Quantum Symmetry of Hopf Actions

Brandon Mather

Algebra Seminar, November 2023

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \frac{1}{2} (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x, \ S(y) = y, \ S(z) = z.$$

Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on $\mathbb{C}_q[x,y]$ where $q^2=-1$ by
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on $\mathbb{C}_q[x,y]$ where $q^2=-1$ by

$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_{\mathcal{O}}[x_1, x_2, x_3, x_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

 H_8 acts on $\mathbb{C}_q[x,y]$ where $q^2=-1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_Q[x_1, x_2, x_3, x_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_{-1}[u,v]$ by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E,
ight.$$
 $KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1 \right
angle$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\left< E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E, \right.$$

$$\left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1} K = 1 \right>$$
with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E,
ight.$$
 $KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1 \right\rangle$

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\left< E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E, \right.$$

$$\left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1} K = 1 \right>$$
with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x,y]$ by the representation

Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x,y]$ by the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x,y]$ by the representation

$$E\mapsto egin{bmatrix} 0 & 1 \ 0 & 0 \end{bmatrix} & F\mapsto egin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix} \\ \mathcal{K}\mapsto egin{bmatrix} q & 0 \ 0 & q^{-1} \end{bmatrix} & \mathcal{K}^{-1}\mapsto egin{bmatrix} q^{-1} & 0 \ 0 & q \end{bmatrix}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode $S: H \to H$ where the bialgebra operations are

A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$ where the bialgebra operations are

$$\nabla: H \otimes H \to H$$
,

so that the following commute:

Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
\downarrow^{id} \otimes \nabla \downarrow & & \downarrow \nabla \\
H \otimes H & \xrightarrow{\nabla} & H
\end{array}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$ where the bialgebra operations are

 $\nabla: H \otimes H \to H$,

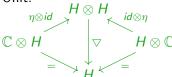
 $\eta:\mathbb{C}\to H$,

so that the following commute:

Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
\downarrow^{id \otimes \nabla} & & \downarrow^{\nabla} \\
H \otimes H & \xrightarrow{\nabla} & H
\end{array}$$

Unit:



A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$ where the bialgebra operations are

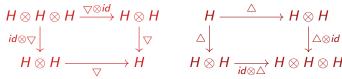
$$\bigtriangledown: H \otimes H \to H,$$

$$\triangle: H \to H \otimes H$$
,

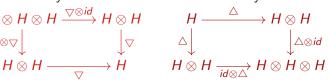
so that the following commute:

Associativity:

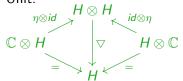
 $\eta:\mathbb{C}\to H$.



Coassociativity:



Unit:



A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$ where the bialgebra operations are

$$\nabla: H \otimes H \to H$$
,

$$\eta:\mathbb{C}\to H$$
,

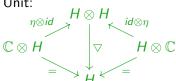
 $\triangle: H \to H \otimes H$. $\varepsilon: H \to \mathbb{C}$

so that the following commute:

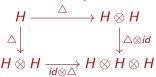
Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
id \otimes \nabla \downarrow & & \downarrow \nabla \\
H \otimes H & \xrightarrow{} & H
\end{array}$$

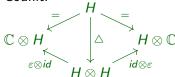
Unit:



Coassociativity:



Counit:



Hopf Algebra Diagrams

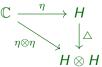
Product and Coproduct compatibility:



Hopf Algebra Diagrams

Product and Coproduct compatibility:

Unit and Counit compatibility:



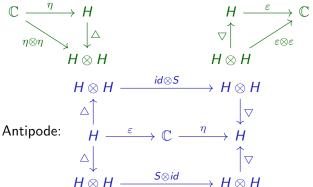


Hopf Algebra Diagrams

Product and Coproduct compatibility:

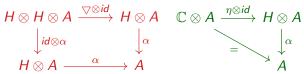
$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} H & \stackrel{\triangle}{\longrightarrow} H \otimes H \\ & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} H \otimes H \otimes H \otimes H \end{array}$$

Unit and Counit compatibility:

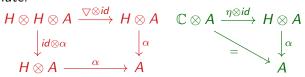


Let H be a Hopf alg and A an alg with a map $\alpha: H \otimes A \to A$. Then we say H acts on A by α if the following the diagrams commute:

Let H be a Hopf alg and A an alg with a map $\alpha: H \otimes A \to A$. Then we say H acts on A by α if the following the diagrams commute:

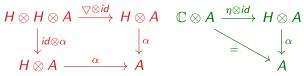


Let H be a Hopf alg and A an alg with a map $\alpha: H \otimes A \to A$. Then we say H acts on A by α if the following the diagrams commute:



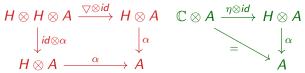
A is called a **module algebra** if the following also commute:

Let H be a Hopf alg and A an alg with a map $\alpha: H \otimes A \to A$. Then we say H acts on A by α if the following the diagrams commute:



A is called a **module algebra** if the following also commute:

Let H be a Hopf alg and A an alg with a map $\alpha: H \otimes A \to A$. Then we say H acts on A by α if the following the diagrams commute:



A is called a **module algebra** if the following also commute:

