Hopf Algebras Acting on Quantum Planes

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1 Hopf Algebra Definitions

Definition 1.1. *A Hopf Algebra $(H, \bigtriangledown, \eta, \bigtriangleup, \varepsilon, S)$ is a bialgebra H over a field $\mathbb{C}$ with an antipode $S : H \to H$ where the bialgebra has product $\bigtriangledown : H \otimes H \to H$, unit $\eta : \mathbb{C} \to H$, coproduct $\bigtriangleup : H \to H \otimes H$, counit $\varepsilon : H \to \mathbb{C}$ such that the following diagrams commute*

Associativity:
$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"] & H \end{tikzcd}$$

Unit:
$$\begin{tikzcd} & H \otimes H \arrow[dd, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

Coassociativity:
$$\begin{tikzcd} H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\ H \otimes H \arrow[r, "id \otimes \bigtriangleup"] & H \otimes H \otimes H \end{tikzcd}$$

Counit:
$$\begin{tikzcd} & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\ \mathbb{C} \otimes H & & H \otimes \mathbb{C} \\ & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] & \end{tikzcd}$$

Coproduct compatibility:
$$\begin{tikzcd} H \otimes H \arrow[r, "\bigtriangledown"] \arrow[d, "\bigtriangleup \otimes \bigtriangleup"] & H \arrow[r, "\bigtriangleup"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"] & & H \otimes H \otimes H \otimes H \arrow[u, "\bigtriangledown \otimes \bigtriangledown"'] \end{tikzcd}$$

Unit compatibility:
$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta \otimes \eta"] & H \arrow[d, "\bigtriangleup"] \\ & H \otimes H \end{tikzcd}$$

Counit compatibility:
$$\begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H \otimes H \arrow[u, "\bigtriangledown"] \arrow[ur, "\varepsilon \otimes \varepsilon"'] & \end{tikzcd}$$

Antipode:
$$\begin{tikzcd} H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \arrow[u, "\bigtriangleup"] \arrow[d, "\bigtriangleup"'] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[u, "\bigtriangledown"'] \end{tikzcd}$$

For the sake of brevity, we write in general $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$, following standard Sweedler notation for Hopf algebras.

Definition 1.2. *A left action of a Hopf algebra on a vector space V is a tuple (α, V) so that $\alpha : H \otimes V \to V$ is a map so that the following diagrams commute*

$$
\begin{tikzcd}
H \otimes H \otimes V \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes V \arrow[d, "\alpha"] \\
H \otimes V \arrow[r, "\alpha"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \otimes V \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes V \arrow[d, "\alpha"] \\
 & V
\end{tikzcd}
$$

In this case, V is called a left H-module.

We also require the following analogue of acting by automorphisms.

Definition 1.3. *When an algebra A is a left H-module with action α, we call it a left H-module algebra if the following diagrams also commute*

$$
\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\
H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\eta"] & \mathbb{C} \arrow[r, "\eta"] & A \\
H \otimes A \arrow[urr, "\alpha"] & &
\end{tikzcd}
$$

We are particularly interested in actions on non-commutative algebras, like quantum polynomial rings:

Definition 1.4. *Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ij}q_{ji}$. A quantum polynomial ring is $\mathbb{C}_Q[v_1, \ldots, v_n] = \mathbb{C}\langle v_1, \ldots, v_n | v_j v_i = q_{ij} v_i v_j \rangle$.*

Definition 1.5. *A quantum group is a Hopf algebra H with a bijective antipode and an element $R \in H \otimes H$ satisfying*

1. $R \left(\sum h_{(1)} \otimes h_{(2)} \right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$
2. $\triangle \otimes id(R) = R_{1,3}R_{2,3}$
3. $id \otimes \triangle(R) = R_{1,3}R_{1,2}$

where, writing $R = \sum R_{(1)} \otimes R_{(2)}$, then $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$.

2 Examples

Example 2.1. *In his seminal book, "Hopf Algebras", Sweedler defined a 4-dimensional non-commutative, non-cocommutative Hopf algebra*

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\triangle g = g \otimes g, \quad \triangle x = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1} \quad S(x) = -xg^{-1}.$$

$\mathbb{C}_{-1}[v_1, v_2]$ *is an* H_4*-module algebra via* $g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
H_4 *is a quantum group with an R-matrix* $R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$.

Example 2.2. *In "Finite Ring Groups", Kac and Paljutkin defined an* 8*-dimensional non-commutative, non-cocommutative Hopf Algebra*

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, xz = zy, yz = zx, z^2 = \tfrac{1}{2}(1 + x + y - xy) \rangle$$

with operations

$$\begin{array}{lll} \triangle(x) = x \otimes x, & \triangle(y) = y \otimes y, & \triangle(z) = \frac{1}{2}(1 \otimes 1 \otimes +1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z) \\ \varepsilon(x) = 1, & \varepsilon(y) = 1, & \varepsilon(z) = 1 \\ S(x) = x, & S(y) = y, & S(z) = z \end{array}$$

H_8 *has as* H_8*-module algebras,* $\mathbb{C}_q[v_1, v_2]$ *with* $q^2 = -1$, $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ *with* $q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}, q_{14}^2 = 1$ *and* $q_{23}^2 = -1$, *and* $\mathbb{C}_{-1}[v_1, v_2]$. H_8 *is a quantum group with* 6 *non-isomorphic quasitriangular structures.*

Example 2.3. *Described by Kulish and Reshetikhin in "Quantum linear problem for the sine-Gordon equation and higher representations",*

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), KEK^{-1} = q^2 E, KFK^{-1} = q^{-2} F,$$

$$KK^{-1} = 1 = K^{-1}K \rangle$$

with operations

$$\begin{array}{llll} \triangle(E) = E \otimes 1 + K \otimes E, & \triangle(F) = F \otimes K^{-1} + 1 \otimes F, & \triangle(K) = K \otimes K, & \triangle(K^{-1}) = K^{-1} \otimes K^{-1}, \\ \varepsilon(E) = 0, & \varepsilon(F) = 0, & \varepsilon(K) = 1, & \varepsilon(K^{-1}) = 1, \\ S(E) = -K^{-1}E, & S(F) = -FK, & S(K) = K^{-1}, & S(K^{-1}) = K \end{array}$$

$\mathbb{C}_q[v_1, v_2]$ *is a* $\mathcal{U}_q(\mathfrak{sl}_2)$*-module algebra where* $q^2 \neq 1$ *via*

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

The R-matrix of $\mathcal{U}_q(\mathfrak{sl}_2)$ *is in a completion of* $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$.

3 Smash Product Algebras

Definition 3.1. *Given a Hopf algebra H and a left Hopf-module algebra A, the smash product algebra $A\#H$ is the algebra where $A\#H = A \otimes H$ as a $\mathbb{C}$-vector space and has product*

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

Definition 3.2. *Let H be a Hopf algebra, $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$ is called the collection of group-like elements and $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$ is called the collection of primitive elements.*

Note that one can show $G(H)$ is a group under the product of H and $P(H)$ is a Lie algebra under the commutator bracket.

Theorem 3.3 (Cartier-Kostant-Milnor-Moore)**.** *Let H be a cocommutative Hopf algebra over $\mathbb{C}$. Then as Hopf algebras,*

$$H \cong \mathcal{U}(P(H))\#\mathbb{C}G(H).$$