Functional Analysis HW2

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Problem 6.

For a nomed linear space V, it equips norm $\|\cdot\|_V$.
Let U be open unit ball and $\bar{U}$ be closed unit ball s.t

$$U = \{x \in V \mid \|x\|_V < 1\}$$

$$\bar{U} = \{x \in V \mid \|x\|_V \leq 1\}$$

For $\forall x_1, x_2 \in U$ and $\forall t \in [0,1]$, $\|tx_1 + (1-t_2)x_2\|_V \leq |t|\|x_1\|_V + |1-t|\|x_2\|_V < 1$.
For $\forall y_1, y_2 \in \bar{U}$ and $\forall t \in [0,1]$, $\|ty_1 + (1-t_2)y_2\|_V \leq |t|\|y_1\|_V + |1-t|\|y_2\|_V \leq 1$.
By triangle inequality. Therefore, U and $\bar{U}$ are convex.

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Problem 7.

Let norm in $C(X)$ be $\|f\|_{C(X)} = \sup_{x \in X} |f(x)|$.
For $f \in C(X)$, $l(f)$ is scalar in non-negative real if $f \geq 0$.
Therefore, l is a linear mapping between two normed spaces.

Then, $\forall f \in C(X)$, $\forall x \in X$, $|f(x)| \leq \|f\|_{C(X)}$ and $\forall x \in X$, $\|f\|_{C(X)} - f(x) \geq 0$ and $\|f\|_{C(X)} + f(x) \geq 0$.
For positive linear functional l,

$$0 \leq l(\|f\|_{C(X)} - f) = \|f\|_{C(X)} l(1) - l(f)$$

$$0 \leq l(\|f\|_{C(X)} + f) = \|f\|_{C(X)} l(1) + l(f)$$

Therefore, $|l(f)| \leq \|f\|_{C(X)} l(1)$. (Because $f = f \cdot 1$ for constant function 1.)

Since l is bounded, l is continuous by Theorem 2.11.

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Problem 8.

Let $x_n \to x$ be a convergent sequence in $\mathcal{H}$ and $Tx_n \to y$ for some $y \in \mathcal{H}$.
Then, $\Gamma(T)$ is closed if $Tx = y$.
For any $z \in \mathcal{H}$,

$$\begin{aligned}
\langle Tx - y, z\rangle &= \langle Tx, z\rangle - \langle y, z\rangle \\
&= \langle x, Tz\rangle - \langle y, z\rangle \\
&= \lim_{n\to\infty} [\langle x_n, Tz\rangle - \langle Tx_n, z\rangle] \\
&= \lim_{n\to\infty} [\langle Tx_n, z\rangle - \langle Tx_n, z\rangle] \\
&= 0
\end{aligned}$$

Thus, $Tx = y$.

Then, the graph $\Gamma(T)$ is closed.
By Closed Graph Theorem, T is continuous.
By theorem 2.11, T is bounded since it is continuous linear mapping between two normed spaces $\mathcal{H}$ and $\mathcal{H}$.

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Problem 9.

For $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\phi(x) - \phi(y)|$ where $\phi(x) = \dfrac{x}{1 + |x|}$, to show d_1 and d_2 induces same topology,

$$\begin{aligned}
d_2(x, y) &= \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| \\
&= \frac{1}{1+|x|}\left|x - \frac{1+|x|}{1+|y|}y\right| \\
&= \frac{1}{1+|x|}\left|x - y - \frac{|x|-|y|}{1+|y|}y\right| \\
&\leq \frac{1}{1+|x|}\left(|x-y| + \frac{|(|x|-|y|)y|}{1+|y|}\right) \\
&\leq \frac{1}{1+|x|}\left(|x-y| + \frac{|x-y||y|}{1+|y|}\right) = \frac{1}{1+|x|}\frac{1+2|y|}{1+|y|}|x-y| \\
&\leq 2|x-y| = 2d_1(x, y)
\end{aligned}$$

Therefore, $\dfrac{1}{2}d_2(x,y) \leq d_1(x,y)$.

$$\begin{aligned}
d_1(x,y) &= |x-y| \\
&= (1+|x|)\left|\frac{x}{1+|x|} - \frac{y}{1+|x|}\right| \\
&= (1+|x|)\left|\frac{x}{1+|x|} - \frac{1+|y|}{1+|x|}\cdot\frac{y}{1+|y|}\right| \\
&= (1+|x|)\left|\frac{x}{1+|x|} - \frac{1+|x|-|x|+|y|}{1+|x|}\cdot\frac{y}{1+|y|}\right| \\
&= (1+|x|)\left|\frac{x}{1+|x|} - \frac{y}{1+|y|} + \frac{|y|-|x|}{1+|x|}\cdot\frac{y}{1+|y|}\right| \\
&\leq (1+|x|)\left[d_2(x,y) + \frac{|y|||x|-|y||}{(1+|x|)(1+|y|)}\right] \\
&\leq (1+|x|)\left[d_2(x,y) + \frac{|y|}{(1+|x|)(1+|y|)}d_1(x,y)\right] \\
&= (1+|x|)d_2(x,y) + \frac{|y|}{1+|y|}d_1(x,y)
\end{aligned}$$

Then,

$$d_1(x,y) \leq (1+|x|)(1+|y|)d_2(x,y)$$

Therefore,

$$\forall x,y \in \mathbb{R}, \frac{1}{2}d_2(x,y) \leq d_1(x,y) \leq d_2(x,y)$$

This is sufficient condition for the statement that d_1 and d_2 is equivalent metric.
In conclusion, these two metrics induces same topology.
(Instead of doing computation, it is enough to use the fact that ϕ is a homeomorphism between $\mathbb{R}$ and $(-1,1)$.)

To show d_2 is not complete, assume it is complete.
Then, every Cauchy sequence has a limit in $\mathbb{R}$.
Let $x_n = n$ be a Cauchy sequence in $\mathbb{R}$.
Then, for $\forall\epsilon > 0$, let $N = \left[\dfrac{2}{\epsilon}\right] + 1$.
Then,

$$d_2(x_n,x_m) = \left|\frac{n-m}{(1+n)(1+m)}\right| < \left|\frac{n-m}{nm}\right| < N^{-2}|n-m| < N^{-2}(|n|+|m|) < 2N^{-1} < \epsilon$$

Thus, it is a cauchy sequence.
However, limit $\lim_{n\to\infty} x_n = \infty$ is not a element of $\mathbb{R}$.
It is a contradiction.
Thus, d_2 is not complete in metric.

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Problem 10.

Refering the given definitions in Hint, for $\forall x, y \in M$ and a scalar α,

$$\lim_{n\to\infty} \Lambda_n(x+y) = \lim_{n\to\infty} [\Lambda_n x + \Lambda_n y] = \lim_{n\to\infty} \Lambda_n x + \lim_{n\to\infty} \Lambda_n y = \Lambda x + \Lambda y,\ x+y \in M$$

$$\lim_{n\to\infty} \Lambda_n(\alpha x) = \lim_{n\to\infty} \alpha \Lambda_n x = \alpha \Lambda x,\ \alpha x \in M$$

Thus, M is a subspace of l^∞.
Actually, l^∞ is a Banach space with a norm $\|x\|_\infty = \sup_n |x_n|$ and complete in metric $\|x-y\|_\infty$ for $x, y \in l^\infty$.

Given subspace M and a seminorm(norm) $\|\cdot\|_\infty$, by Hahn-Banach Theorem, there exists a linear functional Λ such that

$$|\Lambda(x)| \leq \|x\|_\infty \text{ for } x \in l^\infty$$

and for a linear functional f on M such that $\forall x \in M$, $|f(x)| \leq \|x\|_\infty$,

$$\Lambda(x) = f(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \text{ if } x \in M$$

- To show (a),

$$\begin{aligned}
\Lambda(\tau(x)) - \Lambda(x) &= \Lambda(\tau(x) - x) \\
&= \lim_{n\to\infty} \Lambda_n(\tau(x) - x) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} [x_{i+1} - x_i] \\
&= \lim_{n\to\infty} \frac{1}{n} [x_{n+1} - x_1] \\
&= 0
\end{aligned}$$

  Since $x \in l^\infty$ is bounded. Therefore, $\Lambda(\tau(x)) = \Lambda(x)$

- (b) For $x \in l^\infty$, any α, β such that

$$\alpha < \liminf_{n\to\infty} x_n \quad \text{and} \quad \limsup_{n\to\infty} x_n < \beta$$

  $\exists N$ such that $\alpha < x_n < \beta$ for $\forall n > N$.
  Then, $\tau^N(x) - \alpha$ and $\beta - \tau^N(x)$ are positive bounded sequences.
  Moreover,

$$0 < \Lambda(\tau^N(x) - \alpha) = \Lambda(\tau^N(x)) - \alpha = \Lambda(x) - \alpha$$

$$0 < \Lambda(\beta - \tau^N(x)) = \beta - \Lambda(\tau^N(x)) = \beta - \Lambda(x)$$

Thus, $\alpha < \Lambda(x) < \beta$.
Then, we can choose α and β arbitrarily close to liminf and limsup,

$$\liminf_{n\to\infty} x_n \leq \Lambda(x) \leq \limsup_{n\to\infty} x_n$$

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