UNIT-V Multivariable calculus (Integration)

Multiple Integral.

→ Let $y = f(x)$ be a function of one variable defined and bounded on $[a, b]$, Let $[a, b]$ be divided into 'n' sub interval by points $x_0, x_1, x_2 - - x_n$ such that $a = x_0 < x_1 < x_2 - - . < x_n = b$

- The limit is defined to be the definded integral
  $\int_a^b f(x)\,dx$.
- The generalisation of this definition to two dimension is called a double integral and to three dimension is called triple integral :

* Double Integral :

- case - I :- when y_1, y_2 are function of x.
  x_1, x_2 are constant

  $f(x, y)$ is first integrated with respect to y. keep it x fixed b/w limits y_1, y_2 & then the resulting expression is integrated w.r to x within the limits x_1, x_2

$$I = \int_{x_1}^{x_2} \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx$$

- case II :- when x_1, x_2 are function of y. y_1, y_2 are constant

  $f(x, y)$ is I[st] integrated w.r to x. keep it y fixed b/w limits x_1, y_2 & then the resulting expression is integrated w.r to y within the limits y_1, y_2.

$$I = \int_{y_1}^{y_2} \int_{x_1(y)}^{x_2(y)} f(x, y)\, dx\, dy.$$

- case III :- when both pairs of limits are constant the region of integration is rectangle.

$$I = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x,y)\, dx.dy = \int_a^b \int_c^d f(x,y)\, dx\, dy$$

* Properties:

① $\iint_R [f \pm g]\, dxdy = \iint_R f\, dx.dy \pm \iint_R g\, dx\, dy$

② $\iint_R k f\, dx\, dy = k \iint_R f dx.dy$

③ $\iint_R f dx\, dy = \iint_{R_1} f dx\, dy + \iint_{R_2} f dx.dy$

*1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

2) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

③ $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

① Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\, dy\, dx$

sol $I = \int_0^3 \int_1^2 [xy + x^2 y + xy^2]\, dy\, dx$

$= \int_0^3 \left[x \int_1^2 y\, dy + x^2 \int_1^2 y\, dy + x \int_1^2 y^2 dy\right] dx$

$= \int_0^3 x\left[\frac{y^2}{2}\right]_1^2 + x^2 \left[\frac{y^2}{2}\right]_1^2 + x\left[\frac{y^3}{3}\right]_1^2 dx$

$= \int_0^3 \frac{x}{2}\left[2^2 - 1\right] + \frac{x^2}{2}\left[2^2 - 1\right] + \frac{x}{3}\left[2^3 - 1^3\right] dx$

$= \int_0^3 \left[\frac{3}{2}x + \frac{3}{2}x^2 + \frac{7x}{3}\right] dx$

$= \frac{3}{2}\int_0^3 x\, dx + \frac{3}{2}\int_0^3 x^2 dx + \frac{7}{3}\int_0^3 x.dx$

$$= \frac{3}{2}\left[\frac{x^2}{2}\right]_0^3 + \frac{3}{2}\left[\frac{x^3}{\cancel{3}}\right]_0^3 + \frac{7}{3}\left[\frac{x^2}{3}\right]_0^3$$

$$= \frac{3}{4}\left[3^2-0\right] + \frac{\cancel{1}}{2}\left[3^3-0\right] + \frac{7}{6}\left[3^2-0^2\right]$$

$$= \frac{27}{4} + \frac{27}{2} + \frac{63}{6}$$

$$= \frac{123}{4}$$

Ass

② $\int_0^3 \int_1^2 xy(x+y)\,dx\,dy = 24$

Sol) $\int_0^3 \int_1^2 (x^2y + xy^2)\,dx\,dy$

$$= \int_0^3 \left[y\int_1^2 x^2 + y^2 \int_1^2 x\right] dy$$

$$= \int_0^3 y\left[\frac{x^3}{3}\right]_1^2 + y^2\left[\frac{x^2}{2}\right]_1^2$$

$$= \int_0^3 \frac{7}{3}y + \frac{3y^2}{2}$$

$$= \frac{7}{3}\left[\frac{y^2}{2}\right]_0^3 + \frac{3}{2}\left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{7}{6}(9) + \frac{3}{6}[27]$$

$$= \frac{63 + 27(3)}{6} = \frac{63+81}{6}$$

$$= \frac{144}{6}$$

$$= 24$$

***** 2M

② $\int_0^2 \int_0^x y\,dy \cdot dx$

Sol) $\int_0^2 \left[\int_0^x y\,dy\right] dx$

$$= \int_0^2 \left[\frac{y^2}{2}\right]_0^x dx$$

$$= \frac{1}{2}\int_0^2 [x^2 - 0]\,dx$$

$$= \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{6}\left[2^3 - 0^3\right]$$

$$= \frac{8}{6} = \frac{4}{3}$$

③ Evaluate $\int_{y=0}^{2}\int_{x=0}^{3} xy\,dx\,dy$

Sol
$$= \int_0^2 \left[y \int_0^3 x\,dx \right] dy$$

$$= \int_0^2 y \left[\frac{x^2}{2} \right]_0^3 dy$$

$$= \int_0^2 \frac{y}{2} [9-0]\, dy$$

$$= \frac{9}{2} \int_0^2 y\, dy$$

$$= \frac{9}{2} \left[\frac{y^2}{2} \right]_0^2$$

$$= \frac{9}{4} [4-0]$$

$$= \frac{9}{\cancel{4}} \times \cancel{4}$$

$$= 9.$$

H.W

④ Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$

$$I = \int_0^1 \int_x^{\sqrt{x}} [x^2+y^2]\, dy\, dx$$

$$I = \int_0^1 \left[x^2 \int_x^{\sqrt{x}} 1.dy + \int_x^{\sqrt{x}} y^2 dy \right] dx$$

$$= \int_0^1 \left[x^2 [y]_x^{\sqrt{x}} + \left[\frac{y^3}{3} \right]_x^{\sqrt{x}} \right] dx$$

$$= \int_0^1 \left[x^2 [\sqrt{x} - x] + \frac{1}{3} [(\sqrt{x})^3 - x^3] \right] dx$$

$$= \int_0^1 x^2 \left[x^{1/2} - x \right] + \frac{1}{3} \left[x^{3/2} - x^3 \right] \Big] dx$$

* Note :

Note that the limits of the interior integration of function of x. Hence we must understand that they are limits of y. Integration correspond to x

$$= \int_0^1 \left(x^{5/2} - x^3 + \frac{1}{3} x^{3/2} - \frac{1}{3} x^3 \right] dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{1}{3} x^{3/2} - \frac{4}{3} x^3 \right] dx$$

$$= \int_0^1 x^{5/2} dx + \frac{1}{3} \int_0^1 x^{3/2} dx - \frac{4}{3} \int_0^1 x^3 dx$$

$$= \left[\frac{x^{7/2}}{7/2} \right]_0^1 + \frac{1}{3} \left[\frac{x^{5/2}}{5/2} \right]_0^1 - \frac{4}{3} \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{7} [1-0] + \frac{1}{3} \times \frac{2}{5} [1-0] - \frac{1}{3} [1-0]$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{3}{35}$$

① $\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dy\,dx$

Sol $\int_0^a x^2\int_0^b \cancel{y}\,dy + \int_0^b y^2 dy \; dx$

$= \int_0^a \left[x^2 (y)_0^b + \left[\frac{y^3}{3}\right]_0^b\right] dx$

$= \int_0^a x^2 [b] + \frac{1}{3}[b^3]\,dx$

$= \int_0^a \left[bx^2 + \frac{b^3}{3}\right] dx$

$\cancel{= b\int_0^a x^2 dx + \frac{1}{3}\int_0^a b^3 dx}$

$\cancel{= b\left[\frac{x^3}{3}\right]_0^a + \frac{1}{3}\Big[}$

$= b\int_0^a x^2 + \frac{b^3}{3}\int_0^a 1\cdot dx$

$= b\left[\frac{x^3}{3}\right]_0^a + \left[\frac{b^3}{3}[\cancel{x}]\right]_0^a$

$= \frac{b}{3}[a^3] + \frac{b^3}{3}[a]$

$= \frac{a^3b}{3} + \frac{ab^3}{3}$

$= \frac{ab}{3}(a^2+b^2)$

② $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Sol $\int_0^5 \int_0^{x^2} x^3 + xy^2\,dy\,dx$

$= \int_0^5 x^3 \int_0^{x^2} 1\,dy + x\int_0^{x^2} y^2 dy \; dx$

$= \int_0^5 x^3 [y]_0^{x^2} + x\left[\frac{y^3}{3}\right]_0^{x^2} dx$

$= \int_0^5 x^3[x^2-0] + \frac{x}{3}[x^6-0]\,dx$

$= \int_0^5 \left[x^5 + \frac{x^7}{3}\right] dx$

$= \left[\frac{x^6}{6} + \frac{x^8}{24}\right]_0^5$

$= \left[\frac{5^6}{6} + \frac{5^8}{24}\right]$

$= 5^6\left[\frac{1}{6} + \frac{5^2}{24}\right]$

$= 5^6\left[\frac{1}{6} + \frac{25}{24}\right]$

$= 5^6\left[\frac{29}{24}\right]$

③ $\int_{x=0}^{1}\int_{y=0}^{2} y^2\,dy\,dx$

Sol $\int_{x=0}^{1}\int_{y=0}^{2} y^2\,dy\,dx$

$= \int_0^1 \int_0^2 y^2 dy\,dx$

$= \int_0^1 \left[\frac{y^3}{3}\right]_0^2 = \int_0^1 \frac{1}{3}[8] = \frac{8}{3}$

④ $\int_0^1 \int_x^{\sqrt{x}} xy\,dy\,dx$

Sol: $= \int_0^1 x \int_x^{\sqrt{x}} y\,dy\,dx$

$= \int_0^1 x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx = \int_0^1 \frac{x}{2}\left[(\sqrt{x})^2 - x^2\right] dx$

$= \int_0^1 \frac{x}{2}[x-x^2]\,dx$

$= \frac{1}{2}\int_0^1 x^2 - \frac{1}{2}\int_0^1 x^3\,dx$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^1 - \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{6}[1-0] - \frac{1}{8}[1]$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{8-6}{48}$$

$$= \frac{2}{48} = \frac{1}{24}$$

(5) $\int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx$

Sol $= \int_0^1 x \int_0^{\sqrt{x}} y\,dy\,dx$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{x}{2}[x]\,dx$$

$$= \int_0^1 \frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{6}(1-0)$$

$$= \frac{1}{6}$$

(6) $\int_0^2 \int_1^{e^x} dy\,dx$

S:- $\int_0^2 \int_1^{e^x} dy\,dx$

$$= \int_0^2 [y]_1^{e^x} dx$$

$$= \int_0^2 [e^x - \cancel{1}]\,dx$$

$$= [e^x - x]_0^2 = e^2 - 3$$

(7) $\int_0^3 \int_{-x}^{x} xy\,dy\,dx$

Sol $= \int_0^3 x \int_{-x}^{x} y\,dy\,dx$

$$= \int_0^3 x \left[\frac{y^2}{2}\right]_{-x}^{x} dx$$

$$= \int_0^3 \frac{x}{2}[x^2 - x^2]\,dx$$

$$= \int_0^3 \frac{x^3}{2} - \int_0^3 \frac{x^3}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}\right]_0^3 - \frac{1}{2}\left[\frac{x^4}{4}\right]_0^3$$

$$= \frac{1}{8}[81] - \frac{1}{8}[81]$$

$$= \frac{81-81}{8} = 0$$

$$\cancel{= \frac{162}{8}} \quad \cancel{= \frac{81}{4}}$$

(1) (i) $\int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$

S:- $I = \int_0^4 \int_0^{x^2} e^{y/x}\,dy\,dx$

$$= \int_0^4 \left[\int_0^{x^2} e^{y/x}\,dy\right] dx \qquad \left| \int e^{x/a}\,dx = \frac{e^{x/a}}{1/a} + c \right.$$

$$= \int_0^4 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx \quad \{x \text{ is constant}\}$$

$$= \int_0^4 \frac{x}{1}\left[e^{y/x}\right]_0^{x^2} dx$$

$$= \int_0^4 x\left[e^{x^2/x} - e^{0/x}\right] dx$$

$$= \int_0^4 x[e^x - 1]\,dx = \int_0^4 (xe^x - x)\,dx$$

$$= \int_0^4 xe^x - \int_0^4 x\,dx \qquad \left\{ \begin{array}{l} \int xe^x\,dx = xe^x - e^x + c \\ \int u\,dv = uv - u'v_1 + u''v_2 \end{array} \right.$$

$= [xe^{x} - e^{x}]_0^4 - \left[\frac{x^2}{2}\right]_0^4$

$= [(4e^4 - e^4) - (0 - e^0)] - \left[\frac{4^2}{2} - \frac{0^2}{2}\right]$

$= 3e^4 + 1 - \cancel{\frac{16}{2}}\,8$

$= 3e^4 - 7$

(ii) $\int_0^1 \int_0^x (e^{y/x})\, dy\, dx$

Sol $I = \int_0^1 \int_0^x (e^{y/x}) dy\, dx$

$= \int_0^1 \left[\int_0^x e^{y/x} dy \right] dx$

$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^x$

$= \int_0^1 x\,[e^{y/x}]_0^x\, dx$

$= \int_0^1 x\,[e^{x/x} - e^{0/x}]\, dx$

$= \int_0^1 x\,[e^1 - e^0]\, dx$

$= \int_0^1 x\,[e-1]\, dx$

$= \int_0^1 (xe - x)\, dx$

$= \int_0^1 xe - \int_0^1 x\, dx$

$= [xe - e]_0^1 - \left[\frac{x^2}{2}\right]_0^1$

$= [[e-e] - [0-e]] - \left[\frac{1}{2}\right]$

$= 0 + e - \frac{1}{2}$

$= e - \frac{1}{2}$

(2) Evaluate

(i) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

S: $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy$

$= \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2-x^2}\, dx\, dy$

$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-y^2})^2 - x^2}\, dx\, dy$

put $\sqrt{a^2-y^2} = p$

$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{p^2-x^2}\, dx\, dy$

$= \int_0^a \left[\int_0^p \sqrt{p^2-x^2}\, dx\right] dx$

$\left\{ \int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c \right\}$

$= \int_0^a \left\{\frac{x}{2}\sqrt{p^2-x^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{x}{p}\right)\right\}_0^p dy$

$= \int_0^a \left[\frac{p}{2}\sqrt{p^2-p^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{p}{p}\right) - \frac{0}{2}\sqrt{p^2-0^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{0}{p}\right)\right] dy$

$= \int_0^a \left[0 + \frac{p^2}{2}\sin^{-1}(1) - 0 + 0\right] dy$

$\sin^{-1}(0) = 0$
$\sin^{-1}(1) = \pi/2$

$= \int_0^a \frac{p^2}{2} \cdot \frac{\pi}{2}\, dy$

$= \frac{\pi}{4} \int_0^a p^2\, dy$

$= \pi/4 \int_0^a (\sqrt{a^2-y^2})^2\, dy$

$= \pi/4 \left[a^2 \int_0^a 1\, dy - \int_0^a y^2\, dy\right]$

$$= \frac{\pi}{4}\left[a^2[y]_0^a - \left[\frac{y^3}{3}\right]_0^a\right]$$

$$= \frac{\pi}{4}\left[a^2(a) - \frac{a^3}{3}\right]$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right)$$

$$\cancel{= \frac{\pi}{4}\frac{3a^3}{3}}$$

$$= \frac{\pi}{\cancel{4}_2}\frac{2a^3}{3}$$

$$= \frac{\pi a^3}{6}$$

(ii) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \frac{\pi a^3}{6}$$

S:- $$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2 - y^2}\,dy\,dx$$

put $\sqrt{a^2-x^2} = p$

$$= \int_0^a \int_0^p \sqrt{p^2-y^2}\,dy\,dx$$

$$= \int_0^a \left[\frac{y}{2}\sqrt{p^2-y^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{y}{p}\right)\right]_0^p dx$$

$$= \int_0^a \frac{p}{2}\sqrt{p^2-p^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{p}{p}\right) - \frac{0}{2}\sqrt{p^2-0^2} + \frac{p^2}{2}\sin^{-1}\left(\frac{0}{p}\right)$$

$$= \int_0^a \frac{p^2}{2}\frac{\pi}{2}\,dx$$

$$= \frac{\pi}{4}\int_0^a p^2\,dx$$

$$= \frac{\pi}{4}\int_0^a (\sqrt{a^2-x^2})^2\,dx$$

$$= \frac{\pi}{4}\left(a^2\int_0^a 1\,dx - \int_0^a x^2\,dx\right.$$

$$= \frac{\pi}{4}\left(a^2[x]_0^a - \left[\frac{x^3}{3}\right]_0^a\right)$$

$$= \frac{\pi}{4}\left(a^2(a) - \left[\frac{a^3}{3}\right]\right)$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{\pi}{\cancel{4}_2}\left(\frac{2a^3}{3}\right)$$

$$= \frac{\pi a^3}{6}$$

③ ① $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

S:- $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2}\, dy\, dx$

put $\sqrt{1+x^2} = P$

$= \int_0^1 \left(\int_0^P \frac{1}{P^2+y^2}\, dy \right) dx$

$I = \int_0^1 \left[\int_0^P \frac{1}{P^2+y^2}\, dy \right] dx$

$I = \int_0^1 \left[\int_0^P \frac{1}{P^2+y^2}\, dy \right] dx$

$\left\{ \int \frac{1}{a^2+x^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c \right\}$

$I = \int_0^1 \frac{1}{P} \left[\tan^{-1}\left(\frac{y}{P}\right) \right]_0^P dx$

$= \int_0^1 \frac{1}{P} \left[\tan^{-1}\left[\frac{P}{P}\right] - \tan^{-1}\left(\frac{0}{P}\right) \right] dx$

$= \int_0^1 \frac{1}{P} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] dx$

$= \int_0^1 \frac{1}{P} \left[\frac{\pi}{4} - 0 \right] dx$

$= \frac{\pi}{4} \int_0^1 \frac{1}{P}\, dx$ $\qquad \int \frac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1}(x) + c$ or $\log(x+\sqrt{1+x^2}) + c$

$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$

$= \frac{\pi}{4} \left[\sinh^{-1}(x) \right]_0^1$

$= \pi/4 \left[\sinh^{-1}(1) - \sinh^{-1}(0) \right]$

$= \pi/4 \sinh^{-1}(1)$ (or) $\pi/4 \log(1+\sqrt{2})$

② $\int_0^1 \int_0^{\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

S:- $I = \int_0^1 \int_0^{\sqrt{1+y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

$= \int_0^1 \int_0^{\sqrt{1+y^2}} \frac{1}{(\sqrt{1+y^2})^2+x^2}\, dx\, dy$

put $(\sqrt{1+y^2}) = P$

$= \int_0^1 \int_0^P \frac{1}{P^2+x^2}\, dx\, dy$

$= \int_0^1 \left[\int_0^P \frac{1}{P^2+x^2}\, dx \right] dy$

$= \int_0^1 \frac{1}{P} \left[\tan^{-1}\left(\frac{x}{P}\right) \right]_0^P dy$

$= \int_0^1 \frac{1}{P} \left[\tan^{-1}\left[\frac{P}{P}\right] - \tan^{-1}\left[\frac{0}{P}\right] \right] dy$

$= \int_0^1 \left[\frac{1}{P} \tan^{-1}(1) - \tan^{-1}(0) \right] dy$

$= \int_0^1 \frac{1}{P} \left[\frac{\pi}{4} \right] dy$

$= \frac{\pi}{4} \int_0^1 \frac{1}{P}\, dy$

$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+y^2}}\, dy$

$= \frac{\pi}{4} \left(\sinh^{-1}(y) \right]_0^1$

$= \frac{\pi}{4} \left(\sinh^{-1}(1) - \sinh^{-1}(0) \right]$

$= \frac{\pi}{4} \sin^{-1}(1)$

$= \pi/2 \log(1+\sqrt{2})$.

④ $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$

S:- $= \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dx\, dy$

$= \int_0^\infty e^{-y^2} \left[\int_0^\infty e^{-x^2} dx \right] dy$

$= \int_0^\infty e^{-y^2} \left[\frac{\sqrt{\pi}}{2} \right] dy$

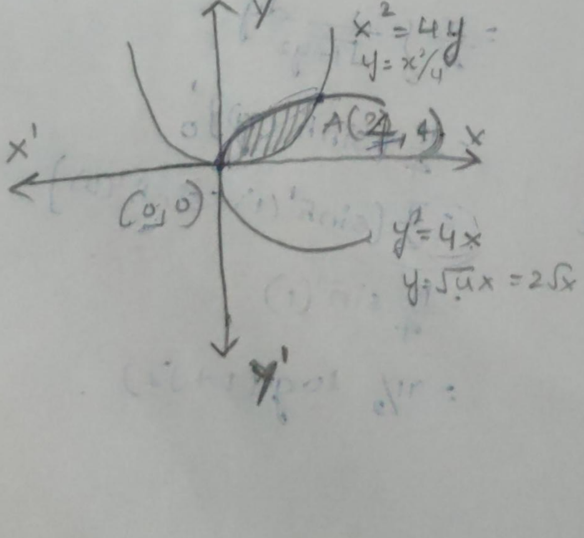
{by using beta function $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ }

$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-y^2} dy$

$= \frac{\sqrt{\pi}}{2} \frac{\sqrt{\pi}}{2}$

$= \pi/4$

(i) $\iint_R y\, dx\, dy$ where R is the region bounded by the paraablos $y^2 = 4x$, $x^2 = 4y$



Sol $y^2 = 4x \rightarrow ①$ $y = 2\sqrt{x}$

$x^2 = 4y \rightarrow ②$ $y = x^2/4$

Solving these two equation

eq② $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

eq① $\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{\cancel{4}^2} = 4x$

$x^4 = 4^3 x$

$x^4 - 4^3 x = 0$

$x(x^3 - 4^3) = 0$

$\boxed{x = 0}$ $x^3 - 4^3 = 0$

$\boxed{x = 4}$

$y = 0$; $y = \frac{4^2}{\cancel{4}}$

$\boxed{y = 4}$

Thus the 2 parabols intersect the pt (0,0) (4,4)

The shaded area b/w the parabola eq① & eq② is the region of integration

$\iint_R y\, dx\, dy = \int_{x=0}^{4} \int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y\, dy\, dx$

$= \int_0^4 \left[\int_{x^2/4}^{2\sqrt{x}} y\, dy \right] dx$

$= \int_0^4 \left[\frac{y^2}{2} \right]_{x^2/4}^{2\sqrt{x}} dx$

$= \int_0^4 \frac{1}{2} \left[(2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2 \right] dx$

$= \int_0^4 \frac{1}{2} \left(4x - \frac{x^4}{16} \right] dx$

$= \int_0^4 \left(2x - \frac{x^4}{32} \right] dx$

$= 2\int_0^4 x\,dx - \frac{1}{32}\int_0^4 x^4\,dx$

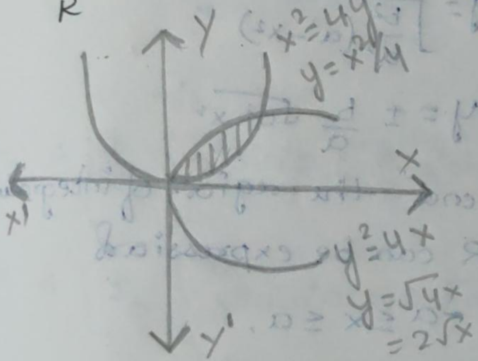
$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_0^4 - \frac{1}{32}\left[\frac{x^5}{5}\right]_0^4$

$= [4^2 - 0^2] - \frac{1}{160}[4^5 - 0^5]$

$= 16 - \frac{4^5}{160}$

$= 48/5$

(ii) $\iint_R y^2\,dx\,dy = \frac{168}{35}$



Sol $y^2 = 4x \rightarrow$ (1)

$x^2 = 4y \rightarrow$ (2)

Solv these two eqⁿ

eq(2) $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$

eq(1) $\left(\frac{x^2}{4}\right)^2 = 4x$

$\frac{x^4}{4^2} = 4x$

$x^4 = 4^3 x$

$x^4 - 4^3 x = 0$

$x(x^3 - 4^3) = 0$

$x = 0;\ x^3 - 4^3 = 0$

$\boxed{x = 4}$

$y = 0;\ y = \frac{4^2}{4}$

$y = 4$

$\iint_R y^2\,dx\,dy = \int_{x=0}^{4}\int_{y=\frac{x^2}{4}}^{2\sqrt{x}} y^2\,dy\,dx$

$= \int_0^4 \left[\frac{y^3}{3}\right]_{x^2/4}^{2\sqrt{x}} dx$

$= \int_0^4 \frac{1}{3}\left[(2\sqrt{x})^3 - \left(\frac{x^2}{4}\right)^3\right] dx$

$= \int_0^4 \frac{1}{3}\left[8\left(\frac{x^{3/2}}{3/2}\right)\right] - \frac{1}{64}\int_0^4 \frac{x^7}{7}$

$= \int_0^4 \frac{24}{3}\frac{2}{3}\left[4^{3/2}\right] - \frac{1}{64\times 7}\left[4^7\right]$

$= \int_0^4 \frac{16}{3}\left[4^{3/2}\right] - \frac{4^5}{4^3\times 7}$

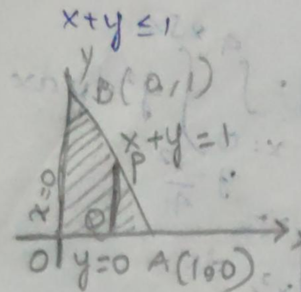
$= \frac{16}{3}[8] - \frac{4^4}{7} = \frac{128}{3} - \frac{256}{7}$

$= \frac{896 - 768}{21}$

$= \frac{128}{21}$

(6) Evaluate $\iint (x^2+y^2)\,dx\,dy$.

$x+y \le 1$



$$\iint (x^2+y^2)\,dx\,dy = \int_0^1\int_0^{1-x} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2\int_0^{1-x} dy + \int_0^{1-x} y^2\,dy\right] dx$$

$$= \int_0^1 x^2\left[y\right]_0^{1-x} + \left[\frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 x^2\left[1-x-0\right] + \frac{1}{3}\left[(1-x)^3 - 0\right] dx$$

$$= \int_0^1 x^2 - x^3 + \frac{1}{3}(1-x)^3\,dx$$

$$= \int_0^1 x^2 - \int_0^1 x^3 + \frac{1}{3}\int_0^1 (1-x)^3.$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \frac{1}{3}\left[\frac{(1-x)^4}{-4}\right]_0^1$$

$$= \frac{1}{3}[1] - \frac{1}{4}[1] + \frac{1}{12}\left[(1-x)^4\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{\cancel{2}}{\cancel{12}\,6} = \frac{1}{6}$$

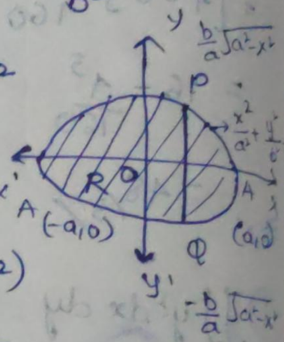
****

(7) $\iint_R (x^2+y^2)\,dx\,dy$ over the area R bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1$$

S:- Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$



$$\frac{y^2}{b^2} = \frac{a^2-x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$y = \sqrt{\frac{b^2}{a^2}(a^2-x^2)}$$

$$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

Hence the region of integration R can be expressed

$$-a \le x \le a,$$

$$-\frac{1}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$$

$$\cancel{\iint_R (x^2+y^2)\,dx\,dy = \int_{x=-a}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx}$$

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{x=-a}^{a}\int_{y=\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$$

$$= 2\int_0^a 2\int_0^{b/a\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx$$

$$= 4\int_0^a \left[\int_0^{b/a\sqrt{a^2-x^2}} [x^2+y^2]\,dy\right] dx$$

$$= 4\int_0^a \left[x^2\int_0^{b/a\sqrt{a^2-x^2}} 1\,dy + \int_0^{b/a\sqrt{a^2-x^2}} y^2\,dy\right] dx$$

$$= 4\int_0^a \left[x^2[y]_0^{b/a\sqrt{a^2-x^2}} + \left[\frac{y^3}{3}\right]^{\frac{b}{a}\sqrt{a^2-x^2}} \right] dx$$

$= \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ of $f(x)$ is even

$= \int_0^a f(x)dx = 0$. if $f(x)$ is odd

$$= 4\int_0^a x^2 \frac{b}{a}\sqrt{a^2-x^2} + \frac{1}{3}\left[\left(\left(\frac{b}{a}(a^2-x^2)^{1/2}\right)^3 - 0^3\right)\right] dx$$

$$= 4\left[\frac{b}{a}\int_0^a x^2\sqrt{a^2-x^2}\,dx + \frac{b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2}dx\right]$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta d\theta$,

when $x=0 \rightarrow 0 = a\sin\theta$
$0 = \sin\theta$
$\sin 0 = \sin\theta$
$\theta = 0°$

when $x = a \rightarrow a - a\sin\theta$
$1 = \sin\theta$
$\sin\pi/2 = \sin\theta$
$\theta = \pi/2$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^3-a^2\sin^2\theta}\,.\,a\cos\theta d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2}(a^2-a^2\sin^2\theta)^{3/2} a\cos\theta d\theta\right)$$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^3\sin^2\theta\cos\theta\sqrt{a^2(1-\sin^2\theta}\,d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2}(a^2)^{3/2}(1-\sin^2\theta)a\cos\theta d\theta\right.$$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^3\sin^2\theta\cos\theta\sqrt{(a\cos\theta)^2}d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2} a^4(\cos^2\theta)^{3/2}\cos\theta d\theta\right.$$

$$= 4\left[\frac{ba^{\cancel{4}\,3}}{\cancel{a}}\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta + \frac{b^3a^{\cancel{4}}}{3a^{\cancel{3}}}\int_0^{\pi/2}\cos^4\theta d\theta\right.$$

$$= 4\left[ba^3\left[\frac{1}{4}\frac{1}{2}\frac{\pi}{2}\right] + \frac{b^3a}{3}\left[\frac{3}{4}\frac{1}{2}\frac{\pi}{2}\right]\right]$$

$$= 4\left[\frac{\pi ba^3}{16} + \frac{\pi b^3a}{16}\right]$$

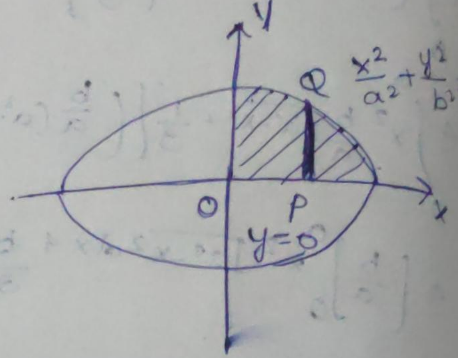
$\left\{ \int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)}\cdot\frac{1}{2}\frac{\pi}{2} \right.$

$$= \frac{\cancel{4}\pi ab}{\cancel{16}\,4}[a^2+b^2]$$

$\int_0^{\pi/2}\cos^n x\,dx = \frac{(n-1)(n-3)\cdots 1}{n(n-2)\quad 2}\frac{\pi}{2}$ { Here n is even)

$$= \pi/4\, ab\,(a^2+b^2)$$

1) Evaluate $\iint (x+y)\,dx\,dy$ over the region ~~of~~ in the +ve quadrant bounded by ellipse



Sol :- $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

$y^2=\frac{b^2}{a^2}(a^2-x^2)$

$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

Hence the region of ~~w~~ integration can be expressed $0\le x\le a,\ y\le\frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R (x+y)\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}}(x+y)\,dy\,dx$$

$$=\int_0^a\left[x\int_0^{b/a\sqrt{a^2-x^2}}1\cdot dy+\int_0^{b/a\sqrt{a^2-x^2}}y\,dy\right]dx$$

$$=\int_0^a\left[x\,[y]_0^{\frac{b}{a}\sqrt{a^2-x^2}}+\left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}}\right]dx$$

$$=\int_0^a\left[x\left[\frac{b}{a}\sqrt{a^2-x^2}-0\right]+\frac{1}{2}\left[\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^2-0^2\right]\right]dx$$

$$=\int_0^a\left[x\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^2}{2a^2}(a^2-x^2)\right]dx$$

$$=\frac{b}{a}\int_0^a x(a^2-x^2)^{1/2}dx+\frac{b^2}{2a^2}\left[\int_0^a(a^2-x^2)dx\right]\qquad\left\{\int f'(x)\left[f(x)\right]^n dx=\frac{f(x)^{n+1}}{n+1}+c\right\}$$

$$=-\frac{b}{2a}\int_0^a(-2x)(a^2-x^2)^{1/2}dx+\frac{b^2}{2a^2}\left[a^2\int_0^a 1\cdot dx-\int_0^a x^2dx\right]$$

$$=-\frac{b}{2a}\left[\frac{(a^2-x^2)^{1/2+1}}{1/2+1}\right]_0^a+\frac{b^2}{2a^2}\left[a^2[x]_0^a-\left[\frac{x^3}{3}\right]_0^a\right]$$

$$=\frac{-b}{2a}\left[\frac{(a^2-x^2)^{3/2}}{3/2}\right]_0^a+\frac{b^2}{2a^2}\left[a^2(a-0)-\left[\frac{a^3}{3}-\frac{a^3}{3}\right]\right]$$

$$=-\frac{b}{2a}\frac{2}{3}\left[(a^2-a^2)^{3/2}-(a^2-0^2)^{3/2}\right]+\frac{b^2}{2a^2}\left[a^3-\frac{a^3}{3}\right]$$

$$=-b/3a\left[0-(a^2)^{3/2}\right]+\frac{b^2}{2a^2}\left(\frac{2a^3}{3}\right)\qquad=\frac{ba^{\cancel{3}}}{3\cancel{a}}+\frac{b^2a^{\cancel{3}}}{3\cancel{a^2}}$$

$= \frac{ba^2}{3} + \frac{b^2 a}{3}$

$= \frac{ab}{3}(a+b)$

formula:-

① $\int \frac{f'(x)}{f(x)} dx = \log f(x) + C$

② $f'(x) \{f^n(x)\} dx = \frac{f^{n+1}(x)}{n+1} + C$

③ $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$

④ $\int \frac{f'(x)}{f^n(x)} dx = \frac{f^{-n+1}(x)}{-n+1} + C$

Find the value of

2. $\iint (xy)\, dy\, dx$ over the positive quadrant bounded by ellipse. $\frac{a^2 b^2}{8}$

③ find $\iint (x+y)^2 dx dy$ over the area bounded by ellipse

2:-Sol $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$

$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

Hence the region of integration can be expressed $0 \le x \le a$, $y \le \frac{b}{a}\sqrt{a^2-x^2}$

$$\iint_R (xy)\, dy\, dx = \int_{x=0}^{a} \int_{y=0}^{b/a\sqrt{a^2-x^2}} (xy)\, dy\, dx$$

$$= \int_0^a [x \int_0^{b/a\sqrt{a^2-x^2}} y\, dy\, dx = \int_0^a [x \left[\frac{y^2}{2}\right]_0^{b/a\sqrt{a^2-x^2}}] dx$$

$$= \int_0^a x \frac{1}{2} \left[\left[\frac{b}{a}\sqrt{a^2-x^2}\right]^2 - 0\right]$$

$$= \int_0^a x \frac{1}{2} \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{b^2}{2a^2} \int_0^a x(a^2 - x^2)\, dx = \frac{b^2}{2a^2} \int_0^a (xa^2 - x^3)\, dx$$

$$= \frac{b^2}{2a^2}\left[a^2 \int_0^a x\, dx - \int_0^a x^3 dx\right] \quad = \frac{b^2}{2a^2}\left[\frac{a^2}{2}[a^2 - 0] - \frac{1}{4}[a^4 - 0]\right]$$

$$= \frac{b^2}{2a^2}\left[a^2 \left[\frac{x^2}{2}\right]_0^a - \left(\frac{x^4}{4}\right]_0^a\right) = \frac{b^2}{2a^2}\left[\frac{1}{2}a^4 - \frac{1}{4}a^4\right]$$

$$= \cancel{\frac{b^2}{2a^2} \frac{a^2}{2}[a^2 - 0] - \frac{1}{4}[a^4 - 0]}$$

$= \frac{b^2}{2a^2} \frac{1}{2} a^4 \left[1 - \frac{1}{2}\right]$

$= \frac{b^2}{2\cancel{a^2}} \frac{\cancel{a^4}^{2}}{2} \left(\frac{1}{2}\right)$

$= \frac{b^2 a^2}{4 \times 2}$

$= \frac{a^2 b^2}{8}$

3 $\iint (x+y)^2 dx\, dy$

Sol:-

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2} \quad y = \sqrt{\frac{b^2}{a^2}(a^2-x^2)}$

$y = \pm \frac{b}{a}\sqrt{a^2-x^2}$

$-a \le x \le a$

$\iint (x+y)^2 dx\, dy = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2) + 2xy \; dy\, dx$

$= 2\int_0^a 2\int_0^{b/a\sqrt{a^2-x^2}} (x^2+y^2) + 2xy \; dy\, dx$

$= 4\int_0^a \left[x^2 \int_0^{b/a\sqrt{a^2-x^2}} 1\, dy + \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dy + 2x \int_0^{b/a\sqrt{a^2-x^2}} y\, dy\right] dx$

$= 4\int_0^a x^2 \left[y\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} + \left[\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}$

$= 4\int_0^a x^2 \left[\frac{b}{a}\sqrt{a^2-x^2} - 0\right] + \left(\frac{b}{a}(a^2-x^2)^{1/3}\right)^3 - 0^3 \,]\, dx$

$= 4 \left[\frac{b}{a}\right] \int_0^a x^2 \sqrt{a^2-x^2}\, dx + \frac{b^3}{3a^3}\int_0^a (a^2-x^2)^{3/2} dx \,]$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$

When
$x = 0 \rightarrow 0 = a\sin\theta$
$\sin 0 = \sin\theta$
$\theta = 0$

where $x = a \rightarrow a = a\sin\theta$
$1 = \sin\theta$
$\sin\frac{\pi}{2} = \sin\theta$
$\theta = \pi/2$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^2\sin^2\theta\sqrt{a^3 - a^2\sin^2\theta}\cdot a\cos\theta\, d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2}(a^2 - a^2\sin^2\theta)^{3/2} a\cos\theta\, d\theta\right)$$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^3\sin^2\theta\cos\theta\sqrt{a^2(1-\sin^2\theta}\, d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2}(a^2)^{3/2}(1-\sin^2\theta)\, a\cos\theta\, d\theta\right.$$

$$= 4\left[\frac{b}{a}\int_0^{\pi/2} a^3\sin^2\theta\cos\theta\sqrt{(a\cos\theta)^2}\, d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2} a^4(\cos^2\theta)^{3/2}\cos\theta\, d\theta\right.$$

$$= 4\left[\frac{b\,a^4}{a}\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{b^3}{3a^3}\int_0^{\pi/2}\cos^4\theta\, d\theta\right.$$

$$= 4\left\{ba^3\left[\frac{1}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right] + \frac{b^3}{3a^3}\left[\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right]\right\}$$

$$= 4\left[\frac{\pi b a^3}{16} + \frac{\pi b^3 a}{16}\right)$$

$$= \frac{\cancel{4}\,\pi ab(a^2+b^2)}{\cancel{16}\,4}$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

* Double integral in polar co-ordinate

(1) $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

Sol: $I = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\,dr\right]d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$

$= \frac{1}{2}\int_0^{\pi}\left[(a\sin\theta)^2 - 0^2\right]d\theta$

$= \frac{1}{2}\int_0^{\pi} a^2\sin^2 d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$

$= \frac{a^2}{2}\int_0^{\pi}\frac{(1-\cos 2\theta)}{2}d\theta$

$= \frac{a^2}{4}\int_0^{\pi}(1-\cos 2\theta)\,d\theta$

$= \frac{a^2}{4}\left[\int_0^{\pi} 1\,d\theta - \int_0^{\pi}\cos 2\theta\,d\theta\right]$

$= \frac{a^2}{4}\left[[\theta]_0^{\pi} - \left[\frac{\sin 2\theta}{2}\right]_0^{\pi}\right]$

$\left\{\begin{array}{l}\int\cos ax\,dx = \frac{\sin ax}{+a} + c\\ \int\sin ax\,dx = \frac{-\cos ax}{a} + c\end{array}\right.$

$= \frac{a^2}{4}\left((\pi - 0) - \left[\frac{\sin 2\pi}{2} - \frac{\sin 0}{2}\right]\right)$

$= \frac{a^2}{4}(\pi - (0-0))$

$= \frac{a^2\pi}{4}$

(2) $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$.

Sol $\left[\int\frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c\right]$

$I = -\frac{1}{2}\int_0^{\pi/4}\left[\int_0^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}dr\right]d\theta$

$= -\frac{1}{2}\int_0^{\pi/4}\left[2\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta$.

$= -\int_0^{\pi/4}\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta} d\theta$.

$= -\int_0^{\pi/4}\left(\sqrt{a^2-a^2\sin^2\theta} - \sqrt{a^2-0}\right)d\theta$

$= -\int_0^{\pi/4}\left(\sqrt{a^2(1-\sin^2\theta)} - \sqrt{a^2}\right)d\theta$

$= -\int_0^{\pi/4}\left(\sqrt{a}\cos\theta - a\right]d\theta$

$= -a\int_0^{\pi/4}(\cos\theta - 1)\,d\theta$

$= -a\left[\int_0^{\pi/4}\cos\theta\,d\theta - \int_0^{\pi/4}1.d\theta\right]$

$= -a\left[[\sin\theta]_0^{\pi/4} - [\theta]_0^{\pi/4}\right]$

$= -a\left((\sin\frac{\pi}{4} - \sin 0) - (\frac{\pi}{4} - 0)\right)$

$= -a\left[\frac{1}{\sqrt{2}} - 0 - \frac{\pi}{4}\right]$

$= a\left[\pi/4 - \frac{1}{\sqrt{2}}\right]$

③ Evaluate $\int_0^{\pi/2}\int_0^{\infty} \frac{r\,dr\,d\theta}{(r^2+a^2)^2}$

$= \frac{1}{2}\int_0^{\pi/2}\left[\int_0^{\infty} \frac{2r}{(r^2+a^2)^2}\,dr\right]d\theta$

$\left\{\int \frac{f'(x)}{f^n(x)}\,dx = \frac{f^{-n+1}}{-n+1} + c\right\}$

$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{(r^2+a^2)^{-2+1}}{-2+1}\right]_0^{\infty} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{(r^2+a^2)^{-1}}{-1}\right]_0^{\infty} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[\frac{-1}{r^2+a^2}\right]_0^{\infty} d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \left[\frac{1}{\infty} - \frac{1}{0^2+a^2}\right] d\theta$

$= -\frac{1}{2}\int_0^{\pi/2} \left(-\frac{1}{a^2}\right) d\theta$

$= \frac{1}{2a^2}\left[\int_0^{\pi/2} 1\,d\theta\right]$

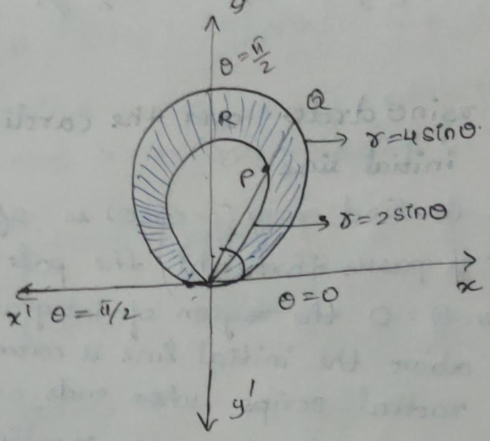
$= \frac{1}{2a^2}\{\theta\}_0^{\pi/2}$

$= \frac{1}{2a^2}\left[\frac{\pi}{2}\right]$

$= \pi/4a^2$ //

*** * mid 2 paka

④ Evaluate $\iint r^3\,dr\,d\theta$ over the area included b/w of the circle $r = 2\sin\theta$ and $r = 4\sin\theta$



The region of integration R is shown shaded. Here r varies from $2\sin\theta$ to $4\sin\theta$ and $\theta = 0$ to π
(P) (Q)

$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{r=4\sin\theta} r^3\,dr\,d\theta$

$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{r=2\sin\theta}^{4\sin\theta} d\theta$

$= \frac{1}{4}\int_0^{\pi} [4^4\sin^4\theta - 2^4\sin^4\theta]\,d\theta$

$= \frac{1}{4}\int_0^{\pi} \sin^4\theta\,[256-16]\,d\theta$

$= \frac{1}{4}\int_0^{\pi} 240\cdot\sin^4\theta\,d\theta$

$= \frac{\cancel{240}^{60}}{\cancel{4}}\int_0^{\pi} \sin^4\theta\,d\theta$

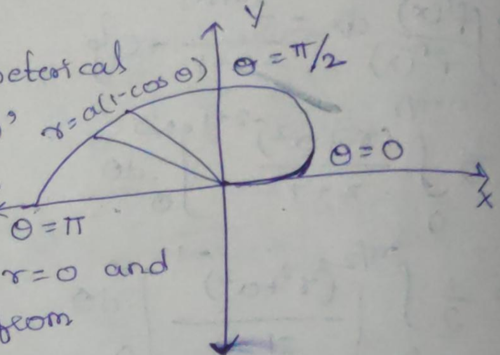
$= 60\int_0^{\pi} \sin^4\theta\,d\theta$ $\left\{\because \int_0^{\pi} f(x)\,dx = 2\int_0^{\pi} f(x)\,dx,\ \text{If } f(2a-x) = f(x)\right\}$

$= 60 \times 2 \int_0^{\pi/2} \sin^4\theta \, d\theta \Rightarrow 120 \int_0^{\pi/2} \sin^4\theta \, d\theta \quad \left\{ \because \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)(n-3)\cdots 1}{(n(1)-2)\cdots 2} \cdot \frac{\pi}{2} \right\}$

$= \cancel{120}^{\cancel{30}\,15} \left[\frac{3}{\cancel{4}_2} \times \frac{1}{\cancel{2}} \times \frac{\pi}{\cancel{2}} \right] = \frac{45\pi}{2}$ //.

① $\iint_R r\sin\theta \, dr\, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line

A:- The cardiod $r = a(1-\cos\theta)$ is symmeterical & it is passes throught the pole 'O' when $\theta = 0$ the region of integration R above the initial line is corered by radial scrips whose ends are $r=0$ and $r = a(1-\cos\theta)$, the scrip starting from $\theta = 0$ end at $\theta = \pi/2$



$$\iint_R r\sin\theta \, dr\, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a(1-\cos\theta)} r\sin\theta \, dr\, d\theta = \int_0^{\pi} \sin\theta \left[\int_0^{a(1-\cos\theta)} r\, dr \right] d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2} \right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a(1-\cos\theta)^2 - 0^2 \right] d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a^2 (1-\cos\theta)^2 \right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \sin\theta \, (1-\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \left[\frac{(1-\cos\theta)^3}{3} \right]_0^{\pi}$$

$$= \frac{a^2}{2\times 3} \left[(1-\cos\pi)^3 - (1-\cos 0)^3 \right]$$

$$= \frac{a^2}{6} \left[(1-(-1))^3 - (1-1)^3 \right].$$

$$= \frac{a^2 \cancel{8}^{4}}{\cancel{6}_{3}}$$

→ change of variable in double integral

Let $x = f(u,v)$, $y = g(u,v)$ be the relation b/w the old variables with the new variable of the new coordinate system then $\iint_R F(x,y)\,dx\,dy = \iint_R F(f,g)\,|J|\,du\,dv \rightarrow (1)$

change of variables from cartesian to polar co-ordinates

In this we have $u = r$, $v = \theta$ and $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

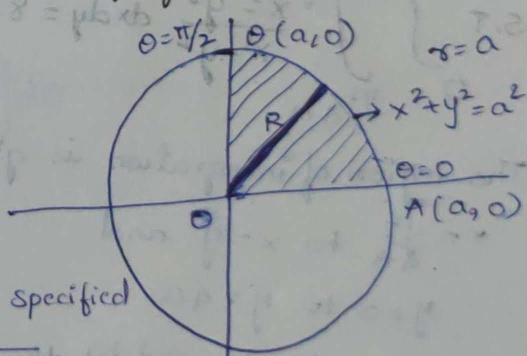
$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta) = r$$

eq (1) becomes

$$\boxed{\iint_R F(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

1. Evaluate $\int_0^a\int_0^{\sqrt{a^2-y^2}} (a^2+y^2)\,dx\,dy$ by changing into polar form on polar cordinate

S:- $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy \rightarrow (1)$



The region R of integration is specified by the inequalities $0 \le x \le \sqrt{a^2-y^2}$ and $0 \le y \le a$

That mean region bounded by the circle $x^2+y^2=a^2$ in the first I^{st} Q.

We reduce eqn 1 into the polar form by putting $x = r\cos\theta$, $y = r\cos\theta$, $dx\,dy = r\,dr\,d\theta$

In the cartesian integral

$$\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} [r^2\cos^2\theta + r^2\sin^2\theta]\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^2[\cos^2\theta + \sin^2\theta]\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^a r^3 (1)\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^a r^3 dr\right] d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4}\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi/2} [1\, d\theta]$$

$$= \frac{a^4}{4}(\theta)_0^{\pi/2}$$

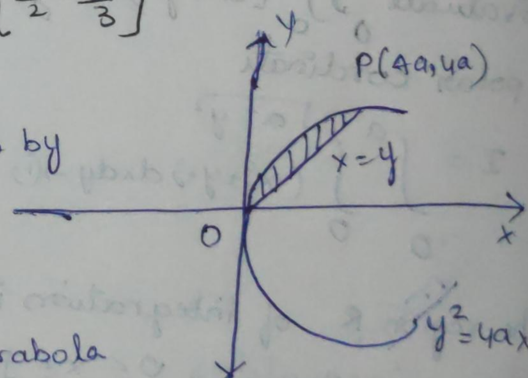
$$= \frac{a^4}{4}\left(\pi/2 - 0\right)$$

$$= \frac{a^4\pi}{8}$$

****

****

② S.T $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy = 8a\left[\frac{\pi}{2} - \frac{5}{3}\right]$



The region of integration is given by

$x = \frac{y^2}{4a}$ to $x = y$ and

$y = 0$ to $y = 4a$

The region is bounded by the parabola $y^2 = 4ax$ & straight line $x = y$.

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Let limits for 'r' are $r = 0$ and 'O' and for P on the parabola. $y^2 = 4ax$

$$r^2\sin^2\theta = 4a\, \cancel{r}\cos\theta$$

$$r = \frac{4a\cos\theta}{\sin^2\theta}$$

for the line $x = y$ for slope $m = 1$

$\tan\theta = 1$

$\theta = \pi/4$

The limits for $\theta = \pi/4 \to \pi/2$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{4} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy \to \int_{\theta=\frac{\pi}{4}}^{\theta=\pi/2}\int^{\frac{4a\sin\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,d\theta\,d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2}\int_0^{\frac{4\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}[\cos^2\theta+\sin^2\theta]}\, r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{(\cos^2\theta-\sin^2\theta)}{1}\, r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}[\cos^2\theta - \sin^2\theta]\left[\int_0^{\frac{4a\cos\theta}{\sin^2\theta}} r\,dr\right]d\theta$$

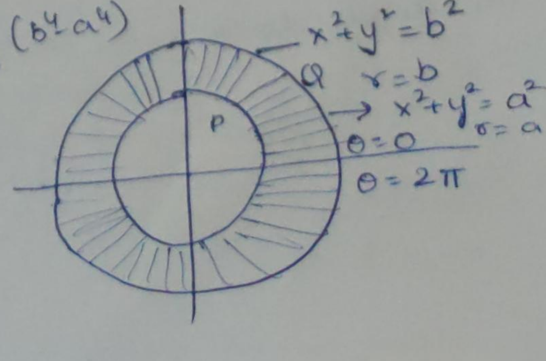
$$= \int_{\pi/4}^{\pi/2}[\cos^2\theta - \sin^2\theta]\left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}} d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/2}\left[\frac{\cos^4\theta}{\sin^4\theta} - \frac{\cos^2\theta\,\cancel{\sin^2\theta}}{\sin^4\theta}\right]d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2}\{\cot^4\theta - \cot^2\theta\}\,d\theta$$

$$= 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$$

③. Evaluate $\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ over the anual region b/w the cle $x^2+y^2=a^2,\ x^2+y^2=b^2\ (b>a) = \frac{\pi}{16}(b^4-a^4)$



Sol:- $x = r\cos\theta,\ y = r\sin\theta$

$dx\,dy = r\,d\theta\,d\theta$

$r=a,\ r=b$

$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy = \int_{\theta=0}^{2\pi}\int_a^b \frac{r^2\cos^2\theta \cdot r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b \frac{r^{\cancel{4}2}(\cos^2\theta\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_a^b \cos^2\theta\sin^2\theta\; r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \cancel{\int_a^b} \cos^2\theta\sin^2\theta \left[\frac{r^4}{4}\right]_a^b d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\sin^2\theta \cdot \frac{1}{4}\left[b^4-a^4\right] d\theta$$

$$= \frac{[b^4-a^4]}{4}\int_0^{2\pi} \cos^2\theta\sin^2\theta\, d\theta$$

$$= \frac{[b^4-a^4)}{4}\int_0^{2\pi} \sin\theta\cos\theta \cdot \sin\theta\cos\theta\, d\theta$$

$$= \frac{b^4-a^4}{4}\int_0^{2\pi} \frac{2\sin\theta\cos\theta}{2} \cdot \frac{2\sin\theta\cos\theta}{2}\, d\theta$$

$$= \frac{b^4-a^4}{16}\int_0^{2\pi} \sin^2 2\theta\, d\theta$$

$$= \frac{b^4-a^4}{16}$$

change of order of integration.

① Identify the variables for limits

② Draw a rough diagram of the given region of integration

③ If we are evaluating the integral with w.r.to to y

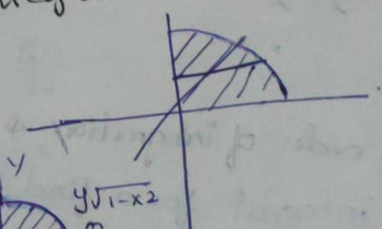
④ Now rotate the scrip by an angle 90° then the angle 90° anti clock wise direction and identify the starting and ending point for scrip. which will be the lower & upper limits of that variable.

⑤ Identify the limits other variable for the region of consideration

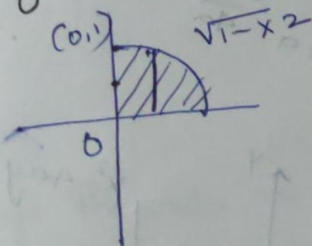
⑥ Evaluate $\iint$ with new order of integration.

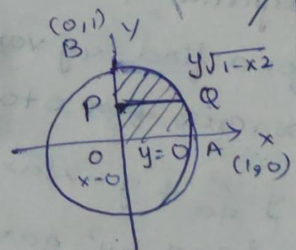
*** 1. By changing the order of integration evaluate.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$$



Sol





The area of integration lies b/w $y=0$ which is the x-axis & which is the circle eq^n.

$y=\sqrt{1-x^2} \rightarrow x^2+y^2=1$

Hence the region of integration is OAB & is divided into vertical strips

For changing the order of integration, we shall divide the region of integration into horizonal strips.

new limits of integration become $x=0$ to $x=\sqrt{1-y^2}$ &

those for y will be $y=0$ to $y=1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx = \int_{y=0}^{1} y^2 dy \int_{x=0}^{\sqrt{1-y^2}} dx$$

$$= \int_0^1 y^2 [x]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2 (\sqrt{1-y^2})dy$$

To change into polar coordinates, put $y = \sin\theta$

Then $dy = \cos\theta\, d\theta$

Also $y=0 \Rightarrow \theta = 0$ and $y=1 \Rightarrow \theta = \pi/2$

$$I = \int_{\theta=0}^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta}\,.\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

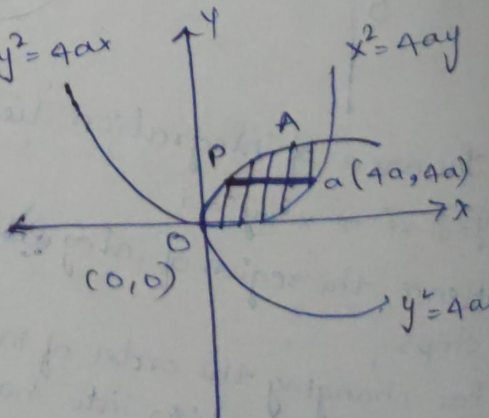
$$I = \frac{\pi}{16}$$

(2) change the order of integration & evaluate $\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$

S:- In the given integral for a find x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ & then x varies from 0 to $4a$

Let us draw the curves $y = \frac{x^2}{4a}$ & $y = 2\sqrt{ax}$

(ie) $x^2 = 4ay$ & $y^2 = 4ax$



The region of integration is the shaded region in figure

The given integral is

$$\int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dy\,dx$$

changing the order of integration we must ~~f(x, y)~~ fix y first for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$

and then y varies from 0 to 4a. Hence the integral is equal to

$$\int_{y=0}^{4a}\int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}}\, dy$$

$$= \int_0^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right]dy$$

$$= 2\sqrt{a}\left[\frac{y^{3/2}}{3/2}\right]_0^{4a} - \left[\frac{y^3}{4a\cdot 3}\right]_0^{4a}$$

$$= 2\sqrt{a}\,\frac{2}{3}\left[(4a)^{3/2}\right) - \frac{1}{12a}\left[(4a)^3 - 0^3\right]$$

$$= \frac{4}{3}a^{1/2}(4)^{3/2}\cdot a^{3/2} - \frac{1}{12a}\left[64a^{\cancel{3}\,2}\right]$$

$$= \frac{4}{3}a^2 2^3 - \frac{64}{12}a^2$$

$$= \frac{32a^2}{3} - \frac{16}{3}a^2$$

$$= \frac{16a^2}{3}$$

③ Change the order the integration in $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the double integral.

S:- This integral is to be written correctly in the form

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$$

The region of integration is given from $y=x^2$ to $y=2-x$ and $x=0$ to $x=1$

Hence we shall draw curve $y=x^2$ on line $y=2-x$

This line $y=2-x$ passes through $(2,0)$ and $(0,2)$ the pt intersection of curve $y=x^2$ and $y=2-x$ are obtain by solving these 2 equations.

$y = 2 - x$

$x^2 = 2 - x$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0$

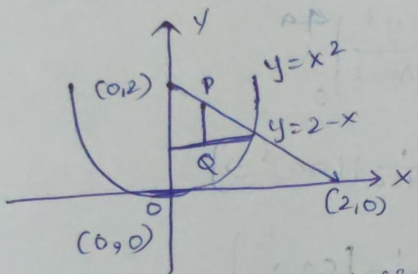
$x = 1, x = -2$

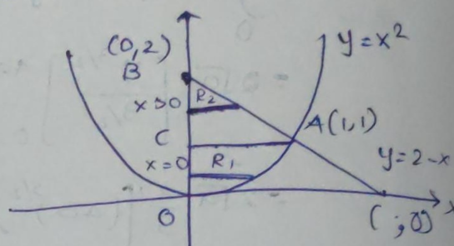
When $x = 2$, $y = (-2)^2 = 4$

when $x = 1$, $y = 1^2 = 1$

Here point of intersection of curve are $(-2, 4)(1, 1)$

Suppose we change of order of integration in the changed order we have to take two horizontal strip





we take region as follows

Area OAB = Area OAC + Area CAB

the 1st region of integration is given limit varies from

by $y = 0$ to $y = 1$

$x = 0$ to $x = \sqrt{y}$

and 2nd region of integration is given by

$y = 1$ to $y = 2$, $x = 0$ to $x = 2 - y$

$$\int_0^1 \int_{x^2}^{y} xy\,dy\,dx = \iint_{R_1} xy\,dx\,dy + \iint_{R} xy\,dx\,dy$$

$$\int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy\,dy\,dx + \int_{y=1}^{2} \int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y \int_{x=0}^{\sqrt{y}} x\,dx\,dy + \int_1^2 y \int_0^{2-y} x\,dx\,dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y \left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2} \int_0^1 y \left[(\sqrt{y})^2 - 0^2\right] dy + \frac{1}{2} \int_1^2 y \left[(2-y)^2 - 0^2\right] dy$$

~~change of the order of integration and solve~~

$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(4+y^2-4y)dy$

$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\int_1^2 [4y+y^3-4y^2]dy$

$= \frac{1}{6}[1-0] + \frac{1}{2}\left[\frac{\cancel{4}^{2}y^2}{\cancel{2}} + \frac{y^4}{4} - 4\frac{y^3}{3}\right]_1^2$

$= \frac{1}{6} + \frac{1}{2}\left[2y^2 + \frac{y^4}{4} - \frac{4}{3}y^3\right]_1^2$

~~$= \frac{1}{6} + \frac{1}{2}\{2(2^2+2)+1$~~

$= \frac{1}{6} + \frac{1}{2}\left[2(2^2) + \frac{2^4}{4} - \frac{4}{3}(2)^3\right] - \left[2 + \frac{1}{4} - \frac{4}{3}\right]$

$= \frac{1}{6} + \frac{1}{2}\left[8 + 4 - \frac{32}{3}\right] - \left[2 - \frac{1}{4} - \frac{4}{3}\right]$

$= \frac{1}{6} + \frac{1}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right]$

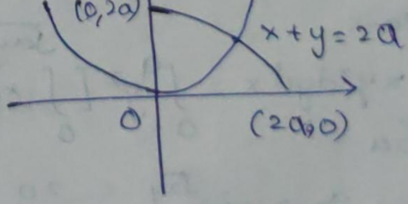
$= \frac{1}{6} + \frac{1}{24}(117-112)$

$= \frac{1}{6} + 5/24$

$= 3/8$

(7) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2 dy dx$ evaluate double integration

S: $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy^2 dy dx$



$y = \frac{x^2}{a}$ on the line $y = 2a-x$

$x=0$ to $x=0$

$y = \frac{x^2}{a}$ to $y = 2a-x$ $\quad x+y=2a$

$R(2a,0)$ $\quad 3(0,2a)$ $\quad y = 2a-x$ $\quad y = \frac{x^2}{a}$

$y = \frac{x^2}{a}$ and $y = 2a-x$ are obtained by

$x^2 = a(2a-x)$

$x^2 = 2a^2 - ax$

$x^2 + ax - 2a^2 = 0$

$x^2 + 2ax - ax - 2a^2 = 0$

$x(x+2a) - a(x+2a) = 0$

$(x-a)(x+2a) = 0$

$x = a, x = -2a,$

when $x = a$, $y = \frac{a^2}{a}$ $y = a$

when $x = -2a$ $y = \frac{4a^2}{a} = 4a$

Here point of intersection of curve are $(-2a, 4a)$ (a, a)

Suppose we change of order of integration: in the changed we have to take two horizontal strip

Area OAB = Area OAC + Area CAB

limit varies from the 1st region of integration is given by

$y = 0$ to $y = a$

$x = 0$ to $x = \sqrt{\pm a}$

and 2nd region of integration is given by $y = a$ to $y = 2a$, $x = 0$ to $x = 2a - y$

$$\int_0^a \int_0^{\sqrt{ay}} xy^2\,dx\,dy + \int_a^{2a} \int_0^{2a-y} xy^2\,dx\,dy$$

$$\int_0^a \int_{x^2/2a}^{2a-x} xy^2\,dy\,dx = \int_0^a y^2 \left[\int_0^{\sqrt{ay}} x\,dx\right] dy + \int_a^{2a} y^2 \left[\int_0^{2a-y} x\,dx\right] dy$$

$$= \int_0^a y^2 \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y^2 \left[\frac{x^2}{2}\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_0^a y^2\left((\sqrt{ay})^2 - a^2\right) dy + \frac{1}{2}\int_a^{2a} y^2\left((2a-y)^2 - a^2\right) dy$$

$$= \frac{1}{2}\int_0^a y^2 ay\,dy + \frac{1}{2}\int_a^{2a} y^2(4a^2 + y^2 - 4ay)\,dx$$

$$= \frac{a}{2}\int_0^a y^3\,dy + \frac{1}{2}\int_0^{2a}\{4a^2y^2 + y^4 - 4ay^3\}\,dy$$

$$\int^{2a}\left(\;\right) \quad \int^{2a} y^4 dy - 4a\int_a^{2a} y^3 dy \;]_0$$

$$= \frac{a}{2}\left[\frac{y^4}{4}\right]_0^a + \frac{1}{2}\left[4a^2\left(\int_a^{2a} y^2 dy\right) + \int_a^{2a} y^4 dy - 4a\int_a^{2a} y^3 dy\right]$$

$$= \frac{a}{2}\left[\frac{a^4}{4}\right] + \frac{1}{2}\left[4a^2\left[\frac{y^3}{3}\right]_a^{2a} + \left[\frac{y^5}{5}\right]_a^{2a} - 4a\left[\frac{y^4}{4}\right]_a^{2a}\right]$$

$$= \frac{a^5}{8} + \frac{\cancel{4}^{2} a^2}{\cancel{2}\times 3}\left[(2a)^3 - a^3\right] + \frac{1}{5}\left[(2a)^5 - a^5\right] - a\left[(2a)^4 - a^4\right]$$

$$= \frac{a^5}{8} + \frac{2a^2}{3}\left(8a^3 - a^3\right) + \frac{1}{5}\left[32a^5 - a^5\right] - a\left[16a^4 - a^4\right]$$

$$= \frac{a^5}{8} + \frac{14a^5}{3} + \frac{31a^5}{5} - \frac{15a^5}{1}$$

$$= \frac{47a^5}{120}$$

Triple Integrals: is evaluated $I = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz$
where the limits of z are z_1, z_2 which are either constants (or) function of x and y
The limits of y are $y_1\; y_2$ which are either constants (or) function of x
The limits of x are x_1, x_2 are constants

$$\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\right]dy\right]dx.$$

① Evaluate $\int_0^1\int_1^2\int_2^3 xyz\;dx\,dy\,dz$

Sol
$$= \int_0^1\int_1^2 yz\left[\int_2^3 x\,dx\right]dy\,dz$$

$$= \int_0^1\int_1^2 yz\left[\frac{x^2}{2}\right]_2^3 dy\,dz$$

$$= \frac{1}{2}\int_0^1\int_1^2 yz\,[9-4]\,dy\,dz$$

$$= \frac{5}{2}\int_0^1\int_1^2 yz\,dy\,dz$$

$$= 5/2\int_0^1 z\left[\int_1^2 y\,dy\right]dz$$

$$= \frac{5}{2}\int_0^1 z\left[\frac{y^2}{2}\right]_1^2$$

$$= \frac{5}{4}\int_0^1 z\,[4-1]\,dy\,dz$$

$$= \frac{15}{4}\int_0^1 z\,dz$$

$$= \frac{15}{4}\left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{15}{8}(1-0) = 15/8$$

2. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dy \, dx$

S:- $I = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\int_0^{\sqrt{1-x^2-y^2}} z \, dz \right] dy \, dx$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{1-x^2-y^2}} dy \, dx$

$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} xy \left[\left(\sqrt{1-x^2-y^2}\right)^2 - 0^2 \right]$

$= \frac{1}{2} \int_0^1 x \int_0^{\sqrt{1-x^2}} \cancel{x} y \, ((1-x^2) - y^2] dy) dx$

$= \frac{1}{2} \int_0^1 x \int_0^{\sqrt{1-x^2}} (1-x^2)y - y^3) \, dy \, dx$

$= \frac{1}{2} \int_0^1 x \left[(1-x^2) \int_0^{\sqrt{1-x^2}} y - y^3 \, dy \, dx \right.$

$= \frac{1}{2} \int_0^1 x (1-x^2) \left[\frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} - \left[\frac{y^4}{4} \right]_0^{\sqrt{1-x^2}} dx$

$= \frac{1}{2} \int_0^1 x \left[\frac{(1-x^2)}{2} \left(\left(\sqrt{1-x^2}\right)^2 - 0^2 \right) - \frac{1}{4} \left[\left((1-x^2)^{1/2} \right)^{\cancel{4}\,2} - 0^4 \right] dx \right.$

$= \frac{1}{2} \int_0^1 x \, (1-x^2)^2 - \left[\frac{1}{2} - \frac{1}{4} \right] dx$

$= \frac{1}{2} \int_0^1 x \, (1-x^2)^2 \left(\frac{1}{4} \right) dx$

$= \frac{1}{8} \int_0^1 x \, (1-x^2)^2 dx$

$= \frac{1}{8} \int_0^1 x \, (1+x^4-2x^2) \, dx$

$= \frac{1}{8} \int_0^1 (x+x^5-2x^3) \, dx$

$= \frac{1}{8} \left[\frac{x^2}{2} + \frac{x^6}{6} - 2\frac{x^4}{4} \right]_0^1 = \frac{1}{8} \left[\cancel{\frac{1}{2}} + \frac{1}{6} - \cancel{\frac{2}{4}} \right]$

$= \frac{1}{48}$ //

③ $I = \int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dz$

Sol $= \int_0^{\log 2} [e^x \int_0^{x} [e^y \int_0^{x+\log y} e^z dz] dy] dx$

$= \int_0^{\log 2} e^x \int_0^{x} e^y \left[e^z\right]_0^{x+\log y} dy\, dz$

$= \int_0^{\log 2} e^x \int_0^{x} e^y [e^{x+\log y} - e^0] dy] dx$ $\{e^{\log_e a} = a\}$

$= \int_0^{\log 2} e^x \int_0^{x} [e^y [e^x . e^{\log y} - 1] dy] dx$

$= \int_0^{\log 2} e^x [\int_0^{x} e^y [e^x y - 1] dy\, dx$

$= \int_0^{\log 2} e^x \int_0^{x} [e^x e^y y - e^y] dy] dx$ $\{\int x e^x dx = xe^x - e^x + c\}$
$\int u dv = uv - u'v + u''v$

$= \int_0^{\log 2} e^x [e^x \int_0^{x} y e^y dy - \int_0^{x} e^y dy) dx$

$= \int_0^{\log 2} e^x [e^x [y e^y - e^y]_0^x - [e^y]_0^x) dx$

$= \int_0^{\log 2} e^x [e^x [x e^x - e^x] - (0 e^0 - e^0) - [e^x - e^0]\, dx$

$= \int_0^{\log 2} e^x [e^x ((x e^x - e^x) + 1) - (e^x - 1) dx$

$= \int_0^{\log 2} e^x [e^x [x e^x - e^x + 1) - (e^x - 1) dx = \int_0^{\log 2} e^x [x e^{2x} - e^{2x} + e^x - \cancel{e^x} + 1] dx$

$= \int_0^{\log 2} [x e^{3x} - e^{3x} + e^x] dx$ $\{\int e^{ax} dx = \frac{e^{ax}}{a} + c\}$

$= \int_0^{\log 2} x e^{3x} dx - \int_0^{\log 2} e^{3x} dx + \int_0^{\log 2} e^x dx$

$= \left[x \left[\frac{e^{3x}}{3}\right] - 1 \cdot \frac{e^{3x}}{3 \times 3}\right]_0^{\log 2} - \left[\frac{e^{3x}}{3}\right]_0^{\log 2} + \left[e^x\right]_0^{\log 2}$

$$= \left[\frac{xe^{3x}}{3} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x}\right]_0^{\log 2}$$

$$= \left\{\log 2 \frac{e^{3\log 2}}{3} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2}\right\} - \left[\frac{0e^{30}}{3} - \frac{e^0}{9} - \frac{e^0}{3} + e^0\right]$$

$$= \log 2\,\frac{8}{3} - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 \quad \left\{ e^{3\log 2} = e^{\log 2^3} = 8 \right\}$$

$$= \log 2\,\frac{8}{3} - \frac{19}{9}$$

① $\int_0^1 \int_y^1 \int_0^{1-x} x\, dz\, dx\, dy = \frac{1}{12}$ ② $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz = \frac{1}{6}$

③ $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x dx\, dy\, dz = \frac{1}{2}(e-3)$ ④ $\int_0^{\log 2} \int_0^{x} \int_0^{x+y} e^{x+y+z} dz\,dy\,dx = \frac{5}{8}$

⑤ $\iiint_V (xy+yz^2+zx)\,dx\,dy\,dz$ where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$ $\quad \frac{33}{2}$

⑥ $\int_{z=-1}^{1} \int_{x=0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$

Sol ①
$$\int_0^1 \int_y^1 \int_0^{1-x} x\, dz\, dx\, dy$$

$$\int_0^1 \int_y^1 x \int_x^{1-x} [1 dz]\, dx\, dy$$

$$= \int_0^1 \int_y^1 x \left[\frac{z^2}{2}\right]_x^{1-x} dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_y^1 x\left[(1-x)^2 - x^2\right] dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_y^1 x\left[(1+x^2-2x) - x^2\right] dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_y^1 x + \cancel{x^3} - 2x^2 - \cancel{x^3}\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x^2}{2} - \frac{2x^3}{3}\right]_y^1 dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{1}{2} - \frac{2}{3}\right] - \left[\frac{y^2}{2} - \frac{2y^3}{3}\right] dy$$

$$= \frac{1}{2}\left[-\frac{1}{6}\right] \int_0^1 - \left[\frac{y^2}{2} - \frac{2y^3}{3}\right] dy$$

$$= -\frac{1}{12} \int_0^1 - y^2\left[\frac{1}{2} - \frac{2y}{3}\right] dy$$

$$= \frac{1}{12}\left[\frac{y^3}{6} - \frac{2y^4}{12}\right]_0^1$$

$$= \frac{1}{12}\left[\frac{1}{6} - \frac{2}{12}\right] - (0-0)$$

$$= \frac{1}{12}$$

(2) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx$

$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1 dz)\,dy\,dx$

$= \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx$

$= \int_0^1 \left(y - xy - \frac{y^2}{2}\right)_0^{1-x} dx$

$= \int_0^1 \left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right) dx$

$= \int_0^1 \left(1 - x - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + \frac{2x}{2}\right) dx$

$= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx$

$= \left(\frac{1}{2}x - x^2 + \frac{1}{6}x^3\right)_0^1$

$= \frac{1}{2} - \frac{1}{2} + \frac{1}{6}$

$= \frac{1}{6}$

(3) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^x\,dx\,dy\,dz$

S:- $\int_0^1 e^x \int_0^{1-x} \int_0^{1-x-y} (1 dz)\,dy\,dx$

$\int_0^1 e^x \int_0^{1-x} [z]_0^{1-x-y}\,dy\,dx$

$\int_0^1 e^x \int_0^{1-x} (1-x-y)\,dy\,dx$

$\int_0^1 e^x \left(y - xy - y^2\right)_0^{1-x} dx$

$= \int_0^1 e^x \left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right] dx$

$= \int_0^1 e^x \left[1 - \cancel{x} - x + x^2 - \frac{1}{2} - \frac{x^2}{2} + \frac{\cancel{2x}}{2}\right] dx$

$= \int_0^1 e^x \left[1 - x + x^2 - \frac{1}{2} - \frac{x^2}{2}\right] dx$

$= \int_0^1 e^x \left[\frac{1}{2} + \frac{1}{2}x^2 - x\right] dx$

$= \left[\frac{e^x}{2} + \frac{e^x}{2}x^2 - xe^x\right]_0^1$

$= \frac{e^x}{2} + x^2e^x - 2xe^x + 2e^x - e^x - xe^x$

④ $$\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$$

S:- $$\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x}e^{y}e^{z}\,dz\,dy\,dx$$

$$\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x}e^{y}\left[e^{z}dz\right]dy\,dx$$

$$\int_0^{\log 2}\int_0^{x} e^{x}e^{y}\left[e^{x+y}-e^{0}\right]dy\,dx$$

$$\int_0^{\log 2}\int_0^{x}\left(e^{x}e^{y}\left(e^{x}e^{y}-1\right)\right)dy\,dx$$

$$\int_0^{\log 2}\int_0^{x} e^{2x}e^{2y}-e^{x}e^{y}\,dy\,dx \Longrightarrow$$

$$\int_0^{\log 2}\int_{\cancel{0}}^{x} e^{2x}\left[\frac{e^{2y}}{2}\right]_0^{x}-e^{x}\left[e^{y}\right]_0^{x}dx$$

$$\int_0^{\log 2} e^{2x}\left[\frac{e^{2x}}{2}-\frac{e^{0}}{2}\right]-e^{x}\left[e^{x}-e^{0}\right]dx \Rightarrow$$

$$\int_0^{\log 2}\left(\frac{e^{4x}}{8}-\frac{e^{2x}}{2}\right)-\left[e^{2x}-e^{x}\right]dx$$

Sol
$$\left[\frac{e^{4x}}{8}-\frac{e^{2x}}{2}\right]_0^{\log 2}-\left[e^{2x}-e^{x}\right]_0^{\log 2}dx$$

$$\left[\frac{e^{4\log 2}}{8}-\frac{e^{2\log 2}}{2}-\frac{e^{0}}{8}+\frac{e^{0}}{2}\right]-\left[e^{2\log 2}-e^{\log 2}-e^{0}+e^{0}\right]$$

$$\left[\frac{\cancel{16}^{\,2}}{\cancel{8}}-\frac{\cancel{4}^{\,2}}{\cancel{2}}-\frac{1}{8}+\frac{1}{2}\right)-\left[4-2-1+1\right]$$

$$\int_0^{\log 2}\left(\frac{e^{2x+2y}}{2}-e^{x+y}\right)_0^{x}dx$$

$$=\int_0^{\log 2}\left(\frac{e^{4x}}{2}-e^{2x}\right)-\left(\frac{e^{2x}}{2}-e^{x}\right)dx$$

$$\cancel{\int_0^{\log 2} e^{2x}\left[\frac{e^{2x}}{2}-\frac{e^{0}}{2}\right]-\left[e^{2x}\right.}$$

$$=\int_0^{\log 2}\left(\frac{e^{4x}}{8}-e^{2x}-\frac{e^{2x}}{2}+e^{x}\right)dx$$

$$=\int_0^{\log 2}\left(\frac{e^{4x}}{8}-\frac{3}{4}e^{2x}+e^{x}\right)dx$$

$$=\left(\frac{e^{4x}}{8}-\frac{3}{4}e^{2x}+e^{x}\right)_0^{\log 2}$$

$$=\left[\frac{e^{4\log 2}}{8}-\frac{3}{4}e^{2\log 2}+e^{\log 2}\right]-\left[\frac{1}{8}-\frac{3}{4}\right.$$

$$=\left(\frac{16}{8}-\frac{3}{4}\times 4+2\right]-\left[\frac{1}{8}-\frac{3}{4}+1\right.$$

$$=1-\frac{3}{8}$$

$$=5/8$$

⑤ $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$ where V is the region of space bounded

$x=0, x=1, y=0, y=2, z=0, z=3$

S:- $= \int_0^3\int_0^2\int_0^1 (xy+yz+zx)\,dx\,dy\,dz$

$= \int_0^3\int_0^2 \left[\frac{x^2y}{2} + xyz + \frac{2x^2}{2}\right]_0^1 dy\,dz = \int_0^3\int_0^2 \left[\frac{y}{2} + yz + \frac{yz}{2}\right] dy\,dz$

$= \int_0^3 \left[\frac{y^2}{4} + \frac{y^2}{2}z + \frac{yz}{2}\right]_0^2 dz = \int_0^3 \left[\frac{4}{4} + \frac{4}{2}z + \frac{2z}{2}\right] dz$

$= \int_0^3 (1+2z+z)\,dz = (1+3z)_0^3\,dz = \left(z + \frac{3}{2}z^2\right)_0^3$

$= 3 + \frac{\cancel{27}}{2}$

$= \frac{6+27}{2} = 33/2$

⑥ $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z} (x+y+z)\,dx\,dy\,dz$.

$z=$ $x=$

$= \int_{-1}^{1}\int_0^z \int_{x-z}^{x+z} \left[(x+y+z)\,dy\right] dx\,dz$

$= \int_{-1}^{1}\int_0^z \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$

$= \int_{-1}^{1}\int_0^z \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z)\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right] dx\,dz$

$= \int_{-1}^{1}\int_0^z \left(x^2+xz + \frac{x^2}{2} + \frac{z^2}{2} + \frac{2xz}{2} + xz + z^2\right] - \left[x^2 - zx + \frac{x^2}{2} + \frac{z^2}{2} - \frac{2xz}{2} + zx - z^2\right] dx\,dz$

$= \int_{-1}^{1}\int_0^z \left\{\cancel{x^2} + xz + \cancel{\frac{x^2}{2}} + \cancel{\frac{z^2}{2}} + \cancel{2}xz + z^2 - \cancel{x^2} + \cancel{zx} - \cancel{\frac{x^2}{2}} - \cancel{\frac{z^2}{2}} + \frac{2xz}{2} - \cancel{zx} + z^2\right\}$

$= 2\int_{-1}^{1}\int_0^z (xz+z^2+xz)\,dx\,dz$ $\quad$ ~~$= \int_{-1}^{1}\int_0^z (2xz+z^2)\,dx)\,dz$~~

$= 2\int_{-1}^{1} \left(\frac{x^2}{2}z + xz^2 + \frac{x^2}{2}z\right)_0^z dz$

$= 2\int_{-1}^{1} \left[\frac{z^3}{2} + z^3 + \frac{z^3}{2}\right] dz \quad = 2\int_{-1}^{1} \left[\frac{\cancel{2}z^3}{\cancel{2}} + z^3\right] = 2\int_{-1}^{1} \left[2z^3\right].$

$= 4 \int_{-1}^{1} z^3 dz = 4\left[\frac{z^4}{4}\right]_{-1}^{1} = \frac{4}{4}[1-1]$

$= \cancel{\frac{4}{4}}[1-1] = 1(1-1)$

$= 0.$

1. Find the area of the region bounded by the parabolas

S: 1. $y^2 = 4ax$ and $x^2 = 4ay$

S:- $y^2 = 4ax \longrightarrow (1)$

$x^2 = 4ay$ To find their points of intersection. solve (1) & (2)

eq 2 : $x^2 = 4ay$

Squaring $x^4 = 16a^2 y^2$

$x^4 = 16a^2(4ax)$

$x^4 = 4^3 a^3 x$

$x^4 - 4^3 a^3 x = 0$

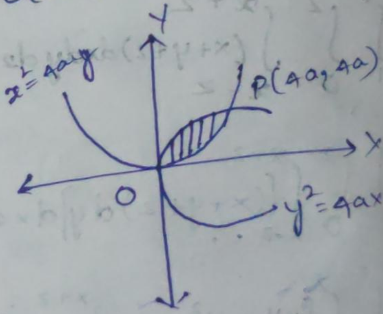
$x = 0 \quad x^3 = (4a)^3$

$x = 4a$

when $x = 0$
$y = 0$

when $x = 4a$
$y = 4a$

Hence the two parabola intersect at $O(0,0)$ & $P(4a, 4a)$



Area $= \iint_R dx\, dy$

The region R can be covered by varying x from the upper curve $x = \frac{y^2}{4a}$ to the lower curve while y varies from 0 to 4a

$y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ (The strip moves from $x=0$ to $x = 4a$)

Thus $A = \int_{y=0}^{4a} \int_{x=\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$

$= \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} = \int_0^{4a} \left[2\sqrt{ay} - \frac{y^2}{4a}\right] dy$

$= 2\sqrt{a}\left(\frac{y^{1/2+1}}{1/2+1}\right)_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$

$= 2\sqrt{a} \cdot \frac{2}{3}\left((4a)^{3/2}\right) - \frac{1}{12a}\left((4a)^3 - 0^3\right)$

$= \frac{4}{3}\sqrt{a}\left[4^3 \cdot a^{3/2}\right] - \frac{1}{12a}(64a^3)$

$= \frac{2\sqrt{6}}{3} a^{2/2} - \frac{64}{12}\frac{a^{32}}{\cancel{a}} = \frac{256}{3}a^2 - \frac{\overset{16}{\cancel{64}}a^2}{\cancel{12}\,3} = \frac{16a^2}{3}$

1. Find the whole area of the lemniscates $r^2 = a^2\cos 2\theta$

S: The curve $r^2 = a^2\cos 2\theta$ is symmetrical about both co-ordinate axes, and it passes through the pole O. It intersects the initial line at $A(a, 0)$ and $B(-a, 0)$

Thus two symmetrical loops are formed by the curve

Also each loop is symmetrical about the intial line

$\therefore$ whole area of the lemniscale = 4 x area enclosed by one of the loops above the intial line

$$A = 4 \times \iint_R r\,dr\,d\theta$$

where R is the region bounded by the lemniscale in the first quadrant

The limits froms $\theta : 0 \to \pi/4$

$r : 0 \to a\sqrt{\cos 2\theta}$

$$A = 4\int_0^{\pi/4}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}}$$

$$= \frac{\cancel{4}^{2} a^2}{\cancel{2}}\int_0^{\pi/2}(\sqrt{\cos 2\theta})^2\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\cos 2\theta\,d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= a^2\left[\sin \cancel{2}\left(\frac{\pi}{\cancel{4}_2}\right) - \sin 2(0)\right]$$

$$= a^2\left[\sin \pi/2 - \sin 0\right]$$

$$= a^2(1-0)$$

$$= a^2$$

Find the area of the loop of the curve $r = a(1+\cos\theta)$

The cardioid $r = a(1+\cos\theta)$ is symmetrical about the initial line and passes through the pole O. It intersects the initial line at the point O and $A(2a, 0)$

As θ varies from 0 to π, r increase from 0 to $2a$

To determine the polar limits of integration for the area A

Hence required area = 2× area of the curve above the initial

θ limits from $\theta = -\pi$ to $\theta = 0$

$r = 0$ to $a(1+\cos\theta)$

$$= 2\iint_A r\,dr\,d\theta$$

$$= 2\int_{-\pi}^{0}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \cancel{2}\int_{-\pi}^{0}\left[\frac{r^2}{\cancel{2}}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2\int_{-\pi}^{0}(1+\cos\theta)^2 d\theta \qquad \left\{\cos^2\theta = \frac{1+\cos 2\theta}{2}\right\}$$

$$= a^2 2^2\int_{-\pi}^{0}\left(\cos^2\frac{\theta}{2}\right)^2 d\theta \qquad \left\{\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right\}$$

$$= 4a^2\int_{-\pi}^{0}\cos^4\frac{\theta}{2}\,d\theta$$

Here $\frac{\theta}{2} = -\phi \rightarrow$ when $\theta = -\pi$

$d\theta = -2d\phi$ $\qquad \frac{-\pi}{2} = -\phi \Rightarrow \phi = \pi/2$

$$= 4a^2\int_{\pi/2}^{0}\cos^4\phi(-2d\phi) \qquad \text{when } \theta = 0,\ \phi = 0$$

$$= 8a^2\int_0^{\pi/2}\cos^4\phi\,d\phi = 8a^2\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2$$

$$= \frac{3a^2\pi}{2}.$$

Evaluate $\iiint xyz\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$

Given sphere is $x^2+y^2+z^2=a^2$

$z = \sqrt{a^2-x^2-y^2}$

The projection of the sphere on the xy-plane is the circle $x^2+y^2=a^2$

So, this circle is covered as y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a

$$\iiint xyz\,dx\,dy\,dz = \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy \int_0^{\sqrt{a^2-x^2-y^2}} z\,dz$$

$$= \int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\,dy \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}}$$

$$= \frac{1}{2}\int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} y\left((\sqrt{a^2-x^2-y^2})^2 - 0^2\right]dy$$

$$= \frac{1}{2}\int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} \left[(a^2-x^2)y - y^3\right]dy$$

$$= \frac{1}{2}\int_0^a x\,dx \left[(a^2-x^2)\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} - \left[\frac{y^4}{4}\right]_0^{\sqrt{a^2-x^2}}\right]$$

$$= \frac{1}{2}\int_0^a x\,dx \left[\frac{1}{2}(a^2-x^2)(a^2-x^2) - \frac{1}{4}\left((\sqrt{a^2-x^2})^4\right)\right]$$

$$= \frac{1}{2}\int_0^a x\,dx\,(a^2-x^2)^2\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{1}{8}\int_0^a x(a^2-x^2)^2\,dx$$

$$= \frac{1}{8}\int_0^a x\left[a^4+x^4-2a^2x^2\right]dx = \frac{1}{8}\int_0^a \left(a^4x+x^5-2a^2x^3\right)dx$$

$$= \frac{1}{8}\left[a^4\frac{x^2}{2}+\frac{x^6}{6}-\cancel{2}a^2\frac{x^4}{\cancel{4}_2}\right]_0^a$$

$$= \frac{1}{8}\left[\frac{a^6}{2}+\frac{a^6}{6}-\frac{a^6}{2}\right]$$

$$= \frac{a^6}{48}$$

Change of variables in a Triple Integral

1. Change of variables from cartesian to spherical polar co-ordinate system

The relations b/w the cartesian co-ordinates x, y, z and spherical polar co-ordinates r, θ, ϕ

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

Change of variables from cartesian to cylindrical co-ordinate system

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\,dr\,d\theta\,dz$$

Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates

S:- changing to spherical polar co-ordinates by putting
$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, we have
$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$
$r^2 = x^2+y^2+z^2$ and

Also the given region of integration is the volume of the sphere $x^2+y^2+z^2=1$ in the positive octant for which r varies from 0 to 1, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \left[\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr \right] \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}}\,dr\right]\sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\int_0^1 \frac{1}{\sqrt{1-r^2}} - \int_0^1 \sqrt{1-r^2}\right]dr\, \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\left(\sin^{-1}(r)\right)_0^1 - \left[\frac{r}{2}\sqrt{1-r^2}+\frac{1}{2}\sin^{-1}r\right]_0^1\right]\sin\theta\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\left[\pi/2 - \frac{1}{2}\frac{\pi}{2}\right]\sin\theta\, d\theta\, d\phi$$

$$= \pi/4 \int_0^{\pi/2}\left[\int_0^{\pi/2}\sin\theta\, d\theta\right]d\phi = \frac{\pi}{4}\int_0^{\pi/2}\left(-\cos\theta\right)_0^{\pi/2} d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} 1\, d\phi$$

$$= \pi^2/8$$

Using cylindrical co-ordinates, find the volume of the sphere $x^2+y^2+z^2=a^2$

S :- Let the eq^n of sphere be $x^2+y^2+z^2=a^2 \longrightarrow (1)$

Now, using cylindrical co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

So $x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

$x^2+y^2+z^2 = a^2 \rightarrow x^2+z^2 = a^2$

limits of z are $-\sqrt{a^2-x^2-y^2}$ to $\sqrt{a^2-x^2-y^2}$ ie $\sqrt{a^2-r^2}$ to $\sqrt{a^2-r^2}$

limits of r are : 0 to a

and limits of θ are : 0 to 2π

Required volume

$= \iiint dx\,dy\,dz = \iiiint (r\,d\theta\,dr)\,dz$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{r=0}^{a} r\,dr \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} dz$$

$$= 2\int_0^{2\pi} d\theta \int_0^a r\,dr\,\left[z\right]_0^{\sqrt{a^2-r^2}}$$

$$= 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2-r^2}\,dr$$

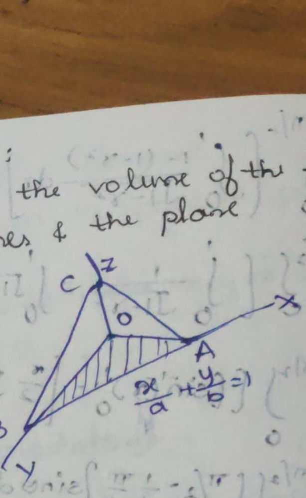
$$= (-1)\int_0^{2\pi} d\theta \int_0^a (-2r)\sqrt{a^2-r^2}\,dr$$

$$= (-1)\int_0^{2\pi} d\theta \left[\frac{(a^2-r^2)^{3/2}}{3/2}\right]_0^a = \frac{-2}{3}\int_0^{2\pi} d\theta\,[0-a^3]$$

$$= \frac{2a^3}{3}\int_0^{2\pi} d\theta = \frac{4\pi a^3}{3}$$

Volume as a double Integral:

1. Using double Integration, find the volume of the tetrahedron bounded by the co-ordinates planes & the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



S:- Given $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \longrightarrow (1)$

$$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$$

$$z=c\left[1-\frac{x}{a}-\frac{y}{b}\right)$$

In the xy-plane $z=0$

Substituting $z=0$ in eq(1), we get $\frac{x}{a}+\frac{y}{b}=1$

$\therefore$ The region R in the xy-plane is a triangle OAB bounded by

$x=0, y=0$ & the $\frac{x}{a}+\frac{y}{b}=1$

In this region x varies from 0 to a, y varies from 0 to $b\left(1-\frac{x}{a}\right)$

Hence the required volume $= \int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)} z\,dy\,dx$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$= c\int_0^a \left(\frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{(-2/b)}\right)_0^{b(1-x/a)} dx$$

$$= -\frac{bc}{2}\int_0^a \left(1-\frac{x}{a}-\frac{b(1-x/a)}{b}\right)^2-\left(1-\frac{x}{a}-0\right)^2 dx$$

$$= -\frac{bc}{2}\int_0^a \left(\left(\left(1-\frac{x}{a}\right)-\left(1-\frac{x}{a}\right)\right)^2-\left(1-\frac{x}{a}\right)^2\right)dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{(1-x/a)^3}{(-3/a)}\right]_0^a$$

$$= -\frac{ab^2c}{6}\left[\left(1-\frac{a}{a}\right)^3-(1-0)^3\right] = -\frac{abc}{6}(0-1)$$

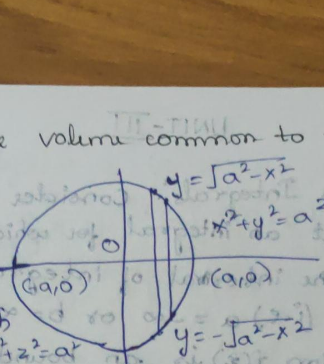
$$= \frac{abc}{6}$$

Find by double integral the volume common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$

Given cylinders are

$x^2+y^2=a^2 \longrightarrow (1)$

$x^2+z^2=a^2 \longrightarrow (2)$



The section of the cylinder with xy plane $z=0$ is the circle $x^2+z^2=a^2$

from eq (2) we have $z=\pm\sqrt{a^2-x^2}$

Each of the surface is symmetrical about the xy-plane

Hence the required volume $2\iint_R z\,dx\,dy$

where $z=\sqrt{a^2-x^2}$ and R is the region bounded by the circle given by (1) on the xy-plane

$-a\le x\le a$ and $-\sqrt{a^2-x^2}\le y\le\sqrt{a^2-x^2}$

$\therefore$ Required volume $= 2\int_{-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z\,dx\,dy$

$$= 2\int_{-a}^{a} dx \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\,\left[y\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}dx$$

$$= 2\int_{-a}^{a}\sqrt{a^2-x^2}\left[\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\right]dx$$

$$= 4\int_{-a}^{a}\left(\sqrt{a^2-x^2}\cdot\sqrt{a^2-x^2}\right)dx$$

$$= 4\int_{-a}^{a}(a^2-x^2)\,dx \longrightarrow 4\cdot 2\int_0^a (a^2-x^2)\,dx$$

$$= 8\left[\int_0^a a^2\,dx-\int_0^a x^2\,dx\right] = 8\left(a^2[x]_0^a-\left[\frac{x^3}{3}\right]_0^a\right)$$

$$= 8\left[a^3-\frac{a^3}{3}\right] = \frac{16a^3}{3}$$