Functional Analysis HW5

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December 14, 2020

Problem 21.

(Late submission) Since $\tau_{\mathbb{K}}$ in $D(\mathbb{K})$ is Frechet space topology (for all compact subset K), any Cauchy sequence has limit in $C_c^\infty(\mathbb{R})$.
Assume $\{\psi_n\}$ is Cauchy sequence.
Let $\psi := \lim_{n\to\infty} \psi_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k}\phi(x-k)$.
Then,

$$\begin{aligned}
\|\psi_n - \psi\|_\alpha &= \max\{|D^\alpha(\psi_n(x) - \psi(x))| : x \in \mathbb{R}\} \\
&= \max\left\{\left|D^\alpha\left(\sum_{k=n+1}^{\infty} \frac{1}{k}\phi(x-k)\right)\right| : x \in \mathbb{R}\right\} \\
&= \max\left\{\left|D^\alpha\left(\sum_{k=n+1}^{\infty} \frac{1}{k}\phi(x-k)\right)\right| : x \in \mathbb{R}\right\} \\
&= \frac{1}{n+1}\max\{D^\alpha\phi(x) : x \in [0,1]\} \to 0
\end{aligned}$$

as $n \to \infty$.
Thus ψ is limit of ψ_n in $D(\mathbb{R})$ (Not enough).
However, ψ is not compactly supported.
Thus it is contradiction.
In conclusion, $\{\psi_n\}$ is not Cauchy sequence in $D(\mathbb{R})$.

(To solve this problem, you may use theorem 6.5 of Rudin.)

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Problem 22.

Since

$$\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon} \frac{\phi(x)}{x}dx = \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} \frac{\phi(x)}{x}dx + \lim_{\epsilon\to 0}\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}dx = \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} \frac{\phi(x)}{x}dx - \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty} \frac{\phi(-x)}{x}dx$$

$$= \lim_{\epsilon \to 0} \int_{|x|\geq\epsilon}^{\infty} \frac{\phi(x) - \phi(-x)}{x} dx$$

And $\dfrac{\phi(x) - \phi(-x)}{x} \leq 2 \sup|\phi'|$, $\lim_{\epsilon\to 0} \int_{|x|\geq\epsilon}^{\infty} \dfrac{\phi(x) - \phi(-x)}{x} dx$ exists.

Thus, $\left| pv\left(\dfrac{1}{x}\right)(\phi) \right| \leq 2a \sup|\phi'|$ while $a = \sup|x|$ for $x \in supp\, \phi$.

Therefore, principal value integration is a distribution. (You can use theorem 6.8 of Rudin)

Let $A_\epsilon = \{x \in \mathbb{R} | |x| \geq \epsilon\}$ and $B_\epsilon = \{x \in \mathbb{R} | |x| = \epsilon\}$.

Then,

$$\int_{A_\epsilon} (\log|x|)'(\phi)\, dx = \int_{A_\epsilon} \phi(x)(\log|x|)'\, dx - \int_{B_\epsilon} \phi'(x) \log|x|\, dx$$

$$= \int_{A_\epsilon} \frac{\phi(x)}{x} dx - \int_{B_\epsilon} \phi'(x) \log|x|\, dx$$

(how do you handle $\int_{B_\epsilon} \phi'(x) \log|x| dx$ and the region $\{|x| < \epsilon\}$?)

If we let $\epsilon \to 0$, $pv(\frac{1}{x})(\phi) = \lim_{\epsilon\to 0} \int_{|x|\geq\epsilon} \dfrac{\phi(x)}{x} dx$ is weak derivative of $\log|x|$.

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Problem 23.

(Late submission) Since $\Delta G_2(x) = 0$ whenever $|x| > 0$, for a fixed $\phi \in D(\mathbb{R}^2)$,

Let $I(r) = \int_{|x|>r} \Delta\phi(x) G(x) dx$.

Since $\phi(x)$ has compactly support, $\exists R > 0$ such that $\phi(x) = 0$ when $|x| > R$.

Define $A_r = \{x | r \leq |x| \leq R\}$. Then,

$$I(r) = \int_{A_r} \Delta\phi(x) G(x) dx = -\int_{A_r} \nabla\phi \cdot \nabla G + \int_{|x|=r} G\nabla\phi \cdot \nu = -\int_{|x|=r} \phi\nabla G \cdot \nu + \int_{|x|=r} G\nabla\phi \cdot \nu$$

where ν is the outward unit normal vector to A_r.

(you also need to consider $\{|x| < r\}$)

On the circle $|x| = r$, $\Delta G \cdot \nu = \dfrac{1}{\pi r}$ and $-\int_{|x|=r} \phi\nabla G \cdot \nu = \int_{|x|=r} \dfrac{\phi(w)}{\pi r} dw$.

By mean value theorem, integral converges to $\phi(0)$ as $r \to 0$.

Also, since $\Delta\phi$ is bounded, there exists a constant $C > 0$ such that

$$\left| \int_{|x|=r} G\Delta\phi \cdot \nu \right| \leq C \int_{|x|=r} |G| = C_r$$

Thus, integral converges to 0 as $r \to 0$.

Therefore, $I(r) \to \phi(0)$ as $r \to 0$.

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Problem 24.

(Late submission) Let ϕ be any test function with compact support.
$\chi'_{[0,1]}(\phi) = -\chi_{[0,1]}(\phi')$.
Assume $\phi'(x) = 0$ when $x \geq R$ for $1 > R > 0$.
Then, $= -\int_0^1 \phi' = -\int_0^R \phi' = \phi(0) - \phi(R) = \phi(0)$
Thus, $\chi'_{[0,1]}(\phi) = \phi(0) = \delta(\phi)$.
Then, $\dfrac{\partial u}{\partial t} = -c\delta(x-ct)$ and $\dfrac{\partial u}{\partial x} = \delta(x-ct)$.
Also, $\dfrac{\partial^2 u}{\partial^2 t} = c^2\delta'(x-ct)$ and $\dfrac{\partial^2 u}{\partial^2 x} = \delta'(x-ct)$.
Therefore, $\dfrac{\partial^2 u}{\partial^2 t} - c^2\dfrac{\partial^2 u}{\partial^2 x} = 0$ in distribution sense.

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Problem 25.

Since set A has positive measure, $\int \chi_A \, d\mu = m(A) > 0$. Then,

$$\begin{aligned}
\int (\chi_A * \chi_{-A})(x)dx &= \int\int \chi_A(y)\chi_{-A}(x-y)\, dydx \\
&= \int\int \chi_A(y)\chi_A(y-x)\, dydx \\
&= \int \chi_A(y) \int \chi_A(y-x)\, dxdy \\
&= m(A) \int \chi_A(y)dy \\
&= m(A)\cdot m(A) > 0
\end{aligned}$$

Since $\chi_A * \chi_{-A}$ is continuous, $S := \{x \in [0,1] \,|\, (\chi_A * \chi_{-A})(x) > 0\} \neq \emptyset$ and $S \subset A - A$.
Thus, $A - A$ contains an open interval.

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