Lecture 5

Math 178

Nonlinear Data Analytics

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OVERVIEW

- **SO(n)**
- **Tangent Plane of SO(n)**
- **Lie algebra of the Lie group of SO(n)**
- **Exponential map**
- **Rigid body Kinematics**

Extension of Euler's formula

A Euclidean vector such as $(2, 3, 4)$ or (a_x, a_y, a_z) can be rewritten as $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ or $a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors representing the three Cartesian axes. A rotation through an angle of θ around the axis defined by a unit vector

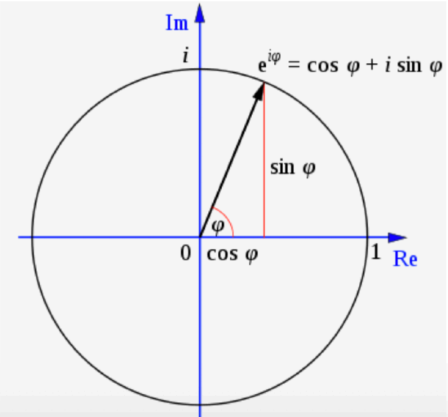
$$\vec{u} = (u_x, u_y, u_z) = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

can be represented by a quaternion. This can be done using an extension of Euler's formula:

$$\mathbf{q} = e^{\frac{\theta}{2}(u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k})} = \cos\frac{\theta}{2} + (u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k})\sin\frac{\theta}{2}$$

Recall: Euler's formula:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$



Euler Parameters from Angle and Axis

Any central rotation in three dimensions is uniquely determined by its axis of rotation (represented by a unit vector $\mathbf{k} = (k_x, k_y, k_z)$) and the rotation angle φ. The Euler parameters for this rotation are calculated as follows.

$$a = \cos\frac{\varphi}{2}$$
$$b = k_x \sin\frac{\varphi}{2}$$
$$c = k_y \sin\frac{\varphi}{2}$$
$$d = k_z \sin\frac{\varphi}{2}$$

Lie Algebra of $SO(n)$

The Lie algebra $so(n)$ of $SO(n)$ is given by

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \left\{X \in M_n(\mathbb{R}) \,\middle|\, X = -X^{\mathrm{T}}\right\} ,$$

and is the space of skew-symmetric matrices of dimension n, see classical group, where $o(n)$ is the Lie algebra of $O(n)$, the orthogonal group. For reference, the most common basis for $so(3)$ is

[A, B] = AB-BA

$$L_{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L_{\mathbf{y}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad L_{\mathbf{z}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Definition of a Lie algebra [edit]

A Lie algebra is a vector space $\mathfrak{g}$ over some field F together with a binary operation $[\,\cdot\,,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket satisfying the following axioms:[nb 1]

- Bilinearity,

$$[ax + by, z] = a[x, z] + b[y, z],$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

  for all scalars *a*, *b* in *F* and all elements *x*, *y*, *z* in $\mathfrak{g}$.

- Alternativity,

$$[x, x] = 0$$

  for all *x* in $\mathfrak{g}$.

- The Jacobi identity,

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

  for all *x*, *y*, *z* in $\mathfrak{g}$.

Rodrigues' Rotation Formula

The Rodrigues' formula provides an algorithm to compute the exponential map from $so(3)$, the Lie algebra of $SO(3)$, to $SO(3)$ without actually computing the full matrix exponential.

If $\mathbf{v}$ is a vector in $\mathbb{R}^3$ and $\mathbf{k}$ is a unit vector describing an axis of rotation about which $\mathbf{v}$ rotates by an angle θ according to the right hand rule, the Rodrigues formula for the rotated vector $\mathbf{v}_rot$ is

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v}\cos\theta + (\mathbf{k}\times\mathbf{v})\sin\theta + \mathbf{k}\ (\mathbf{k}\cdot\mathbf{v})(1-\cos\theta)$$

Exponential Map for $SO(3)$

Connecting the Lie algebra to the Lie group is the exponential map, which is defined using the standard matrix exponential series for e^A For any skew-symmetric matrix A, $\exp(A)$ is always a rotation matrix.

$$
\begin{aligned}
\exp(A) &= \exp\left(\theta(\mathbf{u}\cdot\mathbf{L})\right) \\
&= \exp\left(\begin{bmatrix} 0 & -z\theta & y\theta \\ z\theta & 0 & -x\theta \\ -y\theta & x\theta & 0 \end{bmatrix}\right) \\
&= I + \sin\theta\ \mathbf{u}\cdot\mathbf{L} + (1-\cos\theta)(\mathbf{u}\cdot\mathbf{L})^2,
\end{aligned}
$$

where $\mathbf{u}\cdot\mathbf{L} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$.

Matrix Notation

Representing $\mathbf{v}$ and $\mathbf{k} \times \mathbf{v}$ as column matrices, the cross product can be expressed as a matrix product

$$\begin{bmatrix} (\mathbf{k} \times \mathbf{v})_x \\ (\mathbf{k} \times \mathbf{v})_y \\ (\mathbf{k} \times \mathbf{v})_z \end{bmatrix} = \begin{bmatrix} k_y v_z - k_z v_y \\ k_z v_x - k_x v_z \\ k_x v_y - k_y v_x \end{bmatrix} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Letting $\mathbf{K}$ denote the "cross-product matrix" for the unit vector $\mathbf{k}$,

$\mathbf{K} = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$ the matrix equation is, symbolically,

$$\mathbf{K}\mathbf{v} = \mathbf{k} \times \mathbf{v}$$

Compact Notation

Moreover, since $\mathbf{k}$ is a unit vector, $\mathbf{K}$ has unit 2-norm. The previous rotation formula in matrix language is therefore

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{v} + (\sin\theta)\mathbf{K}\mathbf{v} + (1-\cos\theta)\mathbf{K}^2\mathbf{v}\,, \quad \|\mathbf{K}\|_2 = 1$$

Factorizing the $\mathbf{v}$ allows the compact expression

$$\mathbf{v}_{\mathrm{rot}} = \mathbf{R}\mathbf{v}$$

where

$$\mathbf{R} = \mathbf{I} + (\sin\theta)\mathbf{K} + (1-\cos\theta)\mathbf{K}^2$$

Euler-Rodrigues Formula

A rotation about the origin is represented by four real numbers, a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = 1$. When the rotation is applied, a point at position $\mathbf{x}$ rotates to its new position

$$\mathbf{x}' = \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(bc - ad) & 2(bd + ac) \\ 2(bc + ad) & a^2 + c^2 - b^2 - d^2 & 2(cd - ab) \\ 2(bd - ac) & 2(cd + ab) & a^2 + d^2 - b^2 - c^2 \end{bmatrix} \mathbf{x}$$

Skew Parameters via Cayley's Formula

$$\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \mapsto$$

$$\frac{1}{1+x^2+y^2+z^2} \begin{bmatrix} 1+x^2-y^2-z^2 & 2xy-2z & 2y+2xz \\ 2xy+2z & 1-x^2+y^2-z^2 & 2yz-2x \\ 2xz-2y & 2x+2yz & 1-x^2-y^2+z^2 \end{bmatrix}.$$

Combine a rotation and a translation into one matrix

The Lie group $SE(3)$

$$SE(3) = \left\{ \mathbf{A} \;\middle|\; \mathbf{A} = \left[\begin{array}{c|c} \mathbf{R} & \mathbf{r} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right], \mathbf{R} \in R^{3\times 3}, \mathbf{r} \in R^3, \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}, |\mathbf{R}| = 1 \right\}$$

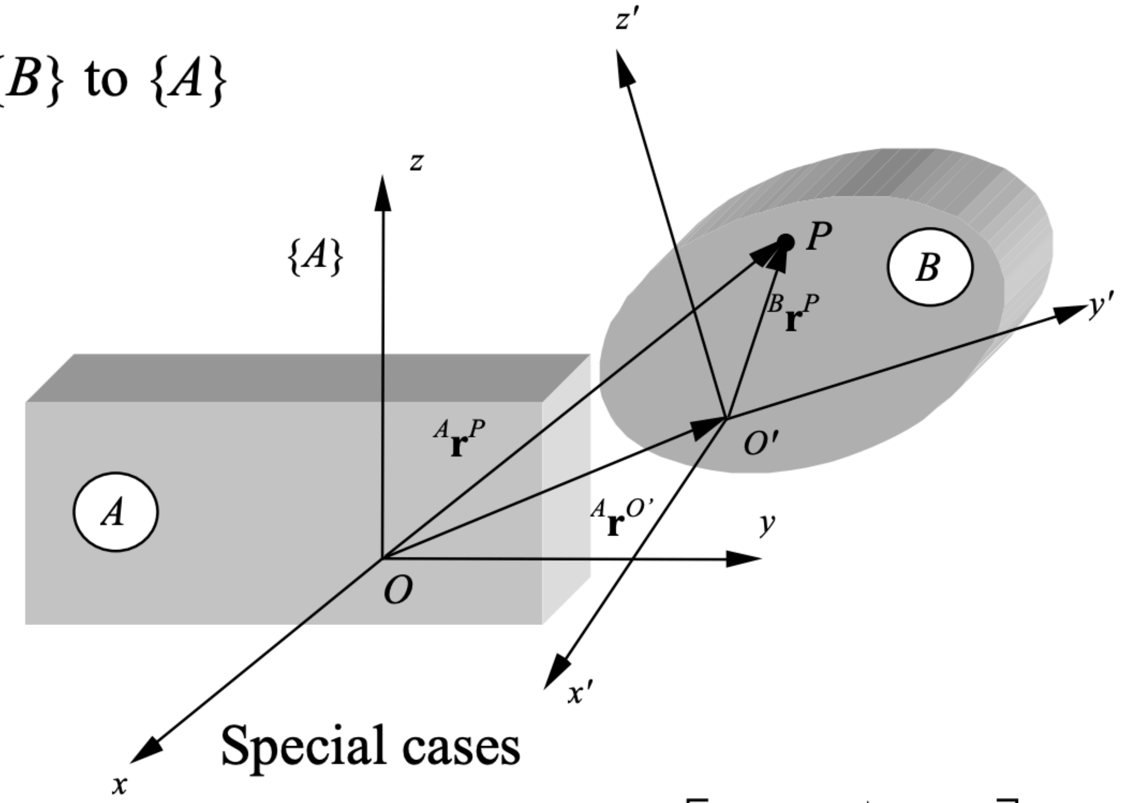
https://www.seas.upenn.edu/~meam620/slides/kinematicsI.pdf

Homogeneous Transformation Matrix

Coordinate transformation from $\{B\}$ to $\{A\}$

$$ {}^{A}\mathbf{r}^{P'} = {}^{A}\mathbf{R}_{B}\,{}^{B}\mathbf{r}^{P} + {}^{A}\mathbf{r}^{O'} $$

$$ \left[\begin{array}{c} {}^{A}\mathbf{r}^{P} \\ \hdashline 1 \end{array}\right] = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] \left[\begin{array}{c} {}^{B}\mathbf{r}^{P} \\ \hdashline 1 \end{array}\right] $$



Homogeneous transformation matrix

$$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right] $$

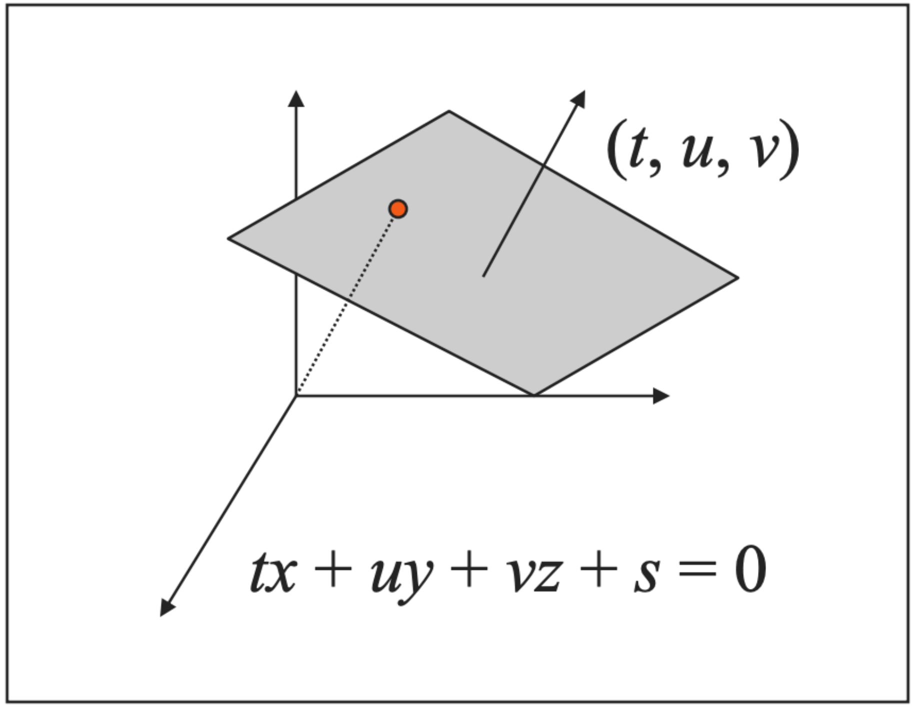
Special cases

1. $ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} {}^{A}\mathbf{R}_{B} & \mathbf{0}_{3\times 1} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]. $

2. $ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c:c} \mathbf{I}_{3\times 3} & {}^{A}\mathbf{r}^{O'} \\ \hdashline \mathbf{0}_{1\times 3} & 1 \end{array}\right]. $

Homogeneous Coordinates

- Description of a point
  (x, y, z)
- Description of a plane
  (t, u, v, s)
- Equation of a circle
  $x^2 + y^2 + z^2 = a^2$



Homogeneous coordinates

- Description of a point
  (x, y, z, w)
- Equation of a plane
  $tx + uy + vz + sw = 0$
- Equation of a sphere
  $x^2 + y^2 + z^2 = a^2 w^2$

Central ideas

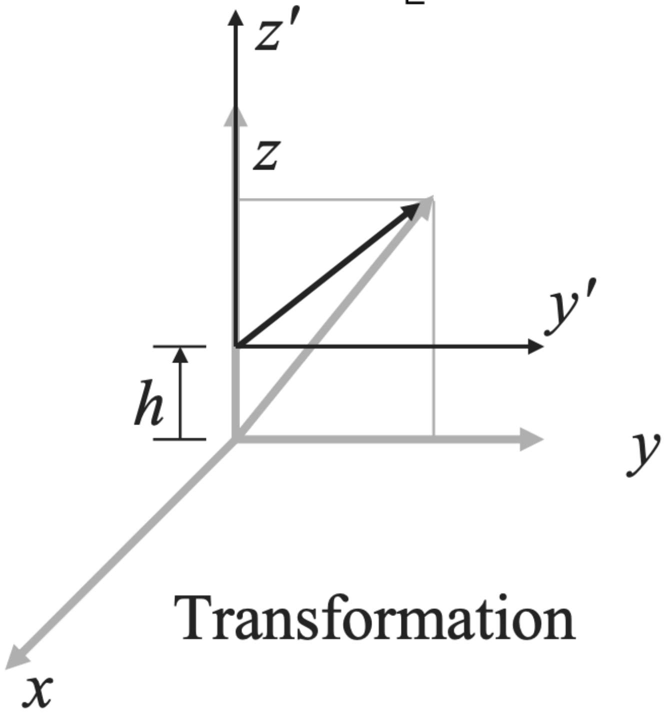
- Equivalence class
- Projective space P^3,
  and not Euclidean space R^3

Mathematical and practical advantages

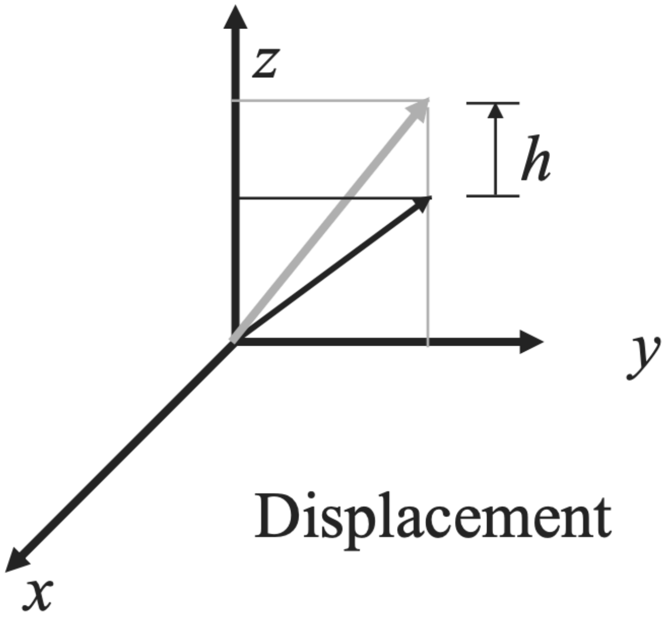
Example: Translation

- Translation along the z-axis through h

$$Trans(z,h)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&h\\0&0&0&1\end{bmatrix} \qquad \begin{bmatrix}x\\y\\z\\1\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&h\\0&0&0&1\end{bmatrix}\begin{bmatrix}x'\\y'\\z'\\1\end{bmatrix}$$



Transformation

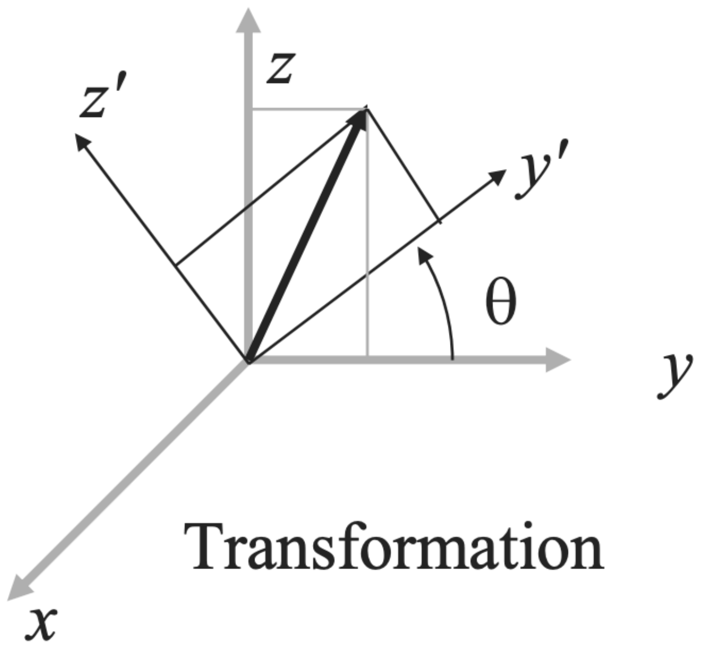


Displacement

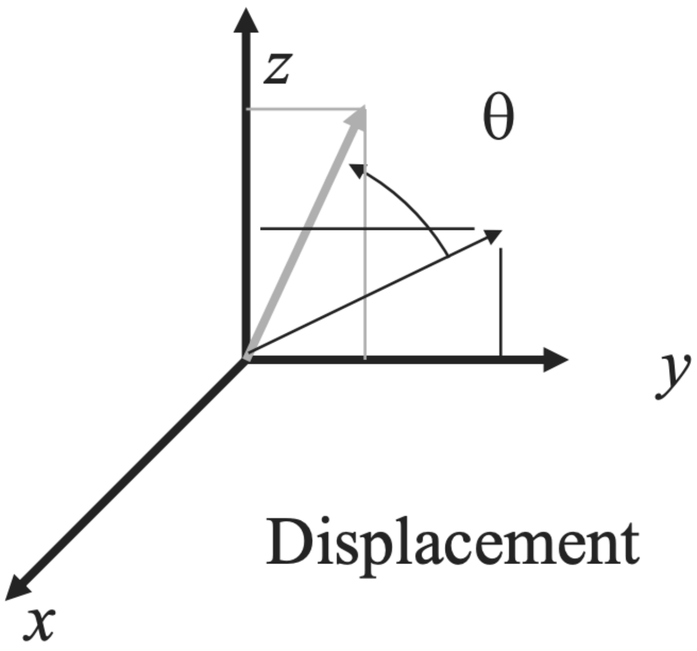
Example: Rotation

- Rotation about the x-axis through θ

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad \begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x'\\ y'\\ z'\\ 1\end{bmatrix}$$



Transformation

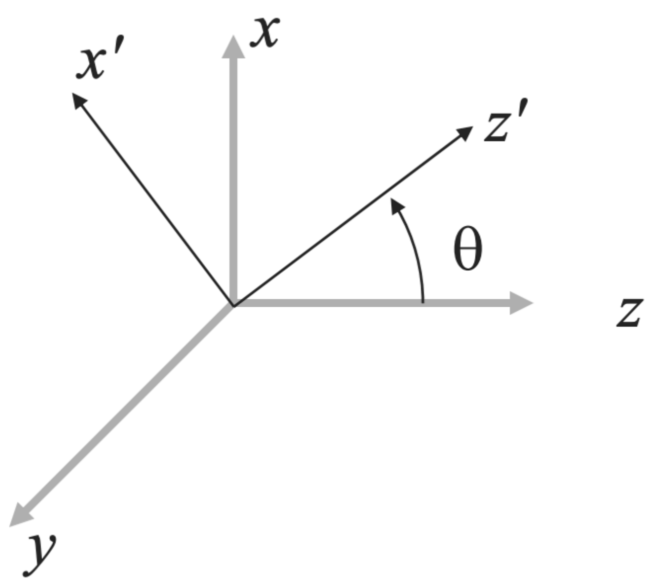


Displacement

Example: Rotation

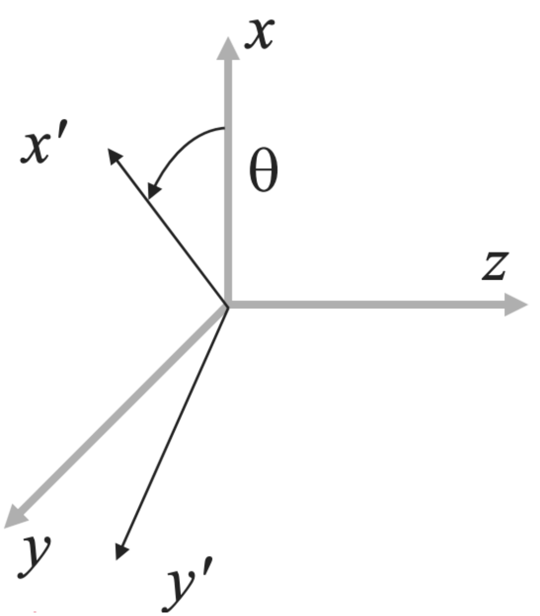
Rotation about the *y*-axis through θ

$$Rot(y,\theta)=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

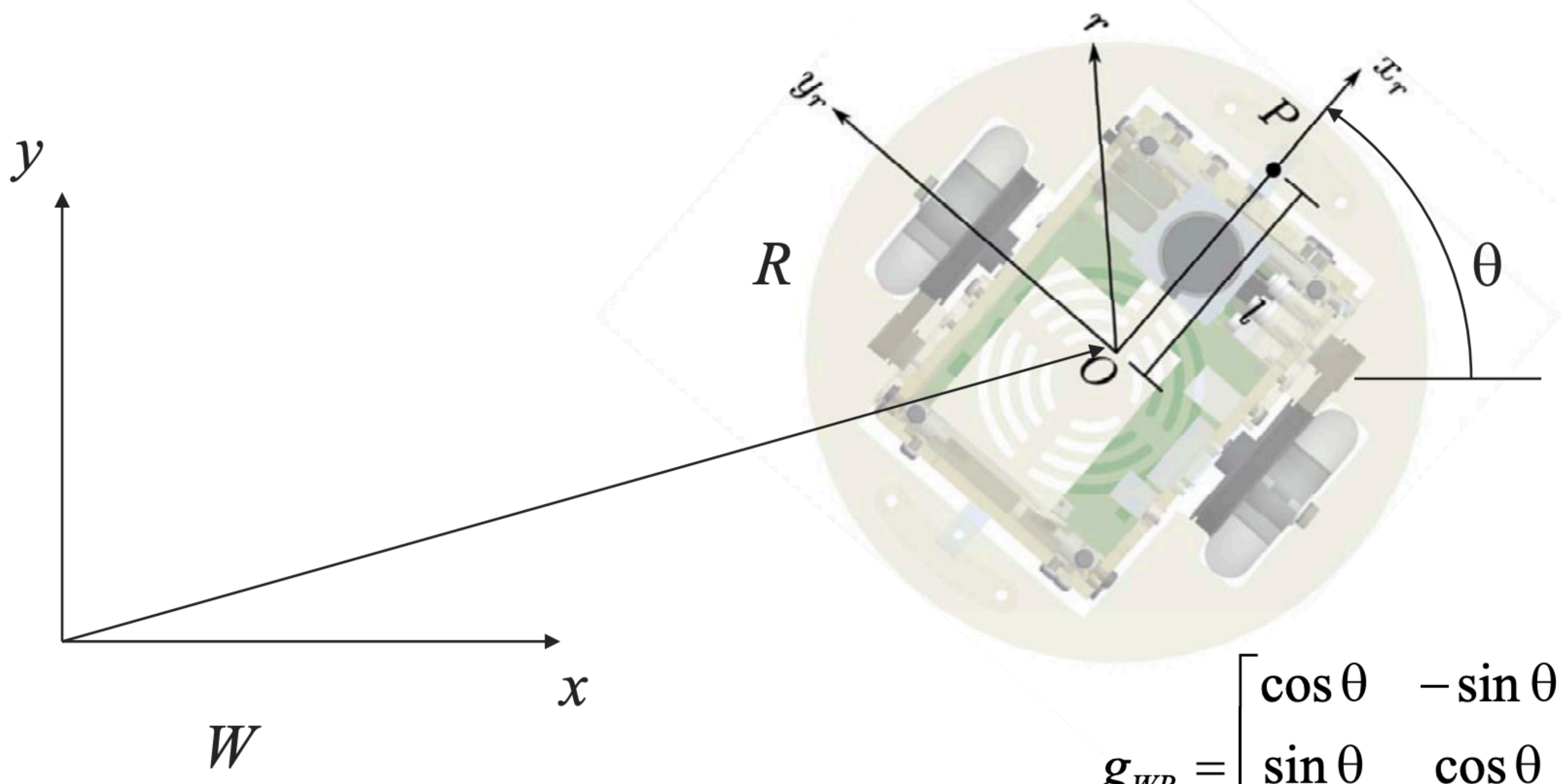


Rotation about the *z*-axis through θ

$$Rot(z,\theta)=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Mobile Robots



$$g_{WR} = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

$^{W}A_{R}$

Example: Displacement of Points

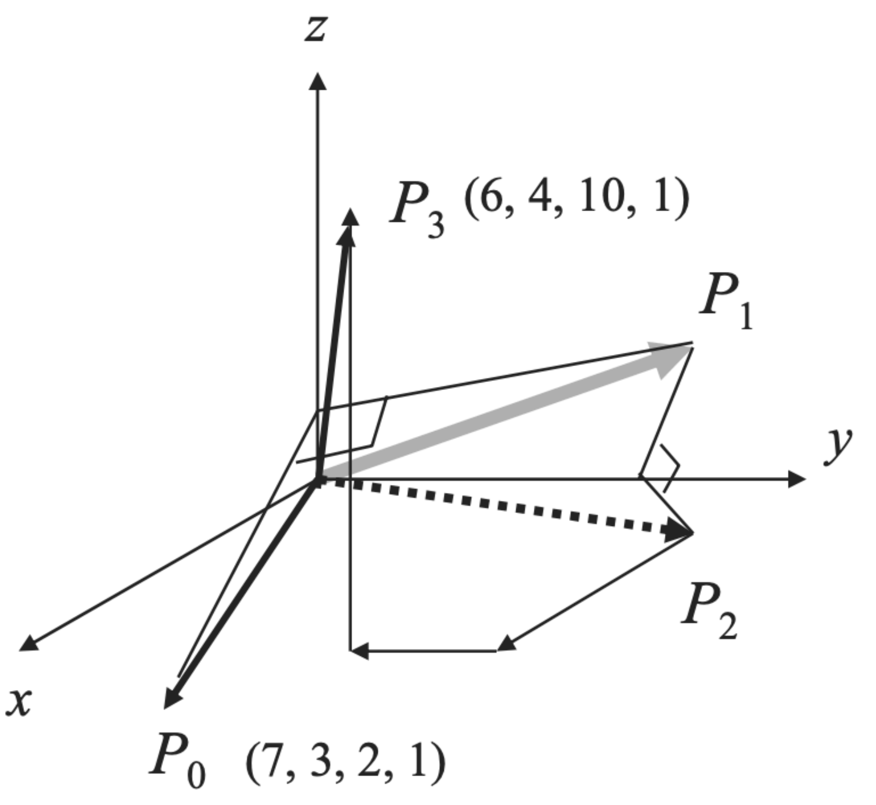
Displace (7, 3, 2) through a sequence of:

1. Rot(z, 90)
2. Rot(y, 90)
3. Trans(4, -3, 7)

in the frame F: (x, y, z):

Trans(4, -3, 7) Rot(y, 90) Rot(z, 90)

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

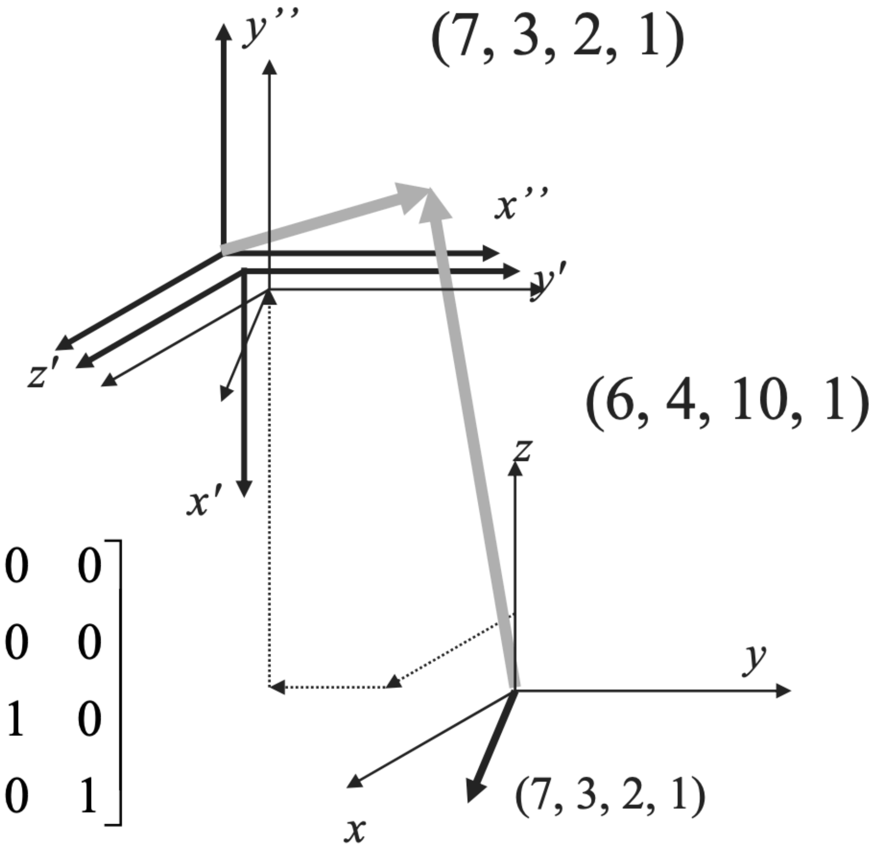


Example: Transformation of Points

Successive transformations of (7, 3, 2):

1. Trans(4, -3, 7)
2. Rot(y, 90)
3. Rot(z, 90)

in a body fixed frame



Trans(4, -3, 7) Rot(y, 90) Rot(z, 90)

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$SE(3)$ is a Lie group

$SE(3)$ satisfies the four axioms that must be satisfied by the elements of an *algebraic group*:

- The set is closed under the binary operation. In other words, if $\mathbf{A}$ and $\mathbf{B}$ are any two matrices in $SE(3)$, $\mathbf{AB} \in SE(3)$.
- The binary operation is associative. In other words, if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any three matrices $\in SE(3)$, then $(\mathbf{AB})\,\mathbf{C} = \mathbf{A}\,(\mathbf{BC})$.
- For every element $\mathbf{A} \in SE(3)$, there is an identity element given by the 4×4 identity matrix, $\mathbf{I} \in SE(3)$, such that $\mathbf{AI} = \mathbf{A}$.
- For every element $\mathbf{A} \in SE(3)$, there is an identity inverse, $\mathbf{A}^{-1} \in SE(3)$, such that $\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$.

$SE(3)$ is a *continuous group*.

- the binary operation above is a continuous operation — the product of any two elements in $SE(3)$ is a continuous function of the two elements
- the inverse of any element in $SE(3)$ is a continuous function of that element.

In other words, $SE(3)$ is a *differentiable manifold*. A group that is a differentiable manifold is called a *Lie group*[Sophus Lie (1842-1899)].

Composition of Displacements

Displacement from $\{A\}$ to $\{B\}$

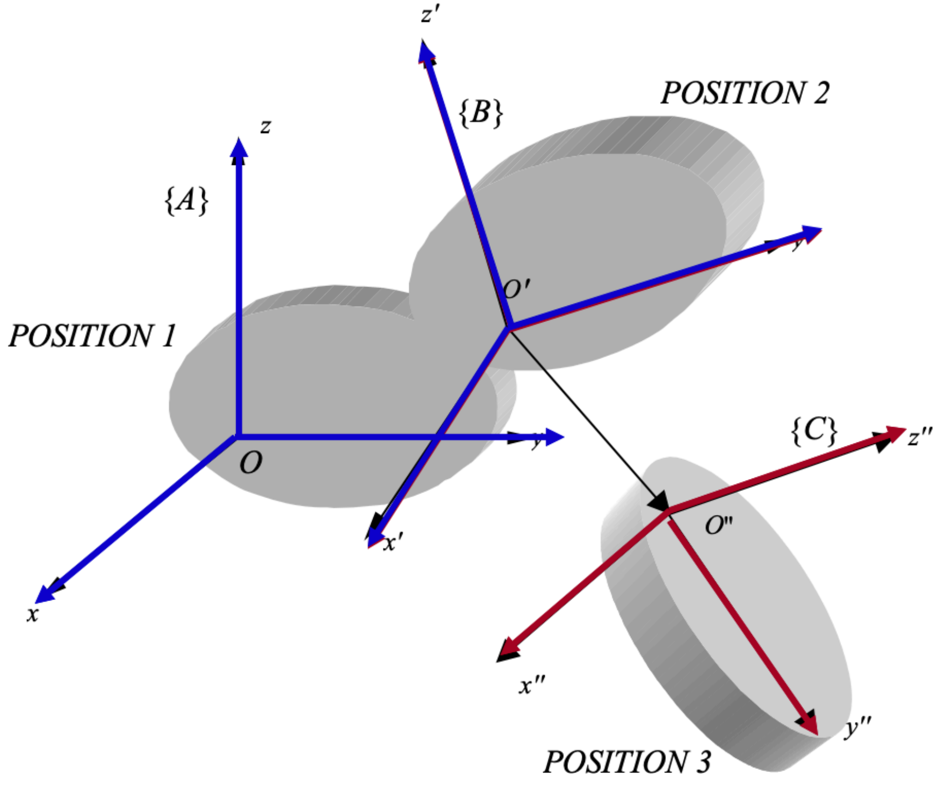
$$ {}^{A}\mathbf{A}_{B} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right], $$

Displacement from $\{B\}$ to $\{C\}$

$$ {}^{B}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0}_{1\times 3} & 1 \end{array}\right], $$

Displacement from $\{A\}$ to $\{C\}$

$$ {}^{A}\mathbf{A}_{C} = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{C} & {}^{A}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} & {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right] \times \left[\begin{array}{c|c} {}^{B}\mathbf{R}_{C} & {}^{B}\mathbf{r}^{O''} \\ \hline \mathbf{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{R}_{C} & {}^{A}\mathbf{R}_{B} \times {}^{B}\mathbf{r}^{O''} + {}^{A}\mathbf{r}^{O'} \\ \hline \mathbf{0} & 1 \end{array}\right] $$

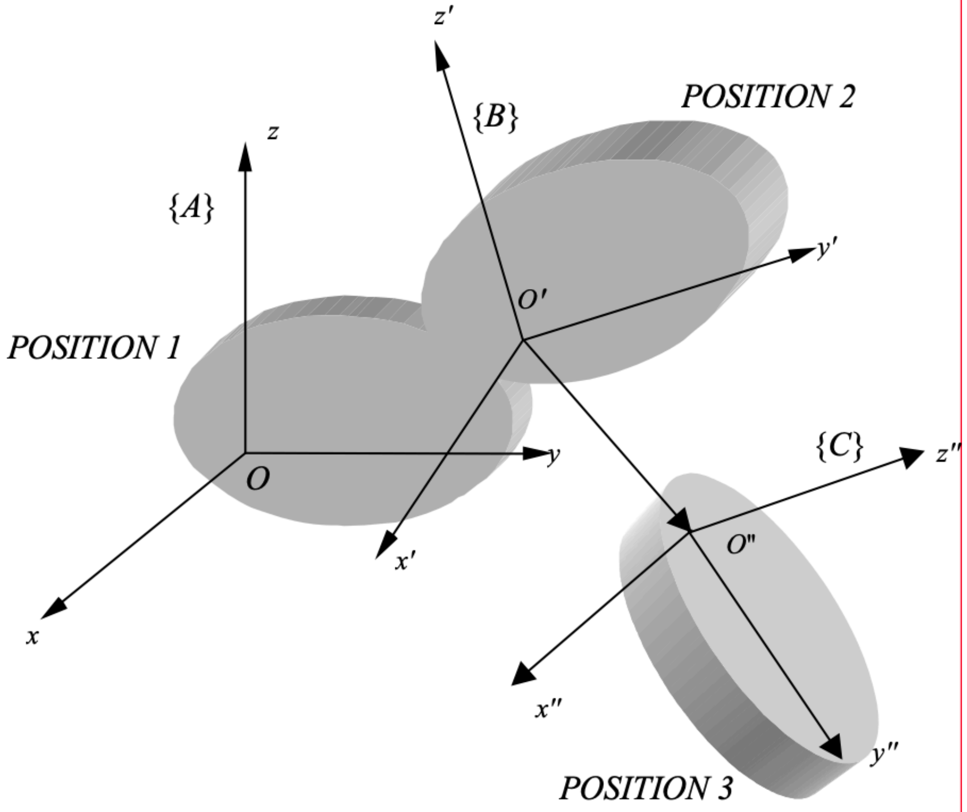


Note ${}^{X}\mathbf{A}_{Y}$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Composition (continued)

Composition of displacements

- Displacements are generally described in a body-fixed frame
- Example: ${}^{B}\mathbf{A}_C$ is the displacement of a rigid body from B to C *relative* to the axes of the "first frame" B.



Composition of transformations

- Same basic idea
- ${}^{A}\mathbf{A}_C = {}^{A}\mathbf{A}_B \; {}^{B}\mathbf{A}_C$

Note that our description of transformations (e.g., ${}^{B}\mathbf{A}_C$) is *relative* to the "first frame" (B, the frame with the leading superscript).

Note ${}^{X}\mathbf{A}_Y$ describes the displacement of the body-fixed frame from $\{X\}$ to $\{Y\}$ in reference frame $\{X\}$

Hints for your HW3

The torus T is a "surface" generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle.

Proof

Let S^1 be the circle in the yz plane with its center on the point $(0, a, 0)$. Then S^1 is given by $(y - a)^2 + z^2 = r^2$.

The points of T are obtained by rotating this circle about the z axis satisfying the equation

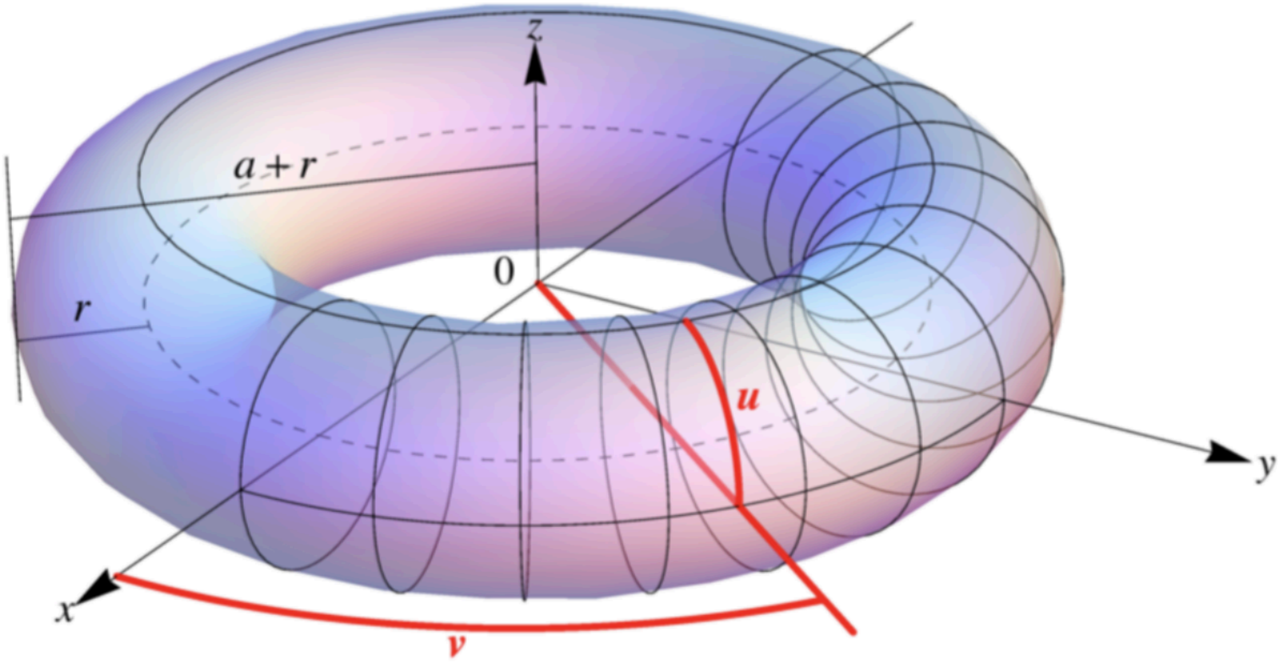
$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = r^2.$$

Example

A parametrization for the torus T of the previous example can be given by

$$\mathbf{x}(u, v) = ((r\cos u + a)\cos v, (r\cos u + a)\sin v, r\sin u),$$

where $0 < u < 2\pi$, $0 < v < 2\pi$.



Proof (cont'd)

Let $f(x,y,z) = (\sqrt{x^2+y^2} - a)^2 + z^2$. Then

$$\frac{\partial f}{\partial z} = 2z, \quad \frac{\partial f}{\partial y} = \frac{2y\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}, \quad \frac{\partial f}{\partial x} = \frac{2x\left(\sqrt{x^2+y^2} - a\right)}{\sqrt{x^2+y^2}}.$$

Hence, $(f_x, f_y, f_z) \neq (0,0,0)$ in $f^{-1}(r^2)$, so r^2 is a regular value. Therefore, the torus is a regular surface.