

Hopf Algebras Acting on Quantum Planes

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1 Hopf Algebra Definitions

Definition 1.1. A Hopf Algebra, $(H, \nabla, \eta, \Delta, \varepsilon, S)$, is a bialgebra H over a field \mathbb{k} with an antipode $S : H \rightarrow H$ where the bialgebra has product $\nabla : H \otimes H \rightarrow H$, unit $\eta : \mathbb{k} \rightarrow H$, coproduct $\Delta : H \rightarrow H \otimes H$, counit $\varepsilon : H \rightarrow \mathbb{k}$ such that the following diagrams commute

$$\begin{array}{c}
 \text{Associativity:} \quad \begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ \downarrow id \otimes \nabla & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Unit:} \quad \begin{array}{ccc} & H \otimes H & \\ \eta \otimes id \nearrow & \downarrow \nabla & \nwarrow id \otimes \eta \\ \mathbb{k} \otimes H & & H \otimes \mathbb{k} \\ \searrow = & & \swarrow = \\ & H & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Coassociativity:} \quad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow \Delta & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Counit:} \quad \begin{array}{ccc} & H & \\ \swarrow = & \downarrow \Delta & \searrow = \\ \mathbb{k} \otimes H & & H \otimes \mathbb{k} \\ \nwarrow \varepsilon \otimes id & & \nearrow id \otimes \varepsilon \\ & H \otimes H & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Coproduct compatibility:} \quad \begin{array}{ccccc} H \otimes H & \xrightarrow{\nabla} & H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow \Delta \otimes \Delta & & & & \uparrow \nabla \otimes \nabla \\ H \otimes H \otimes H \otimes H & \xrightarrow{id \otimes \tau \otimes id} & H \otimes H \otimes H \otimes H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Unit compatibility:} \quad \begin{array}{ccc} \mathbb{k} & \xrightarrow{\eta} & H \\ & \searrow \eta \otimes \eta & \downarrow \Delta \\ & & H \otimes H \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Counit compatibility:} \quad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{k} \\ \uparrow \nabla & \nearrow \varepsilon \otimes \varepsilon & \\ H \otimes H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Antipode:} \quad \begin{array}{ccccc} H \otimes H & \xrightarrow{id \otimes S} & H \otimes H & & \\ \uparrow \Delta & & \downarrow \nabla & & \\ H & \xrightarrow{\varepsilon} & \mathbb{k} & \xrightarrow{\eta} & H \\ \downarrow \Delta & & \uparrow \nabla & & \\ H \otimes H & \xrightarrow{S \otimes id} & H \otimes H & & \end{array}
 \end{array}$$

For the sake of brevity, we write in general that $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, this is called Sweedler notation.

Example 1.2. Let G be any group and denote $\mathbb{k}G = \{\sum_{i=0}^{\infty} a_i g_i \mid a_i \in \mathbb{k}, g_i \in G, n \in \mathbb{N}\}$ The $\mathbb{k}G$ is a Hopf algebra called the group algebra of G . It has the product

$$\left(\sum_{i=0}^n a_i g_i\right) \left(\sum_{j=0}^m b_j g_j\right) = \sum_{i=0}^n \sum_{j=0}^m a_i b_j (g_i g_j)$$

where $a_i \cdot b_j$ is the product in \mathbb{k} and $g_i \cdot g_j$ is the product in the group. The unit is $1_{\mathbb{k}} 1_G$ where $1_{\mathbb{k}}$ is the unit of \mathbb{k} and 1_G is the identity element of the group. The coproduct is defined by $\Delta(g) = g \otimes g$ extended linearly to all of $\mathbb{k}G$, and the counit is $\varepsilon(g) = 1_{\mathbb{k}}$, again extended linearly. Finally, the antipode is $S(g) = g^{-1}$.

Note that group algebras are always cocommutative, in other words $\nabla(h) = \tau \circ \nabla(h)$ for all $h \in \mathbb{k}G$, where $\tau(a \otimes b) = b \otimes a$, and are commutative if and only if G is abelian.

Definition 1.3. For a Hopf algebra H , $G(H) = \{g \in H \mid \Delta g = g \otimes g\}$ is called the set of grouplike elements of H .

Example 1.4. Let \mathfrak{g} be a Lie algebra and $U(\mathfrak{g})$ the corresponding Universal Enveloping algebra. Then $U(\mathfrak{g})$ is naturally an algebra, but also has a Hopf algebra structure. The coproduct is given by $\Delta x = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$.

Definition 1.5. For a Hopf algebra H , $P(H) = \{x \in H \mid \Delta x = x \otimes 1 + 1 \otimes x\}$ is called the set of primitive elements of H . Generally, one can define the skew-primitive elements as $P_{a,b} = \{x \in H \mid \Delta x = x \otimes a + b \otimes x\}$.

Example 1.6. In his seminal book, [16], Sweedler defined a 4-dimensional non-commutative, non-cocommutative Hopf algebra

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1} \quad S(x) = -xg^{-1}.$$

Example 1.7. In [10], Kac and Paljutkin defined an 8-dimensional non-commutative, non-cocommutative Hopf Algebra

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, xz = zy, yz = zx, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$$

with operations

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \Delta(z) &= \frac{1}{2}(1 \otimes 1 \otimes + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z) \\ \varepsilon(x) &= 1, & \varepsilon(y) &= 1, & \varepsilon(z) &= 1 \\ S(x) &= x, & S(y) &= y, & S(z) &= z \end{aligned}$$

Example 1.8.

Definition 1.9. A left action of a Hopf algebra on a vector space V is a tuple (α, V) so that $\alpha : H \otimes V \rightarrow V$ is a map satisfying the diagrams

$$\begin{array}{ccc} H \otimes H \otimes V & \xrightarrow{\nabla \otimes id} & H \otimes V \\ \downarrow id \otimes \alpha & & \downarrow \alpha \\ H \otimes V & \xrightarrow{\alpha} & V \end{array} \quad \begin{array}{ccc} \mathbb{k} \otimes V & \xrightarrow{\eta \otimes id} & H \otimes V \\ & \searrow = & \downarrow \alpha \\ & & V \end{array}$$

In this case, V is called a left Hopf-module.

For the rest of this paper we suppress the action α and instead write $\alpha(h \otimes v) = {}^h v$.

Definition 1.10. A right coaction, ρ , of H on V is a tuple (ρ, V) with $\rho : V \rightarrow V \otimes H$ so that the following commute

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes H \\ \rho \downarrow & & \downarrow id \otimes \Delta \\ V \otimes H & \xrightarrow{\rho \otimes id} & V \otimes H \otimes H \end{array} \quad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes H \\ & \searrow = & \downarrow id \otimes \varepsilon \\ & & V \otimes \mathbb{k} \end{array}$$

In this case, V is called a right Hopf-comodule.

Definition 1.11. When an algebra A is a left Hopf-module with action α , we call it a left Hopf-module algebra if it also satisfies the diagrams

$$\begin{array}{ccc} H \otimes A \otimes A & \xrightarrow{\nabla} & H \otimes A \xrightarrow{\alpha} A \xleftarrow{\nabla} A \otimes A \\ \Delta \otimes id \otimes id \downarrow & & \uparrow \alpha \otimes \alpha \\ H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A \end{array} \quad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{k} \xrightarrow{\eta} A \\ \downarrow \eta & & \nearrow \alpha \\ H \otimes A & & \end{array}$$

Note that a Hopf algebra acts if and only if its dual coacts. (Write why here)

Definition 1.12. And when a coalgebra A is a an Hopf-module with action α , we call it an Hopf-module coalgebra if it also satisfies the diagrams

$$\begin{array}{ccc} H \otimes A & \xrightarrow{\alpha} & A \xrightarrow{\Delta} A \otimes A \\ \Delta \otimes id \otimes id \downarrow & & \uparrow \alpha \otimes \alpha \\ H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\varepsilon} & \mathbb{k} \\ \alpha \uparrow & & \nearrow \varepsilon \otimes \varepsilon \\ H \otimes A & & \end{array}$$

Similar diagrams give the conditions for Hopf-comodule algebras and Hopf-comodule coalgebras.

Definition 1.13. A useful and prevalent construction on Hopf algebras is given a Hopf algebra H and a left Hopf-module algebra A , the smash product algebra $A \# H$ is the algebra where $A \# H = A \otimes H$ as a \mathbb{k} -vector space and has product

$$(a \otimes h)(b \otimes k) = \sum a \ {}^{h_{(1)}}b \otimes h_{(2)}k.$$

Definition 1.14. A Hopf algebra is called pointed if all of its left (right) comodules are 1-dimensional.

Definition 1.15. If $I \subseteq H$ and for any $h \in H$, $hI \subseteq I$, then I is called an ideal of H . If $\Delta(I) \subseteq I \otimes H + H \otimes I$, then I is called a coideal of H . If I is both an ideal and a coideal, it is called a biideal. Finally, if I is a biideal and $S(I) \subseteq I$, then I is called a Hopf ideal of H .

Lemma 1.16. If I is a Hopf ideal of H , then the quotient H/I is a Hopf algebra.

Definition 1.17. If H acts on an algebra A and there is a Hopf ideal I so that $I \cdot A = 0$, then we say the action of H factors through the quotient H/I . In particular, if H/I is isomorphic to a group algebra, we say the action of H factors through a group action.

2 Big Questions

Juan Cuadra, Pavel Etingof and Chelsea Walton have been classifying algebras with Hopf actions for which the action factors through a group action [7]. For example, they have shown that any action by a semi-simple, finite dimensional Hopf algebra on an integral domain always has this property. As well, they have shown that any action by a finite dimensional Hopf algebra on a Weyl algebra also has this property. This is a component of a larger search for algebras on which Hopf algebras act.

Andruskiewitsch and Schneider have been classifying pointed Hopf algebras [1].

Kenneth Chan, Ellen Kirkman, Jim Kuzmanovich, Chelsea Walton, and James Zhang have a series of works on Hopf algebras acting on AS-regular algebras [2] [3] [4] [11]. They have posed the question of when are the coinvariant subrings from these actions Artin-Schelter Gorenstein?

Miriam Cohen and Davida Fishman extended work by Fisher and Montgomery [8] and Cohen and Montgomery [6] to determine when $A \# H$ is semiprime for A an algebra and H a Hopf algebra. Specifically, if H is semi-simple and finite-dimensional and A is semiprime, they ask is $A \# H$ semiprime? [5]

Chelsea Walton and Sarah Witherspoon have been working towards PBW deformation conditions on $B \# H$ where B is a Koszul algebra and H a Hopf algebra. In [18] they are able to provide these conditions when the antipode of H is bijective, B is connected as an H -module algebra, and the action of H preserves the grading on B . In the same paper they pose the question if $H = U_q(\mathfrak{sl}_2)$, are there nontrivial PBW deformations of $B \# H$?

3 Taft Algebras

A Taft algebra, as defined in [2], is a Hopf algebra $T_{n,m} = \langle g, x \mid g^n = 1, x^n = 0, gx = \zeta xg \rangle$ where ζ is a primitive n -th root of unity. $T_{n,m}$ has the maps

$$\begin{aligned} \Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, & S(g) &= g^{-1}, & S(x) &= -xg^{-1}. \end{aligned}$$

A small example is the lowest dimension non-commutative, non-cocommutative Hopf algebra, the 4-dimensional Sweedler algebra. This is given by $H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx \rangle$ with operations

$$\begin{aligned}\Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, & S(g) &= g, & S(x) &= -xg.\end{aligned}$$

This algebra in particular acts on the AS-regular algebra $\mathbb{k}_{-1}[u, v] = \mathbb{k}[u, v]/(uv + vu)$ by

$$g \cdot u = u, \quad g \cdot v = -v, \quad x \cdot u = 0, \quad x \cdot v = v.$$

Let q be a primitive root of unity where $|q^2| = m > 1$, let $\alpha \in \mathbb{k}$ and $n \in \mathbb{Z}^+$ where $|q| \mid n$, then a generalized Taft algebra is a Hopf algebra

$$T_{q,\alpha,n} = \langle g, g^{-1}x \mid gg^{-1} = g^{-1}g = 1, xg = qgx, g^n = 1, x^m = \alpha(g^m - g^{-m}) \rangle$$

with maps

$$\begin{aligned}\Delta(g) &= g \otimes g, & \Delta(x) &= x \otimes g^{-1} + g \otimes x, \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0, & S(g) &= g^{-1}, & S(x) &= -qx.\end{aligned}$$

If $q^m \neq 1$, then $\alpha = 0$, and if $q^m = 1$, $\alpha \in \{0, 1\}$.

Another definition for a generalized Taft algebra given in [1] is to take a Yetter-Drinfeld module V of D_2 -type over the group algebra $k\mathbb{Z}/n\mathbb{Z}$. Then if $\mathcal{B}(V)$ is the Nichols algebra of V , the bosonization $(\mathcal{B}(V) \# k\mathbb{Z}/n\mathbb{Z})^{\text{cop}}$ is a generalized Taft algebra.

Let q be a primitive root of unity with $q^2 \neq 1$, then we can define the quantum polynomial ring $\mathbb{k}_q[u, v] = \mathbb{k}[u, v]/\langle uv - qvu \rangle$. The subring $U = \mathbb{k}u \oplus \mathbb{k}v$ is then a left $T_{q,\alpha,n}$ -module. If as a $T_{q,\alpha,n}$ -module U is not semi-simple, then in [2] it is proved that $T_{q,\alpha,n}$ coacts on $\mathbb{k}_q[u, v]$. This coaction is given by

$$\rho(u) = u \otimes g, \quad \rho(v) = v \otimes g^{-1} + u \otimes x.$$

The coaction induces an action by the finite dual $(T_{q,\alpha,n})^\circ$ on $\mathbb{k}_q[u, v]$.

4 8-dimensional Hopf Algebra

In [10], Kac and Paljutkin define an 8-dimensional, non-commutative, non-cocommutative, semi-simple Hopf algebra

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$$

with operations

$$\begin{aligned}\Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \Delta(z) &= \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), \\ \varepsilon(x) &= \varepsilon(y) = \varepsilon(z) = 1, & S(x) &= x^{-1}, & S(y) &= y^{-1}, & S(z) &= z.\end{aligned}$$

Then in [12], three representations of H_8 acting on quantum polynomial rings are given. For the ring $\mathbb{k}_q[x_1, x_2]$ where $q^2 = -1$, the first representation is given by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the ring $\mathbb{k}_Q[x_1, x_2, x_3, x_4]$ where $Q = (q_{ij})$, $x_j x_i = q_{ij} x_i x_j$ and $q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}$, $q_{14}^2 = 1$, $q_{23}^2 = -1$, we also have the representation

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And finally, for the ring $\mathbb{k}_{-1}[u, v]$, we get the representation

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5 Quantum Enveloping Algebras

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