Koszul Resolution of a Polynomial Ring

1 Koszul Resolution

Let $\Bbbk$ be a field, $A = \Bbbk_q[x_1, x_2] = \Bbbk < x_1, x_2 > /(x_2x_1 - qx_1x_2)$ for some $q \in \Bbbk^*$. Let $A_1 = \Bbbk[x_1]$ and $A_2 = \Bbbk[x_2]$ be subalgebras of A so that A is the twisted tensor product $A_1 \otimes^\tau A_2$ where $\tau : \mathbb{Z}^2 \to \Bbbk^*$ is the bicharacter $\tau(m, n) = q^{-mn}$. We consider the Koszul resolutions of A_1 and A_2

$$0 \to A_1^e \xrightarrow{(x_1\otimes 1 - 1\otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e \xrightarrow{(x_2\otimes 1 - 1\otimes x_2)\cdot} A_2^e \to 0.$$

In order to use a theorem, the differentials of these resolutions need to be graded maps, to this end we shift the grading of the homological degree 1 component of both resolutions up by 1

$$0 \to A_1^e(-1) \xrightarrow{(x_1\otimes 1 - 1\otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e(-1) \xrightarrow{(x_2\otimes 1 - 1\otimes x_2)\cdot} A_2^e \to 0.$$

Then by a theorem proved by Bergh and Oppermann, the total complex of the tensor product of these two resolutions is a projective resolution of A as an A^e-module. This resolution is

$$0 \to A_1^e(-1) \otimes A_2^e(-1) \xrightarrow{\partial_2} A_1^e \otimes A_2^e(-1) \oplus A_1^e(-1) \otimes A_2^e \xrightarrow{\partial_1} A_1^e \otimes A_2^e \to 0$$

where the differentials are given by

$$\partial_2 = \begin{bmatrix} (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id} \\ \mathrm{id} \otimes (1 \otimes x_2 - x_2 \otimes 1)\cdot \end{bmatrix} : x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto \begin{bmatrix} x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d - x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d \\ x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} - x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d \end{bmatrix}$$

$$\partial_1 = \begin{bmatrix} \mathrm{id} \otimes (x_2 \otimes 1 - 1 \otimes x_2)\cdot \\ (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id} \end{bmatrix} :$$

$$\begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} \mapsto x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d - x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1}$$

$$x_1^{r+1} \otimes x_1^s \otimes x_2^u \otimes x_2^v - x_1^r \otimes x_1^{s+1} \otimes x_2^u \otimes x_2^v.$$

Of note is that the last module, $A_1^e \otimes A_2^e$, is isomorphic to A^e as A^e-modules via the map

$$x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto q^{bc} x_1^a x_2^c \otimes x_1^b x_2^d.$$

2 Calculating Cohomology

Apply the functor $Hom_{A^e}(-, A)$ to this resolution to get the complex

$$0 \to Hom_{A^e}(A^e, A) \xrightarrow{d_1} Hom_{A^e}(A_1^e \otimes A_2^e(-1) \oplus A_1^e(-1) \otimes A_2^e, A) \xrightarrow{d_2} Hom_{A^e}(A_1^e(-1) \otimes A_2^e(-1), A) \to 0.$$

The differentials are given by

$$\begin{aligned} d_1(f)\left(\begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix}\right) =& f(x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d) - f(x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1}) \\ & f(x_1^{r+1} \otimes x_1^s \otimes x_2^u \otimes x_2^v) - f(x_1^r \otimes x_1^{s+1} \otimes x_2^u \otimes x_2^v) \end{aligned}$$

$$\begin{aligned} d_2(f)(x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d) = & \; f(x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d, x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1}) \\ & -f(x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d, x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d) \end{aligned}$$

Next, there are isomorphisms as A^e-modules:

$$\begin{aligned} Hom_{A^e}(A^e, A) &\cong A \\ f &\mapsto f(1 \otimes 1) \end{aligned}$$