## Hopf Module Algebras

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UNT Master's

## History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

#### Goal

To understand the actions of Hopf algebras on other algebras

#### Quantum Plane

Notation: 
$$\mathbb{C}[v_1,\ldots,v_n] = \mathbb{C}\langle v_1,\ldots,v_n \mid v_j v_i - v_i v_j \rangle$$

#### Quantum Polynomial Ring

Let  $Q=(q_{ij})$  be an  $n\times n$  matrix of roots of unity where  $q_{ii}=1=q_{ji}q_{ij}$ .  $\mathbb{C}_Q[v_1,\ldots,v_n]=\mathbb{C}\left\langle v_1,\ldots,v_n\mid v_jv_i-q_{ij}v_iv_j\right\rangle$  is called a **quantum polynomial ring**.

Example: 
$$\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 | v_1 v_2 + v_2 v_1 \rangle$$

#### Motivation

• When a grp G acts on a space V by automorphisms, the action can be extended to  $V\otimes V$  by  $g\in G$  acting as  $g\otimes g=\triangle(g)$ . The action naturally induces an action from the grp alg,  $\mathbb{C}G$ , on V. Then  $\triangle$  defines a map  $\mathbb{C}G\to\mathbb{C}G\otimes\mathbb{C}G$  called the coproduct. For arbitrary coproducts,  $\triangle:A\to A\otimes A$ , we call  $g\in A$  grouplike if  $\triangle(g)=g\otimes g$ .

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- ② When a Lie alg  $\mathfrak g$  acts on a space V, the action can be extended to  $V\otimes V$  by  $x\in \mathfrak g$  acting as  $x\otimes 1+1\otimes x=\triangle(x)$ . Again,  $\triangle$  defines a map  $\mathfrak g\to \mathfrak g\otimes \mathfrak g$ . For arbitrary coproducts,  $\triangle:A\to A\otimes A$ , we call  $x\in A$  primitive if  $\triangle(x)=x\otimes 1+1\otimes x$ .

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This notion is dual to an algebra, creating a colagebra structure, C, with coproduct  $\triangle: C \to C \otimes C$  and counit  $\varepsilon: C \to \mathbb{C}$ .

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Let  $\tau$  be the 'flip' over the tensor product, so  $\tau(u \otimes v) = v \otimes u$ . Note that  $\tau \circ \triangle(x) \neq \triangle(x)$ , this is called non-cocommutativity.

### Actions of Sweedler's Algebra

$${\it H}_{4}$$
 acts on  $\mathbb{C}_{-1}[{\it v}_{1},{\it v}_{2}]$  by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

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We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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$$H_8$$
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And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
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Note: You can recover  $\mathcal{U}(\mathfrak{sl}_2)$  by limiting  $q \to 1$ .

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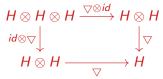
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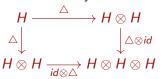
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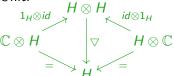
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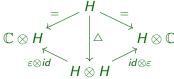
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### Hopf Algebra Diagrams

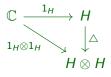
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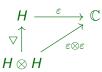
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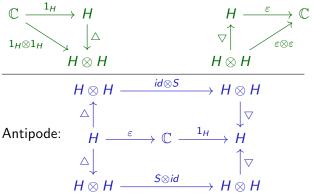


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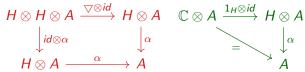
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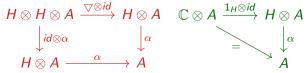


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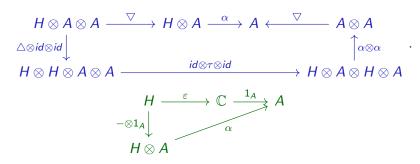
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In words, H acts on A, iff  $\forall h, h' \in H, \forall a \in A$ 

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-module alg iff H acts on A and  $\forall h \in H, \forall a, a' \in A$ 

$$h(aa') = \sum h_i(a) \cdot h_j(a), \quad h(1_A) = \varepsilon(h)1_A$$

where  $\triangle(h) = \sum h_i \otimes h_j$ .

### Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group,  $G \rtimes G'$ . The action can be extended to the group algebras,  $\mathbb{C}G$  and  $\mathbb{C}G'$ . This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G\#\mathbb{C}G'$$

with product  $g'g = (g' \cdot g)g'$ .

### Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is the smash product algebra defined as  $A\otimes H$  as a vector space and with product

$$ha = \sum_{i} (g_i \cdot a) k_i$$

where  $a \in A$ ,  $h \in H$  and  $\triangle(h) = \sum_i g_i \otimes k_i$ .

# Smash Product Algebra

### "Group-like" and "Lie-like"

Let H be a Hopf algebra, define  $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$  and  $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$ .

G(H) is the set of grouplike elements of H and forms a group under the product.

P(H) is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

#### Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over  $\mathbb{C}$ , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over  $\mathbb C$  is isomorphic to a group algebra.

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- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of  $B\#\mathcal{U}_q(\mathfrak{sl}_2)$ ?