Quantum Symmetry of Hopf Actions

Brandon Mather

Algebra Seminar, November 2023

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$$

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Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on $\mathbb{C}_q[u,v]$ where $q^2=-1$ by
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{\mathcal{O}}[v_1, v_2, v_3, v_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
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And on $\mathbb{C}_{-1}[u,v]$ by

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Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E,
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 $\nabla: H \otimes H \to H$,

so that the following commute:

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$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
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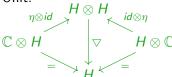
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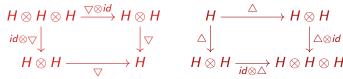
$$\nabla: H \otimes H \to H$$
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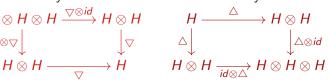
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Coassociativity:



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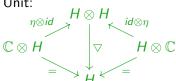
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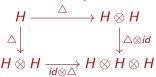
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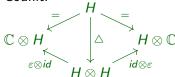
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Hopf Algebra Diagrams

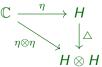
Product and Coproduct compatibility:



Hopf Algebra Diagrams

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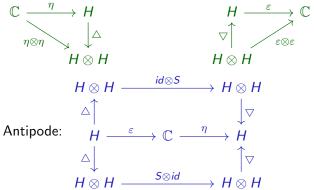


Hopf Algebra Diagrams

Product and Coproduct compatibility:

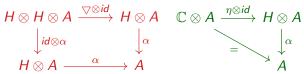
$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} H & \stackrel{\triangle}{\longrightarrow} H \otimes H \\ & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} H \otimes H \otimes H \otimes H \end{array}$$

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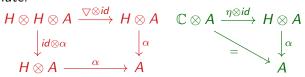


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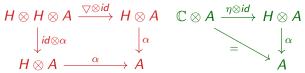


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