Integration of Log/Exp

Monday, February 24, 2025 7:40 PM

Outline for Today's session (2 | 25 | 25)

- Incomplete for now.

As always,

- Yellow highlight : problem / sample problem
- blue highlight : theorem
- purple text : (usually) answer to problem. may not be consistent

Theorem

- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ $\qquad \int \frac{1}{x}\,dx = \ln(x) + C$
- $\frac{d}{dx}(e^x) = e^x$ $\qquad \int e^x\,dx = e^x + C$
- $\frac{d}{dx}(a^x) = \ln(a)\cdot a^x$ $\qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$

May or may not be important

Q : Why is $\frac{d}{dx}(a^x) = (\ln a)\,a^x$?

A : MAIN CONCEPT : Property of log

$\boxed{a^{\log_a b} = b}$

We know $\frac{d}{dx}(e^x) = e^x$.

Also, from properties of log, $a = e^{\ln a}$

So,

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left((e^{\ln a})^x\right)$$ laws of exponents

$$= \frac{d}{dx}\left(e^{(\ln a)x}\right)$$

chain rule

$$= e^{(\ln a)x}\cdot\frac{d}{dx}((\ln a)x)$$

$$= (e^{(\ln a)})^x\cdot \ln a = \boxed{a^x \ln a}$$

Problems

① [PS3 - # 84]

$$\int \frac{x^2}{x^3+4}\,dx = \int \frac{1}{u}\cdot\left(\frac{1}{3}du\right)$$

$\frac{d}{dx}(x^3+4) = 3x^2$

let $u = x^3 + 4$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \frac{1}{3}\cdot\ln u + C$$

$$= \boxed{\frac{1}{3}\ln(x^3+4) + C}$$

② [PS3 - # 92]

$$\int \frac{v(v^2-1)}{v^2+1}\,dv = \int \frac{u-2}{u}\left(\frac{1}{2}du\right)$$

let $u = v^2+1 \rightarrow u-2 = v^2-1$

$du = 2v\,dv$

$\frac{1}{2}du = v\,dv$

$$= \frac{1}{2}\int\left(1-\frac{2}{u}\right)du$$

$$= \frac{1}{2}(u - 2\ln u) + C$$

$$= \boxed{\frac{1}{2}(v^2+1) - \ln(v^2+1) + C}$$

③ [PS3 - # 88]

$$\int \frac{(u^2+1)^2}{u^3}\,du = \int \frac{u^4+2u^2+1}{u^3}\,du$$

~~let $w = u^3$~~

~~$dw = 3u^2\,du$~~

~~let $w = u^2+1$~~

~~$dw = 2u\,du$~~

$$= \int u + 2u^{-1} + u^{-3}\,du$$

$$= \boxed{\frac{u^2}{2} + 2\ln u - \frac{u^{-2}}{2} + C}$$