# Hopf Algebra Actions on Quantum Planes

Brandon Mather

Algebra Seminar November 2023

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## Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on  $\mathbb{C}_q[x,y]$  where  $q^2=-1$  by 
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{\mathcal{O}}[x_1, x_2, x_3, x_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
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$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{-1}[u,v]$  by

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The 16-dimension semisimple Hopf algebras have been classified by Kashina. One such example is

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## Action of $H_{16}$

 $H_{16}$  acts on the algebra

$$\mathbb{C}[t, u, v, w]/(tw + wt, uw + wu)$$

by the representation

$$x \mapsto \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ y \mapsto \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$z \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$egin{aligned} U_q(\mathfrak{sl}_2) = \ & \left\langle {\it E}, {\it F}, {\it K}, {\it K}^{-1} \mid {\it EF-FE} = (q - q^{-1})^{-1} \left( {\it K} - {\it K}^{-1} 
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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$
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A **Hopf algebra** is a bialgebra H over a field with an antipode  $S: H \to H$  where the bialgebra operations are

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$$\nabla: H \otimes H \to H$$
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so that the following commute:

#### Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
\downarrow^{id} \otimes \nabla \downarrow & & \downarrow \nabla \\
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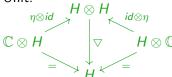
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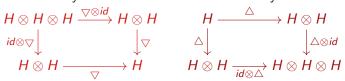
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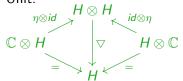
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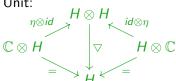
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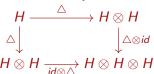
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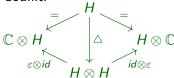
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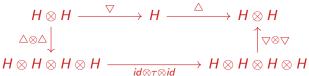


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#### Hopf Algebra Diagrams

Product and Coproduct compatibility:

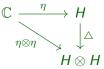


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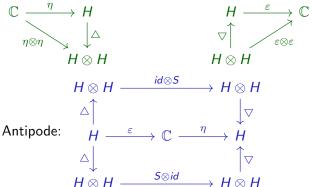


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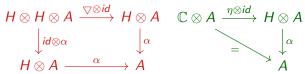
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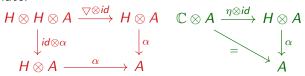


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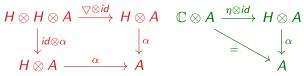


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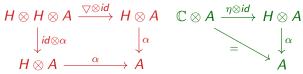
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