Hopf Module Algebras

Brandon Mather

UNT Master's

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation:
$$\mathbb{C}[v_1,\ldots,v_n] = \mathbb{C}\langle v_1,\ldots,v_n \mid v_j v_i - v_i v_j \rangle$$

Quantum Polynomial Ring

Let $Q=(q_{ij})$ be an $n\times n$ matrix of roots of unity where $q_{ii}=1=q_{ji}q_{ij}$. $\mathbb{C}_Q[v_1,\ldots,v_n]=\mathbb{C}\left\langle v_1,\ldots,v_n\mid v_jv_i-q_{ij}v_iv_j\right\rangle$ is called a **quantum polynomial ring**.

Example:
$$\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 | v_1 v_2 + v_2 v_1 \rangle$$

Motivation

• When a grp G acts on a space V by automorphisms, the action can be extended to $V\otimes V$ by $g\in G$ acting as $g\otimes g=\triangle(g)$. The action naturally induces an action from the grp alg, $\mathbb{C}G$, on V. Then \triangle defines a map $\mathbb{C}G\to\mathbb{C}G\otimes\mathbb{C}G$ called the coproduct. For arbitrary coproducts, $\triangle:A\to A\otimes A$, we call $g\in A$ grouplike if $\triangle(g)=g\otimes g$.

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- ② When a Lie alg $\mathfrak g$ acts on a space V, the action can be extended to $V\otimes V$ by $x\in \mathfrak g$ acting as $x\otimes 1+1\otimes x=\triangle(x)$. Again, \triangle defines a map $\mathfrak g\to \mathfrak g\otimes \mathfrak g$. For arbitrary coproducts, $\triangle:A\to A\otimes A$, we call $x\in A$ primitive if $\triangle(x)=x\otimes 1+1\otimes x$.

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This notion is dual to an algebra, creating a colagebra structure, C, with coproduct $\triangle: C \to C \otimes C$ and counit $\varepsilon: C \to \mathbb{C}$.

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Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(x) \neq \triangle(x)$, this is called non-cocommutativity.

Actions of Sweedler's Algebra

$${\it H}_{4}$$
 acts on $\mathbb{C}_{-1}[{\it v}_{1},{\it v}_{2}]$ by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

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We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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Actions of Kac-Paljutkin Algebra

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$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

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Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \to 1$.

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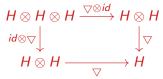
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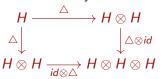
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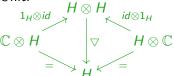
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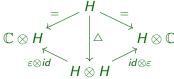
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Hopf Algebra Diagrams

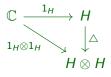
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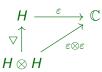
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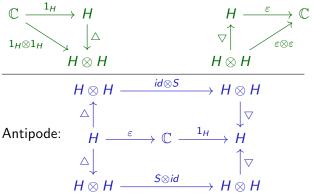


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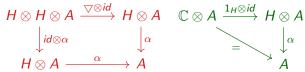
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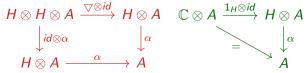


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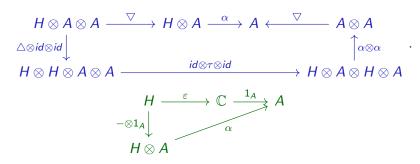
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In words, H acts on A, iff $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-module alg iff H acts on A and $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_i(a) \cdot h_j(a), \quad h(1_A) = \varepsilon(h)1_A$$

where $\triangle(h) = \sum h_i \otimes h_j$.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group, $G \rtimes G'$. The action can be extended to the group algebras, $\mathbb{C}G$ and $\mathbb{C}G'$. This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G\#\mathbb{C}G'$$

with product $g'g = (g' \cdot g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is the smash product algebra defined as $A\otimes H$ as a vector space and with product

$$ha = \sum_{i} (g_i \cdot a) k_i$$

where $a \in A$, $h \in H$ and $\triangle(h) = \sum_i g_i \otimes k_i$.

Smash Product Algebra

"Group-like" and "Lie-like"

Let H be a Hopf algebra, define $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$ and $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$.

G(H) is the set of grouplike elements of H and forms a group under the product.

P(H) is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over \mathbb{C} , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over $\mathbb C$ is isomorphic to a group algebra.

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- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
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- If B is a Koszul algebra, are there nontrivial PBW deformations of $B\#\mathcal{U}_q(\mathfrak{sl}_2)$?