Koszul Resolution of a Skew Polynomial Ring

Brandon Mather

Departments of Mathematics, University of North Texas

October 8, 2024

We consider the skew polynomial ring $A = \mathbb{k}_q[x_1, x_2]$ and construct its Koszul resolution. This allows the computation of its Hochschild cohomology groups.

1 Koszul Resolution

Let $\mathbb{k}$ be a field, $A = \mathbb{k}_q[x_1, x_2] = \mathbb{k}\langle x_1, x_2\rangle/(x_2x_1 - qx_1x_2)$ for some $q \in \mathbb{k}^*$. Let $A_1 = \mathbb{k}[x_1]$ and $A_2 = \mathbb{k}[x_2]$ be subalgebras of A so that A is the twisted tensor product $A_1 \otimes^\tau A_2$ where $\tau : \mathbb{Z}^2 \to \mathbb{k}^*$ is the bicharacter $\tau(m, n) = q^{mn}$. We consider the Koszul resolutions of A_1 and A_2

$$0 \to A_1^e \xrightarrow{(x_1\otimes 1 - 1\otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e \xrightarrow{(x_2\otimes 1 - 1\otimes x_2)\cdot} A_2^e \to 0.$$

In order to construct a resolution of A, the differentials of these resolutions need to be graded maps. To this end, we shift the grading of the homological degree 1 component of both resolutions up by 1:

$$0 \to A_1^e(-1) \xrightarrow{(x_1\otimes 1 - 1\otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e(-1) \xrightarrow{(x_2\otimes 1 - 1\otimes x_2)\cdot} A_2^e \to 0.$$

Then, the differential $(x_1 \otimes 1 - 1 \otimes x_1)\cdot$ maps basis elements as follows

$$(x_1 \otimes 1 - 1 \otimes x_1)(x_1^n \otimes x_1^m) = x_1^{n+1} \otimes x_1^m - x_1^n \otimes x_1^{m+1}.$$

The element on the right has degree $n + m + 1$ in A_1^e and the element $x_1^n \otimes x_1^m$ has shifted degree $n + m + 1$ in $A_1^e(-1)$, so we see the differential is now graded. Similarly, the differential $(x_2 \otimes 1 - 1 \otimes x_2)\cdot : A_2^e(-1) \to A_2^e$ is graded.

Then by a theorem proved by Bergh and Oppermann in "Cohomology of Twisted Tensor Products" (2008), the total complex of the tensor product of these two resolutions is a projective resolution of A as an A^e-module. This resolution is

$$0 \to A_1^e(-1) \otimes A_2^e(-1) \xrightarrow{\partial_2} [A_1^e \otimes A_2^e(-1)] \oplus [A_1^e(-1) \otimes A_2^e] \xrightarrow{\partial_1} A_1^e \otimes A_2^e \to 0$$

where the differentials are given by

$$\partial_2 = \begin{bmatrix} (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id} \\ \mathrm{id} \otimes (1 \otimes x_2 - x_2 \otimes 1)\cdot \end{bmatrix} : x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto \begin{bmatrix} x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d - x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d \\ x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} - x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d \end{bmatrix}$$

$$\partial_1 = \begin{bmatrix} \mathrm{id} \otimes (x_2 \otimes 1 - 1 \otimes x_2)\cdot & (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id} \end{bmatrix} :$$

$$\begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} \mapsto x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d - x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} + x_1^{r+1} \otimes x_1^s \otimes x_2^u \otimes x_2^v - x_1^r \otimes x_1^{s+1} \otimes x_2^u \otimes x_2^v.$$

2 Computing Cohomology

Apply the functor $\mathrm{Hom}_{A^e}(-, A)$ to this resolution to get the complex

$$\begin{aligned} 0 \to \mathrm{Hom}_{A^e}(A_1^e \otimes A_2^e, A) &\xrightarrow{d_1^*} \mathrm{Hom}_{A^e}((A_1^e \otimes A_2^e(-1)) \oplus (A_1^e(-1) \otimes A_2^e)\,, A) \\ &\xrightarrow{d_2^*} \mathrm{Hom}_{A^e}(A_1^e(-1) \otimes A_2^e(-1), A) \to 0. \end{aligned}$$

The differentials are given by $d_i^*(f)(a) = f(\partial_i(a))$.

Next, we want to rewrite this complex in a more familiar form. Let $V = \Bbbk x_1 \oplus \Bbbk x_2$, then we have the A^e-module isomorphisms

$$\begin{gathered} \mathrm{Hom}_A^e(A_1^e \otimes A_2^e, A) \cong A \otimes \textstyle\bigwedge_q^2(V) \\ (1 \otimes 1 \otimes 1 \otimes 1 \mapsto x_1^a x_2^b) \mapsto x_1^a x_2^b \otimes x_1 \wedge x_2 \end{gathered}$$

$$\begin{gathered} \mathrm{Hom}_A^e((A_1^e \otimes A_2^e(-1)) \oplus (A_1^e(-1) \otimes A_2^e(-1))\,, A) \cong A \otimes \textstyle\bigwedge_q^1(V) \\ ((1 \otimes 1 \otimes 1 \otimes 1, 0) \mapsto x_1^a x_2^b, (0, 1 \otimes 1 \otimes 1 \otimes 1) \mapsto x_1^c x_2^d) \mapsto x_1^a x_2^b \otimes x_1 + x_1^c x_2^d \otimes x_2 \end{gathered}$$

$$\begin{gathered} \mathrm{Hom}_A^e(A_1^e(-1) \otimes A_2^e(-1), A) \cong A \otimes \textstyle\bigwedge_q^0(V) \\ (1 \otimes 1 \otimes 1 \otimes 1 \mapsto x_1^a x_2^b) \mapsto x_1^a x_2^b \end{gathered}$$

Ultimately, we have the complex

$$0 \to A \otimes \textstyle\bigwedge_q^2(V) \xrightarrow{d_1} A \otimes \bigwedge_q^1(V) \xrightarrow{d_2} A \otimes \bigwedge_q^0(V) \to 0$$

with differentials given by

$$\begin{aligned} d_1(x_1^a x_2^b \otimes x_1 \wedge x_2) &= (q^a - 1) x_1^a x_2^{b+1} \otimes x_1 + (1 - q^b) x_1^{a+1} x_2^b \otimes x_2 \\ d_2(x_1^a x_2^b \otimes x_1 + x_1^c x_2^d \otimes x_2) &= (1 - q^{b+1}) x_1^{a+1} x_2^b + (1 - q^{c+1}) x_1^c x_2^{d+1} \end{aligned}$$

In particular, for this two-variate casec we note that this complex is quasi-isomorphic to

$$0 \to A \to A \oplus A \to A \to 0.$$