Brandon Mather

Algebra Seminar, November 2023

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation: $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}\langle x_1, \ldots, x_n\rangle / (x_j x_i - x_i x_j)$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ji} q_{ij}$.
$\mathbb{C}_Q[x_1, \ldots, x_n] = \mathbb{C}\langle x_1, \ldots, x_n\rangle / (x_j x_i - q_{ij} x_i x_j)$ is called a **quantum polynomial ring**.

Example: $\mathbb{C}_{-1}[u, v] = \mathbb{C}\langle u, v\rangle / (uv + vu)$

Swwedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by Moss Sweedler (1969):

$H_4 =$

$$\left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

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"Group-like" "Lie-like"

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"Group-like" "Lie-like"

Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(g) = \triangle(g)$ but $\tau \circ \triangle(x) \neq \triangle(x)$, this is what is called non-cocommutativity.

H_4 acts on $\mathbb{C}_{-1}[u, v]$ by

$$g \cdot u = u,\ g \cdot v = -v,\ x \cdot u = 0,\ x \cdot v = u.$$

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Giving the representation on the vector space $\mathbb{C}_{-1}[u, v]$

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

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$$S(x) = x,\ S(y) = y,\ S(z) = z.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[u, v]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_{-1}[u, v]$ by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

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with operations:

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$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$
$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1},\ S(K^{-1}) = K.$$

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$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

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$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \quad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

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$S : H \to H$ where the bialgebra operations are

Hopf Algebra

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$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"'] & H
\end{tikzcd}$$

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Unit:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

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$\nabla : H \otimes H \to H,$ $\qquad \triangle : H \to H \otimes H,$

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$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"'] & H \end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & H \otimes H \arrow[d, "\triangle \otimes id"] \\ H \otimes H \arrow[r, "id \otimes \triangle"'] & H \otimes H \otimes H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} & H \otimes H \arrow[d, "\nabla"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode $S : H \to H$ where the bialgebra operations are

$$\bigtriangledown : H \otimes H \to H, \qquad \bigtriangleup : H \to H \otimes H,$$

$$\eta : \mathbb{C} \to H, \qquad \varepsilon : H \to \mathbb{C}$$

so that the following commute:

Associativity:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"'] & H
\end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd}
H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"'] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\
H \otimes H \arrow[r, "id \otimes \bigtriangleup"'] & H \otimes H \otimes H
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
& H \otimes H \arrow[dd, "\bigtriangledown"] & \\
\mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\
& H &
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
& H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\
\mathbb{C} \otimes H & & H \otimes \mathbb{C} \\
& H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] &
\end{tikzcd}$$

Product and Coproduct compatibility:

$$\begin{tikzcd} & H \otimes H \arrow[r, "\nabla"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[r, "\Delta"] & H \otimes H & \\ H \otimes H \otimes H \otimes H \arrow[rrr, "id \otimes \tau \otimes id"'] & & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle\otimes\triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & H \arrow[d, "\triangle"] \\ & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] & \end{tikzcd}$$

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Unit and Counit compatibility:

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& H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\varepsilon"] & \mathbb{C} \\
H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] &
\end{tikzcd}$$

Antipode:

$$\begin{tikzcd}
H \otimes H \arrow[rr, "id\otimes S"] & & H \otimes H \arrow[d, "\nabla"] \\
H \arrow[u, "\triangle"] \arrow[r, "\varepsilon"] \arrow[d, "\triangle"'] & \mathbb{C} \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "S\otimes id"] & & H \otimes H \arrow[u, "\nabla"']
\end{tikzcd}$$

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

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\end{tikzcd}
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$$\begin{tikzcd} H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\ H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"'] \end{tikzcd}.$$

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

$$
\begin{tikzcd}
H \otimes H \otimes A \arrow[r, "\nabla\otimes id"] \arrow[d, "id\otimes\alpha"] & H \otimes A \arrow[d, "\alpha"] \\
H \otimes A \arrow[r, "\alpha"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \otimes A \arrow[r, "\eta\otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\
 & A
\end{tikzcd}
$$

A is called a **module algebra** if the following also commute:

$$
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H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle\otimes id\otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\
H \otimes H \otimes A \otimes A \arrow[rrr, "id\otimes\tau\otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha\otimes\alpha"']
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\eta"'] & \mathbb{C} \arrow[r, "\eta"] & A \\
H \otimes A \arrow[urr, "\alpha"'] & &
\end{tikzcd}
$$

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group, $G \rtimes G'$. The action can be extended to the group algebras, $\mathbb{C}G$ and $\mathbb{C}G'$. This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product $g'g = (g' \cdot g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then $A \# H$ is the smash product algebra defined as $A \otimes H$ as a vector space and with product

$$ha = \sum_i (g_i \cdot a)k_i$$

where $a \in A$, $h \in H$ and $\triangle(h) = \sum_i g_i \otimes k_i$.

Smash Product Algebra

"Group-like" and "Lie-like"

Let H be a Hopf algebra, define $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$ and $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$.
$G(H)$ is the set of grouplike elements of H and forms a group under the product.
$P(H)$ is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over $\mathbb{C}$, then

$$H \cong \mathcal{U}(P(H))\#\mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over $\mathbb{C}$ is isomorpihc to a group algebra.

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".

Research Directions

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- If H is semisimple and finite-dimensional, and A is semiprime, is $A\#H$ semiprime?

Research Directions

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is $A\#H$ semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B\#\mathcal{U}_q(\mathfrak{sl}_2)$?