1.a) Given,

no. of die, $n = 2$

The total no. of outcomes for 'n' no. of die is given by 6^n

$$n(S) = 6^2 = 36$$

$\{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$

$(2,1), \ldots$

$(3,1), \ldots$

$(4,1), \ldots \ldots (4,6)$

$\ldots \ldots (5,5), \ldots$

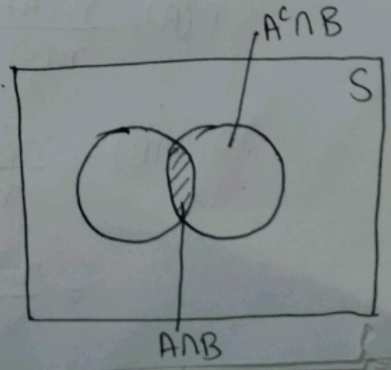
$\ldots \ldots (6,4), \ldots \}$

Required no. of possibilities of getting sum of the numbers on the die is 10, $n(E) = 3$

Required probability, $P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$

1.b) Addition theorem :- For any two events A and B,

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Proof :-

From diagram :

A and $A^c \cap B$ are disjoint events

From axiam 3,

$P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$

$P(A \cup B) = P(A) + P(A^c \cap B)$

$(\because P(A^c \cap B) = P(B) \cancel{+} - P(A \cap B))$ theorem 2

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

For any 3 events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

1.c) Multiplication theorem:-

If A, B are two events in a sample space 'S' such that $p(A) \neq 0$, $p(B) \neq 0$ then

i) $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$

ii) $P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right)$

Proof:-

let $n(A), n(B), n(A \cap B), n(S)$ be the no. of sample point in $A, B, A \cap B, S$ respectively then

$P(A) = \frac{n(A)}{n(S)}$, $P(B) = \frac{n(B)}{n(S)}$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{n(A) \cdot n(A \cap B)}{n(A) \cdot n(S)}$$

$$= \frac{n(A)}{n(S)} \times \frac{n(A \cap B)}{n(A)} = P(A) \cdot P\left(\frac{B}{A}\right)$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{n(B)}{n(S)} \times \frac{n(A \cap B)}{n(B)}$$

$$= P(B)\, P\left(\frac{A}{B}\right)$$

~~∴ P(A∪~~

$\therefore P(A \cap B) = P(A)\, P\left(\frac{B}{A}\right)$

$P(A \cap B) = P(B)\, P\left(\frac{A}{B}\right)$

1.d) Given,

a bag contains 4 white balls, $n(w) = 4$

" " " 6 black " , $n(b) = 6$

Then the sample space , $n(S) = 10$

The probability of getting white ball is,

$\cancel{P(b) = \frac{n(b)}{n(S)}} =$ $P(w) = \frac{n(w)}{n(S)} = \frac{4}{10}$

$P(w) = \frac{2}{5}$ (or) 40%

2.a) i) K

wkt,

$$\sum_{i=1}^{6} P(x_i) = 1$$

$K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$$\boxed{K = \frac{1}{49}}$$

ii) $P(0 < X < 5)$

$P(0 < X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 3K + 5K + 7K + 9K$

$= 24K = \frac{24}{49}$

iii) distribution funcⁿ

$F(x) = P(X \leq x)$

$F_x(0) = \frac{1}{49} \quad (0 \leq x < 1)$

$F_x(1) = \frac{4}{49} \quad (1 \leq x < 2)$

$F_x(2) = \frac{9}{49} \quad (2 \leq x < 3)$

$F_x(3) = \frac{16}{49} \quad (3 \leq x < 4)$

$F_x(4) = \frac{25}{49} \quad (4 \leq x < 5)$

$F_x(5) = \frac{36}{49} \quad (5 \leq x < 6)$

$F_x(6) = 1 \quad (x \geq 6)$

iv) mean

$E(x) = \sum_{x=0}^{6} x\, p(X=x)$

$= K(3+10+21+36+55+78)$

$= K(203)$

$= \frac{203}{49} = 4.14$

Varience

$\sigma^2 = E(X^2) - (E(X))^2$

$= \sum_{x=0}^{6} x^2 p(X=x) - \left(\sum_{x=0}^{6} x\, p(X=x)\right)^2$

$= \frac{973}{49} - \left(\frac{203}{49}\right)^2$

$\sigma^2 = 2.6938$

2b) i) K

$\sum_{x=-2}^{3} p(X=x) = 1$

$0.1 + K + 0.2 + 2K + 0.3 + K = 1$

$4K = 0.4$

$\boxed{K = \frac{1}{10}}$

ii) mean

$\mu = \sum_{x=-2}^{3} x\, p(X=x)$

$= (-2)(0.1) + (-1)(K) + (0)(0.2) + (1)(2K) + (2)(0.3) + (3)(K)$

$= -0.2 - K + 2K + 0.6 + 3K$

$= 4K + 0.4$

$= \frac{4}{10} + 0.4 \quad \boxed{\mu = 0.8}$

iii) varience

$$\sigma^2 = E(x^2) - (E(x))^2$$
$$= 2.8 - (0.8)^2$$
$$= 2.8 - 0.64$$
$$\boxed{\sigma^2 = 2.16}$$

iv) $P(X>1)$

$$P(X>1) = P(X=2) + P(X=3)$$
$$= 0.3 + 0.1$$
$$P(X>1) = 0.4$$

3a) Poisson distribution func^n

$$\boxed{p(x) = \frac{m^x e^{-m}}{x!}}$$

mean,

$$\mu = \sum_{x=0}^{\infty} x \, p(x)$$
$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-m} m^x}{x!}$$
$$= e^{-m} \sum_{x=0}^{\infty} \frac{x \cdot m^x}{x(x-1)!}$$
$$= e^{-m} \sum_{x=0}^{\infty} \frac{m^{x-1} \cdot m}{(x-1)!}$$

$$= m e^{-m} \sum_{x=0}^{\infty} \frac{m^{x-1}}{(x-1)!} = m e^{-m} \left[\frac{m^{-1}}{(-1)!} + \frac{m^0}{0!} + \frac{m^1}{1} + \cdots \right]$$
$$= m e^{-m} (e^m)$$
$$= m$$

Varience,

$$\sigma^2 = \sum_{x=0}^{\infty} x^2 p(x) - \left(\sum_{x=0}^{\infty} x \, p(x) \right)^2$$
$$= \sum_{x=0}^{\infty} x^2 \frac{e^{-m} m^x}{x!} - \mu^2$$

$(\because x^2 = x(x-1) + x)$

$$= \sum_{x=0}^{\infty} (x(x-1) + x) \frac{e^{-m} m^x}{x!} - \mu^2$$
$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-m} m^x}{x!} + \mu - \mu^2$$
$$= e^{-m} \sum_{x=0}^{\infty} \frac{m^{x-2} \cdot m^2}{(x-2)!} + m - m^2$$
$$= e^{-m} m^2 \sum_{x=0}^{\infty} \frac{m^{x-2}}{(x-2)!} + m - m^2$$
$$= e^{-m} m^2 e^m + m - m^2$$
$$= m$$
$$\sigma^2 = m$$

3b) $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Given, $\frac{3}{2}P(X=1) = P(X=3)$

$$\frac{3}{2}\frac{\cancel{e^{-\lambda}}\lambda^1}{1} = \frac{\cancel{e^{-\lambda}}\lambda^3}{3!}$$

$$\lambda^2 = 9$$

$$\boxed{\lambda = 3}$$

i) $P(X \geq 1)$

$P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X=0)$

$= 1 - \frac{e^{-3}3^0}{0!}$

$P(X \geq 1) = 0.9502$

ii) $P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} + \frac{e^{-3}3^3}{3!}$

$= 0.64708$

3.c) <u>Discrete RV</u>

If X is DRV with probability mass func^n $p(X=x_i) = P_i$ then the varience is defined as

$Var(X) = E[(X-\mu)^2] = \sum_i (x_i - \mu)^2 P_i$

$\mu = E(X) = \sum_i x_i P_i$

also written as,

$Var(x) = E(x^2) - (E(x))^2$

<u>continuous RV</u>

If x is a continuous RV with pdf $f(x)$ then the varience defined as

$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx$

$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

also written as,

$Var(X) = E(X^2) - (E(X))^2$

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

3d) Ten coins thrown, $n=10$

probᵗʸ of head, $p = \frac{1}{2}$

" " tail, $q = \frac{1}{2}$

BD,

$$\boxed{P(X=r) = {}^nC_r \, p^r q^{n-r}}$$

i) atleast 7 heads

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{10-7} + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{10-8} + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right)^{10-9} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^{10-10}$$

$$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} \right]$$

$$= \left(\frac{1}{2}\right)^{10} \left[120 + 45 + 10 + 1 \right]$$

$$= \frac{176}{1024}$$

$$\therefore \boxed{P(X \geq 7) = 0.1719}$$

ii) atleast one head

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^{10}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{10} = 0.999$$

4a) we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Given, $f(x) = c e^{-|x|}$

$$\int_{-\infty}^{\infty} c e^{-|x|}\,dx = 1$$

$$c \int_{-\infty}^{\infty} e^{-|x|}\,dx = 1$$

$$|x| = \begin{cases} x > 0, & x \\ x < 0, & -x \end{cases}$$

$$c\left[\int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{\infty} e^{-|x|}\,dx\right] = 1$$

$$c\left[\int_{-\infty}^{0} e^{-(-x)}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right] = 1$$

$$c\left[\int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx\right] = 1$$

$$c\left[\left[e^{x}\right]_{-\infty}^{0} + \left[\frac{e^{-x}}{-1}\right]_{0}^{\infty}\right] = 1$$

$$c\left[\left[e^{0} - e^{-\infty}\right] + \left[-e^{-\infty} + e^{0}\right]\right] = 1$$

$$(e^{-\infty} = 0, \; e^{0} = 1)$$

$$c\left[[1-0] + [0+1]\right] = 1$$

$$\boxed{C = \frac{1}{2}}$$

Mean,

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x c e^{-|x|}\,dx$$

$$= c\left[\int_{-\infty}^{0} x e^{-|x|}\,dx + \int_{0}^{\infty} x e^{-|x|}\,dx\right]$$

$$= c\left[\int_{-\infty}^{0} x e^{x}\,dx + \int_{0}^{\infty} x e^{-x}\,dx\right]$$

$$\boxed{\int uv = uv' - u_1 v''}$$

$$= c\left[\left[x e^{x} - e^{x}\right]_{-\infty}^{0} + \left[\frac{x e^{-x}}{-1} + e^{-x}\right]_{0}^{\infty}\right]$$

$$= c\left[-1 + 1\right]$$

$$= 0$$

$$E(x) = 0$$

Varience,

$$\sigma^2 = E(x^2) - (E(x))^2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x^2 c e^{-|x|}\,dx$$

$= c\left[\int_{-\infty}^{0} x^2 e^{+x}\,dx + \int_{0}^{\infty} x^2 e^{-x}\,dx\right]$

$= c\left[\left[x^2e^{x} - 2xe^{x} + 2e^{x}\right]_{-\infty}^{0} + \left[\frac{x^2e^{-x}}{-1} + 2xe^{-x} - 2e^{-x}\right]_{0}^{\infty}\right]$

$= c\left[2 - (-2)\right] = 4c = 4\left(\frac{1}{2}\right) = 2$

$\cancel{\sigma^2 = 2}$

$E(x^2) = 2$

$\sigma^2 = E(x^2) - (E(x))^2$

$= 2 - 0$

$\sigma^2 = 2$

4b) Given, $\mu = 68$ kgs

$\sigma = 3$ kgs

i) Greater than 72 kgs

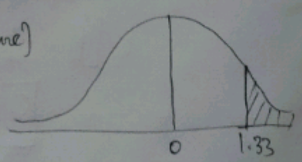
$x = 72$,

$Z = \frac{x-\mu}{\sigma} = \frac{72-68}{3} = \frac{4}{3} = 1.33$

$P(X>72) = P(Z>1.33)$

$= 0.5 - 0.4082$ ($\because$ Z plane)

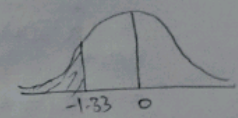
$= 0.0918$



$P(X>72) = 28$ students

ii) $P(X \leq 64)$

$x = 64$, $Z = \frac{64-68}{3} = -1.33$



$P(X \leq 64) = P(Z \leq -1.33)$

$= 0.5 - 0.4082$

$= 0.0918$

$P(X \leq 64) \times 0.0918 = 27.54$

$= 28$ students

4c) $f(x) = \begin{cases} \frac{1}{2}\sin x, & 0 \leq x \leq \pi \\ 0, & \text{elsewhere} \end{cases}$

mean,

$E(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$= \int_{-\infty}^{0} \cancel{x f(x)}\,dx + \int_{0}^{\pi} x f(x)\,dx + \int_{\pi}^{\infty} \cancel{x f(x)}\,dx$

$= \int_{0}^{\pi} x\left(\frac{1}{2}\sin x\right)dx$

$= \frac{1}{2}\int_{0}^{\pi} x\sin x\,dx = \frac{1}{2}\left[x(-\cos x) - (-\sin x)\right]_{0}^{\pi}$

$= \frac{1}{2}\left[-x\cos x + \sin x\right]_{0}^{\pi} = \frac{1}{2}\left[-\pi(-1) + 0\right] = \frac{\pi}{2}$

Varience,

$\sigma^2 = E(x^2) - (E(x))^2$

$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - (E(x))^2$

$= \int_0^{\pi} x^2 \left(\frac{1}{2}\sin x\right) dx - (E(x))^2$

$= \frac{1}{2}\int_0^{\pi} x^2 \sin x\,dx - \left(\frac{\pi}{2}\right)^2$

$= \frac{1}{2}\left[x^2(-\cos x) - (2x)(-\sin x) + 2(-\cos x)\right]_0^{\pi} - \left(\frac{\pi}{2}\right)^2$

$= \frac{1}{2}\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} - \left(\frac{\pi}{2}\right)^2$

$= \frac{1}{2}\left[(-\pi^2(-1) + 2\pi(0) + 2(-1)) - (2)\right] - \left(\frac{\pi}{2}\right)^2$

$= \frac{1}{2}\left[+\pi^2 - 4\right] - \left(\frac{\pi}{2}\right)^2$

$= \frac{2\pi^2 - 8 - \pi^2}{4} = \frac{\pi^2 - 8}{4}$

Now,

$P\left(0 \le X \le \frac{\pi}{2}\right) = \int_0^{\pi/2} f(x)\,dx$

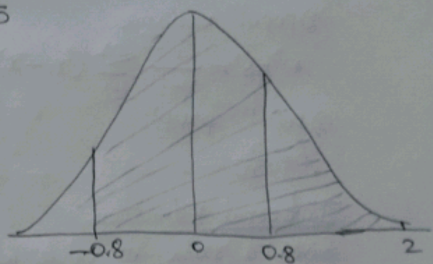
$= \int_0^{\pi/2} \frac{1}{2}\sin x\,dx$

$= \frac{1}{2}\left[-\cos x\right]_0^{\pi/2} = \frac{1}{2}(-0+1) = \frac{1}{2}$

$P\left(0 \le X \le \frac{\pi}{2}\right) = \frac{1}{2}$

4d) Given, $\mu = 30$, $\sigma = 5$

$P(26 \le X \le 40)$



$x = 26$,

$Z = \frac{x-\mu}{\sigma}$

$= \frac{26-30}{5}$

$= -0.8$

$x = 40$

$Z = \frac{40-30}{5}$

$= 2$

$P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$

$= 0.4772 + 0.2881$

$= 0.7653$

$\therefore P(26 \le X \le 40) = 0.7653$

5a) Estimation is the process of using sample data to calculate an approximate value of an unknown population parameter.

Point Estimation:

A single value is used to estimate a population parameter.

i) Population mean

$\mu = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

2) popul^n variance

$\sigma^2 = S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

3) popul^n prop^n

$\hat{p} = \frac{x}{n}$

Interval Estimation:

Gives a range of values within which the parameter lies with certain confidence level.

1) mean (σ known, n large)

$\left(\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} , \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$

2) mean (σ unknown, n small)

$\left(\bar{x} - t_{\alpha/2, n-1} \frac{s}{\sqrt{n}} , \bar{x} + t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}\right)$

3) Proportion

$\left(\hat{p} - z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} , \hat{p} + z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

5b) $n = 400, \bar{x} = 40, \mu = 38, \sigma = 10$

$\alpha = 5\%$

$z_{\alpha/2} = 1.96$

CI is

$\left(\bar{x} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} , \bar{x} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}}\right)$

$\left(40 - (1.96) \frac{10}{\sqrt{400}} , \cancel{\bar{x}}\,40 + (1.96) \frac{10}{\sqrt{400}}\right)$

$\left(40 - (1.96) \frac{1}{2} , 40 + (1.96) \frac{1}{2}\right)$

$(40 - 0.98 , 40 + 0.98)$

$(39.02 , 40.98)$

6a) p = bad apples = 0.13 , n = 500

$q = 0.87$

Standard error prop^n $= \sqrt{pq/n} = \sqrt{(0.13)(0.87)/500}$

$= 0.015$

limits $= p \pm 2.58 \sqrt{pq/n}$

$= 0.13 \pm 2.58 (0.015)$

$= 9.13\%$ & 16.87%