

# Quantum Symmetry of Hopf Actions

Brandon Mather

Algebra Seminar, November 2023

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montgomery writes "Hopf Algebras and Their Actions on Rings"

# Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

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# Actions of Kac-Paljutkin Algebra

$H_8$  acts on  $\mathbb{C}_q[u, v]$  where  $q^2 = -1$  by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$q_{12} = q_{34}^{-1}$ ,  $q_{13} = q_{24}^{-1}$ ,  $q_{14}^2 = 1$ ,  $q_{23}^2 = -1$  by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



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And on  $\mathbb{C}_{-1}[u, v]$  by

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# Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in “Quantum linear problem for the sine-Gordon equation and highest weight representations” (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

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$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

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$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \quad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$



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so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

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Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ \eta \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes \eta & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & & \swarrow = & \\ & & H & & \end{array}$$

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Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \Delta \downarrow & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}$$

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# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H \xrightarrow{\quad \Delta \quad} H \otimes H \\
 \Delta \otimes \Delta \downarrow & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H
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 \qquad
 \begin{array}{ccc}
 H & \xrightarrow{\epsilon} & \mathbb{C} \\
 \nabla \uparrow & & \nearrow \epsilon \otimes \epsilon \\
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Antipode:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad id \otimes S \quad} & H \otimes H & & \\
 \Delta \uparrow & & & & \downarrow \nabla \\
 H & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{\quad \eta \quad} & H \\
 \Delta \downarrow & & & & \uparrow \nabla \\
 H \otimes H & \xrightarrow{\quad S \otimes id \quad} & H \otimes H & & 
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# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ . Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following the diagrams commute:

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$$\begin{array}{ccc}
 H & \xrightarrow{\varepsilon} & \mathbb{C} \xrightarrow{\eta} A \\
 \eta \downarrow & \nearrow \alpha & \\
 H \otimes A & & 
 \end{array}$$