b_1, b_2 \ldots\ldots b_n$ = are known as Resource variable

  LT two variable equation

  $a_1x+b_1y \le c_1$

  $a_2x+b_2y \le c_2$

  where $x_1, x_2 \ge 0$

  objective $Z = c_1x + c_2y$

4. 1 Infinite Solution/Multiple Solution

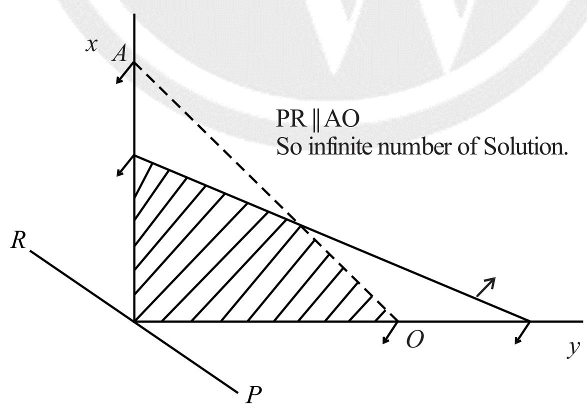

Fig. 4. 1 Infinite Solution

4. 2 Unbounded Solution

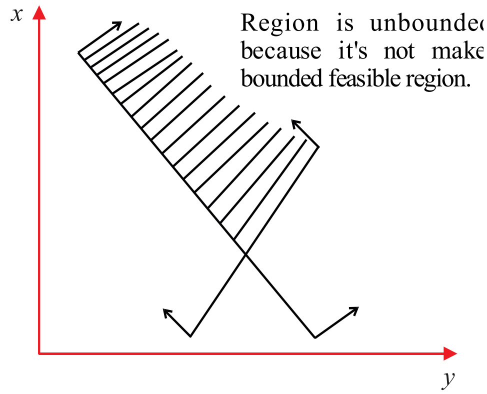

Fig. 2 Unbounded Solution

4. 3 No solution

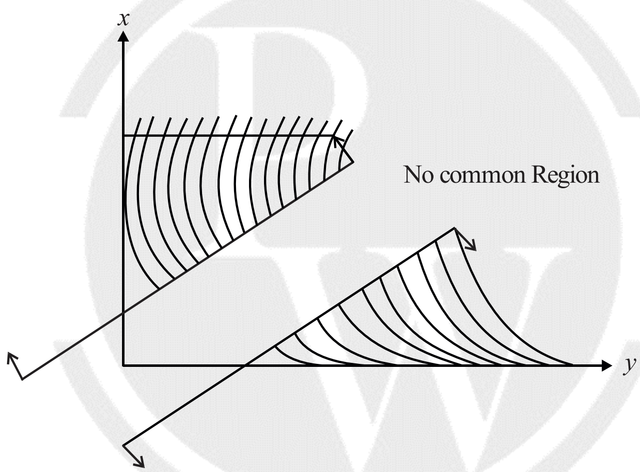

Fig. 3 No Solution

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5 ASSIGNMENT (HUNGARIAN ALGORITHM)

An assignment problem is a special type of transportation problem in which the objective is to assign a number of origins to an equal number of destinations at a minimum cost (or maximum profit).

Formulation of an assignment problem: There are n person and n machine.

Let C_{ij} be the cost if the i^{th} person is assigned to the j^{th} job.

- $C_1, C_2 \ldots\ldots C_n$= are known as profit coefficient.
- $a_1, a_2 \ldots\ldots a_{mn}$ = are known as technological coefficient.
- $b_1, b_2 \ldots. b_n$ = are known as Resource variable

5. 1 Mathematical formulation of the Assignment Problem

$$\text{Minimize}(Z) = \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

Subjected to the restrictions

$$X_{ij} = \begin{cases} 1 \text{ if the } j^{th} \text{ person is assigned } j^{th} \text{ job} \\ 0 \text{ if not} \end{cases}$$

5. 2 Steps for solving Assignment Problem

We make Use of HUNGRIAN ALOGORITHM

(1) Select the minimum entry from each column and subtract that entry from rest of the entries in that column.

(2) Repeat step Number 1 for each Row.

(3) Cover all the Zero's by Drawing minimum number of Lines.

(4) If the number of lines is less than the No. of Rows or columns then select the minimum entry out of all the uncovered entries. Subtract that entry from all the uncovered entry and add that selected entry at the junction.

(5) Repeat step No. 4 and proceed till the No. of lines equal to No. of Rows or Column.

(6) When the No. of lines becomes equal to No. of Rows or Column then select that Row or Column that has minimum No of Zeros and make the corresponding assignment.

5. 3 No of decision variables and constraints →

For $n \times n$ assignment problem → No. of decision variables = n^2

5. 4 For $n \times n$ transportation problem →

No. of constraints = $2n$

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6 INVENTORY CONTROL

6.1 Inventory

Inventory is the raw, component parts, work-in-process, or finished products that are held at a location in the supply chain.

INVENTORY

- Raw material
- Work in process
- Finished item

INVENTORY COST

- Holding cost Or Carrying cost
- Ordering cost Or Setup cost Or Procurement cost
- Shortage cost Or Stock out cost

6.2 Model 1

(1) Deterministic Model (EOQ)

Assumption of this model

(i) Demand D is known with certainty.
(ii) Usage rate is constant.
(iii) Shortage not allowed.
(iv) Lead time constant and known with certainty.
(v) Order cost is fixed "O".
(vi) Holding cost/item/unit time C_h is know
(vii) No quantity discount is offered.

(2) EOQ Model

Where:

D = Annual Demand

Q^* = Optimal order quality

O = Ordering cost

C_h = holding cost

P = purchasing cost

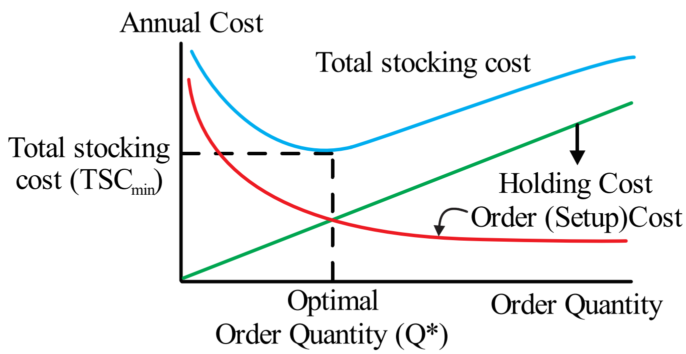

Fig. 6.1 – Basic EOQ

Total stocking cost (TSC) $= \frac{D}{Q^*}O + \frac{Q^*}{2}C_h$

Total material cost (TMC) $= \frac{D}{Q^*}O + \frac{Q^*}{2}C_h + PD$

(3) EOQ Formula:

Where:

D = Annual Demand

Q^* = Optimal order quality

O = ordering cost

Ch = holding cost

P = purchasing cost

$$(TSC)_{min} = \frac{D}{Q^*}O + \frac{Q^*}{2}C_h \quad \text{or} \quad TSC_{min} = \sqrt{2ODC_h}$$

$$(TMC) = \frac{D}{Q^*}O + \frac{Q^*}{2}C_h + PD$$

Economic Order EOQ (Q^*) $\Rightarrow Q^* = \sqrt{\frac{2OD}{C_h}}$

6. 3 Model 2

EOQ Model with Quantity Discount

Supplier commonly offer a quantity discount for sufficiently large number of units purchased at once. The basic objective behind the model is to MINIMIZE TOTAL MATERIAL COST per year.

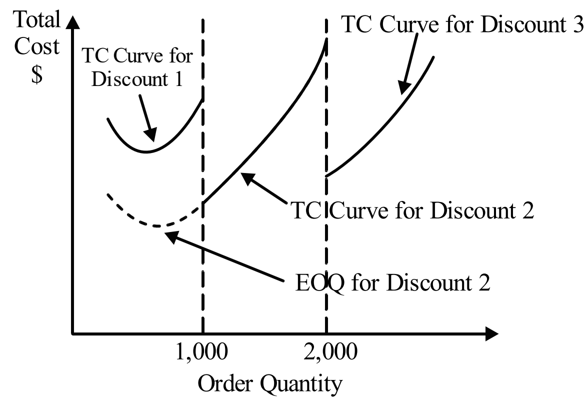

Fig. 6.2 – Discount Model

6. 4 Model 3

Economic Production Quantity (EPQ)

A problem frequently encountered by the manufactures is to determine how many units of products to produce during a production run. This quantity is known as EPQ.

Where

M = manufacturing rate

D = consumption rate

O = ordering cost

Ch = holding cost

I_{max} = maximum inventory

TSC_{min} = minimum total stocking cost

EPQ = economic production quantity

t_1 = production time

$t_1 + t_2$ = cycle time

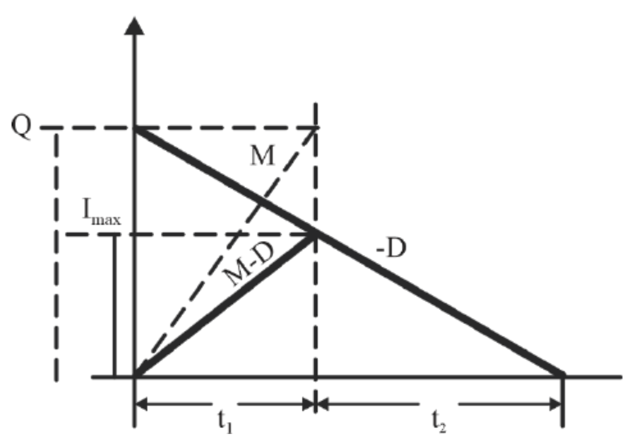

Fig. 6.3 – Production Model

$$I_{max} = (1 - D/M)Q$$

$$\text{EPQ (Optimum lot size)} = \sqrt{\frac{2OD}{C_h\left(1 - \frac{D}{M}\right)}}$$

$$TSC_{min} = \sqrt{2ODC_h\left(1 - \frac{D}{M}\right)}$$

6. 5 Model 4

Shortage Model

- This inventory model taking care of shortage.
- In actual practice shortage may take place and hence shortage cost also need to be considered.
- Shortages may also be allowed for certain advantages:

(i) Shortage increases the cycle time and hence spread the ordering cost over a longer period of time

(ii) Shortage result in decreased net stock inventory resulting in decreased inventory carrying cost.

$$Q^* = \sqrt{\frac{2OD}{C_h}}\sqrt{\frac{C_h + C_s}{C_s}}$$

$$I_{max} = \sqrt{\frac{2OD}{C_h}}\sqrt{\frac{C_s}{C_h + C_s}}$$

$$\text{Min cost / time} = \sqrt{2ODC_h}\sqrt{\frac{C_s}{C_h + C_s}}$$

6. 6 Selective Control System

Inventory management is based on selective control depending on item to be stored, upon the value or the importance. The selective inventory management plays a very important role in the minimize the inventory cost.

6. 6. 1 ABC Analysis

What is ABC analysis?

Always better Control

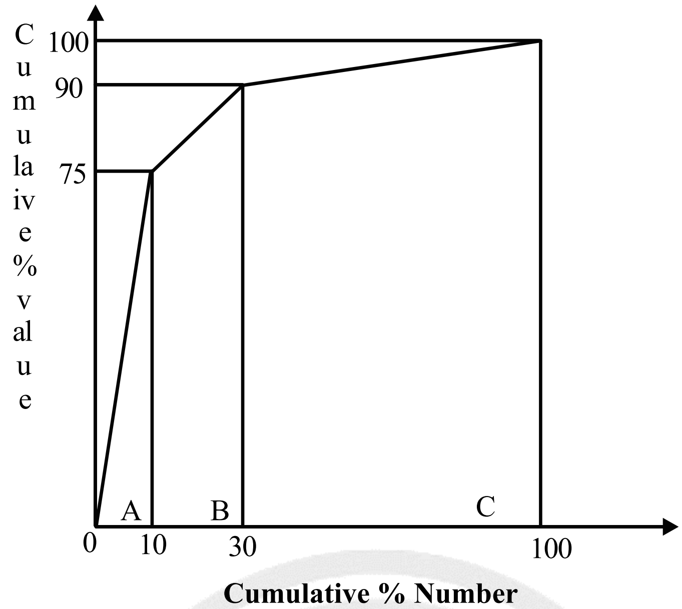

Fig. 6.4 – ABC Analysis

6. 6. 2 VED Analysis

- Vital Essential Desirable
- In this system items are classified on the bases of their importance.

(i) Absence of vital item will stop the system.

(ii) Absence of essential item will reduce the efficiency of system &

(iii) Absence of desirable item has no immediate effect on the system

6. 6. 3 SDE Analysis

This is based on relative case of availability.

(i) S-Scarce item
(ii) D-Difficult item
(iii) E-Easily available

6. 6. 4 FNSD Analysis

This is analysis divided items into four categories in the descending order of their consumption rate.

(i) F-fast
(ii) N-Normal
(iii) S-slow
(iv) D-Dead

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7 TRANSPORTATION

7.1 Objective of Transportation is to Minimize the total Transportation Cost.

From Plant	To Warehouse				Plant Capacity
	S	N	P	M	
J	Xis	Xin	Xip	Xim	100
S	Xss	Xsn	Xsp	Xsm	300
T	Xts	Xtn	Xtp	Xtm	200
Warehouse Demand	150	100	200	150	600

Total plant capacity must equal total warehouse demand.

- If $\sum$ supply = $\sum$ demand = > balance problem
- If $\sum$ supply < $\sum$ demand = > Add dummy factory having unit cost Zero
- If $\sum$ supply > $\sum$ demand = > Add dummy ware house having unit cost Zero

7. 2 Stages of solving transportation problem

(a) To find initial Feasible solution

- **Northwest corner method:** The North West corner rule is a technique for calculating an initial feasible solution for a transportation problem. In this method, we must select basic variables from the upper left cell, i.e., the North-west corner cell
- **Least cost method:** This method consists in allocating as much as possible in the lowest cost cell and then further allocation is done in the cell/cells with second lowest cost and so on.
- **Vogel's approximation method (penalty cost method):** Vogel's Approximation Method (VAM) is **one of the methods used to calculate the initial basic feasible solution to a transportation problem**. However, VAM is an iterative procedure such that in each step, we should find the penalties for each available row and column by taking the least cost and second least cost

(b) After finding the basic feasible solution check Degeneracy

Check for Degeneracy

A = Number of allocations in transportation table

If $m + n - 1 = A \Rightarrow$ No Degeneracy in problem

If $m + n - 1 > A \Rightarrow$ Degeneracy

(c) Getting optimal Solution

(i) Stepping Stone Method

(ii) U-V Method [MODI Method]

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8 PERT & CPM

8.1 PERT and CPM (NETWORK ANALYSIS)

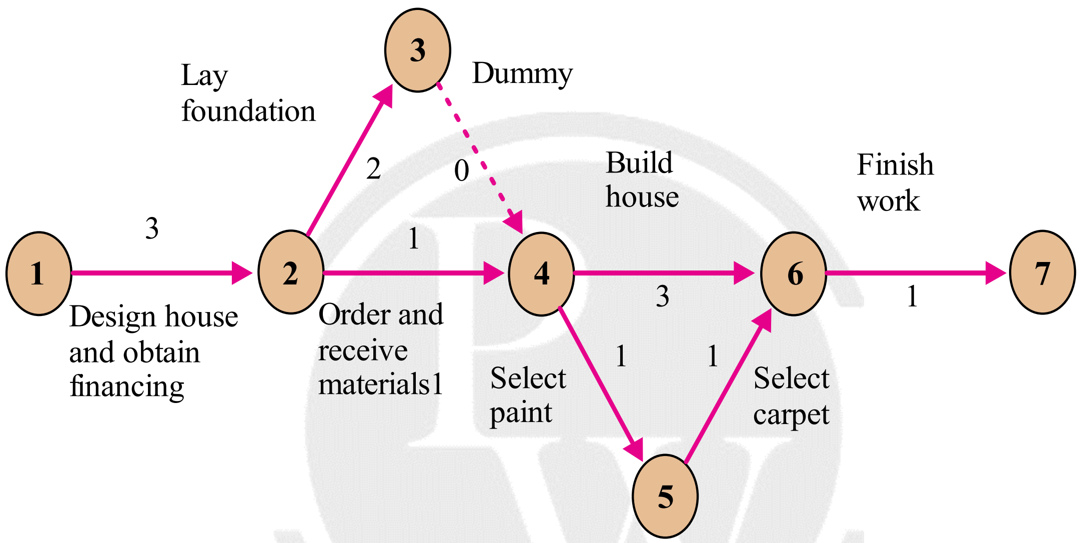
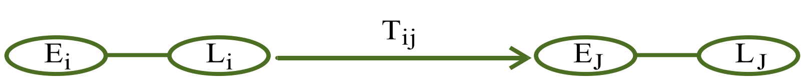


Fig. 8.1 Network Diagram

8.2 Critical Path Method (CPM)

- Deterministic Model
- Activity oriented



E_i Earliest start time for activity

L_i Latest start time for activity

E_J Earliest finished time for activity

L_J Latest finished time for activity

8. 3 Float (Extra time)

- **Total float** represents the maximum time within which an activity can be delayed **without affecting the project completion time**.

  $TF = L_j - E_i - T_{ij}$

- **Free float** is the extra time available with an activity without affecting any successor activity.

  $FF = E_j - E_i - T_{ij}$

- **Safety float** is the extra time available with an activity without affecting the Latest start time of successor activity.

  $SF = L_j - L_i - T_{ij}$

- **Independent float** is the extra time available with an activity *without affecting Earliest & Latest time of other activities*.

  $IF = E_j - L_i - T_{ij}$

8. 4 Project Evaluation Review Technique (PERT)

- Probabilistic model.
- Event oriented time estimate for the activity are based upon three different values:

  (a) Most optimistic time. (t_o)
  (b) Most pessimistic time. (t_p)
  (c) Most likely time. (t_m)

- The distribution followed by t_o, t_p & t_m is β distribution. $T_p \geq t_m \geq t_0$
- The distribution followed by project completion time is normal distribution.
- Expected time $= t_e = \left[\dfrac{t_0 + 4t_m + t_p}{6}\right]$
- Standard deviation $= (\sigma) = \dfrac{t_p - t_o}{6}$
- Variance (V) $= \sigma^2 = \left(\dfrac{t_p - t_o}{6}\right)^2$
- Probability that the project will be completed in a given time. (T)
- The expected completion time (from Critical path) (t_{cp})
- Standard deviation of critical path (σ_{cp})
- Z stands for standard Normal variable.

$$(Z) = \left(\frac{T - t_{cp}}{\sigma_{cp}}\right)$$

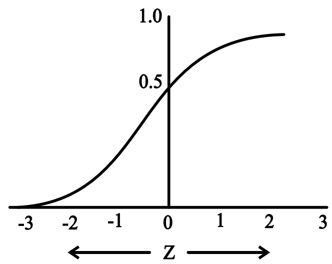


Fig. 8.2 Standard Normal Variable

z = 0, Probability = 50%

z = ± 1, Probability = 68.3%

z = ± 2, Probability = 95.5%

z = ± 3, Probability = 99.7%

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9 QUEUING THEORY

9.1 Queuing Theory

A queue represents items or people awaiting sevice

Fig. 9.1 Queue

9.2 The Queuing Cost Trade-Off

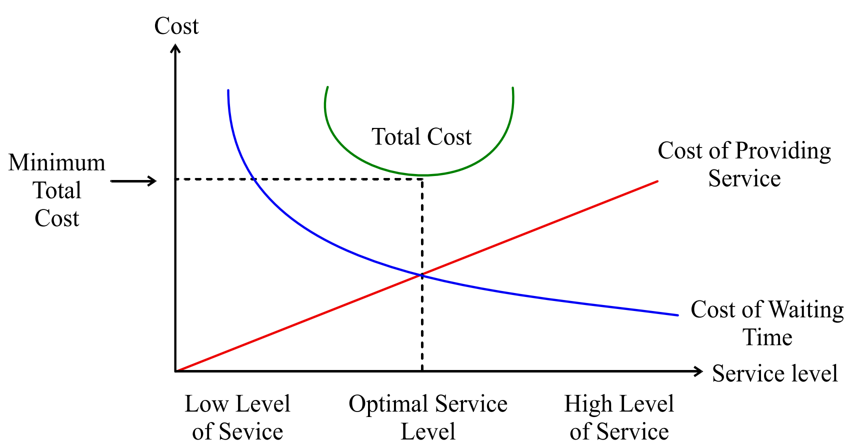

Fig. 9.2 Cost v/s service level

9. 3 Components of a Basic Queuing Process

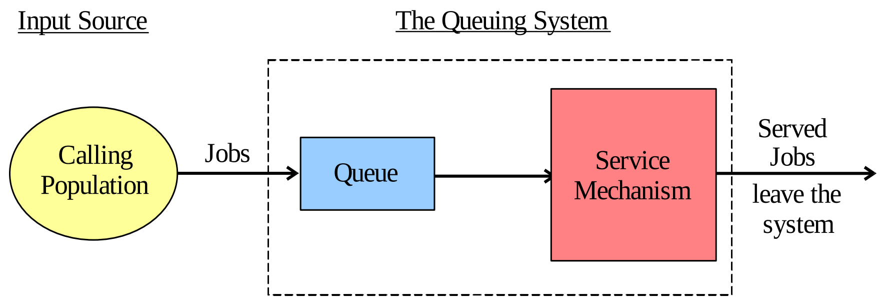

Fig. 9.3 Queuing model

9. 4 Characteristics of Queuing Model

9. 4. 1 Arrival Pattern of Customer

- It can be spaced by equal and unequal time intervals however in most of the cases random arrival is observed which is best described by ***POISSON process***. If the arrivals are governed by Poisson process then the time between successive arrival is ***NEGATIVE EXPONENTIALLY*** distributed.
- The mean arrival rate is denoted by λ

Theorem:

- If n, the number of arrivals in t, follow the Poisson distribution,

$$\frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

- Then T (inter-arrival time) obeys the negative exponential law

$$\lambda e^{-\lambda T}$$

9. 4. 2 Service Pattern

- Unlike arrival process there is no standard probability distribution for service process. In fact, in many cases actual data is used for describing the service time. However, if the service time are ***EXPONENTIALLY DISTRIBUTED*** then the Queuing model become simple.
- If there is infinite number of servers then all the customers are served instantaneously on arrival, and there will be no queue.
- The mean service rate is denoted by **μ**

9. 4. 3 The Queue Discipline

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are

(a) FCFS
(b) FIFO
(c) LIFO
(d) SIRO

9. 4. 4 Customer behavior

(a) Balking
When the customer decided not to enter the queue if the queue is very long.

(b) Reneging
Customer enters the queue but leaves after some time tie without getting service.

(c) Jockeying
Switching of queues

9. 4. 5 Size of population

- In most of the cases the capacity of the system is limited. The entire sample of customer from which a few visit the service facility is known as calling population.
- Size of calling population can be finite or infinite.

9. 5 Kendall's Notation for Representing Queuing Models

(a|b|c) : (d|e)

a = Probability distribution of arrival time

b = probability distribution of service time

c = Number of Servers

d = capacity of the system (size of calling population)

e = Queue Discipline

$$\rho = \frac{\lambda}{\mu}$$

- Traffic intensity factor or utilization factor or channel efficiency or percentage time the server is busy or probability that the customer has to wait.
- The average service rate must always exceed the average arrival rate.

$\mu > \lambda$

- Otherwise, the queue will grow to infinity (∞)

9.6 Model I (M | M | 1): (∞ | FCFS)

- This denotes a Queuing model in which *arrivals* are generated by *Poisson Process* (inter arrival time are exponentially distributed), *service time* are *exponentially distributed* has single server, infinite calling population & service rule FCFS.
- L_s = Expected system/line length i.e., expected number of customers in the system

  $= \frac{\rho}{1-\rho}$
- L_q = Expected queue length i.e., expected number of customers in the Queue

  $L_q = L_s - \rho$
- W_q = Expected waiting time per customer in the Queue.

  $= \frac{\lambda}{\mu(\mu-\lambda)}$
- W_s = Expected waiting time per customer in the system

  $W_s = W_q + \frac{1}{\mu}$
- P_0 = Probability that the service facility is Idle or there are Zero customer in system probability that the customer does not have to wait

  $= 1 - \rho$
- P_0 = Probability of n customer in the system

  $= \rho^n \times P_0$
- L_n = Expected length of Non-empty Queue

  $\frac{1}{1-\rho}$
- $P(Wq \geq t)$ = Probability that the waiting time in the Queue is greater than equal to t

  $= \rho e^{-(\mu-\lambda)t}$
- $P(Ws \geq t)$ = Probability that the waiting time in the system is greater than equal to t

  $e^{-(\mu-\lambda)t}$

9.7 Little's law

$\Rightarrow L = \lambda W$ $\quad\quad$ $L_q = \lambda W_q$

9. 8 Model II (M | M | 1): (N | FCFS)

- This model differs from that of Model I in the sense that the maximum number of customers in the system is limited to N. Arrivals will not exceed N in any case.

$$P_0 = \frac{1-\rho}{1-\rho^{(N+1)}}$$

$$P_n = \left[\frac{1-\rho}{1-\rho^{(N+1)}}\right]\rho^n$$

$$L_s = P_0 \sum_{n=0}^{N} n\rho^n$$

$$L_q = L_s - \rho$$

$$W_s = L_s / \lambda$$

$$W_q = W_s - (1/\mu)$$

9. 9 Model No. III (M | G | 1): (∞/FCFS)

σ = Standard deviation for service tie

$$L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$$

$$W_q = \frac{L_q}{\lambda}$$

$$L_s = L_q + \rho$$

$$W_s = W_q + \frac{1}{\mu}$$

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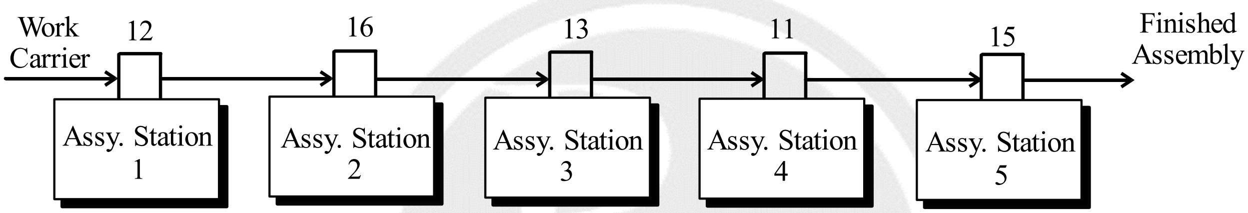
10 PLANT LAYOUT, MRP & CONTROL CHART

10.1 Line Balancing

The problem is to arrange the individual processing and assembly tasks at the workstations so that the total time required at each workstation is approximately the same.

Nearly impossible to reach perfect balance



- Total work content (TWC) = $\sum T_{si}$
- Line efficiency $= \dfrac{TWC}{N(T_c)}$
- Theoretically minimum number of work station

$$N_{min} = \frac{TWC}{(T_c)}$$

- Balance delay = 1 – Line efficiency
- The objective of line balancing is to combine various operation on different work station in such a way that ***the processing time at each work station is almost the same and equal to the cycle time***

$$T_{cycle} \geq T_{system\,max}$$

- Entire job is divided into different element each operation is performed on work piece is known as work element.
- Specific location on the line at which few operations have to be performed is a work station.
- Station time (Tsi) must lie between cycle time and maximum of all work element time (Max $\{T_{iN}\}$):

$$\text{Max}\{T_{iN}\} \leq T_{si} \leq T_c$$

10. 2 Control Chart

- Control charts are statistical tool, showing whether a process is in control or not.
- It is a graphical tool for monitoring the activities of an ongoing process also referred as ***Shewhart*** control charts.

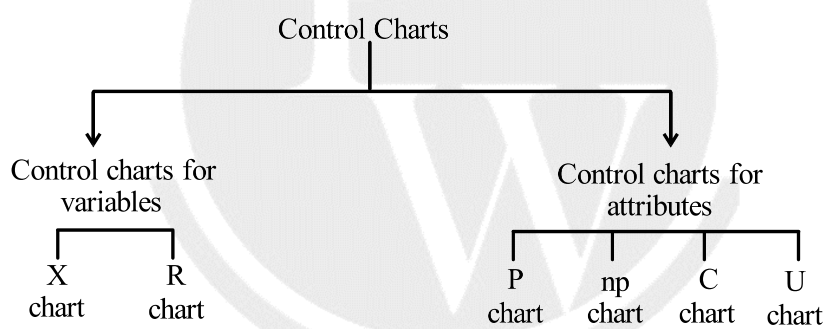
10. 3 Causes of variation in Quality

- **<u>Variation due to chance cause is</u>**: variation due to chance causes conform to normal distribution curve and they are not very high magnitude and also known as RANDOM VARIATION.
- These causes are natural to any manufacturing process and are beyond human control. Like slight variation in temperature, pressure and humidity, etc.
- **<u>Variation due Assignable cause is</u>**: variation due to assignable causes do not conform to any standard distribution. They can be found out and completely eliminated.
- Some of the assignable causes of variation are defective raw material, improper machine setups, worn equipment's, unskilled workers.

10. 4 Types of process data

- **<u>Variable:</u>** continuous data. Things we can measure. Example includes length, weight, time, temperature, diameter etc.
- **<u>Attribute:</u>** discrete data. Things we CAN'T count and can only be classified as good or bad. Examples includes number or percent defective items in a lot, number of defects per item etc.

10. 5 Types of control chart



10.6 Material Requirement Planning

10.6.1 MRP

"MRP constitutes a set of techniques that use bill of material inventory data, and the master production schedule to calculate requirements for materials."

10. 6. 2 Objective of MRP

- Inventory reduction
- Avoid delays
- Realistic commitments
- Increased efficiency

10. 6. 3 Major inputs of MRP System

(a) Bill of Material

- Product structure file
- Determines which component items need to be scheduled

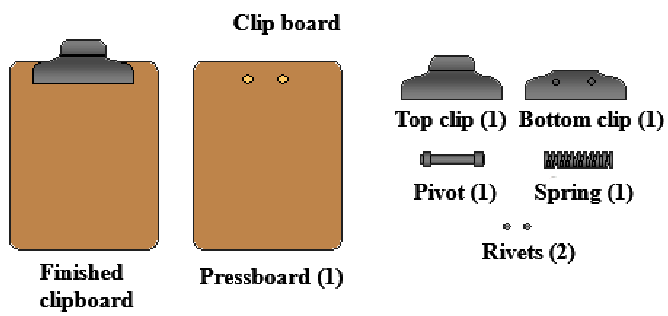


Fig. 10.1

(b) Master Production Schedule (MPS)

- It is the schedule, which shows the number and timing of all end items to be produced over a planning horizon.

	PERIOD				
MPS ITEM	1	2	3	4	5
Clipboard	85	95	120	100	100
Lap desk	0	50	0	50	0
Lapboard	75	120	47	20	17
Pencil Case	125	125	125	125	125

(c) Inventory Record

- Contains an extensive amount of information on every item that is produced, ordered, or inventoried in the system

10. 7 Plant layouts

"Plant layout ideally involves: -

(a) Allocation of space and

(b) Arrangement of equipment in such a manner that overall operating costs are minimized".

Objectives of Layout

The basic objective of layout design is to facilitate a smooth flow of work, material, and information through the system. Supporting objectives generally involve the following:

(a) To use workers and space efficiently.
(b) To avoid bottlenecks.
(c) To minimize material handling costs.
(d) To eliminate unnecessary movements of workers or materials.
(e) To minimize production time or customer service time.
(f) To design for safety

Different types of common layouts

Commonly, the layouts are of the following types:

(a) Product or line layout.

(b) Process or functional layout.

(c) Fixed position layout

(d) Cellular or group technology layout.

10. 8 Product or line layout

(a) The materials move from one workstation to another sequentially without any backtracking or deviation.

(b) Materials are fed into 1st machine and semi-finished goods travel automatically from machine to machine.

(c) The output of one machine becoming input of the next.

Ex: Food Processing Unit, Paper mill

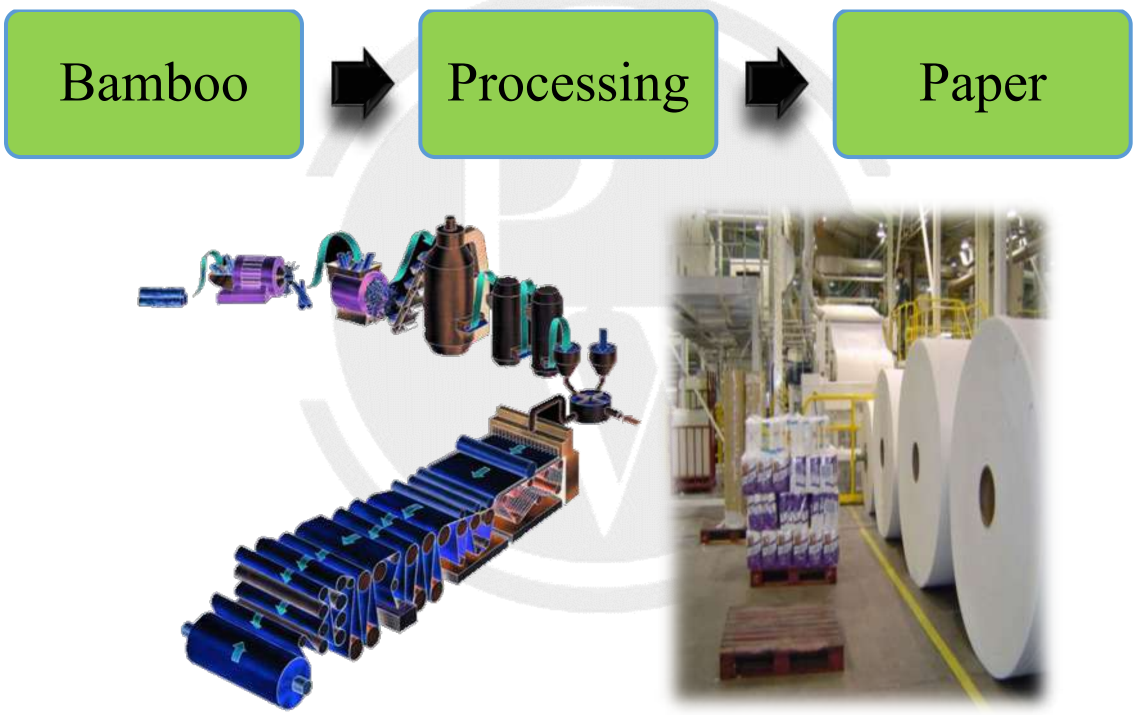


Fig. 10.2 Product layout

10. 9 Process or functional layout

- In this type of layout machines of a similar type are arranged together at one place.
- The work has to be allocated to each department in such a way that no machines are chosen to do as many different jobs as possible.

  Eg: Process oriented layout for a hospital

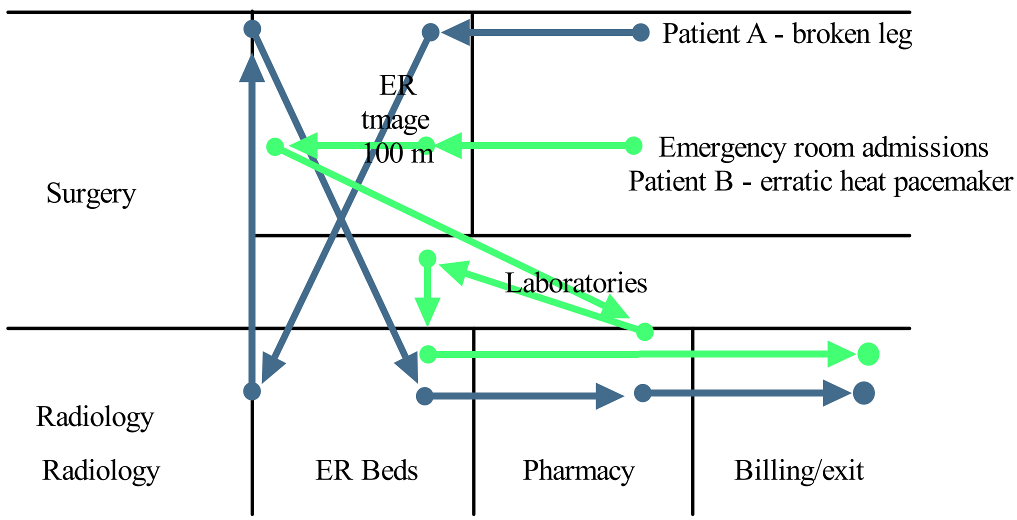


Fig. 10.3 Process layout

10. 10 Fixed Position Layout

- Here, Major products being produced is fixed at one location.
- All other facilities are brought and arranged around the work center.

  Ex: Ship building, Dam construction, flyover construction.

Fig. 10.4 Fixed layout

10. 11 Cellular or Group Layout

- "Group technology is the technique of identifying and bringing together related or similar parts in production process in order to utilize the inherent economy of flow production methods."
- This layout is suitable for a manufacturing environment in which *large variety of products are needed in small volumes* (or batches).
- Every cell contains a group of machines which are dedicated to the production of a family of parts.

10. 12 Process flow before the group technology

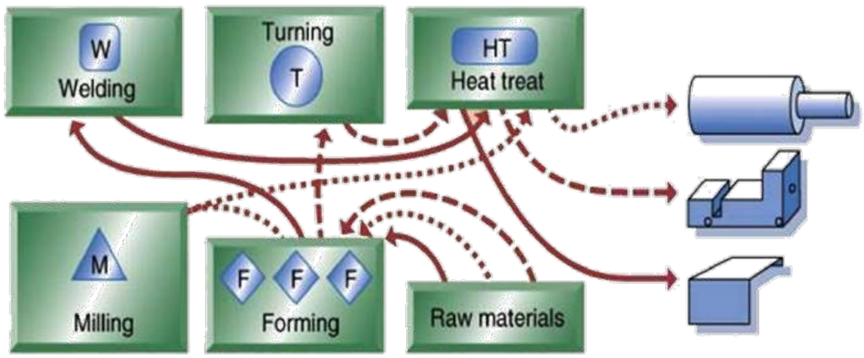


Fig. 10.5

Process flow after the group technology

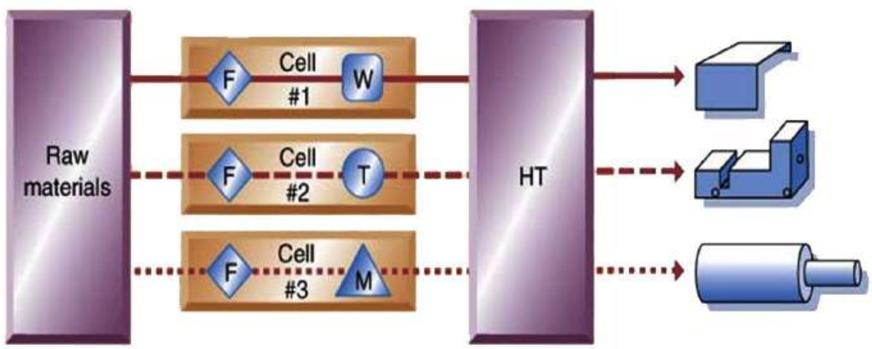


Fig. 10.6

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Manufacturing Engineering

Manufacturing Engineering

INDEX

1 THEORY OF METAL CUTTING

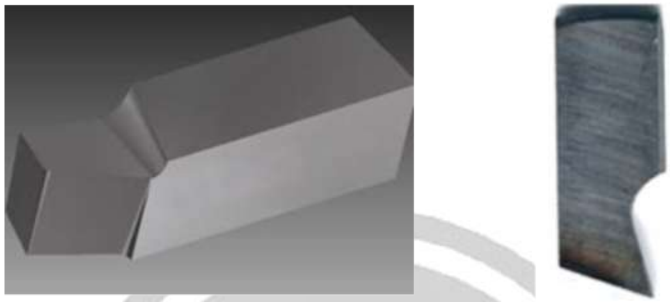


Fig. 1.1 Single point cutting tool

Fig. 1.2 Continues Chip

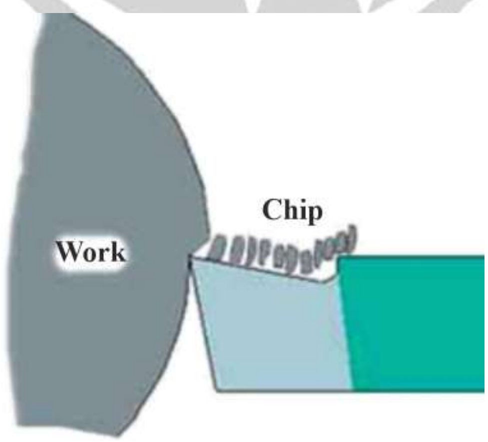


Fig. 1.3 Discontinues Chip

1.1 Machining

Machining is an essential process of finishing by which jobs are produced to

(a) The desired dimensions and

(b) Surface finish

1.2 Orthogonal Machining

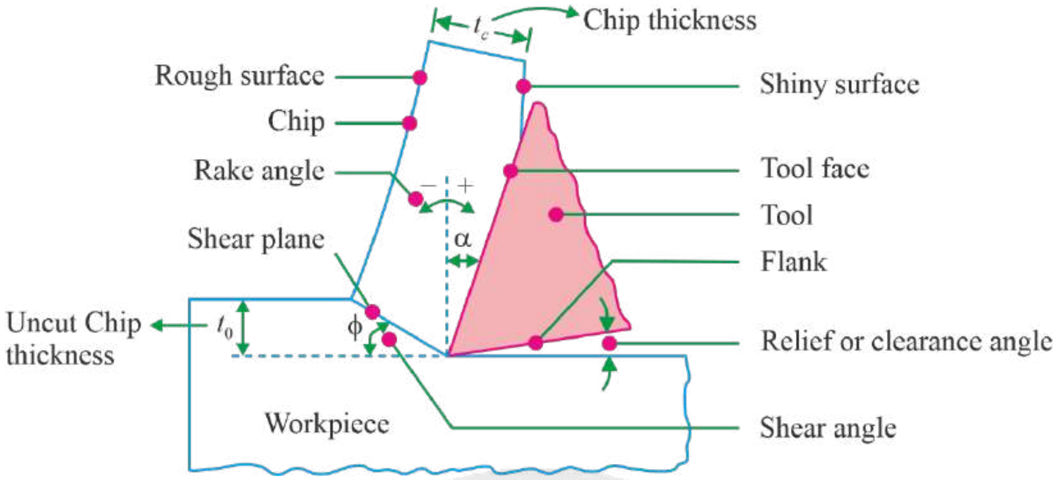


Fig. 1.4. Machining

1.3 Types of Machining

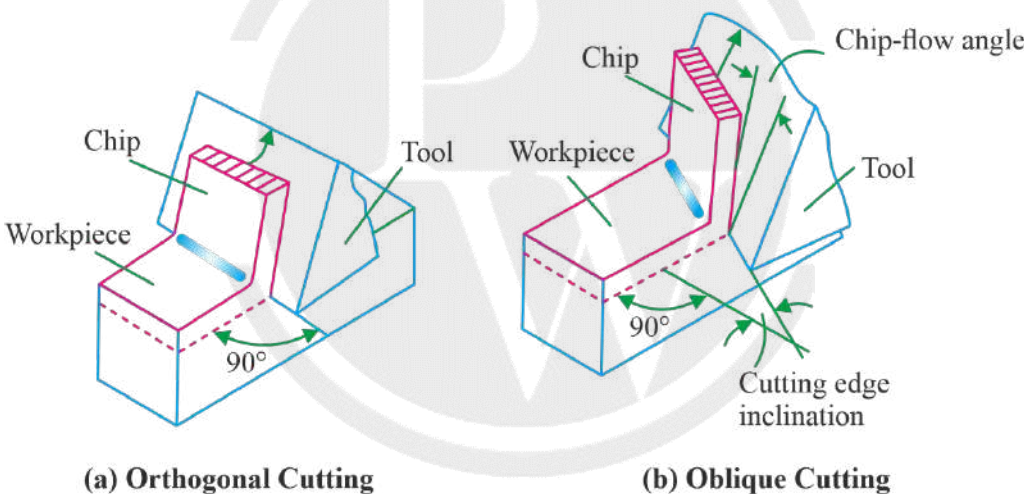


Fig. 1.5 Different types of Cutting Process

1.4 Orthogonal Cutting

1. Cutting edge of the tool is perpendicular to the direction of cutting velocity.
2. The cutting edge is wider than the workpiece width and extends beyond the workpiece on either side. Also, the width of the workpiece is much greater than the depth of cut.
3. The chip generated flows on the rake face of the tool with chip velocity perpendicular to the cutting edge
4. The cutting forces act along two directions only.

1.5 Geometry of single point turning tool

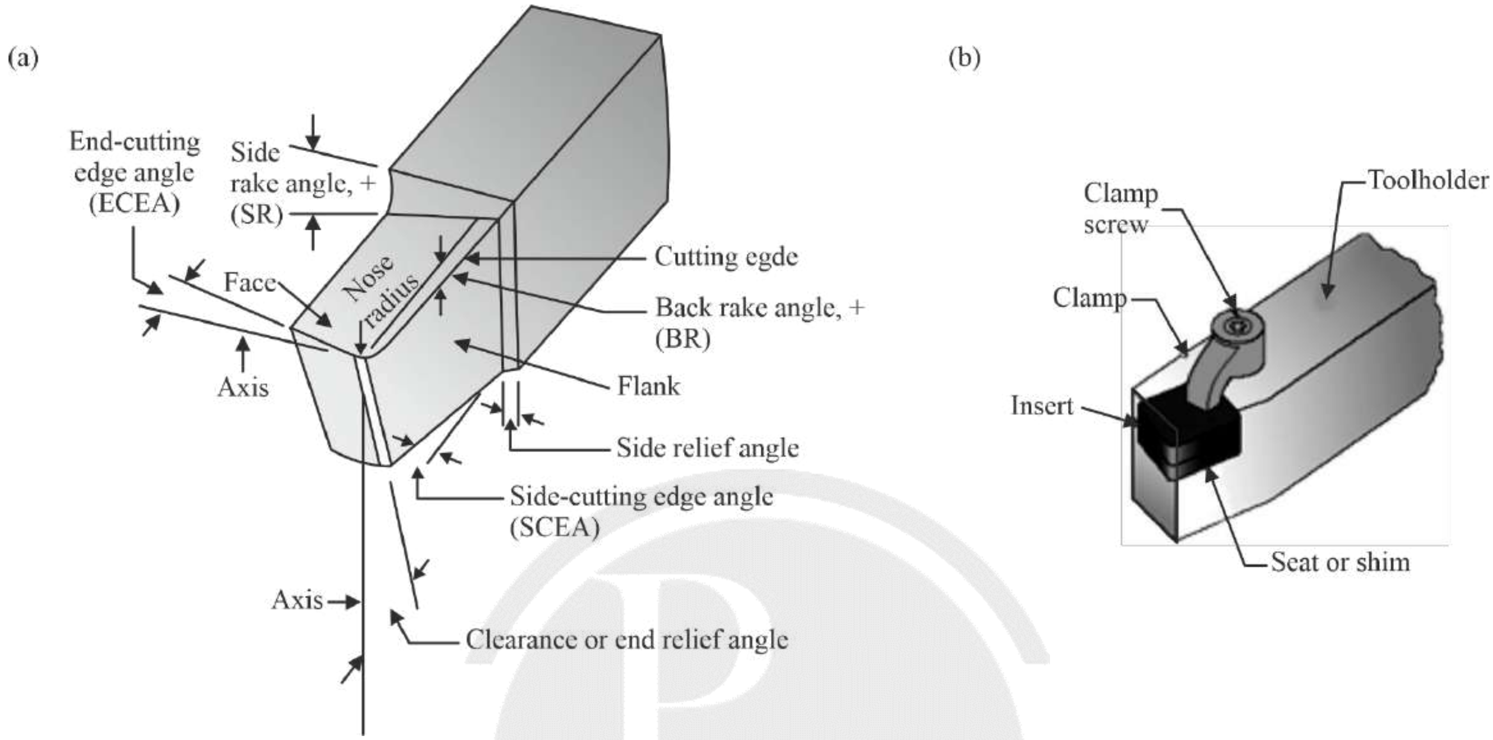


Fig. 1.6 Single point cutting tool

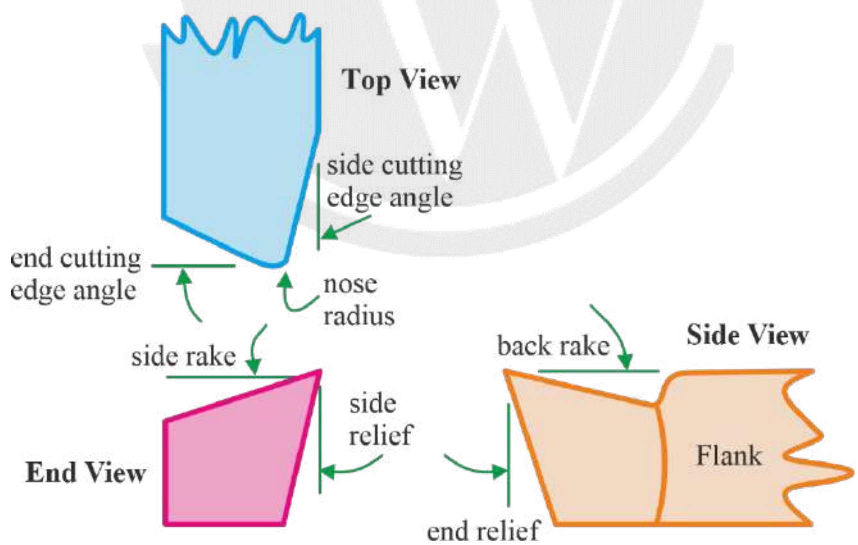


Fig. 1.7 Different type of views

1.6 Tool designation (ANSI) or ASA

1.6.1 To remember easily follow the rule

Rake (α_b α_s), relief (γ_e γ_s), cutting edge (C_e C_s)

Side will come last finish with nose radius (inch)

$$\alpha_b - \alpha_s - \gamma_e - \gamma_s - C_e - C_s - R$$

1.6.2 Orthogonal Rake System (ORS)

$$i - \alpha - \gamma - \gamma_1 - C_e - \lambda - R$$

- Inclination angle (i)
- Orthogonal rake angle (α)
- Side relief angle (γ)
- End relief angle (γ_1)
- End cutting edge angle (C_e)
- Principal cutting edge angle or Approach angle ($\lambda = 90 - C_s$)
- Nose radius (R) (mm)
- For Orthogonal cutting, i = 0
- For Oblique cutting, i $\neq$ 0

1.6.3 Inter conversion between ASA & ORS

$$\tan\alpha = \tan\alpha_s \sin\lambda + \tan\alpha_b \cos\lambda$$

$$\tan\alpha_b = \cos\lambda \tan\alpha + \sin\lambda \tan i$$

$$\tan\alpha_s = \sin\lambda \tan\alpha - \cos\lambda \tan i$$

$$\tan i = -\tan\alpha_s \cos\lambda + \tan\alpha_b \sin\lambda$$

1.7 Shear angle (ϕ)

1.7.1 Chip Thickness Ratio

$$r = \frac{t_0}{t_c} = \frac{l_c}{l} = \frac{V_c}{V} = \frac{\sin\phi}{\cos(\phi - \alpha)} = \frac{1}{h}$$

and $$\tan\phi = \frac{r\cos\alpha}{1 - r\sin\alpha}$$

Where

r = chip thickness ratio or cutting ratio; $r < 1$

$h = 1/r$ = Inverse of chip ratio or chip reduction factor or chip compression ratio; $h > 1$

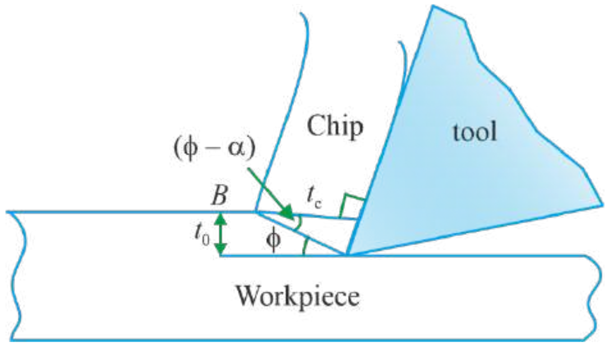


Fig. 1.8

As $\quad \sin\phi = \dfrac{t_0}{AB}$ and $\cos(\phi - \alpha) = \dfrac{t_c}{AB}$

$$\text{Chip thickness ratio (r)} = \frac{t_0}{t_c} = \frac{\sin\phi}{\cos(\phi-\alpha)}$$

1.7.2 Cutting shear strain (ϵ) i.e cutting strain

The magnitude of strain, that develops along the shear plane due to machining action, is called cutting strain (shear). The relationship of this cutting strain, ϵ

$$\varepsilon = \cot\phi + \tan(\phi - \alpha)$$

1.8 Velocity diagram of cutting zone

Need velocities to obtain power estimates

$$V_{\frac{\text{Material}}{\text{Tool}}} + V_{\frac{\text{Chip}}{\text{Material}}} = V_{\frac{\text{Chip}}{\text{Tool}}}$$

$$V_{\frac{\text{Material}}{\text{Tool}}} = \text{Cutting velocity} = V$$

$$V_{\frac{\text{Chip}}{\text{Material}}} = \text{Shear velocity} = V_s$$

$$V_{\frac{\text{Chip}}{\text{Tool}}} = \text{Chip velocity} = V_c$$

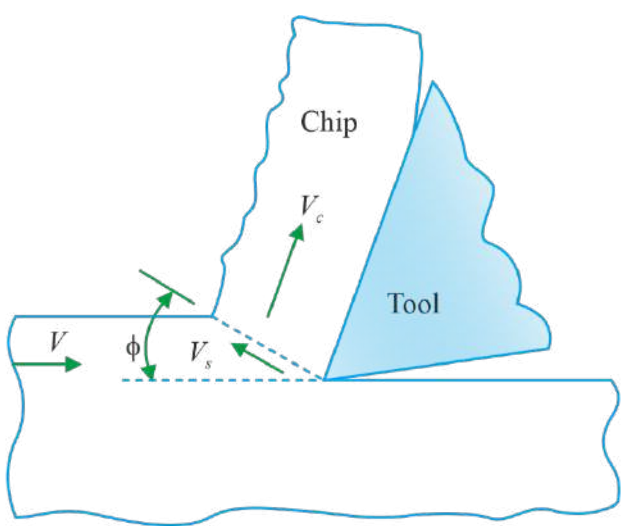


Fig. 1.9

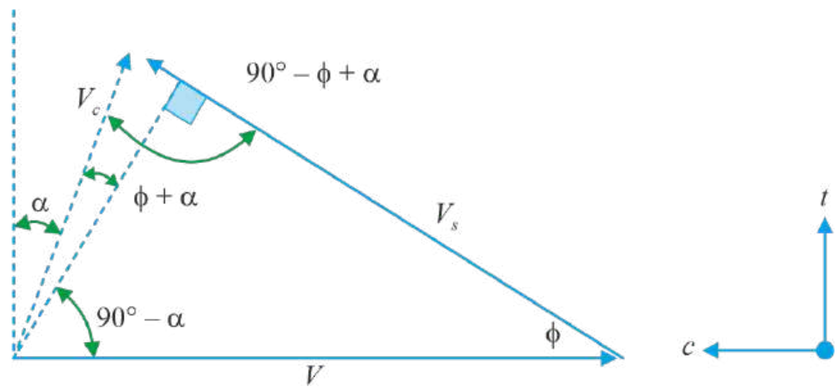


Fig. 1.10 Velocity Triangle

Apply sin rule

$$\frac{v}{\sin\left(\frac{\pi}{2}-\phi+\alpha\right)} = \frac{v_s}{\sin\left(\frac{\pi}{2}-\phi\right)} = \frac{v_c}{\sin(\phi)}$$

$$\frac{v}{\cos(\phi-\alpha)} = \frac{v_s}{\cos(\phi)} = \frac{v_c}{\sin(\phi)}$$

1.9 Shear Strain Rate

$$\dot{\varepsilon} = \frac{d\varepsilon}{dt} = \frac{V_s}{\text{thickness of shear zone}(t_s)}$$

t_s =1/10th or (10%) of shear plane length and its maximum value is 25 microns.

1.10 Determination of Un-deformed chip thickness in Turning:

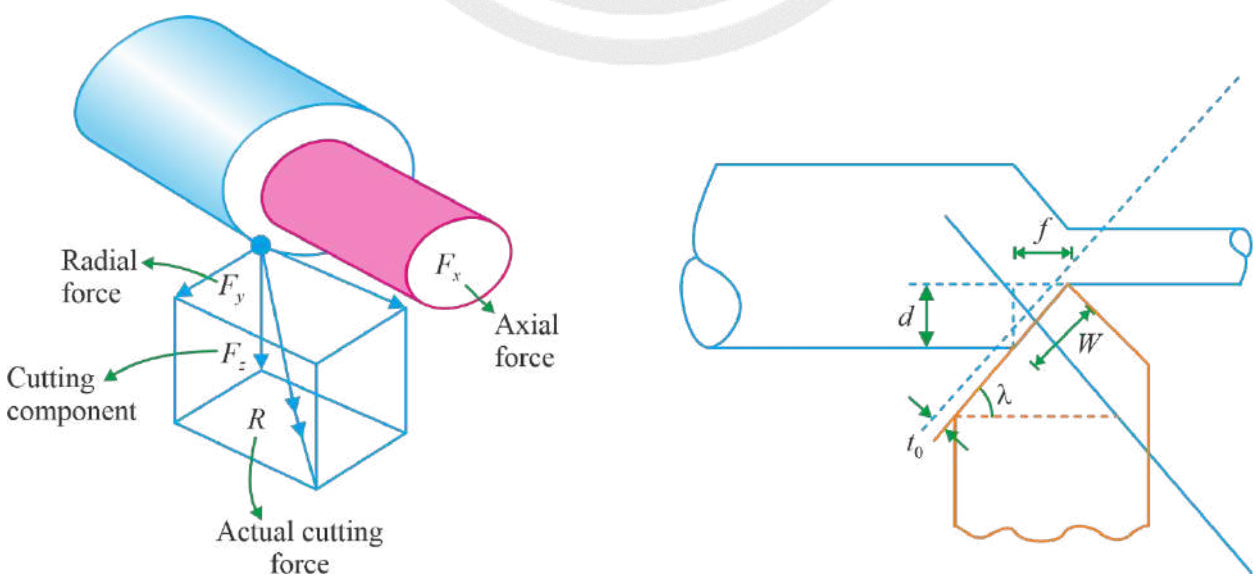


Fig. 1.11 Turning process

$$t_0 = f \sin\lambda,$$

$$w = \frac{d}{\sin\lambda}$$

where

f = feed (mm/rev)

d = depth of cut (mm)

t_0 = uncut chip thickness

W = width of chip

λ = Approach angle

(1) Turning is 3-D cutting $\Rightarrow$ three force comes in picture

(2) Turning is not Orthogonal cutting(2-D)

(3) **t_0 = fsin λ**

(4) **w = d/sin λ**

For orthogonal cutting t_0 = d

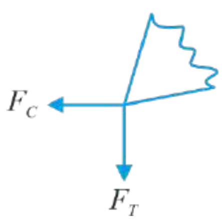
Width of chip = width of cut(w)

1.11 Types of Chip

- Continuous chip
- Discontinuous chip
- Continuous chip with BUE

1.12 Force & Power in Metal Cutting

F_c and F_t



The two orthogonal components (horizontal and vertical) Fc and Ft of the resultant force R can be measured by using a **dynamometer**.

Merchant Force Circle Diagram (MCD)

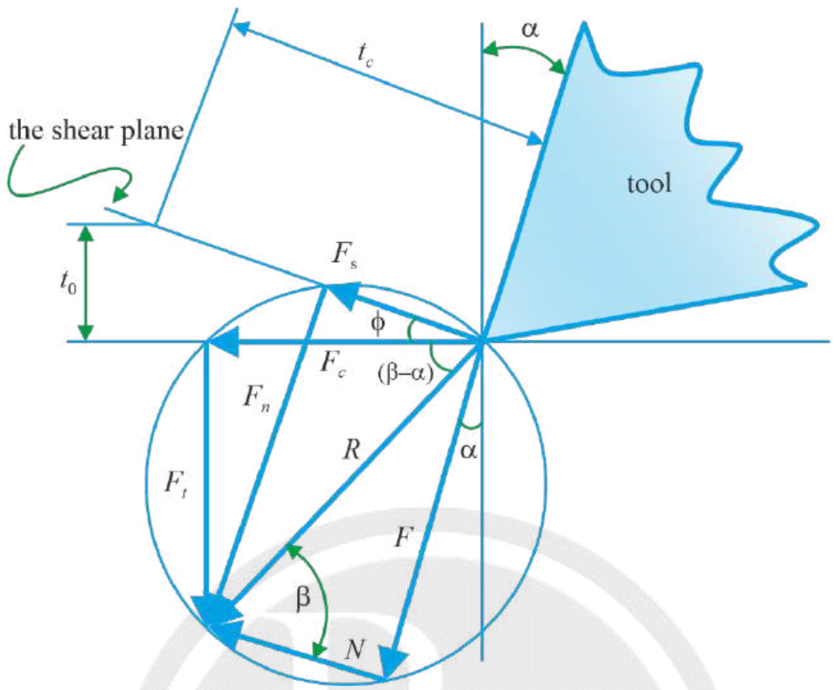


Fig. 1.12 Merchant Circle Diagram

where

f_s = Shear force
F_n = Normal to shear force
F_c = Cutting component
F_t = Thrust component
F = Friction force
N = Normal to friction
β = Friction angle
R = Actual cutting force

For orthogonal cutting only

The force relations

$$F = F_c \sin\alpha + F_t \cos\alpha$$
$$N = F_c \cos\alpha - F_t \sin\alpha$$
$$F_n = F_c \sin\phi + F_t \cos\phi$$
$$F_s = F_c \cos\phi - F_t \sin\phi$$

(a) From Merchant Analysis

$$\phi = \frac{\pi}{4} + \frac{\alpha}{2} - \frac{\beta}{2}$$

(b) Lee and Shaffer

$$\phi = \frac{\pi}{4} + \alpha - \beta$$

(c) Stabler

$$\phi = \frac{\pi}{4} + \frac{\alpha}{2} - \beta$$

1.13 Metal Removal Rate (MRR)

Metal removal rate (MRR) = $A_c.v = wt_0v$(orthogonal cutting) = fdv(turning)

Where

A_c = cross-section area of uncut chip (mm^2)

v = cutting speed = πDN, mm / min

f = feed (mm/rev)

d = depth of cut (mm)

w = width of cut

t_0 = uncut chip thickness

1.14 Heat Distribution in Metal Cutting

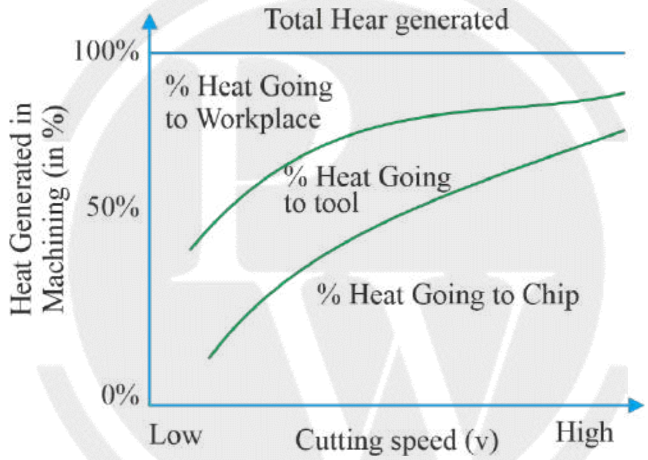


Fig. 1.13 Heat distribution

1.15 Specific cutting pressure

The cutting force, Fc, divided by the cross-section area of the undeformed chip gives the nominal cutting stress or the specific cutting pressure,

$$P_c = \frac{F_c}{bt} = \frac{F_c}{fd}$$

1.16 Tool Wear, Tool Life

1.16.1 Tool Wear

(i) Flank Wear, At low speed → **Slow Death**

(ii) Crater Wear, At high speed → **Slow Death**

(iii) Chipping off of the cutting-edge → **Sudden Death**

1.16.2 Flank Wear: (Wear land)

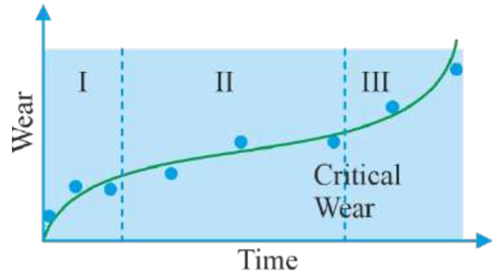


Fig. 1.14 Tool Wear

I = Primary wear zone
II = Secondary wear zone
III = Tertiary wear zone

Note:
(i) In Primary wear zone wear rate is constant.
(ii) Wear gradually increases in tertiary wear zone.

1.17 Tool Life

Taylor's Tool Life Equation

Causes

Sliding of the tool along the machined surface

Temperature rise $VT^n = C$

Where,
V = cutting speed (m/min)
T = Time (min)
n = exponent depends on tool material
C = constant based on tool and work material and cutting condition.

1.18 Extended or Modified Taylor's equation

$VT^n f^a d^b = C$

Where:
d = depth of cut
f = feed rate

$$\frac{a}{n} = \frac{1}{n_1}$$

$$\frac{b}{n} = \frac{1}{n_2}$$

or
$$T = \frac{C^{1/n}}{V^{1/n}.f^{1.n_1}.d^{1/n_2}}$$

$$\frac{1}{n} > \frac{1}{n_1} > \frac{1}{n_2}$$

i.e Cutting speed has the greater effect followed by feed and depth of cut respectively.

1.19 Economics of metal cutting

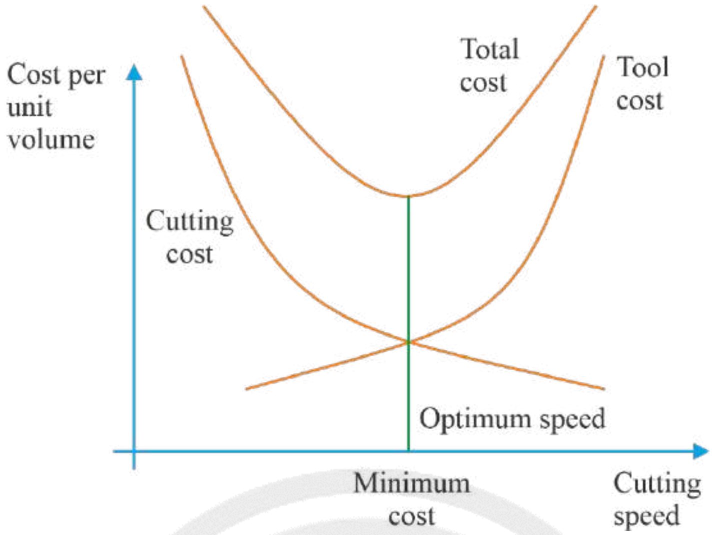


Fig. 1.15 Economics of Metal Cutting

Formula

$$V_0 T_0^n = C$$

(a) Optimum tool life for minimum cost

$$T_o = \left(T_c + \frac{C_t}{C_m}\right)\left(\frac{1-n}{n}\right) \quad \text{if } T_c, C_t \,\&\, C_m \text{ given}$$

$$= \frac{C_t}{C_m}\left(\frac{1-n}{n}\right) \quad \text{if } C_t \,\&\, C_m \text{ given}$$

(b) Optimum tool life for Maximum Productivity

(Minimum production time)

$$T_o = T_c\left(\frac{1-n}{n}\right)$$

Units: T_c – min (Tool changing time)

C_t – Rs./ servicing or replacement (Tooling cost)

C_m – Rs/min (Machining cost)

V – m/min (Cutting speed)

Tooling cost (C_t) = tool regrind cost + tool depreciation per service/ replacement

Machining cost (C_m) = labour cost + overhead cost per min

1.20 Surface Roughness

1.20.1 Ideal Surface (Zero nose radius)

Peak to valley roughness (h) $= \dfrac{f}{\tan C_s + \cot C_e}$

and (Ra) $= \dfrac{h}{4} = \dfrac{f}{4(\tan C_s + \cot C_e)}$

1.20.2 Practical Surface (with nose radius = R)

$$h = \frac{f^2}{8R}$$

and $R_a = \dfrac{f^2}{18\sqrt{3}R}$

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2 NTMM (NON-TRADITIONAL MACHINING METHOD)

2.1 Electro Chemical Machining (ECM)

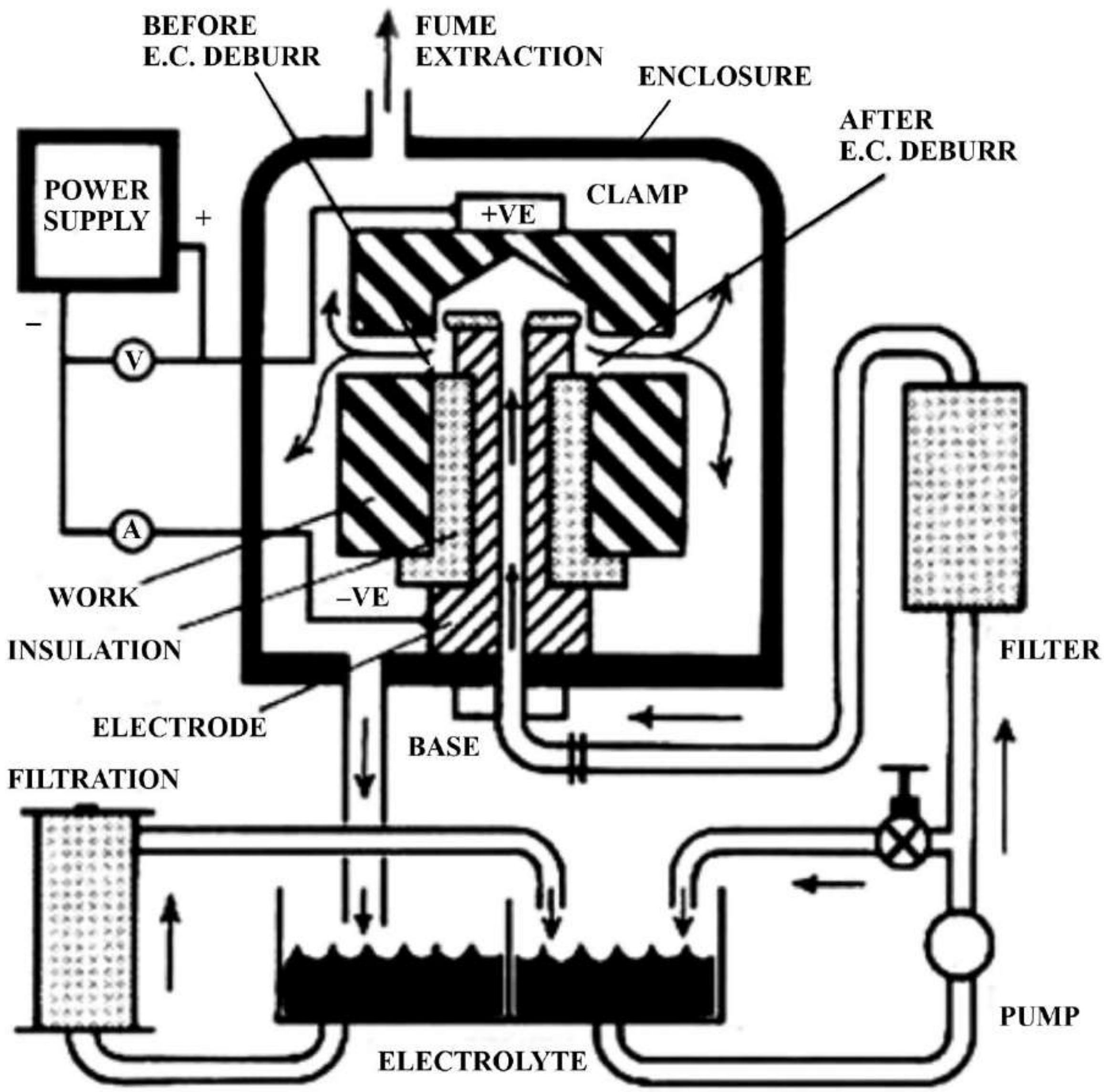


Fig. 2.1 Electro Chemical Machining

2.1.1 Electrochemical Machining

- Electrochemical machining is the *reverse of electro plating*
- The work-piece is made the anode, which is placed in close proximity to an electrode (cathode), and a high-amperage direct current is passed between them through an electrolyte, such as salt water, flowing in the anode-cathode gap.
- **MRR in ECM depends on atomic weight of work material**
- Commercial ECM is carried out at a combination of **low voltage high current**
- ECM has the highest metal removal rate, among the NTMM.

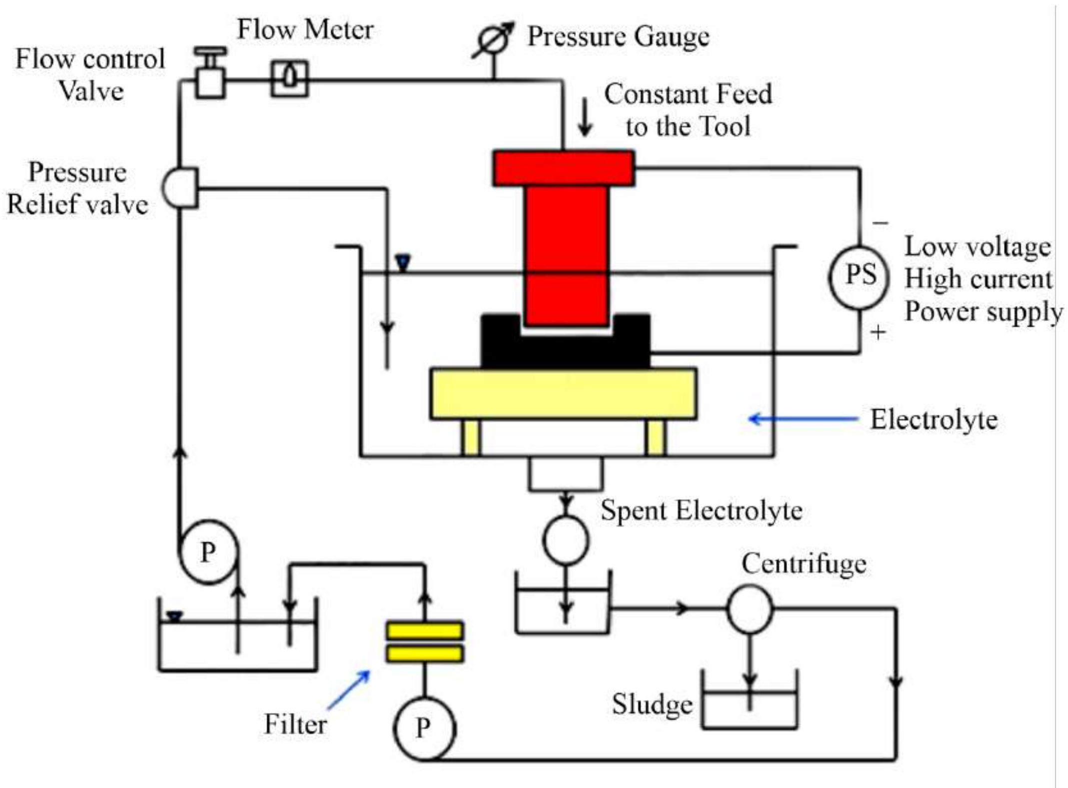


Fig. 2.2 ECM

ECM Formula

$$E = \frac{\text{Atomic Weight}}{\text{Valancty}}$$

Faraday's laws state that,

$$m = \frac{\text{It E}}{\text{F}}$$

Where m = weight (gm) of a material

l = current (A)

t = time (sec)

E = gram equivalent weight of the material

F = constant of proportionality – Faraday (96,500 coulombs)

ECM Calculations

- MRR for pure metal

$$\frac{AI}{\rho vF}\left(\frac{cm^3}{sec}\right)=\frac{EI}{\rho F}\left(\frac{cm^3}{sec}\right)$$

where,
A = Atomic weight
v = Valency

- MRR for Alloy

$$\frac{E_{eq}I}{\rho_{eq}F}\left(\frac{cm^3}{sec}\right)$$

$$\frac{100}{\rho_{eq}F}=\sum_i\left(\frac{x_i}{\rho_i}\right) \quad \text{and} \quad \frac{100}{E_{eq}}=\sum_i\left(\frac{x_i v_i}{A_i}\right)$$

- If the total over voltage at the anode and the cathode is ΔV and the applied voltage is V, the current l is given by,

$$I=\frac{V-\Delta V}{R}$$

$$JS=\frac{(V-\Delta V)}{Y}$$

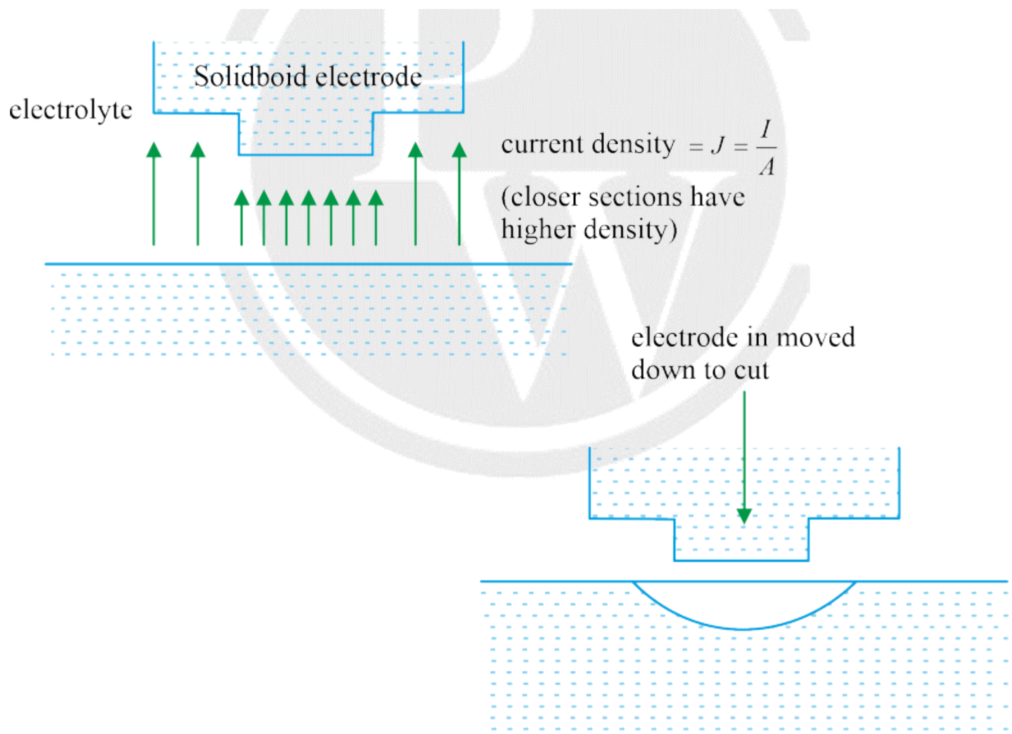


Fig. 2.3 ECM Principal

J = current density $=\frac{I}{A}$

S = specific resistance of electrolyte

I = applied of current

A = Cross sectional area of electrode

Flow analysis

- To calculate the fluid flow, required, match the heat generated to the heat absorbed by the electrolyte.
  The heat generated in the gap, H is given by
  $$H = I^2 \times R$$
  where, I = Current
  R = Resistance of electrolyte in the gap
  Heat absorbed by the electrolyte, H_e is
  $$H_e = q\, \rho_e\, c_e\, (T_f - T_i)$$
- Electing all the heat losses
  $$I^2 R = q\, \rho_e\, c_e\, (T_f - T_i)$$
  where, q= Flow rate of electrolyte
  ρ_e = Density of electrolyte
  c_e= Specific heat of electrolyte
  T_i = Initial temperature
  T_f = Final temperature

2.2 Electrochemical Grinding (ECG)

- In ECG:
  The tool electrode is a rotating, metal bonded, diamond grit grinding wheel.
- As the electric current flows between the workpiece and the wheel, through the electrolyte.
  a.The surface metal is changed to a metal oxide,
  b.Which is ground away by the abrasives.
- ECG is a low-voltage high-current electrical process.
- The abrasive particles are act as an insulating spacer.

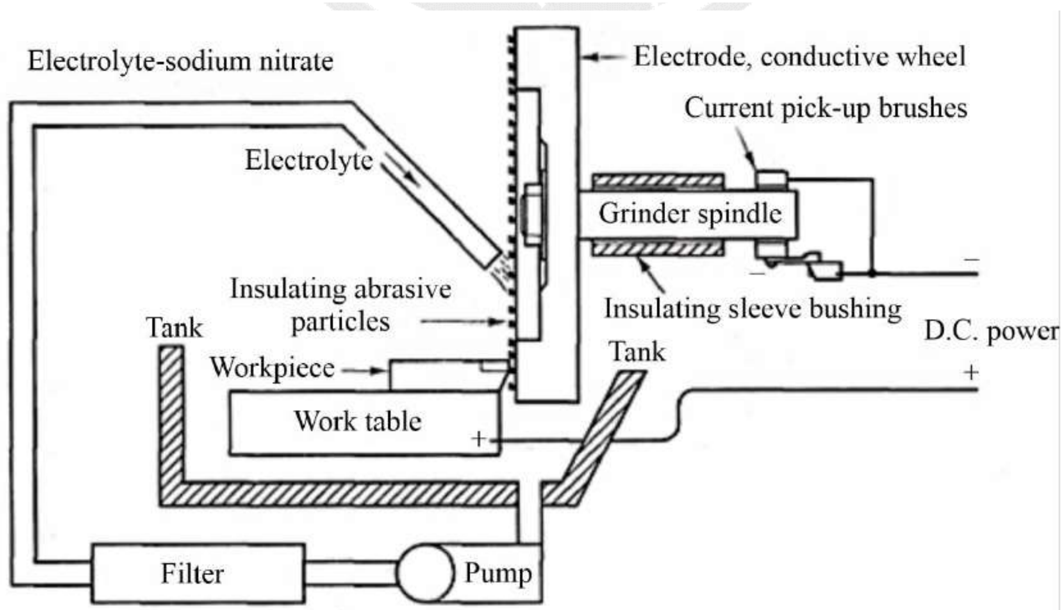


Fig. 2.4 Equipment setup and electrical circuit for electrochemical grinding.

2.3 Electro Discharge Machining (EDM)

Wear Ratio

Tool wear is given in terms of wear ratio which is defined as,

$$\textit{Wear ratio} = \frac{\text{Volume of metal removed work}}{\text{Volume of metal removed tool}}$$

Relaxation circuit

Fig-Relaxation circuit used for generating the pulses in EDM process

$$V_c = V_0\left(1-e^{\frac{1}{RC}}\right)$$

The time constant, τ of the circuit is given by

$$T = R_c \times C$$

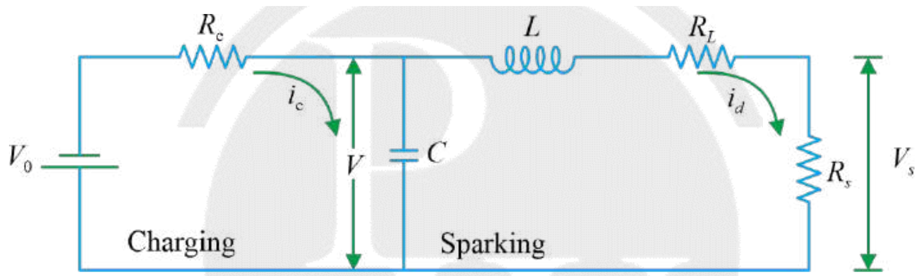


Fig. 2.5

Charging current can then be specified by

$$i_c = \frac{V_0}{R_c}e^{\frac{1}{\tau}}$$

For maximum power

$$V_c = 0.72 \times V_0$$

$$V_c = V_0\left\{1-e^{-\left(\frac{t}{RC}\right)}\right\}$$

V_0 = Open circuit voltage

R = Charging resistance

C = Capacitance

V_c = Instantaneous capacitor voltage during charging

t = voltage at any time

Spark energy

$$E_s = \frac{1}{2}C(V_c)^2 \text{ J / cycle}$$

2.4 Ultrasonic Machining (USM)

Tool of desired shape vibrates at an ultrasonic frequency (19 ~ 25 kHz) with an amplitude of around 15 – 50 μm The tool is pressed downward with a feed force, F.

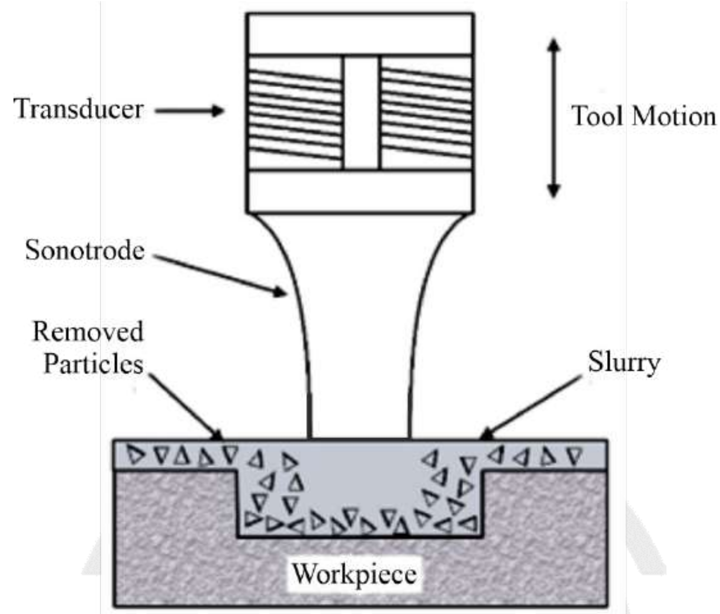


Fig. 2.6 Ultrasonic Machining

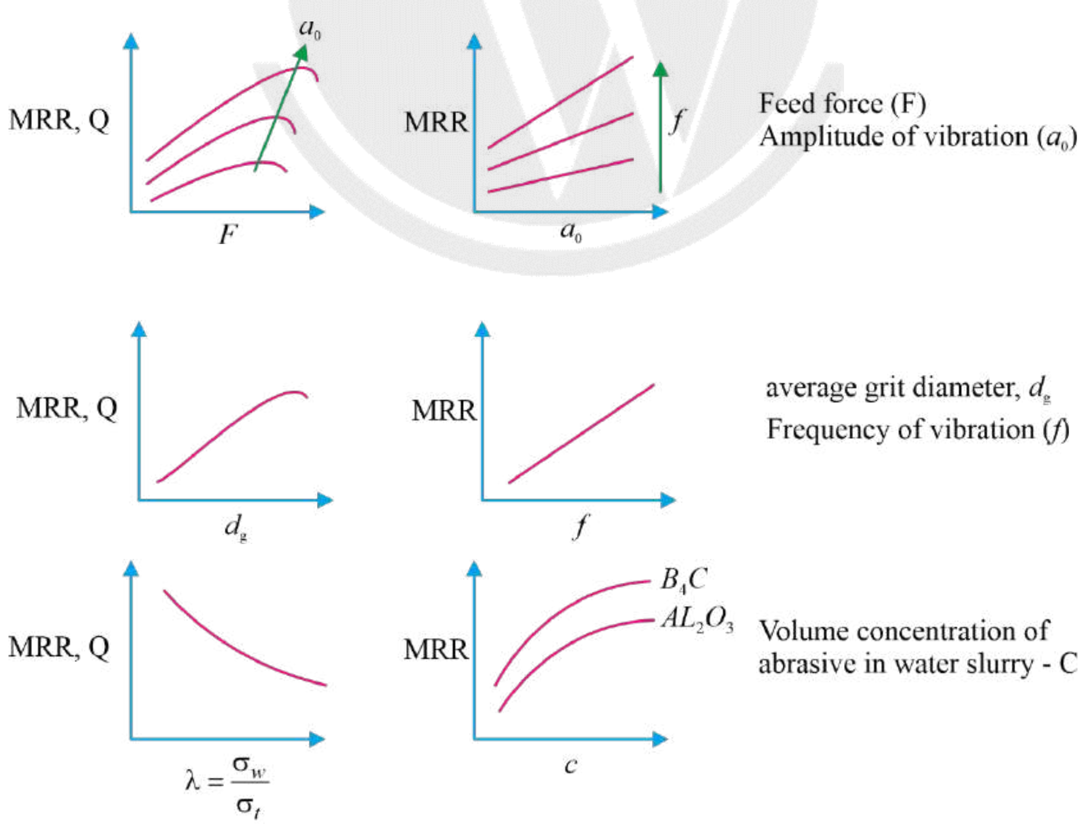


Fig. 2.7 Variation of MRR w.r.t. different Parameters

2.5 Water Jet Machining (WJM)

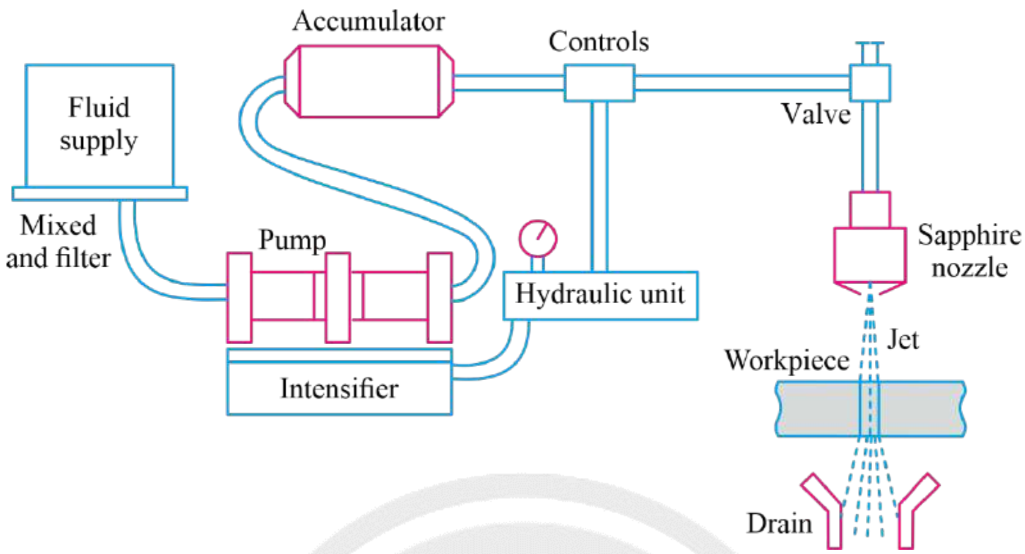


Fig. 2.8 Water Jet Machining

Narrow jet of water directed, at high pressure and velocity, against surface of workpiece

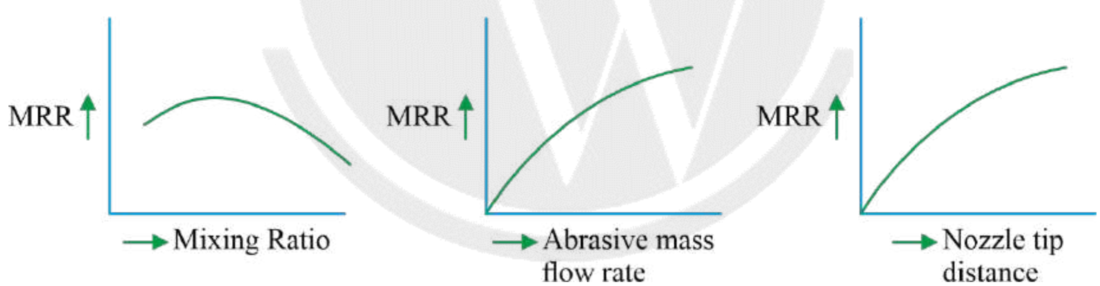


Fig. 2.9 Variation of MRR w.r.t. different Parameters

2.6 Abrasive Jet Machining (AJM)

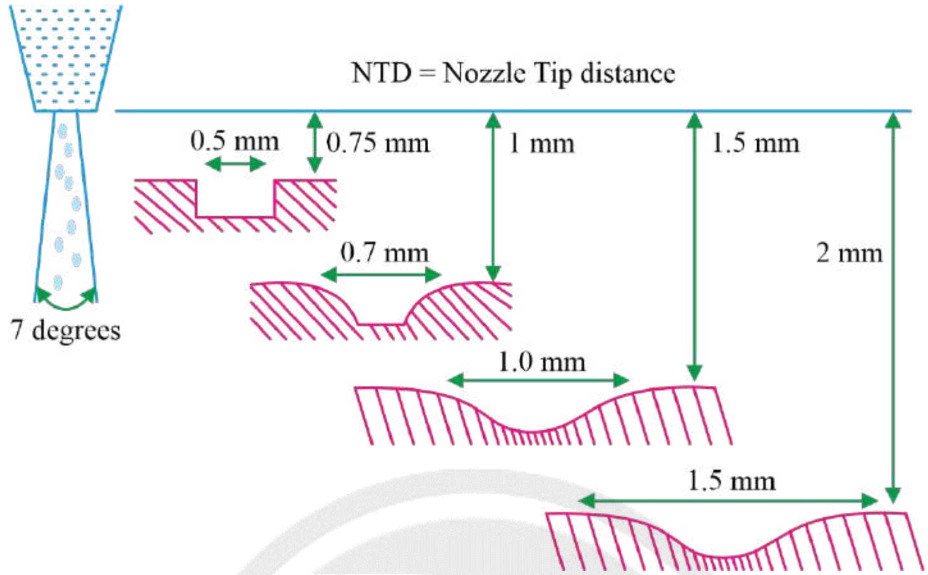


Fig. 2.10 Abrasive jet machining

MRR $\propto QD^3$

Q = flow rate abrasives

D = mean diameter of abrasives

❑❑❑

3 CASTING

3.1 Casting

Process in which molten metal flows by gravity or other force into a mold where it solidifies in the shape of the mold cavity

3.2 Steps in Sand Casting

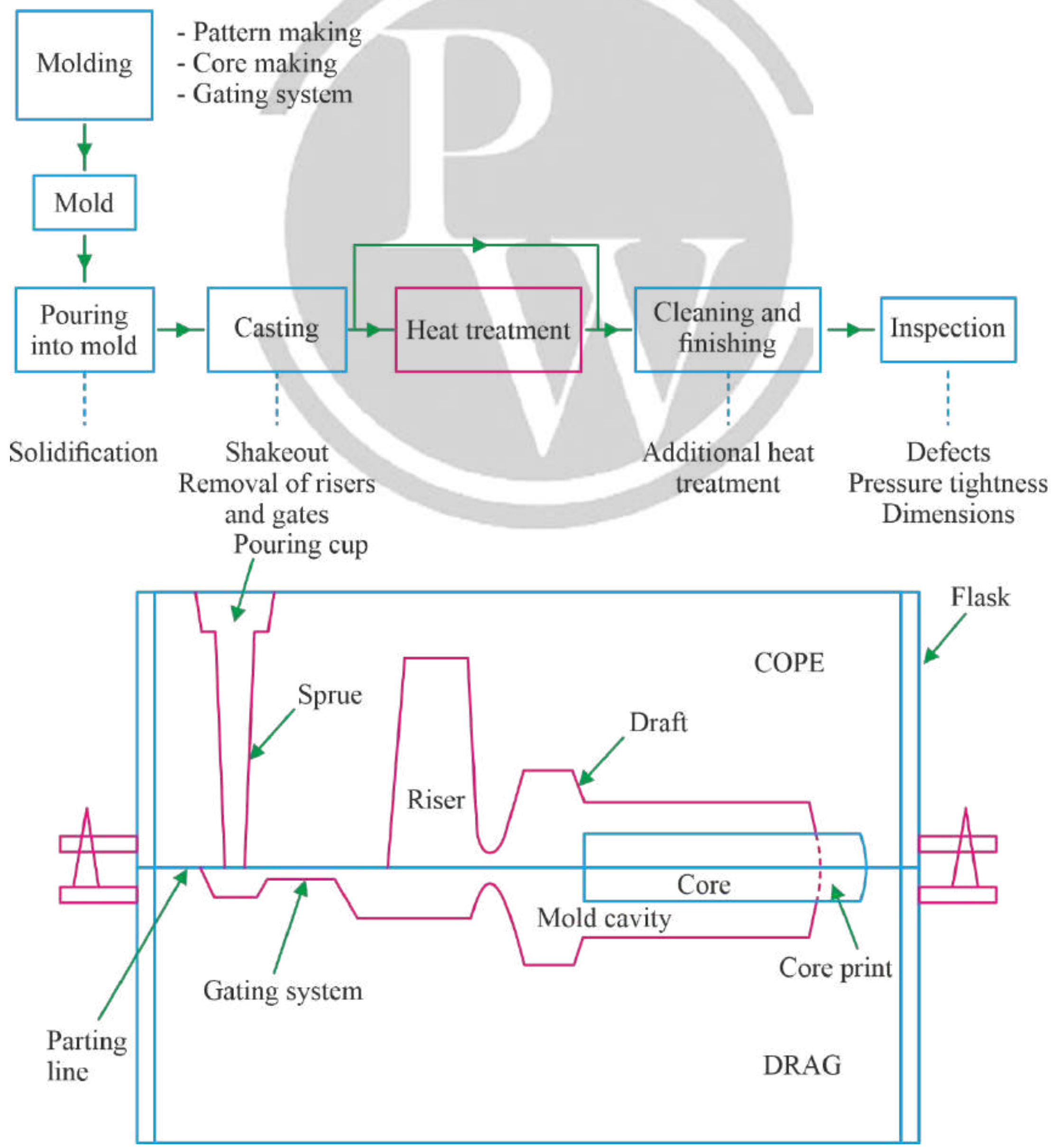


Fig. 3.1 Sand Casting

3.3 Casting Terms

Flask: Flasks have box-like structure made of rectangular walls (sometimes circular also) and without any bottom or top

Cover: These are mostly made of cast iron although wood is also used sometimes

Drag: Lower moulding flask.

Cope: Upper moulding flask.

Cheek: Intermediate moulding flask used in three-piece moulding.

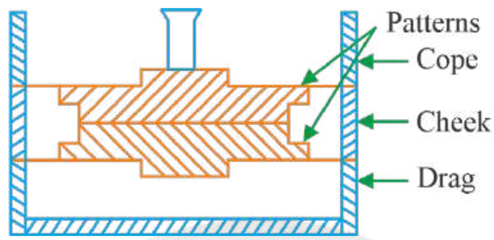


Fig. 3.2 Mould Box

Pattern: Pattern is a replica of the final object to be made with some modifications.

Parting line: This is the dividing line between the two moulding flasks that makes up the sand mould.

Bottom board: This is a board normally made of wood, which is used at the start of the mould making.

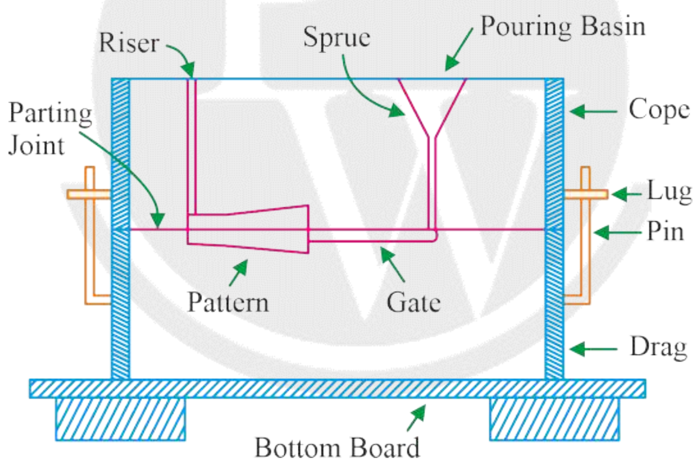


Fig. 3.3 Casting Terms

Moulding Sand: The freshly prepared refractory material used for making the mould cavity. Typical mix: 90% sand, 3% water, and 7% clay

Backing Sand: This is made up of used and burnt sand.

Core: Used for making hollow cavities in castings.

Pouring basin: A small funnel-shaped cavity at the top of the mould into which the molten metal is poured.

Sprue: The passage through which the molten metal from the pouring basin reaches the mould cavity.

Runner: The passage ways in the parting plane through which molten metal flow is regulated before they reach the mould cavity.

Gate: The actual entry point through which molten metal enters the mould cavity in a controlled rate

Chaplet: Chaplets are used to support cores inside the mould cavity.

Chill: Chills are metallic objects, which are placed in the mould to increase the cooling rate of castings.

Riser: It is a reservoir of molten metal provided in the casting so that hot metal can flow back into the mould cavity when there is a reduction in volume of metal due to solidification.

3.4 Pattern

A pattern is a replica of the object to be made by the casting process, with some modifications.

The main modifications are

3.4.1 Pattern Allowances

(1) Shrinkage or contraction allowance
(2) Draft or taper allowance
(3) Machining or finish allowance
(4) Distortion or camber allowance
(5) Rapping allowance/shaken allowance / Negative allowance

3.4.2 Shrinkage allowance

Invar and Bismuth $\rightarrow$ shrinkage is Negligible.

This is because of the inter-atomic vibrations which are amplified by an increase in temperature.

3.4.3 Liquid shrinkage and solid shrinkage

Liquid shrinkage refers to the reduction in volume when the metal changes from liquid to solid state at the solidus temperature. To account for this, risers are provided in the moulds.

Solid shrinkage is the reduction in volume caused, when a metal loses temperature in the solid state. The shrinkage allowance is provided to take care of this reduction.

3.4.4 Pattern Materials

Wood: patterns are relatively **easy to make**. Wood is not very dimensionally stable. Commonly used teak, white pine and mahogany wood.

Metal: patterns are more expensive but are more dimensionally stable and more durable. Commonly used CI, Brass, aluminium and white metal.

Investment casting uses wax patterns.

3.5 Types of Pattern

- **Single Piece Pattern:** These are inexpensive and the simplest type of patterns.
- **Gated Pattern: Gating and runner system are integral with the pattern**. This would eliminate the hand cutting of the runners and gates and help in improving the productivity of a moulding.
- **Split Pattern or Two-Piece Pattern**

  Pattern for intricate castings.

  When the contour of the casting makes its withdrawal from the mould difficult,
- **Cope and Drag Pattern**

  In addition to splitting the pattern, the cope and drag halves of the pattern along with the gating and riser systems are attached separately to the metal or wooden plates along with the alignment pins.
- **Match Plate Pattern**

  The cope and drag patterns along with the gating and the rise ring are mounted on a single matching metal or wooden plate on either side.
- **Loose Piece Pattern**

  This type of pattern is also used when the contour of the part is such that withdrawing the pattern from the mould is not possible.
- **Follow Board Pattern**

  This type of pattern is adopted for those castings where there are some portions, which are structurally weak and if not supported properly are likely to break under the force of ramming.
- **Sweep Pattern**

  These are used for generating large shapes, which are axi-symmetrical or prismatic in nature such as bell-shaped or cylindrical.
- **Skeleton Pattern**

  A skeleton of the pattern made of strips of wood is used for building the final pattern by packing sand around the skeleton

3.6 Cooling curve

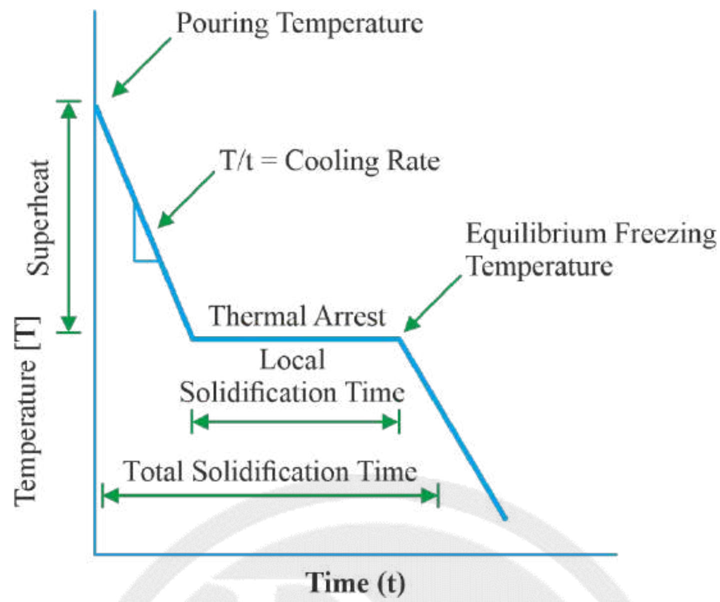


Fig. 3.4 Cooling Curve

3.7 Core

- Used for making cavities and hollow projections.
- Core sand should be of higher strength than the moulding sand.
- **Used clay free silica sand.**
- Binders used are **linseed oil**, core oil, resins, dextrin, molasses, etc.
- The general composition of a core sand mixture could be core oil (1%) and water (2.5 to 6%)

Net Buoyancy Force = Weight of Liquid Metal Displaced – Weight of Core

$$P = Vg\rho_m - Vg\rho_c$$

$$P = Vg(\rho_m - \rho_c)$$

V = Volume of core

ρ_m = Density of molten liquid metal

ρ_c = Density of core material

3.8 Permeability

Gases evolving from the molten metal and generated from the mould may have to go through the core to escape out of the mould. Hence cores are required to have higher permeability.

$$R = \frac{VH}{pAT}$$

R = Permeability Number

V = volume of air = 2000 cm^3

H = height of the sand specimen = 5.08 cm

p = air pressure, g/cm^2 = 10 g/cm^2 (standard)

A = cross sectional area of sand specimen = 20.268 cm^2

T = time in minutes for the complete air to pass through

$$R = \frac{501.28}{p.T}$$

3.9 Grain size number

ASTM (American Society for Testing and Materials) grain size number, defined as

$N = 2^{n-1}$

Low ASTM numbers mean a few massive grains; high numbers refer to many small grains.

3.10 Gate (Ingate) Design

3.10.1 Top Gate

The velocity is more than time taking to fill the mould is minimum that's why the temperature gradient is minimum.

$$t_{top} = \frac{AH}{A_g\sqrt{2gh_t}} \qquad [h_t = h_s + h_c]$$

h_c = Cup height

h_s = Sprue height

h_t = Manometric height

t_{top} = filling time in top gate

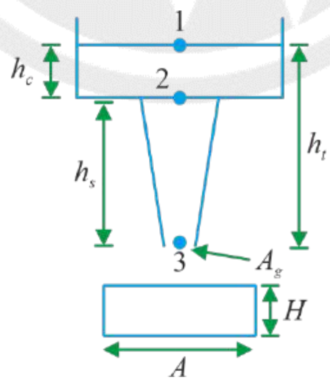


Fig. 3.5 Top Gate

3.10.2 Bottom Gate

$$t_{bottom} = \frac{2A}{A_g}\frac{1}{\sqrt{2g}}\left(\sqrt{h_t} - \sqrt{h_t - H}\right)$$

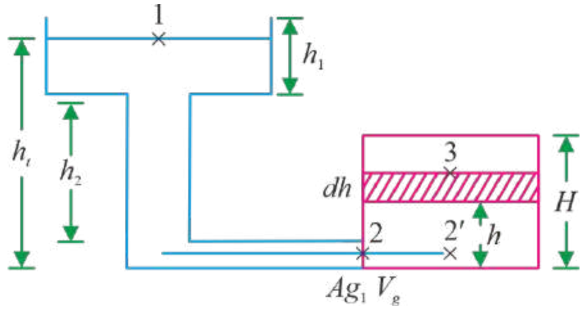


Fig. 3.6 Bottom Gate

Important result

If h_t = H in bottom Gate

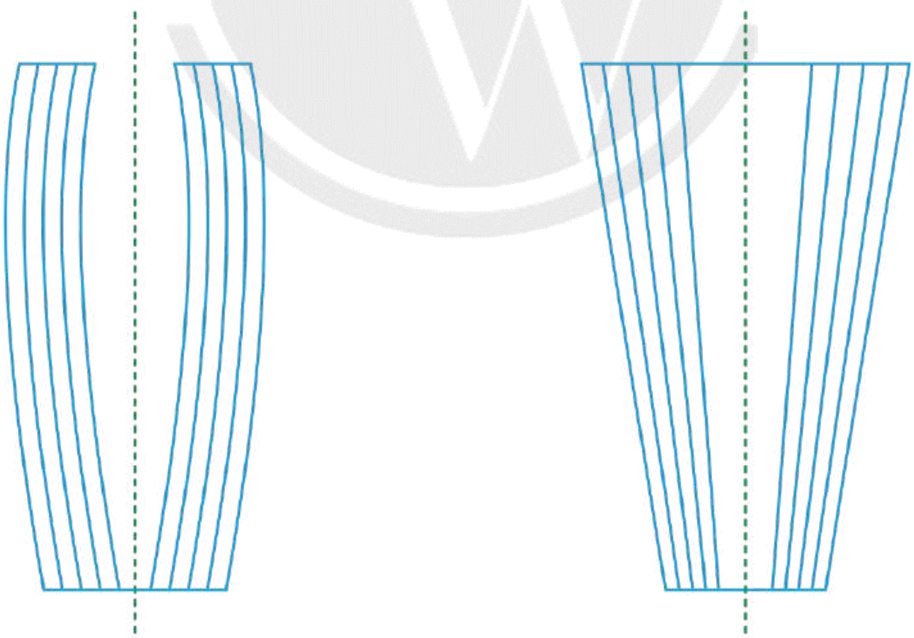
$t_{\text{bottom}} = 2t_{\text{top}}$

3.10.3 Gating ratio

- Gating ratio is defined as: Sprue area: Runner area: Ingate area.

3.10.4 Sprue Design

- **Sprue**: Sprue is the channel through which the molten metal is brought into the parting plane where it enters the runners and gates to ultimately reach the mould cavity.
- To eliminate this problem of air aspiration, the sprue is tapered to gradually reduce the cross section as it moves away from the top of the cope as shown in Figure below(b).



(a) Straight sprue (b) Tapered

Fig. 3.7 Design of Sprue

3.11 Risers and Riser Design

Risers are added reservoirs designed to feed liquid metal to the solidifying casting as a means of compensating for solidification shrinkage.

To perform this function, the risers must solidify after the casting.

According to Chvorinov's rule, a good shape for a riser would be one that has a long freezing time (i.e., a small surface area per unit volume).

Chvorinov's rule

- Total solidification time (t_s) = $B\left(\frac{V}{A}\right)^n$

Where n = 1.5 to 2.0

[Where, B = mould constant and is a function of mould material, casting material, and condition of casting]

n = 2 and $t_{riser} = 1.25\ t_{casting}$.

OR

$$\left(\frac{V}{A}\right)^2_{riser} = 1.25\left(\frac{V}{A}\right)^2_{casting}$$

3.11.1 Important Result

- Compare the solidification times for castings of three different shapes of same volume:

  (i) Cubic (T_{cu})

  (ii) Cylindrical (with height equal to its diameter) (T_{cy})

  (iii) Spherical (T_{sp})

  $\boldsymbol{T_{sp} : T_{cy} : T_{cu} = 1 : 0.763 : 0.649}$

3.11.2 Method of Riser Design

(a) Modulus Method

- It has been empirically established that if the modulus of the riser exceeds the modulus of the casting by a factor of 1.2, the feeding during solidification would be satisfactory.

  MR = 1.2 Mc

- Modulus = Volume/Surface area

(b) Modulus of casting shape

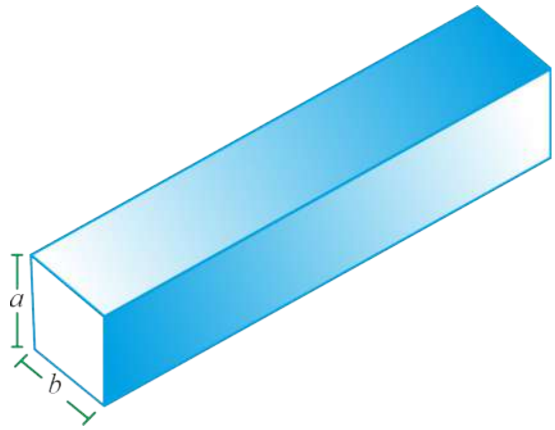
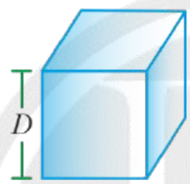
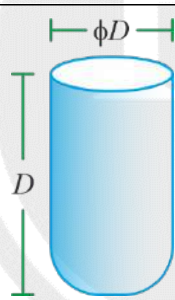
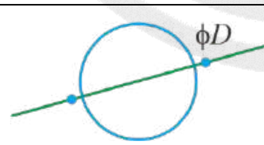
Shape	Figure	Modulus of Casting
Long bar		$\dfrac{ab}{2(a+b)}$
Cube		$\dfrac{D}{6}$
Cylinder		$\dfrac{D}{6}$
Sphere		$\dfrac{D}{6}$

Fig. 3.8 Modulus of different casting shapes

(c) Caine's Method

Freezing ratio = ratio of cooling characteristics of casting to the riser.

$$X = \frac{\left(A/V\right)_{Casting}}{\left(A/V\right)_{Riser}}$$

The riser should solidify last so $x > 1$

According to Caine $\quad x = \dfrac{a}{Y-b} + c$

$$Y = \frac{V_{\text{riser}}}{V_{\text{casting}}} \text{ and } a, b, c \text{ are constant.}$$

(d) Naval research laboratory method (shape factor)

This method, which is essentially a simplification of Caine's method, defines a shape factor to replace the freezing ratio. The shape factor is defined as,

$$SF = \frac{\text{Length} + \text{width}}{\text{thickness}}$$

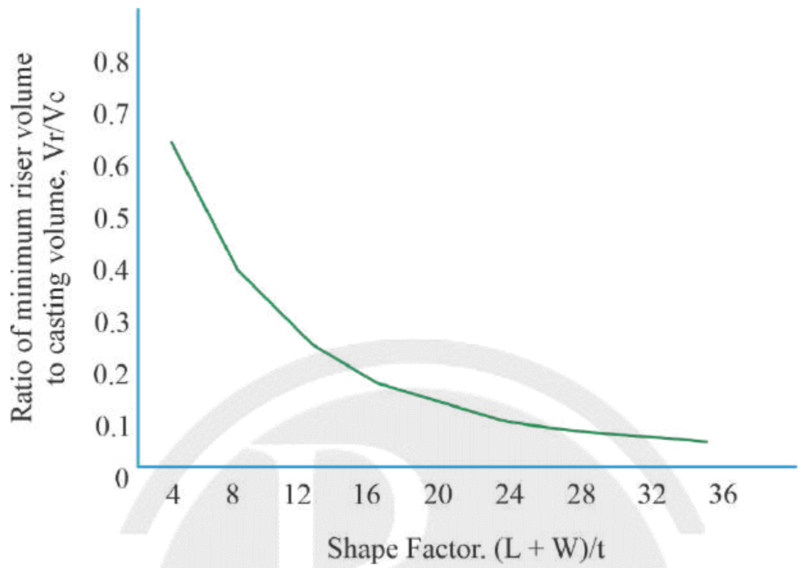


Fig. 3.9 Shape Factor

3.12 Chills

- External chills are masses of high-heat-capacity, high-thermal-conductivity **material that are placed in the mould (adjacent to the casting)** to accelerate the cooling of various regions.

  Chills can effectively promote directional solidification or increase the effective feeding distance of a riser.

- **Internal chills** are **pieces of metal** that are placed within the mould cavity to absorb heat and promote more rapid solidification. Since some of this metal will melt during the operation, it will absorb not only the heat-capacity energy, but also some heat of fusion. Since they ultimately become part of the final casting, internal chills must be made from the same alloy as that being cast.

3.13 Casting Defects

The following are the major defects, which are likely to occur in sand castings:

3.13.1 Gas Defects

- A condition existing in a casting caused by the trapping of gas in the molten metal or by mold gases evolved during the pouring of the casting.
- The defects in this category can be classified into **blowholes** and **pinhole porosity**.

3.13.2 Shrinkage Cavities

- These are caused by liquid shrinkage occurring during the solidification of the casting.
- To compensate for this, proper feeding of liquid metal is required. For this reason, risers are placed at the appropriate places in the mold.

3.13.3 Molding material defects

(i) Cut and washes

- These appear as rough spots and areas of excess metal, and are caused by erosion of molding sand by the flowing metal.
- This is caused by the: (a) Molding sand not having enough strength and (b) The molten metal flowing at high velocity.

(ii) Scab

- This defect occurs when a portion of the face of a mould lifts or breaks down and the recess thus made is filled by metal.
- When the metal is poured into the cavity, gas may be disengaged with such violence as to break up the sand, which is then washed away and the resulting cavity filled with metal.

(iii) Metal Penetration

- When molten metal enters into the gaps between sand grains, the result is a rough casting surface.

(iv) Fusion

- This is caused by the fusion of the sand grains with the molten metal, giving a brittle, glassy appearance on the casting surface.

(v) Swell

- Under the influence of met allostatic forces, the mold wall may move back causing a swell in the dimension of the casting.

(vi) Inclusions

- Particles of slag, refractory materials sand or deoxidation products are trapped in the casting during pouring solidification.

3.13.4 Pouring metal defects

(i) Mis-run

- A mis-run is caused when the metal is unable to fill the mold cavity completely and thus leaves unfilled cavities.

(ii) Cold shut

- A cold shut is caused when two streams while meeting in the mold cavity, do not fuse together properly thus forming a discontinuity in the casting.

3.13.5 Mold Shift

- The mold shift defect occurs when cope and drag or molding boxes have not been properly aligned.

3.14 Special Casting

3.14.1 Shell Casting

- The sand is mixed with a thermosetting resin is allowed to come in contact with a heated metal pattern (2000C).
- A skin (shell) of about 3.5 mm of sand and plastic mixture adhere to the pattern.
- Then the shell is removed from the pattern.
- The cope and drag shells are kept in a flask with necessary backup material and the molten metal is poured into the mold.

Advantages

- Dimensional accuracy.
- Smoother surface finish. (Due to finer size grain used)
- Very thin sections can be cast.
- Very small amount of sand is needed.
- Process is EASY

Limitations

- Expensive pattern
- Small size casting only.
- Highly complicated shapes cannot be obtained.
- More sophisticated equipment is needed for handling the shell moldings.

3.14.2 Investment Casting

Investment casting refers to the ceramics formed around the wax patterns to create a casing for molten metal to be poured. Once the wax patterns are created, they are melted onto a gate system, dipped into slurry and sand to form a layered casing, then replaced with the melted metals such as stainless steel, aluminum, and much more

Advantages

- Exceptional surface polish
- High dimensional precision
- Even the most complicated elements can be cast.
- Casting is possible with almost any metal.

Limitations

- Costly patterns and moulds
- Labour costs can be high
- Limited size

3.14.3 Hot Chamber Die Casting

- Die casting alloy is melted in a furnace located within the equipment
- Casting cycles are significantly shorter, thus has a higher production capacity
- Suitable for low melting point alloys
- Offers longer tool life
- Requires minimum safety measures
- Commonly used metal alloys include Zinc, lead and etc.

3.14.4 Cold Chamber Die Casting

- Dies casting alloy is melted in a separate furnace located outside the equipment.
- Has longer casting cycles; thus, the product capacity is less
- Suitable for high melting point alloys
- Has shorter tool life
- Requires more safety measures
- Commonly used metal alloys include Aluminum, Copper, Brass Magnesium, etc.

3.14.5 Centrifugal Casting

(i) True centrifugal casting

- **Process:** Molten metal is introduced into a rotating sand, metal, or graphite mould, and held against the mould wall by centrifugal force until it is solidified
- A mold is set up and rotated along a vertical (rpm is reasonable), or horizontal (200-1000 rpm is reasonable) axis.
- The mold is coated with a refractory coating.
- During cooling lower density impurities will tend to rise towards the center of rotation.
- ***Important result*:**
  - Mechanical properties of centrifugally cast jobs are better compared to other processes
  - **No cores** are required for making concentric hole

(ii) Semi-centrifugal Casting

- Centrifugal force assists the flow of metal from a central reservoir to the extremities of a rotating symmetrical mold, which may be either expendable or multiple-use Rotational speeds are lower than for true centrifugal casting. Cores can be used to increase the complexity of the product.

(iii) Centrifuging

- Uses centrifuging action to force the metal from a central pouring reservoir into separate mold cavities that are offset from the axis of rotation.

3.14.6 Slush Casting

- Slush casting is a variation of the permanent mold process in which the metal is permitted to remain in the mold only until a shell of the desired thickness has formed. The mold is then inverted and the remaining liquid is poured out.

3.14.7 Squeeze Casting

Molten metal is poured into an open face die. A punch is advanced into the die, and to the metal. Pressure (less than forging) is applied to the punch and die while the part solidifies. The punch is retracted, and the part is knocked out with an ejector pin.

3.14.8 Plaster Casting

A slurry of plaster, water, and various additives is additives is pouted over a pattern and allowed to set. The pattern is removed and the mould is baked to remove excess water. After pouring and solidification, the mould is broken and the casting is removed.

3.14.9 Loam Moulding

Moulding loam is generally artificially composed of common brick-clay, and sharp sand. Loam means mud. Loam Moulding is restricted to forms which cannot be cast conveniently in any other process.

3.15 Type of Furnace

3.15.1 Cupola

- Cupola has been the most widely used furnace for melting cast iron. In hot blast cupola, the flue gases are used to preheat the air blast to the cupola so that the temperature in the furnace is considerably higher than that in a conventional cupola. Coke is fuel and Lime stone ($CaCO_3$) is mostly used flux.

3.15.2 Electric Arc Furnace

- For heavy steel castings, the open-hearth type of furnaces with electric arc or oil fired would be generally suitable in view of the large heat required for melting. Electric arc furnaces are more suitable for ferrous materials and are larger in capacity.

3.15.3 Crucible Furnace

- Smaller foundries generally prefer the crucible furnace.
- The crucible is generally heated by **electric resistance or gas flame**.

3.15.4 Induction Furnace

- The induction furnaces are used for all types of materials, the chief advantage being that the heat source is isolated from the charge and the slag and flux get the necessary heat directly from the charge instead of the heat source.

3.16 Casting Cleaning (Fettling)

Impurities in the molten metal are prevented from reaching the mould cavity by providing a

(i) Strainer
(ii) Bottom well
(iii) Skim bob

❑❑❑

4 WELDING

4.1 Welding

Welding is a process by which two materials, usually metals, are permanently joined together by coalescence, which is induced by a combination of temperature, pressure, and metallurgical conditions.

4.1.1 Types of Joints

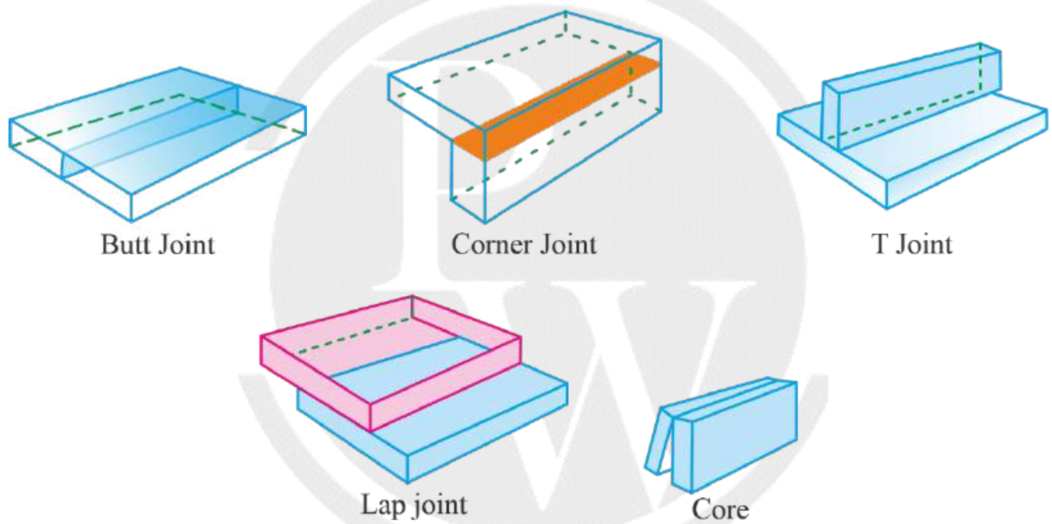


Fig. 4.1 Types of Joints

4.2 Electric Arc Welding

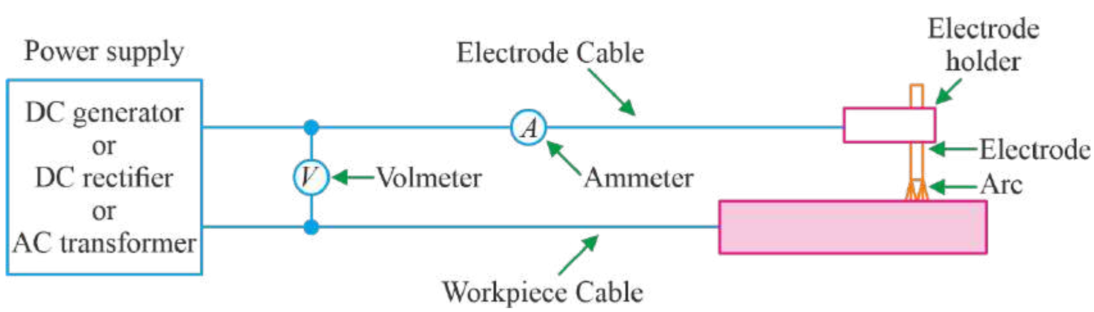


Fig. 4.2 Electric Arc Welding

An arc is generated between cathode and anode when they are touched to establish the flow of current and then separated by a small distance. 65% to 75% heat is generated at the anode.

If DC is used and the work is positive (the anode of the circuit), the condition is known as straight polarity (SPDC). Work is negative and electrode is positive is reverse polarity (RPDC).

Note:
RPDC arc-welding maintain a stable arc and preferred for difficult tasks such as overhead welding

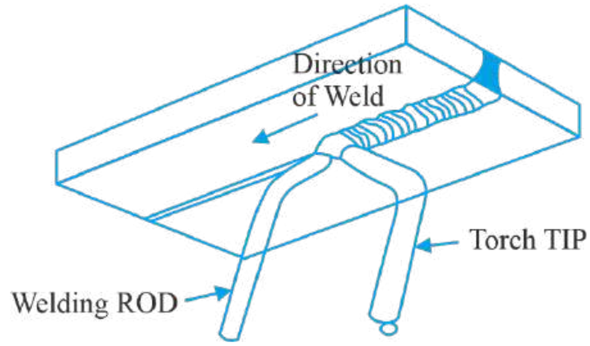


Fig. 4.3 Overhead Welding

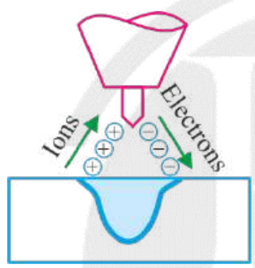
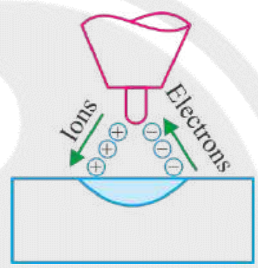
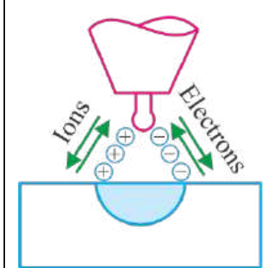
Electrode Polarity	Negative	Positive	AC
Electron and Ion Flow Penetration Characteristics			
Oxide Cleaning Action	No	Yes	Yes-Once Every Half Cycle
Heat Balance In The Arc (Approx.)	70% At Work End 30% At Electrode End	30% At Work End 70% At Electrode End	50% At Work End 50% At Electrode End
Penetration	Deep; Narrow	Shallow; Wide	Medium

Table 4.1

There are three modes of metal transfer (globular, spray and short-circuit).

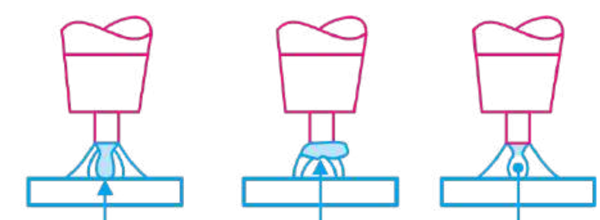


Fig. 4.4 Metal Transfer

Bead is the metal added during single pass of welding. Bead material is same as base metal.

In d.c. welding, the straight polarity (electrode negative) results in Lower deposition rate

4.3 Arc welding equipment's

4.3.1 Droppers: Constant current welding machines

Good for manual welding

$I_{arc} = I_{trnf}$

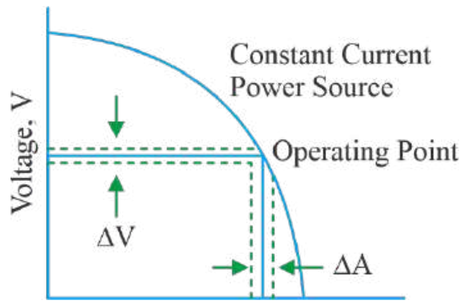


Fig. 4.5

4.3.2 Constant voltage machines

Good for automatic welding

$V_{arc} = V_{trnf}$

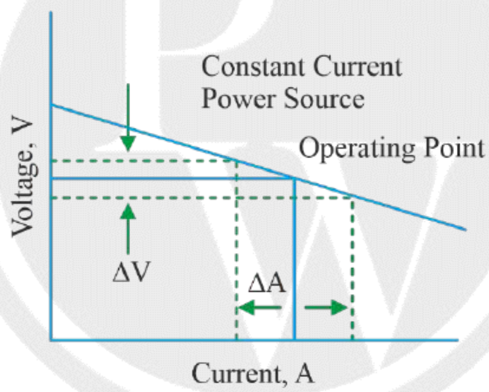


Fig. 4.6

4.4 Constant voltage machines Formula

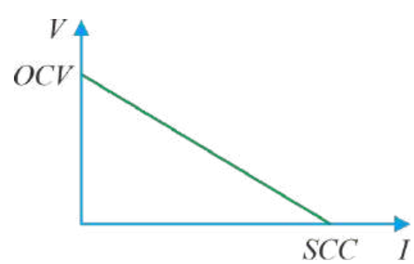


Fig. 4.7

OCV = open circuit voltage

SCC = short circuit current

$$\frac{V}{OCV} + \frac{I}{SCC} = 1$$

OCV: It is the maximum rated voltage between open terminals under no loading conditions.

SCC: It is the maximum rated current that is required during arc generation.

Note:
In arc welding, the arc length should be equal to Rod diameter

In manual arc welding, the equipment should have drooping characteristics in order to maintain Current constant when arc length changes

In arc welding, d.c. reverse polarity is used to bear greater advantage in Overhead welding.

For maximum power applied current(I) and voltage (V) are:

$$I = \frac{SCC}{2}$$
$$V = \frac{OCV}{2}$$

4.4.1 Heat Input (H_{in})

$$\text{Heat input} = \frac{VI}{A_b v} \text{(Joule/mm}^3\text{)}$$

A_b = weld bead area (mm^2)

v = velocity mm/sec

$$\eta_t = \frac{H_m}{H_{in}}$$

Where

H_m = Heat required for melting

H_{in} = Heat input

η_t = Melting efficiency

4.4.2 Duty Cycle

Duty cycle is a welding equipment specification which defines the number of minutes, within a 10 minutes period, during which a given welder can safely produce a particular welding current.

$$\text{Required duty cycle } T_a = \left(\frac{I}{I_a}\right)^2 T$$

Where,

T = rated duty cycle

I = rated current at the rated duty cycle

I_a = Maximum current at the rated duty cycle

Note:
For manual welding a 60% duty cycle is suggested and for automatic welding 100% duty cycle.

4.5 Electrode coating characteristic

1. Provide a protective atmosphere.
2. Stabilize the arc.
3. Provide a protective slag coating to accumulate impurities, prevent oxidation, and slow the cooling of the weld metal.
4. Reduce spatter.
5. Add alloying elements.
6. Affect arc penetration

Note:

The electrodes used in arc welding are coated. This coating is not expected to Prevents electrode from contaminate

The coating material of an arc welding electrode contains which of the following?

1. Deoxidising agent
2. Arc stabilizing agent
3. Slag forming agent

Note:

- Arc Length must be short because
  1. Heat is concentrated.
  2. More stable
  3. More protective atmosphere.
- A long arc has following draw back
  1. Large heat loss into atmosphere.
  2. Unstable arc.
  3. Weld pool is not protected.
  4. Weld has low strength, less ductility, poor fusion and excessive spatter.

4.6 Arc blow in DC arc welding

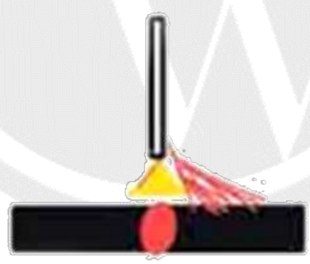


Fig. 4.8

Arc blow occurs during the welding of magnetic materials with DC. The effect is particularly noticeable when welding with bare electrodes or when using currents below or above. Again, the problem of arc blow gets magnified when welding highly magnetic materials

Disadvantage of arc blow:

1. Low heat penetration.
2. Excessive weld spatter.
3. Pinch effect in welding is the result of electromagnetic forces

Note:

Arc blow is more common in D.C. welding with bare electrodes

Pinch effect in welding is the result of Electromagnetic forces

4.7 Gas shields

An inert gas is blown into the weld zone to drive away other atmospheric gases. Gases are argon, helium, nitrogen, carbon dioxide and a mixture of the above gases.

4.8 Tungsten Inert Gas welding (TIG)

Arc is established between a non-consumable tungsten electrode and the workpiece. Arc length is constant, arc is stable and easy to maintain. With or without filler.

Note:
- Gas tungsten arc welding process used non – consumable electrode
- In an inert gas welding process, the commonly used gas is Helium or Argon.

4.9 Gas Metal Arc Welding (GMAW)/MIG

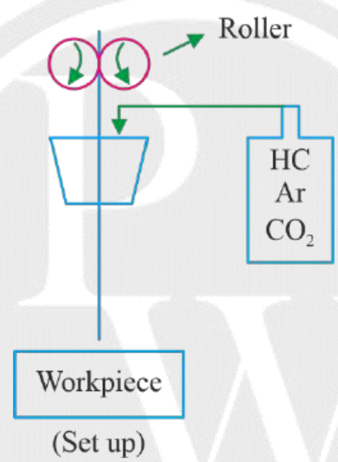


Fig. 4.9 MIG Welding

- Arc is generated between consumable electrode and workpiece. Electrode is in form of small ida (1-2.5mm) will and it will continually supply to workpiece through roller movement. (spray transfer)
- Liquid metal can be protected by inert gas

Note:
- In MIG welding, the metal is transferred into the fine spray of metal.
- MIG welding process uses Consumable electrode D.C. power supply.

4.10 Submerged Arc welding (SAW)

Joining of High thickness object in a single pass, this technique used. Arc is generated between consumable electrode & W/P through welding torch, solid form of **granular flux (CaO, CaF_2)** will be continuously supply. Arc will be submerged under solid flux

- There is No Heat transfer loss, No splashing & weld spatter.
- Thickness-10-50mm, I = 200- 2000A, speed = 5m/min
- High weld deformation rate and High welding speed.
- Limited to Horizontal position.
- Use: Pressure vessel, ship bridges, LPG cylinder etc.

Note:
High welding speeds and High deposition rates are the major characteristics of submerged arc welding.

4.11 Gas Flame processes:

- Oxyacetylene welding, commonly referred to as gas welding, is a process which relies on combustion of oxygen and acetylene. When mixed together in correct proportions within a hand-held torch or blowpipe, a relatively hot flame is produced.
- **Acetylene** is the principal fuel gas employed.

  Combustion of oxygen and acetylene (C_2H_2) in a welding torch produces a temp. in a two-stage reaction.

  In the first stage

  $$C_2H_2 + O_2 \rightarrow 2CO + H_2 + \text{Heat}$$

  In the second stage combustion of the CO and H_2 occurs just beyond the first combustion zone.

  $$2CO + O_2 \rightarrow 2CO_2 + \text{Heat}$$

  $$H_2 + O_2 \rightarrow H_2O + \text{Heat}$$

Note:
Oxygen for secondary reactions is obtained from the atmosphere.

Three types of flames can be obtained by varying the oxygen/acetylene ratio.

4.11.1 Neutral Flame

The ratio (O_2: C_2H_2) is about 1:1, all reactions are carried to completion and a neutral flame is produced. It is chemically neutral and neither oxidizes or carburizes the metal being welded.

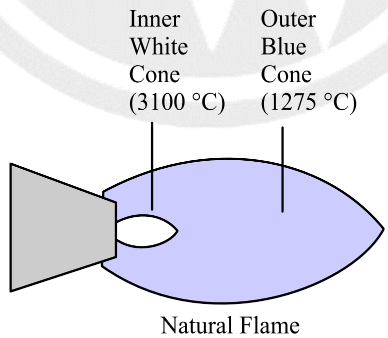
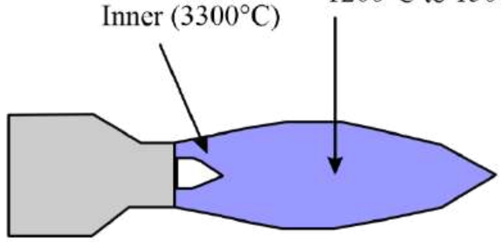


Fig. 4.10

4.11.2 Oxidizing flame

A higher ratio (O_2 > C_2H_2), such as 1.5:1, produces an oxidizing flame, hotter than the neutral flame (about 3300° C). Used when welding some nonferrous alloys such as copper-base alloys and zinc base alloys.

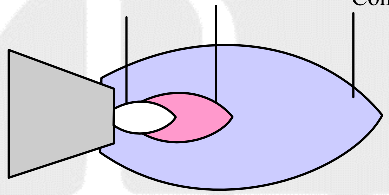


Oxidizing Flame

Fig. 4.11

4.11.3 Carburizing flame

Excess fuel, on the other hand, produces a carburizing flame. Carburizing flame can carburize metal also. The excess fuel decomposes to carbon and hydrogen, and the flame temperature is not as great (about 3000°C).



Carburizing Flame

Fig. 4.12

Note:

- OFW is fusion welding.
- No pressure is involved.
- **Fluxes may** be used
- Flux can be added as a powder, the welding rod can be dipped in a flux paste, or the rods can be pre-coated.
- The ratio between Oxygen and Acetylene gases for neutral flame in gas welding is 1:1.
- In Oxyacetylene gas welding, temperature at the inner cone of the flame is around 3200°C.

4.12 Oxygen Torch Cutting (Gas Cutting)

Iron and steel oxidize (burn) when heated to a temperature between 800°C to 1000°C. High-pressure oxygen jet (300 KPa) is directed against a heated steel plate, the oxygen jet burns the metal and blows it away causing the cut (kerf).

Fig. 4.13 Gas Welding

Note:

- Final microstructure depends on cooling rate.
- Steels with less than 0.3 % carbon cause no problem.
- Cutting CI is difficult, since its melting temp. is lower than iron oxide.

(a) Powder Cutting

- Difficult to cut metals by oxy-fuel cutting process are: Cast iron, stainless steel, and others high alloy steels. So, we can use powder cutting.
- By injecting a finely divided 200-mesh **iron powder** into the flame.

(b) Plasma Cutting

- Uses ionized gas jet (plasma) to cut materials resistant to oxy-fuel cutting, the ionized gas is forced through nozzle (up to 500 m/s), and the jet heats the metal, and blasts the molten metal away.
- HAZ is $\frac{1}{3}^{\text{rd}}$ to $\frac{1}{4}^{\text{th}}$ than oxyfuel cutting.
- Maximum plate thickness = 200 mm

4.13 Resistance Welding

4.13.1 Spot Welding

Resistance welding is the joining of metals by applying pressure and passing current for a length of time through the metal area which is to be joined. The key advantage of resistance welding is that no other materials are needed to create the bond, which makes this process extremely cost effective.

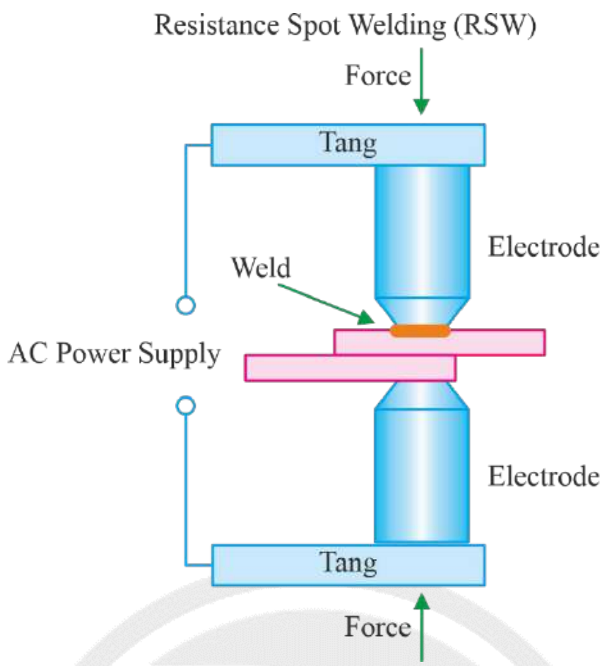


Fig. 4.14 Spot Resistance Welding

Note:

- Overall resistance is very low between the overlapping plate.
- Very high-current (up to 100,000 A) and Very low-voltage (0.5 to 10 V) is used.
- The maximum heat in resistance welding is at the Interface between the two plates being Joined.

4.13.2 Resistance seam welding

Resistance Seam Welding is a subset of Resistance Spot Welding using wheel-shaped electrodes to deliver force and welding current to the parts. The difference is that the workpiece rolls between the wheel-shaped electrodes while weld current is applied.

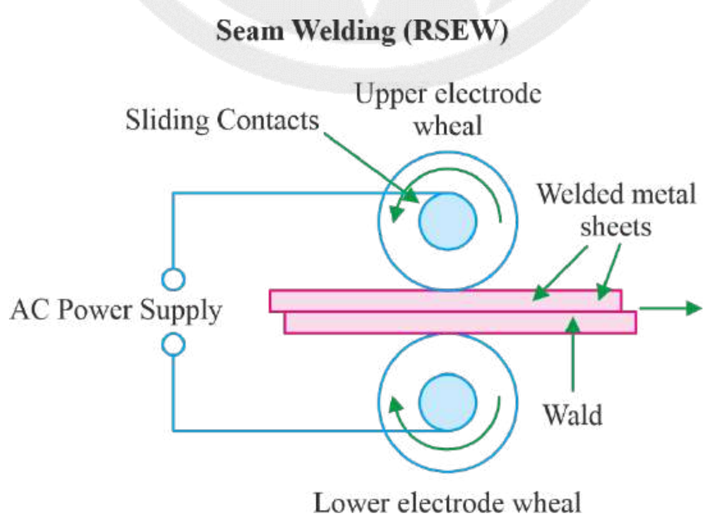


Fig. 4.15 Seam Resistance Welding

Note:
In resistance seam welding, the electrode is in the form of a circular disc.

4.13.3 Projection welding

Like spot welding, the projection welding process relies on heat generated by an electric current to join metal pieces together. Projection electrodes are capable of carrying more current than spot welding electrodes and can, therefore, weld much thicker materials.

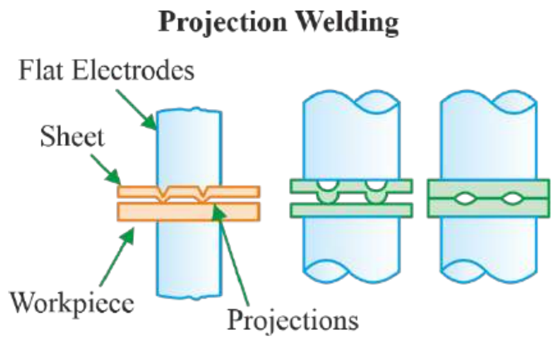


Fig. 4.16 Projection Resistance Welding

4.13.4 Upset welding

Upset welding or resistance butt welding is a welding technique that produces coalescence simultaneously over the entire area of abutting surfaces or progressively along a joint, by the heat obtained from resistance to electric current through the area where those surfaces are in contact.

4.13.5 Flash Welding

It is similar to upset welding except the arc rather than resistance heating.

4.13.6 Percussion Welding

- Similar to flash welding **except arc** power by a rapid discharge of stored electrical energy.
- The arc duration is only **1 to 10 ms,** heat is intense and highly concentrated.

4.14 Thermit Welding

Thermit welding (TW) is a process that uses heat from an **exothermic reaction** to produce **coalescence** between metals. The name is derived from 'thermite' the generic name given to reactions between metal oxides and reducing agents. The thermite mixture consists of **metal oxides** with low heats of formation and metallic reducing agents which, when oxidized, have high heats of formation. The excess heats of formation of

$$8Al + 3Fe_3O_4 \rightarrow 9Fe + 4Al_2O_3 + \text{heat}$$

4.15 Electro Slag Welding

- Electroslag Welding is a welding process, in which the heat is generated by an electric current passing between the consumable electrode (filler metal) and the work piece through a molten slag covering the weld surface.
- Heat, generated by the arc, melts the fluxing powder and forms molten slag. The slag, having low electric conductivity, is maintained in liquid state due to heat produced by the electric current.

Electroslag Welding

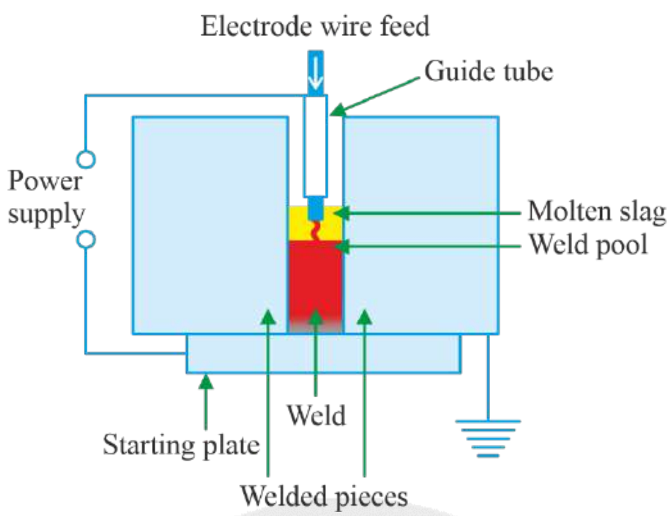


Fig. 4.17 Electroslag Welding

4.16 Electron Beam Welding (EBW)

- A beam of electrons is magnetically focused on the work piece in a vacuum chamber.

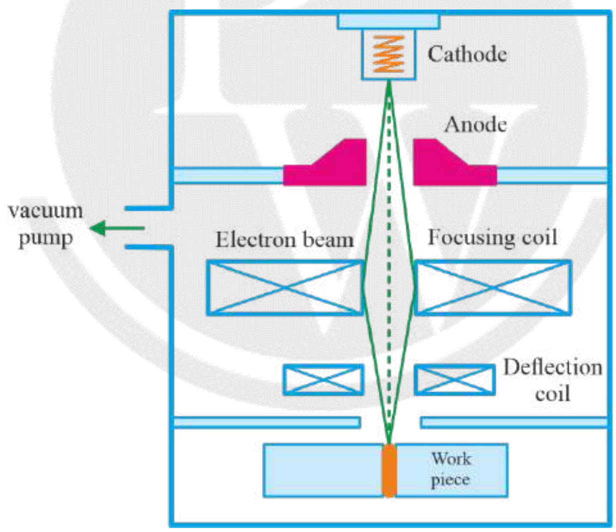


Fig. 4.18 Electron Beam Welding

4.17 Laser Beam Welding (LBW)

- Laser welding utilizes the heat from a high-power concentrated laser beam to melt thin or thick metal interfaces. It is generally used for producing narrow and deep joints of depth to width ratio ranging between 4 and 10

Note:
Increasing order of Heat affected zone (HAZ) are
Laser beam welding < MIG welding < Submerged arc welding < Arc welding

Shielding method	Welding Process
A. Flux coating	1. Shielded metal arc welding
B. Flux granules	2. Submerged arc welding
C. CO_2	3. Gas metal arc welding
D. Vacuum	4. Electron beam welding

4.18 Friction Welding

Friction welding (FRW) is a solid-state welding process that generates heat through mechanical friction between workpieces in relative motion to one another, with the addition of a lateral force called "upset" to plastically displace and fuse the materials.

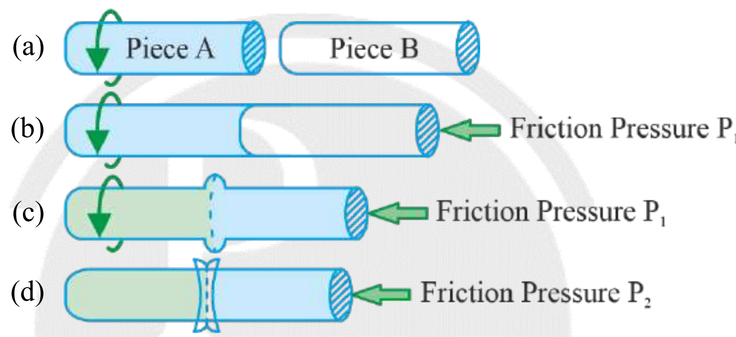


Fig. 4.19 Friction Welding

4.19 Ultrasonic Welding (USW)

- USW is a solid-state welding.
- High-frequency (10 to 200, KHz) is applied.
- Surfaces are held together under light normal pressure.
- Temp. do not exceed one-half of the melting point.
- The ultrasonic transducer is same as ultrasonic machining.

4.20 Explosion Welding

Explosion Welding

- Done at room temperature in air, water or vacuum.
- Surface contaminants tend to be blown off the surface.
- Typical impact pressures are **millions of psi**.
- Well suited to metals that is prone to brittle joints when heat welded, such as,
- Aluminium on steel
- Titanium on steel

Important factors are,

- Critical velocity
- Critical angle
- The cladding plate can be supported with tack welded supports at the edges, or the metal inserts.

4.21 Brazing and Soldering

Fig. 4.20 Brazing and Soldering

4.21.1 Brazing

- Brazing is the joining of metals through the use of heat and a filler metal whose melting temperature is above 450°C; but below the melting point of the metals being joined.
- Fluxes used are combinations of borax, boric acid, chlorides, fluorides, tetra-borates and other wetting agents.

Note:
The strength of a brazed joint Increases up to certain gap between the two joining surfaces beyond which it decreases.

4.21.2 Soldering

- By definition, soldering is a brazing type of operation where the filler metal has a melting temperature **below 450°C**.
- Soldering is used for a **neat leak-proof joint** or a low resistance electrical joint.

❑❑❑

5 METAL FORMING

5.1 Recrystallisation Temperature (Rx)

"The minimum temperature at which the completed recrystallisation of a cold worked metal occurs within a specified period of approximately one hour".

Note:

Rx varies between 1/3 to 1/2 melting point.

Rx = 0.4 x Melting temp.

Rx of Iron is 450°C and for steels around 1000°C

5.1.1 Grain growth

Grain growth follows complete crystallization if the materials left at elevated temperatures.

Heating beyond recrystallization temperature range causes the size of the recrystallized grains to increase, some of the grains grow by consuming others.

Note:

Grain growth is very strongly dependent on temperature.

5.2 Cold working

- Cold working of a metal is carried out below its recrystallisation temperature.
- Although normal room temperatures are ordinarily used for cold working of various types of steel, temperatures up to the recrystallisation range are sometimes used.
- In cold working, recovery processes are not effective.

Advantages of Cold Working

- In cold working processes, smooth surface finish can be easily produced.
- Accurate dimensions of parts can be maintained.
- Strength and hardness of the metal are increased but ductility decreased.
- Since the working is done in cold state, no oxide would form on the surface and Consequently good surface finish is obtained.
- Cold working increases the strength and hardness of the material due to the strain hardening which would be beneficial in some situations.
- There is no possibility of decarburization of the surface.
- Better dimensional accuracy is achieved.

Disadvantages of Cold Working

- Some materials, which are brittle, cannot be cold worked easily.
- Since the material has higher yield strength at lower temperatures, the amount of deformation that can be given to is limited by the capability of the presses or hammers used.
- A distortion of the grain structure is created.
- Since the material gets strain hardened, the maximum amount of deformation that can be given is limited. Any further deformation can be given after annealing.

5.3 Hot working

Plastic deformation of metal carried out at temperature above recrystallization temperature, is called hot working

- Recrystallization temperature = about one half of - melting point on absolute scale
- In practice, hot working usually performed somewhat above 0.5Tm
- Metal continues to soften as temperature increases above 0.5Tm, enhancing advantage of hot working above this level

Advantage of hot working

- Work part shape can be significantly altered
- Lower forces and power required
- Metals that usually fracture in cold working can be hot formed
- Strength properties of product are generally isotropic
- No strengthening of part occurs from work hardening
- Advantageous in cases when part is to be subsequently processed by cold forming.

Dis-advantages of Hot Working

- Heat energy is needed
- It requires expensive tools.
- Poor surface finish of material due to scaling of surface due to the rapid oxidation
- Due to the poor surface finish, close tolerance cannot be maintained.

5.4 Rolling

- Rolling is the process of reducing the thickness or changing the cross section of a long workpiece by compressive forces applied through a set of rolls, as shown in figure.
- Most rolling is carried out by hot working

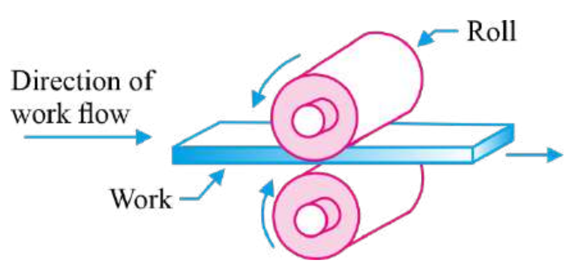


Fig. 5.1 Rolling Process

(a) Continuity Equation

$h_o b_o v_o = h_f b_f v_f$

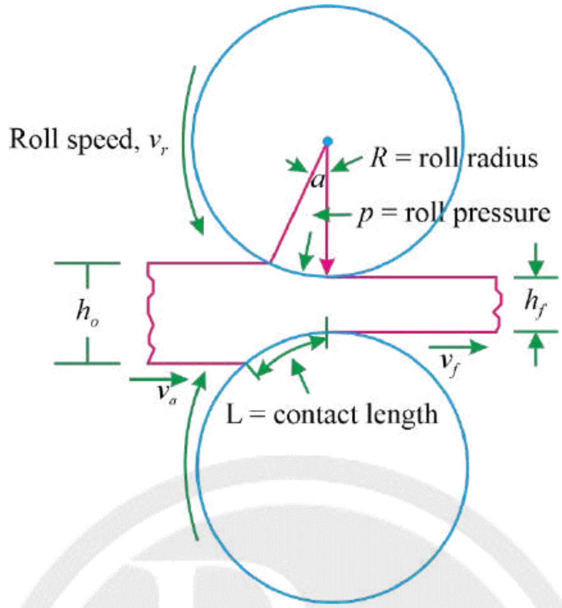


Fig. 5.2 Rolling process

(b) Hot Rolling

- Done above the recrystallization temp.
- Results fine grained structure.
- Surface quality and final dimensions are less accurate.

(c) Cold Rolling

Done below the recrystallization temp.

Products are sheet, strip, foil etc. with good surface finish and increased mechanical strength with close product dimensions

(d) Defects in Rolling

Defects	What is	Cause
Wavy edges	Strip is thinner along its edges than at its Centre.	Due to roll bending edges elongates more and buckle.
Alligatoring	Edge breaks	Non-uniform deformation

(e) Draft/reduction(Δh)

Rolling

R = roll radius

L = contact arc length

L_p = projected arc length

H_f = strip final thickness

V_r = velocity of the roll

V_f = velocity of the strip at roll exit

h_o = strip initial thickness

N = neutral point

a = angle of bite

V_o = velocity of strip at entrance to roll

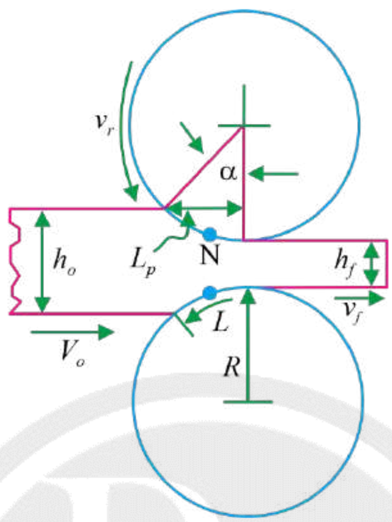


Fig. 5.3 Side view of flat rolling, indicating before and after thickness, work velocities, angle of contact with rolls, and other features.

Maximum Draft Possible (Δh) max

$$(\Delta \mathbf{h})_{\max} = \mu^2 R$$

$$\alpha_{\max} = \tan^{-1}(\mu)$$

Note:

If $\alpha_{\max}$ is larger than this value, the rolls begin to slip,

Number of pass needed

$$n = \frac{\Delta h_{\text{required}}}{\Delta h_{\max}}$$

(f) Elongation Factor or Elongation Co-efficient (E/E^n)

$$E = \frac{L_1}{L_0} = \frac{A_0}{A_1} \text{ for single pass}$$

$$E^n = \frac{L_n}{L_0} = \frac{A_0}{A_n} \text{ for n} - \text{pass}$$

(g) Torque and Power

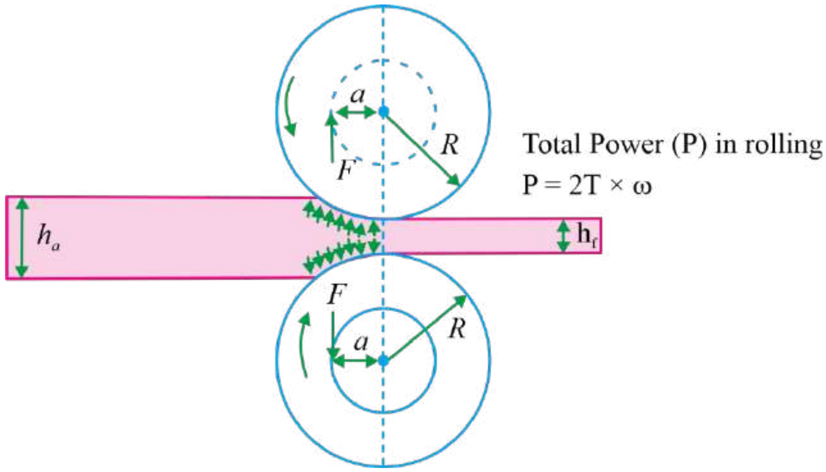


Fig. 5.4

$$\lambda = \frac{a}{L_p} = \frac{a}{\sqrt{R\Delta h}}$$

Where,

L_p = Projected length

R = radius of roller

Δ_h = Draft

T = Torque per roller

F = Roll Separating Force

a = Radius from the center where roll separating force(F) is acting

ω = Angular velocity

Where λ is 0.5 for hot-rolling and 0.45 for cold-rolling.

5.5 Forging

Process in which material is shaped by the application of localized compressive forces exerted manually or with power hammers, presses or special forging machines.

Impression Die forging Here half the impression of the finished forging is sunk or made in the top die and other half of the impression is sunk in the bottom die. In impression die forging, the work piece is pressed between the dies. As the metal spreads to fill up the cavities sunk in the dies, the requisite shape is formed between the closing dies

Open die forging in this, the work piece is compressed between two platens. There is no constraint to material flow in lateral direction. Open die forging is a process by which products are made through a series of incremental deformation using dies of relatively simple shape.

Closed die forging Closed die forging is very similar to impression die forging, but in true closed die forging, the amount of material initially taken is very carefully controlled, so that no flash is formed

Drop forging Drop forging utilizes a closed impression die to obtain the desired shape of the component. The shaping is done by the repeated hammering given to the material in the die cavity. The equipment used for delivering the blows are called drop hammers.

5.5.1 Operations involved in forging

Steps involved in hammer forging

- Fullering or swaging
- Edging or rolling
- Bending
- Drawing or cogging
- Flattening
- Blocking
- Finishing operation
- Trimming or cut off

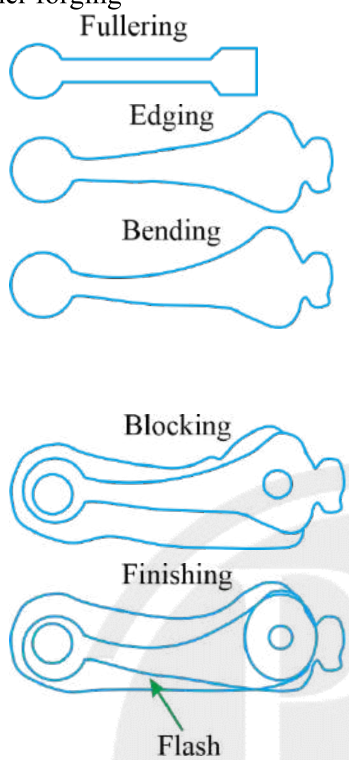


Fig. 5.5 Forging Operations

5.5.2 Die Materials Should have

- Thermal shock resistance
- Thermal fatigue resistance
- High temperature strength
- High wear resistance
- High toughness and ductility
- High hardenability
- High dimensional stability during hardening
- High machinability.
- Die materials: alloyed steels (with Cr, Mo, W, V), tool steels, cast steels or cast iron

Note:
1. Carbon steels with 0.7-0.85% C are appropriate for small tools and flat impressions.
2. Medium-alloyed tool steels for hammer dies.
3. Highly alloyed steels for high temperature resistant dies used in presses and horizontal forging machines.

5.5.3 Typical forging defects

- Incomplete forging penetration- should forge on the press.
- Microstructural differences resulting in pronounced property variation.
- Hot shortness, due to high sulphur concentration in steel and nickel.
- Pitted surface, due to oxide scales occurring at high temperature stick on the dies.
- Buckling, in upsetting forging. Subject to high compressive stress.
- Surface cracking, due to temperature differential between surface and center, or excessive working of the surface at too low temperature.
- Microcracking, due to residual stress.
- Flash line crack, after trimming-occurs more often in thin workpieces. Therefore, should increase the thickness of the flash.
- Cold shut or fold, due to flash or fin from prior forging steps is forced into the workpiece.
- Internal cracking, due to secondary tensile stress.

5.6 Sheet Metal Working

- Shearing is a cutting operation used to remove a blank of required dimension from a large sheet.

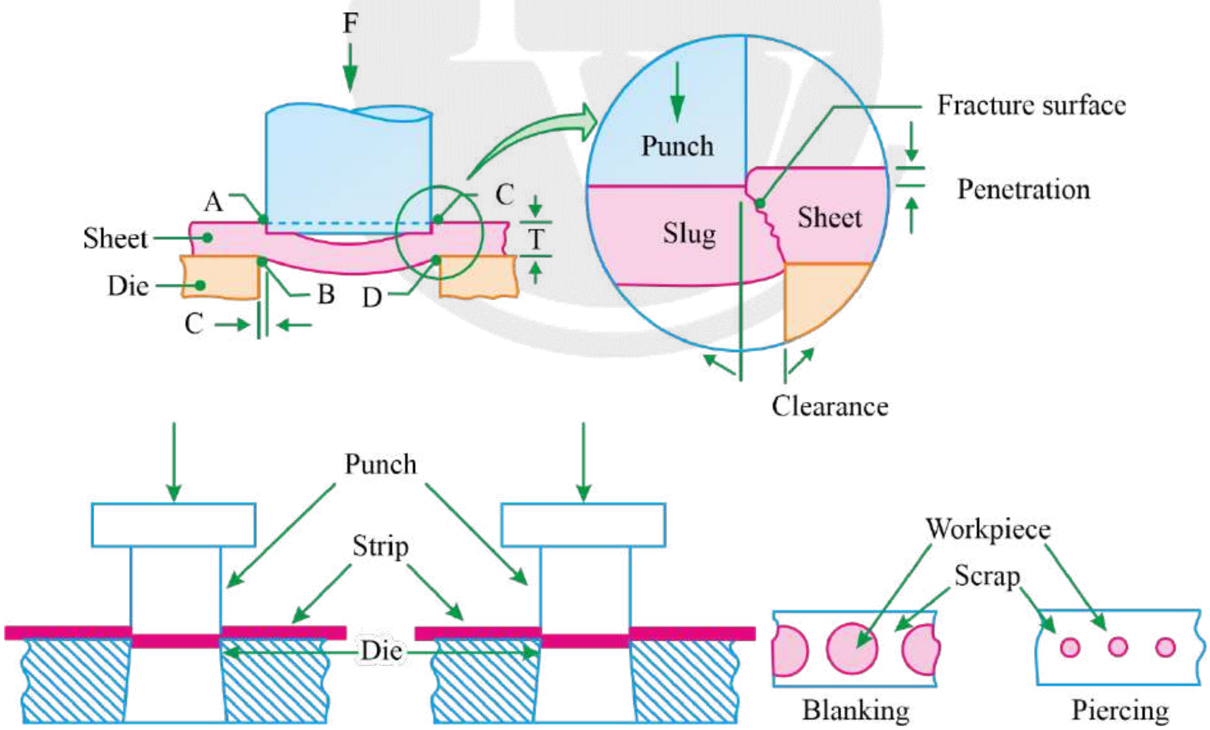


Fig. 5.5 Punching and Blanking operation

In blanking, the piece being punched out becomes the workpiece and any major burrs or undesirable features should be left on the remaining strip.

In piercing (Punching), the punch-out is the scrap, and the remaining strip is the workpiece.

(a) Clearance

- Die opening must be larger than punch and known as 'clearance'.
- Punching

  Punch = size of hole

  Die = punch size +2 clearance

> **Note:** In punching punch is correct size.

(b) Blanking

- Die = size of product
- Punch = Die size –2 clearance

> **Note:**
> In blanking die size will be correct.

Clearance formula

1. $c = 0.0032t\sqrt{\tau}$ or
2. C = allowance (t)
3. C = (x%)t

Where,

t = sheet thickness (mm)

τ = shear strength (N/mm^2)

C = Clearance

(c) Punching Force and Blanking Force

$$F_{\max} = Lt\tau$$

Where

$F_{\max}$ = Maximum force

L = cutting parameter(mm)

t = thickness of sheet (mm)

τ = shear strength (N/mm^2)

The punching force for holes which are smaller than the stock thickness may be estimated as follows:

$$F_{\max} = \frac{\pi d t \sigma}{\sqrt[3]{\frac{d}{t}}}$$

Where

$F_{\max}$ = Maximum force

d = diameter of punch(mm)

t = thickness of sheet (mm)

σ = Tensile strength (N/mm^2)

(d) Shear on Punch

To reduce shearing force, shear is ground on the face of the die or punch. It distributes the cutting action over a period of time.

Note:

Shear only reduces the maximum force to be applied but total work done remains same.

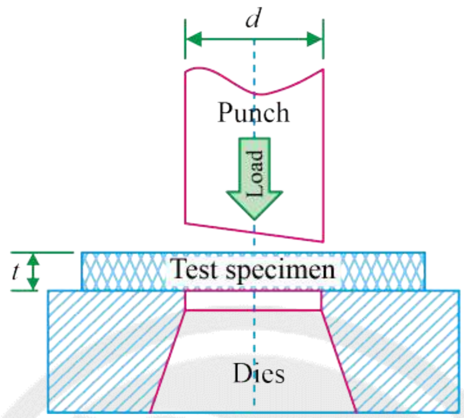


Fig. 5.6

Maximum shear force(F_s)

$$F_s = \frac{F_{max}(Pt)}{Pt + S}$$

If $S > Pt$ then

$$F_s = \frac{F_{max}(Pt)}{S}$$

Where,

F_{max} = Maximum force
P = percentage penetration
t = thickness of sheet (mm)
S = shear height (mm)

5.7 Drawing

Drawing is a plastic deformation process in which a flat sheet or plate is formed into a three-dimensional part with a **depth more than several times the thickness of the metal.**

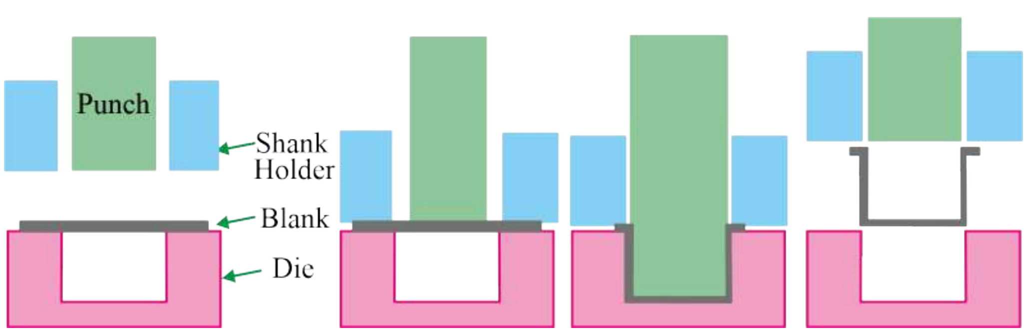


Fig. 5.7 Drawing Operation

Blank Size

$$D = \sqrt{d^2 + 4dh}$$

$$D = \sqrt{d^2 + 4dh} - 0.5r \quad \text{when } 15r \le d \le 20r$$

$$D = \sqrt{(d-2r)^2 + 4d(h-r) + 2\pi r(d-0.7r)} \quad \text{when } d < 10r$$

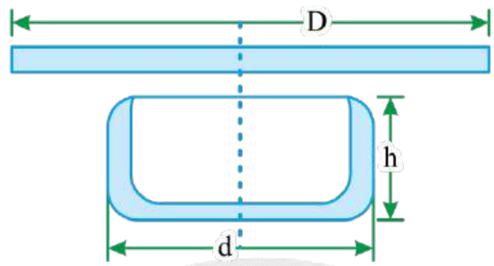


Fig. 5.8

5.8 Deep drawing

Drawing when cup height is more than half the diameter is termed deep drawing. This can be achieved by redrawing the part through a series of dies.

Note:
Deep drawing - is a combination of drawing and stretching.

5.8.1 Deep Drawability

- The ratio of the maximum blank diameter to the diameter of the cup drawn. i.e. D/d.
- The average reduction in deep drawing, thumb rule for reduction

$$\frac{d}{D} = 0.5$$

$$\text{Reduction} = \left(1 - \frac{d}{D}\right) \times 100\% = 50\%$$

Thumb rule:

First draw: Reduction = 50%

Second draw: Reduction = 30%

Third draw: Reduction = 25%

Fourth draw: Reduction = 16%

Fifth draw: Reduction = 13%

5.8.2 Defects in Drawing

- **Wrinkle:** An insufficient blank holder pressure causes wrinkles to develop on the flange, which may also extend to the wall of the cup.

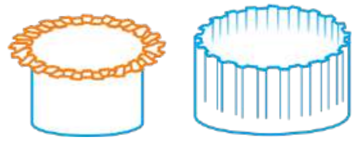


Fig. 5.9 Wrinkle

- **Fracture:** Further, too much of a blank holder pressure and friction may cause a thinning of the walls and a fracture at the flange, bottom, and the corners (if any).

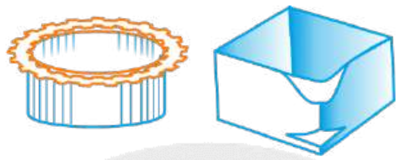


Fig. 5.10 Fracture

- **Earing:** While drawing a rolled stock, ears or lobes tend to occur because of the anisotropy induced by the rolling operation.

Fig. 5.11 Earing

- **Miss strike:** Due to the misplacement of the stock, unsymmetrical flanges may result. This defect is known as miss strike.

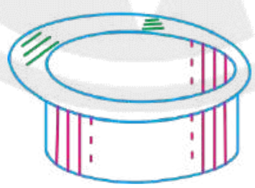


Fig. 5.12 Miss Strike

- **Surface scratches:** Die or punch not having a smooth surface, insufficient lubrication

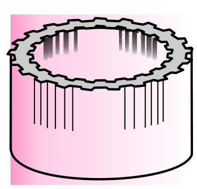


Fig. 5.13 Scratches

5.9 Spinning

- Spinning is a cold-forming operation in which a rotating disk of sheet metal is shaped over a male form, or mandrel. Localized pressure is applied through a simple round-ended wooden or metal tool or small roller, which traverses the entire surface of the part.

  Relation between cone and blank

$$t_c = t_b \sin\alpha$$

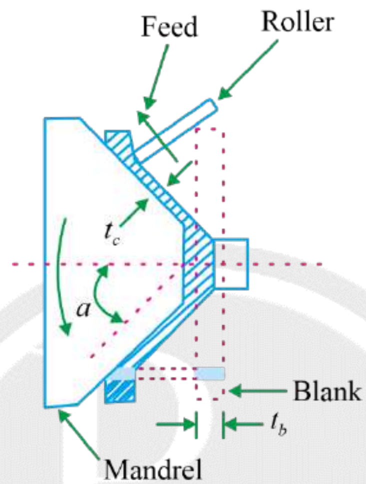


Fig. 5.14 Spinning process

T_c = cone thickness

T_b = blank thickness

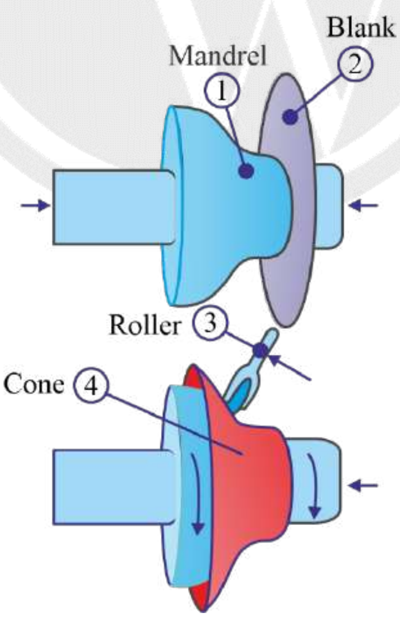


Fig. 5.15 Spinning Process

5.10 Stretch Forming

A sheet of metal is gripped by two or more sets of jaws that stretch it and wrap it around a single form block. Because most of the deformation is induced by the tensile stretching, the forces on the form block are far less than those normally encountered in bending or forming.

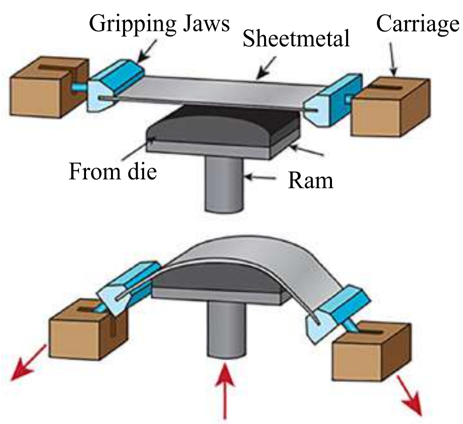


Fig. 5.16

Final thickness(t) formula
For bi-axial stretching of sheets

$$\varepsilon_1 = \text{In}\left(\frac{l_{i1}}{l_{01}}\right) \quad ; \varepsilon_2 = In\left(\frac{l_{i2}}{l_{02}}\right)$$

$$\text{Final thickness} = \frac{\text{Initial thickness(t)}}{e^{\varepsilon_1} \times e^{\varepsilon_2}}$$

where
- l_{i1} = final length in direction 1
- l_{i2} = final length in direction 2
- l_{o1} = initial length in direction 1
- l_{o2} = initial length in direction 2
- ε_1 = strain in direction 1
- ε_2 = strain in direction 2

5.11 Bending

- After basic shearing operation, we can bend a part to give it some shape.
- Bending parts depends upon material properties at the location of the bend.

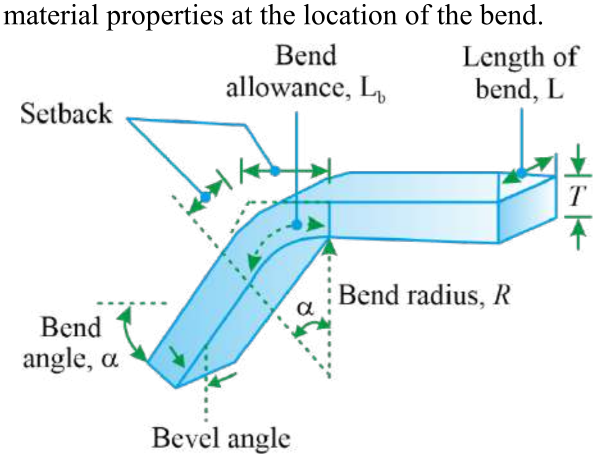


Fig. 5.17 Bending

Bend allowance (L_b), formula

$$L_b = \alpha(R + kt)$$

where

R = bend radius

k = constant (stretch factor)

t = thickness of material

α = bend angle (in radian)

5.12 Extrusion

- Extrusion is a compression process in which the work metal is forced to flow through a die opening to produce a desired cross-sectional shape. As shown in figure

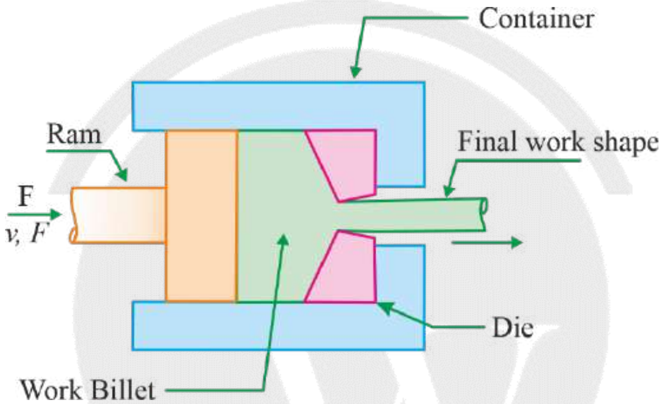


Fig. 5.18

5.12.1 Extrusion Ratio

- Ratio of the cross-sectional area of the billet to the cross-sectional area of the product.

5.12.2 Direct Extrusion

- A solid ram drives the entire billet to and through a stationary die and must provide additional power to overcome the frictional resistance between the surface of the moving billet and the confining chamber.

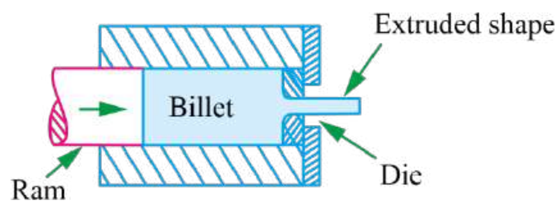


Fig. 5.19 Direct extrusion

5.12.3 Indirect Extrusion

- A hollow ram drives the die back through a stationary, confined billet. Since no relative motion, friction between the billet and the chamber is eliminated.

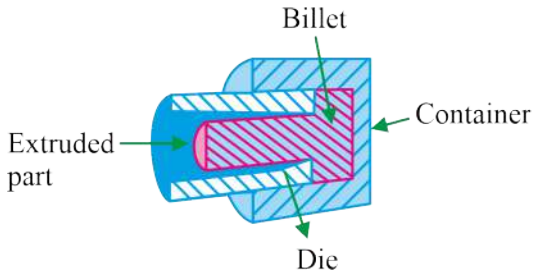


Fig. 5.20 Indirect Extrusion

5.12.4 Hydrostatic Extrusion

In hydrostatic extrusion the container is filled with a fluid. Extrusion pressure is transmitted through the fluid to the billet. Friction is eliminated in this process because of there is no contact between billet and container wall. Brittle materials can be extruded by this process. Highly brittle materials can be extruded into a pressure chamber.

Hydrostatic extrusion is a process in which the billet is completely circumscribed by a pressurized liquid in all the cases, with the exception being the case where billet is in the contact with die. This process can be carried out in many ways including warm, cold or hot but due to the stability of the used fluid, the temperature is limited. Hydrostatic extrusion has to be carried out in a completely sealed cylinder for containing the hydrostatic medium

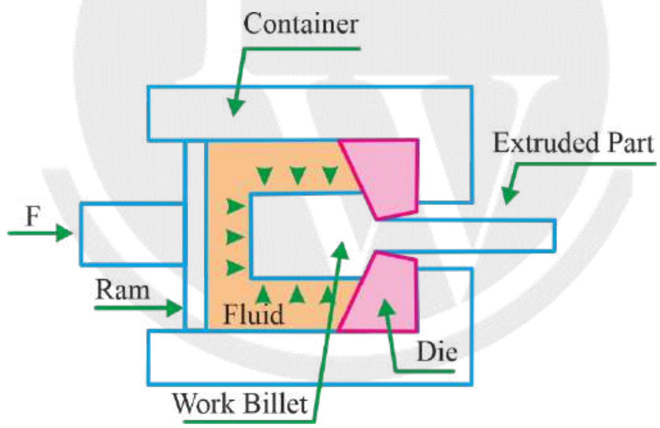


Fig. 5.21 Hydrostatic Extrusion

5.13 Wire Drawing

A cold working process to obtain wires from rods of bigger diameters through a die. For fine wire, the material may be passed through a number of dies, receiving successive reductions in diameter, before being coiled. The wire is subjected to tension only.

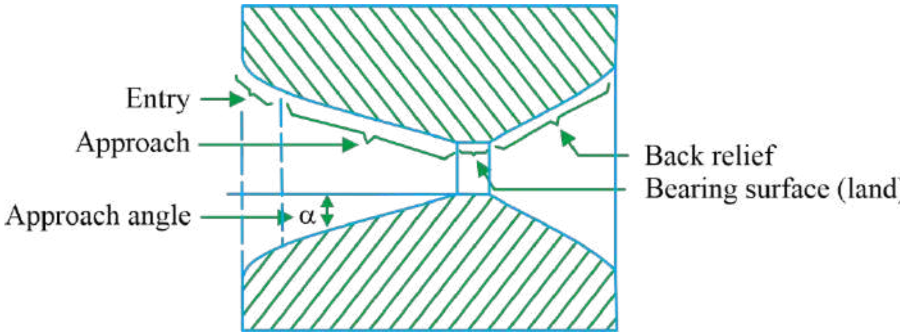


Fig. 5.22 Drawing process

5.13.1 Extrusion Load (F) formula

Extrusion load formula (Uniform deformation, no friction) "work – formula"

$$F = A_o \sigma_o \ln\left(\frac{A_o}{A_f}\right)$$

For real conditions

$$F = KA_o \ln\left(\frac{A_o}{A_f}\right)$$

A_0 = initial cross-sectional area

A_f = final cross-sectional area

σ_0 = yield strength of material

K = extrusion constant

$K = \frac{\sigma_0}{\sqrt{3}}$ (for plane strain)

$= \frac{\sigma_0}{2}$ (for plane stress)

5.13.2 Wire or Tube drawing force(F) formula

- Wire or Tube drawing force formula (Uniform deformation, no friction) "work – formula"

$$F = A_f \sigma_0 \ln\left(\frac{A_o}{A_f}\right)$$

❑❑❑

6 METROLOGY

6.1 Metrology and Inspection

It is the measurement science that includes various aspects like design, manufacture, testing, and applications of various measuring instruments, devices, and techniques. Thus, it facilitates the proper application of the scientific principles in the accurate dimensional control of manufactured components.

6.1.1 Limit System

- **Basic size:** It is the size with reference to which upper or lower limits of size are defined. It is theoretical size of part as suggested by designer.
- **Actual size:** It is the size actually obtained by machining. It is found by actual measurement
- **Tolerance:**
  (a) The difference between the upper limit and lower limit.
  (b) It is the maximum permissible variation in a dimension.
  (c) The tolerance may be unilateral or bilateral.
  (d) It is always positive.
- **Unilateral Limits** occurs when both maximum limit and minimum limit are either above or below the basic size.
- **Bilateral Limits** occur when the maximum limit is above and the minimum limit is below the basic size.

6.1.2 Fit

Fits: It is the relationship that exists between two mating parts, a hole and shaft with respect to their dimensional difference before assembly.

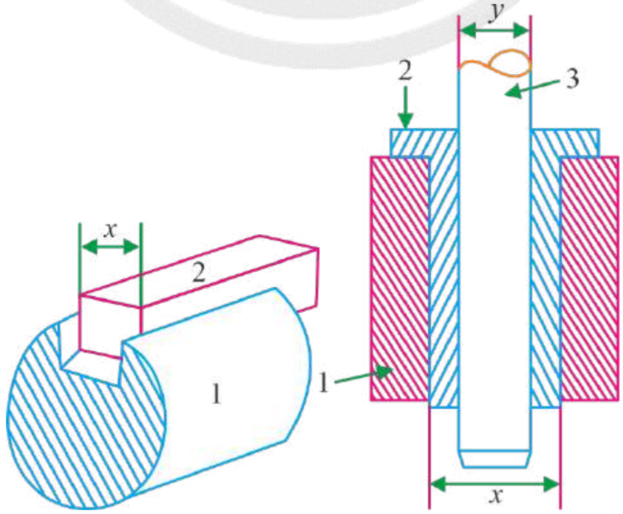


Fig. 6.1 Fit

6.1.3 Allowance

It is Minimum clearance or maximum interference. It is the intentional difference between the basic dimensions of the mating parts. The allowance may be positive or negative.

6.2 Hole basis and Shaft basis System

6.2.1 Basis of Fits - Hole Basis

In this system, the basic diameter of the hole is constant while the shaft size varies according to the type of fit.

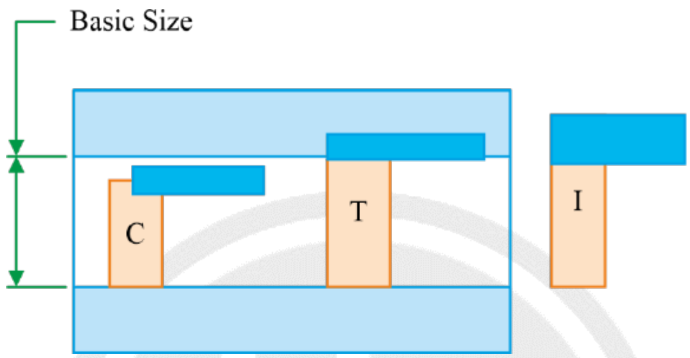


Fig. 6.2 Hole Basis Fits

Hole
Shaft
Tolerance

C - Clearance
T - Transition
I - Interference

6.2.2 Basis of Fits - Shaft Basis

Here the hole size is varied to produce the required class of fit with a basic-size shaft.

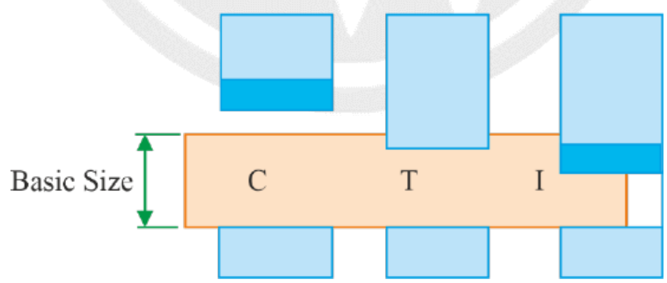


Fig. 6.3 Shaft Basis Fits

Hole
Shaft
Tolerance

C - Clearance
T - Transition
I - Interference

6.3 Limits and Fits

- Limits and fits comprise 18 grades of fundamental tolerances for both shaft and hole, designated as IT01, ITO and IT1 to IT16. These are called standard tolerances. (IS-919) But ISO 286 specify 20 grades up to IT18
- There are 25 (IS 919) and 28 (ISO 286) types of fundamental deviations.

  Hole: A, B, C, CD, D, E, EF, F, FG, G, H, J, JS, K, M, N, P, R, S, T, U, V, X, Y, Z, ZA, ZB, ZC.

  Shaft: a, b, c, cd, d, e, ef, f, fg, g, h, j, js, k, m, n, p, r, s, t, u, v, x, y, z, za, zb, zc.
- A unilateral hole basis system is recommended but if necessary a unilateral or bilateral shaft basis system may also be used.

  'Standard tolerance unit', (i) inμm

$$i = \text{Fundamental tolerance}$$

$$i = 0.45\sqrt[3]{D} + 0.001D$$

(unit tolerance, in μm)

$$D = \sqrt{D_1 D_2}$$

(D_1 and D_2 are the nominal sizes marking the beginning and the end of a range of sizes, in mm)

Value of the Tolerance

IT01 0.3 + 0.008D	IT0 0.5 + 0.012D	IT1 0.8 + 0.02D = a	IT2 $ar^2 = 10^{1/5}$
IT3 ar^2	IT4 ar^3	IT5 $ar^4 = 7i$	IT6 $10(1.6)^{(ITn - IT6)} = 10i$
IT7 $10(1.6)^{(ITn - IT6)} = 16i$	IT8 $10(1.6)^{(ITn - IT6)} = 25i$	IT9 $10(1.6)^{(ITn - IT6)} = 40i$	IT10 $10(1.6)^{(ITn - IT6)} = 64i$
IT11 $10(1.6)^{(ITn - IT6)} = 100i$	IT12 $10(1.6)^{(ITn - IT6)} = 160i$	IT13 $10(1.6)^{(ITn - IT6)} = 250i$	IT14 $10(1.6)^{(ITn - IT6)} = 400i$
IT15 $10(1.6)^{(ITn - IT6)} = 640i$	IT16 $10(1.6)^{(ITn - IT6)} = 1000i$		

Table 6.1 Fundamental Tolerances

6.4 Fundamental Deviation

It is chosen to locate the tolerance zone w.r.t. the zero line

- Holes are designated by capital letter:
- Letters A to G - oversized holes
- Letters P to ZC - undersized holes

Shafts are designated by small letter:

Letters m to zc - oversized shafts
Letters a to g - undersized shafts
H is used for holes and **h** is used for shafts whose fundamental deviation is zero

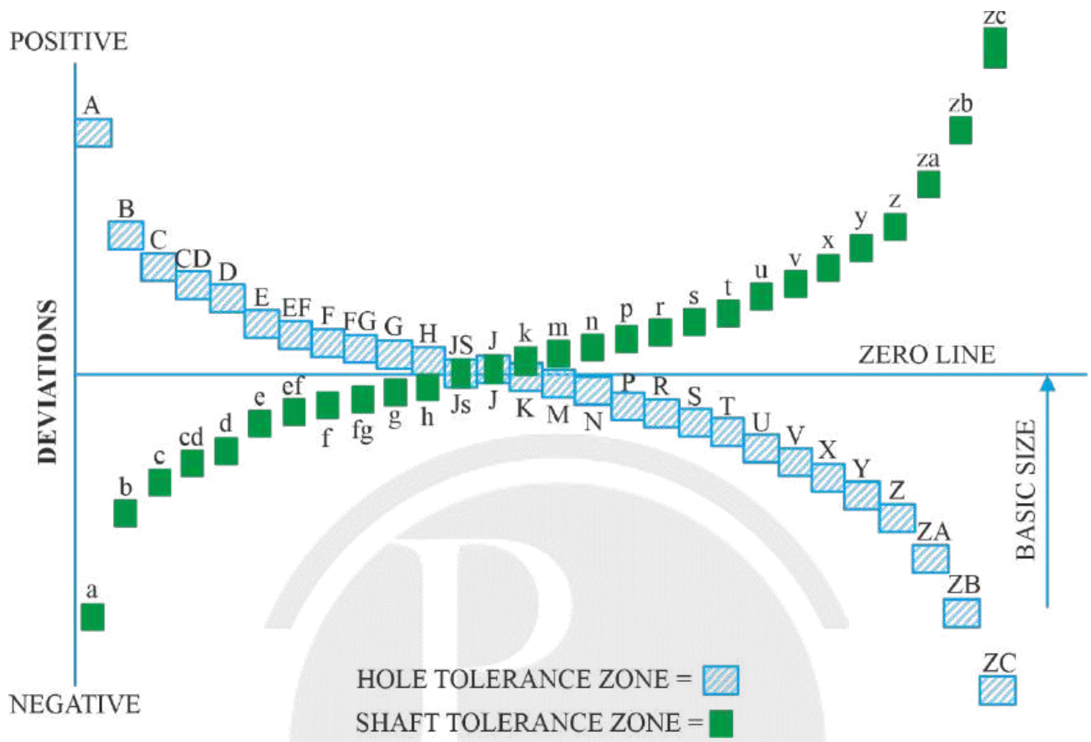


Fig. 6.4 Fundamental Deviations

6.5 Taylor's Principle of Gauging

- **A GO** Gauge will check all the dimension of the work piece in the maximum metal condition (indicating the presence of the greatest amount of material permitted at a prescribed surface). It should check the size of the component also the geometrical shape.
- **NOT GO** Gauges will check only one dimension of the work piece at a time, for the minimum metal conditions (indicating the presence of the least amount of material permitted at a prescribed surface).
- **Plug gauge:** used to check the holes. The GO plug gauge is the size of the low limit of the hole while the NOT GO plug gauge corresponds to the high limit of the hole.
- **Snap, Gap or Ring gauge:** used for gauging the shaft and male components. The Go snap gauge is of a size corresponding to the high (maximum) limit of the shaft, while the NOT GO gauge corresponds to the low (minimum limit).

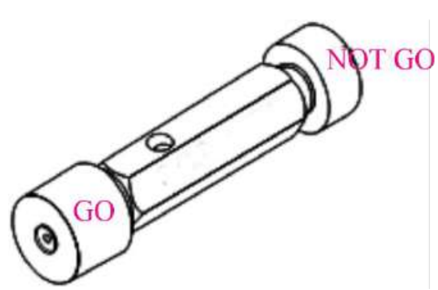


Fig. 6.5 Plug gauge

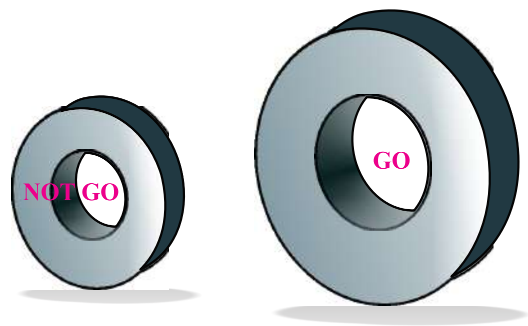


Fig. 6.6 Ring and snap gauges

Note:

1. Wear allowance: GO gauges which constantly rub against the surface of the parts in the inspection are subjected to wear and lose their initial size.
2. The size of go plug gauge is reduced while that of go snap gauge increases.
3. Wear allowance is usually taken as 5% of the work tolerance.

6.6 Process capability Index

$\bar{x} \pm 3\sigma$ Desired Tolerance

$$6\sigma = \text{Process capability}$$
$$C_p > 1 \text{ highly capable}$$
$$= 1 \text{ Just capable}$$
$$< 1 \text{ Not capable}$$

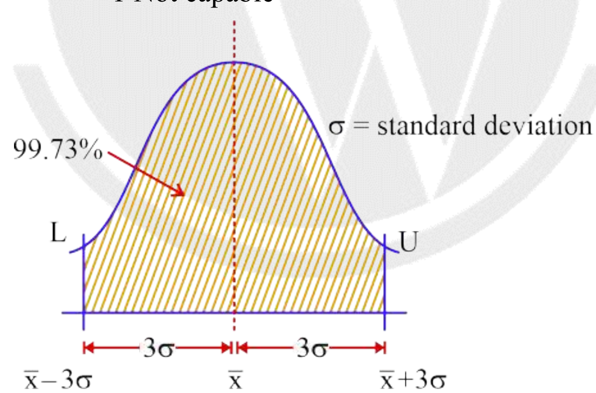


Fig. 6.7

$$\boxed{C_P = \frac{U-L}{6\sigma}} \quad \boxed{C_{B\ell} = \frac{\bar{x}-L}{3\sigma}} \quad \boxed{C_{P_u} = \frac{U-\bar{x}}{3\sigma}}$$

$$C_{P_k} = \text{Min}\left[C_{P\ell}, C_{Pu}\right]$$

Hole & Shaft: 95% cases hole is made first due to standard sizes of drills and reamers available in market.

Accuracy - Closeness to target value (refers to individual [Mode a Median of normal distribution]

Precision - Repeatability (Refers to a group) [Standard Deviation]

Full interchangeability:

In such case the process capability of machine is equal to the desired tolerance.

But practically process capabilities much larger than designed tolerance

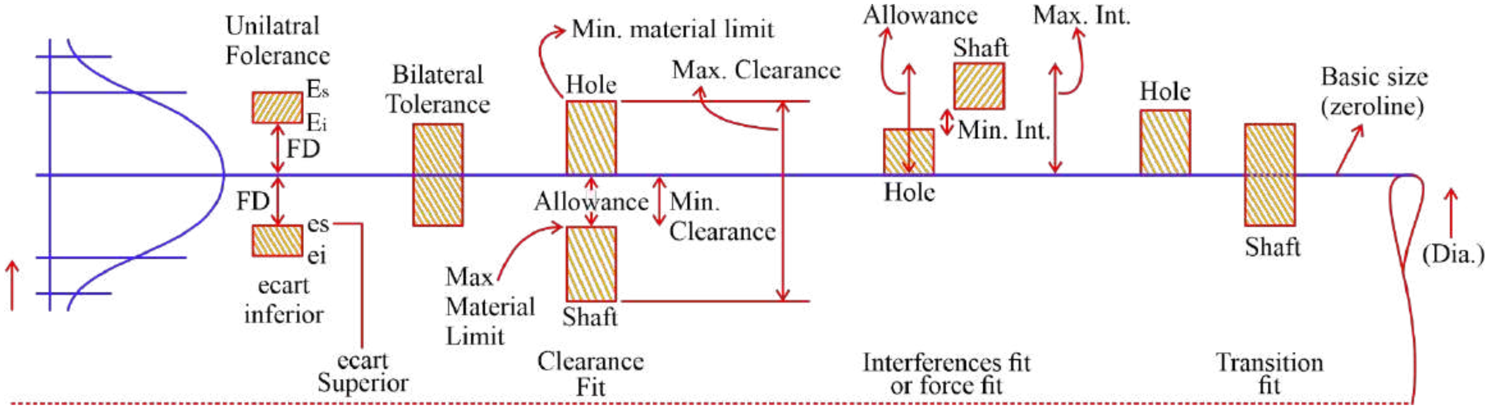


Fig. 6.8

Note:
Negative clearance is called interference and negative interference is called clearance.

Fundamental deviations = 25

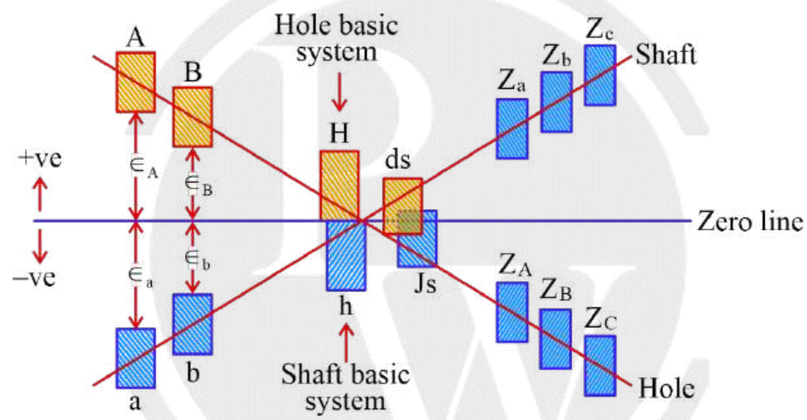


Fig. 6.9

J_s is only bilateral
FD is dist. Between zero line and limit closer to it.

6.7 Grade of Tolerance

$i = 0.45D^{1/3} + 0.001D$; $D = \sqrt{D_1 D_2}$

Diameter range →

D1	–	D2
0	–	3
3	–	6
6	–	10
10	–	18
18	–	30
30	–	50

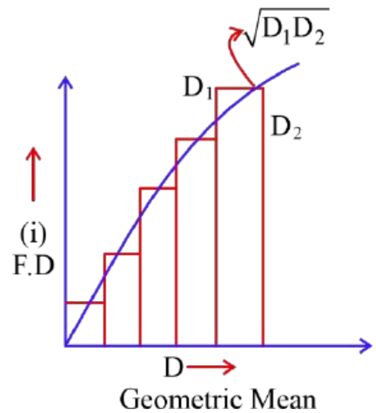


Fig. 6.10

Where D is in mm and tolerance is meter microns.

IT01 IT0 IT1 — Emperical formula

IT2 IT1 — $10^{(1/5)}i$

Sip guage — Buffing, horning

IT4 IT5 IT6 IT7 IT8

Geometric series — $10^{(4/5)}i$ — Preferred series — $10i$ — $10(10)^{1/5}i = 16i$ — $25i$

Grinding — Machining

IT9 IT10 IT11 IT12 IT13

40i 64i 100i 160i 250i

Forming

IT14 IT15 IT16

400i 640i 1000i

Casting

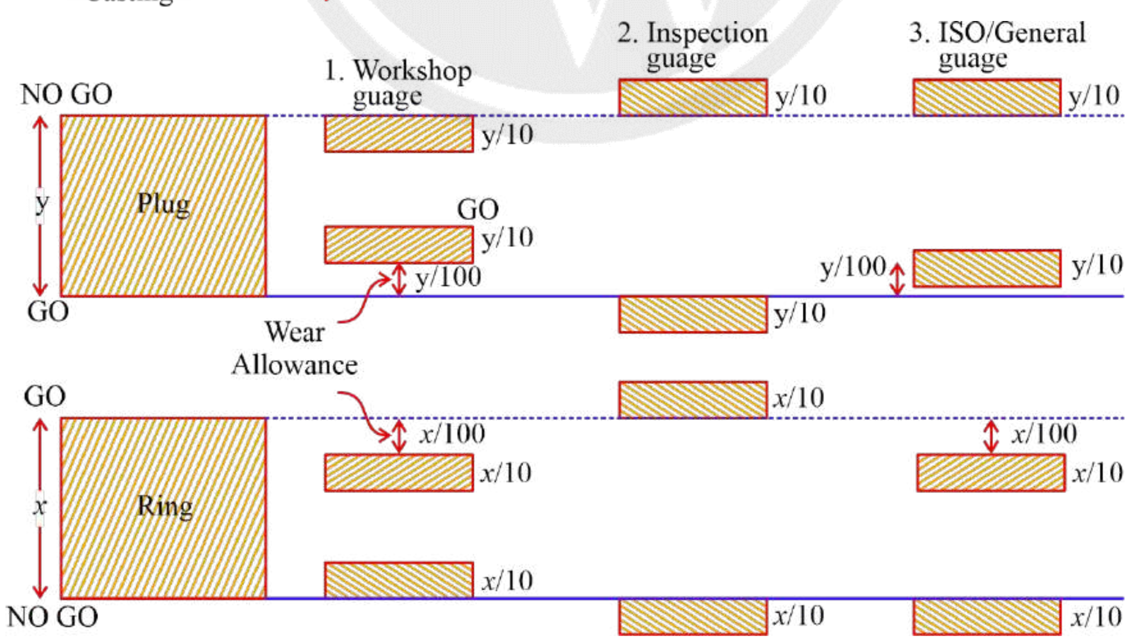


Fig. 6.11

Guage tolerance $= \frac{1}{10}$ (work tolerance)

Wear allowance $= \frac{1}{10}$ (Guage tolerance) $= \frac{1}{100}$ (work tolerance)

Properties of Guages

(1) Hardness

(2) Low α (thermal exp.)

(3) ρ (density)

(4) Corrosion resistant

(5) Machinability

Materials for Guages

(a) En – 24 (High steel)

(b) Invar (36% Ni)

(c) Elinver (42% Ni)

(d) Glass (For guns and rifle)

Tolerance Sink

Tolerance of sink is algebraic summation of all the other tolerances but only like-minded tolerances can be added and it will be least accurate section of assembly.

Micrometer is more accurate instrument than vernier caliper because it has point contact and vernier has area contact so reference plane of measurement keeps on changing.

6.8 Slip Gauges

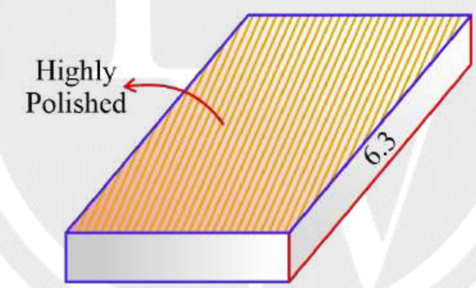


Fig. 6.12 Slip Gauges

Normal set (M45)	Special set (M87)
$1.001 - 1.009 = 9$	$1.001 - 1.009 = 9$
$1.01 - 1.09 = 9$	$1.01 - 1.49 = 49$
$1.1 - 1.9 = 9$	$0.5 - 9.5 = 19$
$1 - 9 = 9$	$10 - 90 = 9$
$10 - 90 = \frac{9}{45}$	$1.005 = \frac{1}{87}$

Minimum number of slips gauges should be used to reduces tolerances.

6.8.1 Angle Blocks

(Used to measure angular datum)

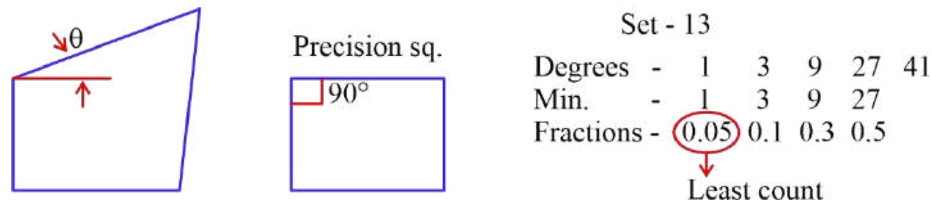
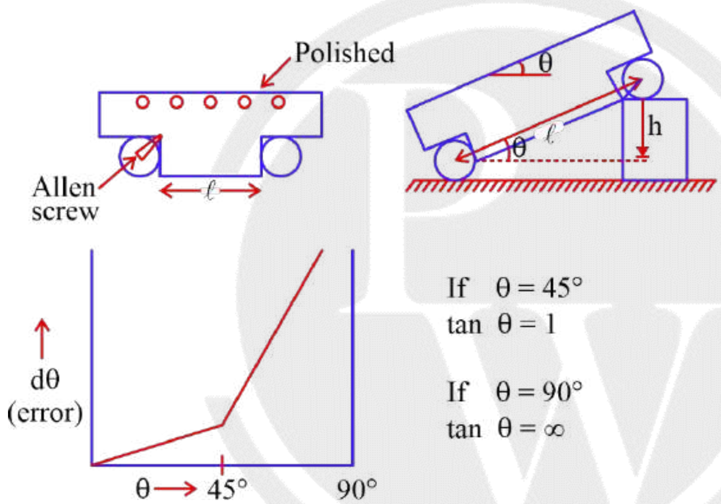


Fig. 6.13

- For adding angles place inclined side of one on flat side of the other.
- For subtracting angles place both incline sides one over other
- For adding and subtracting 90° use precision square.

6.9 Sine Bar



$$\sin\theta = \frac{h}{\ell}$$

$$d\theta = -\tan\theta \frac{d\ell}{\ell}$$

If $\theta = 45°$

$\tan\theta = 1$

If $\theta = 90°$

$\tan\theta = \infty$

Fig. 6.14

That only sine bars are not used beyond 45° angle.

6.10 Precision Ball Measurement

$$\text{cosec}\theta = \frac{h_2 - h_1}{r_2 - r_1} - 1$$

$$\text{cosec}\theta = \frac{dh}{dr} - 1$$

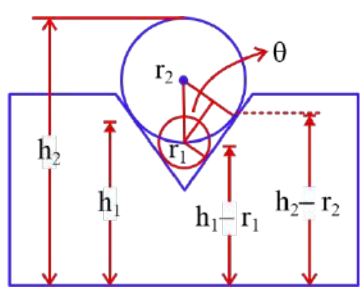


Fig. 6.15

6.11 Taper of Plug Gauge

$$\tan\theta = \frac{2h}{\ell_2 - \ell_1}$$

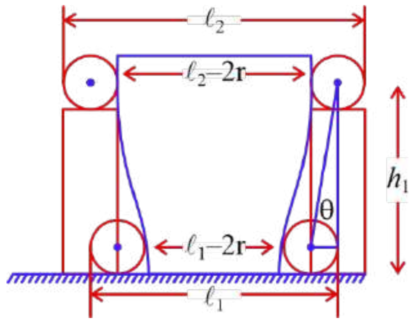


Fig. 6.16

6.12 Types of Error

(1) **System error [Systematic error]:** Follow = pattern & eliminated by calibrating the equipment.

(2) **Short period error:** due to change in environmental conditions taken care by neglecting data.

(3) **Erratic error:** due to maintenance of equipment.

6.12.1 Measurement Errors

(a) Cosine error

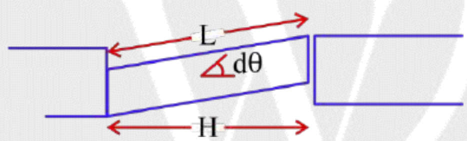


Fig. 6.17 Cosine Error

Error = L – L cos θ, can be neglected.

(b) Sine error

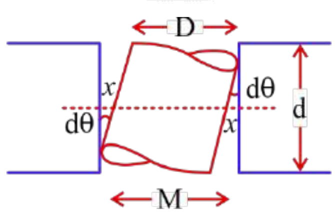


Fig. 6.18 Sine Error

$$\text{Error} = 2x = d\times\delta\theta$$

$$\frac{x}{\frac{d}{2}} = \tan\delta\theta = \delta\theta$$

$$\boxed{x = \frac{d}{2}\delta\theta}$$

6.13 Screw Turned Metrology:

6.13.1 Pitch measurement

The most commonly used methods for measuring the pitch are:

(1) Pitch measuring machine
(2) Tool makers microscope
(3) Screw pitch gauge

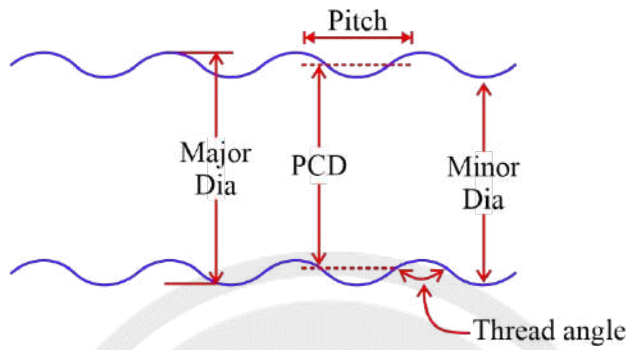


Fig. 6.19

Internal threads are analysed by putting sulphur or wax into the caving (half filled)

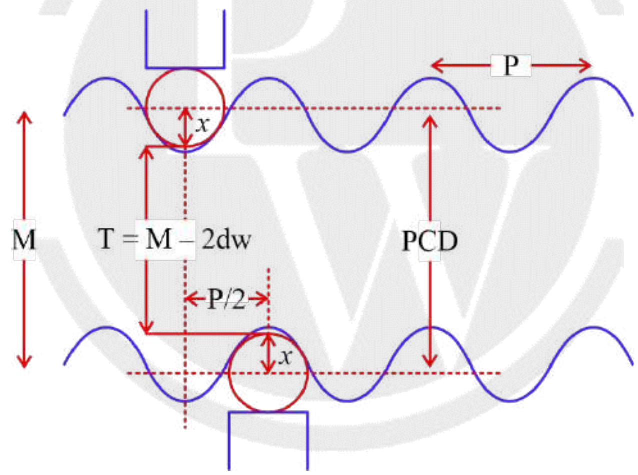


Fig. 6.20

$$PCD = T + 2x$$

$$PCD = T + d\omega(1 - \operatorname{cosec}\theta) + \frac{p}{2}\cot\theta$$

Best wire size is $$dw_b = {}^{p}\!/_{2}\sec\theta$$

6.14 Methods for Qualifying Surface Roughness

6.14.1 Peak to valley height (R_t, R_{max}, H_{max}):

(a) If only f (feed) and R (nose radius) is given

$$H_{max} = \frac{f^2}{8R}$$

(b) If tool signature is given than use

$$H_{max} = \frac{f}{\tan\Psi + \cot\Psi_1} \qquad \begin{array}{l}\Psi = \text{side cutting edge angle} \\ \psi_1 = \text{end cutting edge angle}\end{array}$$

6.14.2 Centre line average value (CLA, R_a):

(a) $\boxed{R_a = \frac{y_1 + y_2 + y_3 .....y_n}{n}}$

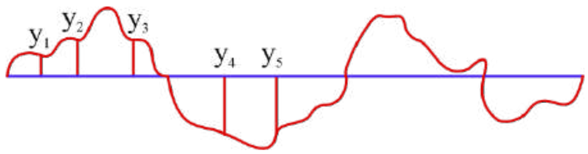


Fig. 6.21

(b) $\boxed{R_a = \frac{\sum a + \sum b}{L}}$

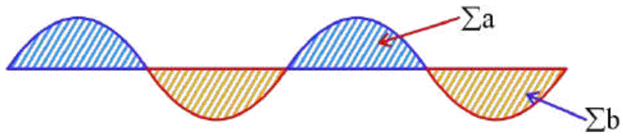


Fig. 6.22

(c) $\boxed{R_a = \frac{1}{L}\int_0^L (y)dx}$ if equation of curve is given eg. $Y = f(x)$

(d) $\boxed{R \approx \frac{H_{max}}{4}}$ No data except $H_{max.}$ is given

$$R_a = \frac{\text{Peak height} \times \text{number of peak} + \text{valley depth} \times \text{number of valleys}}{\text{Number of peak} + \text{number of valleys}}$$

6.14.3 Root mean square value (RMS, Rg):

$$\boxed{R_g = \sqrt{\frac{y_1^2 + y_2^2 + y_3^2 .....y_n^2}{n}}}$$

$$H_{max} > R_g > R_a$$

10 points value (R_z): Average of 5 highest and 5 deepest valleys.

6.15 Representation of Surface Roughness

Machining Symbol

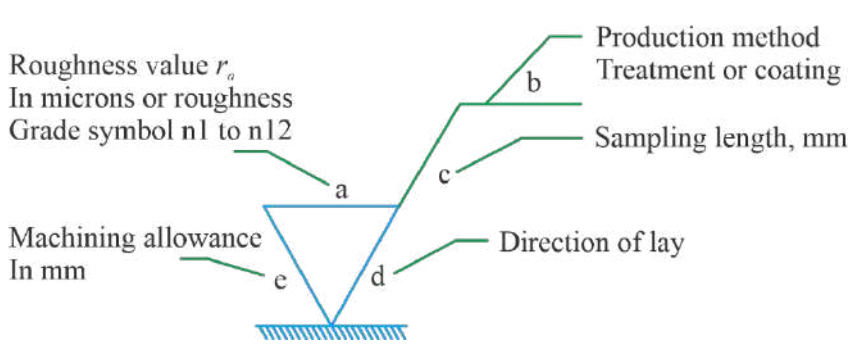


Fig. 6.23 Surface Roughness

6.16 Lay

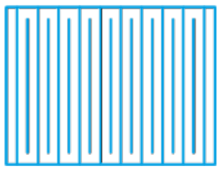
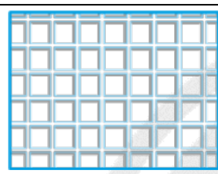
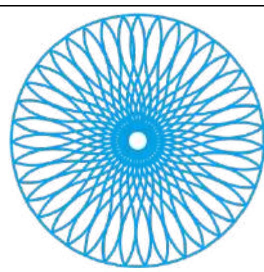
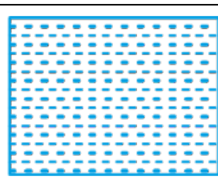
Lay Symbol	Surface pattern	Description
=		**Parallel Lay:** Lay parallel to the Surface. Surface is produced by shaping, planning etc.
⊥		**Perpendicular Lay:** Lay perpendicular to the Surface. Surface is produced by shaping and planning.
×		**Crossed Lay:** Lay angular in both surfaces is produced by knurling, honing.
M		**Multidirectional lay:** Lay multidirectional. Surface is produced by grinding, lapping, super finishing.
C		**Circular lay:** Approximately circular relative to the center. Surface is produced by facing.
R		**Radial lay:** Approximately radial relative to the center to the nominal surface.
P		Lay is particulate, nonchemical of probuberant.

Table 6.2 Different types of Lay

6.17 Optical Flat

$$A'B' \approx B'C$$

$$A'B' = \frac{n\lambda}{2}$$

Where n = number of fringes

λ is wavelength

$$\tan\theta \approx \theta = \frac{A'B'}{L} = \frac{n\lambda}{2L}$$

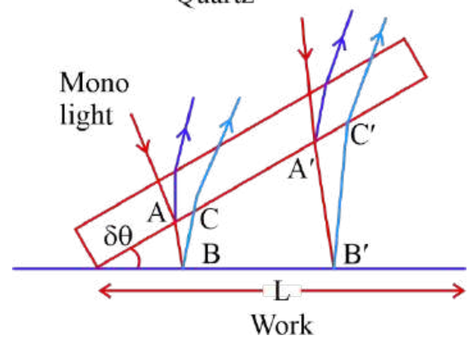


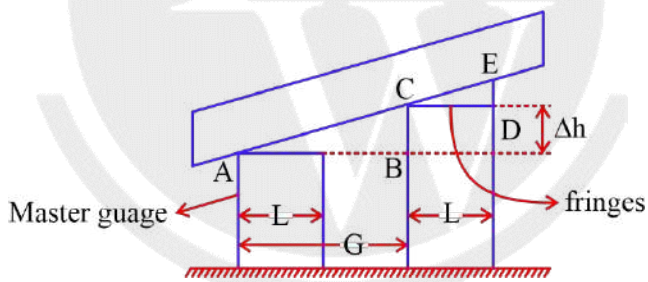
Fig. 6.24 Optical Flat over a Surface

6.18 Optical flat as a Comparator

ΔABC and ΔCDE are similar

$$\boxed{De = \frac{n\lambda}{2}}$$

$$\boxed{\Delta n = \left(\frac{n\lambda}{2}\right)\left(\frac{G}{L}\right)}$$



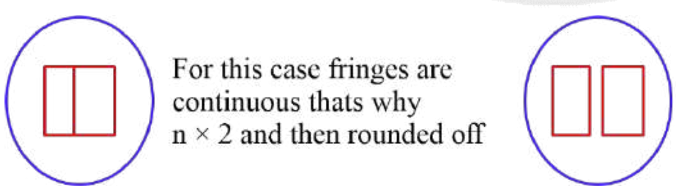


Fig. 6.25

$$\text{Parallelism error} = \frac{h_1 - h_2}{2}$$

Straightness

(1) Straight edge

(2) Sprit level

(3) Auto collimator $(\Delta s = 2f\delta\theta)$, f = focal length

6.19 Comparators:

- Mechanical
  - Sigma comparator
  - Mikrokator
- Pneumatic
  - Rota meter
  - Differential type
  - Back pressure type

Fig. 6.25

Area of control orifice $\quad C = \frac{\pi}{4} dc^2$

Area of measuring orifice $\quad M = \pi d_m \ell$

Characteristic equation $\quad \boxed{\frac{p}{P} = A - b\left(\frac{M}{C}\right)}$

With A, B, continue for higher P/p the value of M/C will be less.

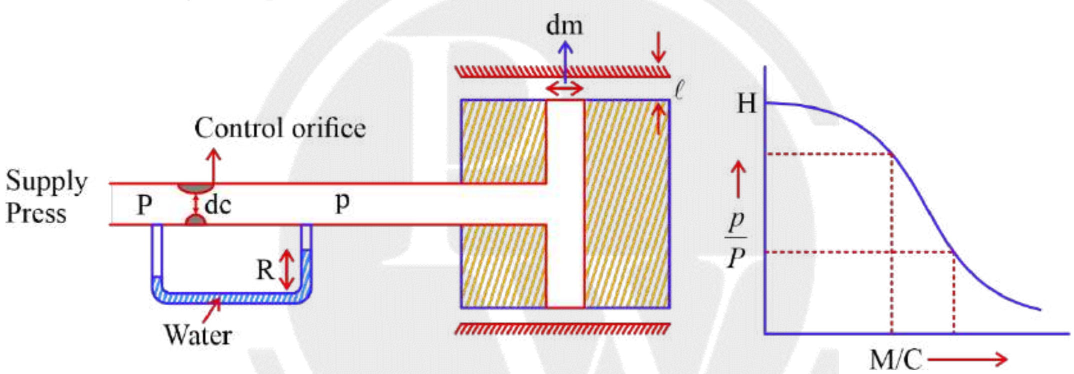


Fig. 6.26

$$M_{max.} - M_{min} = \frac{2}{3} M_{in.} = \text{Range}$$

$$M_{avg.} = \frac{4}{3} M_{min} - 2 \text{ Range}$$

$$\frac{dp}{dM} = \frac{0.4P}{M_{avg}} \quad \text{Pneumatic sensitivity}$$

Magnification factor $= \frac{\text{Output}}{\text{Input}}$

$$= \underset{\substack{\downarrow \\ \text{Indicator} \\ \text{sensitivity}}}{\frac{dp}{dP}} \times \underset{\substack{\downarrow \\ \text{Pneumatic} \\ \text{sensitivity}}}{\frac{dp}{dM}} \times \frac{dM}{d\ell} \rightarrow \text{Measuring head sensitivity} \begin{bmatrix} M = \pi d_m \ell \\ \frac{dM}{d\ell} = \pi d_m \end{bmatrix}$$

- Offset in tail stock offset for producing taper

$$\text{Offset} = \ell \frac{\sin\alpha}{2}$$

❑❑❑

7 ADVANCE MACHINING METHOD

7.1 ADVANCE MACHINING METHOD

- **Numerical Control (NC):** Numerical control is defined as the form of programmable automation, in which the process is controlled by the number, letters, and symbols. In case of the machine tools this programmable automation is used for the operation of the machines.
- **Computer Numerical Control (CNC):** In CNC machines programs are fed in the computer was used to control the operations of the machines. Thus, the control unit used that would read the punched cards in the NC machines was replaced by the microcomputer in the CNC machines.

7.2 CNC Principles

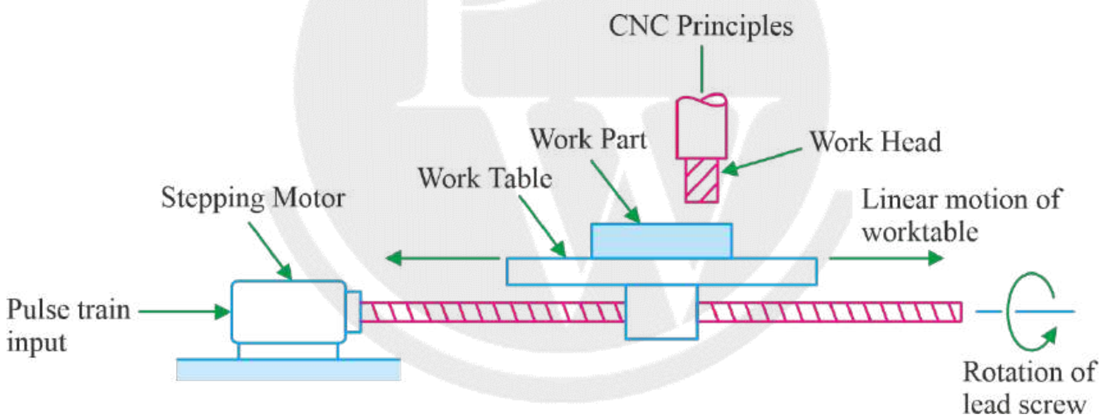


Fig. 7.1

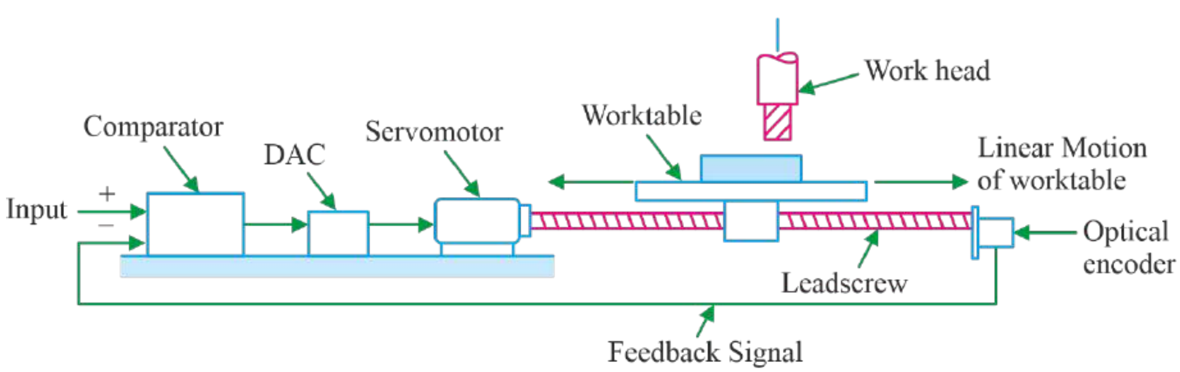


Fig. 7.2

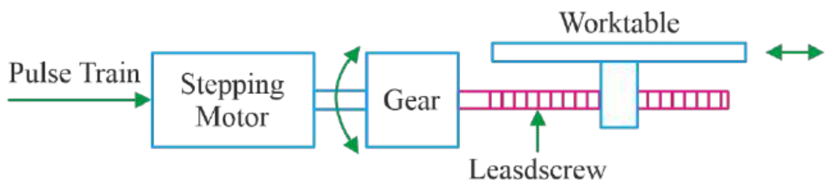


Fig. 7.3

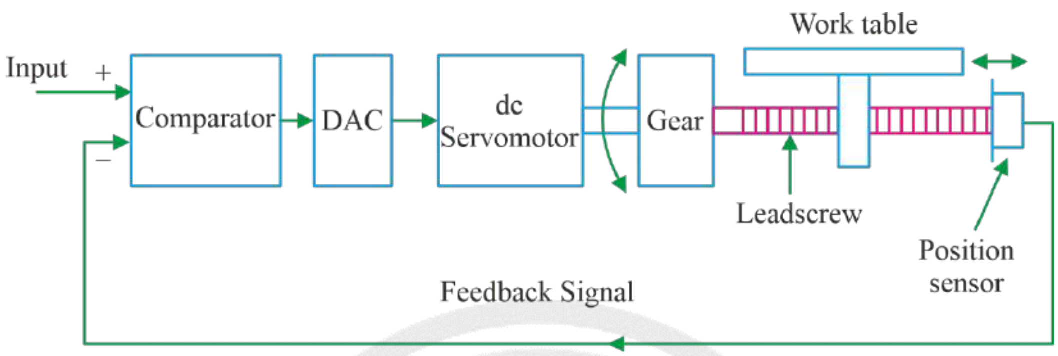


Fig. 7.4

7.3 Types of Motor

- Special motors called servos are used for executing machine movements in closed loop system.

Motor type can be

- AC servos,
- DC servos,
- Hydraulic servos.
- The speed depends on amount of current or hydraulic fluid passing through it.
- Servos are connected to the spindle and they are connected to the machine table through the ball lead screw

7.4 Basic Length Unit (BLU)

In NC machine, the displacement length per one pulse output from machine is defined as a Basic Length Unit (BLU).

(I) $$\text{BLU} = \frac{\text{Pitch of lead screw}}{\text{Number of pulse required for one rotation}}$$

(II) $$\text{BLU} = U \times n \times P \times N$$

$$U = \text{Gear ratio, } n = \text{No. of starts of lead screw}$$

$$P = \text{Pitch of lead screw, } N = \text{No. of revolutions/step}$$

(III) $$\text{Frequency of pulse} = \frac{\text{Table speed}}{\text{BLU}}$$

7.5 Types of tool positioning

7.5.1 Incremental Positioning

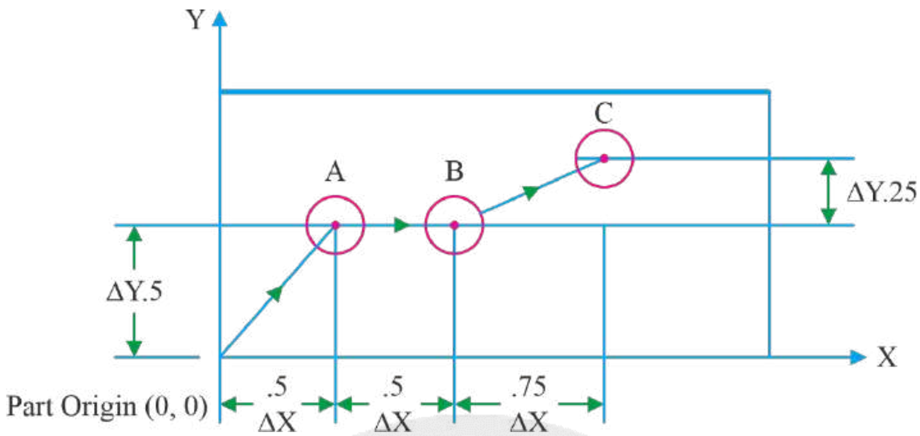


Fig. 7.5

Tool Position	Location	
	ΔX	ΔY
A	.5	.5
B	.5	0
C	.75	.25

7.5.2 Absolute Positioning

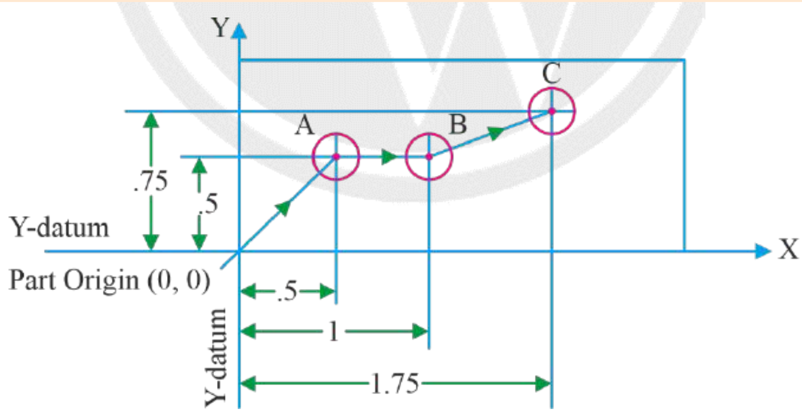


Fig. 7.6

Tool Position	Location	
	ΔX	ΔY
A	.5	.5
B	1	.5
C	1.75	.75

7.6 Automation

Japanese	American
Family	Hire & Fire
Puu	Push
Poka Yoke (O Defect)	Inspection
JIT (Just In The)	Inventories

(i) Routing is the path followed by any raw material inside the production system

(ii) Scheduling in which product will be processed on which machine and within what time period.

(iii) Dispatching is an activity that triggers the production it involves issuing of raw materials tools & sub-assemblies to shop floor.

7.7 Aggregate Planning (Short Forecast)

It is an analysis of how to best balance the total available resources against the demand expected. There are 4 option in the hand of aggregate planner.

(i) Overtime

Demand (Expected) can be achieved by making our existing staff to do overtime. But this option is having two problem.

(a) Worker efficiency less &

(b) As per (ILO) (International Labour organisation) During overtime workers' wages should be doubled.

(ii) Vary the Work Force

If the expected demand is high. I can operation two shifts rather than one for the second shift we can hire separate set of workers. But as per ILO we cannot fire the workers we want.

(iii) Sub Contracting

Although this option looks easy but by subcontracting we are losing our profit marking because it is your product which is selling.

(iv) Building Inventory

The period during which over demand is less will manufacture my products & put those products into my stores whenever there is excess demand. I will release inventory from the stores. But inventory not only block the working capital but also, we have to spent money in maintaining inventory in the stores.

7.8 Master Production Schedule (MPS)

Outcome of aggregate planning is master production schedule. It tells you how many & when final products will be ready for shipments.

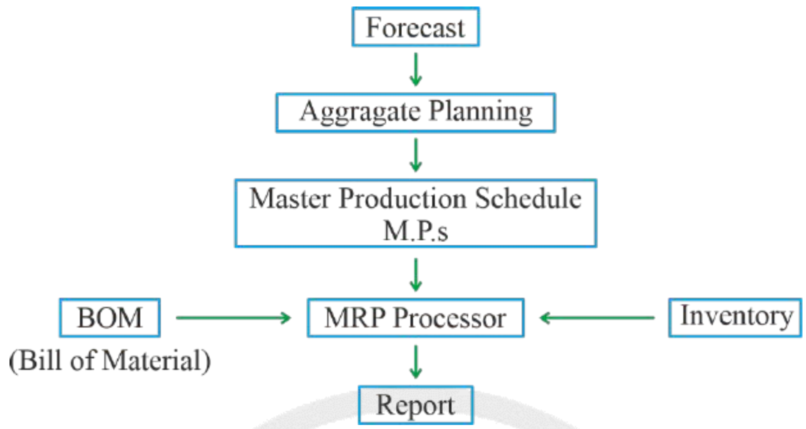


Fig. 7.7 MPS

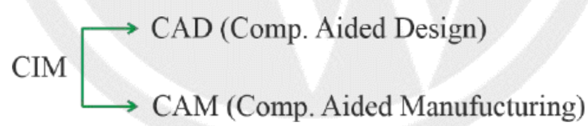
MRP - II : Finance + Marketing + MRPI

(MRP - I): Materials Req^n planning

MRP - I is a computational technique that coverts master schedule for end products into detailed scheduled for raw material & component used in the final product.

Manufacturing Resource planning (MRP – II) is having broader meaning than (MRP – I) it involves all the function of MRP – I & also includes marketing & finance division of company.

7.9 Computer Integrated Manufacturing (CIM)



7.9.1 CAD

Computer Aided design (CAD) can be defined as any design activity that involves effective use of computer to create, modify or document an engineering design. Its uses are

(1) To increase the productivity of designer.
(2) To improve quality of design.
(3) To improve design documentations.
(4) To create manufacturing database.

Finite element Tools

ANSYS
LSDVNA
DEFORM

- Re-engineering is relooking at our own design for betterment & reverse engineering is developing the design from somebody eyes product.

- In reverse engineering we are extracting the data from product & developing a part program that means generating cutter location data (CL Data) & from that we can produce component is mass.
- Any finite element tool will have b-divisions i.e. Preprocessor, solver, Post processor.
- Pre-Processor is almost similar to Autocad or Pro-E in which we are design the product
- Preprocessor does additional function to discretize (divide) the geometry into finite number of element solver solves the governing eq^n element by element. This governing eq^n given by us is preprocessor.
- Post processor is to view the result of solver.

7.9.2 CAM

Computer Aided Manufacturing is a effective use of computer in planning, management & control. Like Computer Aided process planning (Routing, scheduling, dispatching), Computer assistance line balancing, Computer Aided Part programming (CAPP) & virtual manufacturing virtual manufacturing means mimicking the O/P of shop floor over the computer screen.

CIM is having broacher meaning than CAD & CAM. CIM includes all the functions of CAD/CAM & it also includes all the other business function of firm & sometime it also includes the field support.

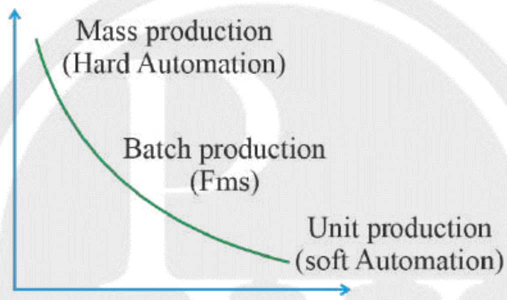


Fig. 7.8

7.10 Flexible Manufacturing System (FMS)

Requirements

(1) NC / CNC
(2) DNC
(3) Automatic Guided Vehicle (AGV)
(4) Simulator support.

When the company is offering frequent design change of the product it is called paperless automation or soft automation. Hard automation is the automation governed by can & follower arrangement & it is means for mass production. Flexible manufacturing system (FIMs) is meant for batch production & in this there is quick & inexpensive change in the design. To implement any FIMs in any unit following are requirements.

(i) All the machines in the unit should be either NC / CNC

(ii) There has to be DNC system (Direct/Distributed NC) DNC is the main frame computer (server). It is not directly involved in manufacturing but it controls no. of NC/CNC installation in plants. Highest level of automation is required in DNC.

Process Planning – Routing, dispatching, scheduling.

(iii) Material handling is done by Automatic Guided Vehicle.

(iv) Simulation support to provide for identifying the bottleneck system.

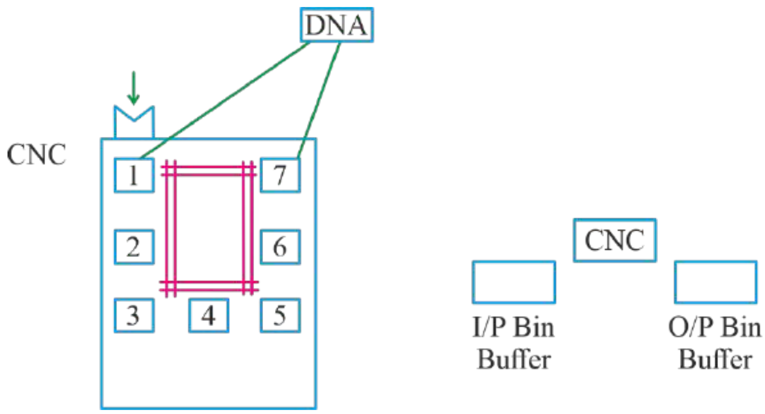


Fig. 7.9

All the NC/CNC are connected with the help of track & AGV is moves over the track. Track is segmented for purpose of Collison control. Each & every CNC will have i/p & o/p Bins & these bins have fixed no. of stations. Robot is working on each & every machine & as per Part entry control strategy robot will pick up raw material from i/p bin & placed it on machining centre. After machining the same robot will pick up semi-finished product & place it in the o/p bin. AGV will pick up the semi finish product from o/p bin of any CNC & placed it in input bin of anyone CNC as per routing.

In CNC program is developed by taking cutter centre as the reference but in machining no. of tools will be used having different diam. & diff. lengths. It is the cutter compensation that Co-ordinates this variation in the program. To minimize the processing time if FMs in number of products are grouped together according to the similarity in design & manufacturing. When this group is presented to FMs unit it decreases the process time because tool changing time will decrease & also lesser time will be consumed in cutter compensation. No. of machines involved in processing of particular group is caused a 'Cell'. Outcome of group technology is cellular manufacturing a cell can be a single CNC or group of CNC's or the entire FMs unit can be cell.

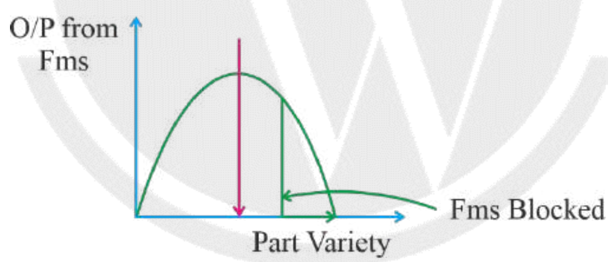


Fig. 7.10

By increasing the part variety output of FMs system also increases but any FMs system can handle maximum part variety. When the part variety exceed beyond some value blocking will start DNC system should be so sensitive to recognize the blocking in beginning otherwise entire FMs system will be blocked.

7.11 NC/CNC

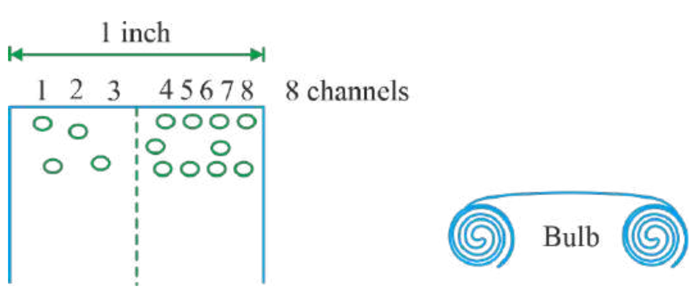


Fig. 7.11

In CNC machine program is street in hard disk & if there is any change in the design it can be add it & use in NC machine program is stored in the Punched Tape & magnetic cassette. If there is a change in a design a separate tape needs to be prepared tape is made out of paper called myler or myler couted plastic. There are 8 channels in the tape & program is punched in binary format below the tape there is bulb & above the tape is a photodiode for each channel.

So, when there is the hole as per the program, through that hole light came out & when the light falls a photodiode signal will be generated.

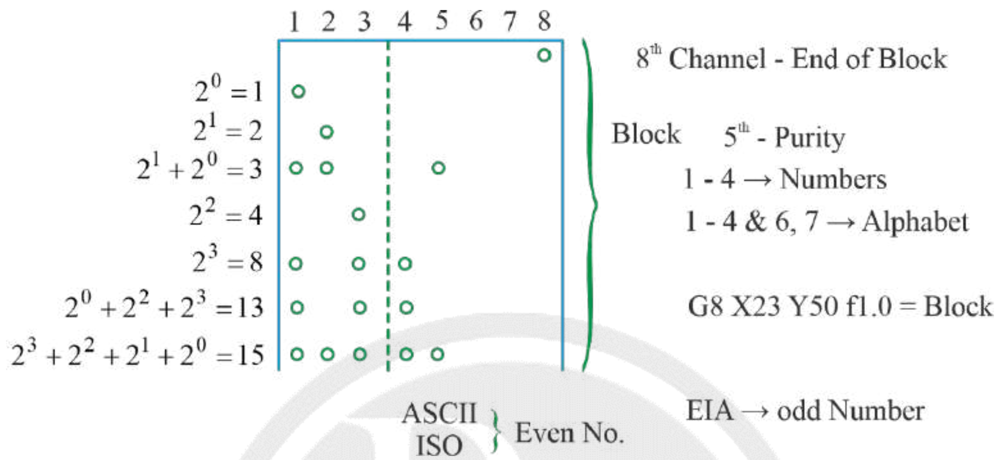


Fig. 7.12

Each line of program corresponds to one block i.e. one piece of information that has to be executed by the machine. But in case of tape it requires multiple rolls to punch the program so 8th channel in tape is reserve for end of block. Program on tape is punch in the binary format.

While punching the program we may experience & difficulties

(i) Removed material may come back & closes the hole,

(ii) Operator, by accident may touch the tape with only hands making it transparent.

(iii) By the repeated use sometime crack may develop bet the two consecutive holes because of these three things a wrong signal may be communicated with the

To take care of this problem there is parity & as per EIA parity is off add number of holes. As per the program if there are even no. of holes in any line we punch additional hole in 5th channel to make the number odd. Before executing the program, tape runs very fest in controller to check this parity & this is called tape proving.

EIA – Electronic Industry Ass.

ASCII – American society code for Info. Interchange

ISO – International organization of standardization.

7.12 Types of Loop System

7.12.1 Close loop system: (Heavy Machine)

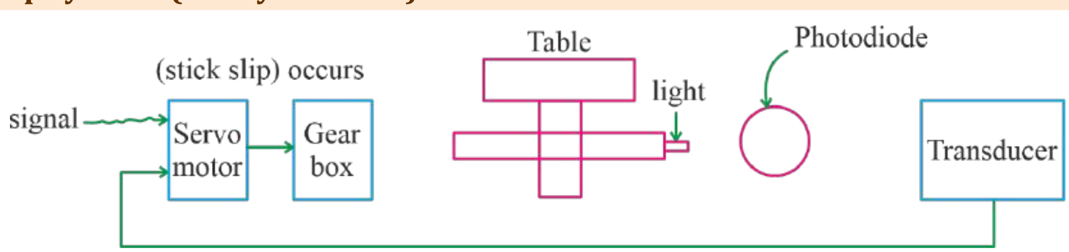


Fig. 7.13

7.12.2 Open loop system: (Small, Medium)

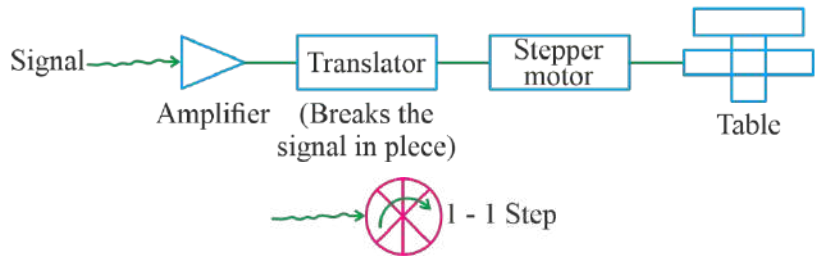


Fig. 7.14

Closed loop System

In this system, control is used transducer which acts as a feedback device. It records the moment of lead screw & if there is any difference bet input & output then electrical drive coil adjust it. This difference may be due to stic slip or backlash.

Open loop System:

This system is used in small & medium capacity machine once the signal is amplified the translator breaks the signal into many pukes. There is small & stepper motor & when 1 pulse received by motor it advanced by 1-slot. There is no stick-slip so feedback is not required.

n_s = no. of slots on stepper

Step angle $(\propto) = \dfrac{360}{n_s}$

P = No. of pulses received by motor

f_P = freq. of pulses by translator.

t = time

$P = f_P \times t$

degrees stepper will move

$= P \bullet \propto$

Degrees $= f_P \bullet t \bullet \propto$

$$\text{RPM of stepper} = \left(\frac{f_P \times \frac{360}{n_s}}{360} \right) \times t \times \frac{60}{t} RPm$$

$$\text{RPM of stepper} = \frac{60 f_P}{n_s}$$

Linear speed of table = rpm × pitch

$$= \text{RPM} \times \text{Pitch}$$

$$= \frac{rev}{min} \times \frac{mm}{rev} = \frac{mm}{min}$$

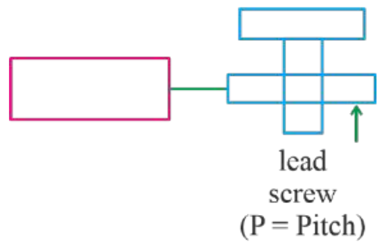


Fig. 7.15

Table resolution (Basic length unit (BLU)

BLU is minimum table movement that can be given to the machine. Corresponding to one pulse received by the motor the minimum advancement of table is called table resolution.

7.13 Interpolation

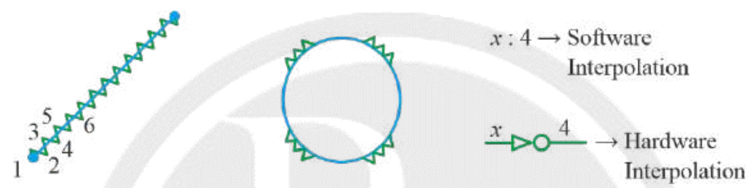


Fig. 7.16

7.13.1 Point to point system:

Interpolation means control of feeds. In PTP system the large number of points are defined around a geometer by either moving x or y variable & the tool moves from one point to another point. Incremental steps are so small that during machining we will not even recognize. When values of these x, y co-ordinates are presented in the form of table & store in the program it is called software interpolation. But where a logic gate relates the value of x & y it is called hardware interpolation. PTP system unnecessarily increases the length of the program. Even today we used PTP system but only in positioning. In PTP system tool moves between path & maximum available speed on the controller.

7.13.2 Linear Interpolation (L – system)

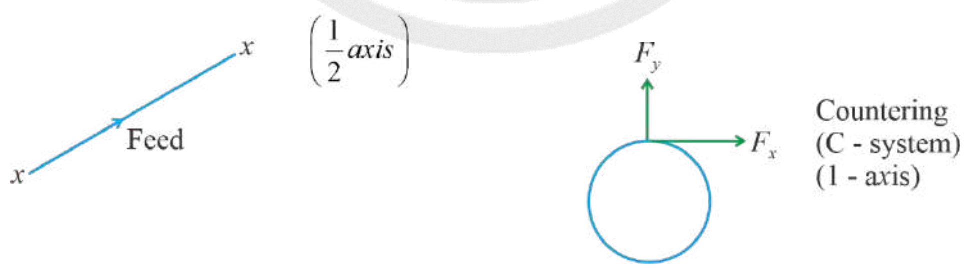


Fig. 7.17

If the machine is having capacity to control feed in 1-direction it is called linear interpolation. L system. When the machine capable to control bi-axial feed in plane it is called countering or C system.

Drill m/c $\rightarrow \frac{1}{2}$axis

Lathe m/c $\rightarrow 1\frac{1}{2}$axis

Milling m/c $\rightarrow 2\frac{1}{2}$axis

In drilling m/c feed can be controlled only in vertical direction so it is half axis machine. In lathe machine, by simultaneous moment of carriage & cross slide one contour can be cut into horizontal plane. Simultaneously, feed can be given by the tail stock. So, it is $1\frac{1}{2}$ axis machine. In case of milling machine contour can be cut in the horizontal & Vertical plane & simultaneous one feed can be given in the vertical direction so it is $2\frac{1}{2}$ axis machine. Higher axis machine can do all the functions of lower axis machine. At present even 12-axis m/c are also available.

7.14 Machining Centre

Any CNC machine having certain facilities is called a machining centre.

(1) Automatic tool changer (ATC)

(2) Automatic Pallet Changer (APC) [pallet-additional table]

(3) Tool transfer in ATC from tool magazine.

(4) Machining centre should be at least $2\frac{1}{2}$ axis .

Horizontal Machining Centre → 20 Tools
→ Movement to tool

Vertical M.C. → 40 Tools
→ Movement to tool

7.15 Part Programming

G – Code
M – Code

7.15.1 G – Code:

G00 – PTP positioning.

G01 – Linear Interpolation.

G02 – CW Circular

G03 – CCW Circular

G04 – Dwell (sharp corner, spot facing etc.)

G 90 – Absolute Programming

G 91 – Incremental Programming.

G 92 – Absolute Preset. (at first line program) as showing this where you are.

In absolute program position of origin doesn’t change in Incremental programming lost machining spot becomes the origin for next matching spot.

Absolute preset is a declaration statement and it tells the machine that at which co-ordinate machine is present in the beginning of executing any program. G 92 will always appear in the first line of program.

G17 – XY

G 18 – XZ

G 19 – YZ

Canned Cycle:

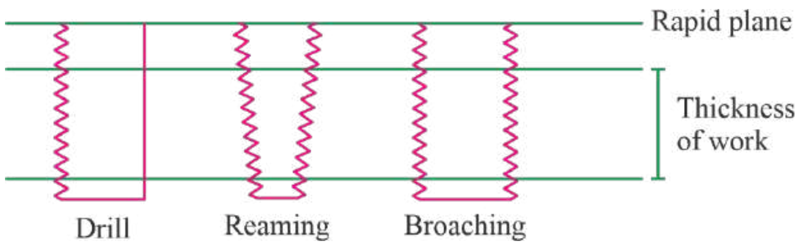


Fig. 7.18

G 81 – Drill

G 85 – Reaming

G 82 – Counter boring

G 86 – Cancel Cannel cycle.

Canned Cycles are like sub routines where sequence of events is represented by a single code. The moment canned cycle is activated. Wherever the tool is from their it will come to rapid plane with the maximum possible speed available on the controller. Feed will start from the rapid plane & after machining tool will come back to rapid plane & wait for the next instructions before executing any other command Canned Cycle needs to be cancel.

7.15.2 M Codes:

$\left.\begin{array}{l}\text{M00}\\\text{M01}\end{array}\right\}$ Optional stops (Inspection)

M02 Last line stop

M02 – Will appear in the last line of program after reading M02 program will be completely terminate.

M00, M01 – are called optional stops & are meant for inspection. If M 00 present in any block, after executing that black m/c will stop. There will be a switch on the controller. When it is present m/c will start executing the instruction from next line on wards.

There will be another switch over controller when it is on only then m/c will stop after reading M 01 otherwise machine will ignore the instruction.

M03 – spindle on CW

M04 – spindle on CCW

M05 – spindle OFF

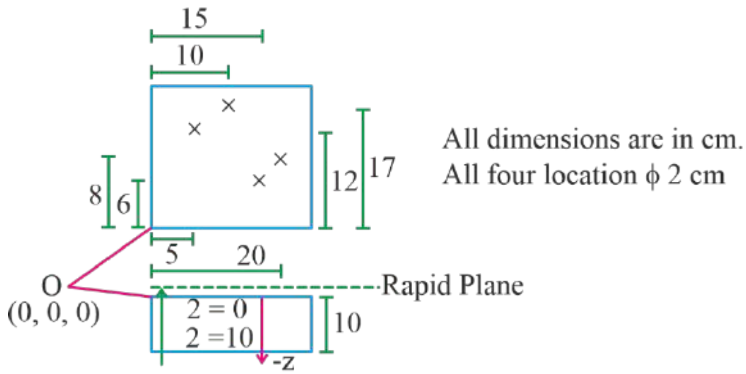
M06 – Tool change

M07 – Coolant No.1 ON

M08 – Coolant No. 2 ON

M09 – Coolant OFF.

Ex.



'Word address format'

Fig. 7.19

```
N01    G92   X0    Y0    Z0
N02    G90   G81   X50   Y120  Z13   R2    M03   M07   S400  F10
N03    X100  Y170
N04    X150  Y60
N05    X200  Y80
N06    G80   G00   X0    Y0    Z100  M05   M09
N07    M02
```

7.16 APT (Automatic Program Tool)

Fig. 7.20

- Geometry Commands
- Auxiliary Statements
- Motion Command

Point:

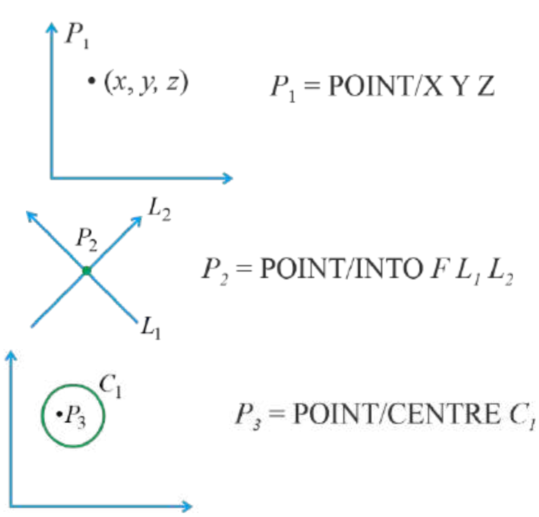
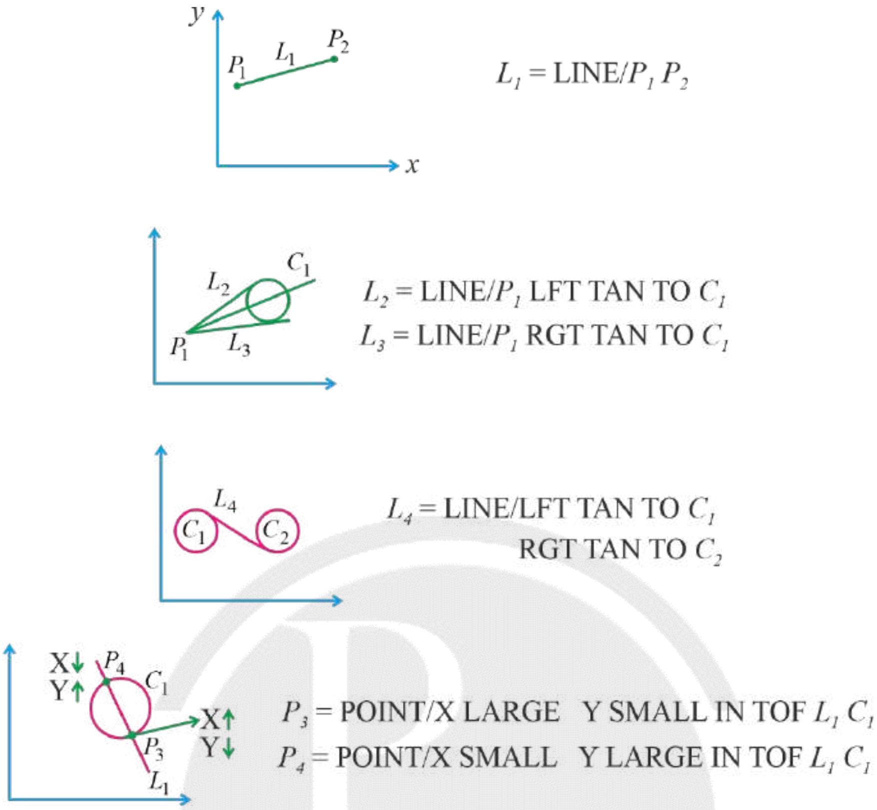


Fig. 7.21

Line:



Circle:

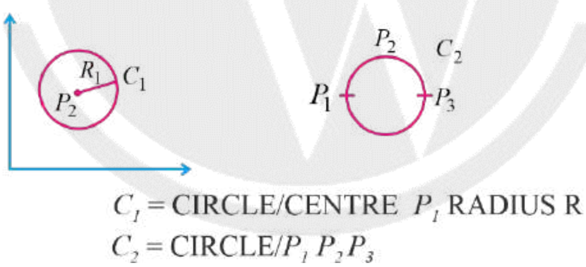


Fig. 7.22

Plane:

$ax + by + cz + d = 0$

P_1 = PLANE/$a\ b\ c\ d$

P_2 = PLANE/ $P_1\ P_2\ P_3$

Auxiliary Statements

Spindle	SPINDL / ON 500 RPM CCWD
Coolant	COLANT / ON
Feed rate	FEDRAT / 2.0 MMPR

Auxiliary Statements

FROM / $P_1 \rightarrow$ G92

GOTO / $P_2 \rightarrow$ Absolute Prog.

GO DLTA/$P_6 \rightarrow$ Incremental

M02 $\rightarrow$ FINI

GO RGT

GO LFT

GO UP

GO DWN

❑❑❑

8 MACHINE TOOL

8.1 Lathe Machine

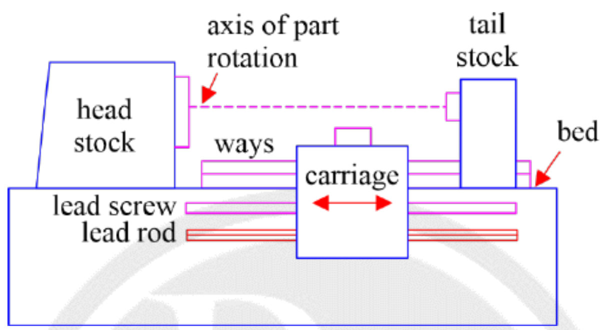


Fig. 8.1 Lathe Machine

- Number of Spindle Speed

Number of spindle speed is in a **geometric progression.**

$$N_1, N_1r, N_1r^2, N_1r^3, ...N_1r^{n-1}$$

$$N_1 = N_{\min} \quad and \quad N_1r^{n-1} = N_{\max}$$

Therefore, **Step Ratio** $(r) = \left(\frac{N_{\max}}{N_{\min}}\right)^{\frac{1}{n-1}}$

Where,

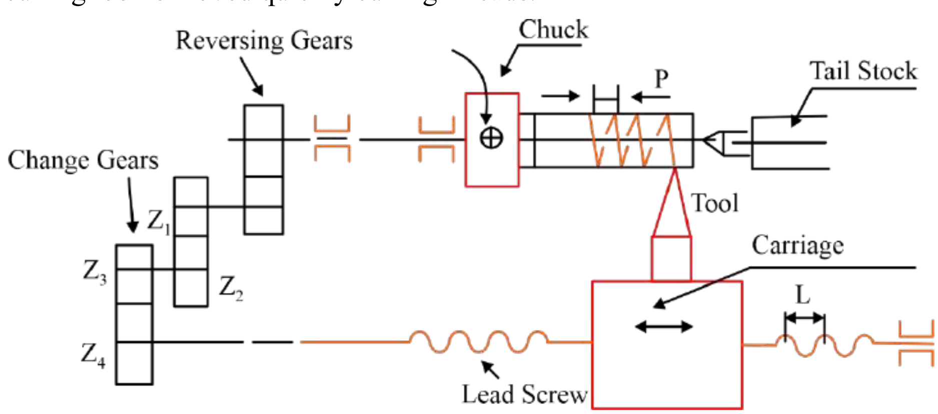
n = number of spindle speed

N_1 = minimum speed

N_2 = maximum speed

8.2 Threading

Threading - The cutting tool is moved quickly cutting threads.



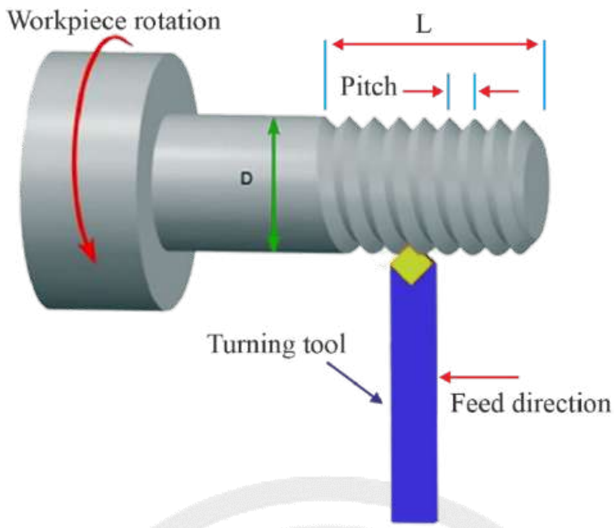


Fig. 8.2 Thread Cutting

In one revolution of the spindle, carriage must travel the pitch of the screw thread to be cut.

$N_s P z_s = N_L L z_L$

P = Pitch of the screw thread to be cut

L = Pitch of the lead screw

z_s = Number of start of the screw thread to be cut

z_L = Number of start of the lead screw

i_{cg} = gear ratio of spindle (N_s) to carriage (N_L) gear train

8.3 Turning

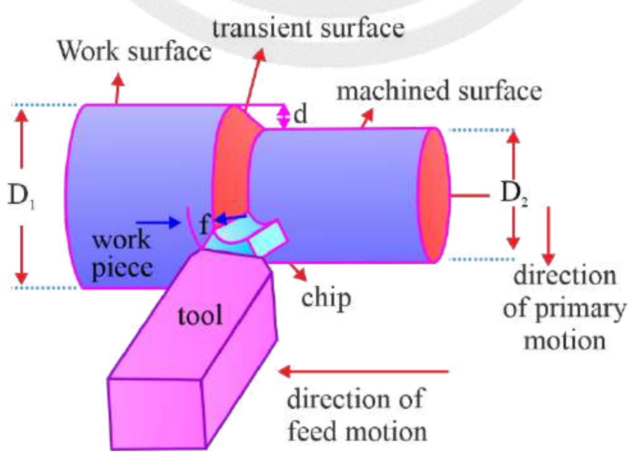


Fig. 8.3 Turning process

Formula for Turning

- Depth of cut, $d = \frac{D_1 - D_2}{2} \ mm$

- Average diameter of workpiece $D_{avg} = \frac{D_1 + D_2}{2}$ mm
- Cutting Time, $= \frac{L + A + O}{fN}$
- Cutting Speed, $V = \frac{\pi D_1 N}{1000}$, m / min

8.3.1 Facing

Facing is the process of removing metal from the end of a workpiece to produce a flat surface. Most often, the workpiece is cylindrical

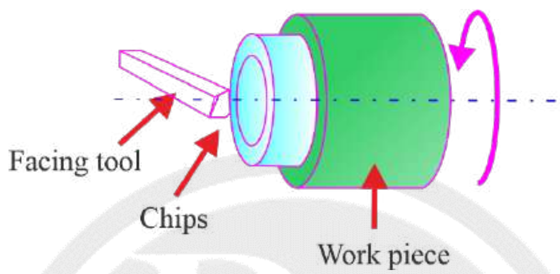


Fig. 8.4 Facing operation

8.3.2 Turning Tapers on Lathes

(a) Using a compound slide,

(b) Using form tools,

(c) Offsetting the tailstock, and

(d) Using taper turning attachment.

Note:

- Compound slide can be employed for turning short internal and external tapers with a large angle of (steep) taper.
- Form tool method is useful for short external tapers

8.3.3 Compound Slide formula

- The angle is determined by $\tan\alpha = \frac{D-d}{2l}$

$\alpha = Half\ taper\ angle$

$D = Diameter\ of\ stock$

$d = smaller\ diameter$

$l = length\ of\ the\ taper$

8.3.4 Offsetting the tailstock formula

- Tailstock offset (h) can be determined by

$$h = \frac{L(D-d)}{2l} \text{ or } h = L\tan\alpha$$

8.4 Drill

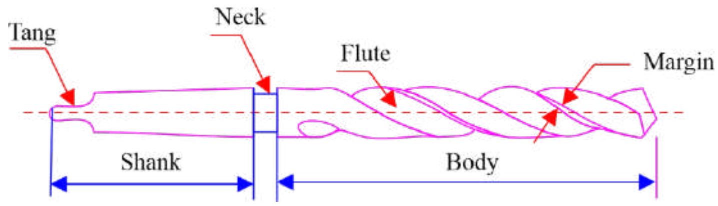


Fig. 8.5 Drill

Note:
- The helical flute in a twist drill provides the necessary Rake angle for the cutting edge and Space for the chip to come out during drilling.
- The rake angle in a twist drill varies from minimum near the dead centre to a maximum value at the periphery.

Cutting Speed (V) in Drilling

The cutting speed in drilling is the surface speed of the twist drill.

$$V = \frac{\pi DN}{1000} \text{ m / min}$$

Machining time (T) = $\frac{L}{fN} m/min$

$$\text{MRR} = \pi D^2 fN / 4$$

Where

D = Diameter of drill(mm)
N = RPM of drill
f = feed(mm/rev)
MRR = Material removal rate
$L = L_1 + L_2 + L_3 + L_4$
L_1 = Depth of hole
L_2 = Approach length
L_3 = Length of tip
L_4 = Over Travel

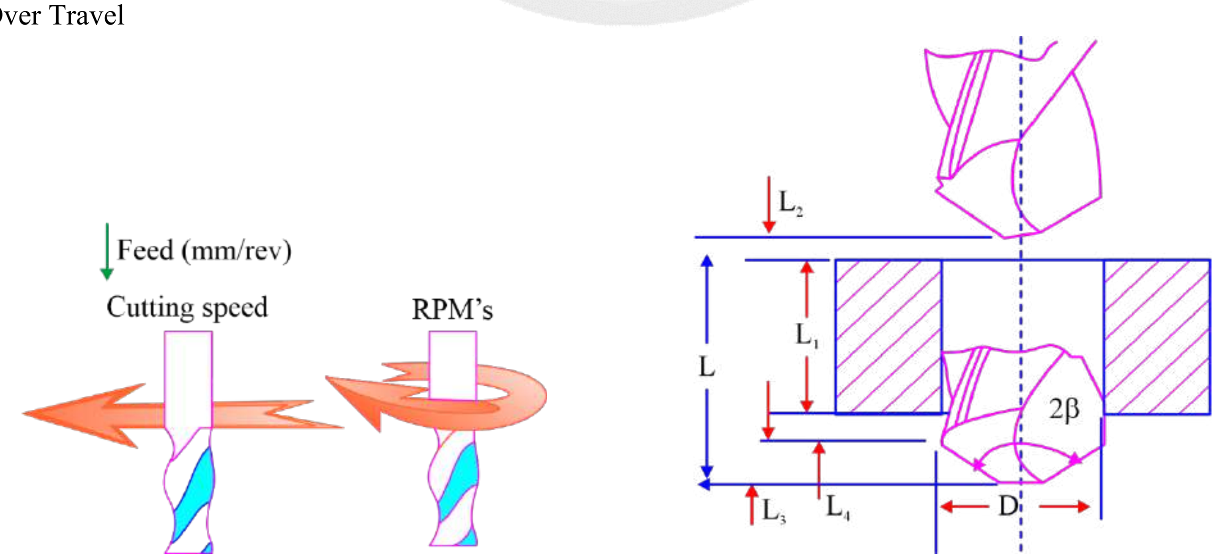


Fig. 8.6 Drilling process

8.5 Milling

Milling is a process performed with a machine in which the cutters rotate to remove the material from the work piece present in the direction of the angle with the tool axis. With the help of the milling machines one can perform many operations and functions starting from small objects to large ones.

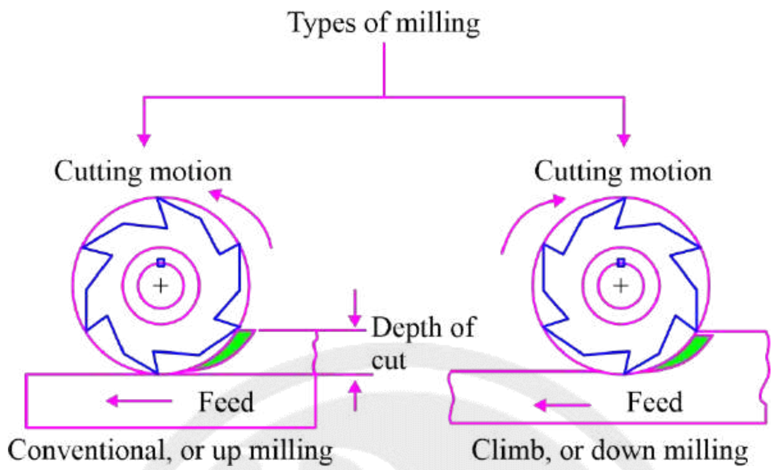


Fig. 8.7 Milling process

Speed of table (mm/min):

feed per tooth = f_z (mm/tooth)

Number of teeth = Z

$$\text{Feed} = f_z Z \frac{\text{-mm}}{\text{tooth}}$$

Speed of table(f) = Feed*N

Milling Velocity (V)

$$V = \frac{\pi DN}{1000}$$

Milling Time (T)

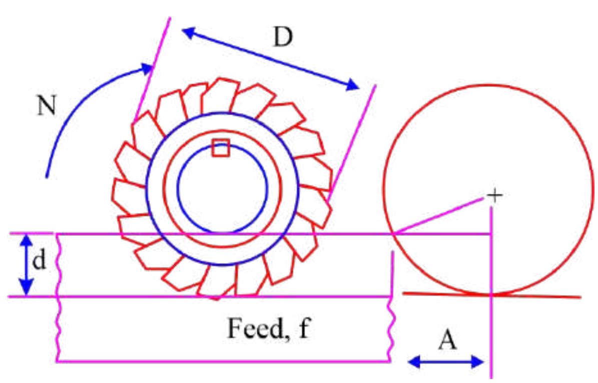


Fig. 8.8

- Time for one pass $= \dfrac{L + 2\times A}{f_z ZN}$ min
- Approach distance $= A = \sqrt{\left(\dfrac{D}{2}\right)^2 - \left(\dfrac{D}{2} - d\right)^2} = \sqrt{d(D-d)}$

$$f = f_z ZN$$

$$f_z = \frac{f}{ZN}$$

Maximum uncut chip thickness $= \dfrac{2f}{ZN}\sqrt{\dfrac{d}{D}}$

Average uncut chip thickness $= \dfrac{f}{ZN}\sqrt{\dfrac{d}{D}}$

Material removal rate (MRR) in Milling

$MRR = w \times d \times F$

where, w = width of cut, d = depth of cut   F = Feed of the table

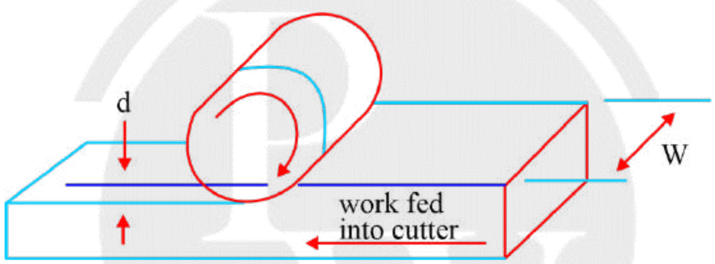


Fig. 8.9

8.6 Grinding

Grinding is an abrasive machining process that uses a grinding wheel or grinder as the cutting tool. Grinding is a subset of cutting, as grinding is a true metal-cutting process

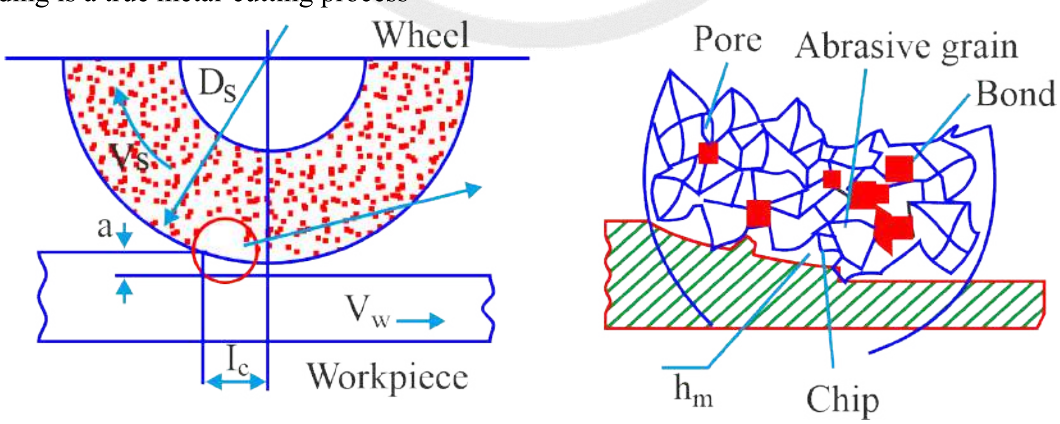


Fig. 8.10 Grinding process

Note:
- Among the conventional machining processes, maximum specific energy is consumed in Grinding
- It is desired to offset the adverse effect of very high negative rake angle of the working grit, to reduce the force per grit as well as the overall grinding force.

(a) G Ratio

The grinding ratio or G ratio is defined as the cubic mm of stock removed divided by the cubic mm of wheel lost.

(b) Parameters for specify a grinding wheel

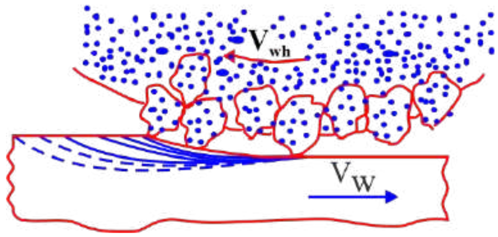


Fig. 8.11

(i) The type of grit material

(ii) The grit sizes

(iii) The bond strength of the wheel, commonly known as wheel hardness

(iv) The structure of the wheel denoting the porosity i.e. the amount of inter grit spacing

(v) The type of bond material

(vi) Other than these parameters, the wheel manufacturer may add their own identification code prefixing or suffixing (or both) the standard code.

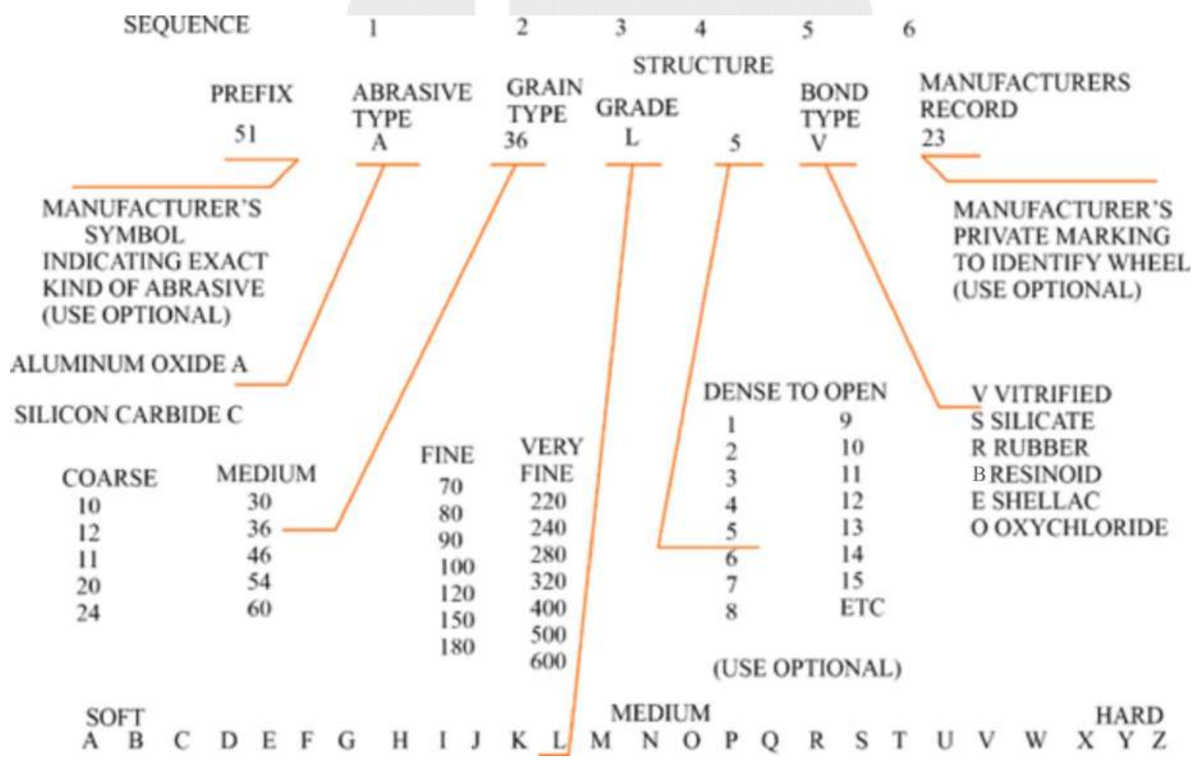


Table 8.1

(c) Creep feed grinding

This machine enables single pass grinding of a surface with a: larger down feed but slower table speed.

8.7 Lapping

- Lapping is basically an abrasive process in which loose abrasives function as cutting points finding momentary support from the laps.

8.8 Honing

- Honing is a finishing process, in which a tool called hone carries out a combined rotary and reciprocating motion while the workpiece does not perform any working motion.

8.9 Buffing

- Buffing is a polishing operation in which the workpiece is brought into contact with a revolving cloth wheel that has been charged with a fine abrasive, such as polishing rough.

Note:
Negligible amount of material is removed in buffing while a very high lustre is generated on the buffed surface.

8.10 Shaper

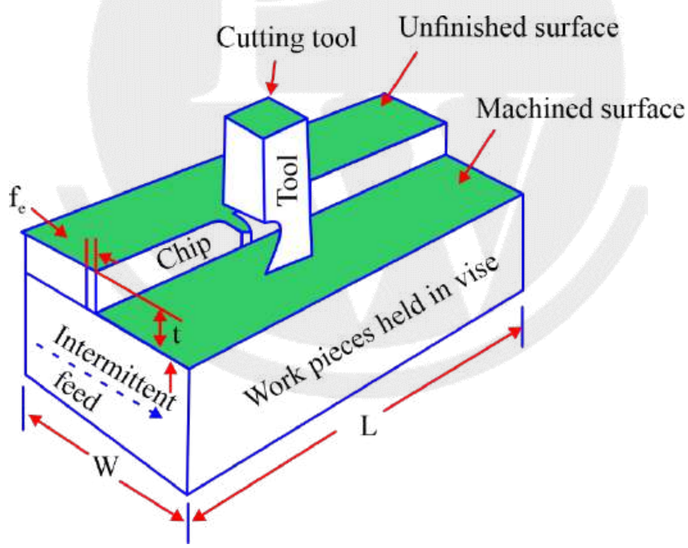


Fig. 8.12 Shaping process

Shaper Machine is a production machine in which the single point cutting tools are attached and the workpiece is fixed and while moving forward the tool cuts the workpiece and in return, there is no cut on the workpiece and used for producing flat and angular surfaces.

Formula

Cutting speed, $V = \dfrac{NL(1+m)}{1000}$

Number of strokes, $N_s = \dfrac{w}{f}$

Time of one stroke, $t = \frac{L(1+m)}{1000V}$ min

Total time, $T = \frac{L(1+m)}{1000v} N_s = \frac{Lw(1+m)}{1000vf}$ min

8.11 Planer

Planning can be used to produce horizontal, vertical, or inclined flat surfaces on workpieces that are too large to be accommodated on shapers.

8.12 Slotter

Slotting machine is basically a vertical axis shaper. Thus, the workpieces, which cannot be conveniently held in shaper, can be machined in a slotter.

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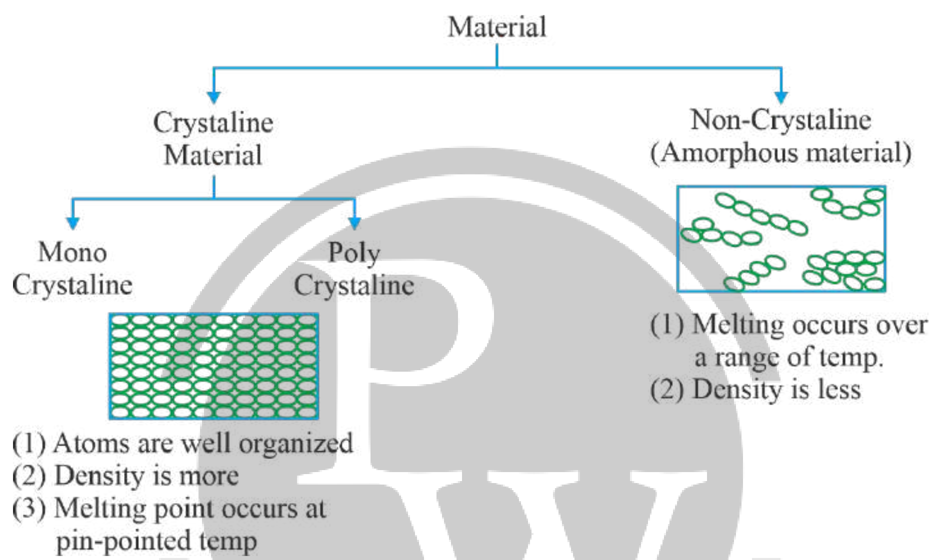
9 MATERIAL SCIENCE

9.1 Crystallography



Important terms:

(a) Unit Cell

A unit cell is defined as the smallest respresentative group of atoms which when repeated in all the crystallography direction for infinity no. of times result in the development of the crystal lattice.

(b) Crystal Lattice

It is a 3D-network of line in space also called as infinity lattice.

(c) Space Latice

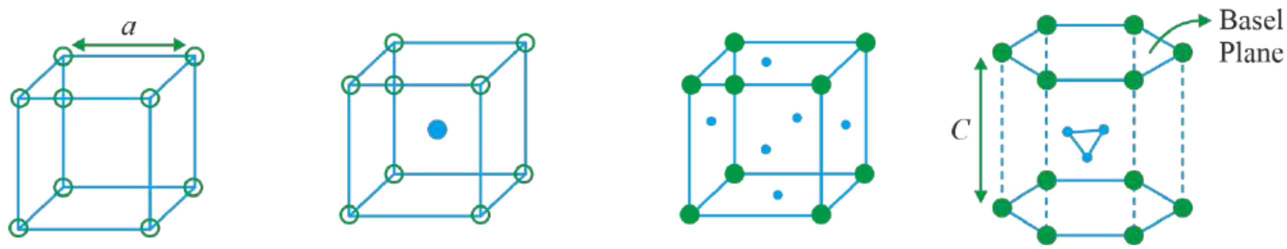
It is a 3-D network of point in space also called at point Latice.

(d) Puimitive Cell

It is smple cubic cell having atoms only at its corners. It is used to define other cubic cell.

(e) Allotropy

The tendency of any material exists in different structure at different temperature and press in known as allotropy of material.



(SSC)
$\text{vol}^w = a^3$
$r = \frac{a}{2}$

(BCC)
$\text{vol}^w = a^3$
$r = \frac{\sqrt{3}\,a}{4}$
Ex-Vanadium molybdenum tungsten chromium Iron (910°-1400°C) except

(FCC)
$\text{vol}^w = a^3$
$r = \frac{\sqrt{2}\,a}{4}$
Ex-copper, Al, Pb, Ag, gold, Ni etc

(HCP)
$\text{vol}^w = \frac{3}{2}\sqrt{3}$
$r = \frac{a}{2}$
$\frac{C}{a} = 1.63$

Fig. 9.1 Unit Cell

9.2 Crystal System

"7 crystal system which are explained in 14 brains lattice".

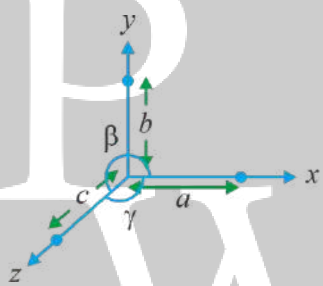


Fig. 9.2 Unit cell angles and length

a, b, c → Interstitial Length
α, β, γ → Interstitial Angle.

	Crystal System	Unit Cell	Nature of Unit Cell
1	Simple cube	SSC, BCC, FCC	$\alpha = \beta = \gamma = 90°$ $a = b = c$
2	Tetragonal	SCC, BCC	$\alpha = \beta = \gamma = 90°$ $a = b \neq c$
3	Ortho-rhombic	SCC, BCC, FCC, HCP	$\alpha = \beta = \gamma = 90°$ $a \neq b \neq c$
4	Monoclinic	SC, BCC	$\alpha = \beta = 90° \neq \gamma$ $a \neq b \neq c$
5	Tri Clinic	SC	$\alpha \neq \beta \neq \gamma \neq 90°$ $a \neq b \neq c$
6.	Rhombohedral (Trigonal)	SC	$\alpha = \beta = \gamma \neq 90°$ $a = b = c$
7	Hexagonal	SC	$\alpha = \beta = 90°,\ \gamma = 120°$ $a = b \neq c$

Table 9.1 Different types of crystal system

9.2.1 No. of Average atoms in a unit cell (N_{avg})

"The no. of atoms originally belonging to unit cell."

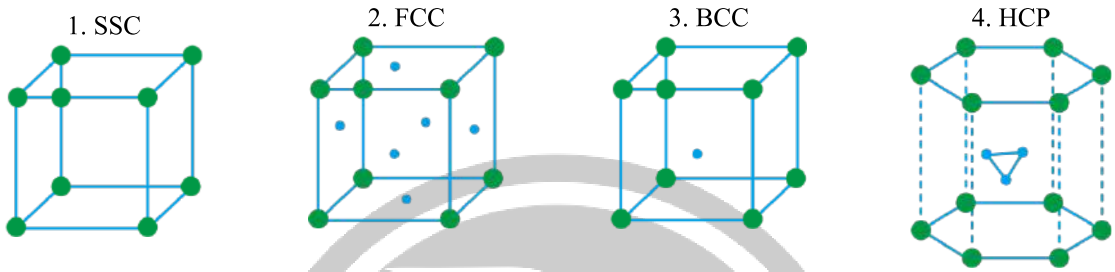
$$N_{avg.} = \frac{N_c}{8} + \frac{N_f}{2} + \frac{N_i}{1} \quad \textbf{(Cubic Cell)}$$

$$N_{avg.} = \frac{N_c}{6} + \frac{N_f}{2} + \frac{N_i}{1} \quad \textbf{(Hexagonal Packing)}$$

$N_C \rightarrow$ No. of atoms at corner.

$N_f \rightarrow$ No. of atoms at face centre

$N_i \rightarrow$ No. of atoms at body centre.



$N_{avg} = \frac{8}{8} = 1$

$N_{avg} = 1$

$N_{avg} = \frac{8}{8} + \frac{6}{2} + \frac{O}{1}$

$N_{avg} = 1 + 3 = 4$

$N_{avg} = \frac{8}{8} + \frac{0}{2} + \frac{1}{1}$

$N_{avg} = 2$

$N_{avg} = \frac{12}{6} + \frac{2}{2} + \frac{3}{1}$

$N_{avg} = \frac{12}{6} + \frac{2}{2} + \frac{3}{1}$

$N_{avg} = 6$

Fig. 9.3 Cubic Cell

9.2.2 Co-ordination No.

"It is defined as no. of nearest and equidistance atoms. surrounded considered atoms".

SCC-6, BCC-8

FCC-12, HCP-12

9.2.3 Atomic Packing factor:

"It is the ratio of volume occupied by avg. No. of atoms to volume of unit cell". "APF is index of density".

S. No.	Properties	SCC	BCC	FCC	HCP
1.	Volume	a^3	a^3	a^3	$\frac{3\sqrt{3}}{2}a^2$
2.	Avg. No. of Atoms	1	2	4	6
3.	Co-ordination No.	6	8	12	12
4.	APF	0.52	0.68	0.74	0.74
5.	Atomic radius	$r = \frac{a}{2}$	$r = \frac{\sqrt{3}}{4}a$	$r = \frac{\sqrt{2}}{4}a$	$r = \frac{a}{2}$

Table 9.2

The tendency of any element to exist in different structure at different pressure and temperature is known as **Allotropy**. In case of most of metals this tendency is thermodynamically reversible. This type of reversible transformation is also called as **polymorphism**.

Allotropic transformation is associated with change in volume and density.

(1) % change in density (BCC-FCC)

$$= \frac{APF_{BCC} - APF_{FCC}}{APF_{BCC}} \times 100$$

$$= \frac{0.68 - 0.74}{0.68} \times 100$$

$$= -8.82\%$$

(2) % change in volume

$$= \frac{2V_{BCC} - V_{FCC}}{2V_{BCC}} \times 100$$

$$= \frac{2 \times a_{BCC}^3 - a_{PCC}^3}{2 \times a_{BCC}^3} \times 100 = \frac{2 \times 2.3^3 - 2.8^3}{2 \times 2.3^3} \times 100$$

$$= 8.14\%$$

9.3 Crystal Plane (Miller Indices)

Miller Indices of a plane is defined as reciprocal of intercepts and written in a bracket without a separating, between them. It is smallest integers.

Steps follows for miller indices:

Step-I $\rightarrow$ Determine Intercepts

Step-II $\rightarrow$ Separate multiplies

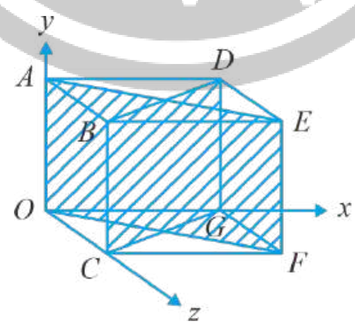
Step-III$\rightarrow$ Take reciprocal of multipliers

Step-IV $\rightarrow$ Convert it into smallest integer.

Note:
- If miller indices are negative then it is given be bar $(-1, \bar{1})$
- If the plane is passing through origin then, we have to shift the origin.

Plane BEFC (001)

	X	Y	Z
1. Intercepts	∞	∞	I
2. Multiplier	∞	∞	1
3. Reciprocal	$\frac{1}{\infty}$	$\frac{1}{\infty}$	$\frac{1}{1}$
4. Miller	0	0	1

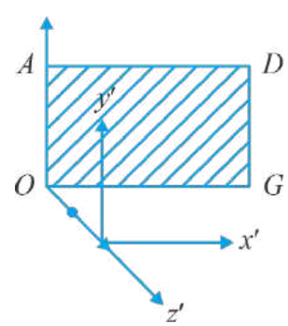


Plane ADGO (001)

x	y	z
∞	∞	-1
$\frac{1}{\infty}$	$\frac{1}{\infty}$	$\frac{-1}{1}$
0	0	$\bar{1}$

Plane BDGC (101)

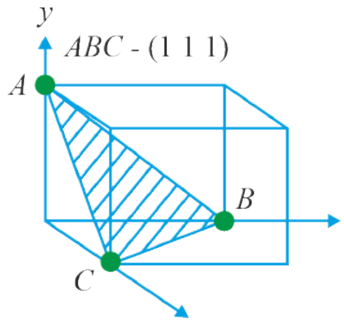
x	y	z
1	∞	1
$\frac{1}{1}$	$\frac{1}{\infty}$	$\frac{1}{1}$
1	0	1



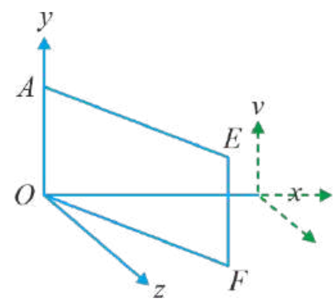
Plane ADEB (010)

x	y	z
∞	1	∞
$\frac{1}{\infty}$	$\frac{1}{1}$	$\frac{1}{\infty}$
0	1	0

Plane ABC – (111)

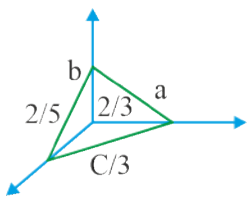


x	y	z
1	1	1
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$

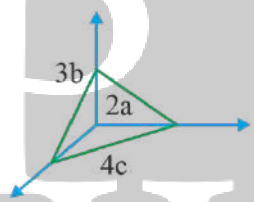


Plane AEFO

x	y	z
–1	∞	1
$\frac{-1}{1}$	$\frac{1}{\infty}$	$\frac{1}{1}$
$\bar{1}$	O	1



$\frac{3}{2}$	$\frac{5}{2}$	3
3	5	6



x	y	z
2	3	4
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
6	4	3

Fig. 9.4 Crystal Plane (Miller Indices)

- Parallel plane will have same Miller Indices Quantitively.
- Only arithmatic sing of non-zero indices will be different.

Interplanar Distance $= \dfrac{a}{\sqrt{h^2+k^2+l^2}}$ for miller indices (h, k, l)

Two Plane are inclined:

$(h_1 \quad k_1 \quad \ell_1)\ (h_2 \quad k_2 \quad \ell_2)$

$$\cos\theta = \frac{h_1h_2+k_1k_2+\ell_1\ell_2}{\sqrt{h_1^2+k_1^2+\ell_1^2}\cdot\sqrt{h_2^2+k_2^2+\ell_2^2}}$$

If two planes are parallel (θ = 90°)

$h_1h_2+k_1k_2+\ell_1\ell_2=0$

Linear density

$$\frac{\text{Number of atoms on any direction}}{\text{Length of direction}}$$

Simple cubic along $100 = \frac{1}{a}$

$$\text{FCC }[110] = \frac{2}{a\sqrt{2}}, \text{ BCC }[111] = \frac{2}{a\sqrt{3}}$$

Family of Direction: $< 100 > = 6 \,;\, < 110 > = 112 \,;\, < 111 > = 8$

Planar Density $= \frac{\text{No. of atoms on a plane}}{\text{Area of plane}}$

$$\text{SC along }(100) = \frac{1}{a^2}, \text{ FC along }(100) = \frac{2}{a^2}, \text{ BC }(110) = \frac{2}{\sqrt{2}a^2}$$

9.3.1 Slip System

Combination of crystal plane & crystal direction along which dislocation will move with an ease

FCC – 12 – Ductile

BCC – 24 – Strong

HCP – 3 – Brittle [Before dislocation, cracks come out]

With increase in slip system ductility increases but in case of BCC due to less atomic packing factor avg. dist. Between atoms decrease due to which dislocations move very difficult]

∴ BCC is always strong.

9.3.2 Critically Resolved shear stress

$\tau = \sigma \cos\phi \cos\lambda$

$\uparrow$

Applied stress

ϕ = angle between to slip plane & the applied stress.

λ = angle between slip & stress direction.

9.3.3 Mechanical Tests

Charpy – Toughness

Knoop – Microhardness

Spiral Test – Fluidity

Cupping Test – Formability

Jominy Test – Hardenability

$$\text{gm/atom} = \frac{\text{Atomic wt.(gm/mole)}}{\text{Avogadro's no.(atoms/mole)}}$$

9.4 Defects in Material:

Defects

- Point defect
  - Vacancies
  - Impurities
  - Interstitial
  - Frankel
  - Schottky
- Line defect
  - Edge Dislocation
  - Screw Dislocation
- Surface defect
  - Low tilt angle
  - Low twist angle
  - Twinning
  - Grain boundary
  - Stactcing fault
- Volume Defect

9.4.1 Point Defects

(a) Vacancies:

It occurs when there is an unoccupied atom site in the crysted structure.

Fig. 9.5 Vacancies Defect

(b) Impurities:

When foreign atoms substitute parent atom in the crystal structure or at interstitial space known as impurities defect.

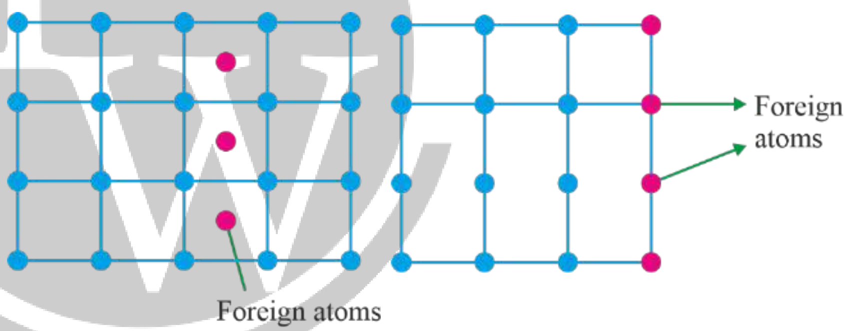


Fig. 9.6 Impurities Defect

(c) Interstitial:

A interstial defect occurs when an atom occupies a definite position in the lattice which is not normally occupied in the ideal crystal.

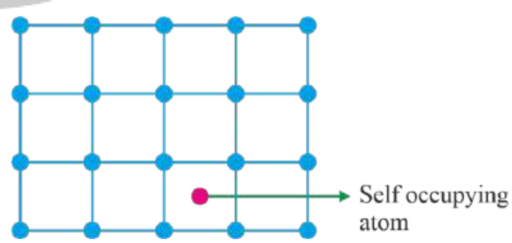


Fig. 9.7 Interstitial Defect

(d) Frankel

When an Ion displace from regular location in crystal lattice to an intertitial location in the crystal lattice. Ionic crystals have two different types of ions which are cation (+ve) and anion (-ve) cations are smaller ions which can easily displaced into the voids in Ionic solid.

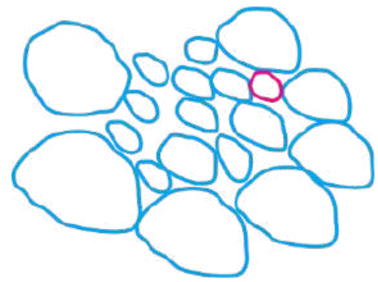


Fig. 9.8

(e) Effect of point defect:

- Strength and hardness can be in or eased due to lattice distortion.
- Electrical resistivity increases
- Phase transfmroation takes places more actively in the presence of point defect.

9.4.2 Line Defect

(a) Edge dislocation:

An edge dislocation is whre an extoal half plane of atoms is introudced midway to the crystal.

Burger Vector:

Burger vector is a vector which characterised the movement of dislocation in practice. Burger vector of ede dislocation is **perpendicular** to edge dislocation line.

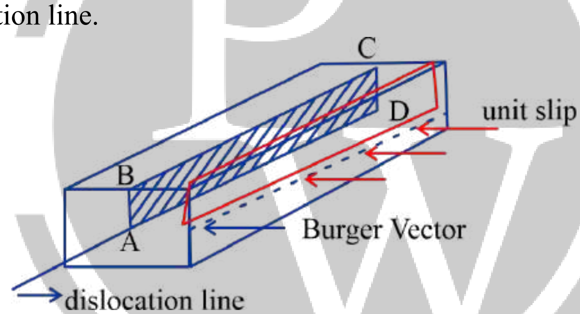
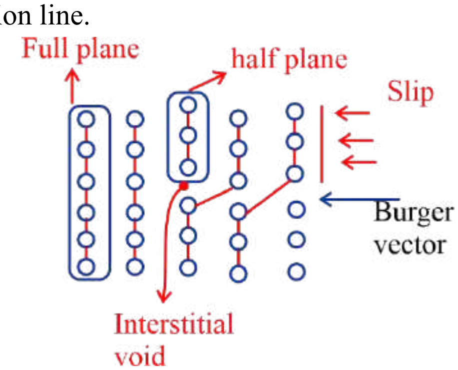


Fig. 9.9 Edge dislocation

(b) Screw dislocation:

It is formed when discreat atomic plane of truly crystaline solid into a surface of solid around dislocation this can be explained by cutting part way through a perfect crystal an then skewing the crystal 1-atom spacing.

Burger vector line **parallel** to dislocation line.



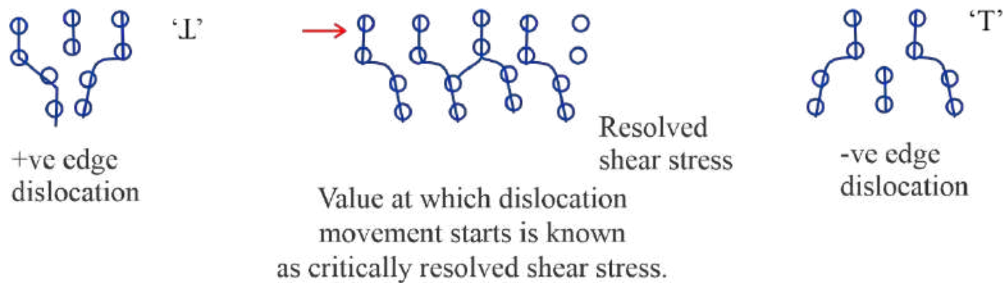


Fig. 9.10 Screw dislocation

(c) Interaction between dislocations:

When two opposite in nature edge dislocation travels towards each other on same slip plane they will attract to each other and combine to become a perfect plane.

When two in similar in nature dislocation travels towares each other on same slip plane they result in repation and due to this strength and hardness of material increases, this is known as **strength hardening**.

9.4.3 Surface Defect

(a) Low tilt angle defect

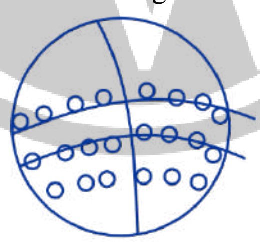
It occurs due to edge dislocaton when two or more than two number of dislocations are exsiting in boundry **one above to another** is results in lattic distortion.

(b) Low twist angle

It occurs due to screw dislocation. When two or more than two number of dislocation one above to other in boundary it results in distortion.

(c) Twinning

Twin boundry is a plane occurs which there is a special minor image disorientation of crystal structure. During anneling process a part of lattice may slipped wrt other of atomic arrangement. Slip part may become mirror image to unslipped part.



Mirror image of grain orientation

Fig. 9.11 Twinning

(d) Grain boundry

Dendride is randomly growing on solid faces. The function between two dendroid randomly is known as grain boundry.Characteristics:Region (High energy, Low melting point, heavy impurity concentrator)

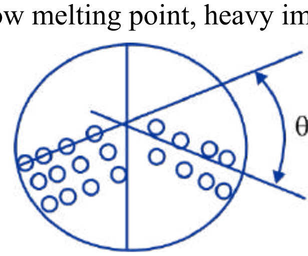


Fig. 9.12 Grain boundry

(e) Stacking

It can be defined as fault in stacking sequence. Stacking fault are surface defect which result in net wise of energy level of lattice.

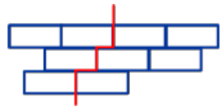


Fig. 9.13 Stacking

9.5 Hume – Rothery Rule

- **Alloy:** Combination of 2 elements whose resulting microstructure reflects the property of individual.
- **Compound:** Resulting product losses the properties of independent element.
- **Intermetallic Inclusions:** Impurities present combines with other element are results into different material with distinct properties.

Before applying this rule crystal structure of both materials should be same. It having 3 conditions:

(i) Difference in atomic radius < 15%

(ii) Valency of both should be same

(iii) Electron negativity (other) & electron affinity (own) should be comparable.

- Alloys have higher strength than the parent metals.

9.5.1 Binary phase Diagram of type– 1: [Isomorphism]

Materials that are completely soluble both in liquid as well as solid state. [Lever Rule only applicable in 2 phase diagram]

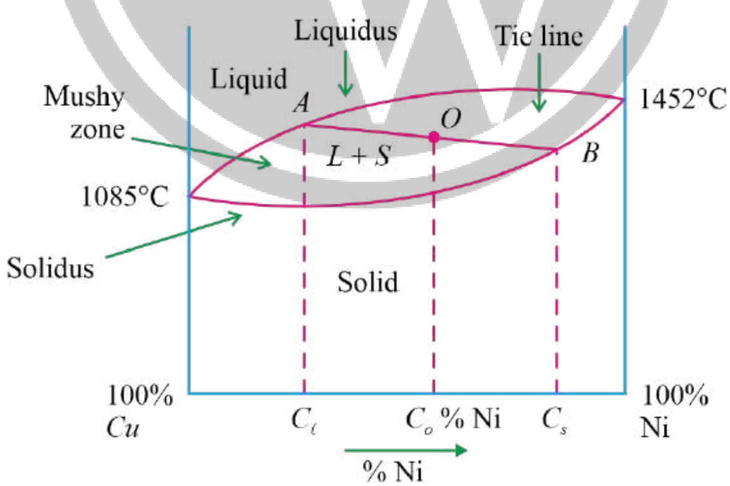


Fig. 9.14 Binary Phase Diagram Type-1

$$\boxed{m_s = \frac{C_0 - C_l}{C_s - Cl}} \quad \boxed{m_2 = \frac{C_S - C_0}{C_s - C_l}} \quad \boxed{m_S + m_l = 1}$$

- Larger Mushy zone exhibit lesser fluidity and more difficult to cast.

Conclusions drawn from level Rule

(1) As temp. decreases, mass fraction of solid phase increases.

(2) With temp. decrease, % of Nickel in solid also decreases.

- Initially diffusion tries to achieve phase stability, but if phase is stable at various temp. Then it tries to achieve chemical homogeneity.

  Slow cooling enables particular orientation of atoms in solidification front and when different solidification fronts fuse together, the region of orientation mismatch are called grain boundaries.

  They can be easily broken that why atmospheric oxygen attacks grain boundary atoms & material starts corroding.

- **Chromium addition** reacts with oxygen producing Cr_2O_3, which does not allow **oxygen to attack** grain boundaries.
- Nickel is added to **stabilize** the **phase mixture** at room temp.
- 18% Cr & 8% Ni material is called "**stainless steel**".
- **Finer is the grain lesser is corrosion.**
- Materials with completely removed grain boundaries are called "**Superalloy**"
- Thin sheets of super alloy is called "**Whisker**".

Note:
Stainless steels are very difficult to weld.

During welding, higher temp. leads to reaction / combining of chromium with oxygen and it appears on surface, which results in corrosion of that part. This phenomenon is called **weld Decay or sensitization of steel**.

- Due to presence of Mushy zone alloys have range of temp. at which liquification & solidification take place.
- Melting point depends upon composition & phase diagram.
- Rapid cooling results into concentration gradient due to no fine for diffusion higher percentage of Ni at centre and decreasing towards the graph boundary. Hot working of such materials, crack is produced due to early melting of grain boundaries leading to brittle fracture. This phenomenon is coring or Miscibility gaps.

9.5.2 Binary Phase diagram of Type-II

Materials that are completely soluble is liquid state but partially soluble in solid state.

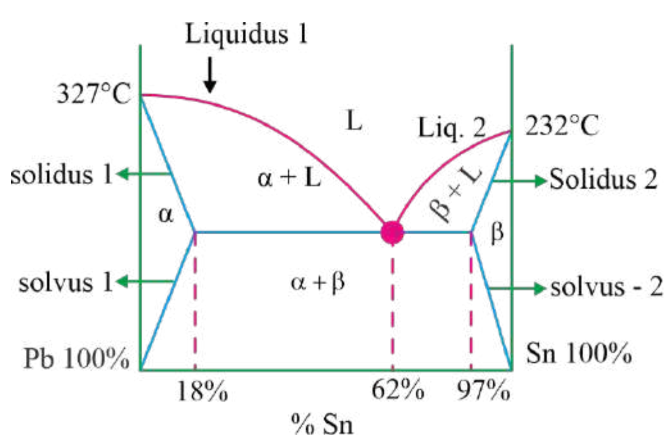


Fig. 9.15 Binary Phase Diagram Type-II

Eutectic Reaction

$$L \xrightleftharpoons[62\% Sn]{183°C} \alpha + \beta$$

Invariant ($F = 0$)

3-phases in equilibrium

9.5.3 Modified Gibbs phase Rule

$$\boxed{P + F = C + 1}$$

Pressure is constant.

Rule of Mixture

$$\boxed{\frac{1}{\rho} = \sum_{i=1}^{n} \frac{x_i}{\rho_i}} \quad \boxed{V_\alpha = \frac{\frac{m_\alpha}{\rho_\alpha}}{\frac{m_\alpha}{\rho_\alpha} + \frac{m_\beta}{\rho_\beta}}}$$

$\therefore$ $$\boxed{V_\alpha + V_\beta = 1}$$

9.5.4 Binary phase diagram of type – III

Materials that are completely soluble in liquid state but completely insoluble in solid state.

Alloy of Bismuth (Bi) shows such characteristics

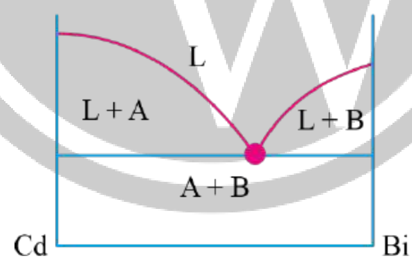


Fig. 9.16 Binary Phase Diagram Type-III

9.6 Cooling Curve of Iron

Phase change is investigated by either change in

(1) Crystal structure

(2) Unit cell size

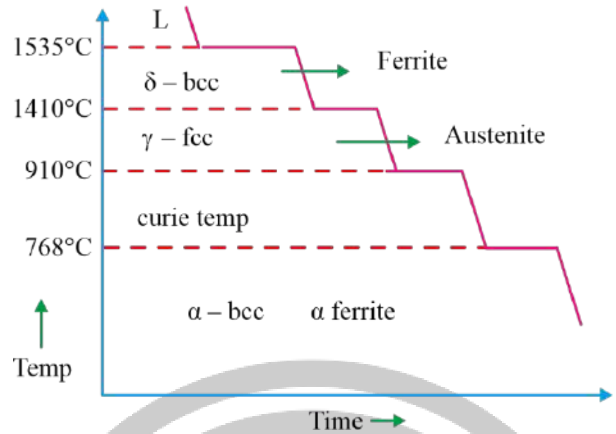


Fig. 9.17 Cooling Curve of Iron

During latent heat transaction at 760°C only magnetic properties are lost.

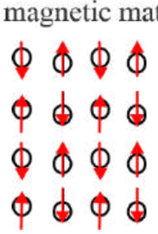
During of periodic table are classified into:

- Randomly oriented dipole
- Soft magnetic material
- Colourful [eg. Cu, Brass, Bronze]

Dipole are exactly opposite to applied field which means all dipoles cancel each other [zero magnetic dipole].

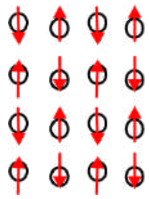
Exception [Ferromagnetic CO, Ni] Iron

- Very strong magnetic dipole
- Hard magnetic material.

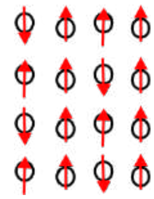


Iron: At room temp. magnetic dipole moments are unidirectional.

Antiferro (Chromium)



Ferrimagnetic



One line of ferro and one line of anti-ferro.

No metals show this behaviour. Only compound exhibit this.

9.7 Iron Carbon Phase Diagram

Fe-Fe₃ C diagram

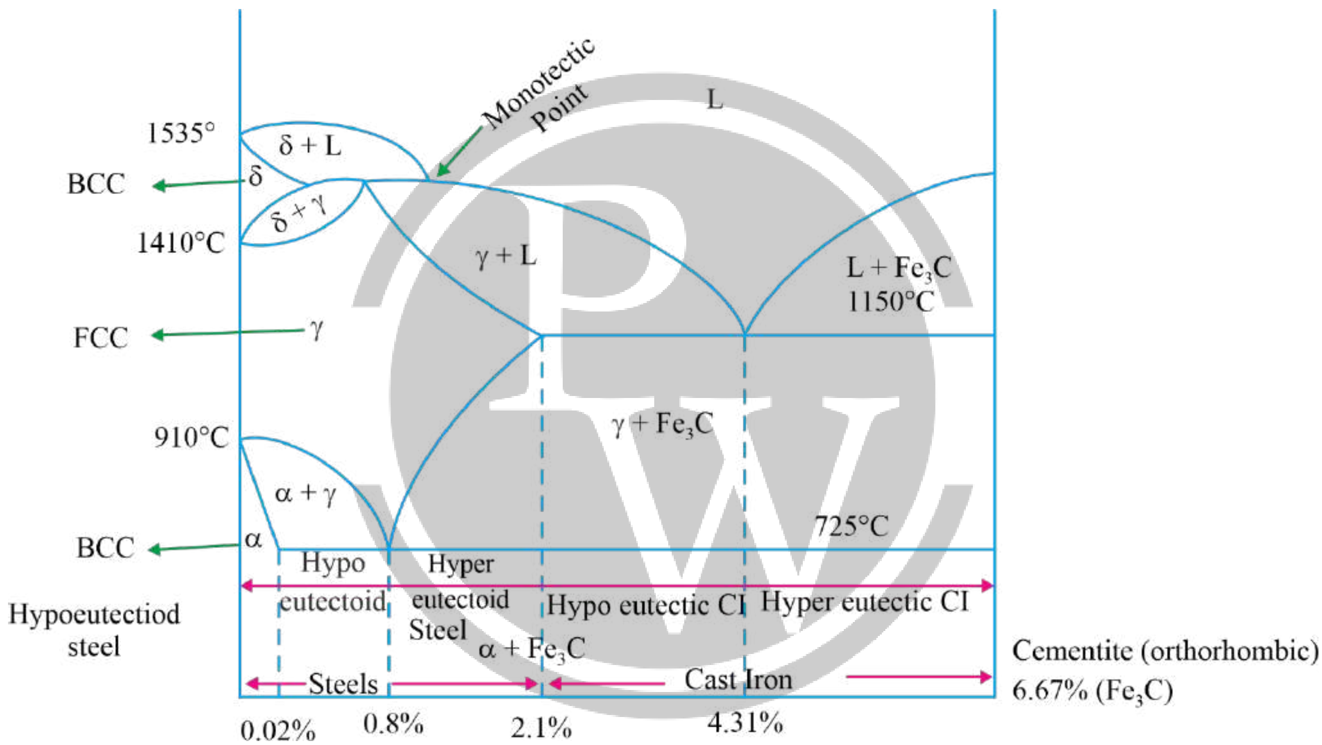


Fig. 9.18 Iron Carbon Diagram

Classification of Carbon

< 0.3% Low C steel (Mild steel)

0.3 – 0.7% Medium steel

> 0.7% High steel

1. $L \xrightleftharpoons[4.3\%]{1150°C} \underbrace{\gamma + Fe_3C}_{\text{Leadburite}} \left[\text{Eutectic Reaction}\right]$

2. $\gamma \xrightleftharpoons[0.8\%C]{725°C} \underbrace{\alpha + Fe_3C}_{\text{Pearlite}} \left[\text{Eutectoid Reaction}\right]$ [γ (Austenite) is not stable in temp below 725°C It becomes Pearlite].

Note:
Pearlite is a phase mixture & is formed purely be diffusion 100% pearlite is formed by eutectoid decomposition.

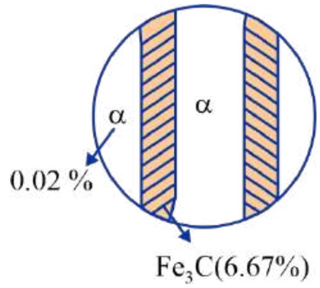


Fig. 9.19 Pearlite Phase

3. $\underset{(s)}{\delta} + \underset{(L)}{L} \underset{0.18\%C}{\overset{1493°C}{\rightleftharpoons}} \underset{(s)}{\gamma}$ [Peritectic Reaction]

   It appears where there is large diff. in melting point of 2 elements.

4. Peritectoid reaction $S_1 + S_2 \overset{\text{cooling}}{\rightleftharpoons} S_3$ [Found Cu-alloy phase diag. Not present in Fe–C phase diag.]

9.7.1 Solubility of carbon in various Cases

(i) Maximum solublity of carbon in δ-ferrite phase in 0.1%
(ii) Maximum solubility of carbon in γ-austenite phase is 2.1%
(iii) Maximum solubility of carbon in α-ferrite is 0.025%

9.7.2 Significance of Some Important Line

A_1 Line → (727°C) when Eutectoid steel are heated above A_1, pertite transformation into Austenite.

A_2 line → It is called Curic point temp line. This line signifies magnetic to non-magnetic transformation on heating

* Carbon % has no influence on Curie point temp.

A_3 line → This line is known as upper ctrical temp. line for Hypo Eutectoid steel.

This line signifies complection of Ferrite to Austenite solution.

A(CM) Line → This line is called upper critical temp. line for Hyper eutectoid steel. It signifes completion of cementite to austenite.

9.8 Classification of Cast Iron

(1) **Gray cast Iron:** Flakes are formed due to excess carbon.

(2) **White cast Iron:** Slow cooling results in combined from of carbon with iron is called white cast iron.

(3) **Chilled cast Iron:** Cast iron of such composition in which it will normally freeze as gray but by rapid cooling it is forced to appears white is called chilled cast Iron Extremely brittle.

(4) **Spheroidal cast CI:** Chilled CI heat treated with Mg or Ce addition results into formation of spheroids of carbon. [High Ductile]

(5) **Nodular CI:** If cooling rate during heat treatment is quite high we get Needle shaped carbon.

9.8.1 Effect of Sulphur & Manganese on the Iron

Mg captures S & produces MgS whose Tm is high. So, it overcomes Hot shortness phenomena. It also improves machinability ➔ (Low shear strength).

Hadfield steel – (12% Mg very strong element, used in Bulldozers).

9.8.2 Effect of silicon on steel

Oxygen is used to remove carbon & last trapped oxygen is removed by using silicon inform of sludge. If oxygen removed is complete then it is called killed steel. When removal is partial it is called semi-killed steel.

9.8.3 Effect of silicon on CI

Addition of SI & P shifts the Iron carbon diagram towards left.

Industrially produced gray CI contains carbon percentage 2.4 – 4% sparkles of red not-graphite on liquid iron surface & it sparkles. This phenomenon is called kish.

Equi-cohesion Temperature

Strength of grain boundary becomes uniform & below this temp. material fail in Brittle manner.

9.8.4 Development of Microstructure in Iron-carbon system

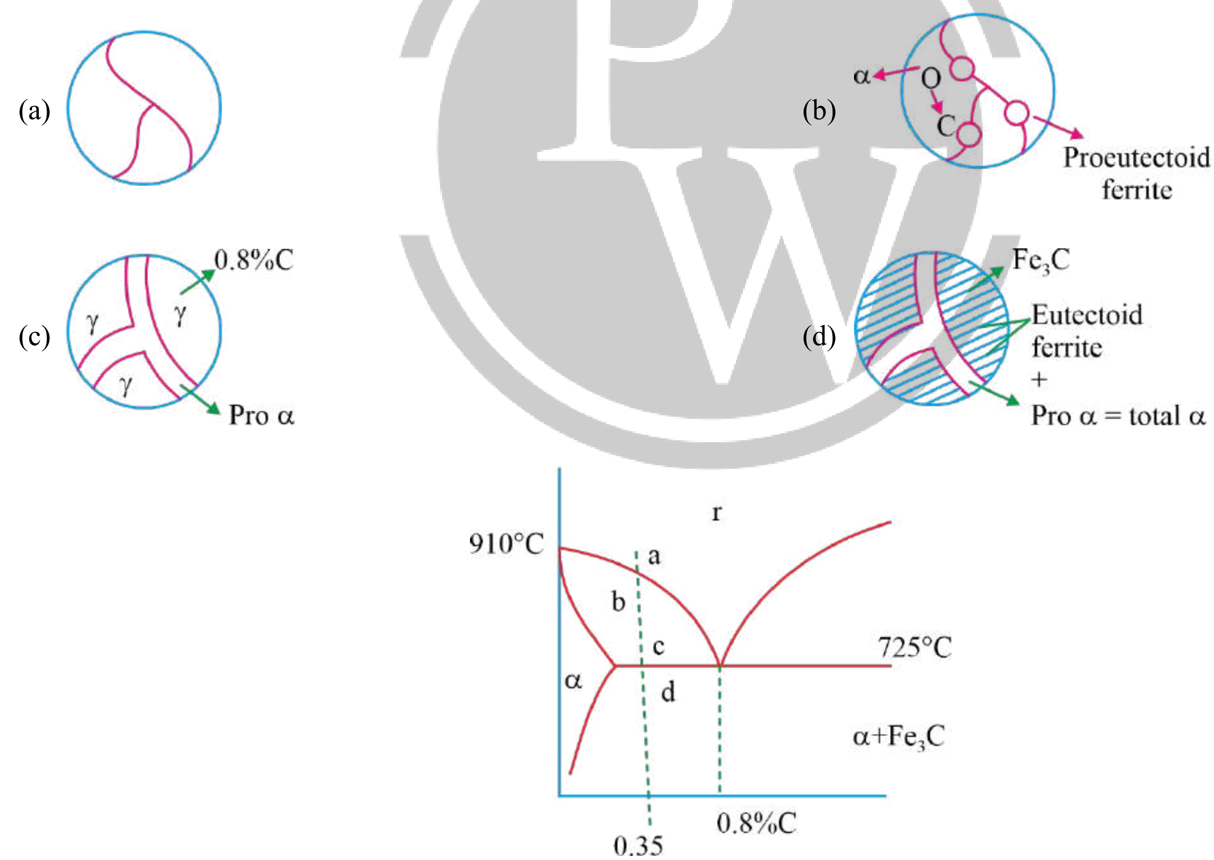


Fig. 9.20 Microstructure of Iron Carbon Diagram

9.9 Yield Point Phenomenon: (YPP)

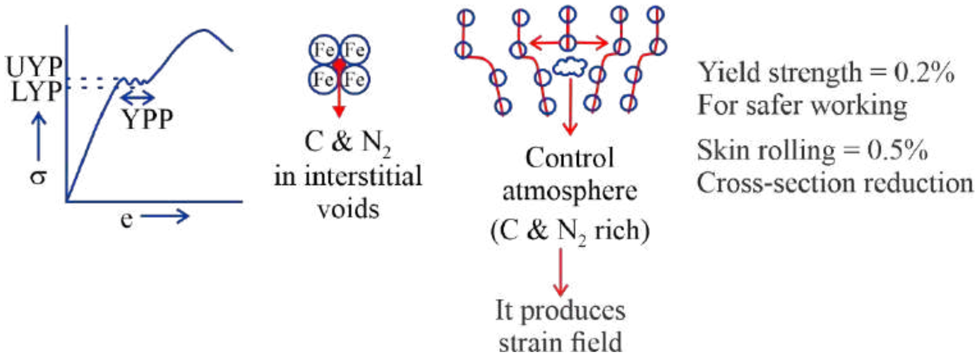


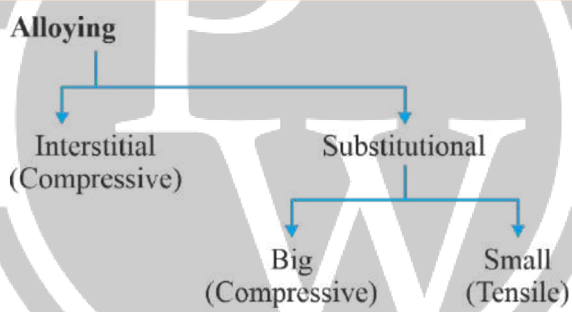
Fig. 9.21 YPP

Time period [$1\frac{1}{2}$ – 2 year] depending upon Carbon percentage

No YPP in medium & high carbon steel. [because carbon is not only percent at dislocation site it is everywhere]

9.10 Strengthening Mechanisms

9.10.1 Alloying



Strain fields are created in host atoms and these creates obstacles in dislocation movt. There by increasing strength.

9.10.2 Grain Refinement

$$\text{Grain density} = \frac{\text{No.of grains}}{\text{unit length}}$$

Finer the grain structure better will be the strength.

Hall petch equation $$\sigma_y = \sigma_0 + \frac{k}{\sqrt{d}}$$

k = constant

d = average grain diameter

$n = 2^{G-1}$

n = grain density

G = ASTM number

9.10.3 Work hardening (strain hardening):

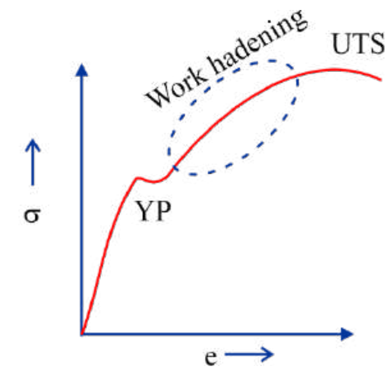


Fig. 9.22 Work Hardening

- Shot blasting
- Sand blasting (Thin section)
- Shot peening (Odd shapes)

} Hard Surface

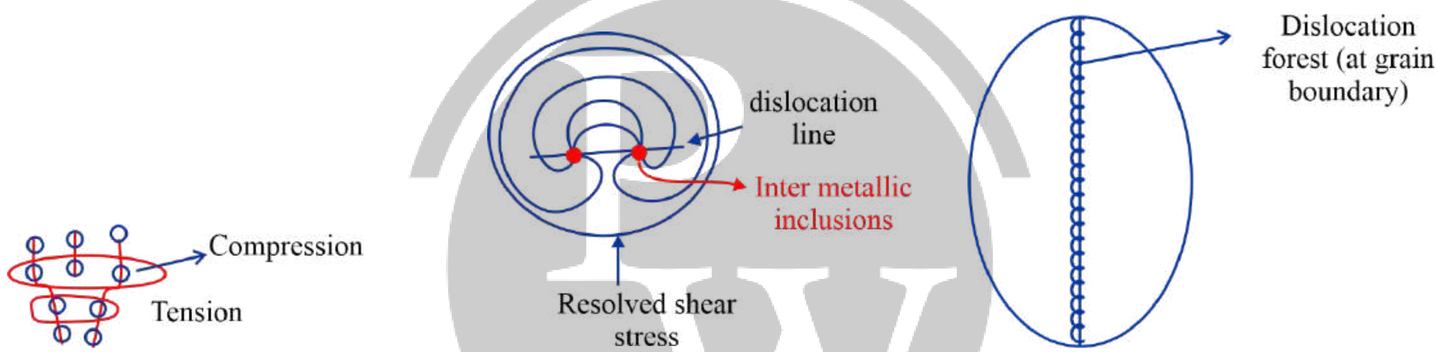


Fig. 9.23

- When more dislocations reach the grain boundary dislocation forest will repel & produces Back stress.
- After unloading from region of work hardening & again reloading yield point will inc. due to back stresses.

9.11 Bauschinger effect

Yield point in compression will appear pre-maturely.

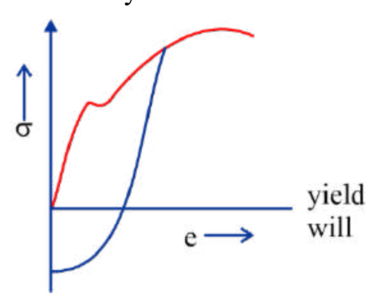


Fig. 9.24 Bauschinger Effect

Nature of stress strain curve in Tension & comp.

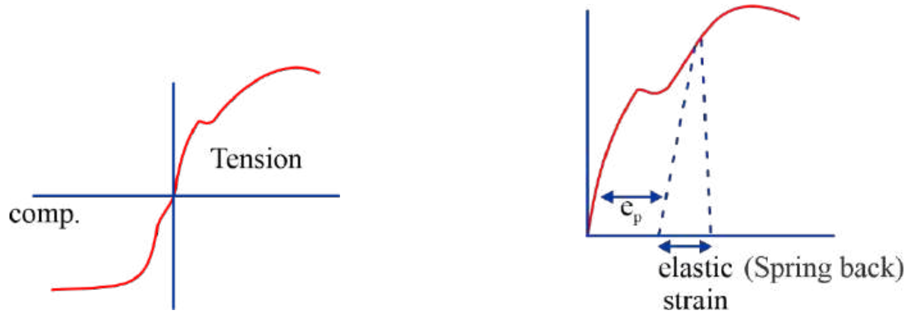


Fig. 9.25 Nature of Stress Strain Curve

Parallel length: Length with uniform cross-section.

Gauge length: Length under observation

Inst. Length & area measured by (Extensometer)

9.12 Various curves with changing strain Rate

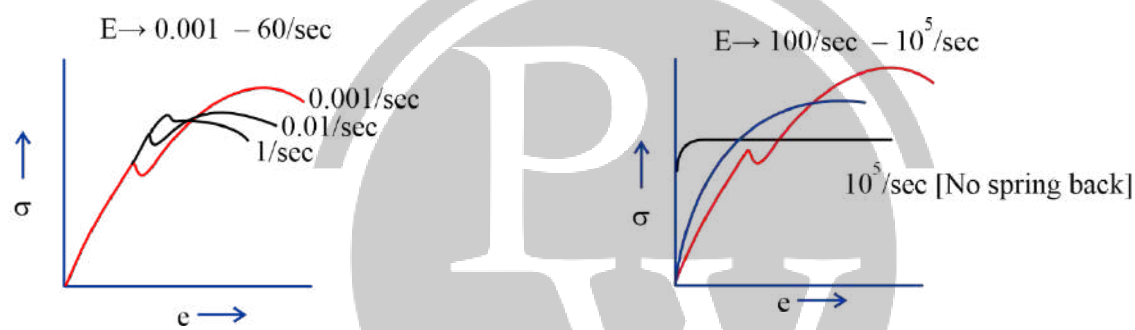


Fig. 9.26 Variation of Strain Rate

- Work hardening decreases due to no time for atoms to respond
- Inc. in yield point
- No elastic region in high strain Rate
- Behave like rigid plastic

9.13 Power Law / Flow curve equation / Holloman equation / Constitutive equation

$\sigma_f = k\varepsilon_T^n$

$\sigma_f \rightarrow$ True stress
k $\rightarrow$ Strength coefficient
$\varepsilon_T \rightarrow$ True strain
n $\rightarrow$ Work hardening exponent
Value of n
for steel = 0.3
for Al = 0.05

$$\sigma_0 = E \in \text{ [Engineering stress \& strain] } \varepsilon = \frac{dl}{\ell_0}$$

$$\sigma_f = K \in_t^n \text{ [True stress \& strain] } \quad \in = \frac{\ell}{\ell_0} = \frac{A_0}{A}$$

$$\sigma_f = \frac{P}{A}[\text{A \& } \ell \text{ inst. Area \& length}]$$

$$\sigma_f = \sigma_0[\in +1] \boxed{\in_t = \ell_n(1+\in)}$$

At UTS

$\boxed{n = \in_t}$

9.14 Age of Precipitation Hardening

Use in air craft (Al – Cu) Duralomin [Cu% -5.5% only]

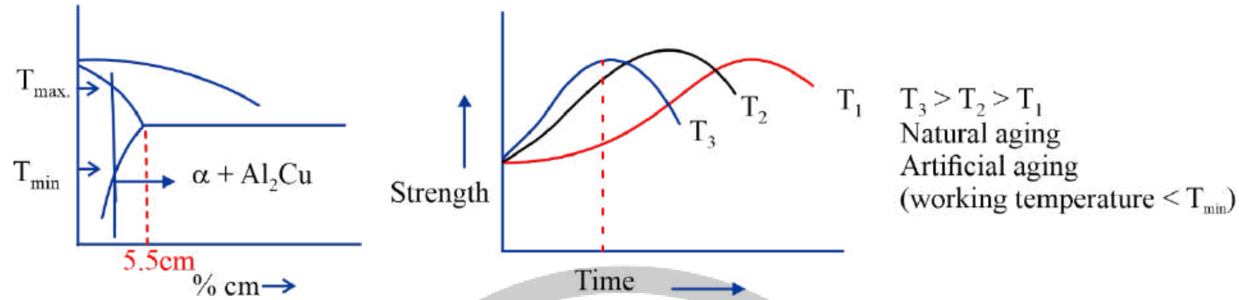


Fig. 9.27

Al – Cu alloy is heated → Then quenched to room temp → microstructure is unstable → slowly with time precipitation occurs creating barriers for dislocation move → strength increases.

Point at which nucleation stops is called peak strength of the material.

9.14.1 Creep

Slow & progressive deformation material over a period of time at constant load or stress at a temp. equal to or greater than recrystallization

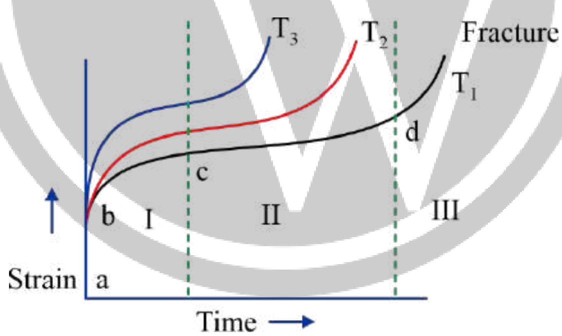


Fig. 9.28

$T_3 > T_2 > T_1$

a – b Instantaneous

b – c transient

c – d steady

d – e accelerated

Region (I) work hardening,

Region (II) Balance b/w work hardening & Recrystallization

Region (III) Recrystallization

- **Tensile Test:** No. of Neck then can appear during tensile test? = 1.

  At what point Neck will appear? = anywhere in parallel length
- **Temp. the transformation curve:** [In equilibrium state] [0.8% C]

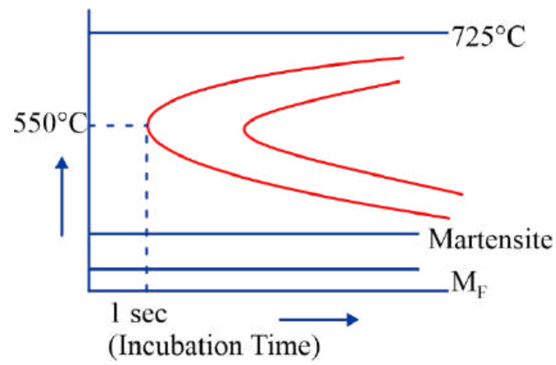
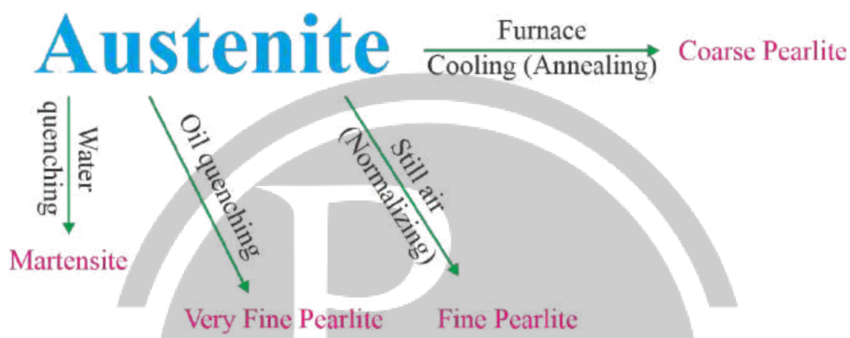


Fig. 9.29

9.15 TTT curve or Bain's curve or S curve or C curve (Non-Equilibrium Diagram)



Hardness Order

(1) Martensite (Max.)

(2) Bainite

(3) Fine Perlite

(4) Coarse Pearlite (Min.)

Period below 725°C during which nothing happens to Austenite is called incubation time.

- Upon decreasing quenching temp. first incubation time decreases, then at 550°C it is 1sec after that it starts increasing.
- Such behaviour is due to
  (a) Driving force
  (b) Diffusion (Atomic Mobility)

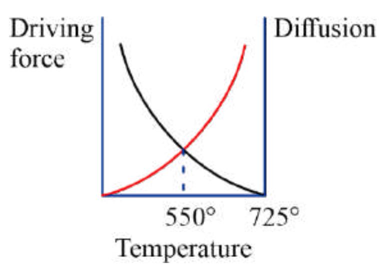


Fig. 9.30

Critical cooling Rate: If cooling rate just touches the nose of TTT diagram is called CCR.

$$\text{CRR} = \frac{750-550}{1\text{sec}} = 200°\text{C/sec}$$

9.15.1 Formation of Martensite (BCT)

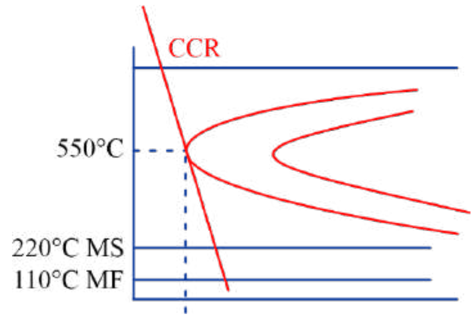


Fig. 9.31

Martensite

- Sub microscopic cementite present in ferrite.
- Hard & most brittle phase of iron
- On TTT curve – Once Austenite converts into same microstructure it never reconverts.

9.15.2 Pearlite

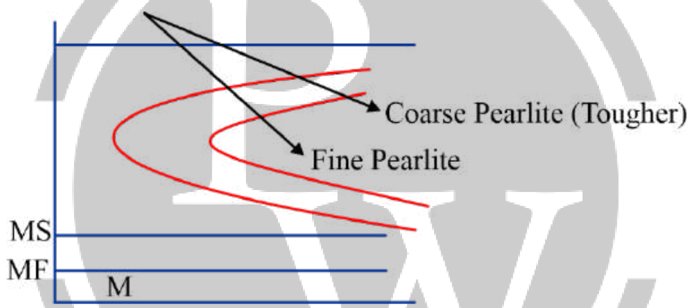


Fig. 9.32 Pearlite

- Low cooling rate allows time for diffusion and result is coarse pearlite.
- Fast cooling rates gives finer Pearlite.
- Generally, with increase in hardness & toughness the ductility decreases [except fine grain].

9.15.3 Austempering

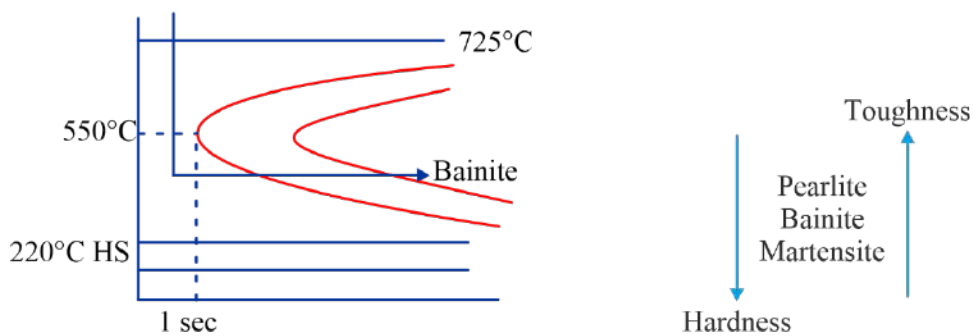


Fig. 9.33

Mech. of formation:

Pearlite – Diffusion

Bainite – Partial (Diffusion + shear)

Martensite – Diffusion less (A thermal)

Bainite has mixed microstructure not uniform like pearlite or Martensite.

9.15.4 TTT diagram for various steels:

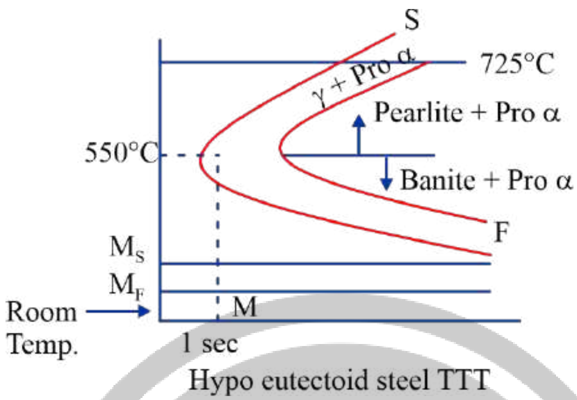


Fig. 9.34

No matter how much percentage carbon is there decomposition line (finish) will be same.

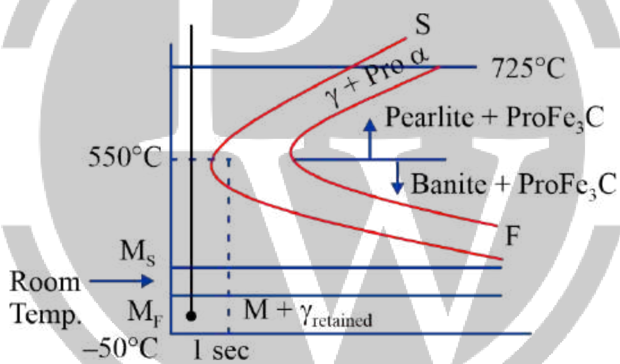


Fig. 9.35 Hyper Eutectoid steel TTT

In Hyper Eutectoid steel TTT diag. we get $\gamma_{retained}$ which is quenched by using Liq. N_2 (77K) i.e. Cryogenic Treatment

9.15.5 Austenitic Stainless steel (ASS)

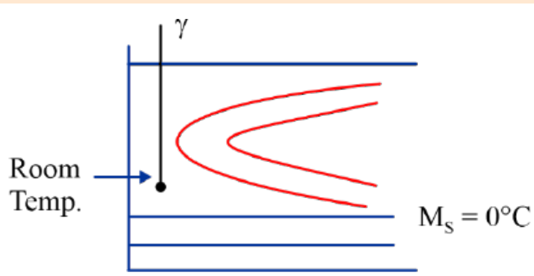


Fig. 9.36

Ni (9%)

Cr (18 – 20%)

In this case to obtain Austenite (stable) we use Nickel & Cr to shift the Martensite start at 0°C.

ASS is used in Nuclear Applications.

- **Maraging steel:** [Used is defence for Missile cover]

  Ni – 17 – 19 %

  Co – 8 – 9 %

  Mo – 3.35 %

  Ti = 0.15 – 0.25%

  Al = 0.05 – 0.15%

  Fe – Balance

9.16 Hardenability – Jominy & Quench Test

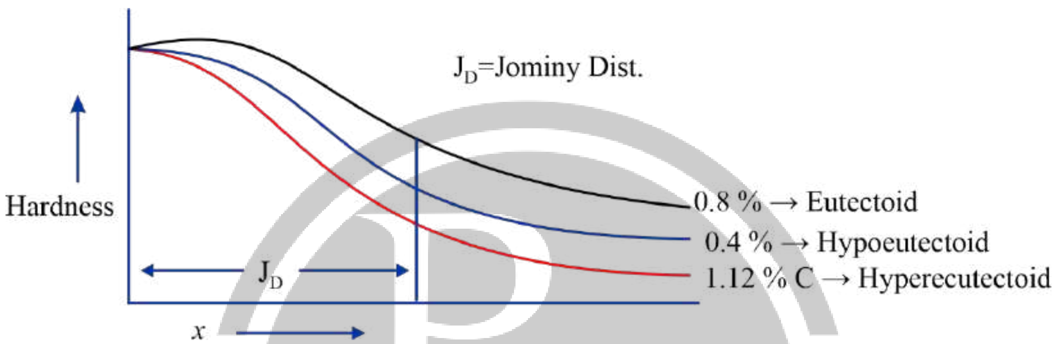


Fig. 9.37

- From bottom to top, there will be variable cooling rate
  At J_D = 50% Pearlite + 50% Martensite
- Hardenability is defined as ease of martensite formation.
- As we deviate from eutectoid comp. hardenability decreases.

9.16.1 Cracks formation on Quenching

During Quenching sample (Austenite) experiences diff temp. at surface & core due to which surface becomes low density (Martensite) & core becomes high density (Austenite). Later when core converts martensite expansion take place & cracks are formed. [Density diff. cause cracks].

Ways to avoid cracks:

(1) Austempering

(2) Martempering (stepped Quenching)

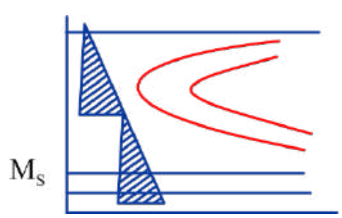


Fig. 9.38

(3) Alloying [Air quenching may produce martensite]

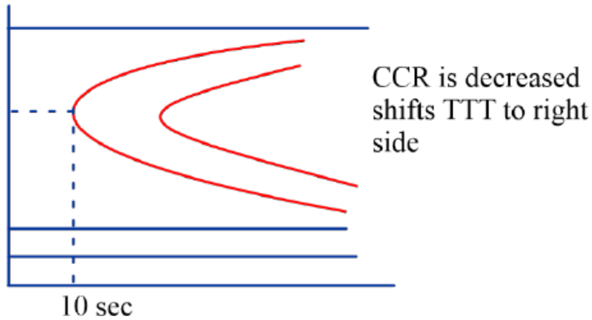
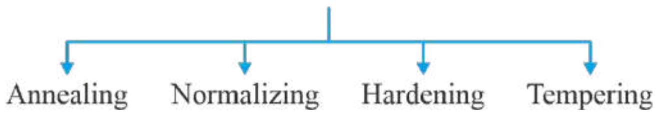


Fig. 9.39

9.17 Heat Treatment Classification



9.17.1 Annealing

1. Full Annealing:

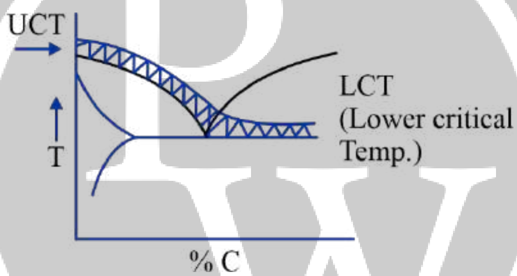


Fig. 9.40 Annealing

- Objective to reduce harness & brittleness & increase ductility
- Result is Coarse pearlite
- Cooling Rate 200°C/hr.

2. Process Annealing:

- Objective stress relieving of low carbon steels.
- Medium & high carbon steels are brittle so they may fracture. Therefore, they are not processed.
- Heated upto Recrystallization Temp. but less than LCT.
- No major change in grain structure [Little finer it becomes]

3. Spheroidize Annealing:

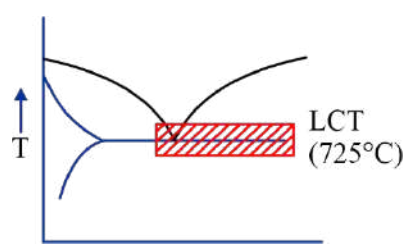


Fig. 9.41

- Cooling rate – 20°C/hr
- Objective to increase machinability of medium & high CS.
- Heated close to lower critical, then cooled slowly in furnace.
- Machinability improves due to formation of spheroids.

4. Diffusion / Homogenizing:

- Performed to make chemical comp. uniform when disturbed by welding.
- Higher temp is used to enable diffusion phenomenon effectively.

9.17.2 Normalizing

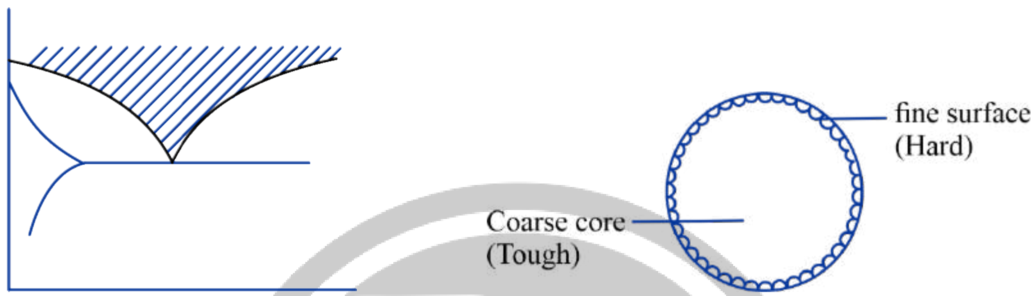


Fig. 9.42

- Air quenched
- Maximum engineering applications
- Considered as the final heat treatment process.

9.17.3 Hardening

Objective: To get martensite structure.

9.18 Quenching Mediums

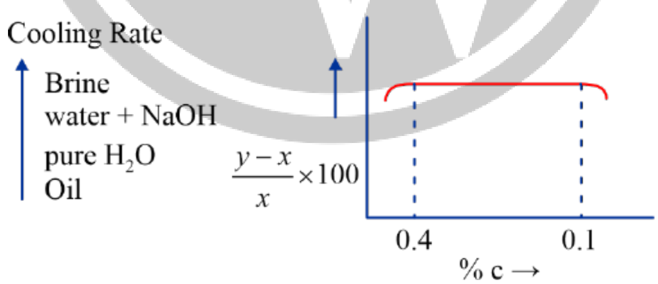


Fig. 9.43

- Water
- 36 % salt (Brine solution)
- Oil baths (Alloy steels)
- Thermoplastic
- Water + NaOH
- Mild steels cannot be treatment by this method as they contain proeutectoid phase [No contribution in Martensite]
- Their strength is increased by Case Hardening.

9.18.1 Case Hardening

Diffusing carbon into MS specimen. Surface is hardened only.

1. Carburizing (Fe_3C)

(a) Pack

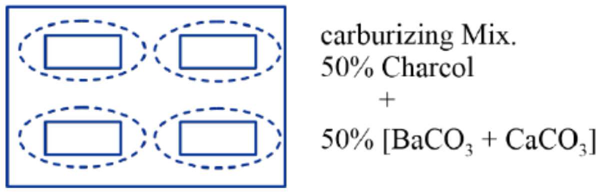


Fig. 9.44

- Cheap & easy to perform
- Poor quality
- Time consuming.

(b) Liquid

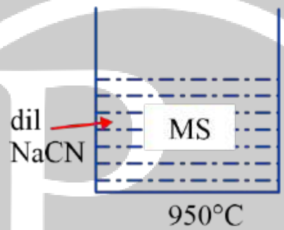


Fig. 9.45

- Good quality cases
- Bath is expensive
- Highly poisonous

(c) Gas carburizing

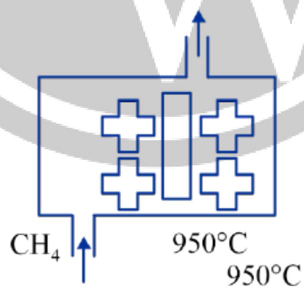


Fig. 9.46

- Thickness of case can be controlled
- Process can be automated.

2. Nitriding

NH_3 + 650°C (extremely brittle)
Mostly it needs further treatment for making tough.

3. Cyaniding

It uses combination of 30% NaCN + NaCl + Na_2CO_3 [at 850°C]

4. Carbo Nitriding (Gas Cyaniding)

$CH_4 + NH_3$ (at temp. 850°C)

- N > CN > C

9.18.2 Flame Hardening

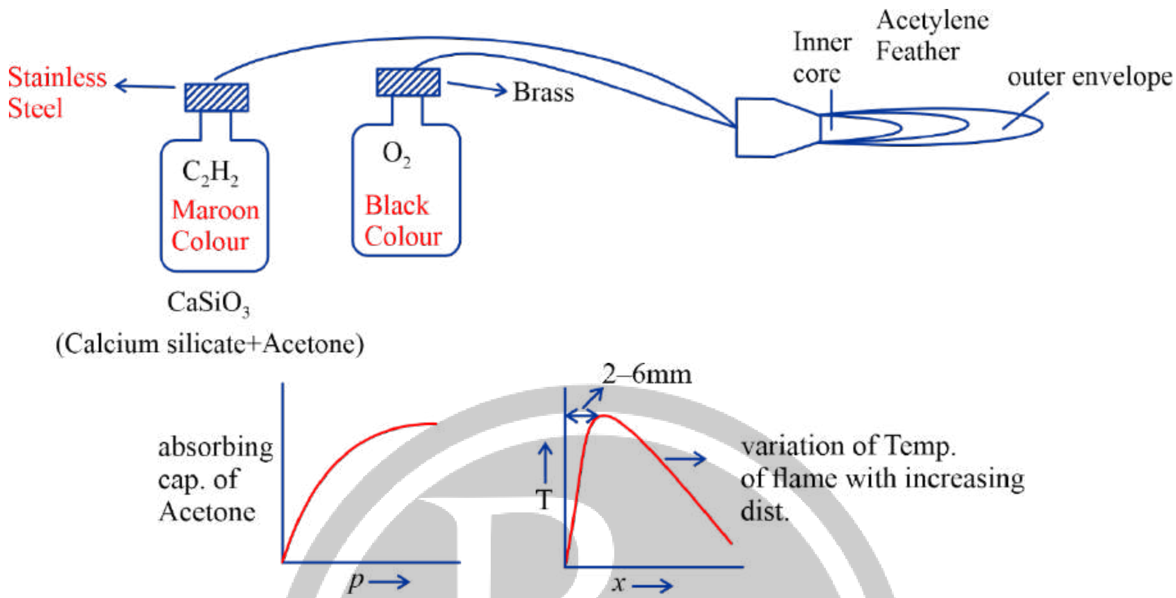


Fig. 9.47 Flame Hardening

1. Carburizing Flame (C)	< 3000°C	Silent	Cast Iron
2. Neutral Flame $C_2H_2 + \frac{5}{2}O_2 \rightarrow 2CO_2 + H_2O$	3150°C	Hissing	Steel
3. Oxidised Flame	3480°C	Roaring	Copper

Flame hardening is used to produce a thin layer of Martensite on surface.

9.18.3 Induction Hardening

[Fastest Method]:

Thickness of case $\quad x = 500\sqrt{\frac{e}{\mu F}}$

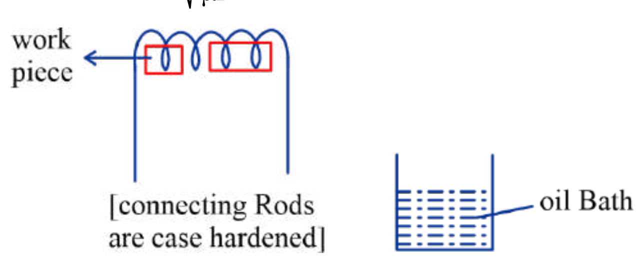


Fig. 9.48

e = electric resistivity

μ = magnetic permeability

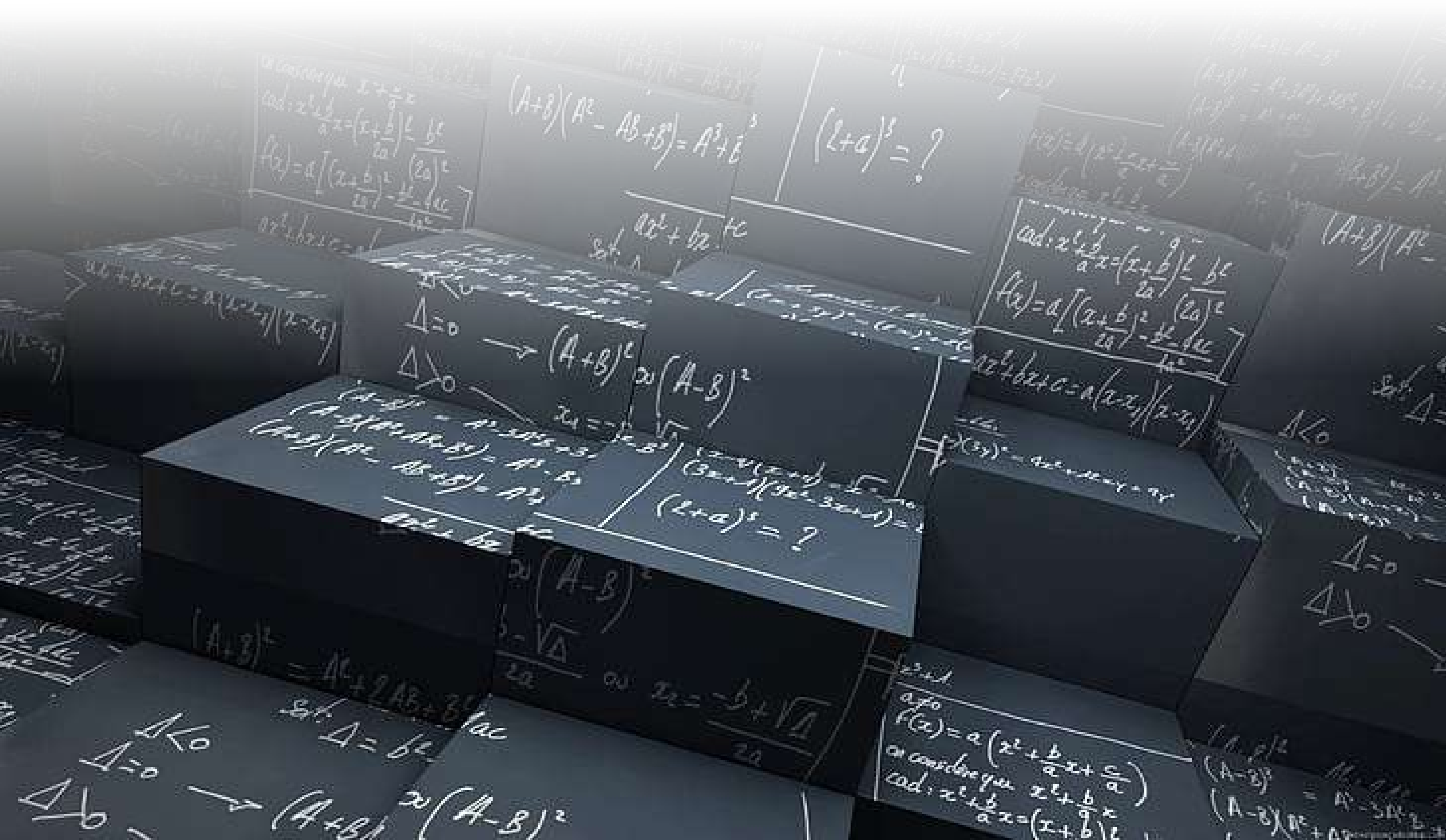
f = frequency

9.18.4 Tempering

High temp. Tempering (500 – 600°C)	Medium Temp. (350 – 500°C)	Low Temp. [250°C]
Coarse structure (Tough)	Use for Finer cementite (Springs)	Only stresses are relieved
Sorbite Microstructure	Troostite Microstructure	Used for Agri. tools, Metrology Tools

❑❑❑

Engineering Mathematics



Engineering Mathematics

INDEX

1 BASIC CALCULUS

1.1 Introduction

1.1.1 Limits, Continuity and Differentiability

a. As x tends to a ($x \to a$) $\Rightarrow$ x is moving towards a

A value l is said to be limit of a function $f(x)$ at $x \to a$ if $f(x) \to l$ as $x \to a$.

It is mathematically defined as

$$\lim_{x \to a} f(x) = l = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

A function f(x) is said to be continuous at x = a if

$$\lim_{x \to a} f(x) = l = f(a) = f(x)|_{x=a}$$

Note:

For $\lim_{x \to a} f(x)$ to exist, the function need not be continuous at $x = a$.

But for $f(x)$ to be continuous at x = a, $\lim_{x \to a} f(a)$ should exist.

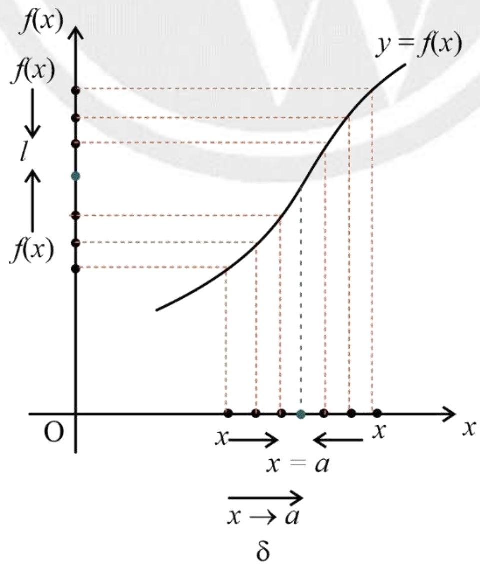


Fig. 1.1

b. Concept of differentiability

A continuous function $f(x)$ is said to be differentiable at $x = a$ if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

$$f'(x)|_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$f'(a) = \tan\theta$ where θ is the angle made by the tangent to the curve at x=a with x – axis.

c. Some Standard Derivatives

(i) $\frac{d}{dx}(x^n) = n.x^{n-1}$

(ii) $\frac{d}{dx}(\sin x) = \cos x$

(iii) $\frac{d}{dx}(\cos x) = -\sin x$

(iv) $\frac{d}{dx}(\tan x) = \sec^2 x$

(v) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(vi) $\frac{d}{dx}(\sec x) = \sec x . \tan x$

(vii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(viii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}; -1 < x < 1$

(ix) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1$

(x) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xi) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xii) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

(xiii) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$; $|x| > 1$

(xiv) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

(xv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(xvi) $\frac{d}{dx}(a^x) = a^x.\log_e a$

(xvii) $\frac{d}{dx}(e^x) = e^x$

(xviii) $\frac{d}{dx}(|x|) = \frac{|x|}{x}(x \neq 0)$

(xix) $\frac{d}{dx}(x^x) = x^x(1 + \log_e x)$

(xx) $\frac{d}{dx}(\sinh x) = \cosh x$

d. Product rule of differentiation

$$\frac{d}{dx}\big(f(x).g(x)\big) = f(x).g'(x) + f'(x).g(x)$$

$$d(uvw) = uvw' + uv'w + u'vw$$

e. Quotient rule of differentiation

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x) - f(x).g'(x)}{(g(x))^2}, (g(x) \neq 0)$$

f. Greatest Integer function / step function / integer part function

$f(x) = [x] = n, \forall\, n \leq x < n + 1 \text{ where } n \in Z$

$\lim\limits_{x\to a}[x] = \nexists$ if a is an integer

L.H.L. = $\lim\limits_{x\to a^-}[x] = a - 1$

R.H.L. = $\lim\limits_{x\to a^+}[x] = a$

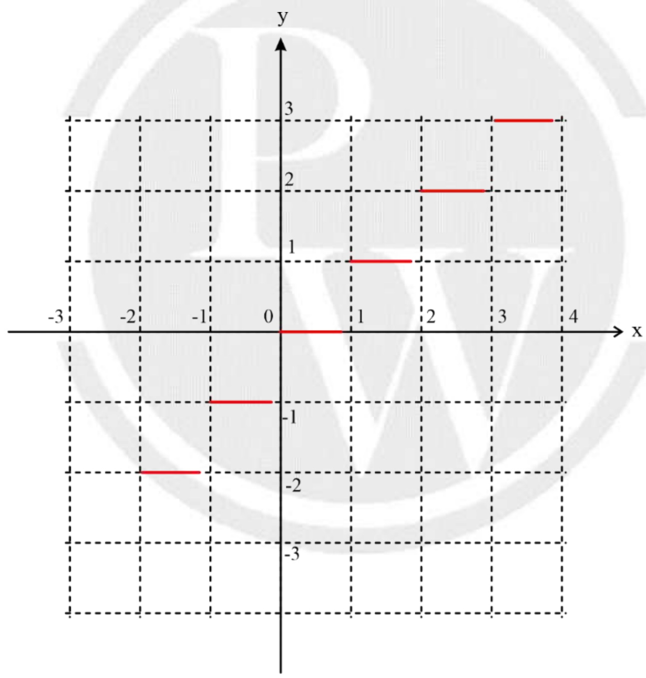


Fig.1. 2 Greatest Integer

g. Properties of Limits

(i) $\lim\limits_{x\to a}(f(x) \pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

(ii) $\lim\limits_{x\to a}\big(f(x).g(x)\big) = \lim\limits_{x\to a} f(x).\lim\limits_{x\to a} g(x)$

(iii) $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\left(\lim\limits_{x\to a} g(x) \neq 0\right)$

(iv) If $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} g(x) = \nexists$, then $\lim\limits_{x\to a} f(x).g(x)$ MAY exist

Ex: let $f(x) = \sin x$, $g(x) = \frac{1}{x}$, $\lim\limits_{x\to 0} f(x) = 0$, $\lim\limits_{x\to 0} \frac{1}{x} = \nexists$

But $\lim\limits_{x\to 0} \sin x . \frac{1}{x} = 1$

(v) If $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)} \neq \left(\frac{0}{0}\right)$

If $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \frac{0}{0}$ (or) $\frac{\infty}{\infty}$, then $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to a} \frac{f''(x)}{g''(x)}$ and so on

(vi) If $\lim\limits_{x\to a}\big(f(x).g(x)\big) = 0 \times \infty \Rightarrow \lim\limits_{x\to a} \frac{f(x)}{\left(\frac{1}{g(x)}\right)} = \frac{0}{0}$ (Apply L- Hospital Rule again)

h. Some Standard Limits

(i) $\lim\limits_{x\to a} \frac{\sin x}{x} = 1$

(ii) $\lim\limits_{x\to a} \frac{\tan x}{x} = 1$

(iii) $\lim\limits_{x\to 0} \frac{1-\cos ax}{x^2} = \frac{a^2}{2}$

(iv) $\lim\limits_{x\to \infty} \frac{\sin x}{x} = 0$

(v) $\lim\limits_{x\to \infty} \frac{\cos x}{x} = 0$

(vi) $\lim\limits_{x\to 0} (1+ax)^{b/x} = e^{ab}$

(vii) $\lim\limits_{x\to \infty} \left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

(viii) $\lim\limits_{x\to 0} \left(\frac{a^x+b^x}{2}\right)^{1/x} = \sqrt{ab}$

(ix) $\lim\limits_{x\to 0} \left(\frac{1^x+2^x+3^x+....+n^x}{n}\right)^{1/x} = \sqrt[n]{n!}$

(x) $\lim\limits_{x\to 0} \frac{a^x-1}{x} = \log_e a$; $\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

(xi) $\lim\limits_{x\to 0} x . \sin\left(\frac{1}{x}\right) = 0$

1.2 Mean Value Theorems

1.2.1 Lagrange's Mean Value Theorem (LMVT):

If $f(x)$ is continuous in [a, b] and it is differentiable in (a, b) then ∃ at least one point 'C' such that C ∈ (a, b) and

$$f'(C) = \frac{f(b) - f(a)}{b - a}$$

Here $f'(C)$ slope of tangent to *f*(*x*) at *x* = *C*.

Tangent at x = c is parallel to the line connecting the points A and B

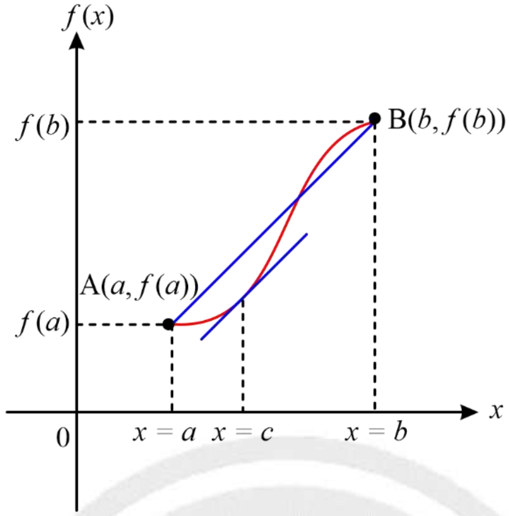


Fig.1.3 LMVT

1.2.2 Rolle's Mean Value Theorem

If $f(x)$ is continuous is [*a*, *b*] and differentiable in (a, b) and *f*(*a*) = *f*(*b*) then ∃ at least one-point C∈ (*a*, *b*) such that $f'(C) = 0$.

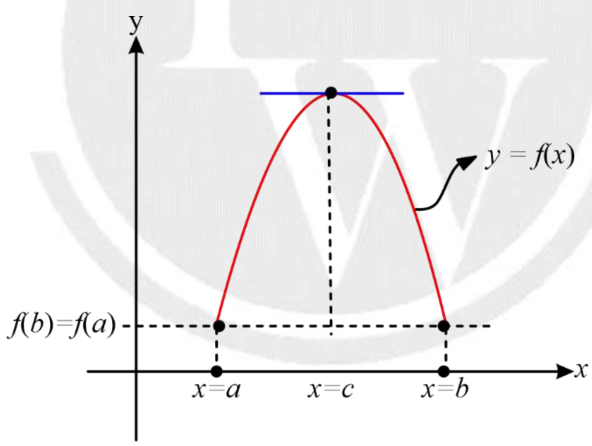


Fig.1.4 Rolle's mean value

1.2.3 Cauchy's Mean Value Theorem

If *f*(*x*) and *g*(*x*) are continuous in [*a*, *b*] and differentiable in (*a*, *b*) then ∃ at least one value of 'C' such that C ∈ (*a*, *b*) and $\frac{g'(C)}{f'(C)} = \frac{g(b)-g(a)}{f(b)-f(a)}$

1.3 Increasing and Decreasing Functions

1.3.1 Increasing Functions

A function *f*(*x*) is said to be increasing if $f(x_1) < f(x_2)\ \forall\ x_1 < x_2$

Or

A function $f(x)$ is said to be increasing if $f(x)$ increases as x increases.

For a function $f(x)$ to be increasing at the point x=a, $f'(a) > 0$.

Example:

e^x, $\log_e x$ $\rightarrow$ Monotonically increasing functions

sin x in (0, π/2) → non-monotonic functions

1.3.2 Decreasing Functions

A function $f(x)$ is said to be a decreasing function if $f(x_1) > f(x_2) \forall x_1 < x_2$

A function $f(x)$ is said to be decreasing function if $f(x)$ decreases as x increases.

Example: e^{-x} →Monotonically decreasing function $\sin x$ in $\left(\frac{\pi}{2}, \pi\right)$

1.4 Concept of Maxima and Minima

Let $f(x)$ be a differentiable function, then to find the maximum (or) minimum of $f(x)$.

(1) Find stationary points from the equation $f'(x) = 0$. Let 'x_0' be the stationary point.

(2) Find the value of $f''(x_0)$

Case (i): If $f''(x_0) < 0$, then the function $f(x)$ has maximum at $x = x_0$ and the maximum value of the function is $f(x_0)$.

Case (ii): If $f''(x_0) > 0$, then the function $f(x)$ has minimum value at $x = x_0$ and the minimum value is $f(x_0)$.

Case (iii): If $f''(x_0) = 0$, then the we cannot comment on the existence of maximum (or) minimum of $f(x)$ at $x = x_0$.

Such points are called points of inflection (or) Critical points.

Example: $x = 0$ is a critical point of $f(x) = x^3$

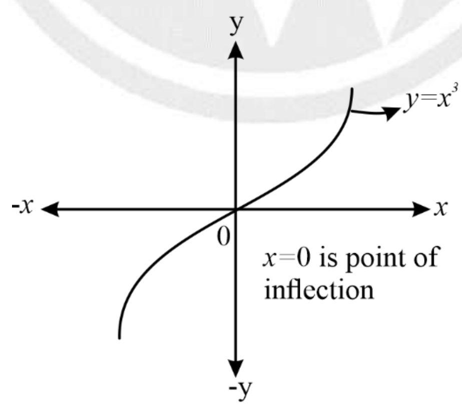


Fig. 1.5 Graph of x^3

$$f(x) = x^3$$

$$\Rightarrow \quad f'(x) = 3x^2 = 0 \quad \Rightarrow \quad x = 0$$

$$f''(x) = 6x \Rightarrow f''(0) = 6(0) = 0$$

1.5 Taylor Series

If *f*(*x*) is continuously differentiable $(f'(x), f''(x), f'''(x), \ldots\ldots$ exists$)$ then the Taylor series expansion of *f*(*x*) about the point *x* = *a* is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3.+\ldots\infty$$

If *a* = 0, then $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!} + \ldots\ldots\infty$

The coefficient of $(x-a)^n$ in the Taylor series expansion of *f*(*x*) is $\frac{f^n(a)}{n!}$.

The general expansion of Taylor series is given by $f(x+h) = f(x) + h.\frac{f'(x)}{1!} + h^2.\frac{f''(x)}{2!} + h^3.\frac{f'''(x)}{3!} + \ldots\ldots\infty$

- Finding the expansion of e^x about *x* = 0

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1; f''(0) = f'''(0) = f''''(0) = \ldots\ldots = 1$

$f(x) = e^x = 1 + (x-0)\frac{1}{1!} + (x-0)^2.\frac{1}{2!} + (x-0)^3.\frac{1}{3!} + \ldots\ldots$

$$\Rightarrow \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots$$

1.6 Integral Calculus

If *F*(*x*) is anti-derivative of *f*(*x*) that is continuous and differentiable in (*a*, *b*), then we write $\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$. Here *f*(*x*) is integrand

If $f(x) > 0\ \forall a \le x \le b, the\ \int_a^b f(x)\,dx$ represents the shaded area in the given figure.

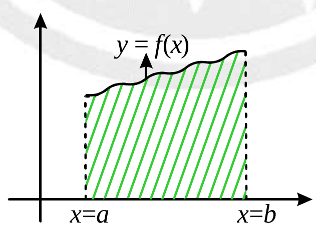


Fig.1.6 Integration of continuous function

1.6.1 Mean Value Theorem of Integration

If *f*(x) is continuous in [*a*, *b*] and differentiable in (*a*, *b*) then '∃' atleast one-point *C* ∈(*a*, *b*) such that

$$f'(C) = \frac{\int_a^b f(x)dx}{(b-a)}$$

-y

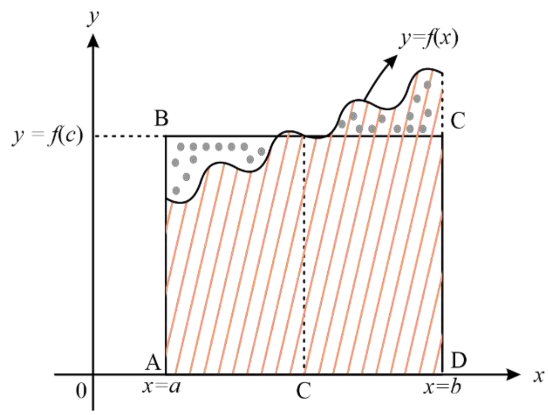


Fig.1.7 Mean value of integration

1.7 Newton-Leibnitz Rule

If $f(x)$ is continuously differentiable and $\phi(x)$, $\Psi(x)$ are two functions for which the 1st derivative exists, then

$$\frac{d}{dx}\left(\int_{\phi(x)}^{\psi(x)} f(x)dx\right) = f\big(\psi(x)\big).\psi'(x) - f\big(\phi(x)\big).\phi'(x)$$

Ex. $\frac{d}{dx}\left(\int_{x}^{x^2} \sin x\, dx\right) = \sin(x^2)\,.\,2x - \sin x\,.\,1 = 2x\sin(x^2) - \sin x$

1.8 Some Standard Integrals

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,(n \neq -1)$

2. $\int \frac{1}{x} dx = \log_e|x| + C$

3. $\int \sin x\, dx = -\cos x + C$

4. $\int \cos x\, dx = \sin x + C$

5. $\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$

6. $\int \tan x\, dx = -\int -\frac{\sin x}{\cos x} dx = -\log_e|\cos x| + C$

$\Rightarrow \int \tan x\, dx = \log_e|\sec x| + C$

7. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \log_e|\sin x| + C = -\log_e|\cos e\, cx| + C$

8. $\int \sec x\, dx = \log_e|\sec x + \tan x| + C$

$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \log_e|\sec x + \tan x| + C$

9. $\int \cos e\, cxdx = \log_e|\cos e\, cx - \cot x| + C$

10. $\int a^x dx = \frac{a^x}{\log_e a} + C$

11. $\int \frac{1}{x.\log_e a} dx = \log_a x + C$

12. $\int x^x (1 + \log_e x)\, dx = x^x + C$

13. $\int f(x)\,.\,f'(x) dx = \frac{1}{2}\left(f(x)\right)^2 + C$

14. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2.\sqrt{f(x)} + C$

15. If $f(x)$, $g(x)$ are two functions. that are differentiable, then

$\int f(x)\,.\,g(x) dx = f(x).\int g(x) - \int f'(x)\,(g(x) dx) dx + C$

Before integrating the product, the functions $f(x)$ and $g(x)$ are to be arranged according to the ILATE Principle.

Here, ILATE stands for INVERSE LOGARITHMIC ALGEBRAIC TRIGONOMETRIC EXPONENTIAL.

1.9 Properties of Definite Integrals

1. If $f(x)$ is differentiable in interval (a, b), then $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

2. If ∃ a point C ∈ (a, b) such that $f(x)$ is not differentiable, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

3. If $f(x)$ is continuously differentiable function,

$$\int_{-a}^{a} f(x)\, dx = 2 \times \int_0^a f(x)\, dx; \text{if } f(-x) = f(x)$$

⇒ $f(x)$" is even function")

$$0;\ \text{if } f(-x) = -f(x) (\Rightarrow f(x) \text{ is odd function})$$

4. $\int_0^{2a} f(x)\, dx = 2 \times \int_0^a f(x)\, dx$ if $f(2a - x) = f(x)$

5. $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

6. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} = \left(\frac{b-a}{2}\right)$

Ex.

(i) $\int_0^{\pi/2} \frac{\sin x}{\sin x+\cos x} = \frac{\pi}{4}$

(ii) $\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}} dx = \int_0^{\pi/2} \frac{1}{1+\left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)} dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}} dx = \frac{\pi}{4}$

(iii) $\int_2^3 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}} = \left(\frac{3-2}{2}\right) = \frac{1}{2}$

(iv) $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}} dx = \frac{\pi}{4}$

7. $\int_0^{\pi/2} \sin^m x\, dx = \int_0^{\pi/2} \cos^m x\ dx = \frac{(m-1)\times(m-3)\times(m-5)}{m\times(m-2)\times(m-4)} \times \ldots \left(\frac{1}{2}\right)$ (or) $\frac{2}{3} \times K$

Where $K = \pi/2$ if m is even

= 1 if m is odd.

8. $\int_0^{\pi} \frac{dx}{a^2 \cos^2 x+b^2 \sin^2 x} = \frac{\pi}{ab}$

9. $\int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x+b^2 \sin^2 x} = \frac{\pi}{2ab}$

1.10 Length of a Curve

If $y = f(x)$ is a differentiable function in (a, b), then the length L of the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

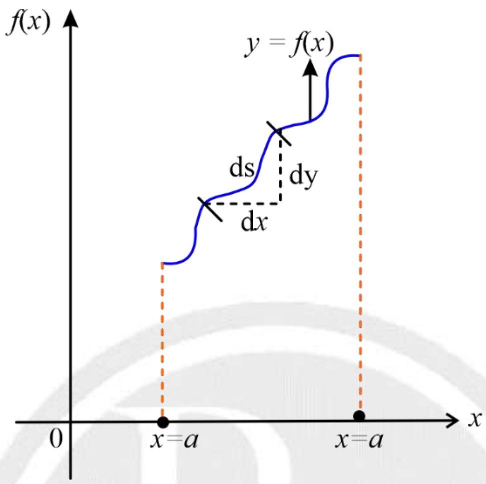


Fig.1.8 Length of the curve

1.11 Surface Area of Solid generated by revolving a curve about a fixed axis.

Elemental Surface Area

$$dA = 2\pi y \times ds = 2\pi y ds$$

$\Rightarrow$ Total surface area = A = $\int_{x=a}^{x=b} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

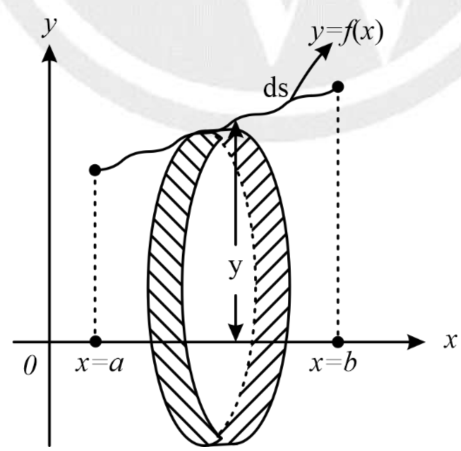


Fig.1.9 Surface area

1.12 Volume of the solid

The volume of the solid obtained by revolving the curve y = $f(x)$ between the lines $x = a$ and $x = b$ is given by

$$V = \int_{x=a}^{x=b} dV = \int_{x=a}^{x=b} \pi y^2 ds = \int_{x=a}^{x=b} \pi y^2 . \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$\Rightarrow$ $$V \approx \int_{x=a}^{x=b} \pi y^2 dx$$

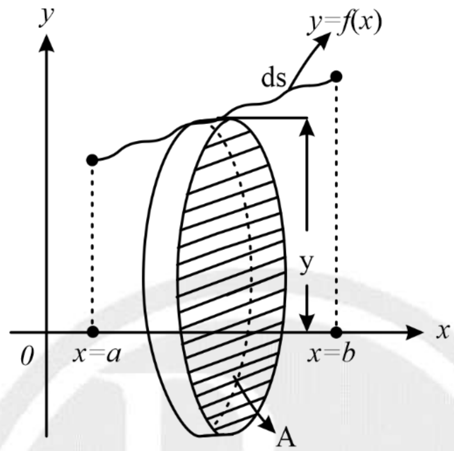


Fig.1.10 Volume of the solid

1.13 Gamma Function

The integral $\int_0^\infty e^{-x}.x^{n-1}\,dx (n > 0)$ is called Gamma function of n. It is denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$.

1.13.1 Properties of Gamma Function

(i) $\Gamma n = (n-1)!$

(ii) $\Gamma(n+1) = (n)!$

(iii) $\Gamma(n+1) = n\Gamma \mathrm{n}$

(iv) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

1.14 Beta Function

The function β $(m, n) = \int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$ $(m, n > 0)$ is called β function of m and n.

1.14.1 Properties of β function

(i) $\beta(m,n) = \frac{\Gamma m.\Gamma n}{\Gamma(\mathrm{m}+n)}$

(ii) $\beta(m,n) = \beta(n,m)$

(iii) $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iv) $sin^p\theta.cos^q\,\theta\,dx = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) (p,q > -1)$

1.15 Area under the curves

If the function f(x) > g(x) for all values of x between x=a and x=b then

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

$\Rightarrow$

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx$$

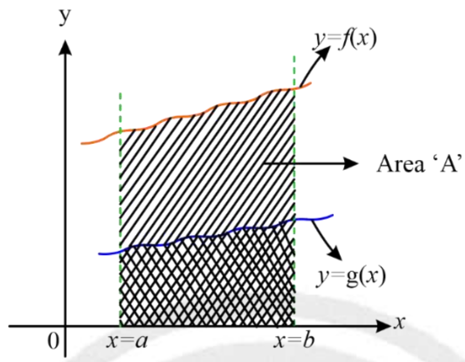


Fig.1.11 Area under curve

1.16 Multi Variable Calculus

a. Continuity of a function

A function *f*(x, y) is said to be continuous at (*a*, *b*) if $\lim\limits_{\substack{x\to a \\ y\to b}} f(x,y) = f(a,b)$

b. Differentiation of a two-variable function

If *f*(*x*, *y*) is a continuous function, then the derivative of *f*(*x*, *y*) with respect to *x* treating *y* as constant is given by p $= \frac{\partial f}{\partial x} = \lim\limits_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

The derivative of *f*(*x*, *y*) with respect to *y* treating *x* as constant is given by $q = \frac{\partial f}{\partial y} = \lim\limits_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$

c. Homogenous Function

A function *f* (x, y) is said to be homogenous function of degree 'n' if $f(kx, ky) = k^n. f(x,y)$.

Ex. $f(x,y) = x^3 - 3x^2y + 3xy^2 + y^3$

$\Rightarrow$ $f(kx,ky) = (kx)^3 - 3(kx)^2(ky) + 3(kx).(ky)^2 + (ky)^3$

= $k^3(x^3 - 3x^2y + 3xy^2 + y^3)$

= $k^3. f(x,y)$ $\Rightarrow$ $f(x,y)$ is a homogenous function of degree '3'.

d. Euler's Theorem

If *f* (*x*, *y*) is a homogeneous function of degree '*n*' then

(i) $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$

(ii) $x^2.\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

If $f(x, y) = g(x, y) + h(x, y) + \phi(x, y)$ where $g(x, y)$, $h(x, y)$ and $\phi(x, y)$ are homogenous functions of degrees m, n and p respectively, then

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = m.g(x,y) + n.h(x,y) + p.\phi(x,y)$$

$$x^2.\frac{\partial^2 t}{\partial x^2} + 2xy.\frac{\partial^2 f}{\partial x \partial y} + y^2.\frac{\partial^2 f}{\partial y^2} = m(m-1).g(x,y) + n(n-1).h(x,y) + p(p-1).\phi(x,y)$$

e. Concept of Maxima and Minima in Two Variables

If $f(x, y)$ is a two-variable differentiable function, then to find the maxima (or) minima.

Step-1: Calculate $p = \frac{\partial f}{\partial x}$ and $q = \frac{\partial f}{\partial y}$ and equate $p = 0$, $q = 0$

Let (x_0, y_0) be a stationary point.

Step-2: Calculate r, s, t where $r = \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)}$; $s = \left.\frac{\partial^2 f}{\partial x.\partial y}\right|_{(x_0,y_0)}$; $t = \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)}$

Case (i): If $rt - s^2 > 0$ and $r > 0$, then the function $f(x, y)$ has minimum at (x_0, y_0) and the minimum value is $f(x_0, y_0)$.

Case (ii): If $rt - s^2 > 0$ and $r < 0$, then the function $f(x, y)$ has maximum at (x_0, y_0) and the maximum value is $f(x_0, y_0)$.

Case (iii): If $rt - s^2 \leq 0$; then we cannot comment on the existence of maxima and minima.

Such stationary points where $rt - s^2 \leq 0$ are called **saddle points**.

f. Concept of Constraint Maxima and Minima (Method of Lagrange's unidentified multipliers).

If $f(x, y, z)$ is a continuous and differentiable function, such that the variables x, y and z are related/constrained by the equation $\phi(x, y, z) = C$ then to calculate the extreme value of $f(x, y, z)$ using Lagrange's Method of unidentified multipliers.

Step-1: Form the function $F(x, y, z) = f(x, y, z) + \lambda\{\phi(x, y, z) - C\}$ where λ is a multiplier.

Step-2: Calculate $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ and equate them to zero

Step-3: Equate the values of λ from the above 3 equations and obtain the relation between the variables x, y and z.

Step-4: Substitute the relation between x, y and z in $\phi(x, y, z) = C$ and get the values of x, y, z. Let they be (x_0, y_0, z_0).

Step-5: Calculate $f(x_0, y_0, z_0)$

The value $f(x_0, y_0, z_0)$ is the extreme value of $f(x, y, z)$.

g. Multiple Integrals

If $f(x, y)$ is continuous and differentiable at every point within a region 'R' bounded by some curves is given by

$$I = \iint_R f(x,y)\,dxdy$$

If there is a double integral,

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(x)} f(x, y) dydx \qquad [\text{Let } \psi(x) > \phi(x)]$$

Then I = area of the region bounded by the curves $y = \phi(x)$; $y = \Psi(x)$, $x = a$ and $x = b$ if $f(x, y) = 1$

Value of x – co-ordinate of centroid of the region bounded by $y = \phi(x)$; $y = \psi(x)$; $x = a$, $x = b$

if $f(x, y) = x$

h. Change of Orders of Integration

$$I = \int_{x=a}^{x=b} \int_{y=\phi(x)}^{y=\psi(r)} f(x, y) dydx \rightarrow I = \int_{x=c}^{x=d} \int_{x=g(y)}^{x=h(y)} f(x, y) dxdy$$

In change of order of Integration, the order of the Integrating variables changes and the limits as well.

❑❑❑

2 DIFFERENTIAL EQUATIONS

2.1 Differential Equation

The equation involving differential coefficients is called a Differential Equation (DE).

1. $x^2.\frac{dy}{dx} + y^2 = 0$

2. $\frac{\partial^2 T}{\partial x^2} = k \cdot \frac{\partial T}{\partial t}$

3. $x^2.\frac{\partial^2 u}{\partial x^2} + y^2.\frac{\partial^2 u}{\partial y^2} = 0$

2.1.1 Ordinary Differential Equations (ODE)

The DEs involving only one independent variable is called ordinary differential equation.

Ex.

(1) $x^2 \frac{dy}{dx} + y^2 = 0;$

(2) $e^{-x} \cdot \frac{dy}{dx} + y^2 = e^x$

2.1.2 Partial Differential Equations

The DEs involving two (or) more independent variables are called Partial Differential Equations (PDEs).

Ex.

$$\frac{\partial^2 u}{\partial x^2} = C^2 \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{K} \cdot \frac{\partial u}{\partial t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.1.3 Order and Degree of a Differential Equation

Order of a DE

The order of the highest derivative that occurs in a DE is called order of a DE.

Ex.

(1) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - y = 0$ → Order = 2

(2) $\frac{dy}{dx} + 2 \cdot \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} - 3x^2 = e^x$ → Order = 3

(3) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 u}{\partial t^2}$ → Order = 2

(4) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\alpha} \frac{\partial u}{\partial t}$ → Order = 2

2.1.4 Degree of a Differential Equation

The Degree of the highest order derivative that occurs in a DE when the DE is free from fractional (or) radical powers.

Ex.

(1) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + 2\left(\frac{dy}{dx}\right)^3 - 3y = 0$ is 1.

(2) The Degree of the DE $\left(\frac{d^2y}{dx^2}\right)^1 + \sqrt{\left(\frac{dy}{dx}\right)^3} + 4y = 0$ is 2

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 = \left(-\sqrt{\left(\frac{dy}{dx}\right)^3} + 4y\right)^2$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^2 = \left(\frac{dy}{dx}\right)^3 + 4y$$

2.2 Formation of Differential Equations

If a solution $y = f(x)$ contains n arbitrary constants in it, then differentiate y for n times and calculate $y', y'', y''', \ldots . y^{(n)}$ So, from the $(n + 1)$ equations available,try to eliminate the arbitrary constants in $y = f(x)$

- The different equation formed for the solution $y = c_1 e^{K_1 x} + C_2 e^{K_2 x}$ where C_1, C_2 are arbitrary constants is

$$\frac{d^2y}{dx^2} - (K_1 + K_2)\frac{dy}{dx} + (K_1 . K_2)y$$

- If the solution is $y = C_1 e^{K_1 x} + C_2 e^{K_2 x} + C_3 e^{K_3 x}$ where C_1, C_2, C_3 are arbitrary constants, then the DE is $y''' - (K_1 + K_2 + K_3)y'' + (K_1K_2 + K_2K_3 + K_3K_1)y' - (K_1K_2K_3)y = 0$

2.2.1 First Order DE

The general form of a 1st order DE is given by $\frac{dy}{dx} = f(x, y)$

If $\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = f(x, y)$

- $N(x, y)dy + M(x, y)dx = 0$
- $Mdx + Ndy = 0$ where M, N are functions of x and y.

2.2.2 Linear ODE:

A DE is said to be linear if it does not contain the higher power terms of dependent variable $\left(y^2, y^3, y^4, \ldots, \left(\frac{dy}{dx}\right)^2, \left(\frac{dy}{dx}\right)^3, \ldots\right)$ and also the terms containing the product of dependent variable and its differential coefficient $\left(y.\frac{dy}{dx}, y^2\frac{dy}{dx}, y\left(\frac{dy}{dx}\right)^2, \ldots\right)$

Ex.

(1) $x^2.\frac{d^2y}{dx^2} - 5x\frac{dy}{dx} + 6y = 0$

(2) $\frac{dy}{dx} = 5y = \sin x$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin y = 0$ $\quad\rightarrow$ Non-linear DE

Here, $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} \ldots$

- $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + \sin x = 4y$ $\quad\rightarrow$ Linear DE

2.3 Solving of Differential Equations

2.3.1 Solving of 1st Order DE

(i) Variable-separable form

If the 1st order DE is given by $\frac{dy}{dx} = \phi(x).\psi(y)$

$\Rightarrow \int \frac{1}{\Psi(y)} dy = \int \phi(x) dx$

On integrating we have solution of the given DE

(ii) Homogenous 1st Order

If the 1st order DE is of the form $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$

Such that both $M(x, y)$ and $N(x, y)$ are homogenous functions of same degree, then we say that the DE is homogeneous.

Ex.

(1) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

(2) $\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$

(a and b are not zero at the same time; a' and b' are not zero at the same time)

If the DE $\frac{dy}{dx} = \frac{M(r,y)}{N(x,y)}$ is a homogeneous DE, then the equation can be converted to Variable Separable form if we substitute $y = Vx$

2.3.2 Exact Differential Equations

The DE $Mdx + Ndy = 0$ where M, N are functions of x and y is said to be an Exact Differential Equation if there exist a function f(x, y) such that $Mdx + Ndy = d\big(f(x,y)\big)$

Mathematical condition to check the Exactness of a differential equation is

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

(i) Solution of an Exact DE

If $M(x, y)$ dx + $N(x, y)$ dy = 0 is an Exact differential Equation, then the solution of the DE is given by $\int_{y=\text{const}} M(x,y)dx + \int(\text{terms not containing } x \text{ in } N)\, dy = C$

(ii) Integrating Factor

The function which, when multiplied to a non-exact DE converts the DE to exact DE.

Ex.

(1) $\frac{1}{y^2}$ is an integrating factor of $ydx - xdy = 0$

(2) $\frac{1}{y}$ is an integrating factor of $x^2dy - xydx = 0$

2.3.3 Methods of Writing the Integrating Factors (I.F.)

(i) If $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous DE, then I.F. $= \frac{1}{Mx+Ny}$ $(Mx + Ny \neq 0)$

(ii)If $Mdx + Ndy = 0$ is of the form $yf(xy)dx + xg(xy)dy = 0$ then I.F. $= \frac{1}{Mx-Ny}$, $(M_x - N_y \neq 0)$

(iii)For a DE, $Mdx + Ndy = 0$, If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then $e^{\int f(x)dx}$ is the integrating factor.

(iv)For the DE, $Mdx + Ndy = 0$, if $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = g(y)$ then $e^{\int g(y)dy}$ is the integrating factor.

2.3.4 Leibnitz Linear Equation

The DE of the form $\frac{dy}{dx} + Py = Q$ where P, Q are functions of x alone, is called Leibnitz Linear Equation

Integrating factor of the equation is $e^{\int Pdx}$

Solution of the Differential Equation is : $y.e^{\int Pdx} = \int Q.(e^{\int Pdx})\, dx$ + C where C is arbitrary constant.

2.3.5. Non-linear Equations Convertible to Leibnitz Linear Form

Bernoulli's Equation

Case-I

$$\frac{dy}{dx} + Py = Q.y^n (n > 1, n \neq 1)$$

(P, Q are functions of x alone)

$\Rightarrow \quad \frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^n}P_{y'} = \frac{Qy^n}{y^n}$

$\Rightarrow \quad y^{-n} \cdot \frac{dy}{dx} + y^{1-n} \cdot P = Q$

Let $\quad y^{1-n} = z$

$\Rightarrow \quad (1-n)y^{-n}\frac{dy}{dz} = \frac{dz}{dx}$

$\Rightarrow \quad y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dz}{dx}$

$\therefore \quad \frac{1}{(1-n)}\frac{dz}{dx} + zP = Q$

$\Rightarrow \quad \frac{dz}{dx} + (1-n)Pz = (1-n)Q$ [Leibnitz Linear Equation]

Case-II

$f'(y)\frac{dy}{dx} + Pf(y) = Q$

where P, Q are functions of x alone.

Let $f(y) = z$

$\Rightarrow \quad f'(y)\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad \frac{dz}{dx} + Pz = Q$ [Leibnitz Linear Equation]

2.3.6 Applications of 1st order DE

Newton's Law of Cooling

The rate of change of temperature of a body placed in an ambience of temperature T_∞ is directional proportional to the temperature difference between the body and the ambient.

$\frac{dT}{dt} \propto -(T - T_\infty)$ where $T_\infty \to$ Ambient Temperature ($T > T_\infty$)

$\frac{dT}{dt} \propto (T_\infty - T)$

$\frac{dT}{dt} = -K(T - T_\infty)$

Radioactive Growth / Decay

The rate of growth/decay on any radioactive substance at any instant is directly proportional to concentration of the substance that is available at that instant.

- $\frac{dN}{dt} \propto N \rightarrow$ For growth
  $\Rightarrow \frac{dN}{dt} = KN$
  $\Rightarrow \int \frac{1}{N} dN = \int K dt$
- $\frac{dN}{dt} \propto -N \rightarrow$ For decay
  $\Rightarrow \frac{dN}{dt} = -KN$
  $\Rightarrow log_e N = Kt + C$
  $\Rightarrow N = e^{Kt} C$

2.4 Higher Order Differential Equations

The general form of Higher order Differential Equations is given by

$$K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X \qquad \ldots(1)$$

If $K_1, K_2, K_3, K_4, \ldots K_n, X$ are functions of x alone then (1) is called Linear Higher Order Linear DE with variable coefficients.

If $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and X is a function of 'x' alone, then (1) is called Higher Order Linear DE with constant coefficients.

2.5 Higher Order Linear Differential Equations with Constant Coefficients

The DE $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + K_3 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + K_n y = X$ …. (1) is said to be a higher order linear DE with constant coefficients if $K_1, K_2, K_3, K_4, \ldots K_n$ are constants and 'X' is a function of x alone.

If $X = 0$, then (1) is called Homogeneous DE

If $X \neq 0$, then (1) is called Non-Homogeneous DE.

2.5.1 Solution of Higher order Linear Differential Equation

$$Y = y_c + y_p$$

$y_c \rightarrow$ Complimentary function; $y_p \rightarrow$ Particular Integral

(Solution of homogeneous part; ($X = 0$); (Solution of Non-Homogeneous Part; ($X \neq 0$))

If $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + K_{n-1} \frac{dy}{dx} + K_n y = X$ …. (1) is a linear DE with constant coefficients.

2.5.2 Rules for Writing the Complete Solution of $\big(f(D)\big)y = X$:

(i) Form the auxiliary equation of $\big(f(D)\big)y = X$ i.e. $f(M) = 0$

(ii) Depending on the roots of the auxiliary equation $(f(M) = 0)$, we write the complimentary function.

(iii) Calculate the Particular Integral $y_P = \frac{1}{(f(D))}X$.

(iv) Write the total solution of the equation $y = y_C + y_P$.

2.5.3 Rules for Writing the Complementary Function

(i) If the roots of $f(M) = 0$ are $M_1, M_2, M_3, \ldots$ $(M_1, M_2, M_3, \ldots \in$ Rational)

Then $y_C = C_1e^{M_1x} + C_2e^{M_2x} + C_3e^{M_3x} + \ldots$ where C_1, C_2, C_3,…. Are arbitrary constants)

(ii) If the roots of $f(M) = 0$ are $M_1, M_1, M_3 \ldots\ldots (M_1, M_3 \ldots\ldots \in$ Rational)

Then $y_C = (C_1x + C_2)e^{M_1x} + C_3.e^{M_3x} + \ldots$ Where C_1, C_2, C_3, …. are arbitrary constants).

(iii) If the roots of $f(M) = 0$ are $M_1, M_1, M_1, M_4 \ldots$ (Where $\in M_1, M_4 \ldots \in$ Rational)

Then $y_C = (C_1x^2 + C_2x + C_3)e^{M_1x} + C_4e^{M_4x} + \ldots$ (where C_1, C_2, C_3 …. Are arbitrary constants)

(iv) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, M_3, M_4 \ldots$ then $y_C = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x) + C_3e^{M_3x} + C_4e^{M_4x} + \ldots$

(v) If the roots of $f(M) = 0$ are $\alpha + i\beta, \alpha - i\beta, \alpha + i\beta, \alpha - i\beta, M_5, M_6, \ldots$ then

$y_c = e^{\alpha x}((C_1x + C_2)\cos\beta x + (C_3x + C_4)\sin\beta x) + C_5e^{M_5x} + C_6e^{M_6x} + \ldots$

(vi) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}$, $\alpha - \sqrt{\beta}$, M_3, M_4, …. then $y_C = e^{\alpha x}\{C_1 \sinh\sqrt{\beta}\,x + C_2 \cosh\sqrt{\beta}\,x\} + C_3e^{M_3x} + C_4e^{M_4x} + \ldots$

(vii) If the roots of $f(M) = 0$ are $\alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, \alpha + \sqrt{\beta}, \alpha - \sqrt{\beta}, M_5, M_6, \ldots$ then $y_C = e^{\alpha x}\{(C_1x + C_2)\sinh\sqrt{\beta}\,x + (C_3x + C_4)\cosh\sqrt{\beta}\,x\} + C_5e^{M_5x} + C_6.e^{M_6x} + \ldots$

2.5.4 Rules for writing the particular Integral

(i) If $X = e^{ax}$,

$y_P = \frac{1}{f(0)}e^{ax} = \frac{1}{f(a)}e^{aX}$ (if $f(a) \neq 0$)

If $f(a) = 0$, then $y_P = x\frac{1}{f'(a)}e^{ax}$ (if $f'(a) \neq 0$)

If $f'(a) = 0$, then $y_p = x^2.\frac{1}{f''(a)}e^{ax}$ (if $f''(a) \neq 0$) and so on.

Solve $\frac{d^2y}{dx^2} - 5.\frac{dy}{dx} + 6y = e^{2x}$

Sol. Aux. $Eq^n \rightarrow M^2 - 5M + 6 = 0 \Rightarrow M = 2,3$

$y_C = C_1e^{2x} + C_2e^{3x}$

$y_P = \frac{1}{D^2-5D+6}e^{2x}$ since $f(2) = 0$

$\Rightarrow y_P = x\frac{1}{(2D-5)}e^{2x} = x.\frac{1}{(2(2)-5)}e^{2x}$

$\frac{x}{-1}e^{2x} = -x.e^{2x}$

(ii) If $X = sin(ax + b)\ (or)\ \cos(ax + b)$

$y_P = \frac{1}{f(D)} sin(ax + b)$

Replace D^2 by $-a^2$ in $f(D)$

If the denominator is the form $CD + d$ then rationalize the denominator and replace D^2 by $-a^2$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = sin(2x + 3)$

$y_p = \frac{1}{D^2-5D+6} \cdot sin(2x + 3)$

$a = 2 \Rightarrow -a^2 = -4$

$\Rightarrow\ y_p = \frac{1}{-4-5D+6} sin(2x + 3)$

$\Rightarrow\ y_p = \frac{1}{2-5D} \times \frac{2+5D}{2+5D} . sin(2x + 3)$

$\Rightarrow\ y_p = \frac{2+5D}{4-25D^2} sin(2x + 3) = \frac{2+5D}{4-25(-4)} \sin(2x + 3) = \frac{1}{104}(2. \sin(2x + 3) + 10. \cos(2x + 3))$

(iii) If $X = x^m$

$y_P = \frac{1}{f(D)} x^m$

$\Rightarrow\ y_P = [f(D)]^{-1} x^m$

Calculate y_P for the DE $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$

$y_P = \frac{1}{D^2-5D+6} x^2$

$= \frac{1}{6\left(1+\left(\frac{D^2-5D}{6}\right)\right)} x^2$

$= \frac{1}{6}\left(1 + \left(\frac{D^2-5D}{6}\right)\right)^{-1} x^2$

$= \frac{1}{6}.\left\{1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 - \left(\frac{D^2-5D}{6}\right)^3 + \ldots\ldots\right\} x^2$

$= \frac{1}{6}\left\{x^2 - \frac{1}{6}(2 - 5(2x)) + \frac{1}{36}\{25(2)\}\right\}$

$= \frac{1}{6}x^2 - \frac{1}{18} + \frac{5x}{18} + \frac{25}{108}$

$= \frac{1}{6}x^2 + \frac{5x}{18} + \frac{19}{108}$

(iv) If $X = e^{ax}V$, then

$y_P = \frac{1}{f(D)} \cdot e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V$

2.6 Method of Variation of Parameters

If the second order linear DE with constant coefficients is given by $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + qy = X$, and if $y_C = C_1y_1 + C_2y_2$ then y_P (Particular integral of the DE) is given by

$$y_P = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

Where W $\rightarrow$ Wronskian of the solution, $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

2.7 Euler Cauchy Equation (Higher order linear DE with Variable Coefficients)

The DE of the form $x^n \frac{d^ny}{dx^n} + K_1 \cdot x^{n-1} \cdot \frac{d^{n-1}y}{dx^{n-1}} + k_2 \cdot x^{n-2} \cdot \frac{d^{n-2}y}{dx^{n-2}} + \ldots.. + K_{n-1} \cdot x \cdot \frac{dy}{dx} + K_n y = X$ Where $K_1, K_2, K_3, \ldots. K_n$ are constants is called Euler-Cauchy Equation

2.7.1 Procedure to solve Euler Cauchy Equations

Let $x^n \cdot \frac{d^ny}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + K_2 \cdot x^{n-2} \cdot \frac{d^{n-2}y}{dx^{n-2}} + \ldots.. + K_{n-1} \cdot x \cdot \frac{dy}{dx} + K_n y = X$ ...(1)

$\left(x^n \cdot \frac{d^n}{dx^n} + K_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + K_2 \cdot x^{n-2} \cdot \frac{d^{n-2}}{dx^{n-2}} + \ldots.. + K_{n-1} \cdot x \cdot \frac{d}{dx} + K_n\right) y = X$

Let $x = e^z \;\Rightarrow\; z = \log_e x$

$$x \frac{d}{dx} = \frac{d}{dz} = D$$

$$x^2 \cdot \frac{d^2}{dx^2} = \frac{d}{dz}\left(\frac{d}{dz} - 1\right) = D(D-1)$$

$x^3 \cdot \frac{d^3}{dx^3} = D(D-1)(D-2)$ and so, on

$(1) \Rightarrow \{D(D-1)(D-2) \ldots \ldots . (D-(n-1) + K_1 D(D-1)(D-2) \ldots .. D-(n-2) + \ldots .. K_{n-1}D + K_n)\}y = X$

Where $D = \frac{d}{dz}$

$\Rightarrow \quad (f(D))y = z \rightarrow$ Higher order linear DE with constant

$f(D) \rightarrow$ Polynomial in terms of D with constant coefficients.

2.8 Partial differential equation

2.8.1 The general form of a 2nd order Partial differential equation

The general form of a 2nd order Partial differential equation is given by

$$A.\frac{\partial^2 u}{\partial x^2} + B \bullet \frac{\partial^2 u}{\partial x dy} + \bullet C \frac{\partial^2 u}{\partial y^2} + D \bullet \frac{\partial u}{\partial x} + E \bullet \frac{\partial u}{\partial y} + F \bullet u = G$$

For the nature of the above equation to be

(a) Elliptic $\rightarrow$ $B^2 - 4AC < 0$

(b) Parabolic $\rightarrow$ $B^2 - 4AC = 0$

(c) Hyperbolic $\rightarrow$ $B^2 - 4AC > 0$

2.9 Heat Equation

The heat equation in 1 – D is of the form, $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \bullet \frac{\partial u}{\partial t}$. Where 'c' is a constant.

Solution of heat equation is given by u (x, t) = (c_1 cos Px + c_2. s in Px). $e^{-c^2 pt^2}$

2. 10 Laplace equation

The Laplace equation is 2-D is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

2.10.1 Possible solution of Laplace equation

Possible solution of Laplace equation is given by

$$u\,(x,\,y) = \left(c_1 \bullet e^{Px} + c^2 \bullet e^{-px}\right).(c_3 \cos Py + c_u \bullet \sin Py)$$

$$u\,(x,\,y) = \left(c_1 \cos Px + c_2 \sin px\right).(c_3.e^{py} + cy.e^{-Py})$$

$$u\,(x,\,y) = \left(c_1 x + c_2\right).(c_3.y + c_u)$$

Where c_1, c_2, c_3, c_4 are arbitrary constants and the solution is picked depending on boundary conditions.

❑❑❑

3 VECTOR CALCULUS

3.1 Vector Product / Cross Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the cross product of the two vectors is denoted by $\vec{a} \times \vec{b}$ and it is given by $\vec{a} \times \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot sin\,\theta \cdot \hat{n}$

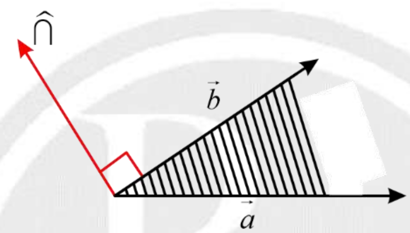


Fig. 3.1 Cross product

$\hat{n} \rightarrow$ unit vector passing through the point of intersection of $\vec{a}$ and $\vec{b}$ and lying perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

3.2 Dot / Scalar Product

If $\vec{a}$ and $\vec{b}$ are two vectors, then the dot /scalar product of the two vectors is denoted by $\vec{a}.\vec{b}$ and it is given by $\vec{a}.\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot cos\,\theta$ where $\theta$is the angle between the vectors $\vec{a}\ and\ \vec{b}$.

Note:

$$\left|(\vec{a} \cdot \vec{b})\right|^2 + \left|\vec{a} \times \vec{b}\right|^2 = |\vec{a}|^2 \cdot \left|\vec{b}\right|^2 . cos^2\,\theta + |\vec{a}|^2 \cdot \left|\vec{b}\right|^2 \cdot sin^2\,\theta = |\vec{a}|^2 \cdot \left|\vec{b}\right|^2$$

If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

$\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then $\vec{a} \cdot (\vec{b} \times \vec{c}) = [\vec{a}\vec{b}\vec{c}] \Rightarrow\ [\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3.3 Differentiation of Vector Point functions

If $\vec{R}(t)$ is a vector point function, then the derivative of $\vec{R}(t)$ is given by

$$\frac{d\vec{R}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t)-\vec{R}(t)}{\Delta t}$$

If $\vec{R}(t) = f(t)\,\hat{\imath} + g(t)\,\hat{\jmath}$ then $\frac{d\vec{R}(t)}{dt} = f'(t)\hat{\imath} + g'(t)\hat{\jmath}$

Ex.

If $\vec{R}(t) = \sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \Rightarrow \frac{d\vec{R}(t)}{dt} = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath}$

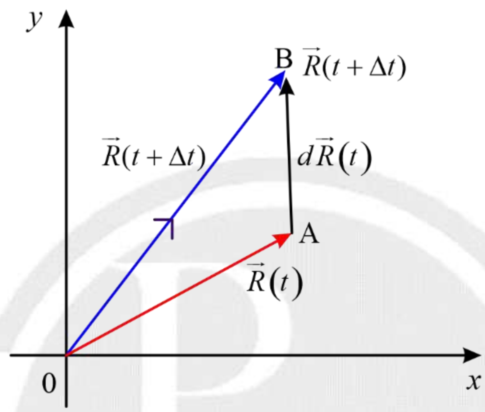


Fig. 3.2

3.3.1 Differentiation of Product of two vectors

$$\frac{d}{dt}\left(\vec{a}(t)\cdot\vec{b}(t)\right) = \vec{a}(t)\cdot\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}.\vec{b}(t)$$

$$\frac{d}{dt}\left(\vec{a}(t)\times\vec{b}(t)\right) = \vec{a}(t)\times\frac{d\vec{b}(t)}{dt} + \frac{d\vec{a}(t)}{dt}\times\vec{b}(t)$$

If $\vec{F}(t)$ is a vector point function with constant magnitude, then $\vec{F}(t)\cdot\frac{d}{dt}\vec{F}(t) = 0$.

If $\vec{F}(t)$ is a vector point function with constant direction, then $\vec{F}(t)\times\frac{d}{dt}\vec{F}(t) = \vec{0}$.

3.4 Del operator

The Vector operator $\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$ is called the differential operator in vector and it denoted as Del (or) ∇

$$\nabla = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}.$$

3.4.1 Gradient of a Scalar Point Function

If $\phi(x,y,z)$ is a Scalar Point function, then the gradient (change) of $\phi(x,y,z)$ is denoted by grad $\phi (or) \nabla\phi$ and it is given by

$$\nabla\phi = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\vec{\imath} + \frac{\partial\phi}{\partial y}\vec{\jmath} + \frac{\partial\phi}{\partial z}\vec{k}.$$

Note:

If $\vec{F}(x, y, z)$ is irrotational vector field ($\nabla \times \vec{F} = \vec{0}$), then definitely there exists a scalar point function $\emptyset(x, y, z)$ such that $\vec{F}(x, y, z) = grad\ \emptyset$.

If $\phi(x, y, z) = c$ is a level surface then $\nabla\phi|_{P(x_0, y_0, z_0)}$ gives the gradient of $\phi(x, y, z)$ at Point ' P '.

$$|\nabla\phi|_P| = \sqrt{\left(\frac{\partial\phi}{\partial x}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial y}\Big|_P\right)^2 + \left(\frac{\partial\phi}{\partial z}\Big|_P\right)^2}.$$

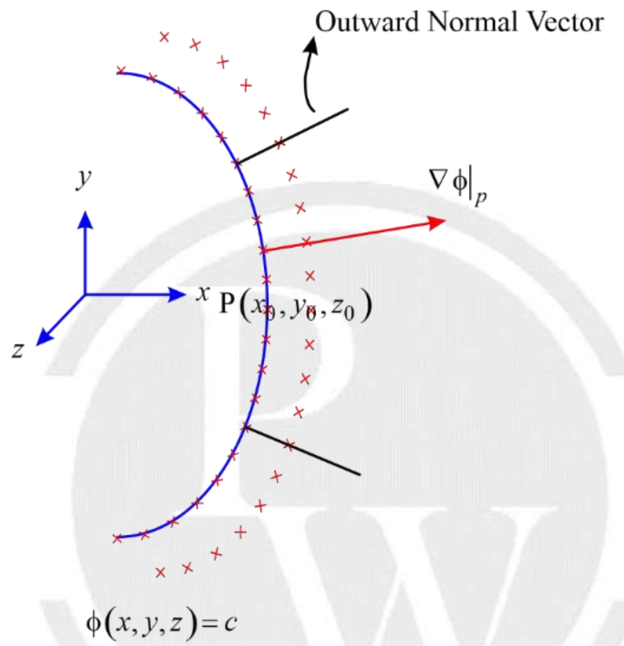


Fig. 3.3

→ $\nabla\phi|_p$ gives the change of $\phi(x, y, z)$ in the direction Normal to the surface $\phi(x, y, z) = c$ at P(x, y, z).

3.5 Directional Derivative

If $\phi(x, y, z) = c$ is a level surface, then the derivative of $\phi(x, y, z)$ at Point ' P ' in the direction of $\vec{a}$ is called Directional Derivative of $\phi(x, y, z)$ in the direction of $\vec{a}$.
It is given by

Direction Derivative = $\nabla\phi|_p \cdot \hat{a}$

$$= \nabla\phi\Big|_p \cdot \frac{\vec{a}}{|\vec{a}|} = \left|\nabla\phi|_p\right| \cdot |\vec{a}| \cdot \frac{\cos\theta}{|\vec{a}|} = \left|\nabla\phi|_p \cdot \cos\theta\right|$$

Directional Derivative of $\phi(x, y, z)$ at P in the direction of $\vec{a}$ is

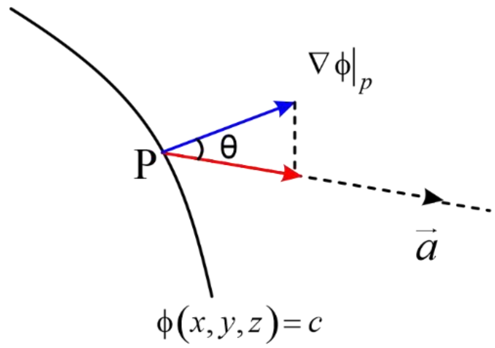


Fig.3.4

Directional derivative

$DD = |\nabla\phi|_p| \cdot \cos\theta$ where ' θ ' is angle between $\nabla\phi\ |_p$ and $\vec{a}$.

For Directional derivative to be maximum $\cos\theta = 1 \Rightarrow \theta = 0°$

$\Rightarrow$ The change of $\phi(x,y,z)$ at Point ' P ' is Maximum in the direction of Normal to $\phi(x,y,z)$

Maximum Change of $\phi(x,y,z)$ at $'p' = |\nabla\phi|_p|$

3.5.1 Del operated-on Vector Point functions

If Del is a differential operator and $\vec{F}(x,y,z)$ is a vector Point function then the Del operator is operated on $\vec{F}(x,y,z)$ in two Ways.

(i) $\nabla \cdot \vec{F} \rightarrow$ Divergence

(ii) $\nabla \times \vec{F} \rightarrow$Curl

(i) Divergence of a Vector Point function:

If $\vec{F}(x,y,z)$ is a Vector Point function, then the divergence of $\vec{F}(x,y,z)$ is denoted by div $\vec{F}$(or)$\nabla \cdot \vec{F}$ and for any $\vec{F}(x,y,z) = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ the divergence is given by

$$\text{div} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}\right) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

If div $\cdot \vec{F} = 0$, then $\vec{F}(x,y,z)$ is called Solenoidal (or) Incompressible flow Vector.

(ii) Curl of a Vector Point Function:

If $\vec{F}(x,y,z)$ is a Vector Point function, then the curl of $\vec{F}(x,y,z)$ is denoted by $Curl\ \vec{F}(or)\nabla \times \vec{F}$ and for any $\vec{F}(x,y,z) = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$, the curl of $\vec{F}(x,y,z)$is given by

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \times \left(F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}\right) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

If curl $\vec{F} = \vec{0}$; then $\vec{F}$ is called Irrotational Flow Vector.

(iii) Properties of div, Curl & Grad:

If $\phi(x,y,z)$ and $\vec{F}(x,y,z)$ are a scalar point function and a vector point function respectively, then

(a) curl (grad ϕ)=$\vec{0}$

(b) div(curl $\vec{F}$) $= 0$

(c) div(grad ϕ) $= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$

(iv) Angle between two Intersecting Surfaces:

If $\phi_1(x,y,z) = c_1$ & $\phi_2(x,y,z) = c_2$ are two surfaces intersecting at 'P', then the angle of Intersection 'θ' is given by

$$\cos\theta = \frac{\nabla\phi_1|_p \cdot \nabla\phi_2|_p}{|\nabla\phi_1|_p|\cdot|\nabla\phi_2|_p|}$$

3.6 Vector Integration

3.6.1 Line Integrals

If $\vec{F}(x,y,z) = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point along the path C , then the Integral of $\vec{F}(x,y,z)$ from Point 'A' to point 'B' along a path is given by $\int_{A,C}^{B} \vec{F}\cdot d\vec{r}$

where $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

$$\int_{A,C}^{B} \vec{F}\cdot d\vec{r} = \int_{A,C}^{B} (F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k})\cdot(dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k})$$

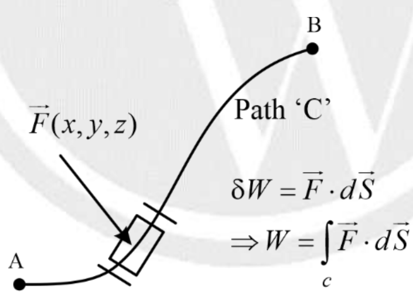


Fig.3.5 Line Integral

If $\vec{F}(x,y,z)$ is Irrotational Vector Point Function, (i.e., Curl $\vec{F} = \vec{0}$) then $\int_A^B \vec{F}\cdot d\vec{r}$ is independent of the path followed between the points A and B.

If $\vec{F}(x,y,z)$ is Irrotational Vector Point Function, then

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla\phi\cdot d\vec{r} \text{ where} \vec{F} = \nabla\phi$$

$$= \phi|_B - \phi|_A$$

3.6.2 Surface Integral

If $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point on a surface 'S', then the surface integral of $\vec{F}(x,y,z)$ on the surface 'S' is given by $\int_S \vec{F} \cdot d\vec{s}$

Where $d\vec{s} = ds.\hat{n}$ and $\hat{n}$ is the outward unit normal vector to the surface at ds and

$$ds = \frac{dx.dy}{|\hat{n}.\hat{k}|} = \frac{dy.dz}{|\hat{n}.\hat{i}|} = \frac{dx.dz}{|\hat{n}.\hat{j}|}:$$

3.6.3 Volume Integral

If $\vec{F}(x,y,z) = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$ is a continuous & differentiable Vector Point function at every point over a volume V, then the volume integral of $\vec{F}(x,y,z)$ on the volume 'V' is given by $\int_V \vec{F} \cdot dv$.

3.6.4 Greens Theorem: (Connects closed line Integral to surface Integral)

If $\vec{F}(x,y) = F_x\vec{i} + F_y\vec{j}$ and if the first order derivatives of F_x& $F_y$are continuous at every point with in a region 'R' bounded by a closed path 'C', then

$$\oint_C \vec{F} d\vec{r} = \oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dxdy$$

$$\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

3.6.5 Gauss – Divergence Theorem: (Connects closed surface integral to a Volume Integral)

If 'S' is a closed surface enclosing a volume 'V' and $\vec{F}$ is continuous and differentiable at every point on the closed surface 'S', then the closed surface integral $\oint_s \vec{F}.d\vec{s} = \iiint_V div\,\vec{F}.dV$

3.6.7 Stokes Theorem: (Connect Closed line integral to surface Integral)

If $\vec{F}$ is continuous and differentiable at every point within a region 'R' (on a surface S) bounded by a closed path 'C', then

$$\oint_C \vec{F} d\vec{r} = \iint_R curl\,\vec{F}.d\vec{S}$$

❑❑❑

4 LINEAR ALGEBRA

4.1 Matrix

An array of elements in horizontal lines (Rows) and Vertical Lines (Columns) is called a Matrix.

Ex.
$$A = \begin{bmatrix} i & n & d & i & a \\ j & a & p & a & n \end{bmatrix}$$

4.1.1 Size of Matrix

If a matrix has 'm' rows and 'n' columns, then we say that the size of the matrix is m × n (read as m by n)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots\dots\dots a_{1n} \\ a_{21} & a_{22} & a_{23} \dots\dots\dots a_{2n} \\ . & . & .. \\ . & . & .. \\ a_{m1} & a_{m2} & a_{m3} \dots\dots\dots a_{mn} \end{bmatrix}; \; A = \left[a_{ij}\right]_{m\times n} \text{ such that } 1 \le i \le m, 1 \le j \le n \text{ and } a_{ij} = f(i,j)$$

4.1.2 Addition of Matrices

(i) Two matrices $A = \left[a_{ij}\right]_{m\times n}$ & $B = \left[b_{ij}\right]_{p\times q}$ can be added only if m = p & n = q.

(ii) Matrix Addition is commutative (A + B = B + A)

(iii) Matrix Addition is Associative. A + (B + C) = (A + B) + C

4.1.3 Multiplication of Matrices

The multiplication of two matrices $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{p\times q}$ $\left(\Rightarrow AB_{m\times q}\right)$ is feasible only if n = P.

$$A_{m\times n} \cdot B_{p\times q} = C$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

$$A_{3\times 3}B_{3\times 2} = \begin{bmatrix} a_{11}.b_{11} + a_{12}\cdot b_{21} + a_{13}.b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}.b_{11} + a_{22}.b_{21} + a_{23}.b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31.}b_{11} + a_{32}\cdot b_{21} + a_{33}\cdot b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3\times 2}$$

4.1.4 Properties of Multiplication of Matrices

(i) Matrix Multiplication Need not be commutative.

(ii) Matrix Multiplication is Associative (A(BC)) = ((AB)C)

(iii) Matrix Multiplication is distributive (A(B + C) = AB + AC)

(iv) The product of two Matrices $A_{m\times n}, B_{n\times q}$ (i.e. $AB_{m\times q}$) can be a zero matrix even if $A \neq O \& B \neq O$.

Ex. $A = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- For the multiplication of two matrices $A_{m\times n}$ & $B_{n\times q}$
  (i) The No. of Multiplications required = m n q
  (ii) The No. of Additions required = m (n –1) q

4.2 Types of Matrices

(1) Upper triangular Matrix: A matrix $A = [a_{ij}]; 1 \leq i, j \leq n$ is said to be an upper triangular matrix if

$$a_{ij} = 0 \ \forall \ i > j$$

(2) Lower Triangular Matrix: A matrix $A = [a_{ij}]_{n\times n}; 1 \leq i, j \leq n$ is said to be a lower Triangular Matrix

if $a_{ij} = 0 \forall i < j$

(3) Diagonal Matrix: A matrix $A = [a_{ij}], \forall\ 1 \leq i, j \leq n$ is said to be diagonal matrix if $a_{ij} = O \forall i \neq j$

Ex. $A = \begin{bmatrix} d_1 & O & O \\ O & d_2 & O \\ O & O & d_3 \end{bmatrix}$. Diagonal Matrix is also denoted as $A = diag[d_1, d_2, d_3]$

(4) Scalar Matrix: A Matrix 'A' = $[a_{ij}]$; $1 \leq i, j \leq n$is said to be a scalar Matrix if $a_{ij} = \begin{cases} k; i = j \\ O; 1 \neq j \end{cases}$

If K = 1, then A ⟶ Identity Matrix,

If K = O, then A ⟶ Null Matrix.

(5) Idempotent Matrix:

A Matrix '$A_{n\times n}$' is said to be an idempotent matrix if $A^2 = A$.

Ex. $A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

$\Rightarrow A^2 A \cdot A = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix} = A$

(6) Nilpotent Matrix: A non-zero matrix '$A_{n\times n}$' is said to be Nilpotent Matrix. if ∃ a value 'n' such that $n \in Z^+$ and $A^n = O$.

Ex. $\begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix} = A \Rightarrow A^2 = \begin{bmatrix} 4 & -1 \\ 16 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow A^2 = O$

The least of 'n' for which $A^n = 0$ is called Index of the Nilpotent Matrix.

(7) Orthogonal Matrix: A matrix A is said to be orthogonal if $\mathrm{A}.\mathrm{A}^{\mathrm{T}} = \mathrm{I}$

Ex. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = A$

(8) Involutory Matrix: A matrix A is said to be involutory if $\mathrm{A}^2 = \mathrm{I}$

Ex. $\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = A$

4.3 Transpose of a Matrix

For a given matrix $A = [a_{ij}]$; $1 \le i \le m, 1 \le j \le n$, we say that 'B' where $B = [b_{ij}]$, $i \le i \le n$ $i \le j \le m$ is transpose of the Matrix 'A' if $a_{ij} = b_{ji}$

4.3.1 Properties of Transpose of a Matrix

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T \cdot A^T$

(iii) $(KA)^T = KA^T$ where 'K' is a scalar.

4.4 Determinant

The summation of product of element of a row(or) column of a matrix with their corresponding Co-factors.

$$A \cdot adj(A) = |A| \cdot \mathrm{I}$$

4.4.1 Properties of Determinants

(i) If 'A' is a Square Matrix of size ' $n \times n$ ' and 'k' is a Scalar then

(a) $|K \cdot A_{n\times n}| = K^n \cdot |A_{n\times n}|$

(b) $|adj(A)| = |A|^{(n-1)}$

(c) $\left|adj\left(adj(A)\right)\right| = (|A|)^{(n-1)^2}$

(ii) $|AB| = |A| \cdot |B|$

(iii) $|(AB)^T| = |B^T| \cdot |A^T|$

(iv) If two rows (or) two columns of a determinant are interchanged, then determinant changes its sign.

(v) The determinant of an upper triangular Matrix/a lower triangular Matrix/a diagonal Matrix is product of the principal diagonal elements of the Matrix.

(vi) The determinant of Every Skew-Symmetric Matrix of odd order $(A_{n\times n})('n'isodd)$ is zero

(vii) The determinant of an orthogonal Matrix $A_{n\times n}$ is ± 1

(viii) The determinant of an Idempotent Matrix is either 0 (or) 1.

(ix) The determinant of an Involuntary Matrix is ± 1

(x) The determinant of a Nilpotent Matrix is always zero.

(xi) If the product of two Non-zero Matrices $A_{n\times n} \neq O; B_{n\times n} \neq O$ is a zero Matrix $((AB)_{n\times n} = O)$, then both $|A| = 0$ & $|B| = 0$.

(xii) If two rows (or) two columns of a Matrix are either equal or Proportional, then the determinant of the Matrix is equal to zero.

(xiii) The number of terms in the general expansion of a 'n × n' determinant is $\underline{n!}$

4.5 Rank of a Matrix

A real Number 'r' is said to be rank of a matrix '$A_{m\times n}$' if

(1) All minors of order $(r+1)\times(r+1)$ and above are zeros and

(2) '∃' atleast one Non-zero minor of order '$r\times r$' of the matrix 'A'.

It is mathematically denoted by $\rho(A)=r$

4.5.1 Properties of Rank of a Matrix

(i) $\rho(A_{m\times n})\le(m,n)$

(ii) $\rho(AB)\le min\ \{\rho(A),\rho(B)\}$

4.5.2 Row Echleon Form

A Matrix $A_{m\times n}$is said to be in row-echleon form if

(i) Number of zeroes before the 1st Non-zero element in any row is less then number of such zeroes in its succeeding row.

(ii) Zero rows (if any) should lie at the bottom of the Matrix.

$$\rho(A_{m\times n}) = \text{Number of non-zero rows in Row-Echleon form of A.}$$

4.6 System of Equations

The given system of equations

$$a_{11}x_1+a_{12}x_{12}+a_{13}x_3=b_1$$

$$a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2$$

$$a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3$$

can be written in Matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}=\boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

Ax = B

Coefficient Matrix | Variable Matrix | Constants Matrix

The system Ax = B is said to be homogeneous system if B = 0.

The system of Ax = B is said to be non-homogeneous system if $B\neq 0$.

4.6.1 Consistency of a non-homogeneous system of Equations

For above system of non – homogeneous equations, Ax = B; Augmented Matrix = [A/B] = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

(i) If $\rho(A) = \rho(A/B) =$ Number of unknowns, then the system Ax = B has unique solution.

(ii) If $\rho(A) = \rho(A/B) <$ Number of unknowns, then the system has infinitely many solutions.

(iii) If $\rho(A) \neq \rho(A/B)$, then the system has no solution.

No. of linearly independent solutions for a system of 'n' equations given by Ax = B is $n - \rho(A)$

4.6.2 Consistency of Homogeneous System of Equations

$$a_{11}x + a_{12}y + a_{13}z = 0$$
$$a_{21}x + a_{22}y + a_{23}z = 0$$
$$a_{31}x + a_{32}y + a_{33}z = 0$$

Ax = 0 ⇒ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $[A/B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{bmatrix}_{3\times 4}$

If $\rho(A) = \rho(A/B) = n$ $(i.e\ |A| \neq 0)$; the system has unique solution.

(Trivial solution; x = 0, y = 0, z = 0)

If $\rho(A) = \rho(A/B) < n(|A| = 0)$; the system has infinitely many solutions (Non-trivial solution exists for the system)

4.7 Linear Combination of Vectors

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' rows vectors, then the combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots\ldots + k_nx_n$ is called linear combination of $x_1, x_2, \ldots., x_n (k_1, k_2, k_3, \ldots.. k_n$ are scalars)

(1) The linear combination $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n$ is said to be linearly dependent if $k_1x_1 + k_2x_2 + k_3x_3 + \ldots.. + k_nx_n = 0$ when $k_1, k_2, k_3, \ldots.., k_n$ (NOT All zeroes).

If $x_1 = [a_1 \quad b_1 \quad c_1]; x_2[a_2 \quad b_2 \quad c_2]; x_3 = [a_3 \quad b_3 \quad c_3]$, then the vectors x_1, x_2, x_3 are said to be linearly dependent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

(2) The combination $k_1x_1 + k_2x_2 + \ldots\ldots + k_nx_n$ is said to be linearly independent if $k_1x_1 + k_2x_2 + \ldots\ldots.. + k_nx_n = 0$ when $k_1 = k_2 = k_3 = \ldots.. k_n = 0$

4.7.1 Eigen Values and Eigen Vectors

For any square Matrix $A_{n\times n}$, the equation $|A - \lambda I| = 0$ where 'λ' is a scalar is called the characteristic equation. The roots of the characteristic equation of a Matrix are called Eigen Values.

4.7.2 Properties of Eigen Values

(i) If $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$ are 'n' Eigen Values of $A_{n\times n}$, then

(a) Sum of Eigen Values of 'A' = $\lambda_1, \lambda_2 + \lambda_3 + \ldots + \lambda_n = \sum_{i=1}^{n} \lambda_i = tr(A)$ = Sum of Principal diagonal elements

(b) Product of all the Eigen Values of 'A' = $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \ldots\ldots \lambda_n = \prod_{i=1}^{n} \lambda_i = |A|$

(c) Eigen Values of A^m are $\lambda_1^m, \lambda_2^m, \lambda_3^m, \ldots\ldots \lambda_n^m$

(d) Eigen Values of adj(A) are $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots\ldots, \frac{|A|}{\lambda_n}$

(e) Eigen Values of A & A^T are same.

(f) Eigen Values of $k_1A + k_2I$ (Where k_1 and k_2 are scalar) are

$$k_1\lambda_1 + k_2, k_1\lambda_2 + k_2, k_1\lambda_3 + k_2, k_1\lambda_4 + k_2, \ldots\ldots\ldots k_1\lambda_n + k_2$$

(ii) '0' is always an Eigen Value of an odd order Skew-Symmetric Matrix.

(iii) Eigen Values of Real Symmetric Matrix are always real.

(iv) Eigen Values of Skew-Symmetric Matrix are either zero (or) purely Imaginary.

(v) The Eigen values of an Orthogonal Matrix are of unit modulus.

(vi) If sum of all the elements in a row (or Column) is constant (= k) for all the rows (or columns) in the matrix respectively, then 'k' is an Eigen Value of the Matrix.

Ex. If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and if $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = a_3 + b_3 + c_3 = k,$

then 'k' is an Eigen Value of 'A'.

(vii) The Eigen Values of an upper triangular Matrix, a lower triangular Matrix, a diagonal Matrix are the Principal diagonal elements of the Matrix.

4.8 Eigen Vector

A non-zero column vector $X_{n\times 1}$ is said to be an Eigen Vector of $A_{n\times n}$ corresponding to the Eigen Value 'λ', if $AX = \lambda X (X \neq 0)$.

4.8.1 Properties of Eigen Vectors

(i) Eigen Vectors of A & A^T are not same.
(ii) Eigen Vectors of A & A^M are same.
(iii) The Eigen Vectors of a Real Symmetric Matrix are always orthogonal.
(iv) The number of linearly independent Eigen Vectors of '$A_{n\times n}$' is equal to number of distinct Eigen Values of '$A_{n\times n}$'.

4.8.2 Cayley Hamilton Theorem

Every Matrix satisfies its own characteristic equation.

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5 PROBABILITY AND STATISTICS

5.1 Random Experiment

The experiment in which the outcome is uncertain is called a Random Experiment (RE).

Ex. Flipping a coin, rolling a pair of dice, Picking a ball from a bag.

5.1.1 Sample Space

The set containing of all the possible outcomes of a random experiment. It is denoted by 'S'.

If RE is flipping a coin, S = {Head, Tail}

If RE is rolling a dice, S = {1,2,3,4,5,6}

5.2 Event

Any subset of sample space 'S' is called as Event.

Ex. If RE is flipping a coin, then occurring of a Head is an Event.

If RE is rolling a dice, then getting an odd number is an Event.

5.2.1 Probability of an Event

If 'A' is any event with in the sample space 'S' of a Random experiment, then the probability of event 'A' is given by

$$P(A) = \frac{No. of outcomes favouring event 'A' to happen}{Total\ number\ of\ elements\ in\ 'S'} = \frac{n(A)}{n(S)}$$

Probability of getting an Even Number when a dice is rolled.

P(Even Number) = $\frac{3}{6} = 0.5$

S = {1,2,3,4,5,6}, A = {2,4,6}

5.2.2 Axioms Probability

(i) If 'A' is any event with in the sample space 'S' of a RE, then $0 \le P(A) \le 1$

$$\frac{0}{n(S)} \le \boxed{\frac{n(A)}{n(S)}} \le \frac{n(S)}{n(S)}$$

$$\boxed{0 \le P(A) \le 1}$$

(ii) P(S) = 1

When a RE is conducted the experiment yields a possible outcome.

5.2.3 Types of Events

(i) Mutually Exclusive Events:

If A, B are two events within a sample space 'S', then A & B are said to be mutually exclusive if $A \cap B = \phi$.

Ex. If 'A' is the event of getting a prime number when a dice is rolled and 'B' is the event of getting a composite number when a dice is rolled then

$S = \{1,2,3,4,5,6\}$, $A = \{2,3,5\}$, $B = \{4,6\} \Rightarrow A \cap B = \phi \Rightarrow P(A \cap B) = 0$

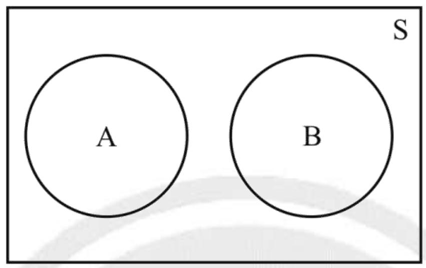

Fig. 5.1 Mutually exclusive event

(ii) Mutually Exhaustive Events:

If 'A', 'B' are two events with in a sample space 'S', then 'A' & 'B' are said to be mutually exhaustive if $A \cup B = S$

Ex. If 'A' is the event of getting an odd number when a dice is rolled and 'B' is the event of getting an Even Number, then

$A \cup B = S$

$S = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$, $B = \{2,4,6\}$

$\Rightarrow A \cup B = S$

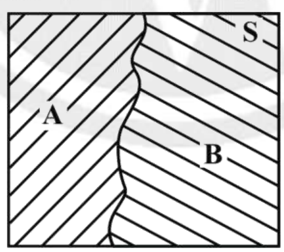

Fig. 5.2 Mutually exhaustive event

(iii) Independent Events:

Two events 'A' & 'B' with in the sample space 'S' (or) with in two different sample spaces 'S_1' & 'S_2' are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$.

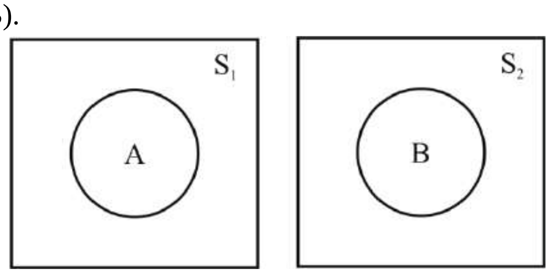

Fig. 5.3 Independent event

(iv) Impossible Event (ϕ):

The event with zero probability is called on Impossible Event P(ϕ) = 0.

5.3 Addition Theorem of Probability

If A, B are two events with a sample space 'S' of a Random Experiment, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

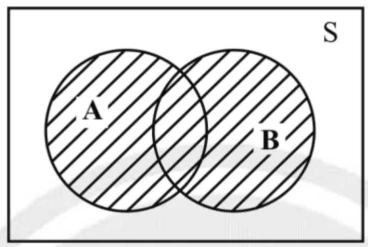

Fig. 5.4 Addition theorem

⇒ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

When A, B are mutually exclusive event, $A \cap B = \phi$.

⇒ $P(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

- If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive events ($E_i \cap E_j = \phi$), then $P(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$ $= P(E_1) + P(E_2) + P(E_3) + \ldots\ldots p(E_n)$

5.3.1 Conditional Probability

The probability of heppening of event 'A' when it is known that event 'B' has already occurred is given by P(A/B)

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)}$$

5.3.2 Multiplication Theorem of Probability

If A, B are two events within a sample space 'S', then $P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$

$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A/B) \cdot P(B) \rightarrow (1)$

$P(B/A) = \frac{P(B \cap A)}{P(A)} \Rightarrow P(B \cap A) = P(B/A) \cdot P(A) \rightarrow (2)$

From (1) & (2)

$$P(A/B) \cdot P(B) = P(B/A) \cdot P(A)$$

5.3.3 Total Theorem of Probability

If $E_1, E_2, E_3, \ldots\ldots E_n$ are 'n' mutually exclusive $(E_i \cap E_j = \phi; \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$P(A) = P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + \ldots\ldots + P(E_n) \cdot P(A/E_n)$$

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i) \bullet P(A/E_i)}$$

5.3.4 Baye's Theorem

If $E_1, E_2, E_3, \ldots\ldots E_n$ are mutually exclusive $(E_i \cap E_j = \phi \forall i \neq j)$ and collectively exhaustive event $(E_1 \cup E_2 \cup E_3 \cup \ldots\ldots \cup E_n = S)$ and 'A' is any event with in the sample space 'S', then

$$\boxed{P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}}$$

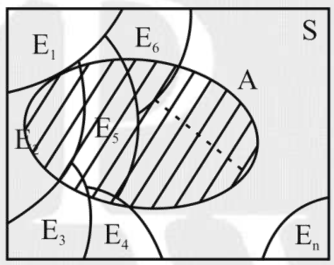

Fig. 5.5 Baye's theorem

5.4 Statistics

Statistics $\rightarrow$ Collection and Analysis of Data

Data —
- → Ungrouped data / Row data
- → Grouped Data

5.4.1 Analysis of Ungrouped Data

If $x_1, x_2, x_3, \ldots\ldots, x_n$ are 'n' observations, then

(1) The range of the data = R = max$\{x_1, x_2, \ldots\ldots, x_n\}$ – min$\{x_1, x_2, x_3, \ldots, x_n\}$

(2) Arithmetic mean **:** Mean of the data is equal to sum of observaions divided by the total number of observations.

$$\bar{x}(or)\mu = \frac{x_1 + x_2 + \ldots\ldots + x_n}{n} = \boxed{\frac{\sum_{i=1}^{n} x_i}{n}} = \bar{x} = \mu$$

- The mean of 1st 'n' natural numbers = $\frac{\left(\frac{n(n+1)}{2}\right)}{n} = \frac{n+1}{2}$
- The mean of 1st 'n' odd numbers = $\frac{n^2}{n} = n$
- The mean of 1st 'n' even numbers = n +1

5.4.2 Median

The middle most observation of the data $(x_1, x_2, x_3, \ldots.. x_n)$ When the data is arranged in either ascending or descending order.

If $x_1, x_2, x_3, x_4, \ldots\ldots. x_n$ are 'n' observations that are arranged in ascending/descending order then

(i) Median of the Data = $\left(\frac{n+1}{2}\right)^{th}$ observation, if 'n' is odd.

(ii) Median of the Data = Mean of $\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$ observations, if 'n' is even.

5.4.3 Mode

The observation with highest frequency is called mode.

Any Data with two Modes is called → Bimodel Data

If $x_1, x_{2,} x_3, \ldots\ldots, x_n$ are 'n' data points, $\bar{x} = \mu = \frac{x_1+x_2+\ldots\ldots+x_n}{n}$

Mean Deviation of the observation $(x_i) = d_i = x_i - \bar{x}$

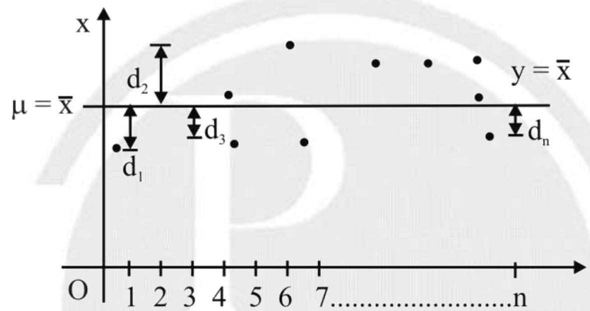

Fig. 5.6 Discrete data

Sum of derivations of all the observations = $\Sigma d_i = (x_1 - \bar{x}) + (x_2 - \bar{x}) + \ldots\ldots 0 + (x_n - \bar{x})$

$$= \Sigma d_i = (x_1 + x_2 + \ldots.. + x_n) - n\bar{x}$$

$$\boxed{\Sigma d_i = 0}$$

The sum of mean deviations of all the observations is equal to zero.

5.4.4 Absolute Mean Deviation

If $x_1, x_2, x_3, \ldots\ldots., x_n$ are 'n' data points with Mean = $\bar{x}$, then the absolute mean deviation of x_i about $\bar{x}$ is given by $|d_i| = |x - \bar{x}|$

The sum of absolute mean derivations of given data is not zero.

$$(\Sigma|d_i| \neq 0) \Rightarrow (|x_1 - \bar{x}| + |x_2 - \bar{x}| + \ldots\ldots.. + |x_n - \bar{x}| \neq 0)$$

5.4.5 Standard Deviation

If $x_1, x_2, x_3, \ldots\ldots, x_n$ ('n' is very large), then the standard deviation of the data is given by

Population Standard Deviation $\sigma = \sqrt{\frac{1}{n}\Sigma(x_i - \bar{x})^2}$, n → size of population

Sample Standard derivation: $\sigma = \sqrt{\frac{1}{(n-1)}\Sigma(x_i - \bar{x})^2}$, n→ size of sample

Generally (n > 29 → population) (n < 29 →sample)

5.5 Random Variables

The variable that connects the outcome of a Random Experiment to a real number.

Ex. 'x' is the value of the number that a dice shows when it is rolled.

Random Variable
- → Discrete RV → The RV whose value is obtained by counting
- → Continuous RV → The RV whose value is obtained by Measuring

- If a data consists of 'f$_1$' datapoints with value $'x_i', 'f_2'$ data points with value $'x_2' \ldots\ldots 'f_n'$ data point with value $'x_n'$, then
  (i) Expectation of 'x' $= E(x) = \sum_{i=1}^{n} x_i P(x = x_i)$
  (ii) Variance of 'x' = $\sigma^2 = E(x^2) - (E(x))^2$ and σ is the standard deviation.

5.5.1 Continuous RV

The value of the Random Variable is obtained by Measuring.

5.6 Probability distribution Function (Pdf)

A continuous & differentiable function P(x) is said to be a probability distribution/density function of a continuous random variable 'x' if $P(a \le x \le b) = \int_a^b P(x)dx$

5.6.1 Mean (or) Expectation

If P(x) is a probability distribution/density function of a continuous Random Variable 'x' then the Mean of 'x' = E(x) = $\int_{-\infty}^{\infty} x \cdot P(x)dx$

5.6.2 Median

The value of 'x' for which the total probability is exactly divided into two equal halves is called Median.

5.6.3 Mode

The valueof 'x' at which P(x) is maximum is called mode.

5.6.4 Variance

V = $\sigma^2 = E(x^2) - (E(x))^2$

$$\boxed{\Rightarrow \sigma^2 = \int_{-\infty}^{\infty} x^2 \cdot P(x)dx - \left(\int_{-\infty}^{\infty} x \cdot P(x)dx\right)}$$

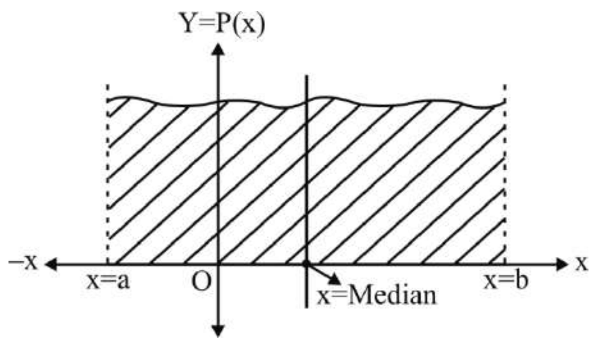

Fig.5.7 Continuous random variables

5.7 Continous RV Distributions

(1) Gaussian/Normal Distributon:

If 'x' is a continuous Random variable with mean 'μ' and standard deviation 'σ', then the probability distribution/density function of normally distributed variable 'x' is given by

$$P(x)=\frac{1}{\sigma\sqrt{2\pi}}.e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)}$$

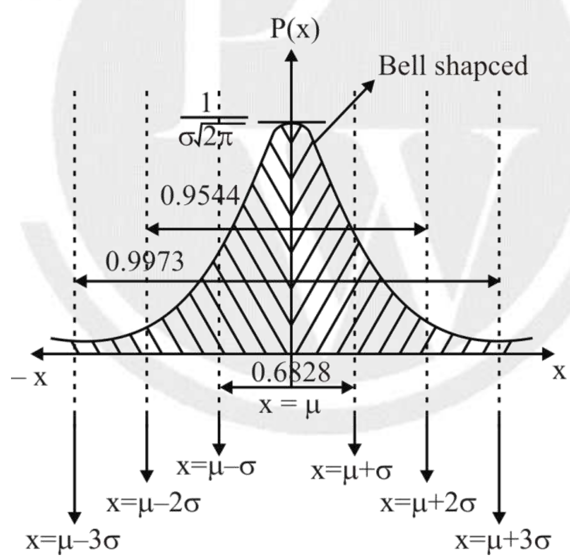

Fig.5.8 Normal distribution

Mean = Median = Mode = μ

$$P(\mu-\sigma \le x \le \mu+\sigma) = 0.6828$$
$$P(\mu-2\sigma \le x \le \mu+2\sigma) = 0.9544$$
$$P(\mu-3\sigma \le x \le \mu+3\sigma) = 0.9973$$
$$P(x) = \frac{1}{\sigma\cdot\sqrt{2\pi}}.e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(2) Standard Normal Distribution:

Assuming $z = \frac{x-\mu}{\sigma}; \mu = 0; \sigma = 1, P(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-z^2}{2}}$

$$P(-1 \le z \le 1) = 0.6828$$

$$P(-2 \le z \le 2) = 0.9544$$

$$P(-3 \le z \le 3) = 0.9973$$

Note:

1. The normal distribution curve is bell shaped curve
2. The points of infelection of the normal distribution curve are at $x = \mu + \sigma \ and \ x = \mu - \sigma$.
3. The cumulative function graph is of 'S' Shape.
4. For a given normal distribution, Mean = median = Mode

(3) Uniform Distribution:

If 'x' is a uniformly distrbuted random variable such that $a \le x \le b$ then the Pdf is given by

$$P(x) = \frac{1}{(b-a)}$$

Mean = $\int_a^b x \cdot P(x)dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{(b-a)} \int_a^b x \cdot dx$

$$\boxed{\left(\frac{b+a}{2}\right) = \text{Mean}}$$

$\Rightarrow$ $Variance = \sigma^2 = \frac{(b-a)^2}{12}$

Std.deviation = $\sigma = \frac{(b-a)}{\sqrt{12}}$

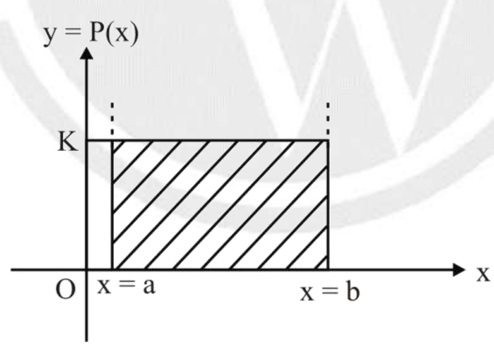

Fig.5.9 Uniform distribution

5.7.1 Properties of Mean and Variance

$E(ax + by) = a \cdot E(x) + b \cdot E(y)$

$V(ax + by) = a^2 \cdot V(x) + b^2 \cdot V(y) - 2abCOV(x, y) \ where \ COV(x, y) = E(xy) - E(x) \cdot E(y)$

If x,y are indpendent random variables, then $E(xy) = E(x) \cdot E(y) \Rightarrow COV(x, y) = 0$

(1) Exponential Dirtibution:

If 'x' is a contrinous random variable with mean as $\frac{1}{\lambda}$ then the exponential distribution of 'x' is given by the function

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & ; x \geq 0 \\ 0 & : otherwise \end{cases}$$

Mean = $\frac{1}{\lambda}$

$\sigma^2 = \frac{1}{\lambda^2}$

$\boxed{\text{Mean = Standard Deviation} = \frac{1}{\lambda}}$

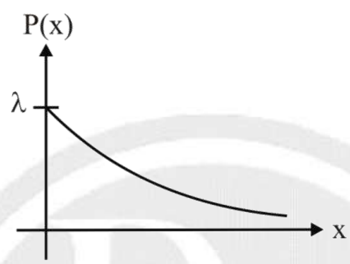

Fig.5.10 Exponential distribution

5.8 Discrete Random Variable Distributions

5.8.1 Binomial Distribution

If a Random experiment has **only two Possible outcomes**, (one is Success & other is failure) and the Probability of Success doesn't depend on time, then the probability of occuring of exactly 'r-successes' in 'n-trials' is given by

$$P(X = r) = {}^{n}C_r \cdot P^r \cdot q^{n-r}$$

Where, P → Probability of Success,

q → Probability of Failure

p + q = 1

Mean = np, Variance = npq = σ^2, standard deviation = $\sigma = \sqrt{npq}$

5.8.2 Poisson Distribution

If a random experiment has only two possible outcomes, and the average number of successes in a given time 't' is λ, then the probability that exactly 'r' successes occur within the same time 't' given by

$P(x = r) \frac{e^{-\lambda}.\lambda^r}{r!}$

Mean = λ.

Mean = $\sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}.\lambda^x}{x!(x-1)!}$ $\qquad$ $E(x^2) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}.\lambda^x}{x!}$

$=e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!} \Rightarrow E(x^2)$

$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \left\{1 + \frac{\lambda}{1!} + \frac{\lambda}{2!} + \frac{\lambda}{3!} + \ldots..\right\}$

$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda = E(x)$

$\sigma^2 = E(x^2) - (E(x))^2$

$= \lambda^2 + \lambda - \lambda^2 = \lambda$

$\Rightarrow \sigma^2 = \lambda$

$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{x^2 \cdot \lambda^x}{x!(x-1)!}$

$= e^{-\lambda} \cdot \sum_{x=0}^{\infty} \frac{\lambda \cdot x \cdot \lambda^{x-1}}{(x-1)!}$

$= e^{-\lambda} \cdot \lambda \cdot \sum_{x=0}^{\infty} \left\{\frac{(x-1)\lambda^{x-1}}{(x-1)!} + \frac{\lambda^{x-1}}{(r-1)!}\right\}$

$= e^{-\lambda} \cdot \lambda\left\{\lambda \cdot e^{\lambda} + e^{\lambda}\right\} = \lambda^2 + \lambda$

For Poisson distribution,

Mean = Variance = λ

$\Rightarrow \sigma = \sqrt{\lambda}$

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6 NUMERICAL METHODS

6.1 Solving System of Linear Equations

6.1.1 Gauss Elimination Method

Let the given system of linear equations be $\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \Rightarrow AX = B$

Augmented Matrix = $[A|B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$

By Row transformations, we convert the above matrix to row echelon form and then we go backward substitution.

$$\rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^1 & a_{23}^1 \\ 0 & 0 & a_{33}^{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^1 \\ b_3^{11} \end{bmatrix}$$

$$\Rightarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$
$$a_{22}^1x_2 + a_{23}^1x_3 = b_2$$
$$a_{33}^{11}x_3 = b_3$$

By solving the above equations, we get the values of the variables.

Note:

While applying row transformations, zero pivot elements are avoided by swapping the rows of augmented matrix.

LU Decomposition Method:

Let the given system of linear equations be $\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \Rightarrow AX = B$

The coefficient matrix A, is split into the product of an upper triangular and a lower triangular Matrix.

A = LU

⇒ LUX = B

Taking UX = Y we get AY = B

On Backward substitution , we get the elements of the matrix Y.

Since $UX = Y$, on forward substituion,we get the value of X.

6.2 Solution of Algebraic Equations (Non-Linear)

6.2.1 Bracketing Methods

In all the bracketing methods, to find the root of an equation $f(x) = 0$, we assume an interval [a, b] such that $f(a).f(b) < 0$, then we develop an iterative scheme for evaluating the iterates.

Method	Iteration formulae
(1) Bisection Method	(1) $c = \frac{a+b}{2}$
(2) Secant Method	(2) $c = \frac{a.f(b)-b.f(a)}{f(b)-f(a)}$

6.2.2 Newton Raphson Method

To find the root of the equation $f(x) = 0$, we assume an initial guess $x = x_0$ and the iterative scheme for the iterates is given by the equation

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Note

For the convergence of Newton Raphson method, $|f(x).f''(x)| < |f'(x)|^2$.

6.2.3 Rate of Convergence Values of iterative Methods

(1) Bisection Method -1

(2) Regula – Falsi Method -1

(3) Secant Method – 1.618

(4) Newton – Raphson Method -2

6.3 Numerical Integration

6.3.1 Trapezoidal Rule

If f(x) is continuous on [a,b] and differentiable on (a,b) then the value of $\int_a^b f(x)dx$ evaluated using 'n' sub intervals is given by $\int_a^b f(x) = dx = \frac{h}{2}[(f(a) + f(b)) + 2(f(a+h) + f(a+2h) + \ldots\ldots + f(a+(n-1))h)]$

$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots + y_{n-1})]$$

Salient Points

(i) The order of the fitting polynomial is 1.

(ii) The error involved in Trapezoidal Rule is $= \frac{nh^3}{12}$| Max. of f"(x) in [a, b] |$= \frac{n(b-a)^3}{12n^3}$ | max. of $f''(x)$ in [a, b] |

(iii) Trapezoidal Rule gives exact results for Polynomials of order ≤ 1

(iv) As number of subintervals increases, the accuracy of the result increases.

6.3.2 Simpson's 1/3rd Rule

If f(x) is continuous on [a,b] and differentiable on (a,b) then the value of $\int_a^b f(x)dx$ evaluated using 'n' even number of sub intervals then the value of the integration is given by

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_2 + 4y_4 + y_6 + \ldots\ldots + y_{n-2})]$$

Salient Points:

(i) The order of fitting polynomial → 2 (Quadratic)

(ii) Simpons rule gives the exact results for all the polynomials of degree ≤ 2.

(iii) Error in volved in the integration = $\frac{h^4}{100}|\max \ of\ f^4(x) in[a,b]|$ where f^4 (x) is the fourth derivative of f (x)

6.4 Numerical Solutions of a 1st Order DE

6.4.1 Explicit Euler Method (or) Forward Euler Method

For a given differential Equation, $\frac{dy}{dx} = f(x,y)$

$\left.\frac{dy}{dx}\right|_{x_i} = f'(x)|_{x_i} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$ (Forward difference)

We have $\left.\frac{dy}{dx}\right|_i = f(x,y)|_{(x_i,y_i)}$

$\Rightarrow \quad \frac{y_{i+1}-y_i}{h} = f(x_i, y_i)$

$\boxed{\Rightarrow y_{i+1} = y_i + h \cdot f(x_i, y_i)}$

Explicit Euler Method

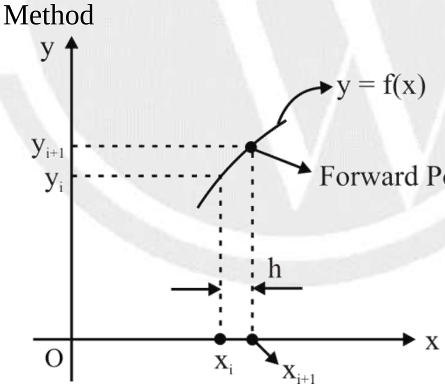

Fig.6.1 Forward Euler method

6.4.2 Implicit Euler Metohd (or) Backward Euler Method

$\left.\frac{dy}{dx}\right|_{x_{i+1}} = f(x_{i+1}, y_{i+1})$

$\left.\frac{dy}{dx}\right|_{x_{i+1}} = f(x,y)|_{(x_{i+1},y_{i+1})} = \frac{y_{i+1}-y_i}{x_{i+1}-x_i}$

$\Rightarrow \frac{y_{i+1}-y_i}{h} = f(x_{i+1}, y_{i+1})$

$\boxed{\Rightarrow y_{i+1} = y_i + h \cdot \{f(x_{i+1}, y_{i+1})\}}$

Implicit Euler Method

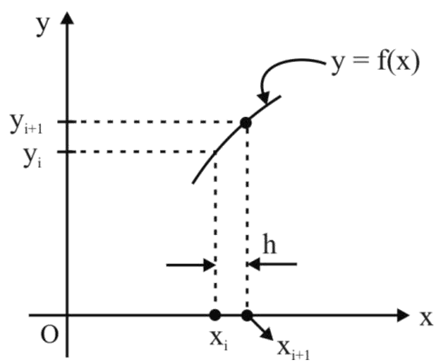

Fig.6.2 Backward Euler method

6.4.3 Modified Euler Method

For a given differential Equation, $\frac{dy}{dx} = f(x, y)$

$y_{i+1}^{(c)} = y_i + \frac{h}{2} \cdot \left\{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(p)})\right\}$ where $y_{i+1}^{(p)} is\ the$ predicted value.

And the predicted value is calclated using one of the Explicit (or) Implicit Euler Methods. (Mostly Forward method id used)

6.4.4 Runge – Kutta (R-K) Methods

For a given differential Equation, $\frac{dy}{dx} = f(x, y)$ with the condition $f(x_0) = y_0$,

(i) **R-K 1st order Method** : Forward Euler Method

(ii) **R-K 2nd order Method** : Modified Euler Method

(iii) **R-K 3rd order Method** :The iterative Scheme is given by $y_{i+1} = y_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$

Where

$k_1 = f(x_i, y_i)$

$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$

$k_3 = f(x_i + h, y_i + k_2)$

(iv) **R-K 4th order Method :**

The iterative Scheme is given by $y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Where

$k_1 = f(x_i, y_i)$

$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right)$

$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right)$

$k_4 = f(x_i + h, y_i + k_3)$

Multi step methods Includes Adams- Bashforth Methods

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7 COMPLEX CALCULUS

A number of the form $z = x + iy$ where $x, y \in R$ is called a complex number.

x is called real part of $z.x = Re(z)$

y is called real part of $zy = Im(z)$

7.1 Modulus – Amplitude form of a Complex Number

Every Complex number $z = x + iy$ can be written as $z = r.e^{i\theta}$ where

$$r = \sqrt{x^2 + y^2} \text{ is called the modulus of the complex number and}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ is called the amplitude (or) argument of the complex number.}$$

$$e^{i\theta} = \cos\theta + i.\sin\theta = \cos\theta \text{ and}$$

$$e^{-i\theta} = \cos\theta - i.\sin\theta$$

7.2 Arithmetic Operations with Complex Numbers

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers then

(i) $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$

(ii) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1y_2 + x_2y_1)$

(iii) $\dfrac{z_1}{z_2} = \dfrac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$

(iv) $|z_1 + z_2| \leq |z_1| + |z_2|$

(v) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

If r_1, θ_1 are modulus and amplitude of a complex number z_1 and r_2, θ_2 are modulus and amplitude of a complex number z_2 respectively, then

(i) The modulus of $z_1 \cdot z_2$ is $r_1 \cdot r_2$ and the amplitude of $z_1 \cdot z_2$ is $\theta_1 + \theta_2$

(ii) The modulus of $\dfrac{z_1}{z_2}$ is $\dfrac{r_1}{r_2}$ and the amplitude of $\dfrac{z_1}{z_2}$ is $\theta_1 - \theta_2$.

If $z = x + iy$ is a complex number, then the conjugate of the complex number is given by z^* (or) $\bar{z} = x - iy$.

$$Re(z) = \frac{z + z^*}{2}$$

and

$$Im(z) = \frac{z - z^*}{2i}$$

$$z \cdot z^* = |z|^2$$

7.3 De-moivers Theorem

If $z = r \cdot \cos\theta$ is a complex number, then

(i) $z^n = r^n \cdot \cos n\theta$ if 'n' is an integer.

(ii) One of the values of $z^n = r^n \cdot \cos n\theta$ if 'n' is a fraction.

If $n = \frac{p}{q}$, then the n values of z^n are given by $r^n \cdot \cos(2n\pi + \theta)\left(\frac{p}{q}\right)$ where $n = 0,1,2,3 \ldots\ldots q\text{-}1$.

The cube roots of unity are given by 1, ω, ω^2 where $\omega = -\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$

The cube roots of unity when plotted on an argand plane form an equilateral triangle.

7.4 Standard Complex Functions

If $z = x + iy$ is a complex number, then

(i) $\ln z = \frac{1}{2} \cdot \ln(x^2 + y^2) + i \cdot \tan^{-1}\left(\frac{y}{x}\right)$

(ii) $\exp(z) = e^x \cdot (\cos y + i \sin y)$

7.5 Periodic function

A complex function $f(z)$ is a periodic function if there exists a complex number 'k' such that $f(z) = f(z + k)$

Ex. The function $f(z) = e^z$ is a periodic function with period $2\pi i$.

7.5.1 Analytic Functions

A function $f(z)$ is said to analytic at a point $z = z_0$ if the fucntion $f(z)$ is differentiable at the pount $z = z_0$ and also at every point in the neighbourhood of z_0.

The mathematical conditions for a function $f(z) = u(x, y) + i.v(x, y)$ to be analytic at a point $z_0 = x_0 + i\, y_0$ is

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and differentiable at (x_0, y_0)

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These set of euations are called Cauchy – Riemann (C-R) Equations .

Note:

If the function $f(z) = u(x, y) + i.v(x, y)$ is analytic then

(i) Both $u(x, y)$ and $v(x, y)$ satisfy laplace equation.

i.e. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(ii) The family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal to each other.

Cauchy – Riemann Equations in polar form for the function $f(z) = u(x, y) + i.v(x, y)$ are given by

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$$

and

$$\frac{\partial u}{\partial \theta} = -r\cdot\frac{\partial v}{\partial r}$$

7.6 Complex Integration

If $f(z) = u + iv$ is continuous and differentiable at every point along a path 'C' , then the evaluation of $f(z)$ along the pathe 'C' is given by

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (u\,dy + v\,dx)$$

Note:

If the function $f(z)$ is analytic, then the integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path connecting the complex numbers z_1 and z_2.

7.6.1 Cauchy Integral Theorem

If the function $f(z)$ is analytic at everypoint with in a closed path 'C' then $\oint_C f(z)dz = 0$.

Note:

If the function $f(z)$ is analytic at everypoint with in a closed path 'C', except at the point $z = z_0$, then

$$\oint_C (z-z_0)^n dz = \begin{cases} 0, \text{ if } n \neq -1 \\ 2\pi i, \text{ if } n = -1 \end{cases}$$

7.6.2 Cauchy Integral formula

If $f(z) = \dfrac{\phi(z)}{(z-z_0)^{n+1}}$ is analytic at every point with in a closed path 'C' except at the point $z = z_0$, then

$$\oint_C f(z)dz = \oint_C \frac{\phi(z)}{(z-z_0)^{n+1}}dz = 2\pi i\cdot\frac{(\phi^n(z_0))}{n!}$$

Where $(\phi^n(z_0))$ is the n^{th} derivative of $\phi(z)$ at the point $z = z_0$.

7.7 Taylor Series and Laurentz Series

(i) Taylor series: If the function $f(z)$ is analytic at every point with in a circle with centre at $z = z_0$, then for any point z with in the circle,

$$f(z) = \sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n$$

where

$$a_n = \left(\frac{1}{2\pi i}\right)\cdot\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

(ii) Laurentz Series: If the function $f(z)$ is analytic at every point with in a region bounded by two concetric circles C and C_1 with radii r, r_1 respectively $(r > r_1)$ with centre at $z = z_0$, then for any point z with in the region,

$$f(z) = \sum_{n=0}^{\infty} a_n\cdot(z-z_0)^n + \sum_{n=1}^{\infty} b_n\cdot(z-z_0)^{-n}$$

where $$a_n = \left(\frac{1}{2\pi i}\right)\cdot\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

and $$b_n = \left(\frac{1}{2\pi i}\right)\cdot\oint_{C_1} \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

Note:

All the formulaes above for the cyclic integrals are for counter clockwise sense by default, if the questions are asked for clockwise sense, the answer evaluted using above formulaes should be written with sign change.

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GENERAL APTITUDE

General Aptitude

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1 PERCENTAGES

1.1. Understanding Percentages

The word percent can be understood as follows:

Per cent ⇒ for every 100.

So, when percentage is calculated for any value, it means that that you calculate the value for every 100 of the reference value.

Why Percentage?

Percentage is a concept evolved so that there can be a uniform platform for comparison of various things. (Since each value is taken to a common platform of 100.)

Example:

- To compare three different students depending on the marks they scored we cannot directly compare their marks until we know the maximum marks for which they took the test. But by calculating percentages they can directly be compared with one another.
- Before going deeper into the concept of percentage, let u have a look at some basics and tips for faster calculations:

1.2. Calculation of Percentage

$$\text{Percentage} = \left(\frac{\text{Value}}{\text{Total value}}\right)\times 100$$

Example: 50 is what % of 200?

Solution: $\text{Percentage} = \left(\frac{50}{200}\right)\times 100 = 25\%$.

1.2.1. Calculation of Value:

$$\text{Value} = \left(\frac{\text{Percentage}}{100}\right)\times \text{ total value}$$

Example: What is 20% of 200?

Solution: $\text{Value} = \left(\frac{20}{100}\right)\times 200$

Note: Percentage is denoted by "%", which means "/100".

Example: What is the decimal notation for 35%?

Solution: $35\% = \frac{35}{100} = 0.35$.

For faster calculations we can convert the percentages or decimal equivalents into their respective fraction notations.

1.3. Percentages – Fractions Conversions:

The following is a table showing the conversions of percentages and decimals into fractions:

Percentage	Decimal	Fraction
10%	0.1	$\frac{1}{10}$
12.5%	0.125	$\frac{1}{8}$
16.66%	0.1666	$\frac{1}{6}$
20%	0.2	$\frac{1}{5}$
25%	0.25	$\frac{1}{4}$
30%	0.3	$\frac{3}{10}$
33.33%	0.3333	$\frac{1}{3}$
40%	0.4	$\frac{2}{5}$
50%	0.5	$\frac{1}{2}$
60%	0.6	$\frac{3}{5}$
62.5%	0.625	$\frac{5}{8}$
66.66%	0.6666	$\frac{2}{3}$

Similarly, we can go for converting decimals more than 1 from the knowledge of the above cited conversions as follows:

We know that $12.5\% = 0.125 = \frac{1}{8}$

Then, $1.125 = \frac{[8(1)+1]}{8} = \frac{9}{8}$ (i.e., the denominator will add to numerator once, denominator remaining the same.

Also, $2.125 = \frac{[8(2)+1]}{8} = \frac{17}{8}$ (here the denominator is added to numerator twice)

$3.125 = \frac{[8(3)+1]}{8} = \frac{25}{8}$ and so on.

Thus we can derive the fractions for decimals more than 1 by using those les than 1.

We will see how use of fractions will reduce the time for calculations:

Example: What is 62.5% of 320?

Solution: $\text{Value} = \left(\frac{5}{8}\right) \times 320 \left(\text{since } 62.5\% = \frac{5}{8}\right) = 200$.

1.4. Percentage Change

A change can be of two types – an increase or a decrease.

When a value is changed from initial value to a final value,

% change = (Difference between initial and final value/initial value) × 100

Example: If 20 changes to 40, what is the % increase?

Solution: $\%\ increase = \frac{(40-20)}{20} \times 100 = 100\%$.

Note:

1. If a value is doubled the percentage increase is 100.
2. If a value is tripled, the percentage change is 200 and so on.

1.5. Percentage Difference

% Difference = (Difference between values/value compared with) × 100.

Example: By what percent is 40 more than 30?

Solution: $\%\text{ difference} = \frac{(40-30)}{30} \times 100 = 33.33\%$

(Here 40 is compared with 30. So 30 is taken as denominator)

Example: By what % is 60 more than 30?

Solution: $\%\text{ difference} = \frac{(60-30)}{30} \times 100 = 100\%$.

(Here is 60 is compared with 30.)

Hint: To calculate percentage difference the value that occurs after the word "than" in the question can directly be used as the denominator in the formula.

1.6. Important Points to Note

1. When any value increases by

(a) 10%, it becomes 1.1 times of itself. (since 100+10 = 110% = 1.1)

(b) 20%, it becomes 1.2 times of itself.

(c) 36%, it becomes 1.36 times of itself.

(d) 4%, it becomes 1.04 times of itself.

Thus we can see the effects on the values due to various percentage increases.

2. When any value decreases by

(a) 10%, it becomes 0.9 times of itself. (Since 100 – 10 = 90% = 0.9)

(b) 20%, it becomes 0.8 times of itself

(c) 36%, it becomes 0.64 times of itself

(d) 4%, it becomes 0.96 times of itself.

Thus, we can see the effects on a value due to various percentage decreases.

Note:
1. When a value is multiplied by a decimal more than 1 it will be increased and when multiplied by less than 1 it will be decreased.
2. The percentage increase or decrease depends on the decimal multiplied.

Example: 0.7 $\Rightarrow$ 30% decrease, 0.67 $\Rightarrow$ 33% decrease, 0. 956 $\Rightarrow$ 4.4% decrease and so on.

Example: When the actual value is x, find the value when it is 30% decreased.

Solution: 30% decrease $\Rightarrow$ 0.7 x.

Example: A value after an increase of 20% became 600. What is the value?

Solution: 1.2x = 600 (since 20% increase)

$\Rightarrow$ x = 500.

Example: If 600 is decrease by 20%, what is the new value?

Solution: New value = 0.8 × 600 = 480. (Since 20% decrease)

Thus, depending on the decimal, we can decide the % change and vice versa.

Example: When a value is increased by 20%, by what percent should it be reduced to get the actual value?

Solution: (It is equivalent to 1.2 reduced to 1 and we can use % decrease formula)

$$\% \text{ decrease} = \frac{(1.2 - 1)}{1.2} \times 100 = 16.66\%.$$

3. When a value is subjected multiple changes, the overall effect of all the changes can be obtained by multiplying all the individual factors of the changes.

Example: The population of a town increased by 10%, 20% and then decreased by 30%. The new population is what % of the original?

Solution: The overall effect = 1.1 × 1.2 × 0.7 (Since 10%, 20% increase and 30% decrease)

= 0.924 = 92.4%.

Example: Two successive discounts of 10% and 20% are equal to a single discount of ___

Solution: Discount is same as decrease of price.

So, decrease = 0.9 × 0.8 = 0.72 ⇒ 28% decrease (Since only 72% is remaining).

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2 AVERAGES & AGES

2.1. What is Average?

The concept of average is equal distribution of the overall value among all the things or persons present there. So the formula for finding the average is as follows:

Average, $$A = \frac{\text{Total of all things, } T}{\text{Number of things, } N}$$

Therefore, Total, $T = AN$

If any person joins a group with more value than the average of the group then the overall average increases. This is because the value in excess than the average will also be distributed equally among all the members.

Similarly when any value less than the average joins the group the overall group decreases as the deficit is divided equally among all the people present there.

Example:

Consider three people A, B and C with total of Rs. 30/-. Their average becomes Rs. 10/- for each. If another person D joins them with Rs. 50/- then he has Rs. 40/- more than actual average of Rs. 10/-.

So this Rs. 40/- will get distributed among those four and each gets Rs. 10/-. Thus the average becomes Rs. 20/- each.

Example:

The average age of a class of 30 students is 12. If the teacher is also included the average becomes 13 years. Find the teacher's age.

Solution:

- When the teacher is included there are totally 31 members in the class and the average is increased by 1 year. This means that everyone got 1 extra year after distributing the extra years of the teacher.
- So extra years of the teacher are as follow: $31 \times 1 = 31$ years.
- Age of the teacher = actual avg + extra years = 12 + 31 = 43 years.

3 PROFIT AND LOSS

3.1. What is Profit?

When a person does a business transaction and gets more than what he had invested, then he is said to have profit. The profit he gets will be equal to the additional money he gets other than his investment.

So, profit can be understood as the extra money one gets other than what he had invested.

Example: A person bought an article for Rs. 100 and sold it for Rs. 120. Then he got Rs. 20 extra and so his profit is Rs. 20.

3.2. What is Loss?

When a person gets an amount less than what he had invested, then he is said to have a loss. The loss will be equal to the deficit he got than the investment.

Example: A person bought an article at Rs. 100 and sold it for Rs. 90. Then he got a deficit of Rs. 10 and so his loss is Rs. 10.

3.3. Cost Price (CP)

- The money that the trader puts in his business is called Cost Price. The price at which the articles are bought is called Cost Price.
- In other words, Cost Price is nothing but the investment in the business.

3.4 Selling Price (SP)

- The price at which the articles are sold is called the Selling Price. The money that the trader gets from the business is called Selling Price.
- In other words, Selling Price is nothing but the returns from a business.

3.5. Marked/Market/List Price (MP):

- The price that a trader marks or lists his articles to is called the Marked Price.
- This is the only price known to the customer.

3.6. Discount

The waiver of cost from the Marked Price that the trader allows a customer is called Discount.

Note:

1. Profit or loss percentage is to be applied always to the Cost Price only.
2. Discount percentage is to be applied always to the Marked Price only.

3.7. Relationship Among CP, SP and MP:

A trader adds his profit to the investment and sells it at that increased price.
Also he allows a discount on Marked Price and sells at the discounted price.
So, we can say that,

- SP = CP + Profit. (CP applied with profit is SP)
- SP = MP – Discount. (MP applied with discount is SP)

3.8. Understanding Profit and Loss:

So, by now we came to know that if CP is increased and sold it would result in profit and vice versa.

Also whatever increase is applied to CP, that increase itself is the profit.

For Rs. 10 profit, CP is to be increased by RS. 10 and the increased price becomes SP.

For 10% profit, CP is to be increased by 10% and it is the SP.

(From previous chapter we know that any value increased by 10% becomes 1.1 times.)

So, for 10% profit, CP increased by 10% $\Rightarrow$ 1.1CP = SP.

- $SP = 1.1CP \Rightarrow \dfrac{SP}{CP} = 1.1 \Rightarrow$ 10% profit
- $SP = 1.07CP \Rightarrow \dfrac{SP}{CP} = 1.07 \Rightarrow$ 7% profit
- $SP = 1.545CP \Rightarrow \dfrac{SP}{CP} = 1.545 \Rightarrow$ 54.5% profit and so on.

Similarly,

- $SP = 0.9CP \Rightarrow \dfrac{SP}{CP} = 0.9 \Rightarrow$ 10% loss (Since 10% decrease)
- $SP = 0.7CP \Rightarrow \dfrac{SP}{CP} = 0.76 \Rightarrow$ 24% loss and so on.

So, to calculate profit % or loss %, it is enough for us to find the ratio of SP to CP.

Note:

1. If SP/CP > 1, it indicates profit.
2. If SP/CP < 1, it indicates loss.

3.9. Multiple Profits or Losses

A trader may sometimes have multiple profits or losses simultaneously. This is equivalent to having multiple changes and so all individual changes are to be multiplied to get the overall effect.

Examples: A trader uses a 800gm weight instead of 1 kg. Find his profit %.

Solution: (He is buying 800 gm but selling 1000 gm.

So, CP is for 800 gm and SP is for 1000 gm.)

$$\frac{SP}{CP}=\frac{1000}{800} = 1.25 \Rightarrow 25\% \text{ profit.}$$

Examples: A trader uses 1 kg weight for 800 gm and increases the price by 20%. Find his profit/loss %.

Solution: 1 kg weight for 800 gm ⇒ loss (decrease) ⇒ 800/1000 = 0.8

20% increase in price ⇒ profit (increase) ⇒ 1.2

So, net effect = (0.8) × (1.2) = 0.96 ⇒ 4% loss.

Examples: A milk vendor mixes water to milk such that he gains 25%. Find the percentage of water in the mixture.

Solution: To gain 25%, the volume has to be increased by 25%.

So, for 1 lt of milk, 0.25 lt of water is added ⇒ total volume = 1.25 lt

$$\% \text{ of water} = \frac{0.25}{1.25}\times 100 = 20\%.$$

Examples: A trader bought an item for Rs. 200. If he wants a profit of 22%, at what price must he sell it?

Solution: CP=200, Profit = 22%.

So, SP = 1.22CP = 1.22 × 200 = 244/-.

Examples: A person buys an item at Rs. 120 and sells to another at a profit of 25%. If the second person sells the I tem to another at Rs. 180, what is the profit % of the second person?

Solution: SP of 1^{st} person = CP of 2^{nd} person = 1.25 × 120 = 150.

SP of 2^{nd} person = 180.

$$\text{Profit \%} = \frac{SP}{CP} = \frac{180}{150} = 1.2 \Rightarrow 20\%.$$

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4 RATIOS AND PROPORTIONS

4.1. What is a Ratio?

A ratio is a representation of distribution of a value present among the persons present and is shown as follows:

If a total is divided among A, B and C such that A got 4 parts, B got 5 parts and C got 6 parts then it is represented in ratio as A:B:C = 4:5:6.

So, 4:5:6 means that the total value is divided into 4+5+6 = 15 equal parts and then distributed as per the ratio.

Example 1: Divide Rs. 580 between A and B in the ratio of 14:15.

Solution: A:B = 14:15 $\Rightarrow$ 580 is divided into 29 equal parts $\Rightarrow$ each part = Rs. 20.

So A's share = 14 parts = 14 × 20 = Rs. 280

B's share = 15 parts = Rs. 300.

Example 2: If A:B = 2:3 and B:C = 4:5 then find A:B:C.

Solution: To combine two ratios the proportions common for them shall be in equal parts. Here the common proportion is B for the given ratios.

Making B equal in both ratios they become 8:12 and 12:15 $\Rightarrow$ A:B:C = 8:12:15.

Example 3: Three numbers are in the ratio of 3: 4 : 8 and the sum of these numbers is 975. Find the three numbers.

Solution: Let the numbers be $3x$, $4x$ and $8x$. Then their sum = $3x + 4x + 8x = 15x = 975 \Rightarrow x = 65$.

So the numbers are $3x = 195$, $4x = 260$ and $8x = 520$.

Example 4: Two numbers are in the ratio of 4 : 5. If the difference between these numbers is 24, then find the numbers.

Solution: Let the numbers be $4x$ and $5x$. Their difference = $5x - 4x = x = 24$ (given).

So the numbers are $4x = 96$ and $5x = 120$.

Example 5: Given two numbers are in the ratio of 3 : 4. If 8 is added to each of them, their ratio is changed to 5 : 6. Find two numbers.

Solution: Let the numbers be a and b.

$A:B = 3:4 \Rightarrow \frac{A}{B} = \frac{3}{4}$. Also, $\frac{(A+8)}{(B+8)} = \frac{5}{6}$.

Solving we get, A=12 and B = 16.

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5 TIME AND DISTANCE

5.1. Speed

We have the relation between speed, time and distance as follows:

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

So the distance covered in unit time is called speed.

This forms the basis for Time and Distance. It can be re-written as Distance = Speed X Time or

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}.$$

5.1.1. Units of Speed

The units of speed are kmph (km per hour) or m / s.

$$1 \text{ kmph} = \frac{5}{18} \text{ m/s}$$

$$1 \text{ m/s} = \frac{18}{5} \text{ kmph}$$

5.1.2. Average Speed

When the travel comprises of various speeds then the concept of average speed is to be applied.

$$\text{a Average Speed} = \frac{\text{Total distance covered}}{\text{Total time of travel}}$$

Note: In the total time above, the time of rest is not considered.

Example 1: If a car travels along four sides of a square at 100 kmph, 200 kmph, 300 kmph and 400 kmph find its average speed.

Solution: $\text{Average Speed} = \frac{\text{Total distance}}{\text{Total time}}.$

Let each side of square be x km. Then the total distance = $4x$ km.

The total time is sum of individual times taken to cover each side.

To cover x km at 100 kmph, time = $\frac{x}{100}$.

For the second side time $= \frac{x}{200}$.

Using this we can write average speed = $\frac{4x}{\left(\frac{x}{100}+\frac{x}{200}+\frac{x}{300}+\frac{x}{400}\right)} = 192$ kmph.

Example 2: A man if travels at $\frac{5}{6}$ th of his actual speed takes 10 min more to travel a distance. Find his usual time.

Solution: Let s be the actual speed and t be the actual time of the man.

Now the speed is $\left(\frac{5}{6}\right)$s and time is (t +10) min. But the distance remains the same.

So distance 1 = distance 2 $\Rightarrow s \times t = \left(\frac{5}{6}\right) s \times (t+10) \Rightarrow t = 50$ min.

Example 3: If a person walks at 30 kmph he is 10 min late to his office. If he travels at 40 kmph then he reaches to his office 5 min early. Find the distance to his office.

Solution: Let the distance to his office be d. The difference between the two timings is given as 15 min = $\frac{1}{4}$ hr.

Now if d km are covered at 30 kmph then time = d/30. Similarly, second time = d/40.

So, $\frac{d}{30} - \frac{d}{40} = \frac{1}{4} \Rightarrow d = 30$ km.

Note: When two objects move with speeds s_1 and s_2

- In opposite directions their combined speed = $s_1 + s_2$
- In same direction their combined speed = $s_1 \sim s_2$.

Example 4: Two people start moving from the same point at the same time at 30 kmph and 40 kmph in opposite directions. Find the distance between them after 3 hrs.

Solution: Speed = 30 + 40 = 70 kmph (since in opposite directions)

Time = 3 hrs

So distance = speed × time = 70 × 3 = 210 km.

Example 5: A starts from X to Y at 6 am at 40 kmph and at the same time B starts from Y to X at 50 kmph. When will they meet if X and Y are 360 km apart?

Solution: Distance = 360 km, Speed = 40 + 50 = 90 kmph.

Time = $\frac{\text{distance}}{\text{speed}} = \frac{360}{90} = 4$hrs from 6 am $\Rightarrow$ 10 am.

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6 TIME AND WORK

6.1. Introduction

If a person can complete a work in ‘n’ days then he can do $\frac{1}{n}$ part of the work in one day.

The amount of work done be a person in 1 day is called his efficiency.

Example: A can do a work in 10 days. Then the efficiency of A is given by $A = \frac{1}{10}$.

Note: Number of days required to do a work = work to be done/work per day.

Example 1: If A can do a work in 10 days, B can do it in 20 days and C in 30 days in how many days will the three together do it?

Solution: The efficiencies are $A = \frac{1}{10}, B = \frac{1}{20}$ $and\ C = \frac{1}{30}$

So work done per day by the three = $\frac{1}{10}+\frac{1}{20}+\frac{1}{30} = \frac{11}{60} \Rightarrow No\ of\ days = \frac{60}{11} = 5.45\ days$.

Example 2: If A and B can do a work in 10 days, B and C can do it in 20 days and C and A can do it in 40 days in what time all the three can do it?

Solution: $A+B = \frac{1}{10}$

$B+C = \frac{1}{20}$

$C+A = \frac{1}{40}$

Adding all the three we get $2(A+B+C) = \frac{7}{40} \Rightarrow A+B+C = \frac{7}{80} \Rightarrow No\ of\ days = \frac{80}{7}\ days$.

Note: If all the people do not work for all the time then the principle below can be used:

$$mA + nB + \boldsymbol{oC} = \mathbf{1}. \quad \text{(1 is the total work)}$$

Here, m = no of days A worked

n = no of days B worked

o = no of days C worked

A, B, C = efficiencies

Example 3: If A can do a work in 12 days, B can do it in 18 days and C in 24 days. All the three started the work. A left after two days and C left three days before the completion of the work. How many days are required to complete the work?

Solution: Let the total no of days be x.

A worked only for 2 days, B worked for x days and C worked for x – 3 days.

So, mA + nB + oC = 1

$$\Rightarrow 2\left(\frac{1}{12}\right) + x\left(\frac{1}{18}\right) + (x-3)\left(\frac{1}{24}\right) = 1$$

$\Rightarrow$ 12 + 4x + 3(x-3) = 72

$$\Rightarrow x = \frac{69}{7}\ days.$$

Note: The ratio of dividing wages = ratio of efficiencies = ratio of parts of work done

Example 4: A can do a work in 10 days and B can do it in 30 days and C in 60 days. If the total wages for the work is Rs. 1800 what is the share of A?

Solution: Ratio of wages = $\frac{1}{10} : \frac{1}{30} : \frac{1}{60} = 6 : 2 : 1$ (Multiplying each term by LCM 60)

So total 9 equal parts in Rs. 1800 $\Rightarrow$ each part = Rs. 200 $\Rightarrow$ share of A = 6 parts = Rs. 1200.

Note: When pipes are used filling the tank they are treated similar to the men working but some outlet pipes emptying the tank are present whose work will be considered negative.

Example 5: A pipe can fill a tank in 5 hrs but because of a leak the bottom it takes 1 hr extra. In what time can the leak alone empty the tank?

Solution: Let the filling pipe be A.

$$A = \frac{1}{5}$$

But with the leak L, $A - L = \frac{1}{6}$ (A-L because leak is outlet)

So, $\frac{1}{L} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30} \Rightarrow$ Leak can empty the tank in 30 hrs.

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7 CLOCKS

7.1. Introduction

In a clock the most important hands are the minutes hand and the hours hand. Whatever may be the shape of the dial they move in a circular track.

The total angle of 360 degrees in a watch is divided into 1 sector, one for each hour.

So one hour sector = $\frac{360}{12} = 30$ degrees.

For every one hour (60 min),

- The minutes hand moves through 360 deg.
- The hours hand moves through 30 deg.

So for every minute,

- The minutes hand moves through 6 deg
- The hours hand moves through 0.5 deg.

They move in same direction. So their relative displacement for every minute is 5.5 deg.

This 5.5 deg movement constitutes the movements of both the hands.

So for every minute both the hands give a displacement of 5.5 deg.

Note:

1. Between every two hours i.e., between 1 and 2, 2 and 3 and so on the hands of the clock coincide with each other for one time except between 11, 12 and 12, 1.

   In a day they coincide for 22 times.
2. Between every two hours they are perpendicular to each other two times except between 2, 3 and 3, 4 and 8, 9 and 9, 10.

   In a day they will be perpendicular for 44 times.
3. Between every two hours they will be opposite to each other one time except between 5, 6 and 6, 7.

   In a day they will be opposite for 22 times.

Examples: At what time between 5 and 6 will the hands of the clock coincide?

Solution: At 5 the angle between the hands is 150 deg.

To coincide, they collectively have to travel this distance. Every minute they travel 5.5 deg.

So no. of minutes required to coincide = $\frac{150}{5.5} = \frac{300}{11} = 27\frac{3}{11}\ min$.

Examples: At what time between 6 and 7 will the hands be perpendicular?

Solution: At 6 the angle between the hands is 180 deg.

To form 90 deg they have to cover 90 deg (out of 180 if 90 is covered 90 will remain)

So no. of minutes required = $\frac{90}{5.5} = \frac{180}{11} = \frac{164}{11} min$.

But they will be perpendicular for two times. The second one will happen after the minutes hand crosses the hours hand and then for 90 deg.

So it has to travel 180 + 90 = 270 deg.

So time = $\frac{270}{5.5} = \frac{540}{11} = 49\frac{1}{11} min$.

Examples: What is the angle between the hands of the clock at 3.45?

Solution: At 3, the angle between the hands = A = 90 deg.

In 45 min the hands will move angle of B = 45 × 5.5 deg (since 5.5 deg for 1 min)

B = 247.5 deg.

Required angle = A ~ B = 157.5 deg.

Examples: What is the angle between the hands at 4.40?

Solution: At 4 the angle between the hands, A = 120 deg.

In 40 min, B = 40 × 5.5 = 220 deg.

The required angle = A ~ B = 100 deg.

Examples: A clock loses 5 min for every hour and another gains 5 min for every hour. If they are set correct at 10 am on Monday then when will they be 12 hrs apart?

Solution: For every hour watch A loses 5 min and watch B gains 5 min.

So for every hour they will differ by 10 min.

For 12 hrs (720 min) difference between them the time required = $\frac{720}{10} = 72\ hrs$

So they will be 12 hrs apart after 3 days i.e., at 10 am on Thursday.

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8 CALENDARS

8.1. Calendars

Here you mainly deal in finding the day of the week on a particular given date.

The process of finding this depends on the number of odd days.

Odd days are quite different from the odd numbers.

- **Odd Days:** The days more than the complete number of weeks in a given period are called odd days.
- **Ordinary Year:** An year that has 365 days is called Ordinary Year.
- **Leap Year:** The year which is exactly divisible by 4 (except century) is called a leap year.

Example: 1968, 1972, 1984, 1988 and so on are the examples of Leap Years.

1986, 1990, 1994, 1998, and so on are the examples of non-leap years.

Note: The Centuries divisible by 400 are leap years.

Important Points:

- An ordinary year has 365 days = 52 weeks and 1 odd day.
- A leap year has 366 days = 52 weeks and 2 odd days.
- Century = 76 Ordinary years + 24 Leap years.
- Century contain 5 odd days.
- 200 years contain 3 odd days.
- 300 years contain 1 odd day.
- 400 years contain 0 odd days.
- Last day of a century cannot be Tuesday, Thursday or Saturday.
- First day of a century must be Monday, Tuesday, Thursday or Saturday.

Explanation:

100 years = 76 ordinary years + 24 leap years

= 76 odd days + 24 × 2 odd days

= 124 odd days = 17 weeks + 5 days

∴ 100 years contain 5 odd days.

No. of odd days in first century = 5

∴ Last day of first century is Friday.

No. of odd days in two centuries = 3

∴ Wednesday is the last day.

No. of odd days in three centuries = 1

∴ Monday is the last day.

No. of odd days in four centuries = 0

∴ Sunday is the last day.

Since the order is continually kept in successive cycles, the last day of a century cannot be Tuesday, Thursday or Saturday.

So, the last day of a century should be Sunday, Monday, Wednesday or Friday.

Therefore, the first day of a century must be Monday, Tuesday, Thursday or Saturday.

8.2. Working Rules

Working rule to find the day of the week on a particular date when reference day is given:

Step 1: Find the net number of odd days for the period between the reference date and the given date (exclude the reference day but count the given date for counting the number of net odd days).

Step 2: The day of the week on the particular date is equal to the number of net odd days ahead of the reference day (if the reference day was before this date) but behind the reference day (if this date was behind the reference day).

Working rule to find the day of the week on a particular date when no reference day is given

Step 1: Count the net number of odd days on the given date

Step 2: Write:

For 0 odd days – Sunday

For 1 odd day – Monday

For 2 odd days – Tuesday

⋮ ⋮ ⋮

For 6 odd days - Saturday

Examples: If 11^{th} January 1997 was a Sunday then what day of the week was on 10^{th} January 2000?

Solution: Total number of days between 11^{th} January 1997 and 10^{th} January 2000

= (365 – 11) in 1997 + 365 in 1998 + 365 in 1999 + 10 days in 2000

= (50 weeks + 4 odd days) + (52 weeks + 1 odd day) + (52 weeks + 1 odd day) + (1 week + 3 odd days)

Total number of odd days = 4 + 1 + 1 + 3 = 9 days = 1 week + 2 days

Hence, 10^{th} January, 2000 would be 2 days ahead of Sunday i.e. it was on Tuesday.

Examples: What day of the week was on 10^{th} June 2008?

Solution: 10^{th} June 2008 = 2007 years + First 5 months up to May 2008 + 10 days of June

2000 years have 0 odd days.

Remaining 7 years has 1 leap year and 6 ordinary years $\Rightarrow$ 2 + 6 = 8 odd days

So, 2007 years have 8 odd days.

No. of odd days from 1^{st} January 2008 to 31^{st} May 2008 = 3+1+3+2+3 = 12

10 days of June has 3 odd days.

Total number of odd days = 8+12+3 = 23

23 odd days = 3 weeks + 2 odd days.

Hence, 10^{th} June, 2008 was Tuesday.

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9 BLOOD RELATIONS

9.1. Introduction

The standard definitions of relations are given below

A/An ⇓	is related to a PERSON as ⇓
Grandfather	The father of his/her mother or father
Grandmother	The mother of his/her mother or father
Grandson	The son of his/her daughter/son
Granddaughter	The daughter of his/her daughters/son
Uncle	The brother of his/her mother or father
Aunt	The sister of his/her mother or father
Nephew	The son of his/her brother or sister
Cousin	The son or daughter of his/her aunt of uncle
Niece	The daughter of his/her brother or sister
Spouse	as her husband or his wife
Father-in-law	the father of his/her spouse
Mother-in-law	the mother of his/her spouse
Sister-in-law	the sister of his/her spouse
Brother-in-law	the brother of his/her spouse
Son-in-law	the spouse of his/her daughter
Daughter-in-law	the spouse of his/her son

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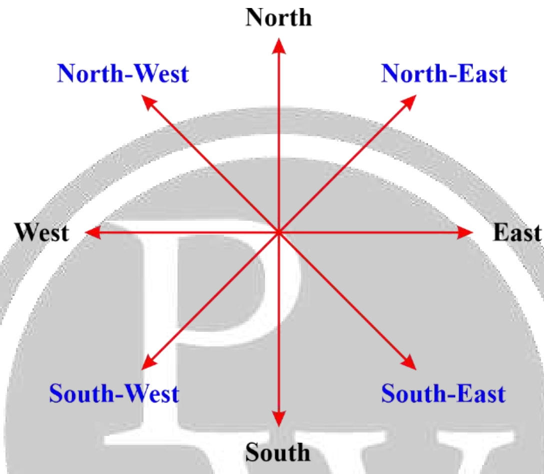
10 DIRECTIONS

10.1. Introduction

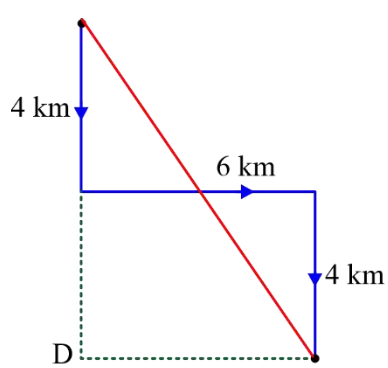


Examples:

Example 1: Ravi traveled 4 km straight towards south. He turned left and traveled 6 km straight, then turned right and traveled 4 km straight. How far is he from the starting point?

(a) 8 km (b) 10 km (c) 18 km (d) 12 km

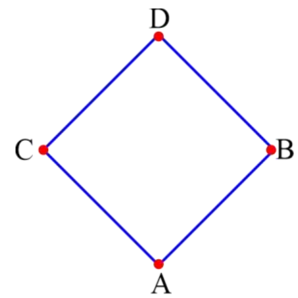
Solution. 10 km. B is the finishing point and A is starting point. The distance of A from B is



Example 2. A is to the South-East of C, B is to the East of C and North-East of A. If D is to the North of A and North-West of B. In which direction of C is D located?

(a) North-West (b) South-West (c) North-East (d) South-East

Solution. North-East D is located to the North-East of C.

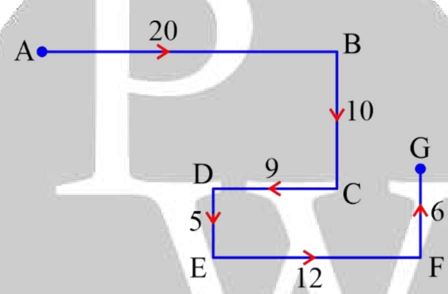


Example 3. A rat runs 20' towards East and turns to right, runs 10' and turns to right, runs 9' and again turns to left, runs 5' and then runs to left, runs 12' and finally turns to left and runs 6'. Now, which direction is the rat facing?

(a) East (b) West

(c) North (d) South

Solution. North. The movements of the rat from A to G are as shown in the fig. It is clear, rat is walking in one direction FG, *i.e.*, North.

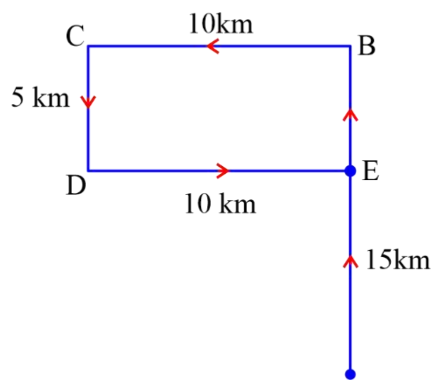


Example 4. From his house, Lokesh went 15 km to the North. Then he turned west and covered 10 km, then he turned South and covered 5 km. Finally, turning to East, he covered 10 km. In which direction is he from his house?

(a) East (b) South

(c) North (d) West

Solution. North. Starting point is A and ending point is E. E is to the north of his house at A.



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11 DATA INTERPRETATION

11.1. Data Interpretation

- It deals with careful reading, understanding, organizing and interpreting the data provided so as to derive meaningful conclusions.
- Mostly used tools for interpretation of a data are
  - Ratio
  - Percentage
  - Rate
  - Average

11.2. Types of Data Interpretation:

The numerical data pertaining to any event can be presented by any one or more of the following methods.

1. Tables
2. Line Graphs
3. Bar Graphs or Bar Charts
4. Pie Charts or Circle Graphs

11.2.1. Tables

It is the systematic presentation of data in tabular form to understand the given information and to make clear the problem in a certain field of study. It has six elements namely:

- **Title:** It is the heading of the table.
- **Stule:** It is the section of the table containing row headings
- **Column Captions:** It is the heading of each column
- **Body:** It consists the numerical figures
- **Footnotes:** It is for further explanation of the table
- **Source:** It is the authority of the data

Example: Study the following table and answer the questions given below it.

Annual Income of Five Schools

Figures in ’00 rupees

Sources of Income	School A	School B	School C	School D	School E
Tuition Fee	120	60	210	90	120
Term Fee	24	12	45	24	30
Donations	54	21	60	51	60
Funds	60	54	120	42	55
Miscellaneous	12	3	15	3	15
Total	270	150	450	210	280

Example: The income by way of donation to school D is what per-cent of its miscellaneous?

Solution: Required percentage = $\frac{5100}{300} = 27\%$

11.2.2. Line Graph

A line graph indicates the variation of a quantity w.r.t two parameters calibrated on X and Y-axis respectively.

Note:

1. Any part of the line graph parallel to X-axis represents no change in the value of Y parameter w.r.t the value of X parameter.
2. The steepest or maximum part of the line graph indicates maximum percentage change of the value during the two-consecutive period in which the related part lies.
3. If the steepest part is a rise slope, then it is the highest percentage growth.
4. If the steepest part is a decline slope, it will represent a maximum percentage fall of the value calibrated in the other axis.

11.2.3. Bar Graph

Bar graphs are diagrammatic representation of a discrete data.

Types of Bar Graphs:

- **Simple Bar Graphs:** A simple bar graph relates to only one variable. The values of the variables may relate to different years or different terms.
- **Sub-divided Bar Graph:** It is used to represent various parts of sub-classes of total magnitude of the given variable.
- **Multiple Bar Graphs:** In this type, two or more bars are constructed adjoining each other, to represent either different components of a total or to show multiple variables.

11.2.4. Pie Chart

In this method of representation, the total quantity is distributed over a total angle of 360° which is one complete circle or pie.

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12 DATA SUFFICIENCY

12.1. Introduction

Data sufficiency questions are designed to measure your ability to analyze a quantities problem, recognize which given information is relevant, and determine at what point there is sufficient information to solve a problem. In these questions, you are to classify each problem according to the five or four fixed answer choice, rather than find a solution to the problem.

Each Data sufficiency question consists of a question, often accompanied by some initial information, and two statements, labeled (1) and (2), which contain additional information. You must decide whether the information in each statement is sufficient to answer the question or- if neither statement provides enough information –whether the information in the two statements together is sufficient. It is also possible that the statements in combination do not give enough information answer the question.

Begin by reading the initial information and the question carefully. Next, consider the first statement. Does the information provided by the first statement is sufficient to answer the question? Go on the statement. Try to ignore the information given in the first statement when you consider the second statement. Now you should be able to say, for each statement, whether it is sufficient to determine the answer.

Next, consider the two statements in tandem. Do they, together, enable you to answer the question?

Give our answers as per the following statements

A Statement (1) alone is sufficient but
Statement (2) alone is not sufficient

B Statement (2) alone is sufficient but
Statement (1) alone is not sufficient

C Both statements together are sufficient
but neither statement alone is sufficient

D Each statement alone is sufficient

E Both statements together are still not sufficient.

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Thank you