# Hopf Algebras Acting on Quantum Planes

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## 1 Hopf Algebra Definitions

**Definition 1.1.** A Hopf Algebra,  $(H, \nabla, \eta, \triangle, \varepsilon, S)$ , is a bialgebra H over a field  $\mathbb C$  with an antipode  $S: H \to H$  where the bialgebra has product  $\nabla: H \otimes H \to H$ , unit  $\eta: \mathbb C \to H$ , coproduct  $\Delta: H \to H \otimes H$ , counit  $\varepsilon: H \to \mathbb C$  such that the following diagrams commute

$$Associativity: \begin{array}{c} H \otimes H \otimes H \\ \downarrow id \otimes \nabla \\ H \otimes H \end{array} \begin{array}{c} \downarrow \\ \downarrow \downarrow id \otimes \nabla \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \downarrow \downarrow \\ \downarrow \downarrow \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \end{pmatrix} \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \\ \downarrow \\ \end{pmatrix} \begin{array}{c} \downarrow \\ \\ \downarrow \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \downarrow \\ \end{array} \begin{array}{c} \downarrow \\ \\ \downarrow \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \downarrow \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \downarrow \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}$$

For the sake of brevity, we write in general that  $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$ , this is called Sweedler notation.

**Definition 1.2.** A left action of a Hopf algebra on a vector space V is a tuple  $(\alpha, V)$  so that  $\alpha: H \otimes V \to V$  is a map satisfying the diagrams

In this case, V is called a left H-module.

**Definition 1.3.** When an algebra A is a left H-module with action  $\alpha$ , we call it a left H-module algebra if the following diagrams also commute

**Definition 1.4.** Let  $Q = (q_{ij})$  be an  $n \times n$  matrix of roots of unity where  $q_{ii} = 1 = q_{ij}q_{ji}$ . A quantum polynomial ring is  $\mathbb{C}_Q[v_1,\ldots,v_n]=\mathbb{C}\langle v_1,\ldots,v_n|v_iv_i=q_{ij}v_iv_j\rangle$ .

**Definition 1.5.** A quantum group is a Hopf algebra H with a bijective antipode and an element  $R \in H \otimes H$  satisfying

1. 
$$R\left(\sum h_{(1)} \otimes h_{(2)}\right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

2. 
$$\triangle \otimes id(R) = R_{1,3}R_{2,3}$$

3. 
$$id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where, writing  $R = \sum R_{(1)} \otimes R_{(2)}$ , then  $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$ ,  $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$ , and  $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}.$ 

#### 2 **Examples**

**Example 2.1.** In his seminal book, "Hopf Algebras", Sweedler defined a 4-dimensional noncommutative, non-cocommutative Hopf algebra

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations

$$\triangle g = g \otimes g, \quad \triangle x = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

$$S(g) = g^{-1} \quad S(x) = -xg^{-1}.$$

$$\mathbb{C}_{-1}[v_1,v_2] \text{ is an $H_4$-module algebra via } g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
 
$$H_4 \text{ is a quantum group with an $R$-matrix } R = 1 \otimes 1 - 2 \frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x \frac{1-g}{2} \otimes x \frac{1-g}{2} - 2x \otimes x \frac{1-g}{2}.$$

**Example 2.2.** In "Finite Ring Groups", Kac and Paljutkin defined an 8-dimensional non-commutative, non-cocommutative Hopf Algebra

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, xz = zy, yz = zx, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$$

with operations

 $H_8$  has as  $H_8$ -module algebras,  $\mathbb{C}_q[v_1, v_2]$  with  $q^2 = -1$ ,  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  with  $q_{12} = q_{34}^{-1}$ ,  $q_{13} = q_{24}^{-1}, q_{14}^2 = 1$  and  $q_{23}^2 = -1$ , and  $\mathbb{C}_{-1}[v_1, v_2]$ .  $H_8$  is a quantum group with 6 non-isomorphic quasitriangular structures.

**Example 2.3.** Described by Kulish and Reshetikhin in "Quantum linear problem for the sine-Gordon equation and higher representations",

$$\mathcal{U}_q(\mathfrak{sl}_2) = \langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}(K - K^{-1}), KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F, KK^{-1} = 1 = K^{-1}K \rangle$$

 $with \ operations$ 

$$\triangle(E) = E \otimes 1 + K \otimes E, \quad \triangle(F) = F \otimes K^{-1} + 1 \otimes F, \quad \triangle(K) = K \otimes K, \quad \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = 0, \qquad \qquad \varepsilon(F) = 0, \qquad \qquad \varepsilon(K) = 1, \qquad \qquad \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, \qquad S(F) = -FK, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K$$

 $\mathbb{C}_q[v_1, v_2] \text{ is a } \mathcal{U}_q(\mathfrak{sl}_2)\text{-module algebra where } q^2 \neq 1 \text{ via } E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$ 

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}, K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

The R-matrix of  $\mathcal{U}_q(\mathfrak{sl}_2)$  is in a completion of  $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$ .

### 3 Smash Product Algebras

**Definition 3.1.** Given a Hopf algebra H and a left Hopf-module algebra A, the smash product algebra A#H is the algebra where  $A\#H = A\otimes H$  as a  $\mathbb{C}$ -vector space and has product

$$(a \otimes h)(b \otimes k) = \sum a(h_{(1)} \cdot b) \otimes h_{(2)}k.$$

**Definition 3.2.** Let H be a Hopf algebra,  $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$  is called the collection of group-like elements and  $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$  is called the collection of primitive elements.

**Theorem 3.3** (Cartier-Kostant-Milnor-Moore). Let H be a cocommutative Hopf algebra over  $\mathbb{C}$ , then as Hopf algebras

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H).$$