

# Quantum Group Actions and Hopf Algebras

Brandon Mather

University of North Texas Master's Defense February 28th 2024  
Department of Mathematics

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book “Hopf Algebras”
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montgomery writes “Hopf Algebras and Their Actions on Rings”

## Goal

To understand the actions of Hopf algebras on other algebras

Notation:  $\mathbb{C}[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - v_i v_j \rangle$

## Quantum Polynomial Ring

Let  $Q = (q_{ij})$  be an  $n \times n$  matrix of roots of unity where

$$q_{ii} = 1 = q_{ji} q_{ij}.$$

A **quantum polynomial ring** is

$$\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - q_{ij} v_i v_j \rangle.$$

Example:  $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \rangle$

# Motivation

- ① When a grp  $G$  acts on a space  $V$  over  $\mathbb{C}$  linearly, the action can be extended to  $V \otimes V$  by  $g \in G$  acting as

$$g \otimes g = \Delta(g).$$

Then  $\Delta$  defines a coproduct map

$$\Delta : \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,

$$\Delta : A \rightarrow A \otimes A,$$

we call  $g \in A$  **grouplike** if

$$\Delta(g) = g \otimes g.$$

# Motivation

- ① When a grp  $G$  acts on a space  $V$  over  $\mathbb{C}$  linearly, the action can be extended to  $V \otimes V$  by  $g \in G$  acting as

$$g \otimes g = \Delta(g).$$

Then  $\Delta$  defines a coproduct map

$$\Delta : \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,

$$\Delta : A \rightarrow A \otimes A,$$

we call  $g \in A$  **grouplike** if

$$\Delta(g) = g \otimes g.$$

- ② When a Lie alg  $\mathfrak{g}$  acts on a space  $V$  over  $\mathbb{C}$ , the action can be extended to  $V \otimes V$  by  $x \in \mathfrak{g}$  acting as

$$x \otimes 1 + 1 \otimes x = \Delta(x).$$

Then  $\Delta$  defines a coproduct map

$$\Delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts,

$$\Delta : A \rightarrow A \otimes A,$$

we call  $x \in A$  **primitive** if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$$



# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$

$$S(g) = g^{-1}, \quad S(x) = -x$$

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

$$\Delta(g) = g \otimes g,$$

$$\Delta(x) = x \otimes 1 + g \otimes x$$

$$\varepsilon(g) = 1,$$

$$\varepsilon(x) = 0$$

$$S(g) = g^{-1},$$

$$S(x) = -x$$

---

Group-like

---

$(1, g)$  – Primitive

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x$$

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0$$

$$S(g) = g^{-1}, \quad S(x) = -x$$

---

Group-like

---

$(1, g)$  – Primitive

Let  $\tau$  be the 'flip' over the tensor product, so  $\tau(u \otimes v) = v \otimes u$ .  
Note that  $\tau \circ \Delta(x) \neq \Delta(x)$ , so  $H$  is non-cocommutative.

# Actions of Sweedler's Algebra

$H_4$  acts on  $\mathbb{C}_{-1}[v_1, v_2]$  by

$$g \cdot v_1 = v_1, \quad g \cdot v_2 = -v_2, \quad x \cdot v_1 = 0, \quad x \cdot v_2 = v_1.$$

# Actions of Sweedler's Algebra

$H_4$  acts on  $\mathbb{C}_{-1}[v_1, v_2]$  by

$$g \cdot v_1 = v_1, \quad g \cdot v_2 = -v_2, \quad x \cdot v_1 = 0, \quad x \cdot v_2 = v_1.$$

We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

# Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

with operations

# Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

with operations

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$



# Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y,$$

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x, S(y) = y, S(z) = z.$$

# Actions of Kac-Paljutkin Algebra

$H_8$  acts on  $\mathbb{C}_q[v_1, v_2]$  where  $q^2 = -1$  via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

# Actions of Kac-Paljutkin Algebra

$H_8$  acts on  $\mathbb{C}_q[v_1, v_2]$  where  $q^2 = -1$  via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

---

And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$q_{12} = q_{34}^{-1}$ ,  $q_{13} = q_{24}^{-1}$ ,  $q_{14}^2 = 1$ ,  $q_{23}^2 = -1$  via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

# Actions of Kac-Paljutkin Algebra

$H_8$  acts on  $\mathbb{C}_q[v_1, v_2]$  where  $q^2 = -1$  via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

---

And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$q_{12} = q_{34}^{-1}$ ,  $q_{13} = q_{24}^{-1}$ ,  $q_{14}^2 = 1$ ,  $q_{23}^2 = -1$  via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

---

And on  $\mathbb{C}_{-1}[v_1, v_2]$  via the rep

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K, \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$



# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K, \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}, S(K^{-1}) = K.$$

# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\Delta(K) = K \otimes K, \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}, S(K^{-1}) = K.$$

Note: You can recover  $\mathcal{U}(\mathfrak{sl}_2)$  by limiting  $q \rightarrow 1$ .

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[v_1, v_2]$  by the representation

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[v_1, v_2]$  by the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[v_1, v_2]$  by the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

# Hopf Algebra

A **Hopf algebra** is a bialgebra  $H$  over a field with an antipode  
 $S : H \rightarrow H$  where the bialgebra operations are

# Hopf Algebra

A **Hopf algebra** is a bialgebra  $H$  over a field with an antipode

$S : H \rightarrow H$  where the bialgebra operations are

$$\nabla : H \otimes H \rightarrow H,$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

# Hopf Algebra

A **Hopf algebra** is a bialgebra  $H$  over a field with an antipode

$S : H \rightarrow H$  where the bialgebra operations are

$$\nabla : H \otimes H \rightarrow H,$$

$$1_H : \mathbb{C} \rightarrow H,$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ 1_H \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes 1_H & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & \nwarrow = & & \end{array}$$



# Hopf Algebra

A **Hopf algebra** is a bialgebra  $H$  over a field with an antipode

$S : H \rightarrow H$  where the bialgebra operations are

$$\nabla : H \otimes H \rightarrow H,$$

$$\Delta : H \rightarrow H \otimes H,$$

$$1_H : \mathbb{C} \rightarrow H,$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \Delta \downarrow & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}$$

Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ 1_H \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes 1_H & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & & \swarrow = & \\ & & H & & \end{array}$$

# Hopf Algebra

A **Hopf algebra** is a bialgebra  $H$  over a field with an antipode

$S : H \rightarrow H$  where the bialgebra operations are

$$\nabla : H \otimes H \rightarrow H,$$

$$1_H : \mathbb{C} \rightarrow H,$$

$$\Delta : H \rightarrow H \otimes H,$$

$$\varepsilon : H \rightarrow \mathbb{C}$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \Delta \downarrow & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}$$

Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ 1_H \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes 1_H & \\ \mathbb{C} \otimes H & & & & H \otimes \mathbb{C} \\ & \searrow = & & \swarrow = & \\ & & H & & \end{array}$$

Counit:

$$\begin{array}{ccccc} & & H & & \\ \varepsilon \otimes id \nwarrow & & \downarrow \Delta & \nearrow id \otimes \varepsilon & \\ \mathbb{C} \otimes H & & & & H \otimes \mathbb{C} \\ & \searrow = & & \swarrow = & \\ & & H \otimes H & & \end{array}$$

# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H & & 
 \end{array}$$

# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\nabla} & H & \xrightarrow{\Delta} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{id \otimes \tau \otimes id} & H \otimes H \otimes H \otimes H & & 
 \end{array}$$

Unit and Counit compatibility:

$$\begin{array}{ccc}
 \mathbb{C} & \xrightarrow{1_H} & H \\
 1_H \otimes 1_H \searrow & & \downarrow \Delta \\
 & & H \otimes H
 \end{array}
 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{\epsilon} & \mathbb{C} \\
 \nabla \uparrow & \nearrow \epsilon \otimes \epsilon & \\
 H \otimes H & & 
 \end{array}$$

# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H & & 
 \end{array}$$

Unit and Counit compatibility:

$$\begin{array}{ccc}
 \mathbb{C} & \xrightarrow{1_H} & H \\
 1_H \otimes 1_H \searrow & & \downarrow \Delta \\
 & & H \otimes H
 \end{array}
 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{\epsilon} & \mathbb{C} \\
 \nabla \uparrow & & \nearrow \epsilon \otimes \epsilon \\
 H \otimes H & & 
 \end{array}$$

Antipode:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad id \otimes S \quad} & H \otimes H & & \\
 \Delta \uparrow & & & & \downarrow \nabla \\
 H & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{1_H} & H \\
 \Delta \downarrow & & & & \uparrow \nabla \\
 H \otimes H & \xrightarrow{\quad S \otimes id \quad} & H \otimes H & & 
 \end{array}$$

Common philosophy holds that a **quantum group** might be defined as a Hopf alg with a bijective antipode,  $S$ , and an invertible element  $R \in H \otimes H$  satisfying

$$(1) R \left( \sum h_{(1)} \otimes h_{(2)} \right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

$$(2) \Delta \otimes id(R) = R_{1,3} R_{2,3}$$

$$(3) id \otimes \Delta(R) = R_{1,3} R_{1,2}$$

where  $R = \sum R_{(1)} \otimes R_{(2)}$ ,  $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$ ,  $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$ , and  $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$ . The element  $R$  witnesses how close being cocommutative the quantum group is. As well, one can show that

$$R_{1,2} R_{1,3} R_{2,3} = R_{2,3} R_{1,3} R_{1,2}.$$

Hence,  $R$  is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

# Quantum Group Examples

The three Hopf algs we have already discussed are all quantum groups.

Sweedler's algebra,  $H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$ , has the universal R-matrix

$$R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$$

The Kac-Paljutkin algebra,

$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$ , has 6 non-isomorphic quasitriangular structures.

And the ambiguity of what is a quantum group can be best seen with  $\mathcal{U}_q(\mathfrak{sl}_2)$ . This is undoubtedly a quantum group as it was the inspiration for the concept, but its universal R-matrix exists only in a completion of the tensor product  $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$ , and so is a formal power series of pure tensors.

# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .

Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:



# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .

Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

$$\begin{array}{ccc} H \otimes H \otimes A & \xrightarrow{\nabla \otimes id} & H \otimes A \\ \downarrow id \otimes \alpha & & \downarrow \alpha \\ H \otimes A & \xrightarrow{\alpha} & A \end{array} \quad \begin{array}{ccc} \mathbb{C} \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\ & \searrow = & \downarrow \alpha \\ & & A \end{array}$$

# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .

Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

$$\begin{array}{ccc} H \otimes H \otimes A & \xrightarrow{\nabla \otimes id} & H \otimes A \\ \downarrow id \otimes \alpha & & \downarrow \alpha \\ H \otimes A & \xrightarrow{\alpha} & A \end{array} \quad \begin{array}{ccc} \mathbb{C} \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\ & \searrow = & \downarrow \alpha \\ & & A \end{array}$$

$A$  is called an  $H$ -**module algebra** if the following also commute:

# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .  
Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

$$\begin{array}{ccc}
 H \otimes H \otimes A & \xrightarrow{\nabla \otimes id} & H \otimes A \\
 \downarrow id \otimes \alpha & & \downarrow \alpha \\
 H \otimes A & \xrightarrow{\alpha} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 C \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\
 & \searrow = & \downarrow \alpha \\
 & & A
 \end{array}$$

$A$  is called an  **$H$ -module algebra** if the following also commute:

$$\begin{array}{ccccc}
 H \otimes A \otimes A & \xrightarrow{id \otimes \nabla} & H \otimes A & \xrightarrow{\alpha} & A \xleftarrow{\nabla} A \otimes A \\
 \Delta \otimes id \otimes id \downarrow & & & & \uparrow \alpha \otimes \alpha \\
 H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & 
 \end{array}$$

# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .  
Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

$$\begin{array}{ccc}
 H \otimes H \otimes A & \xrightarrow{\nabla \otimes id} & H \otimes A \\
 \downarrow id \otimes \alpha & & \downarrow \alpha \\
 H \otimes A & \xrightarrow{\alpha} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{C} \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\
 & \searrow = & \downarrow \alpha \\
 & & A
 \end{array}$$

$A$  is called an  **$H$ -module algebra** if the following also commute:

$$\begin{array}{ccccc}
 H \otimes A \otimes A & \xrightarrow{id \otimes \nabla} & H \otimes A & \xrightarrow{\alpha} & A \xleftarrow{\nabla} A \otimes A \\
 \Delta \otimes id \otimes id \downarrow & & & & \uparrow \alpha \otimes \alpha \\
 H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & 
 \end{array}$$
  

$$\begin{array}{ccc}
 H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{1_A} A \\
 id \otimes 1_A \downarrow & & \nearrow \alpha \\
 H \otimes A & & 
 \end{array}$$

In words,  $H$  **acts on**  $A$  iff you can multiply in  $H$  and then act on  $A$  or act on  $A$  consecutively,  $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And  $A$  is an  $H$ -**module alg** iff  $H$  acts on  $A$  and  $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a'), \quad h(1_A) = \varepsilon(h)1_A$$

where  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ .

# Semidirect Product

Let  $G$  and  $G'$  be groups where  $G'$  acts on  $G$  by automorphisms, giving the semidirect product group,  $G \rtimes G'$ .

The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product  $g'g = g'(g)g'$ .

## Smash Product Algebra

If  $H$  is a Hopf algebra and  $A$  an  $H$ -module algebra, then  $A \# H$  is defined as  $A \otimes H$  as a vector space and with product

$$(a' \otimes h)(a \otimes h') = \sum_i a' h_{(1)}(a) \otimes h_{(2)} h'$$

where  $a, a' \in A$ ,  $h, h' \in H$  and  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ .

# Smash Product Algebra

## "Group-like" and "Lie-like"

For Hopf alg  $H$ , define

$$G(H) = \{h \in H \mid \Delta(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

## Cartier-Kostant-Milnor-Moore Theorem

Let  $H$  be a cocommutative Hopf algebra over  $\mathbb{C}$ , then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dimpt'l Hopf alg over  $\mathbb{C}$  is iso to a grp alg.

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".



- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If  $H$  is semisimple and finite-dimensional, and  $A$  is semiprime, is  $A \# H$  semiprime?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If  $H$  is semisimple and finite-dimensional, and  $A$  is semiprime, is  $A \# H$  semiprime?
- If  $B$  is a Koszul algebra, are there nontrivial PBW deformations of  $B \# \mathcal{U}_q(\mathfrak{sl}_2)$ ?