Lecture 7

Math 178

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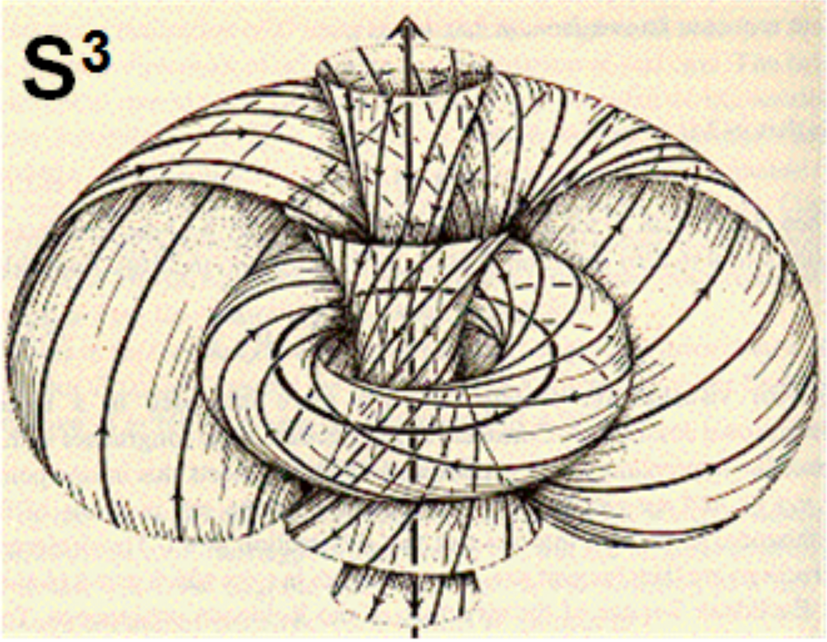
Today's topics

- Geometry and topology of SO(3)
- Subgroups of SE(3)
- **The one-parameter subgroup in SE(3)**
- **Logarithm of a Matrix**
- **Property of rigid motions**
- **View things from different points of view, especially for rotations**
- **Derivation of Rodrigues Formula and its application**

S^3 is a double cover of SO(3)

- Last time: Very pair of unit quaternions q & -q represents a rotation.
- The set of unit quaternions form a sphere S^3 in R^4.
- We can show that S^3 is a double cover of SO(3).

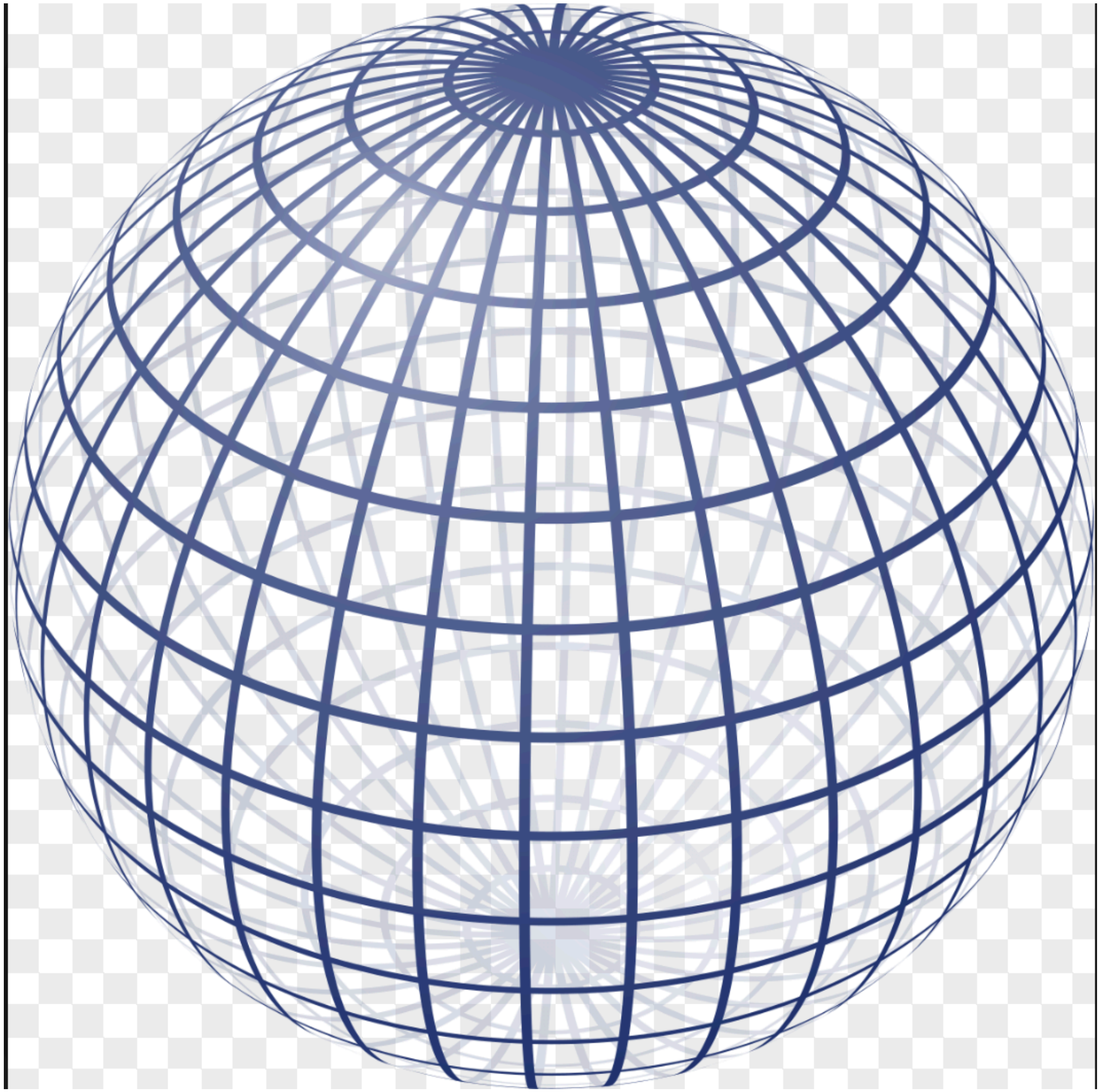
How does S^3 look like?



By identify antipodal points

$$M = SO(3) = RP^3$$

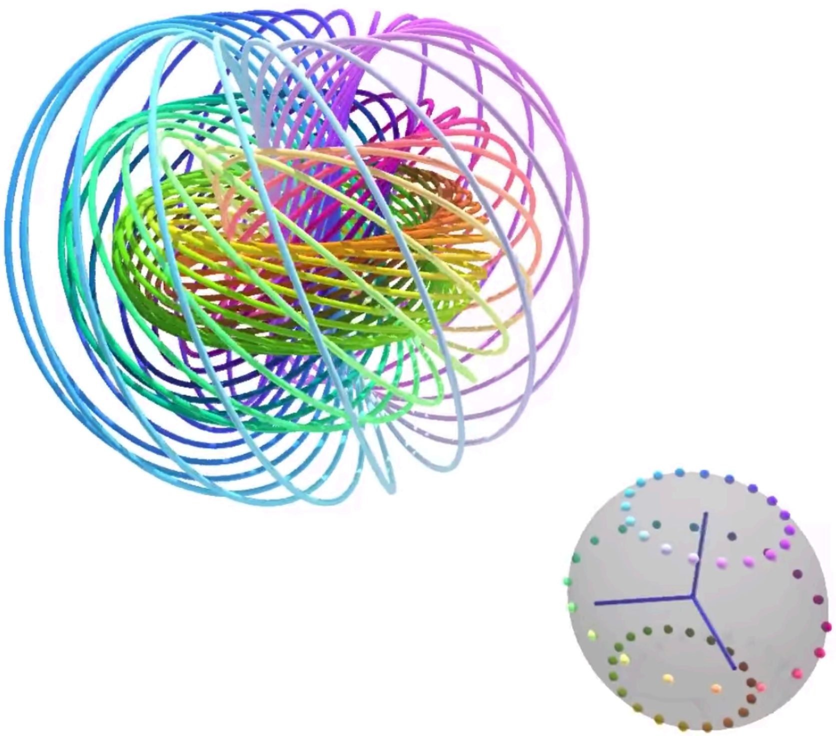
Recall: There are a lot of circles on S^2



There are also a lots of circles on S^3

Stereographic projection of the hypersphere's parallels (red), meridians (blue) and hypermeridians (green). Because this projection is conformal, the curves intersect each other orthogonally (in the yellow points) as in 4D. All curves are circles: the curves that intersect $\langle 0,0,0,1 \rangle$ have infinite radius (= straight line). In this picture, the whole 3D space maps the *surface* of the hypersphere, whereas in the previous picture[*clarification needed*] the 3D space contained the *shadow* of the bulk hypersphere.

Hopf Fibration



- https://www.youtube.com/watch?v=AKotMPGFJYk

Rigid Body Kinematics

Recall:

The Lie group $SE(3)$

$$SE(3)=\left\{\mathbf{A}\ \middle|\ \mathbf{A}=\left[\begin{array}{c|c}\mathbf{R} & \mathbf{r}\\ \hline \mathbf{0}_{1\times 3} & 1\end{array}\right], \mathbf{R}\in R^{3\times 3}, \mathbf{r}\in R^{3}, \mathbf{R}^{T}\mathbf{R}=\mathbf{R}\mathbf{R}^{T}=\mathbf{I}, |\mathbf{R}|=1\right\}$$

- https://www.seas.upenn.edu/~meam620/slides/kinematicsI.pdf

The important subgroups of SE(3)

Subgroup	Notation	Definition	Significance
The group of rotations in three dimensions	$SO(3)$	The set of all proper orthogonal matrices. $SO(3) = \{\mathbf{R} \mid \mathbf{R} \in R^{3\times 3}, \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}\}$	All spherical displacements. Or the set of all displacements that can be generated by a spherical joint (S-pair).
Special Euclidean group in two dimensions	$SE(2)$	The set of all 3×3 matrices with the structure: $\begin{bmatrix} \cos\theta & \sin\theta & r_x \\ -\sin\theta & \cos\theta & r_y \\ 0 & 0 & 1 \end{bmatrix}$ where θ, r_x, and r_y are real numbers.	All planar displacements. Or the set of displacements that can be generated by a planar pair (E-pair).
The group of rotations in two dimensions	$SO(2)$	The set of all 2×2 proper orthogonal matrices. They have the structure: $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, where θ is a real number.	All rotations in the plane, or the set of all displacements that can be generated by a single revolute joint (R-pair).

The important subgroups of SE(3) (continue)

The group of translations in n dimensions.	$T(n)$	The set of all $n\times1$ real vectors with vector addition as the binary operation.	All translations in n dimensions. $n = 2$ indicates planar, $n = 3$ indicates spatial displacements.
The group of translations in one dimension.	$T(1)$	The set of all real numbers with addition as the binary operation.	All translations parallel to one axis, or the set of all displacements that can be generated by a single prismatic joint (P-pair).
The group of cylindrical displacements	$SO(2)\times T(1)$	The Cartesian product of $SO(2)$ and $T(1)$	All rotations in the plane and translations along an axis perpendicular to the plane, or the set of all displacements that can be generated by a cylindrical joint (C-pair).
The group of screw displacements	$H(1)$	A one-parameter subgroup of $SE(3)$	All displacements that can be generated by a helical joint (H-pair).

The Group of Rotations

A rigid body B is said to rotate relative to another rigid body A, when a point of B is always fixed in $\{A\}$. Attach the frame $\{B\}$ so that its origin O' is at the fixed point. The vector ${}^{A}\mathbf{r}^{O'}$ is equal to zero in the homogeneous transformation in Equation (1).

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

The set of all such displacements, also called *spherical displacements*, can be easily seen to form a subgroup of $SE(3)$.

If we compose two rotations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

$$
= \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & \mathbf{0}_{3\times 1} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

Notice that only the 3×3 submatrix of the homogeneous transformation matrix plays a role in describing rotations. Further, the binary operation of multiplying 4×4 homogoneous transformation matrices reduces to the binary operation of multiplying the corresponding 3×3 submatrices. Thus, we can simply use 3×3 rotation matrices to represent spherical displacements. This subgroup, is called the special orthogonal group in three dimensions, or simply $SO(3)$:

$$
SO(3) = \left\{ \mathbf{R} \mid \mathbf{R} \in R^{3\times 3},\ \mathbf{R}^{T}\mathbf{R} = \mathbf{R}\mathbf{R}^{T} = \mathbf{I} \right\} \tag{4}
$$

Locally exponential down to form data manifold

- Suppose we have data points in R^n and clustered in different areas. In each area find a center c of the data points and view them as a vectors in R^n. Use exponential map to map this tangent space at c to form a manifold.
- Cover the data using those areas and then then exponential down all of them.
- The collections of those exponential down curved pieces could form a manifold?

Decompose a Rotation to 3 Successive Rotations

It is well known that any rotation can be decomposed into three finite successive rotations, each about a different axis than the preceding rotation. The three rotation angles, called Euler angles, completely describe the given rotation. The basic idea is as follows. If we consider any two reference frames $\{A\}$ and $\{B\}$, and the rotation matrix ${}^{A}\mathbf{R}_{B}$, we can construct two intermediate reference frames $\{M\}$ and $\{N\}$, so that

$$
{}^{A}\mathbf{R}_{B} = {}^{A}\mathbf{R}_{M} \times {}^{M}\mathbf{R}_{N} \times {}^{N}\mathbf{R}_{B}
$$

where

1. The rotation from $\{A\}$ to $\{M\}$ is a rotation about the x axis (of $\{A\}$) through ψ;
2. The rotation from $\{M\}$ to $\{N\}$ is a rotation about the y axis (of $\{M\}$) through ϕ; and
3. The rotation from $\{N\}$ to $\{B\}$ is a rotation about the z axis (of $\{N\}$) through θ.

$$
{}^{A}\mathbf{R}_{B} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix} \times \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \times \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Thus any rotation can be viewed as a composition of these three elemental rotations except for rotations at which the Euler angle representation is singular[1]. This in turn means all rotations in an open neighborhood in $SO(3)$ can be described by three real numbers (coordinates). With a little work it can be shown that there is a 1-1, continuous map from $SO(3)$ onto an open set in R^3. This gives $SO(3)$ the structure of a three-dimensional differentiable manifold, and therefore a Lie group.

The rotations in the plane, or more precisely rotations about axes that are perpendicular to a plane, form a subgroup of $SO(3)$, and therefore of $SE(3)$. To see this, consider the *canonical form* of this set of rotations, the rotations about the z axis. In other words, connect the rigid bodies A and B with a revolute joint whose axis is along the z axis in Figure 1. The homogeneous transformation matrix has the form:

$$^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where θ is the angle of rotation. If we compose two such rotations, $^{A}\mathbf{A}_{B}$ and $^{B}\mathbf{A}_{C}$, through θ_1 and θ_2 respectively, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) & 0 & 0 \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

All matrices in this subgroup are the same periodic function of one real variable, θ, given by:

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This subgroup is called $SO(2)$. Further, since $\mathbf{R}(\theta_1) \times \mathbf{R}(\theta_2) = \mathbf{R}(\theta_1 + \theta_2)$, we can think of the subgroup as being locally isomorphic[2] to R^1 with the binary operation being addition.

The group of translations

A rigid body B is said to translate relative to another rigid body A, if we can attach reference frames to A and to B that are always parallel. The rotation matrix ${}^{A}\mathbf{R}_{B}$ equals the identity in the homogeneous transformation in Equation (1).

$$
{}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
$$

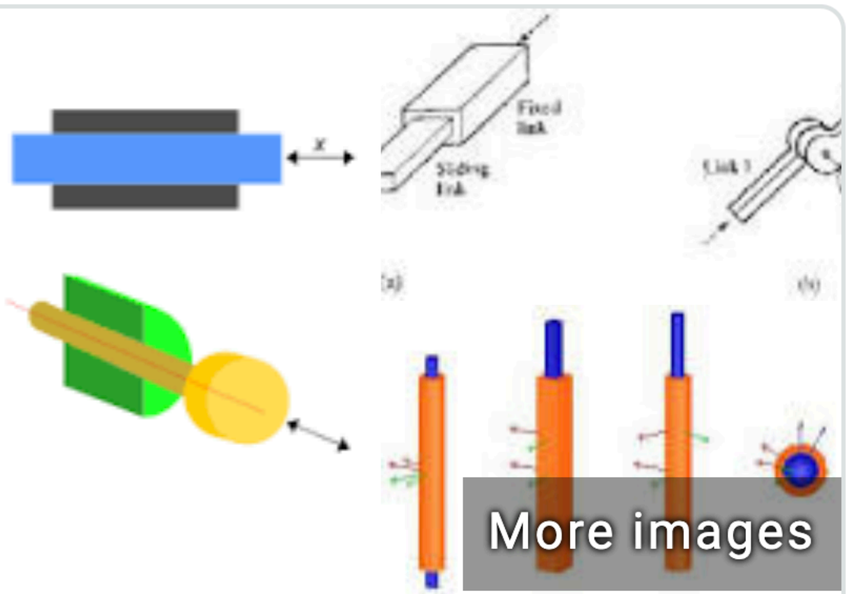
The set of all such homogeneous transformation matrices is the group of translations in three dimensions and is denoted by $T(3)$.

If we compose two translations, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
\begin{aligned}
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} &= \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \times \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{B}\mathbf{r}^{O''} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \\
&= \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} + {}^{B}\mathbf{r}^{O''} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]
\end{aligned}
$$

Notice that only the 3×1 vector part of the homogeneous transformation matrix plays a role in describing translations. Thus we can think of a element of $T(3)$ as simply a 3×1 vector, ${}^{A}\mathbf{r}^{O'}$. Since the composition of two translations is captured by simply adding the two corresponding 3×1 vectors, ${}^{A}\mathbf{r}^{O'}$ and ${}^{B}\mathbf{r}^{O''}$, we can define the subgroup, $T(3)$, as the real vector space R^3 with the binary operation being vector addition.

Similarly, we can describe the two subgroups of $T(3)$, $T(1)$ and $T(2)$, the group of translations in one and two dimensions respectively. Because they are subgroups of $T(3)$, they are also subgroups of $T(3)$. It is worth noting that $T(1)$ consists of all translations along an axis and this is exactly the set of displacements that can be generated by connecting A and B with a single prismatic joint.



Prismatic joint

A prismatic joint provides a linear sliding movement between two bodies, and is often called a slider, as in the slider-crank linkage. A prismatic pair is also called as sliding pair. A prismatic joint can be formed with a polygonal cross-section to resist rotation. Wikipedia

The special Euclidean group in two dimensions

If we consider all rotations and translations in the plane, we get the set of all displacements that are studied in *planar kinematics*. These are also the displacements generated by the *Ebene*-pair, the planar *E*-pair. If we let the rigid body B translate along the x and y axis and rotate about the z axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & {}^{A}r_x^{O'} \\ -\sin\theta & \cos\theta & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation, and ${}^{A}r_x^{O'}$ and ${}^{A}r_y^{O'}$ are the two components of translation of the origin O'. If we compose two such displacements, ${}^{A}\mathbf{A}_{B}$ and ${}^{B}\mathbf{A}_{C}$, the product is given by:

$$
{}^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & {}^{A}r_x^{O'} \\ -\sin\theta_1 & \cos\theta_1 & 0 & {}^{A}r_y^{O'} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & {}^{B}r_x^{O''} \\ -\sin\theta_2 & \cos\theta_2 & 0 & {}^{B}r_y^{O''} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos(\theta_1 + \theta_2) & \sin(\theta_1 + \theta_2) & 0 & \left({}^{A}r_x^{O'} + {}^{B}r_x^{O''}\cos\theta_1 + {}^{B}r_y^{O''}\sin\theta_1\right) \\ -\sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 & \left({}^{A}r_y^{O'} - {}^{B}r_x^{O''}\sin\theta_1 + {}^{B}r_y^{O''}\cos\theta_1\right) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Because the set of matrices can be continuously parameterized by three variables, θ, ${}^{A}r_x^{O'}$, and ${}^{A}r_y^{O'}$, $SE(2)$ is a differentiable, three-dimensional manifold.

The one-parameter subgroup in SE(3)

The group of cylindrical motions is the group of motions admitted by a *cylindrical pair*, or a *C*-pair. If we let the rigid body B translate along and rotate about the z axis relative to the frame $\{A\}$, we get the canonical set of homogeneous transformation matrices of the form:

$$
{}^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where θ is the angle of rotation and k is the translation. The set of such matrices is continuously parameterized by these two variables. Thus, this subgroup is a two-dimensional Lie group. In fact, it is nothing but the Cartesian product $SO(2) \times T(1)$. Physically this means we can realize the cylindrical pair by arranging a revolute joint and a prismatic joint in series (in any order) along the same axis.

A one-dimensional subgroup is obtained by coupling the translation and the rotation so that they are proportional. The canonical homogeneous transformation matrix is of the form:

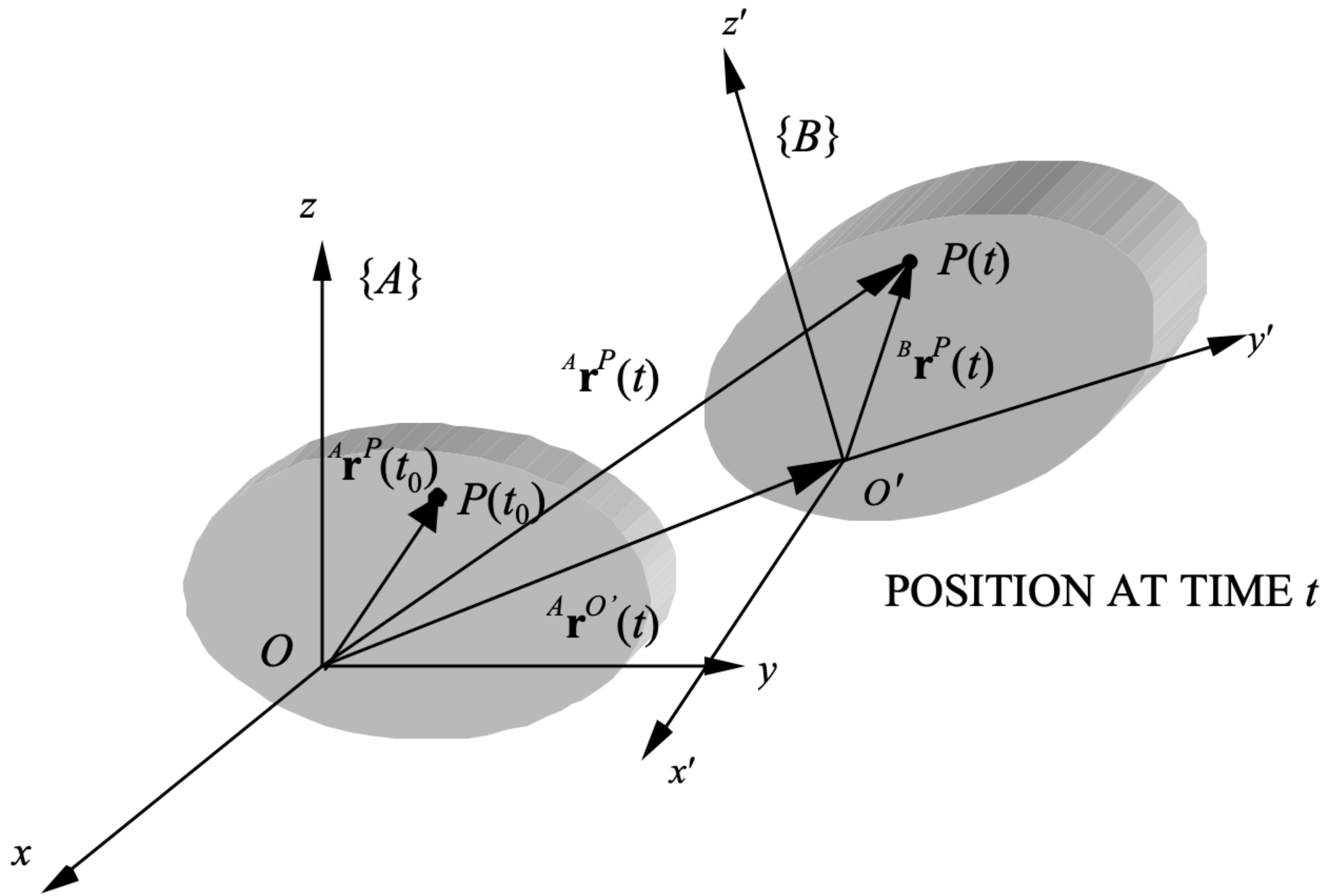
$$^{A}\mathbf{A}_{B} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & h\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where h is a scalar constant called the pitch. Because the displacement involves a rotation and a co-axial translation that is linearly coupled to the rotation, this displacement is called a *screw displacement*. It is exactly the displacement generated by a *helical pair*, or the *H*-pair.

$$^{A}\mathbf{A}_{B} \times {}^{B}\mathbf{A}_{C} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & h\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & h\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 0 \\ 0 & 0 & 1 & h(\theta_1+\theta_2) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The set of all screw displacements about the z-axis can be described by a matrix function $\mathbf{A}(\theta)$, with the property $\mathbf{A}(\theta_1) \times \mathbf{A}(\theta_2) = \mathbf{A}(\theta_1 + \theta_2)$. Thus this one-dimensional subgroup is isomorphic to the set R^1 with the binary operation of addition. Such one-dimensional subgroups



Now All the controls can be done in the Lie algebra of the Lie group SE(3)
In general: Definition of Lie Algebra

a *Lie algebra* is a vector space $\mathfrak{g}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$; $(X, Y) \mapsto [X, Y]$ called a *Lie bracket*, such that:

(a) **Antisymmetry**: $[Y, X] = -[X, Y]$
(b) **Bilinearity**: for all $a, b \in \mathbb{F}$ and for all $X, Y, Z \in \mathfrak{g}$
  (i) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
  (ii) $[X, aY + bZ] = a[X, Y] + b[X, Z]$.
(c) **The Jacobi Identity**: for all $X, Y, Z \in \mathfrak{g}$, $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$.

[X, Y] = XY-YX
$[X, Y]^t = (XY-YX)^t = Y^tX^t$-XtYt = (-Y)(-X) –(-X)(-Y) = - (XY-YX) = - [X, Y]

Get used to see things from different points of view, especially for rotations

Example 3.2 (Rotation of a coordinate frame and rotation of a vector) *Consider the 2D example in Figure 3.3, where on the left, a vector x is rotated clockwise by an angle α to $x_\star$. This is equivalent to (on the right) rotating the coordinate frame v counterclockwise by an angle α. Note that* $x_\star^v = x^u$.

Figure 3.3: Left: clockwise rotation α of the vector x to the vector $x_\star$. Right: counterclockwise rotation α of the coordinate frame v to the coordinate frame u.

In Figure 3.4, a vector x is rotated an angle α around the unit vector n. We denote the rotated vector by $x_\star$. Suppose that x as expressed in the coordinate frame v is known (and denoted x^{v}) and that we want to express $x_\star^{\mathrm{v}}$ in terms of x^{v}, α and n.

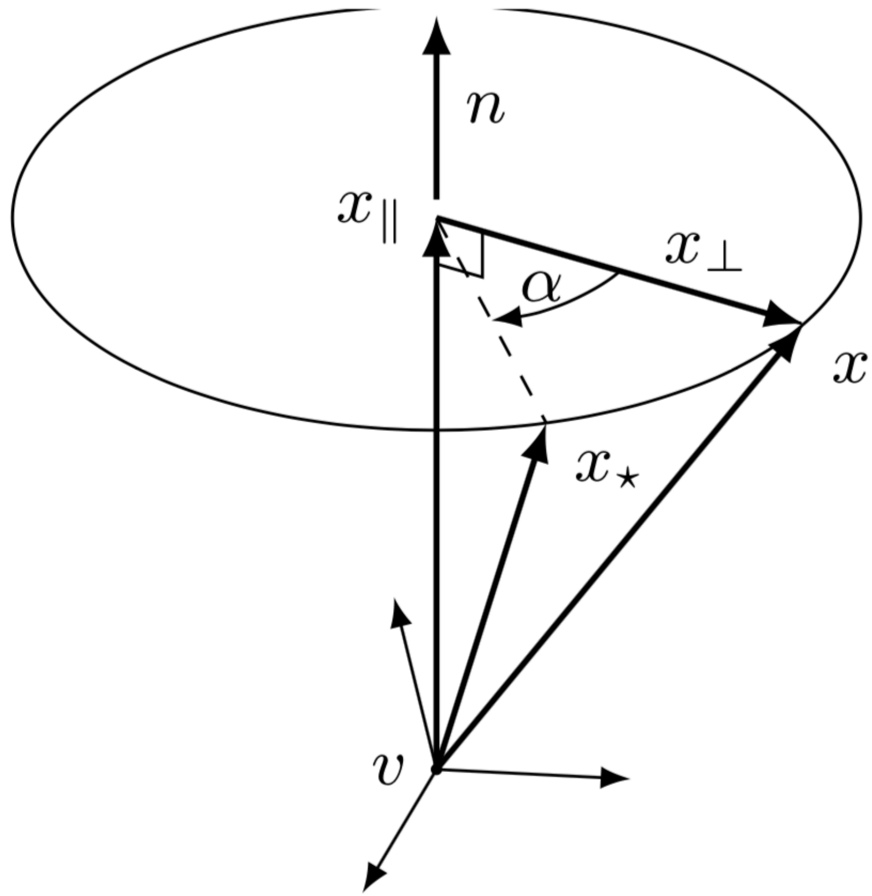
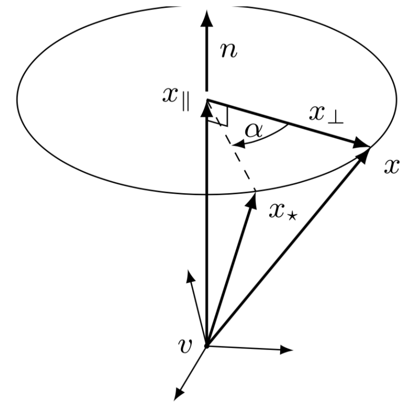


Figure 3.4: Clockwise rotation of a vector x by an angle α around the unit vector n. The rotated vector is denoted by $x_\star$. The vector x is decomposed in a component $x_{\|}$ that is parallel to the axis n, and a component $x_{\perp}$ that is orthogonal to it.

It can first be recognized that the vector x can be decomposed into a component parallel to the axis n, denoted $x_{\|}$, and a component orthogonal to it, denoted $x_{\perp}$, as



$$x^{\mathrm{v}} = x_{\|}^{\mathrm{v}} + x_{\perp}^{\mathrm{v}}. \tag{3.10a}$$

Based on geometric reasoning we can conclude that

$$x_{\|}^{\mathrm{v}} = (x^{\mathrm{v}} \cdot n^{\mathrm{v}})\, n^{\mathrm{v}}, \tag{3.10b}$$

where $\cdot$ denotes the inner product. Similarly, $x_{\star}^{\mathrm{v}}$ can be decomposed

$$x_{\star}^{\mathrm{v}} = (x_{\star}^{\mathrm{v}})_{\|} + (x_{\star}^{\mathrm{v}})_{\perp}, \tag{3.11a}$$

where

$$(x_{\star}^{\mathrm{v}})_{\|} = x_{\|}^{\mathrm{v}}, \tag{3.11b}$$

$$(x_{\star}^{\mathrm{v}})_{\perp} = x_{\perp}^{\mathrm{v}} \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha. \tag{3.11c}$$

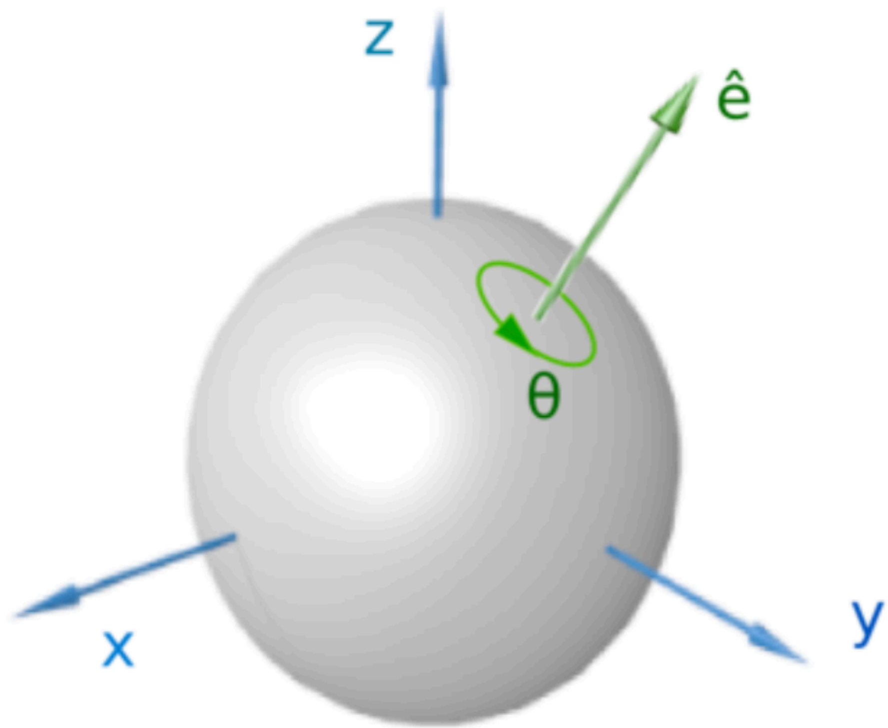
Hence, $x_{\star}^{\mathrm{v}}$ can be expressed in terms of x^{v} as

$$\begin{aligned} x_{\star}^{\mathrm{v}} &= (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}} + (x^{\mathrm{v}} - (x^{\mathrm{v}} \cdot n^{\mathrm{v}})n^{\mathrm{v}}) \cos\alpha + (x^{\mathrm{v}} \times n^{\mathrm{v}}) \sin\alpha \\ &= x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \end{aligned} \tag{3.12}$$

Denoting the rotated coordinate frame the u-frame and using the equivalence between $x_{\star}^{\mathrm{v}}$ and x^{u} as shown in Example 3.2, this implies that

$$x^{\mathrm{u}} = x^{\mathrm{v}} \cos\alpha + n^{\mathrm{v}}(x^{\mathrm{v}} \cdot n^{\mathrm{v}})(1 - \cos\alpha) - (n^{\mathrm{v}} \times x^{\mathrm{v}}) \sin\alpha. \tag{3.13}$$

This equation is commonly referred to as the *rotation formula* or *Euler's formula* [135]. Note that the combination of n and α, or $\eta = n\alpha$, is denoted as the *rotation vector* or the *axis-angle parameterization*.



- Visualizing a rotation represented by an Euler axis and angle.

Recall: Extension of Euler's formula

A Euclidean vector such as $(2, 3, 4)$ or (a_x, a_y, a_z) can be rewritten as $2\,\mathbf{i} + 3\,\mathbf{j} + 4\,\mathbf{k}$ or $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors representing the three Cartesian axes. A rotation through an angle of θ around the axis defined by a unit vector

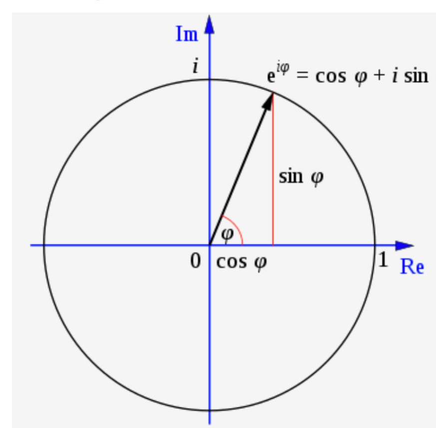
$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Recall: Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Inverse and Composition

$$\mathbf{q}^{-1} = e^{-\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} - (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}.$$

It follows that conjugation by the product of two quaternions is the composition of conjugations by these quaternions: If $\mathbf{p}$ and $\mathbf{q}$ are unit quaternions, then rotation (conjugation) by $\mathbf{pq}$ is

$$\mathbf{pq}\vec{v}(\mathbf{pq})^{-1} = \mathbf{pq}\vec{v}\mathbf{q}^{-1}\mathbf{p}^{-1} = \mathbf{p}(\mathbf{q}\vec{v}\mathbf{q}^{-1})\mathbf{p}^{-1},$$

which is the same as rotating (conjugating) by $\mathbf{q}$ and then by $\mathbf{p}$. The scalar component of the result is necessarily zero.

Look at Euler Angles one more time

Rotation can also be defined as a consecutive rotation around three axes in terms of so-called *Euler angles*. We use the convention (z, y, x) which first rotates an angle ψ around the z-axis, subsequently an angle θ around the y-axis and finally an angle ϕ around the x-axis. These angles are illustrated in Figure 3.5. Assuming that the v-frame is rotated by (ψ, θ, ϕ) with respect to the u-frame as illustrated in this figure, the rotation matrix R^{uv} is given by

$$\begin{aligned} R^{\mathrm{uv}} &= R^{\mathrm{uv}}(e_1,\phi)R^{\mathrm{uv}}(e_2,\theta)R^{\mathrm{uv}}(e_3,\psi) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}, \end{aligned} \tag{3.20}$$

where we make use of the notation introduced in (3.17) and the following definition of the unit vectors

$$e_1 = \begin{pmatrix}1 & 0 & 0\end{pmatrix}^{\mathsf{T}}, \qquad e_2 = \begin{pmatrix}0 & 1 & 0\end{pmatrix}^{\mathsf{T}}, \qquad e_3 = \begin{pmatrix}0 & 0 & 1\end{pmatrix}^{\mathsf{T}}. \tag{3.21}$$

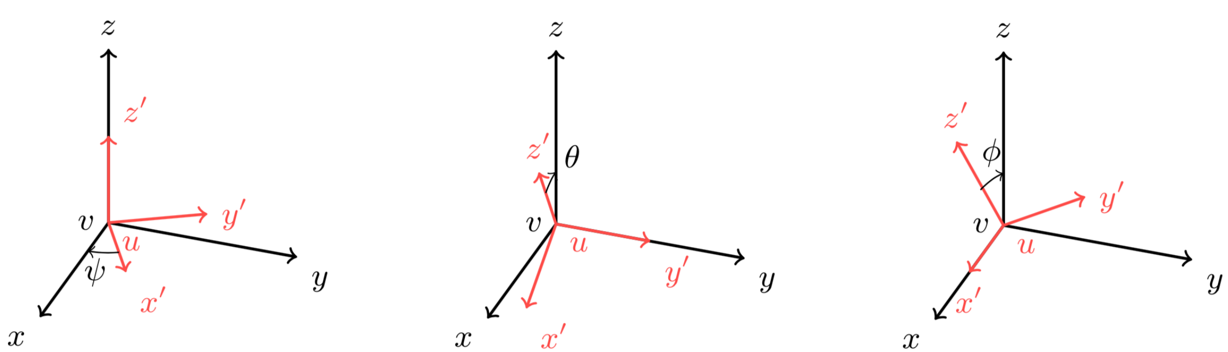


Figure 3.5: Definition of Euler angles as used in this work with left: rotation ψ around the z-axis, middle: rotation θ around the y-axis and right: rotation ϕ around the x-axis.

Gimbal Lock

Similar to the rotation vector, Euler angles parametrize orientation as a three-dimensional vector. Euler angle representations are not unique descriptions of a rotation for two reasons. First, due to wrapping of the Euler angles, the rotation $(0,0,0)$ is for instance equal to $(0,0,2\pi k)$ for any integer k. Furthermore, setting $\theta = \frac{\pi}{2}$ in (3.20), leads to

$$R^{\mathrm{uv}} = \begin{pmatrix} 0 & 0 & -1 \\ \sin\phi\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi & 0 \\ \cos\phi\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\psi - \sin\phi\cos\psi & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -1 \\ \sin(\phi-\psi) & \cos(\phi-\psi) & 0 \\ \cos(\phi-\psi) & -\sin(\phi-\psi) & 0 \end{pmatrix}. \tag{3.22}$$

Hence, only the rotation $\phi - \psi$ can be observed. Because of this, for example the rotations $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2}, -\frac{\pi}{2})$, $(\pi, \frac{\pi}{2}, \frac{\pi}{2})$ are all three equivalent. This is called *gimbal lock* [31].

- We will analyze further using quaternions next.

Key property of any rigid motion: preserves the distance!

$$d(g(\mathbf{X}), g(\mathbf{Y}))^2 = d(\mathbf{X}, \mathbf{Y})^2.$$

$$\begin{aligned} d(\mathbf{X}, \mathbf{Y})^2 &= (X_1 - Y_1)^2 + (X_2 - Y_2)^2 + \ldots + (X_n - Y_n)^2 \\ &= (\mathbf{X} - \mathbf{Y}) \cdot (\mathbf{X} - \mathbf{Y}). \end{aligned}$$

For the Euclidean distance d and

a rigid transformation g:$\mathrm{R}^n \rightarrow \mathrm{R}^n$

- Now let’s extend the concepts and results from SO(3) to SO(n).

Claim: The tangent plane of the Lie group SO(n) at the identity is the set of skew-symmetric matrices.

Work out details with the students on the board.

Even we are interested in n=3. All the derivations can be extended to dim=n.

$$SO(n) = \{Q \in \mathbb{R}^{n\times n} : Q^\top = Q^{-1},\ \det(Q) = 1\}.$$

The algebra of skew-symmetric matrices is denoted by

$$so(n) = \{S \in \mathbb{R}^{n\times n} : S^\top = -S\}$$

so(n) is called the Lie algebra of the Lie group SO(n).

What is a rigid motion/transformation?

A rigid transformation is formally defined as a transformation that, when acting on any vector **v**, produces a transformed vector $T(\mathbf{v})$ of the form

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

where $R^{\mathsf{T}} = R^{-1}$ (i.e., R is an orthogonal transformation), and **t** is a vector giving the translation of the origin.

A proper rigid transformation has, in addition,

$$\det(R) = 1$$

which means that R does not produce a reflection, and hence it represents a rotation (an orientation-preserving orthogonal transformation). Indeed, when an orthogonal transformation matrix produces a reflection, its determinant is -1.

Note: Rotation and Translation do not commute

$$T(\mathbf{v}) = R\,\mathbf{v} + \mathbf{t}$$

- This can be viewed as first rotating the vector v by using rotation matrix R, then translate by t.
- What if we first translate the vector v by t and then do the rotation by R?

Recall:

Given a square matrix $X \in \mathbb{R}^{n\times n}$,
the exponential of X is given by the absolute convergent power series

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}.$$

Exponential a skew symmetric matrix, we get a rotation matrix

- Same as for SO(3) case.

SE(n) also forms a Lie Group

$$SE(n) = \left\{ \begin{bmatrix} Q & \mathbf{u} \\ \mathbf{0} & 1 \end{bmatrix} : Q \in SO(n), \mathbf{u} \in \mathbb{R}^{n\cancel{\times 1}} \right\}.$$

While $SE(n)$ describes configurations, its Lie algebra $se(n)$, defined by

$$se(n) = \left\{ \begin{bmatrix} S & \mathbf{v} \\ \mathbf{0} & 0 \end{bmatrix} : S \in so(n), \mathbf{v} \in \mathbb{R}^{n\cancel{\times 1}} \right\},$$

- Note: Here we used the homogeneous representation of rotation and translation.

Geometric aspect of the exponential and logarithm

To any vector $\mathbf{u} = (u_1,\ u_2,\ u_3) \in \mathbb{R}^3$ one can associate the 3×3 skew-symmetric matrix

$$S_{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

It is easy to see that, for every $\mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = S_{\mathbf{u}}\,\mathbf{v},$$

where $\times$ stands for the cross product.

Matrix Rotation

Representing $\mathbf{v}$ and $\mathbf{k} \times \mathbf{v}$ as column matrices, the cross product can be expressed as a matrix product

$$\begin{bmatrix} (\mathbf{k} \times \mathbf{v})_x \\ (\mathbf{k} \times \mathbf{v})_y \\ (\mathbf{k} \times \mathbf{v})_z \end{bmatrix} = \begin{bmatrix} k_y v_z - k_z v_y \\ k_z v_x - k_x v_z \\ k_x v_y - k_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}.$$

Letting $\mathbf{K}$ denote the "cross-product matrix" for the unit vector $\mathbf{k}$,

$$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix},$$

the matrix equation is, symbolically,

$$\mathbf{K}\mathbf{v} = \mathbf{k} \times \mathbf{v}$$

for any vector $\mathbf{v}$. (In fact, $\mathbf{K}$ is the unique matrix with this property. It has eigenvalues 0 and $\pm i$).

Iterating the cross product on the right is equivalent to multiplying by the cross product matrix on the left, in particular

$$\mathbf{K}(\mathbf{K}\mathbf{v}) = \mathbf{K}^2\mathbf{v} = \mathbf{k} \times (\mathbf{k} \times \mathbf{v}).$$

Lie-Algebraic derivation of Rodrigues' Rotation Formula

Because $\mathbf{K}$ is skew-symmetric, and the sum of the squares of its above-diagonal entries is 1, the characteristic polynomial $P(t)$ of $\mathbf{K}$ is $P(t) = \det(\mathbf{K} - t\mathbf{I}) = -(t^3 + t)$. Since, by the Cayley–Hamilton theorem, $P(\mathbf{K}) = 0$, this implies that

$$\mathbf{K}^3 = -\mathbf{K}\,.$$

As a result, $\mathbf{K}^4 = -\mathbf{K}^2$, $\mathbf{K}^5 = \mathbf{K}$, $\mathbf{K}^6 = \mathbf{K}^2$, $\mathbf{K}^7 = -\mathbf{K}$.

This cyclic pattern continues indefinitely, and so all higher powers of $\mathbf{K}$ can be expressed in terms of $\mathbf{K}$ and $\mathbf{K}^2$. Thus, from the above equation, it follows that

$$R = I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\mathbf{K} + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)\mathbf{K}^2\,,$$

that is,

$$R = I + (\sin\theta)\mathbf{K} + (1 - \cos\theta)\mathbf{K}^2\,.$$

This is a Lie-algebraic derivation, in contrast to the geometric one in the article Rodrigues' rotation formula.[1]

Exponential map from $\mathfrak{so}(3)$ to SO(3)

The exponential map effects a transformation from the axis-angle representation of rotations to rotation matrices,

$$\exp\colon \mathfrak{so}(3) \to \mathrm{SO}(3)\,.$$

Essentially, by using a Taylor expansion one derives a closed-form relation between these two representations. Given a unit vector $\boldsymbol{\omega} \in \mathfrak{so}(3) = \mathbb{R}^3$ representing the unit rotation axis, and an angle, $\theta \in \mathbb{R}$, an equivalent rotation matrix R is given as follows, where $\mathbf{K}$ is the cross product matrix of $\boldsymbol{\omega}$, that is, $\mathbf{K}\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$ for all vectors $\mathbf{v} \in \mathbb{R}^3$,

$$R = \exp(\theta\mathbf{K}) = \sum_{k=0}^{\infty} \frac{(\theta\mathbf{K})^k}{k!} = I + \theta\mathbf{K} + \frac{1}{2!}(\theta\mathbf{K})^2 + \frac{1}{3!}(\theta\mathbf{K})^3 + \cdots$$

Log map from SO(3) to $\mathfrak{so}(3)$

Let $\mathbf{K}$ continue to denote the 3 × 3 matrix that effects the cross product with the rotation axis $\boldsymbol{\omega}$: $\mathbf{K}(\mathbf{v}) = \boldsymbol{\omega} \times \mathbf{v}$ for all vectors $\mathbf{v}$ in what follows.

To retrieve the axis–angle representation of a rotation matrix, calculate the angle of rotation from the trace of the rotation matrix

$$\theta = \arccos\left(\frac{\operatorname{Tr}(R) - 1}{2}\right)$$

and then use that to find the normalized axis,

$$\boldsymbol{\omega} = \frac{1}{2\sin\theta}\begin{bmatrix} R(3,2) - R(2,3) \\ R(1,3) - R(3,1) \\ R(2,1) - R(1,2) \end{bmatrix}.$$

Note also that the Matrix logarithm of the rotation matrix R is

$$\log R = \begin{cases} 0 & \text{if } \theta = 0 \\ \dfrac{\theta}{2\sin\theta}\left(R - R^{\mathsf{T}}\right) & \text{if } \theta \neq 0 \text{ and } \theta \in (-\pi, \pi) \end{cases}$$

An exception occurs when R has eigenvalues equal to −1. In this case, the log is not unique. However, even in the case where $\theta = \pi$ the Frobenius norm of the log is

$$\| \log(R) \|_{\mathrm{F}} = \sqrt{2}|\theta| \, .$$

Recall: ||A||^2 = sum of (a_{ij})^2 = tr(AA^t)

Logarithm of a Matrix

The exponential of a matrix *A* is defined by

$$e^A \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

Given a matrix *B*, another matrix *A* is said to be a **matrix logarithm** of B if $e^A = B$. Because the exponential function is not one-to-one for complex numbers (e.g. $e^{\pi i} = e^{3\pi i} = -1$), numbers can have multiple complex logarithms, and as a consequence of this, some matrices may have more than one logarithm, as explained below.

If *B* is sufficiently close to the identity matrix, then a logarithm of *B* may be computed by means of the following power series:

$$\log(B) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(B-I)^k}{k} = (B-I) - \frac{(B-I)^2}{2} + \frac{(B-I)^3}{3} \cdots.$$

Specifically, if $\|B - I\| < 1$, then the preceding series converges and $e^{\log(B)} = B$.[1]

Log B = log (I + B-I). Recall log (1+x) = x– x^2/2 – x^3/3 +… when |x| less than 1

Example:
Logarithm of rotations in the plane

The rotations in the plane give a simple example. A rotation of angle α around the origin is represented by the 2×2-matrix

$$A = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

For any integer n, the matrix

$$B_n = (\alpha + 2\pi n)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

is a logarithm of A. Thus, the matrix A has infinitely many logarithms. This corresponds to the fact that the rotation angle is only determined up to multiples of 2π.

In the language of Lie theory, the rotation matrices A are elements of the Lie group SO(2). The corresponding logarithms B are elements of the Lie algebra so(2), which consists of all skew-symmetric matrices. The matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is a generator of the Lie algebra so(2).