# Linear Algebre

Study of vector space $V = \mathbb{C}^n$

$A : V \to V$

$$v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = |\psi\rangle \text{ ket}$$

$(z_1 \dots z_n) = \langle\psi| \text{ Bra}$

$A : V \to W$

$$A = \sum_i a_{ij} |w_i\rangle\langle v_j|$$

Hermitian operators $A^\dagger = A$

Eigenvalues are all real.

$$\underset{n \times n}{A} = \sum_i a_i |i\rangle\langle i|$$

$\{|i_1\rangle, |i_2\rangle \dots |i_n\rangle\}$ eigenvector

$$A = \begin{pmatrix} a_1 & & 0 \\ & a_2 & & 0 \\ 0 & & \ddots \\ & 0 & & a_n \end{pmatrix}$$

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Unitary Operators

$U^{\dagger} U = I$

Inner product

$(|v\rangle, |w\rangle) = \langle w | v \rangle = \lambda \in C$

$V \quad |v_i\rangle \quad |w_i\rangle$

$|w_i\rangle = U|v_i\rangle$

$U = \sum_i |w_i\rangle\langle v_i|$

$(|v\rangle, |w\rangle) \Rightarrow \underline{(U|v\rangle, U|w\rangle)}$

$\langle w | U^{\dagger} U | v \rangle$

$$U^\dagger U = I \Rightarrow$$

$$\left( U|v\rangle, U|w\rangle \right) = \left( |v\rangle, |w\rangle \right)$$

$$A_{ij} = \langle v_i | A | v_j \rangle \quad \{ |v_i\rangle \}$$

$$A'_{ij} = \langle w_i | A | w_j \rangle \quad \{ |w_i\rangle \}$$

$$|w_i\rangle = U |v_i\rangle$$

Ex 2.20

$$A'_{ij} = \langle w_i | A | w_j \rangle$$

$$= \langle v_i | U^\dagger A U | v_j \rangle$$

$$\left( A'_{ij} \right) = \left( U^\dagger A U \right)_{ij}$$

$$\underline{A' = U^\dagger A U}$$

Tensor Products:

$V, W$ Hilbert spaces (equipped with an inner product)

$V \otimes W$

The elements of $(V \otimes W)$ of the form $(|v\rangle \otimes |w\rangle)$

$\underline{|v\rangle \in V}$ $\underline{|w\rangle \in W}$

$\{|i\rangle\} = V$ $|j\rangle \in W$

$|i\rangle \otimes |j\rangle$ is a basis for $V \otimes W$

$V = C^2$

$\{|0\rangle, |1\rangle\}$.

$V \otimes V = C^2 \otimes C^2$

$= C^4$

$|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$

$|1\rangle \otimes |1\rangle$

$|0\rangle \otimes |0\rangle = |00\rangle$

$z(|v\rangle \otimes |w\rangle) \qquad z \in \mathbb{C}$.

$= z|v\rangle \otimes |w\rangle$

or

$|v\rangle \otimes z|w\rangle$

$\otimes$ is distributive over vector addition.

$|v_1\rangle, |v_2\rangle \in V \qquad |w\rangle \in W$

$(|v_1\rangle + |v_2\rangle) \otimes |w\rangle$

$= (|v_1\rangle \otimes |w\rangle) + (|v_2\rangle \otimes |w\rangle)$

Operators of $V \otimes W$.

$|v\rangle, |w\rangle \in V, W \qquad A: V \to V$

$B: W \to W$

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

$$(A \otimes B)\left(\sum_i a_i |v_i\rangle \otimes |w_i\rangle\right) = \sum_i a_i A|v_i\rangle \otimes B|w_i\rangle$$

$A : V \to V'$

$B : W \to W'$

$C : (V \otimes W) \to V' \otimes W'$

$$C = \sum_i c_i A_i \otimes B_i$$

$A_i, B_i$ are operators

of $V \to V'$ $W \to W'$

Inner product of $V \otimes W$

$$\left( \sum_i a_i |v_i\rangle \otimes |w_i\rangle, \sum_j b_j |v_j'\rangle \otimes |w_j'\rangle \right)$$

$$= \sum_{ij} a_i^* b_j \underbrace{\langle v_i | v_j' \rangle}\underbrace{\langle w_i | w_j' \rangle}$$

Matrix representation:

$A \otimes B$

$A$ $m \times n$ matrix

$B$ $p \times q$ matrix

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ \\ \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{pmatrix}$$

$V = C^2$

$\{|0\rangle, |1\rangle\}$ $\quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} 0Y & -iY \\ iY & 0Y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

N - Qubit system.

$V = \mathbb{C}^2$ single qubit system.

$$V^N = (\mathbb{C}^2)^{\otimes N} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \mathbb{C}^2$$

$$|\psi\rangle^{\otimes k} = \overbrace{|\psi\rangle \otimes |\psi\rangle \otimes \cdots |\psi\rangle}^{k\text{- factors.}}$$

$$|\psi\rangle^{\otimes 2} = |\psi\rangle \otimes |\psi\rangle$$

Properties of tensor products.

$$(A \otimes B)^* = A^* \otimes B^*$$

$$(A \otimes B)^T = A^T \otimes B^T$$

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

$U$ unitary operator.

$$U : V \to V$$

$$V^2 = V \otimes V$$

$$U|v_i\rangle = |w_i\rangle$$

$$|v_i\rangle \otimes |v_j\rangle$$

$$(U \otimes U)\, |v_i\rangle \otimes |v_j\rangle$$

$$= U|v_i\rangle \otimes U|v_j\rangle$$

$U \otimes U$ unitary.

$$(U \otimes U)^\dagger = U^\dagger \otimes U^\dagger$$

$$\underline{(U^\dagger \otimes U^\dagger)(U \otimes U)}$$

$$U^\dagger U \otimes U^\dagger U = \mathbb{1} \otimes \mathbb{1}$$
$$= \mathbb{1}.$$

$$(A \otimes B)(V \otimes W)$$

$$(A' \otimes B')$$

$$(A' \otimes B')(A \otimes B) : V \otimes W \to V' \otimes W'$$

$$A'A \otimes B'B$$

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Projective operator.

$$P^2 = P$$

$$(P \otimes P)^2 = (P \otimes P)(P \otimes P)$$
$$= P^2 \otimes P^2 = P \otimes P$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \Big[ (|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1| \Big]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H^{\otimes n}$$

$$\underbrace{|0\rangle \otimes |0\rangle \otimes |0\rangle \cdots |0\rangle}_{n \text{ factors}}$$

$$H^{\otimes n}(|0\rangle \otimes |0\rangle \cdots |0\rangle)$$

$$= H|0\rangle \otimes H|0\rangle \otimes H|0\rangle \cdots \otimes H|0\rangle$$

$$= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \cdots \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= |00\cdots0\rangle + |000\cdots1\rangle$$

$$+ \cdots |111111\rangle$$

$$|1111111\rangle = |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes \cdots |1\rangle$$

single qubit

$$\{|0\rangle, |1\rangle\}.$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\underline{|0\rangle + |1\rangle}\right)$$

$$H|0\rangle = \frac{1}{\sqrt{2}} |0\rangle + |1\rangle.$$

$$H^\dagger H = \underline{\mathbb{1}}$$ Unitary transformation

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$$|v\rangle \otimes |w\rangle \in V \otimes W$$

$$A : V \to V' \qquad A|v\rangle = |v'\rangle$$

$$B : W \to W' \qquad B|w\rangle = |w'\rangle$$

$$(A \otimes B)(|v\rangle \otimes |w\rangle)$$

$$= \left( A|v\rangle \otimes B|w\rangle \right)$$

Functions of Operators:

$$A = \sum_a a\,|a\rangle\langle a|$$

$|a\rangle$ kets

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$$

Spectral decomposition.

Normal Operator; $A$

$$AA^\dagger = A^\dagger A$$

Any normal operator on a vector space is diagonal w.r.t a basis.

$$A = \begin{pmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$$A = \sum_i a_i |a_i\rangle\langle a_i|$$

$$f(A) = \sum_i f(a_i)\, |a_i\rangle\langle a_i|$$

$$f(A) = \mathbb{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!}$$

$$= \mathbb{1} + \sum_i a_i |a_i\rangle\langle a_i| + \frac{1}{2!} \sum_i a_i^2 |a_i\rangle\langle a_i|$$

$$+ \cdots$$

$$= \sum_i f(a_i)\, |a_i\rangle\langle a_i|$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z^2 = \mathbb{1}$$

$$\exp(\theta Z) = \begin{pmatrix} e^{\theta} & 0 \\ 0 & e^{-\theta} \end{pmatrix}$$

$$= \mathbb{1} + \theta Z + \frac{\theta^2 Z^2}{2!} + \frac{\theta^3 Z^3}{3!} + \cdots$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\exp(i \vec{v}\cdot\vec{\sigma}\,\theta) = \cos(\theta)\mathbb{1} + i\sin(\theta)\vec{v}\cdot\vec{\sigma}$$

X Y Z

$$\vec{v}\cdot\vec{\sigma} = v_x \sigma_x + v_y \sigma_y + v_z \sigma_z$$

Trace of an operator:

$$Tr(A) = \sum_i A_{ii}$$

Properties

1. Cyclic

$Tr(AB) = Tr(BA)$

2. linear

$Tr(A+B) = Tr(A) + Tr(B)$

3. $Tr(zA) = z\,Tr(A)$

$$A' \rightarrow UAU^{\dagger}$$

$A \quad |v_i\rangle$

$A' \quad |w_i\rangle$

$$Tr(A') = Tr(UAU^{\dagger}) = Tr(A\underbrace{U^{\dagger}U}) = Tr(A)$$

$\Rightarrow Tr(A)$ is basis independent.

$$A = \sum_a a\,|a\rangle\langle a|$$

$$Tr(A) = \sum_a a = \text{sum of eigenvalues.}$$

$\mathrm{tr}(A|\psi\rangle\langle\psi|)$

$|\psi\rangle \Rightarrow |\psi\rangle\langle\psi|$

$\mathrm{tr}(A|\psi\rangle\langle\psi|) = \underline{\langle\psi|A|\psi\rangle}$

$\mathrm{tr}(A|\psi\rangle\langle\psi|) = \sum_i \langle i|A|\psi\rangle\langle\psi|i\rangle$

$= \langle\psi|A|\psi\rangle$

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Commutator:

$A, B$

$[A,B] = AB - BA$

Anti-commutator

$\{A,B\} = AB + BA$

$[A,B] = 0$ $A, B$ commute

$A^\dagger = A$

$B^\dagger = B$

Simultaneously diagonlizable

Share a common set of eigenvectors

$$A = \sum_i a_i |i\rangle\langle i|$$

$|i\rangle$ Eigenvectors

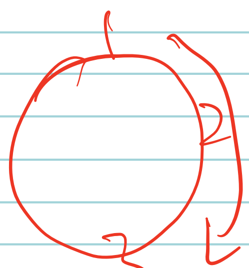
$$B = \sum_i b_i |i\rangle\langle i|$$

$$[X, Y] = 2iZ \qquad [Y, Z] = 2iX$$

$$[Z, X] = 2iY.$$

$$[\sigma_i, \sigma_j] = 2i \underbrace{\varepsilon_{ijk}} \sigma_k$$

$\sigma_i, \sigma_j, \sigma_k$ rank-3 antisymmetric tensor.



$\varepsilon_{123} = 1$

$\varepsilon_{132} = -1$

$X, Y, Z$

$\{\sigma_i, \sigma_j\}$ $\quad i, j = 1, 2, 3$

$$[\sigma_i, \sigma_j] = 2i \varepsilon_{ijk} \sigma_k$$

$$\{\sigma_i, \sigma_j\} = 0$$

$$\Rightarrow \sigma_i^2 = \mathbb{1} \qquad X^2 = \mathbb{1} = Y^2 = Z^2$$

Ex 2.42 → 2.47 in Neilsen and Chuang

Postulates of QM

1. Associated to any isolated physical system there is a complex vector space. And the state of the system is described by the state vector $|\psi\rangle$ which is a unit vector.

Qubit $V = \mathbb{C}^2$

$$\{|0\rangle, |1\rangle\}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$|a|^2 + |b|^2 = 1.$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle - |1\rangle$$

2. Evolution of a "closed" system

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