Hopf Algebras Acting on Quantum Planes

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1   Hopf Algebra Definitions

Definition 1.1. *A Hopf Algebra, $(H, \triangledown, \eta, \triangle, \varepsilon, S)$, is a bialgebra H over a field $\mathbb{C}$ with an antipode $S : H \to H$ where the bialgebra has product $\triangledown : H \otimes H \to H$, unit $\eta : \mathbb{C} \to H$, coproduct $\triangle : H \to H \otimes H$, counit $\varepsilon : H \to \mathbb{C}$ such that the following diagrams commute*

$Associativity:$

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\triangledown \otimes id} & H \otimes H \\ \downarrow{\scriptstyle id \otimes \triangledown} & & \downarrow{\scriptstyle \triangledown} \\ H \otimes H & \xrightarrow{\triangledown} & H \end{array}$$

$Unit:$

Diagram with $H \otimes H$ at top, $\mathbb{C} \otimes H$, $H \otimes \mathbb{C}$, H; arrows $\eta \otimes id$, $id \otimes \eta$, $\triangledown$, and two $=$.

$Coassociativity:$

$$\begin{array}{ccc} H & \xrightarrow{\triangle} & H \otimes H \\ \downarrow{\scriptstyle \triangle} & & \downarrow{\scriptstyle \triangle \otimes id} \\ H \otimes H & \xrightarrow{id \otimes \triangle} & H \otimes H \otimes H \end{array}$$

$Counit:$

Diagram with H at top, $\mathbb{C} \otimes H$, $H \otimes \mathbb{C}$, $H \otimes H$; arrows $=$, $=$, $\triangle$, $\varepsilon \otimes id$, $id \otimes \varepsilon$.

$Coproduct\ compatibility:$

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\triangledown} & H & \xrightarrow{\triangle} & H \otimes H \\ \downarrow{\scriptstyle \triangle \otimes \triangle} & & & & \uparrow{\scriptstyle \triangledown \otimes \triangledown} \\ H \otimes H \otimes H \otimes H & & \xrightarrow{id \otimes \tau \otimes id} & & H \otimes H \otimes H \otimes H \end{array}$$

$Unit\ compatibility:$

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\eta} & H \\ & {\scriptstyle \eta \otimes \eta} \searrow & \downarrow{\scriptstyle \triangle} \\ & & H \otimes H \end{array}$$

$Counit\ compatibility:$

$$\begin{array}{ccc} H & \xrightarrow{\varepsilon} & \mathbb{C} \\ {\scriptstyle \triangledown} \uparrow & \nearrow{\scriptstyle \varepsilon \otimes \varepsilon} & \\ H \otimes H & & \end{array}$$

$Antipode:$

$$\begin{array}{ccccc} H \otimes H & & \xrightarrow{id \otimes S} & & H \otimes H \\ {\scriptstyle \triangle} \uparrow & & & & \downarrow{\scriptstyle \triangledown} \\ H & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & H \\ {\scriptstyle \triangle} \downarrow & & & & \uparrow{\scriptstyle \triangledown} \\ H \otimes H & & \xrightarrow{S \otimes id} & & H \otimes H \end{array}$$

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For the sake of brevity, we write in general that $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$, this is called Sweedler notation.

Definition 1.2. *A left action of a Hopf algebra on a vector space V is a tuple (α, V) so that $\alpha : H \otimes V \to V$ is a map satisfying the diagrams*

$$
\begin{array}{ccc}
H \otimes H \otimes V & \xrightarrow{\triangledown \otimes id} & H \otimes V \\
{\scriptstyle id \otimes \alpha} \downarrow & & \downarrow {\scriptstyle \alpha} \\
H \otimes V & \xrightarrow{\alpha} & V
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{C} \otimes V & \xrightarrow{\eta \otimes id} & H \otimes V \\
& {\scriptstyle =} \searrow & \downarrow {\scriptstyle \alpha} \\
& & V
\end{array}
$$

In this case, V is called a left H-module.

Definition 1.3. *When an algebra A is a left H-module with action α, we call it a left H-module algebra if the following diagrams also commute*

$$
\begin{array}{ccccc}
H \otimes A \otimes A & \xrightarrow{\triangledown} & H \otimes A & \xrightarrow{\alpha} & A & \xleftarrow{\triangledown} & A \otimes A \\
{\scriptstyle \triangle \otimes id \otimes id} \downarrow & & & & & & \uparrow {\scriptstyle \alpha \otimes \alpha} \\
H \otimes H \otimes A \otimes A & & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & & & H \otimes A \otimes H \otimes A
\end{array}
\qquad
\begin{array}{ccccc}
H & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & A \\
{\scriptstyle \eta} \downarrow & & & {\scriptstyle \alpha} \nearrow & \\
H \otimes A & & &
\end{array}
$$

Definition 1.4. *Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ij}q_{ji}$. A quantum polynomial ring is $\mathbb{C}_Q[v_1, \ldots, v_n] = \mathbb{C}\langle v_1, \ldots, v_n | v_j v_i = q_{ij} v_i v_j \rangle$.*

Definition 1.5. *A quantum group is a Hopf algebra H with a bijective antipode and an element $R \in H \otimes H$ satisfying*

1. *$R \left(\sum h_{(1)} \otimes h_{(2)} \right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$*

2. *$\triangle \otimes id(R) = R_{1,3} R_{2,3}$*

3. *$id \otimes \triangle(R) = R_{1,3} R_{1,2}$*

where, writing $R = \sum R_{(1)} \otimes R_{(2)}$, then $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$.

2 Examples

Example 2.1. *In his seminal book, "Hopf Algebras", Sweedler defined a 4-dimensional non-commutative, non-cocommutative Hopf algebra*

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\triangle g = g \otimes g, \quad \triangle x = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1} \quad S(x) = -xg^{-1}.$$

$\mathbb{C}_{-1}[v_1, v_2]$ is an H_4-module algebra via $g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

H_4 is a quantum group with an R-matrix $R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$.

Example 2.2. *In "Finite Ring Groups", Kac and Paljutkin defined an 8-dimensional non-commutative, non-cocommutative Hopf Algebra*

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, xz = zy, yz = zx, z^2 = \tfrac{1}{2}(1 + x + y - xy) \rangle$$

with operations

$$\triangle(x) = x \otimes x, \quad \triangle(y) = y \otimes y, \quad \triangle(z) = \tfrac{1}{2}(1 \otimes 1 \otimes +1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$$
$$\varepsilon(x) = 1, \quad \varepsilon(y) = 1, \quad \varepsilon(z) = 1$$
$$S(x) = x, \quad S(y) = y, \quad S(z) = z$$

H_8 has as H_8-module algebras, $\mathbb{C}_q[v_1, v_2]$ with $q^2 = -1$, $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ with $q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}, q_{14}^2 = 1$ and $q_{23}^2 = -1$, and $\mathbb{C}_{-1}[v_1, v_2]$. H_8 is a quantum group with 6 non-isomorphic quasitriangular structures.

Example 2.3. *Described by Kulish and Reshetikhin in "Quantum linear problem for the sine-Gordon equation and higher representations",*

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), KEK^{-1} = q^2 E, KFK^{-1} = q^{-2}F,$$

$$KK^{-1} = 1 = K^{-1}K \rangle$$

with operations

$$\triangle(E) = E \otimes 1 + K \otimes E, \quad \triangle(F) = F \otimes K^{-1} + 1 \otimes F, \quad \triangle(K) = K \otimes K, \quad \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\varepsilon(E) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(K) = 1, \quad \varepsilon(K^{-1}) = 1,$$
$$S(E) = -K^{-1}E, \quad S(F) = -FK, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K$$

$\mathbb{C}_q[v_1, v_2]$ is a $\mathcal{U}_q(\mathfrak{sl}_2)$-module algebra where $q^2 \neq 1$ via $E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$

$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$
The R-matrix of $\mathcal{U}_q(\mathfrak{sl}_2)$ is in a completion of $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$.

3 Smash Product Algebras

Definition 3.1. *Given a Hopf algebra H and a left Hopf-module algebra A, the smash product algebra $A\#H$ is the algebra where $A\#H = A \otimes H$ as a $\mathbb{C}$-vector space and has product*

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

Definition 3.2. *Let H be a Hopf algebra, $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$ is called the collection of group-like elements and $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$ is called the collection of primitive elements.*

Theorem 3.3 (Cartier-Kostant-Milnor-Moore)**.** *Let H be a cocommutative Hopf algebra over $\mathbb{C}$, then as Hopf algebras*

$$H \cong \mathcal{U}(P(H))\#\mathbb{C}G(H).$$