Functional Analysis HW4

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Problem 16.

Since the left-invariant Haar measure of the G is $\frac{1}{x^2}dxdy$ or $\frac{1}{x^2}dydx$ and right Haar measure is $\frac{1}{|x|}dxdy$ or $\frac{1}{|x|}dydx$, G is not unimodular. (notation overlapping. why? how do you get them?)

Furthermore, let adjoint representation of $g_1, g_2 \in G$ be $ad_{g_1}g_2 = g_1g_2 - g_2g_1$.
Then, for $g_1 = \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} z & w \\ 0 & 1 \end{bmatrix}$, $ad_{g_1}g_2 = \begin{bmatrix} 0 & xw + y - yz - w \\ 0 & 0 \end{bmatrix}$.
Then, $|\det ad_{g_1}g_2| = 0$ for $\forall g_1, g_2 \in G$.
Thus, it is not unimodular.
(Your argument is not sufficient.)

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Problem 17.

For $f \in B_1$, define function $F : \mathbb{R} \to \mathbb{R}, t \mapsto \int_0^t |f(x)|dx$.
Then, for $a, b \in [0, 1]$,

$$|F(b) - F(a)| = \left| \int_a^b |f(x)|dx \right| \leq \|f\|_{L^1([0,1])}|b - a|$$

Moreover, $\|F\|_{L(\mathbb{R})} \leq \|f\|_{L^1([0,1])} < \infty$.

Using Intermediate Value Theorem, $\exists t \in [0, 1]$ such that $F(t) = \frac{1}{2}\|f\|_{L^1}$. (separate the trivial case $f = 0$ in your argument.)
Then, $2f|_{X[0,t]}, 2f|_{X[t,1]} \in B_1$ and $2f|_{X[0,t]} \neq f \neq 2f|_{X[t,1]}$.
However, $f = f|_{X[0,t]} + f|_{X[t,1]}$.
Thus f cannot be extreme point.
In conclusion, there are no extreme points in B_1.

Therefore, the assumption that $L^1([0,1])$ is a dual space of a Banach Space is wrong since if it was dual space of Banach space, weak* topology of B_1 is compact by Banach-Alaoglu Theorem.

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Problem 18.

Let T be a compact operator and (u_n) be a weakly convergent sequence converging to u.
Since u_n is weakly convergent, it follows that it is bounded and T is compact.
Thus, for any subsequence (Tu_{n_s}) converge strongly in Y to Tu.
Then any subsequence of (Tu_n) has a convergent subsequence with limit Tu (why?).
By topological lemma, $Tu_n \to Tu$ in Y.
Please write more detail of the proof.

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Problem 19.

Given $\epsilon > 0$, let $x_0 \in \mathcal{H}$ and $\|x_0\|_{\mathcal{H}} \leq 1$ such that $\|Tx_0\|_{\mathcal{H}} \geq \|T\|_{op} - \epsilon$.
If we extend x_0 to orthonormal basis e_i,

$$\|T\|_{op}^2 = \sup_{\|x\|\leq 1} \|Tx\|_{\mathcal{H}}^2 \leq \|Tx_0\|_{\mathcal{H}}^2 + \epsilon$$

Since $x_0 = \sum_i \alpha_i e_i$ for some constant $\{\alpha_i\}$ such that $\forall i$, $|\alpha_i| \leq 1$ and orthnormal basis $\{e_i\}$.

$$\|Tx_0\|_{\mathcal{H}}^2 = \left\|\sum_i T(\alpha_i e_i)\right\|_{\mathcal{H}}^2 = \sum_i \|T(\alpha_i e_i)\|_{\mathcal{H}}^2$$

By Pythagoras's theorem.

$$\sum_i \|T(\alpha_i e_i)\|_{\mathcal{H}}^2 = \sum_i \alpha_i^2 \|T(e_i)\|_{\mathcal{H}}^2 \leq \sum_i \|Te_i\|_{\mathcal{H}}^2 = \|T\|_{HS}^2$$

Then, since we can choose ϵ arbitarily small, $\|T\|_{op}^2 \leq \|T\|_{HS}^2$ and $\|T\|_{op} \leq \|T\|_{HS}$

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Problem 20.

Since $\|I - A^{-1}B\| \leq \|A - B\|\|A^{-1}\| = \|B - A\|\|A^{-1}\| \leq 1$.
By theorem in Analysis, $I - (I - A^{-1}B) = A^{-1}B$ is invertible. (you should explain what you mean clearly with writing down the right argument. Do not rely on the high level language. I know what you're saying but cannot give you full credit.)
Also, BA^{-1} is invertible similarly.

Assume B is not invertible and thus, not bijective.
Then, neither $A^{-1}B$ nor BA^{-1} is bijective.
This is contradiction.
Therefore, B is invertible.
(Do not skip the detail.)

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