# Hopf Module Algebras

Brandon Mather

# History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

#### Goal

To understand the actions of Hopf algebras on other algebras

## Quantum Plane

Notation: 
$$\mathbb{C}[x_1,\ldots,x_n] = \mathbb{C}\langle x_1,\ldots,x_n\rangle/(x_jx_i-x_ix_j)$$

#### Quantum Polynomial Ring

Let  $Q = (q_{ij})$  be an  $n \times n$  matrix of roots of unity where  $q_{ii} = 1 = q_{ii}q_{ii}$ .

 $\mathbb{C}_Q[x_1,\ldots,x_n] = \mathbb{C}\langle x_1,\ldots,x_n\rangle/(x_jx_i-q_{ij}x_ix_j)$  is called a **quantum polynomial ring**.

Example: 
$$\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 \rangle / (v_1 v_2 + v_2 v_1)$$

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by Moss Sweedler (1969):

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Let  $\tau$  be the 'flip' over the tensor product, so  $\tau(u \otimes v) = v \otimes u$ . Note that  $\tau \circ \triangle(g) = \triangle(g)$  but  $\tau \circ \triangle(x) \neq \triangle(x)$ , this is what is called non-cocommutativity.

## Actions of Sweedler's Algebra

$$\mathcal{H}_4$$
 acts on  $\mathbb{C}_{-1}[\mathit{v}_1,\mathit{v}_2]$  by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

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Giving the representation on the vector space  $\mathbb{C}_{-1}[v_1,v_2]$ 

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \right\rangle$$

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## Actions of Kac-Paljutkin Algebra

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$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

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Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left( K - K^{-1} \right), KEK^{-1} = q^2 E, 
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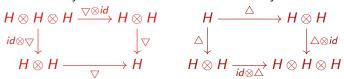
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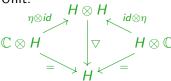
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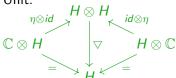
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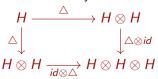
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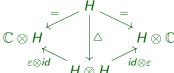
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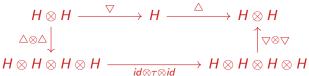


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## Hopf Algebra Diagrams

Product and Coproduct compatibility:

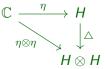


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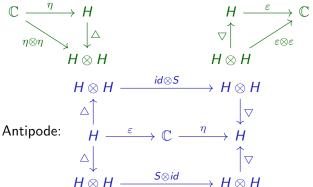


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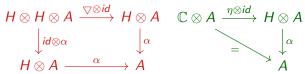
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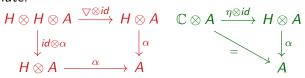


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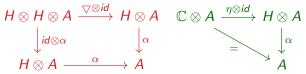


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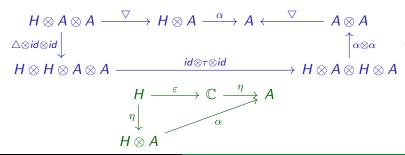
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### Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group,  $G \rtimes G'$ . The action can be extended to the group algebras,  $\mathbb{C}G$  and  $\mathbb{C}G'$ . This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G\#\mathbb{C}G'$$

with product  $g'g = (g' \cdot g)g'$ .

#### Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is the smash product algebra defined as  $A\otimes H$  as a vector space and with product

$$ha = \sum_{i} (g_i \cdot a) k_i$$

where  $a \in A$ ,  $h \in H$  and  $\triangle(h) = \sum_i g_i \otimes k_i$ .

## Smash Product Algebra

### "Group-like" and "Lie-like"

Let H be a Hopf algebra, define  $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$  and  $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$ .

G(H) is the set of grouplike elements of H and forms a group under the product.

P(H) is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

#### Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over  $\mathbb{C}$ , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over  $\mathbb C$  is isomorpihe to a group algebra.

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- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of  $B\#\mathcal{U}_q(\mathfrak{sl}_2)$ ?