

Hopf Algebras Acting on Quantum Planes

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1 Hopf Algebra Definitions

Definition 1.1. A Hopf Algebra $(H, \nabla, \eta, \Delta, \varepsilon, S)$ is a bialgebra H over a field \mathbb{C} with an antipode $S : H \rightarrow H$ where the bialgebra has product $\nabla : H \otimes H \rightarrow H$, unit $\eta : \mathbb{C} \rightarrow H$, coproduct $\Delta : H \rightarrow H \otimes H$, counit $\varepsilon : H \rightarrow \mathbb{C}$ such that the following diagrams commute

$$\begin{array}{c}
 \text{Associativity:} \quad \begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ \downarrow id \otimes \nabla & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \text{Unit:} \quad \begin{array}{ccccc} & & H \otimes H & & \\ \eta \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes \eta & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & \downarrow = & \nearrow = & \\ & & H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Coassociativity:} \quad \begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow \Delta & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \text{Counit:} \quad \begin{array}{ccccc} & & H & & \\ = \swarrow & & \downarrow \Delta & \searrow = & \\ \mathbb{C} \otimes H & & H \otimes H & & H \otimes \mathbb{C} \\ \nwarrow \varepsilon \otimes id & & \downarrow id \otimes \varepsilon & \nearrow id \otimes \varepsilon & \\ & & H \otimes H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Coproduct compatibility:} \quad \begin{array}{ccccc} H \otimes H & \xrightarrow{\nabla} & H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow \Delta \otimes \Delta & & & & \downarrow \nabla \otimes \nabla \\ H \otimes H \otimes H \otimes H & \xrightarrow{id \otimes \tau \otimes id} & H \otimes H \otimes H \otimes H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Unit compatibility:} \quad \begin{array}{ccc} \mathbb{C} & \xrightarrow{\eta} & H \\ \searrow \eta \otimes \eta & & \downarrow \Delta \\ & & H \otimes H \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \text{Counit compatibility:} \quad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{C} \\ \nwarrow \varepsilon \otimes \varepsilon & & \uparrow \nabla \\ H \otimes H & & \end{array}
 \end{array}$$

$$\begin{array}{c}
 \text{Antipode:} \quad \begin{array}{ccccc} H \otimes H & \xrightarrow{id \otimes S} & H \otimes H & & \\ \uparrow \Delta & & \downarrow \nabla & & \\ H & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & H \\ \downarrow \Delta & & \uparrow \nabla & & \\ H \otimes H & \xrightarrow{S \otimes id} & H \otimes H & & \end{array}
 \end{array}$$

For the sake of brevity, we write in general $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, following standard Sweedler notation for Hopf algebras.

Definition 1.2. A left action of a Hopf algebra on a vector space V is a tuple (α, V) so that $\alpha : H \otimes V \rightarrow V$ is a map so that the following diagrams commute,

$$\begin{array}{ccccc} H \otimes H \otimes V & \xrightarrow{\nabla \otimes id} & H \otimes V & \mathbb{C} \otimes V & \xrightarrow{\eta \otimes id} & H \otimes V \\ \downarrow id \otimes \alpha & & \downarrow \alpha & \searrow = & & \downarrow \alpha \\ H \otimes V & \xrightarrow{\alpha} & V & & & V \end{array}$$

In this case, V is called a left H -module.

We also require the following analogue of acting by automorphisms.

Definition 1.3. When an algebra A is a left H -module with action α , we call it a left H -module algebra if the following diagrams also commute

$$\begin{array}{ccccccc} H \otimes A \otimes A & \xrightarrow{\nabla} & H \otimes A & \xrightarrow{\alpha} & A & \xleftarrow{\nabla} & A \otimes A \\ \Delta \otimes id \otimes id \downarrow & & & & & & \uparrow \alpha \otimes \alpha \\ H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & & & H \otimes A \end{array} \quad \begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{\eta} A \\ \downarrow \eta & \nearrow \alpha & \\ H \otimes A & & \end{array}$$

We are particularly interested in actions on non-commutative algebras, like quantum polynomial rings:

Definition 1.4. Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ij}q_{ji}$. A quantum polynomial ring is $\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C}\langle v_1, \dots, v_n \mid v_j v_i = q_{ij} v_i v_j \rangle$.

Definition 1.5. A quantum group is a Hopf algebra H with a bijective antipode and an element $R \in H \otimes H$ satisfying

1. $R \left(\sum h_{(1)} \otimes h_{(2)} \right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$
2. $\Delta \otimes id(R) = R_{1,3} R_{2,3}$
3. $id \otimes \Delta(R) = R_{1,3} R_{1,2}$

where, writing $R = \sum R_{(1)} \otimes R_{(2)}$, then $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$.

2 Examples

Example 2.1. In his seminal book, "Hopf Algebras", Sweedler defined a 4-dimensional non-commutative, non-cocommutative Hopf algebra

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\Delta g = g \otimes g, \quad \Delta x = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1} \quad S(x) = -xg^{-1}.$$

$\mathbb{C}_{-1}[v_1, v_2]$ is an H_4 -module algebra via $g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

H_4 is a quantum group with an R -matrix $R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$.

Example 2.2. In "Finite Ring Groups", Kac and Paljutkin defined an 8-dimensional non-commutative, non-cocommutative Hopf Algebra

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, xz = zy, yz = zx, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$$

with operations

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \Delta(z) &= \frac{1}{2}(1 \otimes 1 \otimes + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z) \\ \varepsilon(x) &= 1, & \varepsilon(y) &= 1, & \varepsilon(z) &= 1 \\ S(x) &= x, & S(y) &= y, & S(z) &= z \end{aligned}$$

H_8 has as H_8 -module algebras, $\mathbb{C}_q[v_1, v_2]$ with $q^2 = -1$, $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ with $q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}$, $q_{14}^2 = 1$ and $q_{23}^2 = -1$, and $\mathbb{C}_{-1}[v_1, v_2]$. H_8 is a quantum group with 6 non-isomorphic quasitriangular structures.

Example 2.3. Described by Kulish and Reshetikhin in "Quantum linear problem for the sine-Gordon equation and higher representations",

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F,$$

$$KK^{-1} = 1 = K^{-1}K \rangle$$

with operations

$$\begin{aligned} \Delta(E) &= E \otimes 1 + K \otimes E, & \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, & \Delta(K) &= K \otimes K, & \Delta(K^{-1}) &= K^{-1} \otimes K^{-1}, \\ \varepsilon(E) &= 0, & \varepsilon(F) &= 0, & \varepsilon(K) &= 1, & \varepsilon(K^{-1}) &= 1, \\ S(E) &= -K^{-1}E, & S(F) &= -FK, & S(K) &= K^{-1}, & S(K^{-1}) &= K \end{aligned}$$

$\mathbb{C}_q[v_1, v_2]$ is a $\mathcal{U}_q(\mathfrak{sl}_2)$ -module algebra where $q^2 \neq 1$ via

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

The R -matrix of $\mathcal{U}_q(\mathfrak{sl}_2)$ is in a completion of $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$.

3 Smash Product Algebras

Definition 3.1. *Given a Hopf algebra H and a left Hopf-module algebra A , the smash product algebra $A\#H$ is the algebra where $A\#H = A \otimes H$ as a \mathbb{C} -vector space and has product*

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

Definition 3.2. *Let H be a Hopf algebra, $G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$ is called the collection of group-like elements and $P(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ is called the collection of primitive elements.*

Note that one can show $G(H)$ is a group under the product of H and $P(H)$ is a Lie algebra under the commutator bracket.

Theorem 3.3 (Cartier-Kostant-Milnor-Moore). *Let H be a cocommutative Hopf algebra over \mathbb{C} . Then as Hopf algebras,*

$$H \cong \mathcal{U}(P(H))\#CG(H).$$