Hopf Algebras Acting on Quantum Planes

Brandon Mather

1 Hopf Algebra Definitions

Definition 1.1. *A Hopf Algebra, $(H, \bigtriangledown, \eta, \bigtriangleup, \varepsilon, S)$, is a bialgebra H over a field $\Bbbk$ with an antipode $S : H \to H$ where the bialgebra has product $\bigtriangledown : H \otimes H \to H$, unit $\eta : \Bbbk \to H$, coproduct $\bigtriangleup : H \to H \otimes H$, counit $\varepsilon : H \to \Bbbk$ such that the following diagrams commute*

Associativity:
$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"] & H
\end{tikzcd}$$

Unit:
$$\begin{tikzcd}
 & H \otimes H \arrow[dd, "\bigtriangledown"] & \\
\Bbbk \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \Bbbk \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\
 & H &
\end{tikzcd}$$

Coassociativity:
$$\begin{tikzcd}
H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\
H \otimes H \arrow[r, "id \otimes \bigtriangleup"] & H \otimes H \otimes H
\end{tikzcd}$$

Counit:
$$\begin{tikzcd}
 & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\
\Bbbk \otimes H & & H \otimes \Bbbk \\
 & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] &
\end{tikzcd}$$

Coproduct compatibility:
$$\begin{tikzcd}
H \otimes H \arrow[r, "\bigtriangledown"] \arrow[d, "\bigtriangleup \otimes \bigtriangleup"] & H \arrow[r, "\bigtriangleup"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"] & & H \otimes H \otimes H \otimes H \arrow[u, "\bigtriangledown \otimes \bigtriangledown"']
\end{tikzcd}$$

Unit compatibility:
$$\begin{tikzcd}
\Bbbk \arrow[r, "\eta"] \arrow[dr, "\eta \otimes \eta"] & H \arrow[d, "\bigtriangleup"] \\
 & H \otimes H
\end{tikzcd}$$

Counit compatibility:
$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] & \Bbbk \\
H \otimes H \arrow[u, "\bigtriangledown"] \arrow[ur, "\varepsilon \otimes \varepsilon"'] &
\end{tikzcd}$$

Antipode:
$$\begin{tikzcd}
H \otimes H \arrow[rr, "id \otimes S"] & & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \arrow[u, "\bigtriangleup"] \arrow[r, "\varepsilon"] \arrow[d, "\bigtriangleup"'] & \Bbbk \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[u, "\bigtriangledown"]
\end{tikzcd}$$

For the sake of brevity, we write in general that $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$, this is called Sweedler notation.

One can note that these diagrams are self-dual, changing the directions of the morphisms gives another diagram. Then an immediate question is when is the dual of a Hopf algebra again a Hopf algebra?

Definition 1.2. *Let V be a $\Bbbk$ vector space and V^* its corresponding dual, then they determine a non-degenerate bilinear form $\langle , \rangle : V^* \otimes V \to \Bbbk$ by $\langle \phi, v \rangle = \phi(v)$.*

Definition 1.3. *If V and W are $\Bbbk$ vector spaces and $f : V \to W$ is $\Bbbk$-linear, then the transpose of f is $f^* : W^* \to V^*$ given by*

$$f^*(\phi)(v) = f(\phi(v)).$$

Definition 1.4. *Let $(C, \triangle, \varepsilon)$ be a coalgebra, then C^* is an algebra with multiplication $\triangle^* : C^* \otimes C^* \to C^*$ and unit $\varepsilon^* : \Bbbk \to C^*$.*

Note that $\triangle^*$, by definition 1.3, maps from $(C \otimes C)^*$, but we can restrict the map to the domain $C^* \otimes C^*$ to meet the criteria of being a product.

In a similar vein, if we start with an algebra $(A, \bigtriangledown, \eta)$, then the transpose of the product $\bigtriangledown$ is $\bigtriangledown^* : A^* \to (A \otimes A)^*$. But unless A is finite dimensional, we cannot know that $\bigtriangledown^*(A^*) \subseteq A^* \otimes A^*$, which is required for $\bigtriangledown^*$ to be a coproduct. This is exactly the requirement for A^* to be a coalgebra. This motivates the following definition.

Definition 1.5. *The finite dual of an algebra H is $H^\circ = \{f \in H^* \mid f(I) = 0$ for some ideal I of A where $\dim H/I < \infty\}$.*

If H is finite-dimensional, then H° is exactly H^*.

Proposition 1.6. *If A is an algebra, then A° is a coalgebra with coproduct $\bigtriangledown^* : A^\circ \to (A \otimes A)^\circ = A^\circ \otimes A^\circ$ and counit $\eta^* : A^\circ \to \Bbbk$.*

Proposition 1.7. *As proved in [13], if H is a Hopf algebra, H° is also a Hopf algebra with product, unit, coproduct, counit and antipode $\triangle^*$, ε^*, $\bigtriangledown^*$ η^*, S^* respectively. Explicitly, $\forall \phi, \psi \in H^\circ$ and all $h, g \in H$,*

$$\langle \bigtriangledown^*(\phi\psi), h \rangle = \langle \phi \otimes \psi, \triangle(h) \rangle, \;\; \langle 1, h \rangle = \varepsilon(h), \;\; \langle \triangle^*(\phi), h \otimes g \rangle = \langle \phi, \bigtriangledown(hg) \rangle, \;\; \varepsilon^*(\phi) = \langle \phi, 1 \rangle,$$

$$\langle S^* \phi h \rangle = \langle \phi, Sh \rangle.$$

Example 1.8. *Let G be any group and denote $\Bbbk G = \{\sum_{i=0}^{\infty} a_i g_i \mid a_i \in \Bbbk, g_i \in G, n \in \mathbb{N}\}$ The $\Bbbk G$ is a Hopf algebra called the group algebra of G. It has the product*

$$\left(\sum_{i=-0}^{n} a_i g_i \right) \left(\sum_{j=0}^{m} b_j g_j \right) = \sum_{i=o}^{n} \sum_{j=0}^{m} a_i b_j (g_i g_j)$$

where $a_i \cdot b_j$ is the product in $\Bbbk$ and $g_i \cdot g_j$ is the product in the group. The unit is $1_\Bbbk 1_G$ where $1_\Bbbk$ is the unit of $\Bbbk$ and 1_G is the identity element of the group. The coproduct is defined by $\triangle(g) = g \otimes g$ extended linearly to all of $\Bbbk G$, and the counit is $\varepsilon(g) = 1_\Bbbk$, again extended linearly. Finally, the antipode is $S(g) = g^{-1}$.

Note that group algebras are always cocommutative, in other words $\bigtriangledown(h) = \tau \circ \bigtriangledown(h)$ for all $h \in \Bbbk G$, where $\tau(a \otimes b) = b \otimes a$, and are commutative if and only if G is abelian.

Definition 1.9. *For a Hopf algebra H, $G(H) = \{g \in H \mid \triangle g = g \otimes g\}$ is called the set of grouplike elements of H.*

Example 1.10. *Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ the corresponding Universal Enveloping algebra. Then $U(\mathfrak{g})$ is naturally an algebra, but also has a Hopf algebra structure. The coproduct is given by $\triangle x = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$.*

Definition 1.11. *For a Hopf algebra H, $P(H) = \{x \in H \mid \triangle x = x \otimes 1 + 1 \otimes x\}$ is called the set of primitive elements of H. Generally, one can define the skew-primitive elements as $P_{a,b} = \{x \in H \mid \triangle x = x \otimes a + b \otimes x\}$.*

Example 1.12. *In his seminal book, [16], Sweedler defined a 4-dimensional, non-commutative, non-cocommutative Hopf algebra*

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\triangle g = g \otimes g, \;\; \triangle x = x \otimes 1 + g \otimes x, \;\; \varepsilon(g) = 1, \;\; \varepsilon(x) = 0,$$
$$S(g) = g^{-1} \;\; S(x) = -xg^{-1}.$$

We will see in section 3 a generalization of this to Taft algebras, which were introduced by Taft in [17].

Definition 1.13. *If H is a Hopf algebra, then H^{op} is a Hopf algebra with the same structure except the opposite multiplication, $\bigtriangledown^{op}(hg) = \bigtriangledown(gh)$. As well, H^{cop} is a Hopf algebra with the same structure as H but with the opposite coproduct, $\triangle^{cop}(h) = \sum h_{(2)} \otimes h_{(1)}$.*

Definition 1.14. *A left action of a Hopf algebra on a vector space V is a tuple (α, V) so that $\alpha : H \otimes V \to V$ is a map satisfying the diagrams*

$$\begin{tikzcd} H \otimes H \otimes V \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes V \arrow[d, "\alpha"] \\ H \otimes V \arrow[r, "\alpha"] & V \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes V \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes V \arrow[d, "\alpha"] \\ & V \end{tikzcd}$$

In this case, V is called a left Hopf-module.

For the rest of this paper we suppress the action α and instead write $\alpha(h \otimes v) = {}^h v$.

Definition 1.15. *A right coaction, ρ, of H on V is a tuple (ρ, V) with $\rho : V \to V \otimes H$ so that the following commute*

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes H \arrow[d, "id \otimes \triangle"] \\ V \otimes H \arrow[r, "\rho \otimes id"'] & V \otimes H \otimes H \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[dr, "="'] & V \otimes H \arrow[d, "id \otimes \varepsilon"] \\ & V \otimes \Bbbk \end{tikzcd}$$

In this case, V is called a right Hopf-comodule.

Definition 1.16. *When an algebra A is a left Hopf-module with action α, we call it a left Hopf-module algebra if it also satisfies the diagrams*

$$\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\
H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\eta"] & \Bbbk \arrow[r, "\eta"] & A \\
H \otimes A \arrow[rru, "\alpha"] & &
\end{tikzcd}$$

Note that a Hopf algebra acts if and only if its dual coacts. (Write why here)

Definition 1.17. *And when a coalgebra A is a an Hopf-module with action α, we call it an Hopf-module coalgebra if it also satisfies the diagrams*

$$\begin{tikzcd}
H \otimes A \arrow[r, "\alpha"] \arrow[d, "\triangle \otimes id \otimes id"'] & A \arrow[r, "\triangle"] & A \otimes A \\
H \otimes H \otimes A \otimes A \arrow[rr, "id \otimes \tau \otimes id"] & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\varepsilon"] & \Bbbk \\
H \otimes A \arrow[u, "\alpha"] \arrow[ru, "\varepsilon \otimes \varepsilon"'] &
\end{tikzcd}$$

Similar diagrams give the conditions for Hopf-comodule algebras and Hopf-comodule coalgebras.

Definition 1.18. *A useful and prevalent construction on Hopf algebras is given a Hopf algebra H and a left Hopf-module algebra A, the smash product algebra $A\#H$ is the algebra where $A\#H = A \otimes H$ as a $\Bbbk$-vector space and has product*

$$(a \otimes h)(b \otimes k) = \sum a \ {}^{h_{(1)}}b \otimes h_{(2)}k.$$

Definition 1.19. *A Hopf algebra is called pointed if all of its left (right) comodules are 1-dimensional.*

Definition 1.20. *If $I \subseteq H$ and for any $h \in H$, $hI \subseteq I$, then I is called an ideal of H. If $\triangle(I) \subseteq I \otimes H + H \otimes I$, then I is called a coideal of H. If I is both an ideal and a coideal, it is called a biideal. Finally, if I is a biideal and $S(I) \subseteq I$, then I is called a Hopf ideal of H.*

Lemma 1.21. *If I is a Hopf ideal of H, then the quotient H/I is a Hopf algebra.*

Definition 1.22. *If H acts on an algebra A and there is a Hopf ideal I so that $I \cdot A = 0$, then we say the action of H factors through the quotient H/I. In particular, if H/I is isomorphic to a group algebra, we say the action of H factors through a group action.*

2 Big Questions

Juan Cuadra, Pavel Etingof and Chelsea Walton have been classifying algebras with Hopf actions for which the action factors through a group action [7]. For example, they have shown that any action by a semi-simple, finite dimensional Hopf algebra on an integral domain always has this property. As well, they have shown that any action by a finite dimensional Hopf algebra on a Weyl algebra also has this property. This is a component of a larger search for algebras on which Hopf algebras act.

Andruskiewitsch and Schneider have been classifying pointed Hopf algebras [1].

Kenneth Chan, Ellen Kirkman, Jim Kuzmanovich, Chelsea Walton, and James Zhang have a series of works on Hopf algebras acting on AS-regular algebras [2] [3] [4] [11]. They have posed the question of when are the coinvariant subrings from these actions Artin-Schelter Gorenstein?

Miriam Cohen and Davida Fishman extended work by Fisher and Montogomery [8] and Cohen and Montogomery [6] to determine when $A\#H$ is semiprime for A and algebra and H a Hopf algebra. Specifically, if H is semi-simple and finite-dimensional and A is semiprime, they ask is $A\#H$ is semiprime? [5]

Chelsea Walton and Sarah Witherspoon have been working towards PBW deformation conditions on $B\#H$ where B is a Koszul algebra and H a Hopf algebra. In [18] they are able to provide these conditions when the antipode of H is bijective, B is connected as an H-module algebra, and the action of H preserves the grading on B. In the same paper they pose the question if $H = U_q(\mathfrak{sl}_2)$, are there nontrivial PBW deformations of $B\#H$?

3 Taft Algebras

A Taft algebra, as defined in [2], is a Hopf algebra $T_{n,m} = \langle g, x \mid g^n = 1, x^n = 0, gx = \zeta xg\rangle$ where ζ is a primitive n-th root of unity. $T_{n,m}$ has the maps

$$\triangle(g) = g \otimes g, \quad \triangle(x) = 1 \otimes x + x \otimes g$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g^{-1}, \quad S(x) = -xg^{-1}.$$

A small example is the lowest dimension non-commutative, non-cocommutative Hopf algebra, the 4-dimensional Sweedler algebra. This is given by $H_4 = \langle g, x \mid g^2 = 1, x^=0, xg = -gx\rangle$ with operations

$$\triangle(g) = g \otimes g, \quad \triangle(x) = 1 \otimes x + x \otimes g$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g, \quad S(x) = -xg.$$

This algebra in particular acts on the AS-regular algebra $\Bbbk_{-1}[u, v] = \Bbbk[u, v]/(uv + vu)$ by

$$g \cdot u = u, \quad g \cdot v = -v, \quad x \cdot u = 0, \quad x \cdot v = v.$$

Let q be a primitive root of unity where $|q^2| = m > 1$, let $\alpha \in \Bbbk$ and $n \in \mathbb{Z}^+$ where $|q|\big|n$, then a generalized Taft algebra is a Hopf algebra

$$T_{q,\alpha,n} = \left\langle g, g^{-1}x \mid gg^{-1} = g^{-1}g = 1, xg = qgx, g^n = 1, x^m = \alpha(g^m - g^{-m})\right\rangle$$

with maps

$$\triangle(g) = g \otimes g, \quad \triangle(x) = x \otimes g^{-1} + g \otimes x,$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g^{-1}, \quad S(x) = -qx.$$

If $q^m \neq 1$, then $\alpha = 0$, and if $q^m = 1$, $\alpha \in \{0, 1\}$.

Another definition for a generalized Taft algebra given in [1] is to take a Yetter-Drinfeld module V of D_2-type over the group algebra $k\mathbb{Z}/n\mathbb{Z}$. Then if $\mathcal{B}(V)$ is the Nichols algebra of V, the bosonization $(\mathcal{B}(V)\#k\mathbb{Z}/n\mathbb{Z})^{\mathrm{cop}}$ is a generalized Taft algebra.

Let q be a primitive root of unity with $q^2 \neq 1$, then we can define the quantum polynomial ring $\Bbbk_q[u,v] = \Bbbk[u,v]/\langle uv - qvu\rangle$. The subring $U = \Bbbk u \oplus \Bbbk v$ is then a left $T_{q,\alpha,n}$-module. If as a $T_{q,\alpha,n}$-module U is not semi-simple, then in [2] it is proved that $T_{q,\alpha,n}$ coacts on $\Bbbk_q[u,v]$. This coaction is given by

$$\rho(u) = u \otimes g \quad \rho(v) = v \otimes g^{-1} + u \otimes x.$$

The coaction induces an action by the finite dual $(T_{q,\alpha,n})^{\circ}$ on $\Bbbk_q[u,v]$.

4 8-dimensional Hopf Algebra

In [10], Kac and Paljutkin define an 8-dimensional, non-commutative, non-cocommutative, semi-simple Hopf algebra

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy)\rangle$$

with operations

$$\triangle(x) = x \otimes x, \;\; \triangle(y) = y \otimes y, \;\; \triangle(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1, \;\; S(x) = x^{-1}, \;\; S(y) = y^{-1}, \;\; S(z) = z.$$

Then in [12], three representations of H_8 acting on quantum polynomial rings are given. For the ring $\Bbbk_q[x_1, x_2]$ where $q^2 = -1$, the first representation is given by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the ring $\Bbbk_Q[x_1, x_2, x_3, x_4]$ where $Q = (q_{ij})$, $x_j x_i = q_{ij} x_i x_j$ and $q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}$, $q_{14}^2 = 1$, $q_{23}^2 = -1$, we also have the representation

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And finally, for the ring $\Bbbk_{-1}[u,v]$, we get the representation

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

5 Quantum Enveloping Algebras

References

[1] Andruskiewitsch, N., and H. -J. Schneider. "Pointed Hopf Algebras," 2001. https://doi.org/10.48550/ARXIV.MATH/0110136.

[2] Chan, Kenneth, Ellen Kirkman, Chelsea Walton, and James Zhang. "Quantum Binary Polyhedral Groups and Their Actions on Quantum Planes." arXiv, July 2, 2014. http://arxiv.org/abs/1303.7203.

[3] Chan, Kenneth, Ellen Kirkman, Chelsea Walton, and James Zhang. "McKay Correspondence for Semisimple Hopf Actions on Regular Graded Algebras, II." arXiv, October 2, 2017. http://arxiv.org/abs/1610.01220.

[4] Chan, Kenneth, Ellen Kirkman, Chelsea Walton, and James Zhang. "McKay Correspondence for Semisimple Hopf Actions on Regular Graded Algebras, I." arXiv, May 12, 2018. http://arxiv.org/abs/1607.06977.

[5] Cohen, Miriam, and Davida Fishman. "Hopf Algebra Actions." Journal of Algebra 100, no. 2 (May 1986): 363–79. https://doi.org/10.1016/0021-8693(86)90082-7.

[6] Cohen, M, and S Montgomery. "GROUP-GRADED RINGS, SMASH PRODUCTS, AND GROUP ACTIONS," n.d.

[7] Etingof, Pavel, and Chelsea Walton. "Finite Dimensional Hopf Actions on Deformation Quantizations." arXiv, July 2, 2016. http://arxiv.org/abs/1602.00532.

[8] Fisher, Joe W, and Susan Montgomery. "Semiprime Skew Group Rings." Journal of Algebra 52, no. 1 (May 1978): 241–47. https://doi.org/10.1016/0021-8693(78)90272-7.

[9] Halbig, Sebastian. "Generalised Taft Algebras and Pairs in Involution." Communications in Algebra 49, no. 12 (December 2, 2021): 5181–95. https://doi.org/10.1080/00927872.2021.1939043.

[10] Kac, G. I., and V. G. Paljutkin. "Finite Ring Groups." Trudy Moskov. Mat. Obšč. 15 (1966): 224–61.

[11] Kirkman, E., J. Kuzmanovich, and J.J. Zhang. "Gorenstein Subrings of Invariants under Hopf Algebra Actions." Journal of Algebra 322, no. 10 (November 2009): 3640–69. https://doi.org/10.1016/j.jalgebra.2009.08.018.

[12] Kirkman, E., J. Kuzmanovich, and J.J. Zhang. "Gorenstein Subrings of Invariants under Hopf Algebra Actions." Journal of Algebra 322, no. 10 (November 2009): 3640–69. https://doi.org/10.1016/j.jalgebra.2009.08.018.

[13] Majid, Shahn. Foundations of Quantum Group Theory. Cambridge University Press 1995.

[14] Masuoka, A., Semisimple Hopf algebras of dimension 6, 8, Israel J. Math. 92 (1–3) (1995) 361–373

[15] Montgomery, Susan. Hopf Algebras and Their Actions on Rings. Vol. 82. Providence, R.I.; 4: Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, 1993.

[16] Sweedler, Moss E. Hopf Algebras. New York: W. A. Benjamin, 1969.

[17] Taft, Earl J. "The Order of the Antipode of Finite-Dimensional Hopf Algebra." Proceedings of the National Academy of Sciences 68, no. 11 (November 1971): 2631–33. https://doi.org/10.1073/pnas.68.11.2631.

[18] Walton, Chelsea, and Sarah Witherspoon. "Poincaré–Birkhoff–Witt Deformations of Smash Product Algebras from Hopf Actions on Koszul Algebras." Algebra & Number Theory 8, no. 7 (October 21, 2014): 1701–31. https://doi.org/10.2140/ant.2014.8.1701.