

Quantum Group Actions and Hopf Algebras

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- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book “Hopf Algebras”
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montgomery writes “Hopf Algebras and Their Actions on Rings”

Goal

To understand the actions of Hopf algebras on other algebras

Notation: $\mathbb{C}[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - v_i v_j \rangle$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where

$$q_{ii} = 1 = q_{ji} q_{ij}.$$

A **quantum polynomial ring** is

$$\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - q_{ij} v_i v_j \rangle.$$

Example: $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \rangle$

Motivation

- ① When a grp G acts on a space V over \mathbb{C} linearly, the action can be extended to $V \otimes V$ by $g \in G$ acting as

$$g \otimes g = \Delta(g).$$

Then Δ defines a coproduct map

$$\Delta : \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,

$$\Delta : A \rightarrow A \otimes A,$$

we call $g \in A$ **grouplike** if

$$\Delta(g) = g \otimes g.$$

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- ② When a Lie alg \mathfrak{g} acts on a space V over \mathbb{C} , the action can be extended to $V \otimes V$ by $x \in \mathfrak{g}$ acting as

$$x \otimes 1 + 1 \otimes x = \Delta(x).$$

Then Δ defines a coproduct map

$$\Delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts,

$$\Delta : A \rightarrow A \otimes A,$$

we call $x \in A$ **primitive** if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

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with operations $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

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Group-like

$(1, g)$ – Primitive

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$(1, g)$ – Primitive

Let τ be the ‘flip’ over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \Delta(x) \neq \Delta(x)$, so H is non-cocommutative.

Actions of Sweedler's Algebra

H_4 acts on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$g \cdot v_1 = v_1, \quad g \cdot v_2 = -v_2, \quad x \cdot v_1 = 0, \quad x \cdot v_2 = v_1.$$

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We can express this action on the generators v_1, v_2 as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

with operations

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Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}$, $q_{14}^2 = 1$, $q_{23}^2 = -1$ via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{-1}[v_1, v_2]$ via the rep

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups:

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

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with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

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Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \rightarrow 1$.

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$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

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$$\nabla : H \otimes H \rightarrow H,$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

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Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ 1_H \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes 1_H & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & \nwarrow = & & \end{array}$$

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Coassociativity:

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Counit:

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Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H & &
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 \mathbb{C} & \xrightarrow{1_H} & H \\
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 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{\epsilon} & \mathbb{C} \\
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Antipode:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad id \otimes S \quad} & H \otimes H & & \\
 \Delta \uparrow & & & & \downarrow \nabla \\
 H & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{1_H} & H \\
 \Delta \downarrow & & & & \uparrow \nabla \\
 H \otimes H & \xrightarrow{\quad S \otimes id \quad} & H \otimes H & &
 \end{array}$$

Common philosophy: A **quantum group** is a Hopf alg H with a bijective antipode, and some invertible $R \in H \otimes H$ witnessing how close H is to being cocommutative and satisfying:

$$(1) R \left(\sum h_{(1)} \otimes h_{(2)} \right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

$$(2) \Delta \otimes id(R) = R_{1,3} R_{2,3}$$

$$(3) id \otimes \Delta(R) = R_{1,3} R_{1,2}$$

where $R = \sum R_{(1)} \otimes R_{(2)}$, $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$,
 $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$.
Hence, R is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}.$$

Quantum Group Examples

The three Hopf algs we have already discussed are all quantum groups.

Sweedler's algebra, $H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$, has the universal R-matrix

$$R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$$

The Kac-Paljutkin algebra,

$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$, has 6 non-isomorphic quasitriangular structures.

And the ambiguity of what is a quantum group can be best seen with $\mathcal{U}_q(\mathfrak{sl}_2)$. This is undoubtedly a quantum group as it was the inspiration for the concept, but its universal R-matrix exists only in a completion of the tensor product $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$, and so is a formal power series of pure tensors.

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 H \otimes A & \xrightarrow{\alpha} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{C} \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\
 \searrow = & & \downarrow \alpha \\
 & & A
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Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \rightarrow A$.
Then we say H **acts** on A by α if the following diagrams commute:

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 \end{array}$$

$$\begin{array}{ccc}
 H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{1_A} A \\
 id \otimes 1_A \downarrow & \nearrow \alpha & \\
 H \otimes A & &
 \end{array}$$

In words, H **acts on** A iff you can multiply in H and then act on A or act on A consecutively, $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H -**module alg** iff H acts on A and $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a'), \quad h(1_A) = \varepsilon(h)1_A$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms, giving the semidirect product group, $G \rtimes G'$.

The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product $g'g = g'(g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H -module algebra, then $A \# H$ is defined as $A \otimes H$ as a vector space and with product

$$(a' \otimes h)(a \otimes h') = \sum_i a' h_{(1)}(a) \otimes h_{(2)} h'$$

where $a, a' \in A$, $h, h' \in H$ and $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

Smash Product Algebra

"Group-like" and "Lie-like"

For Hopf alg H , define

$$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over \mathbb{C} , then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dimpt'l Hopf alg over \mathbb{C} is iso to a grp alg.

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- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is $A \# H$ semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B \# \mathcal{U}_q(\mathfrak{sl}_2)$?