Functional Analysis HW3

Kim Juhyeong

November 8, 2020

Problem 11.

For $X = L^2(\mathbb{R})$, dual space is $X^* = L^2(\mathbb{R})$, too.
The norm defined in X^* is for some $f \in X^*$, $\|f\|_{X^*} = (\int_{\mathbb{R}} |f|^2 \, d\mu)^{\frac{1}{2}}$.

Define $u_n \in L^2(\mathbb{R})$ such that:

$$x \in \mathbb{R}, u_n(x) = \begin{cases} n^{\frac{1}{2}}, & \text{if } x \in [0, 1/n] \\ 0, & \text{otherwise} \end{cases}$$

Then,

$$\forall n, \|u_n\|_X = \left(\int_0^{\frac{1}{n}} |n^{\frac{1}{2}}|^2 \, d\mu \right)^{\frac{1}{2}} = \left(\int_0^{\frac{1}{n}} n \, d\mu \right)^{\frac{1}{2}} = n \left(\frac{1}{n} - 0 \right) = 1$$

However,

$$\forall f \in X^*, f(u_n) = \int_0^{\frac{1}{n}} f u_n \, d\mu = \int_0^{\frac{1}{n}} f \sqrt{n} \, d\mu \to 0 \text{ as } n \to \infty \; \cancel{\text{(In almost everywhere sense.)}}$$

(it's much better to add some explanation supporting the above assertion, e.g. Cauchy-Schwarz inequality.)
which implies $u_n \rightharpoonup 0$.

Therefore, u_n is weakly convergent but not strongly convergent to constant function 0.

10/10

Problem 12.

$\psi_n \rightharpoonup \psi$, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N$, $\forall l \in \mathcal{H}^*$, $\exists \{l_k\}_{k=1}^{\infty} \subset \mathcal{H}$ s.t $\|l_k - l\|_{\mathcal{H}} \leq \epsilon$ for some $\epsilon > 0$. Then,

$$\begin{aligned} |l(\psi_n) - l(\psi)| &\leq |l(\psi_n) - l_k(\psi_n)| + |l_k(\psi_n) - l_k(\psi)| + |l_k(\psi) - l(\psi)| \\ &\leq \|l - l_k\|_{\mathcal{H}^*} \|\psi_n\|_{\mathcal{H}} + |l_k(\psi_n) - l_k(\psi)| + \|l_k - l\|_{\mathcal{H}^*} \|\psi\|_{\mathcal{H}} \\ &\leq \epsilon M + \epsilon \|\psi\|_{\mathcal{H}} + |l_k(\psi_n) - l_k(\psi)| \end{aligned}$$

If we take limit $n \to \infty$,

$$\begin{aligned} \limsup_{n\to\infty} |l(\psi_n) - l(\psi)| &\leq \epsilon M + \epsilon \|\psi\|_{\mathcal{H}} + \lim_{n\to\infty} |l_k(\psi_n) - l_k(\psi)| \\ &= \epsilon M + \epsilon \|\psi\|_{\mathcal{H}} \end{aligned}$$

Since we can choose ϵ arbitrarily close to 0, $l(\psi_n) \to l(\psi)$ as $n \to \infty$ which implies $\psi_n \rightharpoonup \psi$. Thus, if $\psi_n^{(\alpha)} \to \psi^{(\alpha)}$ for each α, $\psi_n \rightharpoonup \psi$.

Assume $\|\psi_n\|$ is bounded with some constant $M > 0$. i.e, $\|\psi_n\|_{\mathcal{H}} \leq M$.

If orthonormal basis $\{\varphi_\alpha\}_\alpha$ for $\mathcal{H}$ exists, for $\psi_n \in \mathcal{H}$, infinite linear combination of the orthonormal basis exists and its coefficient is projection with each element in basis.

$$\psi_n = \sum_\alpha \langle \psi_n, \varphi_\alpha \rangle \varphi_\alpha, \ \{\varphi_\alpha\}_\alpha \text{ orthonomal}$$

Then,

$$\text{for } \{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}, \forall l \in \mathcal{H}^*, \ l(\psi_n) - l(\psi) = l(\psi_n - \psi) = \sum_\alpha \langle \varphi_\alpha, \psi_n - \psi \rangle \, l(\varphi_\alpha)$$

Thus, if $l(\psi_n) - l(\psi) \to 0$, this implies $\langle \varphi_\alpha, \psi_n - \psi \rangle = 0$ for every α, since l can be arbitrary linear functional.(ex. identity mapping)

$6/10$

Please read your proof by yourself and consider how to reorganize and correct it. In the first part of your proof, the condition "Assume $\|\psi_n\|$ is bounded with some constant $M > 0$. i.e, $\|\psi_n\|_{\mathcal{H}} \leq M$" should appear at the beginning. Also, you should clarify l_k is the finite combination of $\{\langle \varphi_\alpha, \cdot \rangle\}$. For the second part, you should prove $\{\|\phi_n\|\}$ is bounded (if $\psi_n \rightharpoonup \psi$). In fact, if $\psi_n \rightharpoonup \psi$, it follows that $\psi_n^{(\alpha)} \to \psi^{(\alpha)}$ directly from the definition. You can shorten your reasoning.

Problem 13.

The weak-* topology on $L^1(\mathbb{R}) = (L^\infty(\mathbb{R}))^*$ can be defined as a family of seminorms$\{\|\cdot\|_\lambda : \lambda \in L^\infty(\mathbb{R})\}$ where $\|f\|_\lambda = |f(\lambda)| = \left|\int_{\mathbb{R}} f(x)\lambda(x)\, d\mu(x)\right|$ for $\forall \lambda \in L^\infty(\mathbb{R})$.

Since $\|f_n\|_\lambda = \left|\int_0^{\frac{1}{n}} n\lambda(x)\, d\mu(x)\right|$,

$$\lim_{n\to\infty}\|f_n\|_\lambda = \lim_{n\to\infty}|f_n(\lambda)| = |\lambda(0)| \quad \text{(could you justify it for arbitrary } \lambda \in L^\infty(\mathbb{R})\text{?)}$$

Let $\lambda_0 \in L^\infty(\mathbb{R})$ such that $\lambda_0(x) = 1$ for $\forall x \in \mathbb{R}\backslash\{0\}$ and $\lambda_0(0) = 0$.
Then, $\|f_n\|_{\lambda_0} = \left|\int_0^{\frac{1}{n}} n\, d\mu(x)\right| = 1$ for $\forall n \in \mathbb{R}$.
But $\lim_{n\to\infty}\|f_n\|_\lambda = 0$.
Thus, it is not sequentially compact.
Therefore, $\{f_n\}$ has no convergent subsequence in weak-* topology.

6/10

It would be better to consider $\lambda = 1_K$ where K is a compact set in $\mathbb{R}$ and $0 \notin K$.

Problem 14.

Let T be bounded linear operator from X to Y.
And T^* be Banach space adjoint of T.
For $\forall \Lambda \in Y^*$,

$$\|T^*\Lambda\|_{X^*} = \|\Lambda T\|_{\mathcal{L}(X,Y)} \le \|\Lambda\|_{Y^*}\|T\|_{\mathcal{L}(X,Y)}$$

Since Λ is bounded linear functional on X and T is bounded linear operator on X.
Then,

$$\frac{\|T^*\Lambda\|_{X^*}}{\|\Lambda\|_{Y^*}} \le \|T\|_{\mathcal{L}(X,Y)}$$

which implies $\|T^*\|_{\mathcal{L}(Y^*,X^*)} \le \|T\|_{\mathcal{L}(X,Y)}$.

Let $x_0 \in X$ with $\|x_0\|_X = 1$.
By the one of the Corollaries of the Hahn-Banach thorem, there exists an element $\Lambda_0 \in Y^*$, $\|\Lambda_0\|_{Y^*} = 1$ such that $\Lambda_0 T x_0 = \|Tx_0\|_Y$.

Therefore,

$$\|T^*\|_{\mathcal{L}(Y^*,X^*)} = \sup_{\|\Lambda\|=1}\|T^*\Lambda\|_{X^*} \ge \|T^*\Lambda\|_{X^*} = \|\Lambda T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1}|(\Lambda T)x| \ge |(\Lambda T)x_0| = \|Tx_0\|_Y$$

Thus, $\|Tx_0\| \le \|T^*\|$.
Therefore, for $\forall x \in X$ with $x \neq 0$,

$$\|T^*\|_{\mathcal{L}(Y^*,X^*)} \ge \left\|T\frac{x}{\|x\|_X}\right\|_Y \ge \frac{\|Tx\|_Y}{\|x\|_X}$$

which implies $\|T\|_{\mathcal{L}(X,Y)} \le \|T^*\|_{\mathcal{L}(Y^*,X^*)}$.
Thus, $\|T\|_{\mathcal{L}(X,Y)} = \|T^*\|_{\mathcal{L}(Y^*,X^*)}$.

10/10

Problem 15.

If T is self-adjoint, $\forall x \in \mathcal{H}$,

$$\langle x, Tx \rangle = \langle T^* x, x \rangle = \langle Tx, x \rangle$$

Also, since $\langle Tx, x \rangle = \overline{\langle x, Tx \rangle}$, $\langle x, Tx \rangle \in \mathbb{R}$ for $\forall x \in \mathcal{H}$.

If $\forall x \in \mathcal{H}$, $\langle x, Tx \rangle = \langle Tx, x \rangle$,

$$\langle x, Tx \rangle = \langle Tx, x \rangle = \langle x, T^* x \rangle$$

Thus, $T = T^*$ and T is self-adjoint.

5/10