Functional Analysis HW6

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Problem 26.

If we apply fourier transform to g_λ,

$$\begin{aligned}\hat{f}(k) &= \int_{\mathbb{R}} \exp(-2\pi i k x)\exp(-\pi\lambda|x|^2)\,dx \\ &= \exp\left(-\frac{\pi k^2}{\lambda}\right)\int_{\mathbb{R}} \exp\left(-\lambda\pi\left(x^2 + 2\frac{ki}{\lambda}x - \frac{k^2}{\lambda^2}\right)\right)dx \\ &= \exp\left(-\frac{\pi k^2}{\lambda}\right)\int_{\mathbb{R}} \exp\left(-\lambda\pi\left(x + \frac{ki}{\lambda}\right)^2\right)dx \end{aligned} \tag{1}$$

Since

$$\int \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx = 1$$

$$\int \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)dx = \sqrt{2\pi\sigma^2}$$

It is not enough because we should deal with some complex (number) integral.

Then,

$$(1) = \exp\left(-\frac{\pi k^2}{\lambda}\right)\int_{\mathbb{R}} \exp\left(-\frac{1}{2}\frac{x^2}{\frac{1}{2\lambda\pi}}\right)dx = \sqrt{2\pi\cdot\frac{1}{2\lambda\pi}}\exp\left(-\frac{\pi k^2}{\lambda}\right) = \frac{1}{\sqrt{\lambda}}\exp\left(-\frac{\pi k^2}{\lambda}\right)$$

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Problem 27.

Let $g := \widehat{f}$.

Then, $-\Delta f = (\|x\|^2 g(x))^\vee$ and $\|x\|^2 g(x) \in L^2(\mathbb{R}^3)$.

By Parseval's formula, $\|-\Delta f\|_{L^2} = \left\|\|x\|^2 g(x)\right\|_{L^2}$ and $\|f\|_{L^2} = \|g\|_{L^2}$.

Then, let $C := \left\|(1+\|x\|^2)^{-1}\right\|_{L^2} = \left|\int_{\mathbb{R}^d} \left|(1+\|x\|^2)^{-1}\right|^2 dx\right|^{\frac{1}{2}} < \infty$.
By the Hölder's inequality,

$$\|g\|_{L^1} = \left\|(1+\|x\|^2)^{-1}(1+\|x\|^2)g(x)\right\|_{L^1} \leq C\left\|(1+\|x\|^2)g(x)\right\|_{L^2} \leq C(\left\|\|x\|^2 g(x)\right\|_{L^2}+\|g\|_{L^2})$$

Thus, $g \in L^1(\mathbb{R}^d)$. Therefore, $f = g^\vee \in L^\infty(\mathbb{R}^d)$ and $\|f\|_\infty \leq \|g\|_{L^1}$. (Hausdorff-Young)
In conclusion, $\|f\|_\infty \leq C(\|-\Delta f\|_{L^2} + \|f\|_{L^2})$.

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Problem 28.

Since $\widehat{(f*g)}(x) = \widehat{f}(k)\widehat{g}(k)$ by Theorem 6.2, $\widehat{(\widehat{f}*\widehat{g})}(x) = \widehat{\widehat{f}}(k)\widehat{\widehat{g}}(k) = f(-k)g(-k)$.
By applying Fourier inversion for both sides,

$$(\widehat{f}*\widehat{g})(x) = \int_{\mathbb{R}^n} e^{2\pi ikx} f(-k)g(-k)dk = \int_{\mathbb{R}^n} e^{-2\pi itx} f(t)g(t)dt = \widehat{fg}(x)$$

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Problem 29.

For $f \in L^2(\mathbb{R}^n)$, by Plancherel theorem,

$$\begin{aligned}\left\|e^{t\triangle}f - f\right\|_{L^2} &= \left\|\widehat{(e^{t\triangle}f)} - \widehat{f}\right\|_{L^2} \\ &= \left\|g\widehat{f} - \widehat{f}\right\|_{L^2} \\ &= \left\|(g-1)\widehat{f}\right\|_{L^2} \\ &\leq \|g-1\|_{L^2}\left\|\widehat{f}\right\|_{L^2}\end{aligned}$$

(g is Gaussian.)
Thus, $\left\|e^{t\triangle}f - f\right\|_{L^2} \to 0$ as $t \to 0$ since $g \to 1$ as $t \to 0$ by dominated convergence.

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Problem 30.

$$
\begin{aligned}
\frac{1}{\pi}\frac{y}{x^2+y^2} &= \frac{1}{2\pi}\left[\frac{1}{x+iy}+\frac{1}{-x+iy}\right] \\
&= \int_0^\infty e^{-2\pi ky}e^{2\pi ikx}\,dk + \int_0^\infty e^{-2\pi ky}e^{-2\pi ikx}\,dk \\
&= \int_0^\infty e^{-2\pi ky}e^{2\pi ikx}\,dk + \int_{-\infty}^0 e^{2\pi ky}e^{2\pi ikx}\,dk \\
&= \int_{-\infty}^\infty e^{-2\pi|k|y}e^{2\pi ikx}\,dk \\
&= (e^{-2\pi|k|y})^\vee
\end{aligned}
$$

Thus, $\widehat{P_y}(x) = e^{-2\pi|k|y}$.

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