Lecture 3

Math 178

Nonlinear Data Analytics

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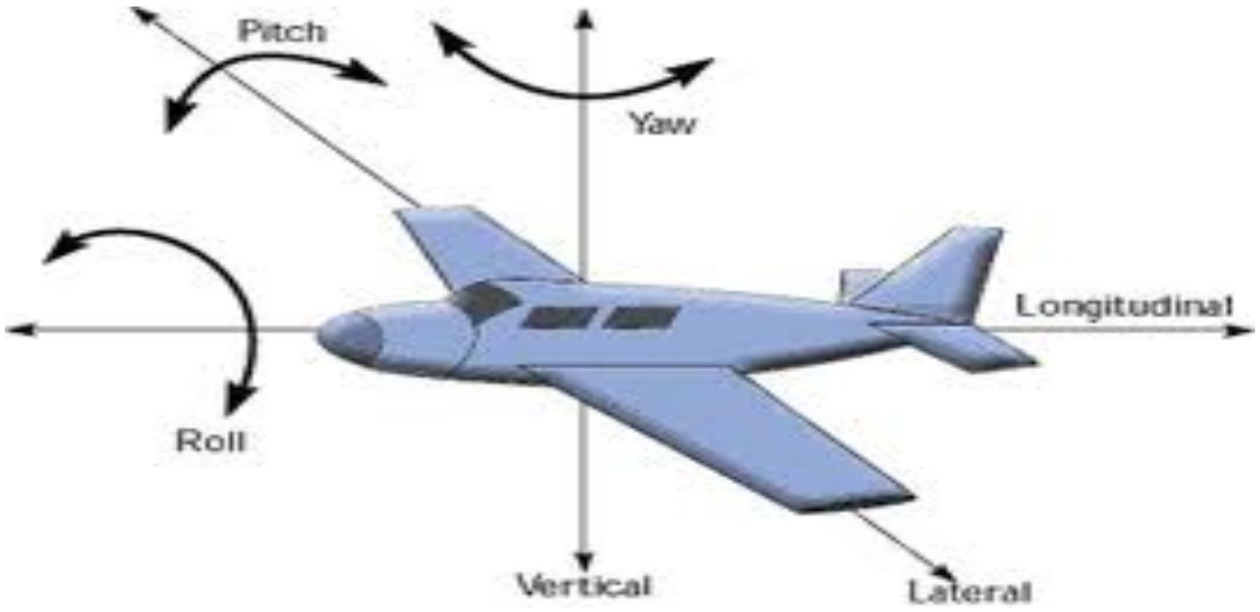
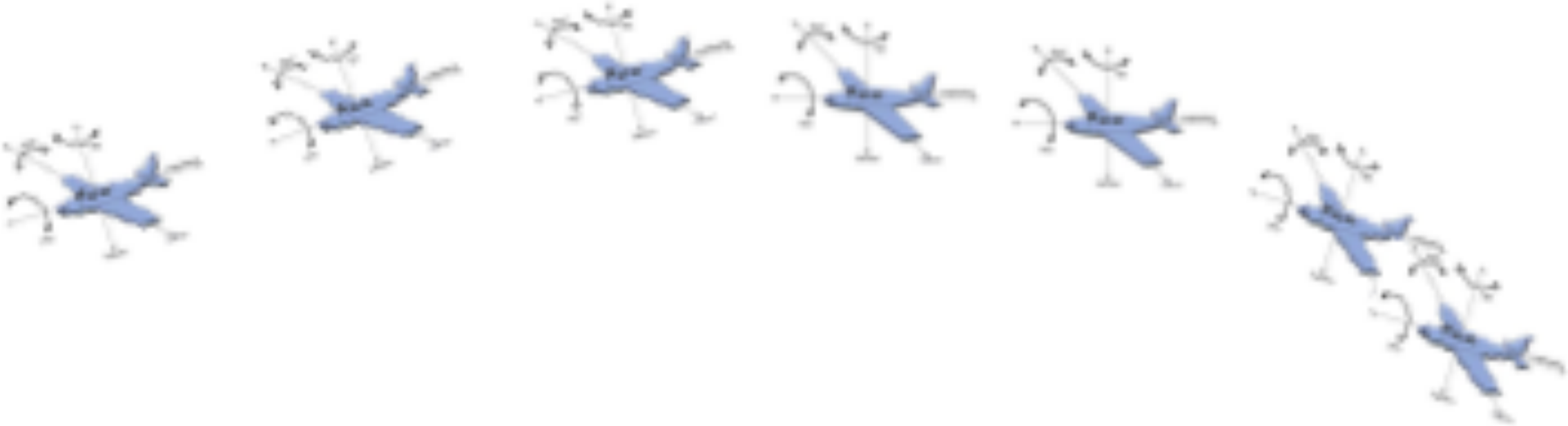
Configuration space of a robot

Definition: The **configuration space** of a **robot** is the **space** of possible positions the **robot** may attain.

- The configuration space of any auto car moving on R^2 consists two parts, 1) the set of translations in R^2 , and 2) the set of rotations in R^2
- The configuration space of any UVA consists two parts, 1) the set of translations in R^3 , and 2) the set of rotations in R^3
- The configuration space of any cell phone consists two parts, 1) the set of translations in R^3 , and 2) the set of rotations in R^3

- **How to model and capture the dynamics and kinematics of an UAV?**





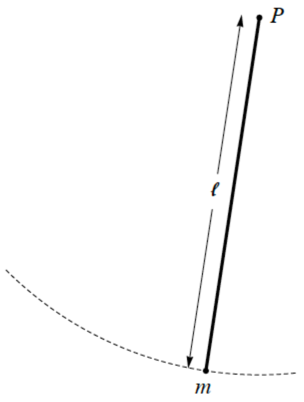
Example: Configuration space usually is a manifold

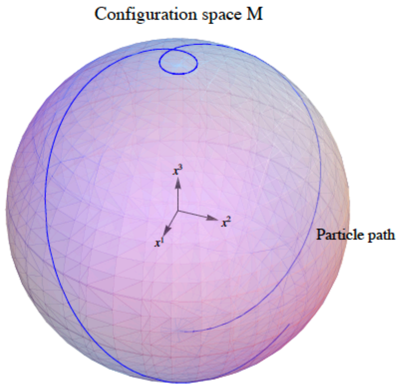
Mathematical Models and Physical Systems

When we wish to describe a physical system in a "mathematical" way we try to construct some sort of mathematical structure which, in some sense, "represents" those aspects of the system which are of interest to us. This structure is then a "mathematical model" of the physical system.

Example

A mass m is fixed on the end of a rigid rod of negligible mass having length ℓ. One end of the rod is fixed at a point P in space so that the mass can move about about P subject to the condition that it always be a distance ℓ from P. The sphere M (a *regular surface* or *manifold*) of all possible positions for m is called the *configuration space* of the system.





How to model the velocity of a robot?

Example (cont'd)

Suppose we are only interested in the motion of the particle. Then we take, as the state of the particle, the pair of three-dimensional vectors (x, v), $x = (x^1, x^2, x^3)$, $v = (v^1, v^2, v^3)$, where x is the position vector of m and v is the velocity vector of m (with respect to some Cartesian coordinate system).

Since the mass must stay on the sphere M, we see v must be tangent to M. Thus our *state space* S does not consist of all pairs of 3-vectors but, instead, we have the *tangent bundle* of M (which can also be viewed as a manifold);

$$S = \{(x, v) \mid x \in M \text{ and } v \text{ is tangent to } M \text{ at } x\}.$$

Although S is not a Euclidean space, nor an open set in one, we shall see that S is a space on which notions such as tangent vector, vector field, and time-dependent vector field have meaning. If we have a force field then the force field will determine a vector field on the state space S.

A simple anomaly detection example:

1. How do the "headings" of the following flight path look like?

- Only consider UAV heading

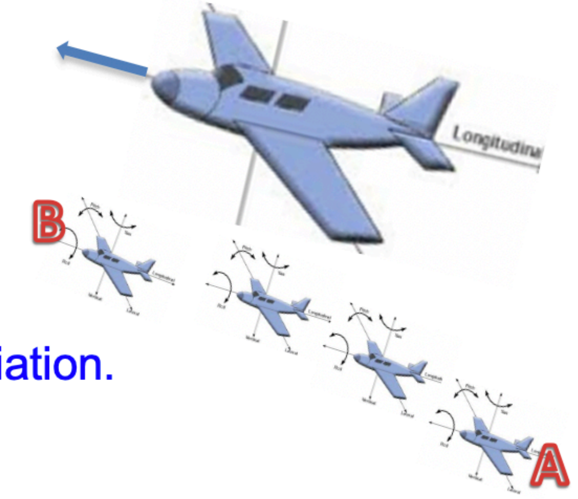
1. How do the "headings" of the following flight path look like?

- Only consider UAV heading Directions (depend on time)

2. Mission Objective: Transport a missal from site *A* to site *B* as soon as possible.

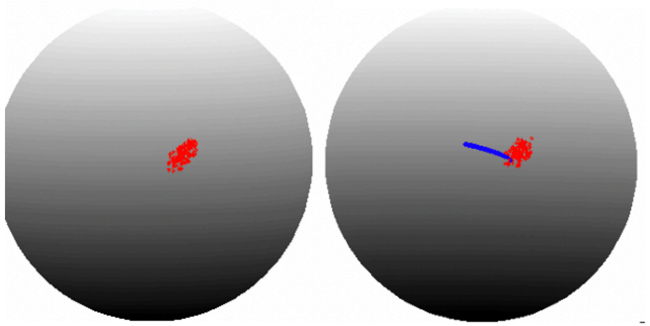
Q: *Dose the pilot have to constantly monitor the UAV?*

3. How to detect anomalies? (See figures below}.



4. Back-identify: a strong wind just started causing the deviation.

Figure L: Normal neighborhood of UAV headings in a specified direction;
Figure R: Deviation beyond bdry of normal nbhd considered anomalous



5. A warning auto issued

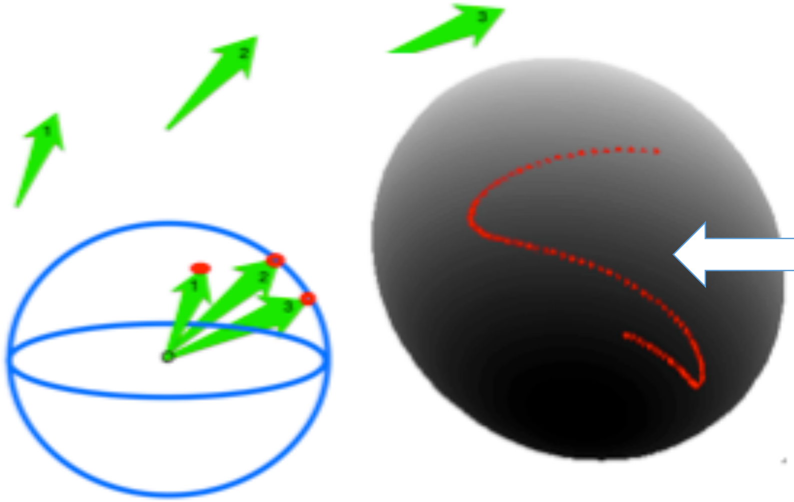
6. The operator corrected UAV's deviation from its mission path.

Note: Smaller the neighborhood, Less mission cost!

You may wonder: How to use manifold to study UAV data?
Simplest case: drawing a curve on a sphere
Try to capture characteristics of flight controls

- **For example: Only look at UAV "headings"**
- **All possible headings for all UAVs form a sphere.**

Only consider UAV heading directions here, but works for any other UAV characteristics



- ***Key: Developed a dimension-reduction technique for large nonlinear data.***

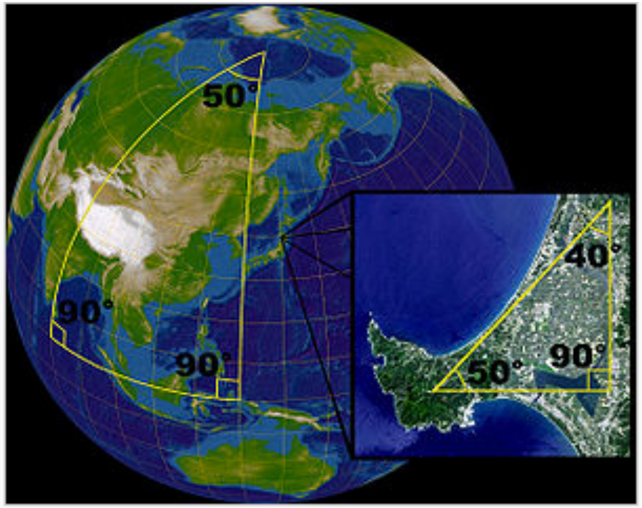
Just recording the heading while a UAV is flying gives a heading-behavior curve.

Rigid Body Kinematics

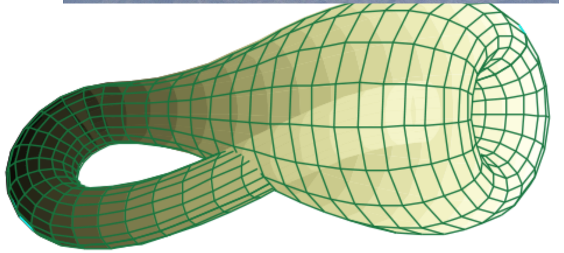
- The set of all 3-dimensional rotations is denoted by SO(3)
- Claim: SO(3) is a manifold, in fact SO(3) is also a group.
- A manifold structure + A group structure = Lie group
- Nonlinear data is in SO(3)
- Work out details with students on the board

What is a manifold?

- An n-dimensional manifold locally "looks like" a piece of $\mathbf{R}^n$.
- For examples, sphere and torus.
- Key features of a **manifold**: **curved**



The sphere (surface of a ball) is a two-dimensional manifold since it can be represented by a collection of two-dimensional maps.



- Only manifolds can capture UAV's dynamical behaviors

From Regular Surface to Manifold

Definition

A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

1. $\mathbf{x}$ is differentiable (so we can use calculus).

2. $\mathbf{x}$ is a homeomorphism (so we can use analysis)

3. $\mathbf{x}$ is regular (so we can use linear algebra)

Remark

In contrast to our treatment of curves, we have *defined a surface as a subset* S of $\mathbb{R}^3$, and not as a map. This is achieved by covering S with the traces of parametrizations which satisfy conditions 1, 2, and 3.

Exact meanings:

$\mathbf{x}$ is differentiable

This means that if we write

$$\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in U,$$

the functions $x(u, v)$, $y(u, v)$, and $z(u, v)$ have continuous partial derivatives of all orders.

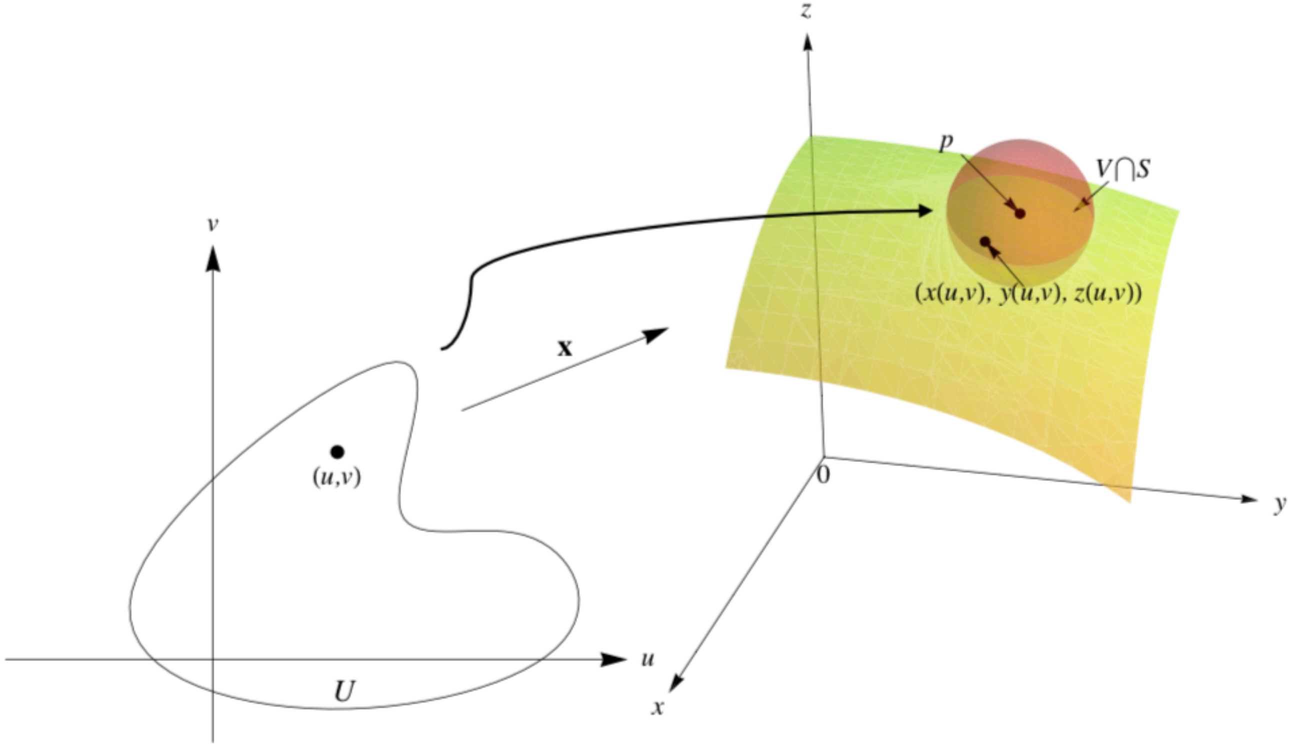
$\mathbf{x}$ is a homeomorphism

Since $\mathbf{x}$ is continuous by condition 1, this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

$\mathbf{x}$ is regular

For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

A Parametrization and a coordinate neighborhood



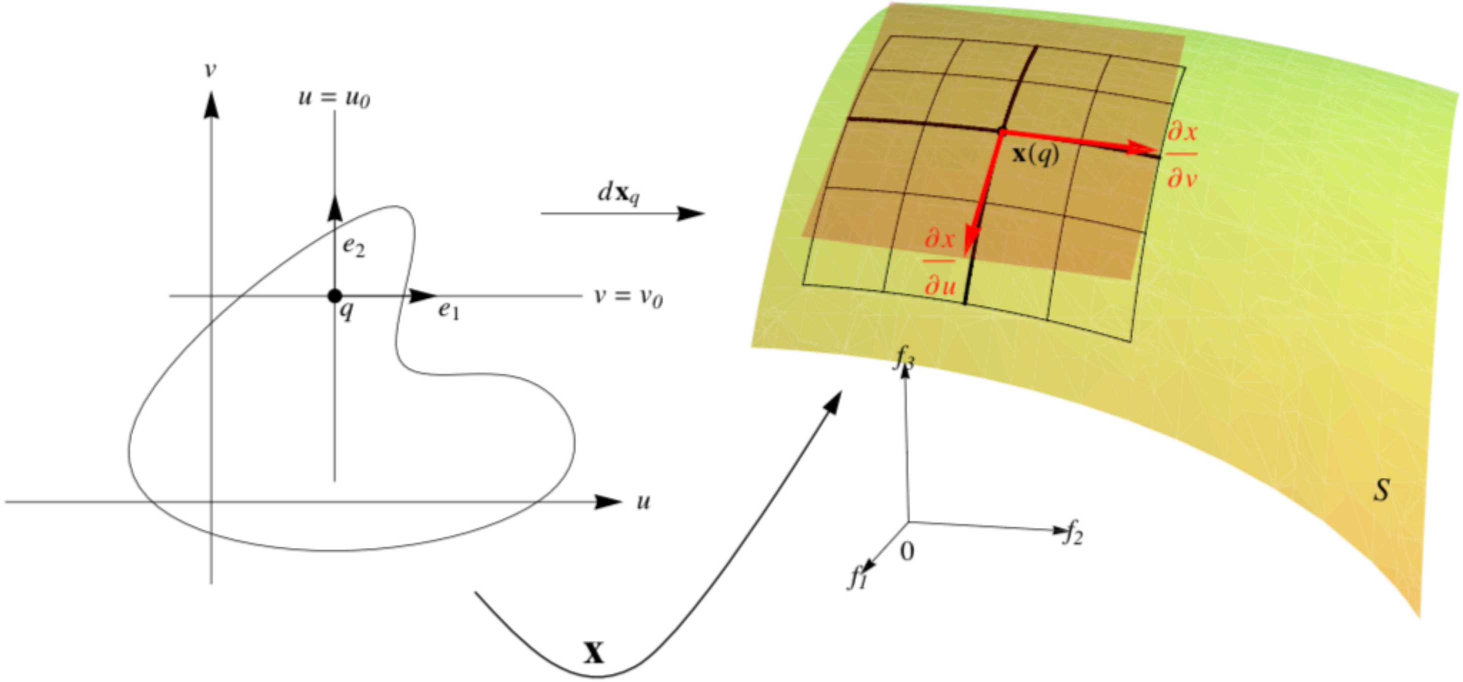
Definition

The mapping **x** is called a *parametrization* or a *system of (local) coordinates* in (a neighborhood of) p. The neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood*.

The Regularity Condition

An Illustrative Example

To give condition 3 a more familiar form, let us compute the matrix of the linear map $d\mathbf{x}_q$ in the canonical bases $e_1 = (1, 0)$, $e_2 = (0, 1)$ of $\mathbb{R}^2$ with coordinates u, v and $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$, $f_3 = (0, 0, 1)$ of $\mathbb{R}^3$, with coordinates (x, y, z).



The Regularity Condition

An Illustrative Example (cont'd)

Thus, the matrix of the linear map $d\mathbf{x}_q$ in the referred (standard) basis is

$$d\mathbf{x}_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}.$$

Condition 3 may now be expressed by requiring the two column vectors of this matrix to be linearly independent; or, equivalently, that the vector product $\partial \mathbf{x}/\partial u \wedge \partial \mathbf{x}/\partial v \neq 0$; or, in still another way, that one of the minors of order 2 of the matrix $d\mathbf{x}_q$, that is, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(x,z)}{\partial(u,v)},$$

be nonzero at q.

The Three Conditions

- Condition 1 is very natural if we expect to do some differential geometry on S.
- The one-to-oneness in condition 2 has the purpose of preventing self-intersections in regular surfaces. This is clearly necessary if we are to speak about, say, *the* tangent plane at a point $p \in S$. The continuity of the inverse in condition 2 has a more subtle purpose. For the time being, we shall mention that this condition is essential to proving that certain objects defined in terms of a parametrization do not depend on this parametrization but only on the set S itself.
- Finally, condition 3 will guarantee the existence of a "tangent plane" at all points of S.

Proving that a Set is a Regular Surface

Example

Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

Method 1: Using Cartesian Coordinates

We first verify that the map $\mathbf{x}_1 : U \in \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{x}_1(x, y) = (x, y, +\sqrt{1 - (x^2 + y^2)}), \quad (x, y) \in U,$$

where $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ and
$U = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ is a parametrization of S^2.

Proving that a Set is a Regular Surface

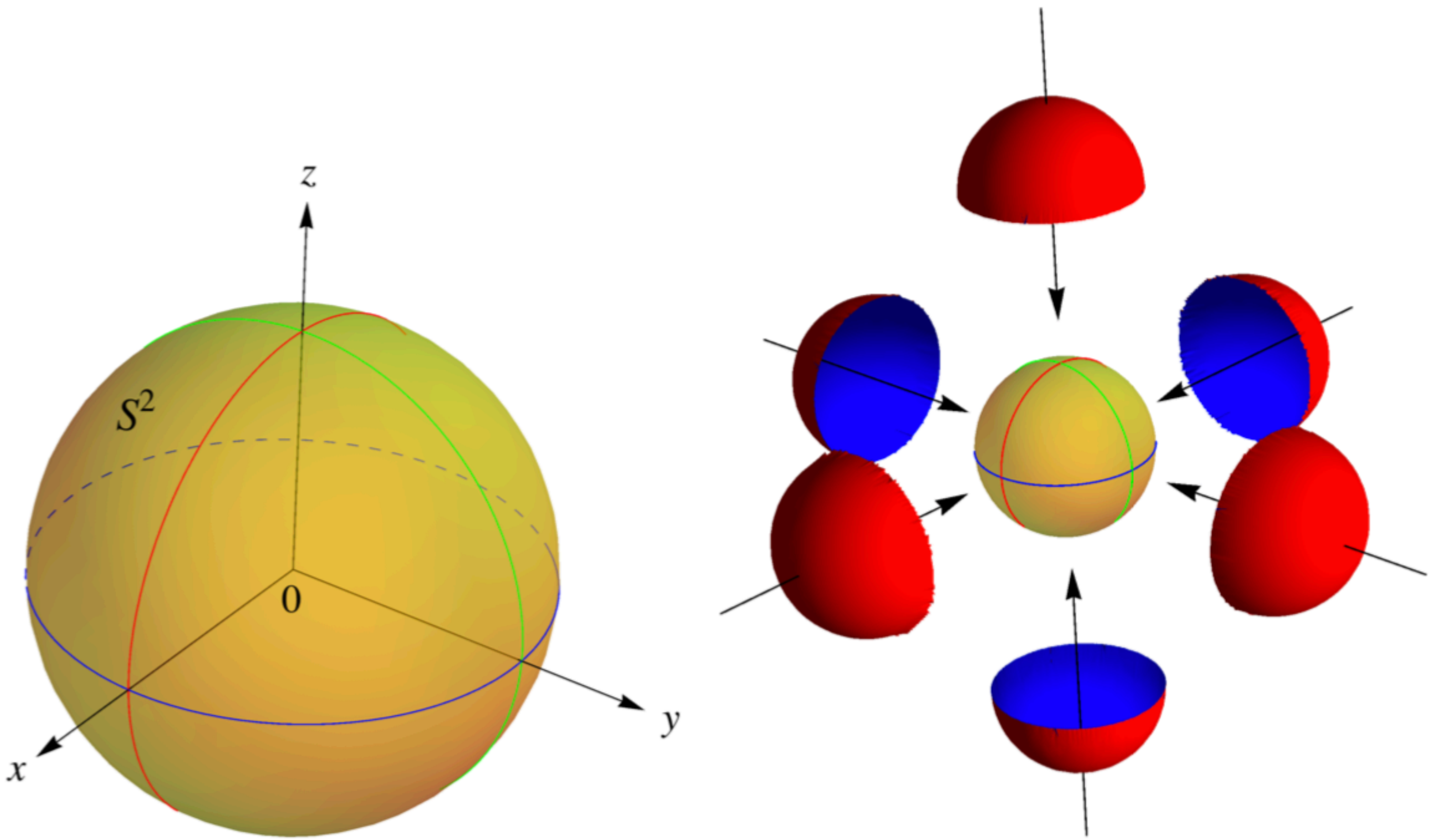
We shall now cover the whole sphere with similar parametrizations as follows. we define $\mathbf{x}_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{x}_2(x, y) = (x, y, -\sqrt{1 - (x^2 + y^2)}),$$

check that $\mathbf{x}_2$ is a parametrization, and observe that $\mathbf{x}_1(U) \cup \mathbf{x}_2(U)$ covers S^2 minus the equator $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$. Then, using the xz and zy planes, we define the parametrization

$$\begin{aligned}
\mathbf{x}_3(x, z) &= (x, +\sqrt{1 - (x^2 + z^2)}, z),\\
\mathbf{x}_4(x, z) &= (x, -\sqrt{1 - (x^2 + z^2)}, z),\\
\mathbf{x}_5(y, z) &= (+\sqrt{1 - (y^2 + z^2)}), y, z),\\
\mathbf{x}_6(y, z) &= (-\sqrt{1 - (y^2 + z^2)}), y, z),
\end{aligned}$$

which, together with $\mathbf{x}_1$ and $\mathbf{x}_2$, cover S^2 completely and shows that S^2 is a regular surface.



z
S^2
0
x
y

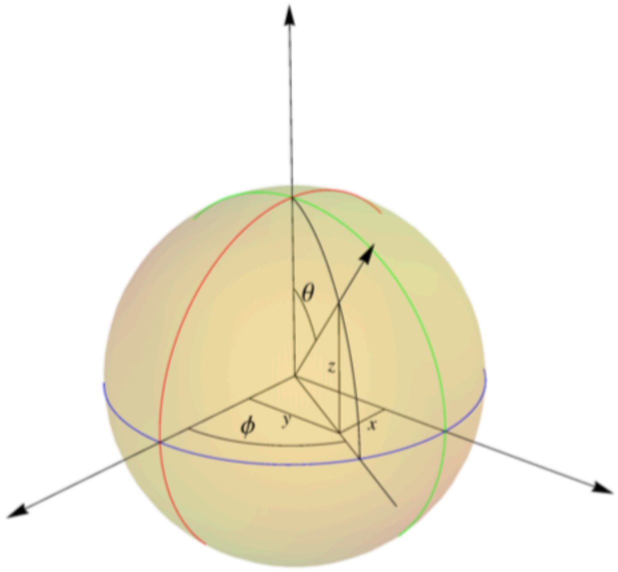
Proving that a Set is a Regular Surface

Method 2: Using Spherical Coordinates

For most applications, it is convenient to relate parametrizations to the geographical coordinates on S^2. Let $V = \{(\theta, \varphi) \mid 0 < \theta < \pi, 0 < \varphi < 2\pi\}$ and let $\mathbf{x} : V \to \mathbb{R}^3$ be given by

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

Clearly, $\mathbf{x}(V) \subset S^2$.



Proving that a Set is a Regular Surface

We shall prove that $\mathbf{x}$ is a parametrization of S^2.

Next, we observe that given $(x, y, z) \in S^2 \setminus C$, where C is the semicircle $C = \{(x, y, z) \in S^2 \mid y = 0, x \geq 0\}$, θ is uniquely determined by $\theta = \cos^{-1} z$, since $0 < \theta < \pi$. By knowing θ, we find $\sin\varphi$ and $\cos\varphi$ from $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, and this determines φ uniquely $(0 < \varphi < 2\pi)$. It follows that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1}$. To complete the verification of condition 2, we should prove that $\mathbf{x}^{-1}$ is continuous. However, since we shall soon prove that this verification is not necessary provided we already know that the set S is a regular surface, we shall not do that here.

We remark that $\mathbf{x}(V)$ only omits a semicircle of S^2 (including the two poles) and that S^2 can be covered with the coordinate neighborhoods of two parametrizations of this type.

HW1: Show that a sphere is a regular surface using spherical coordinates.

- Reference: Differential geometry of curves and surfaces, by do Carmo.

Two Shortcuts

The last example in the previous lecture shows that deciding whether a given subset of $\mathbb{R}^3$ is a regular surface directly from the definition may be quite tiresome.

Shortcut 1

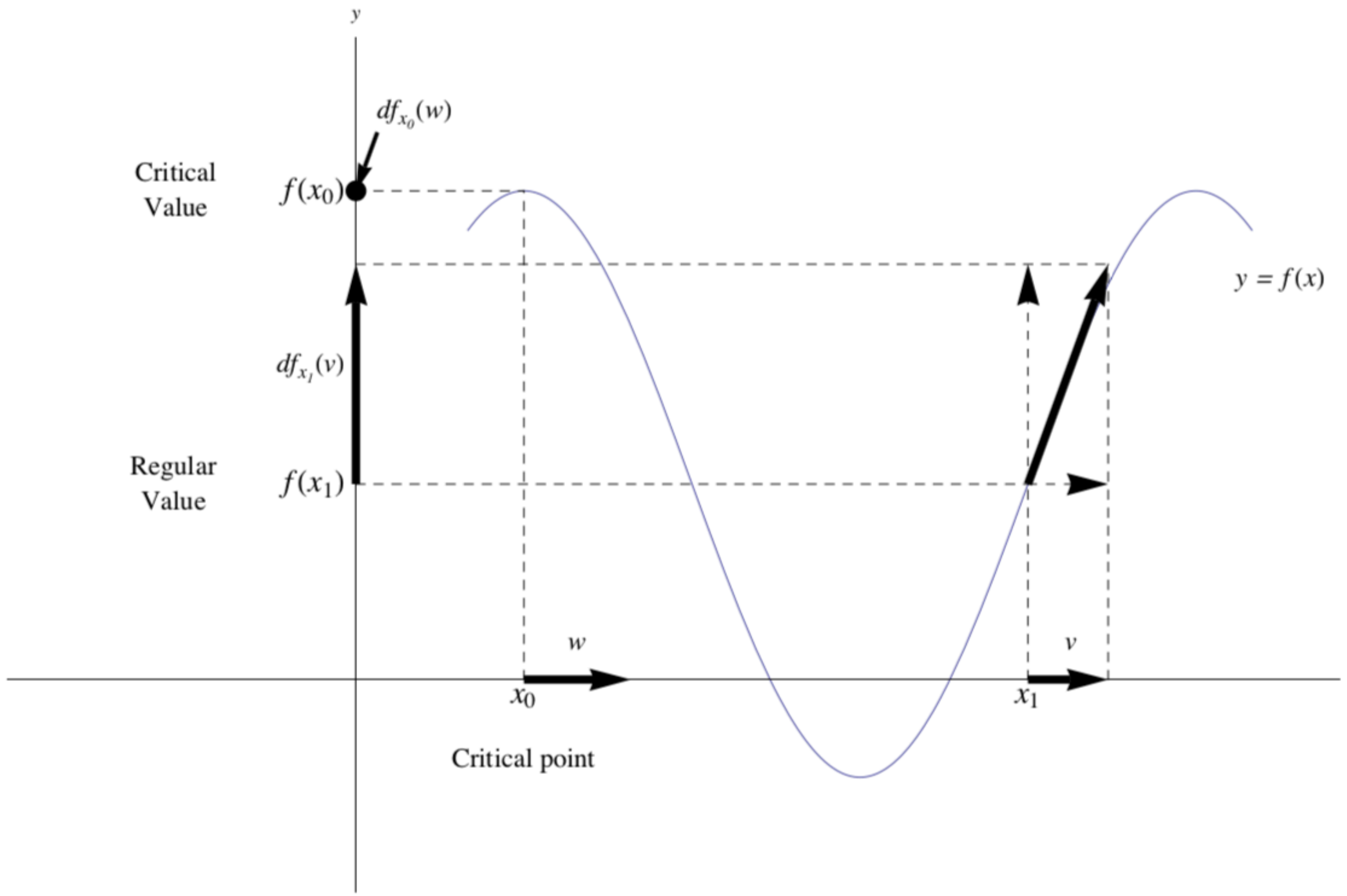
If $f : U \to \mathbb{R}$ is a differentiable function in an open set U of $\mathbb{R}^2$, then the graph of f, that is, the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ for $(x, y) \in U$, is a regular surface

Critical Points and Values

Definition

Given a differentiable map $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ defined in an open set U of $\mathbb{R}^n$ we say that $p \in U$ is a *critical point* of F if the differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not a surjective (or onto) mapping. The image $F(p) \in \mathbb{R}^m$ of a critical point is called a *critical value* of F. A point of $\mathbb{R}^m$ which is not a critical value is called a *regular value* of F.

The terminology is evidently motivated by the particular case in which $f : U \subset \mathbb{R} \to \mathbb{R}$ is a real-valued function of a real variable. A point $x_0 \in U$ is critical if $f'(x_0) = 0$, that is, if the differential df_{x_0} carries all the vectors in $\mathbb{R}$ to the zero vector. Notice that any point $a \notin f(U)$ is trivially a regular value of f.



y
$df_{x_0}(w)$
Critical
Value
$f(x_0)$
$y = f(x)$
$df_{x_1}(v)$
Regular
Value
$f(x_1)$
w
v
x_0
x_1
Critical point

Critical Points and Values

Remark

If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function, then

$$df_p = (f_x, f_y, f_z).$$

Note, in this case, that to say that df_p is not surjective is equivalent to saying that $f_x = f_y = f_z = 0$ at p. Hence, $a \in f(U)$ is a regular value of $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ if and only if f_x, f_y, and f_z do not vanish simultaneously at any point in the inverse image

$$f^{-1}(a) = \{(x, y, z) \in U \mid f(x, y, z) = a\}.$$

Two Shortcuts

Shortcut 2

If $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

Examples

Example

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface.

The examples of regular surfaces presented so far have been connected subsets of $\mathbb{R}^3$. A surface $S \subset \mathbb{R}^3$ if said to be *connected* if any two of its points can be joined by a continuous curve in S. In the definition of a regular surface we made no restrictions on the connectedness of the surfaces, and the following example shows that the regular surfaces given by Shortcut 2 may not be connected.

WTS:

$$\underbrace{\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}_{f(x,y,z)}=1$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$

$$(x,y,z) \longmapsto f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

$$\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right)=(0,0,0)$$

$$\Rightarrow \begin{cases} x=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow \text{the critical pt is only } (0,0,0).$$

The only

$\Rightarrow$ critical value is $f(0,0,0) = \frac{0^2}{a^2} + \frac{0^2}{b^2} + \frac{0^2}{c^2} = \underline{\underline{0}}$

$\Rightarrow$ so all the values "a" s.t $a \neq 0$ are regular values. $\Rightarrow$ 1 is an regular value.

$f^{-1}(1) = \{ (x, y, z) \in \mathbb{R}^3 \mid \underbrace{f(x, y, z)}_{x^2/a^2 + y^2/b^2 + z^2/c^2} = 1 \}$ be a regular surface.

Show S^3 is a manifold.

- Work out details with the students on the iPad.

Now we can show S^3 is a manifold in $\mathbb{R}^4$.

$$S^3 = \{ \underbrace{(x, y, z, w)}_{\in \mathbb{R}^4} \mid x^2 + y^2 + z^2 + w^2 = 1 \}$$

Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$(x, y, z, w) \longmapsto x^2 + y^2 + z^2 + w^2$

$$\nabla f = (2x, 2y, 2z, 2w) = (0, 0, 0, 0)$$

$\Rightarrow (x, y, z, w) = (0, 0, 0, 0)$ is the only critical pt of f.

$\Rightarrow$ The only critial value is $f(0, 0, 0, 0) = 0$

$\Rightarrow$ The only critial value is $f(0,0,0,0) = 0$

$\Rightarrow$ 1 is a regular value

$$\Rightarrow f^{-1}(1) = \{(x,y,z,w) \mid \underbrace{f(x,y,z,w)}_{x^2+y^2+z^2+w^2} = 1\} = S^3$$

is a regular 3-D manifold.

Q: Why there are 3 variables x, y, and z, but S^2 is two dimensional?

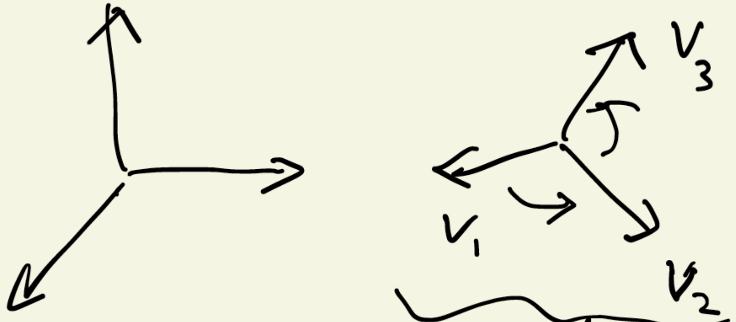
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

3 (variables) − 1 (constraint) = 2

What is SO(3)?

- The set of rotation matrix in R^3 is denoted by SO(3).

$$SO(3) = \left\{ A = (\vec{V}_1, \vec{V}_2, \vec{V}_3) \;\middle|\; \begin{array}{c} A^TA = AA^T = I \\ \det A = 1 \end{array} \right\}$$



$$\underbrace{\text{right hand sided}}$$ $\Rightarrow \det A = 1$

$$\left\{ \begin{array}{l} \|V_i\| = 1, \quad i = 1,2,3 \\ V_i \circ V_j = \delta_{ij} \end{array} \right.$$

$\cancel{\vec{V}_1 \cdot \vec{V}_2} \to 0$

$$A^TA = \underbrace{\begin{bmatrix} \vec{V}_1^{\,T} \\ \vec{V}_2^{\,T} \\ \vec{V}_3^{\,T} \end{bmatrix}}_{A^T} \underbrace{\left[\vec{V}_1, \vec{V}_2, \vec{V}_3 \right]}_{A} = \underbrace{\begin{bmatrix} \cancel{\vec{V}_1^{\,T} \vec{V}_1} \to 1 & \cancel{\vec{V}_1^{\,T} \vec{V}_2} \to 0 & \cancel{\vec{V}_1^{\,T} \vec{V}_3} \to 0 \\ \cancel{\vec{V}_2^{\,T} \vec{V}_1} \to 0 & \cancel{\vec{V}_2^{\,T} \vec{V}_2} \to 1 & \cancel{\vec{V}_2^{\,T} \vec{V}_3} \to 0 \\ \cancel{\vec{V}_3^{\,T} \vec{V}_1} \to 0 & \cancel{\vec{V}_3^{\,T} \vec{V}_2} \to 0 & \cancel{\vec{V}_3^{\,T} \vec{V}_3} \to 1 \end{bmatrix}}_{I}$$

$$(a_1, b_1, c_1) \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = (a_1, b_1, c_1) \bullet (a_2, b_2, c_2)$$

Show SO(3) is a manifold.

- Work out details with the students on the iPad.

<u>Claim</u>: $SO(3) = S^3/\{\pm 1\}$

$\underline{\underline{\text{Pf}}}$: Let $q \in \mathbb{H}$ w/ $\|q\| = 1$. (quaternion number)

For each q, I am going to define a map, call R_q.

$R_q : \operatorname{Im}\mathbb{H} \to \operatorname{Im}\mathbb{H}$

R_q can be viewed from $\underbrace{\operatorname{Im}\mathbb{H}}_{} \to \underbrace{\operatorname{Im}\mathbb{H}}_{}$

$R_q : \mathbb{R}^3 \to \mathbb{R}^3$

by first define $R_q : \mathbb{H} \to \mathbb{H}$

$$x \longmapsto q x q^*$$

Recall: $q^* = a - bi - cj - dk$

$q = a + bi + cj + dk$

Note: R_q fixes all real numbers r.

$r \longmapsto \underbrace{q r q^*}_{r q q^*}$, $\because r \in \mathbb{R}$, $\underbrace{r q q^*}_{r\|q\|^2}$, $\underbrace{\|q\|^2}_{1}$

$\Downarrow$

$R_q(r) = r$

<u>Claim</u>: R_q is a <u>rotation</u> in $\mathbb{R}^3$.

That is to say: we need to show

$R_q \in SO(3)$

1) R is Linear map: $\mathbb{R}^3 \to \mathbb{R}^3$.
2) R keeps length.

} will emply that R is an orthogonal map.

3) $\det R = 1$.

To show 1), let $R_q(ax+by) = q(ax+by)q^*$, $a, b \in \mathbb{R}$

$$= qaxq^* + qbyq^*$$

$$= a\,qxq^* + b\,qyq^* = a R_q(x) + b R_q(y)$$

$\Rightarrow R$ is linear ✓

To show 2) $\|R_q(x)\| = \|qxq^*\| = \underbrace{\|q\|}_{1}\|x\|\underbrace{\|q^*\|}_{1}$
$= \|x\|$.

Note:

The set of orthonormal matrices have two components.

$$\left\{ A \in M_3(\mathbb{R}) \;\middle|\; \underbrace{A^T A}_{\shortparallel \atop AA^T} = I \right\} = O(3)$$

If $A^TA = I$, then determinant both sides

$$\det(A^TA) = \det I$$

$$\Rightarrow \underbrace{\det A^T}_{(\det A)} \det A = 1$$

$$\Rightarrow (\det A)^2 = 1$$

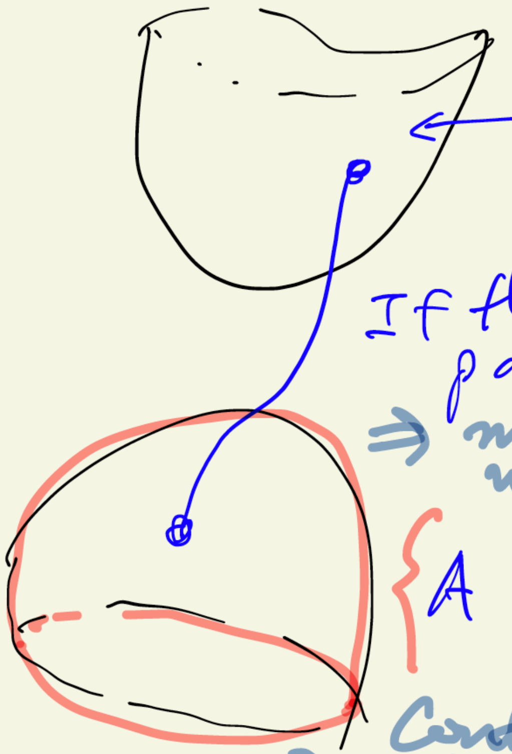
$$\Rightarrow \det A = 1$$

or $\det A = -1$

Ca use analysis to prove it.

But det is a continuous fcn

$\Rightarrow O(3)$ must be not (Path) connected.



$\{A \mid \det A = 1\}$

If there were path connected $\Rightarrow$ must $\exists$ some matrix B s.t. $\det B = 0$

$\{A \mid \det A = -1\}$

Contradiction

$\Rightarrow O(3)$ must disconnected.

Note: unit quaternion q and –q correspond to the same rotation.

Note:

$$R_q = R_{-q}$$

$\forall x \in \mathbb{H}$

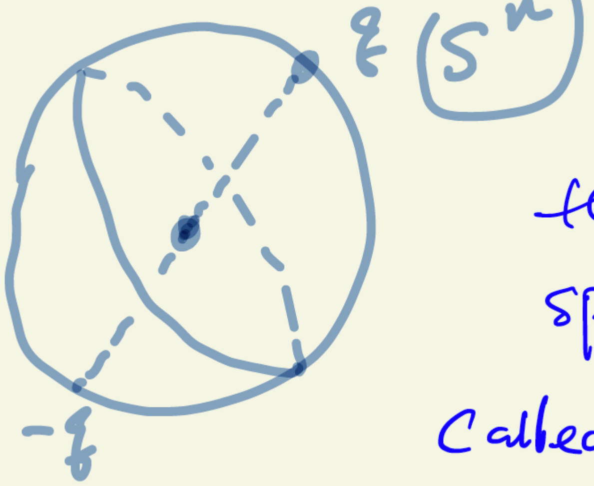
$$R_q(x) = q x q^*, \quad q^* \text{ is a conjugate of } q.$$

$$R_{-q}(x) = (-q) x (-q)^* = q x q^*$$

$$\Rightarrow R_q = R_{-q}.$$

$$\underbrace{q = \underline{a} + bi + cj + dk}_{1} \quad \Rightarrow R_q \in SO(3).$$

From manifold point view, SO(3) is the manifold RP^3



Identify ξ w/ $-\xi$ on S^n, the resulting space is called a real projective space, denoted by $\mathbb{R}P^n$.

$$S^n/\{\pm 1\} \cong \mathbb{R}P^n$$

$$SO(3) = S^3/\{\pm 1\} = \mathbb{R}P^3$$

Thm: Any Projective space $\mathbb{R}P^n$ is a manifold of dim n.

So $SO(3) = \mathbb{R}P^3$ is a <u>mfld</u>.

Why? locally it is just a piece of S^3 after identification under q w/ $-q$;

$\Rightarrow$ any piece of S^3 locally "looks like" $\mathbb{R}^3$.

Now Show SO(3) is also a group.

- Work out details with the students on the iPad.

In fact, SO(3) is also a group.

Now we show that SO(3) is also a gp.

(HW).

Hint:

i) $\forall A, B \in SO(3)$

W.T.S. $AB \in SO(3)$.

why: $(AB)(AB)^T$

$= A\cancel{B}\cancel{B^T} A^T$

$= AA^T = I$ ✓

$\det(AB) = 1$ ✓

$\underbrace{}_{\det A \cdot \det B}$

$\det A = 1$, $\det B = 1$

$I \in SO(3)$? ✓

$A \in SO(3)$

$A^{-1} \overset{?}{\in} SO(3)$

SO(3) has both manifold and group structures! Moreover those two structures are compatible! In such a case we say SO(3) is a Lie Group!

Now SO(3) is both a $\underset{\text{structure}}{\underline{\text{manifold}}}$, also a $\underset{\text{group structure}}{\underline{\text{gp.}}}$

We must make sure the two structures being compatible.

$$\left.\begin{array}{l} (AB)' \\ (A^{-1})' \end{array}\right\} \text{both exist.}$$

We will give a rigorous definition of a Lie group in our next lecture.