## Quantum Group Actions and Hopf Algebras

Brandon Mather

University of North Texas Master's Defense Department of Mathematics February 28th 2024

## History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

#### Goal

To understand the actions of Hopf algebras on other algebras

#### Quantum Plane

Notation:  $\mathbb{C}[v_1,\ldots,v_n] = \mathbb{C}\langle v_1,\ldots,v_n \mid v_j v_i - v_i v_j \rangle$ 

#### Quantum Polynomial Ring

Let  $Q = (q_{ij})$  be an  $n \times n$  matrix of roots of unity where

$$q_{ii}=1=q_{ji}q_{ij}.$$

A quantum polynomial ring is

$$\mathbb{C}_{Q}[v_{1},\ldots,v_{n}] = \mathbb{C}\left\langle v_{1},\ldots,v_{n} \mid v_{j}v_{i} - q_{ij}v_{i}v_{j}\right\rangle.$$

Example:  $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 | v_1 v_2 + v_2 v_1 \rangle$ 

#### Motivation

• When a grp G acts on a space V over  $\mathbb C$  linearly, the action can be extended to  $V\otimes V$  by  $g\in G$  acting as

$$g \otimes g = \triangle(g)$$
.

Then  $\triangle$  defines a coproduct map

$$\triangle: \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,  $\triangle: A \rightarrow A \otimes A$ , we call  $g \in A$  grouplike if  $\triangle(g) = g \otimes g$ .

#### Motivation

• When a grp G acts on a space V over  $\mathbb C$  linearly, the action can be extended to  $V\otimes V$  by  $g\in G$  acting as

$$g \otimes g = \triangle(g)$$
.

Then  $\triangle$  defines a coproduct map

$$\triangle: \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,  $\triangle: A \rightarrow A \otimes A$ , we call  $g \in A$  grouplike if  $\triangle(g) = g \otimes g$ .

When a Lie alg  $\mathfrak g$  acts on a space V over  $\mathbb C$ , the action can be extended to  $V\otimes V$  by  $x\in \mathfrak g$  acting as

$$x \otimes 1 + 1 \otimes x = \triangle(x)$$
.

Then  $\triangle$  defines a coproduct map

$$\triangle: \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts,  $\triangle: A \rightarrow A \otimes A$ , we call  $x \in A$  **primitive** if  $\triangle(x) = x \otimes 1 + 1 \otimes x$ .

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\triangle: H_4 \to H_4 \otimes H_4, \varepsilon: H_4 \to \mathbb{C}, S: H_4 \to H_4$ 

$$\triangle(g) = g \otimes g,$$
  $\triangle(x) = x \otimes 1 + g \otimes x$ 

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\triangle: H_4 \to H_4 \otimes H_4, \varepsilon: H_4 \to \mathbb{C}, \mathcal{S}: H_4 \to H_4$ 

$$\triangle(g) = g \otimes g,$$
  $\triangle(x) = x \otimes 1 + g \otimes x$   
 $\varepsilon(g) = 1,$   $\varepsilon(x) = 0$ 

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\triangle: H_4 \to H_4 \otimes H_4, \varepsilon: H_4 \to \mathbb{C}, S: H_4 \to H_4$ 

$$\triangle(g) = g \otimes g,$$
  $\triangle(x) = x \otimes 1 + g \otimes x$   
 $\varepsilon(g) = 1,$   $\varepsilon(x) = 0$   
 $S(g) = g^{-1},$   $S(x) = -x$ 

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations  $\triangle: H_4 \to H_4 \otimes H_4, \varepsilon: H_4 \to \mathbb{C}, S: H_4 \to H_4$ 

$$\triangle(g) = g \otimes g,$$
  $\triangle(x) = x \otimes 1 + g \otimes x$   
 $\varepsilon(g) = 1,$   $\varepsilon(x) = 0$   
 $S(g) = g^{-1},$   $S(x) = -x$ 

Group-like

(1,g) – Primitive

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

with operations  $\triangle: H_4 \to H_4 \otimes H_4, \varepsilon: H_4 \to \mathbb{C}, S: H_4 \to H_4$ 

$$\triangle(g) = g \otimes g,$$
  $\triangle(x) = x \otimes 1 + g \otimes x$   
 $\varepsilon(g) = 1,$   $\varepsilon(x) = 0$   
 $S(g) = g^{-1},$   $S(x) = -x$ 

Group-like

(1,g) – Primitive

Let  $\tau$  be the 'flip' over the tensor product, so  $\tau(u \otimes v) = v \otimes u$ . Note that  $\tau \circ \triangle(x) \neq \triangle(x)$ , so H is non-cocommutative.

## Actions of Sweedler's Algebra

$$\mathcal{H}_4$$
 acts on  $\mathbb{C}_{-1}[\mathit{v}_1,\mathit{v}_2]$  by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

## Actions of Sweedler's Algebra

 $H_4$  acts on  $\mathbb{C}_{-1}[v_1, v_2]$  by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

We can express this action on the generators  $v_1, v_2$  as

$$g\mapsto \begin{bmatrix}1&0\\0&-1\end{bmatrix},\; x\mapsto \begin{bmatrix}0&1\\0&0\end{bmatrix}.$$

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$
$$\triangle(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \frac{1}{2} (1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x, \ S(y) = y, \ S(z) = z.$$

## Actions of Kac-Paljutkin Algebra

 $H_8$  acts on  $\mathbb{C}_q[v_1,v_2]$  where  $q^2=-1$  via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

## Actions of Kac-Paljutkin Algebra

 $H_8$  acts on  $\mathbb{C}_q[v_1,v_2]$  where  $q^2=-1$  via the representation

$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on  $\mathbb{C}_{\mathcal{O}}[v_1, v_2, v_3, v_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

## Actions of Kac-Paljutkin Algebra

 $H_8$  acts on  $\mathbb{C}_q[v_1,v_2]$  where  $q^2=-1$  via the representation

$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on  $\mathbb{C}_{\mathcal{O}}[v_1, v_2, v_3, v_4]$  for

$$q_{12}=q_{34}^{-1}, \ \ q_{13}=q_{24}^{-1}, \ \ q_{14}^2=1, \ \ q_{23}^2=-1 \ \text{via the rep}$$

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And on  $\mathbb{C}_{-1}[v_1, v_2]$  via the rep

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:  $\mathcal{U}_q(\mathfrak{sl}_2)=$ 

$$\left\langle E,F,K,K^{-1}\mid EF-FE=(q-q^{-1})^{-1}\left(K-K^{-1}
ight),KEK^{-1}=q^{2}E,
ight.$$
  $KFK^{-1}=q^{-2}F,KK^{-1}=K^{-1}K=1
ight
angle$ 

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:  $\mathcal{U}_q(\mathfrak{sl}_2)=$ 

$$\left\langle E,F,K,K^{-1}\mid EF-FE=(q-q^{-1})^{-1}\left(K-K^{-1}\right),KEK^{-1}=q^{2}E,
ight.$$
  $\left. KFK^{-1}=q^{-2}F,KK^{-1}=K^{-1}K=1
ight
angle$ 

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$
$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:  $\mathcal{U}_q(\mathfrak{sl}_2)=$ 

$$\left\langle E,F,K,K^{-1}\mid EF-FE=(q-q^{-1})^{-1}\left(K-K^{-1}
ight),KEK^{-1}=q^{2}E,
ight.$$
  $KFK^{-1}=q^{-2}F,KK^{-1}=K^{-1}K=1
ight
angle$ 

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:  $\mathcal{U}_q(\mathfrak{sl}_2)=$ 

$$\left\langle E,F,K,K^{-1}\mid EF-FE=(q-q^{-1})^{-1}\left(K-K^{-1}
ight),KEK^{-1}=q^{2}E,
ight.$$
  $KFK^{-1}=q^{-2}F,KK^{-1}=K^{-1}K=1
ight
angle$ 

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:  $\mathcal{U}_q(\mathfrak{sl}_2)=$ 

$$\left\langle E,F,K,K^{-1}\mid EF-FE=(q-q^{-1})^{-1}\left(K-K^{-1}
ight),KEK^{-1}=q^{2}E,
ight.$$
  $KFK^{-1}=q^{-2}F,KK^{-1}=K^{-1}K=1
ight
angle$ 

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K, \ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0, \ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

Note: You can recover  $\mathcal{U}(\mathfrak{sl}_2)$  by limiting  $q \to 1$ .

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[\mathit{v}_1,\mathit{v}_2]$  via the representation

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[v_1,v_2]$  via the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

# Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

 $\mathcal{U}_q(\mathfrak{sl}_2)$  acts on  $\mathbb{C}_q[v_1,v_2]$  via the representation

$$E\mapsto egin{bmatrix} 0 & 1 \ 0 & 0 \end{bmatrix} \ \mathcal{K}\mapsto egin{bmatrix} q & 0 \ 0 & q^{-1} \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode  $S: H \to H$  where the bialgebra operations are

A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$  where the bialgebra operations are

 $\nabla: H \otimes H \to H$ ,

so that the following commute:

Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
id \otimes \nabla \downarrow & & \downarrow \nabla \\
H \otimes H & \xrightarrow{\nabla} & H
\end{array}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

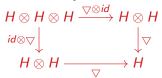
 $S: H \rightarrow H$  where the bialgebra operations are

 $\nabla: H \otimes H \to H$ ,

 $1_H:\mathbb{C}\to H$ ,

so that the following commute:

#### Associativity:



#### Unit:



A **Hopf algebra** is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$  where the bialgebra operations are

 $\nabla: H \otimes H \to H$ ,

 $\triangle: H \to H \otimes H$ ,

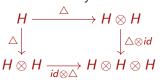
 $1_H:\mathbb{C}\to H$ ,

so that the following commute:

#### Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
id \otimes \nabla \downarrow & & \downarrow \nabla \\
H \otimes H & \xrightarrow{\nabla} & H
\end{array}$$

#### Coassociativity:



#### Unit:



A  $\operatorname{Hopf}$  algebra is a bialgebra H over a field with an antipode

 $S: H \rightarrow H$  where the bialgebra operations are

 $\nabla: H \otimes H \to H$ ,

 $1_H:\mathbb{C}\to H$ ,

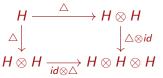
 $\triangle: H \to H \otimes H,$  $\varepsilon: H \to \mathbb{C}$ 

so that the following commute:

Associativity:

 $\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
id \otimes \nabla \downarrow & & \downarrow \nabla \\
H \otimes H & \xrightarrow{\qquad} & H
\end{array}$ 

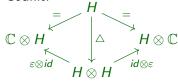
Coassociativity:



Unit:



Counit:



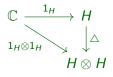
## Hopf Algebra Diagrams

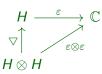
Product and Coproduct compatibility:

## Hopf Algebra Diagrams

Product and Coproduct compatibility:

Unit and Counit compatibility:



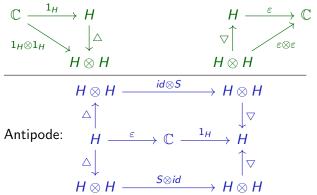


# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} & H & \stackrel{\triangle}{\longrightarrow} & H \otimes H \\ & & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} & H \otimes H \otimes H \otimes H \end{array}$$

Unit and Counit compatibility:



# Quantum Group

Common philosophy: A **quantum group** is a Hopf alg H with a bijective antipode, and some invertible  $R \in H \otimes H$  witnessing how close H is to being cocommutative and satisfying:

$$(1)R\left(\sum h_{(1)} \otimes h_{(2)}\right)R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

$$(2)\triangle \otimes id(R) = R_{1,3}R_{2,3}$$

$$(3)id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where  $R = \sum R_{(1)} \otimes R_{(2)}$ ,  $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$ ,  $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$ , and  $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$ . Hence, R is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}.$$

# Quantum Group Examples

The three Hopf algs examples above are all quantum groups. Sweedler's algebra,  $H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$ , has the univ R-matrix

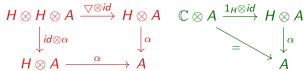
$$R=1\otimes 1-2\tfrac{1-g}{2}\otimes \tfrac{1-g}{2}+x\otimes x+2x\tfrac{1-g}{2}\otimes x\tfrac{1-g}{2}-2x\otimes x\tfrac{1-g}{2}.$$

The Kac-Paljutkin algebra,

 $H_8=\left\langle x,y,z\mid x^2=y^2=1,xy=yx,zx=yz,zy=xz,z^2=rac{1}{2}(1+x+y-xy)\right\rangle$ , has 6 non-iso quasitriangular structures. And the ambiguity of what is a quantum group can be best seen with  $\mathcal{U}_q(\mathfrak{sl}_2)$ . This is undoubtedly a quantum group as it was the inspiration for the concept, but its univ R-matrix exists only in a completion of the tensor product  $\mathcal{U}_q(\mathfrak{sl}_2)\otimes\mathcal{U}_q(\mathfrak{sl}_2)$ , and so is a formal power series of pure tensors.

Let H be a Hopf alg and A an alg with a map  $\alpha: H \otimes A \to A$ . Then we say H acts on A by  $\alpha$  if the following diagrams commute:

Let H be a Hopf alg and A an alg with a map  $\alpha: H \otimes A \to A$ . Then we say H acts on A by  $\alpha$  if the following diagrams commute:



Let H be a Hopf alg and A an alg with a map  $\alpha: H \otimes A \to A$ . Then we say H acts on A by  $\alpha$  if the following diagrams commute:

$$H \otimes H \otimes A \xrightarrow{\nabla \otimes id} H \otimes A \quad \mathbb{C} \otimes A \xrightarrow{1_H \otimes id} H \otimes A$$

$$\downarrow^{id \otimes \alpha} \qquad \downarrow^{\alpha} \qquad \downarrow^{\alpha}$$

$$H \otimes A \xrightarrow{\alpha} A$$

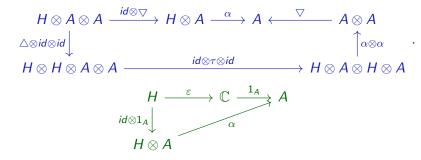
A is called an H-module algebra if the following also commute:

Let H be a Hopf alg and A an alg with a map  $\alpha: H \otimes A \to A$ . Then we say H acts on A by  $\alpha$  if the following diagrams commute:

A is called an H-module algebra if the following also commute:

Let H be a Hopf alg and A an alg with a map  $\alpha: H \otimes A \to A$ . Then we say H acts on A by  $\alpha$  if the following diagrams commute:

A is called an H-module algebra if the following also commute:



In words, H acts on A iff you can multiply in H and then act on A or act on A consecutively,  $\forall h, h' \in H, \forall a \in A$ 

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-module alg iff H acts on A and  $\forall h \in H, \forall a, a' \in A$ 

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a), \quad h(1_A) = \varepsilon(h)1_A$$

where 
$$\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$$
.

### Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms, giving the semidirect product group,  $G \rtimes G'$ .

The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product g'g = g'(g)g'.

### Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is defined as  $A\otimes H$  as a vector space and with product

$$(a'\otimes h)(a\otimes h')=\sum_i a'h_{(1)}(a)\otimes h_{(2)}h'$$

where  $a, a' \in A$ ,  $h, h' \in H$  and  $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$ .

# Smash Product Algebra

### "Group-like" and "Lie-like"

For Hopf alg H, define

$$G(H) = \{h \in H \mid \triangle(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

#### Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over  $\mathbb{C}$ , then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dimpt'l Hopf alg over  $\mathbb C$  is iso to a grp alg.

 Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of B#U<sub>q</sub>(sl<sub>2</sub>)?