Quantum Group Actions and Hopf Algebras

Brandon Mather

University of North Texas Master's Defense
Department of Mathematics
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History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation: $\mathbb{C}[v_1, \ldots, v_n] = \mathbb{C}\left\langle v_1, \ldots, v_n \mid v_j v_i - v_i v_j \right\rangle$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where

$$q_{ii} = 1 = q_{ji} q_{ij}.$$

A **quantum polynomial ring** is

$$\mathbb{C}_Q[v_1, \ldots, v_n] = \mathbb{C}\left\langle v_1, \ldots, v_n \mid v_j v_i - q_{ij} v_i v_j \right\rangle.$$

Example: $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C}\left\langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \right\rangle$

Motivation

1. When a grp G acts on a space V over $\mathbb{C}$ linearly, the action can be extended to $V \otimes V$ by $g \in G$ acting as

   $$g \otimes g = \triangle(g).$$

   Then $\triangle$ defines a coproduct map

   $$\triangle : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$

   For arbitrary coproducts,
   $\triangle : A \to A \otimes A$,
   we call $g \in A$ **grouplike** if
   $\triangle(g) = g \otimes g$.

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2. When a Lie alg $\mathfrak{g}$ acts on a space V over $\mathbb{C}$, the action can be extended to $V \otimes V$ by $x \in \mathfrak{g}$ acting as

$$x \otimes 1 + 1 \otimes x = \triangle(x).$$

Then $\triangle$ defines a coproduct map

$$\triangle : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts, $\triangle : A \to A \otimes A$, we call $x \in A$ **primitive** if $\triangle(x) = x \otimes 1 + 1 \otimes x$.

We want an algebra that synthesizes group actions and actions by Lie algebras.

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

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with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$$\triangle(g) = g \otimes g, \qquad \triangle(x) = x \otimes 1 + g \otimes x$$

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Group-like $\qquad\qquad$ $(1, g) - \text{Primitive}\textit{verticalspacing}$

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Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(x) \neq \triangle(x)$, so H is non-cocommutative.

Actions of Sweedler's Algebra

H_4 acts on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$g \cdot v_1 = v_1,\ g \cdot v_2 = -v_2,\ x \cdot v_1 = 0,\ x \cdot v_2 = v_1.$$

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We can express this action on the generators v_1, v_2 as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

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$$S(x) = x,\ S(y) = y,\ S(z) = z.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{-1}[v_1, v_2]$ via the rep

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:
$\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

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Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \to 1$.

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$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \qquad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

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$\nabla : H \otimes H \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"'] & H \end{tikzcd}$$

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Unit:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\nabla"] & \\ \mathbb{C} \otimes H \arrow[ur, "1_H \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes 1_H"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

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Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "\Delta \otimes id"] \\ H \otimes H \arrow[r, "id \otimes \Delta"'] & H \otimes H \otimes H \end{tikzcd}$$

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$\bigtriangledown : H \otimes H \to H$, $\bigtriangleup : H \to H \otimes H$,

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$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"'] & H \end{tikzcd}$$

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Counit:

$$\begin{tikzcd} & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\ \mathbb{C} \otimes H & & H \otimes \mathbb{C} \\ & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] & \end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\bigtriangledown"] \arrow[d, "\triangle\otimes\triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\bigtriangledown\otimes\bigtriangledown"']
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$$\begin{tikzcd}
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\end{tikzcd}
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\end{tikzcd}$$

Antipode:

$$\begin{tikzcd}
H \otimes H \arrow[rr, "id\otimes S"] & & H \otimes H \arrow[d, "\triangledown"] \\
H \arrow[u, "\triangle"] \arrow[d, "\triangle"'] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "1_H"] & H \\
H \otimes H \arrow[rr, "S\otimes id"] & & H \otimes H \arrow[u, "\triangledown"']
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$$(3)\ id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where $R = \sum R_{(1)} \otimes R_{(2)}$ and

$$R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$$

$$R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$$

$$R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}.$$

Quantum Group

Common philosophy: A **quantum group** is a Hopf alg H with a bijective antipode, and some invertible $R \in H \otimes H$ witnessing how close H is to being cocommutative and satisfying:

$$(1)\ R\left(\sum h_{(1)} \otimes h_{(2)}\right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

$$(2)\ \triangle \otimes id(R) = R_{1,3}R_{2,3}$$

$$(3)\ id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where $R = \sum R_{(1)} \otimes R_{(2)}$ and

$$R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$$

$$R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$$

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Hence, R is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}.$$

Quantum Group Examples

The three Hopf algs examples above are all quantum groups.

And the ambiguity of what is a quantum group can be best seen with $\mathcal{U}_q(\mathfrak{sl}_2)$. This is undoubtedly a quantum group as it was the inspiration for the concept, but its univ R-matrix exists only in a completion of the tensor product $\mathcal{U}_q(\mathfrak{sl}2) \otimes \mathcal{U}_q(\mathfrak{sl}2)$, and so is a formal power series of pure tensors.

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- Sweedler's algebra, $H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$, has the univ R-matrix

$$R = 1 \otimes 1 - 2\tfrac{1-g}{2} \otimes \tfrac{1-g}{2} + x \otimes x + 2x\tfrac{1-g}{2} \otimes x\tfrac{1-g}{2} - 2x \otimes x\tfrac{1-g}{2}.$$

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- The Kac-Paljutkin algebra, $H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$, has 6 non-iso quasitriangular structures. And the ambiguity of what is a quantum group can be best seen with $\mathcal{U}_q(\mathfrak{sl}_2)$. This is undoubtedly a quantum group as it was the inspiration for the concept, but its univ R-matrix exists only in a completion of the tensor product $\mathcal{U}_q(\mathfrak{sl}2) \otimes \mathcal{U}_q(\mathfrak{sl}2)$, and so is a formal power series of pure tensors.

Hopf Algebra Actions

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\end{tikzcd}
\qquad
\begin{tikzcd}
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$$\begin{tikzcd} H \otimes A \otimes A \arrow[r, "id \otimes \nabla"] \arrow[d, "\Delta \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & & A \otimes A \arrow[ll, "\nabla"'] \\ H \otimes H \otimes A \otimes A \arrow[rrrr, "id \otimes \tau \otimes id"] & & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"'] \end{tikzcd} .$$

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H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}.$$

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "id \otimes 1_A"'] & \mathbb{C} \arrow[r, "1_A"] & A \\
H \otimes A \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Hopf Algebra Actions

In words, H **acts on** A iff you can multiply in H and then act on A or act on A consecutively, $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-**module alg** iff H acts on A and $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a), \quad h(1_A) = \varepsilon(h)1_A$$

where $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.
This generalizes the notion of acting by automorphisms.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms, giving the semidirect product group, $G \rtimes G'$.
The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product $g'g = g'(g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then $A\#H$ is defined as $A \otimes H$ as a vector space and with product

$$(a' \otimes h)(a \otimes h') = \sum_i a' h_{(1)}(a) \otimes h_{(2)} h'$$

where $a, a' \in A$, $h, h' \in H$ and $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.

Smash Product Algebra

"Group-like" and "Lie-like"

For Hopf alg H, define

$$G(H) = \{h \in H \mid \triangle(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over $\mathbb{C}$, then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dim'l Hopf alg over $\mathbb{C}$ is iso to a grp alg.

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- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on pther noncommutative algebras?
- If H is semisimple and finite-dimensional, and A is semiprime, is $A\#H$ semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B\#\mathcal{U}_q(\mathfrak{sl}_2)$?