Functional Analysis HW1

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Problem 1.

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If $f = 0$ almost everywhere, then the theorem is trivial.

Since $1 \le p, q \le \infty$ and $1/p + 1/q = 1$, if $g \in \mathrm{L}^q(\mathrm{S}, d\mu)$ and $\|g\|_q = 1$, by Hölder's inequality,

$$\left|\int_S fg\,d\mu\right| = \|fg\|_1 \le \|f\|_p \|g\|_q = \|f\|_p$$

Thus, if $g \in \mathrm{L}^q(\mathrm{S}, d\mu)$ and $\|g\|_q = 1$,

add complex conjugate

$$\|f\|_p \ge \left|\int_S fg\,d\mu\right|$$

If $\cancel{g}$ f $\neq 0$, let $g = \frac{|f|^{p-1} sgn(f)}{\|f\|_p^{p-1}}$. Then, $\|g\|_q = 1$.

not well-defined in the case $p = \infty$. consider x s.t. $|f(x)| > 1$.

$$\left|\int_{\mathrm{S}} fg\,d\mu\right| = \left|\int_{\mathrm{S}} \frac{|f|^p}{\|f\|_p^{p-1}}\,d\mu\right| = \frac{\|f\|_p^p}{\|f\|_p^{p-1}} = \|f\|_p$$

Therfore,

$$\|f\|_p = \sup\left\{\left|\int_{\mathrm{S}} fg\,d\mu\right| : g \in \mathrm{L}^q(\mathrm{S}, d\mu), \|g\|_q = 1\right\}$$

Problem 2.

in "a.e" sense.

Let $f \in \mathrm{L}^p(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$.
Then, $|f|^r = |f|^{r-p}|f|^p \le \|f\|_\infty^{\cancel{q-p}\ r-p}|f|^p$. By Hölder's inequality,

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$$\int_{\mathbb{R}} |f|^{\cancel{q}\ r}\,d\mu \le \int_{\mathbb{R}} \|f\|_\infty^{\cancel{q-p}\ r-p}|f|^p\,d\mu = \|f\|_\infty^{\cancel{q-p}\ r-p}\int_{\mathbb{R}} |f|^p\,d\mu$$

$$\|f\|_{\cancel{q}\ r}^{\cancel{q}\ r} \le \|f\|_\infty^{\cancel{q-p}}\|f\|_p^p < \infty$$

$$\|f\|_{\cancel{q}\ r} \le \|f\|_\infty^{1-p/\cancel{q}\ r}\|f\|_p^{p/\cancel{q}\ r} < \infty \quad \text{since} \quad 0 < \frac{p}{\cancel{q}\ r} < 1$$

Thus, $f \in \mathrm{L}^q(\mathbb{R})$.

need to check it is well-defined or not.

Problem 3.

Let $\|\cdot\|_{\mathbb{C}_0(\mathbb{R}^n)}$ be a norm in $\mathbb{C}_0(\mathbb{R}^n)$ such that for $f \in \mathbb{C}_0(\mathbb{R}^n)$, $\|f\|_{\mathbb{C}_0(\mathbb{R}^n)} = \max_{x\in\mathbb{R}^n}|f(x)|$. While absolute value for complex number is defined as $a+bi \in \mathbb{C}$, $|a+bi| = \sqrt{a^2+b^2}$. Then,

- Non-negativity : Since $|a+bi| = \sqrt{a^2+b^2} > 0$ and $\sqrt{a^2+b^2} = 0$ iff $a = b = 0$, $\|\cdot\|_{\mathbb{C}_0(\mathbb{R}^n)} > 0$ and equal to 0 iff $f = 0$.

- Scalar multiple : Since $|c(a+bi)| = |c||a+bi|$ for $c \in \mathbb{C}$, $\|cx\|_{\mathbb{C}_0(\mathbb{R}^n)} = |c|\|x\|_{\mathbb{C}_0(\mathbb{R}^n)}$ since $\max_x |cf(x)| = |c| \max_x |f|$.

- Triangle Inequality : Since $|f(x)+g(x)| \leq |f(x)| + |g(x)|$ for $\forall x \in \mathbb{R}^n$, $\max_x |(f+g)(x)| = \max_x |f(x)+g(x)| \leq \max_x |f(x)| + \max_x |g(x)|$. Thus $\|f+g\|_{\mathbb{C}_0(\mathbb{R}^n)} \leq \|f\|_{\mathbb{C}_0(\mathbb{R}^n)} + \|g\|_{\mathbb{C}_0(\mathbb{R}^n)}$ for $\forall f, g \in \mathbb{C}_0(\mathbb{R}^n)$.

Thus, $\mathbb{C}_0(\mathbb{R}^n)$ is normed space.

why is it well-defined?

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To show $\mathbb{C}_0(\mathbb{R}^n)$ is complete, let $\{f_n\}_{n\in\mathbb{N}}$ be a norm Cauchy sequence in $\mathbb{C}_0(\mathbb{R}^n)$.
We can find a measurable set E with $\mu(\mathrm{E}^c) = 0$ such that for $\forall x \in \mathrm{E}$, $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\mathbb{C}_0(\mathbb{R}^n)}$ for $\forall n, m \in \mathbb{N}$.
This implies that $\{f_n(x)\}$ is a uniform Cauchy sequence on E.
Define $f(x) = \lim_{n\to\infty} f_n(x)$ for $x \in \mathrm{E}$ and $f(x) = 0$ for $x \in \mathrm{E}^c$.
Then f measurable and $|f_n - f|$ converges uniformly to zero on E. why?
Thus, $\|f_n - f\|_{\mathbb{C}_0(\mathbb{R}^n)} \to 0$ as $n \to \infty$.
Since $|f(x)| \leq |f_n(x)| + |f(x) - f_n(x)| \leq \|f_n\|_{\mathbb{C}_0(\mathbb{R}^n)} + \|f - f_n\|_{\mathbb{C}_0(\mathbb{R}^n)}$ a.e, $f \in \mathbb{C}_0(\mathbb{R}^n)$ and $\mathbb{C}_0(\mathbb{R}^n)$ is complete.

not enough.

In conclusion, $\mathbb{C}_0(\mathbb{R}^n)$ is Banach Space.

Check continuity of f & the condition $f(x) \to 0$ as $|x| \to \infty$

Problem 4.

- To show uniform continuity for $x \mapsto \langle x, y\rangle$ given fixed $y \in \mathcal{H}$, by Schwartz inequality, $\forall x_1, x_2 \in \mathcal{H}$, $|\langle x_1, y\rangle - \langle x_2, y\rangle| = |\langle x_1 - x_2, y\rangle| \leq \|x_1 - x_2\|_{\mathcal{H}}\|y\|_{\mathcal{H}}$
  if $\|x_1 - x_2\|_{\mathcal{H}} < \frac{\epsilon}{\|y\|_{\mathcal{H}}}$. → cover the case $y \equiv 0$ separately.
  Thus, map $x \mapsto \langle x, y \rangle$ is uniformly continuous with respect to $\|\cdot\|_{\mathcal{H}}$.

- To show uniform continuity for $x \mapsto \|x\|_{\mathcal{H}}$,

$$\|x_1\|_{\mathcal{H}} = \|x_1 - x_2 + x_2\|_{\mathcal{H}} \leq \|x_1 - x_2\|_{\mathcal{H}} + \|x_2\|_{\mathcal{H}}$$

$$|\|x_1\|_{\mathcal{H}} - \|x_2\|_{\mathcal{H}}| \leq \|x_1 - x_2\|_{\mathcal{H}}$$

  Thus $\|\cdot\|_{\mathcal{H}}$ is uniformly continuous.

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Problem 5.

Let kernel of l be $\mathrm{K}(l) = \{h \in \mathcal{H} \quad \text{s.t} \quad l(h) = 0\}$.
$\forall h \in \mathcal{H}$, $h = k + p$ for some $k \in \mathrm{K}(l)$ and $p \in \mathrm{K}(l)^{\perp}$.
~~Let $p_0 \in \mathrm{K}(l)^{\perp}$ such that $l(p_0) = 1$.~~ Why is it possible to choose such p_0?
Then, $l(h - l(h)p_0) = l(h) - l(h)\cancel{l(p_0)} = l(h)(1 - \cancel{l(p_0)}) = 0$.
Thus, $h - l(h)p_0 \in \mathrm{K}(l)$. Since $p_0 \in \mathrm{K}(l)^{\perp}$,

$$\langle h - l(h)p_0, p_0 \rangle_{\mathcal{H}} = 0$$

$$\langle h, p_0 \rangle_{\mathcal{H}} - l(h)\langle p_0, p_0 \rangle_{\mathcal{H}} = 0$$

$$l(h) = \left\langle h, \frac{p_0}{\|p_0\|_{\mathcal{H}}^2} \right\rangle_{\mathcal{H}}$$

$$\therefore \quad g = \frac{p_0}{\|p_0\|_{\mathcal{H}}^2}$$

Consider the case $K(l) = \mathcal{H}$ separately.

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