

# Koszul Resolution of a Polynomial Ring

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We consider the polynomial ring  $A = \mathbb{k}[x_1, \dots, x_n]$  and construct its Koszul resolution. This allows the computation of its Hochschild cohomology groups.

## 1 Koszul Resolution

Let  $\mathbb{k}$  be a field,  $A = \mathbb{k}[x_1, \dots, x_n]$ , and  $V = \mathbb{k}x_1 \oplus \dots \oplus \mathbb{k}x_n$ . Fix a basis of  $A^e = \mathbb{k}[x_1, \dots, x_n] \otimes \mathbb{k}[x_1, \dots, x_n]$  as a  $\mathbb{k}$ -vector space

$$\{x_1^{i_1} \dots x_n^{i_n} \otimes x_1^{j_1} \dots x_n^{j_n} = \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \mid \vec{i}, \vec{j} \in \mathbb{N}^n\}.$$

Also fix a basis of  $(A^e)^n$  as an  $A^e$ -module

$$\left\{ e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \mid 0 \leq i \leq n, \text{ 1 in } i\text{th row} \right\}.$$

Finally, fix a basis of  $V$  as  $\mathbb{k}$ -vector space

$$\{x_1, \dots, x_n\}.$$

The sequence  $(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n)$  is clearly regular in  $A^e$ , so we get a free resolution from the Koszul complex

$$0 \rightarrow A^e \rightarrow \bigwedge^{n-1} (A^e)^n \rightarrow \dots \rightarrow \bigwedge^2 (A^e)^n \rightarrow (A^e)^n \rightarrow A^e \rightarrow 0$$

of  $A^e/(x_1 \otimes 1 - 1 \otimes x_1, \dots, x_n \otimes 1 - 1 \otimes x_n) \cong A$ . The differentials are given by

$$\begin{aligned} \partial_k : \bigwedge^k (A^e)^n &\rightarrow \bigwedge^{k-1} (A^e)^n \\ \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} e_{m_0} \wedge \dots \wedge e_{m_{k-1}} &\mapsto \sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) e_{m_0} \wedge \dots \wedge \widehat{e_{m_t}} \wedge \dots \wedge e_{m_{k-1}} \end{aligned}$$

where  $\widehat{e_{m_t}}$  means ommitting that term. Note:  $\bigwedge^k (A^e)^n$  is being taken over  $A^e$ , and so we are identifying  $\bigwedge^n (A^e)^n \cong A^e$  and  $\bigwedge^1 (A^e)^n \cong (A^e)^n$ .

## 2 Isomorphic Resolution

There is an isomorphism of  $A^e$ -modules

$$\bigwedge^k (A^e)^n \cong A^e \otimes \bigwedge^k V$$

$$\vec{x}^i \otimes \vec{x}^j e_{m_0} \wedge \cdots \wedge e_{m_{k-1}} \mapsto \vec{x}^i \otimes \vec{x}^j \otimes x_{m_1} \wedge \cdots \wedge x_{m_{k-1}}.$$

Note that  $\bigwedge^k V$  is being taken over  $\mathbb{k}$ . This isomorphism induces a chain map isomorphism from the Koszul resolution to the resolution

$$0 \rightarrow A^e \rightarrow A^e \otimes \bigwedge^{n-1} V \rightarrow \cdots \rightarrow A^e \otimes \bigwedge^2 V \rightarrow A^e \otimes V \rightarrow A^e \rightarrow 0$$

where we identify  $A^e \otimes \bigwedge^n V \cong A^e$ ,  $A^e \otimes \bigwedge^1 V \cong A^e \otimes V$ . The induced differentials of this resolution are given by

$$\partial'_k : A^e \otimes \bigwedge^k V \rightarrow A^e \otimes \bigwedge^{k-1} V$$

$$\vec{x}^i \otimes \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto \sum_{t=0}^{k-1} (-1)^t \vec{x}^i \otimes \vec{x}^j (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}}.$$

Henceforth, this resolution will be identified as the Koszul resolution of  $A$ .

## 3 Calculating Cohomology

Recall that as  $A$  is a  $\mathbb{k}$ -vector space, and so a free  $\mathbb{k}$ -module,  $HH^k(A) = Ext_{A^e}^k(A, A)$ . This can be computed as the  $k$ -th homology group of the complex given by from applying the functor  $Hom_{A^e}(-, A)$  to the Koszul resolution. This complex is

$$0 \rightarrow Hom_{A^e}(A^e, A) \rightarrow Hom_{A^e}(A^e \otimes V, A) \rightarrow \cdots \rightarrow Hom_{A^e}(A^e \otimes \bigwedge^{n-1} V, A) \rightarrow Hom_{A^e}(A^e, A) \rightarrow 0$$

with differentials

$$d_k : Hom_{A^e}(A^e \otimes \bigwedge^k V, A) \rightarrow Hom_{A^e}(A^e \otimes \bigwedge^{k-1} V, A)$$

$$d_k(g)(\vec{x}^i \otimes \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge x_{m_k}) = \sum_{t=0}^k (-1)^t \vec{x}^i \otimes \vec{x}^j (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) g(1 \otimes 1 \otimes x_{m_1} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_k}).$$

Note that we are using the fact that  $g$  is  $A^e$ -linear to factor out  $\vec{x}^i \otimes \vec{x}^j (x_{m_t} \otimes 1 - 1 \otimes x_{m_t})$ . As well,  $g(1 \otimes 1 \otimes x_{m_1} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_k}) \in A = \mathbb{k}[x_1, \dots, x_n]$ , so we can write it as  $p(x_1, \dots, x_n)$ . But

$$(x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \cdot p(x_1, \dots, x_n) = x_{m_t} p(x_1, \dots, x_n) - p(x_1, \dots, x_n) x_{m_t} = 0$$

for all  $t$  and any  $p(x_1, \dots, x_n) \in A$ . Hence,  $d_k(g) = 0$  for all  $g \in Hom_{A^e}(A^e \otimes \bigwedge^k V, A)$  and for all  $k$ , so every differential is the 0 map. Then, the homology groups are  $Ext_{A^e}^k(A, A) = Hom_{A^e}(A^e \otimes \bigwedge^k V, A)$  for all  $n \leq k \leq 0$ .

There is an isomorphism of  $A^e$ -modules

$$A \otimes \bigwedge^k V \cong \text{Hom}_{A^e}(A^e \otimes \bigwedge^k V, A)$$

$$\vec{x}^i \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto (1 \otimes 1 \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto \vec{x}^i).$$

Therefore,  $HH^k(\mathbb{k}[x_1, \dots, x_n]) = \mathbb{k}[x_1, \dots, x_n] \otimes \bigwedge^k (\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n)$ .

## 4 Calculating Homology

As with the cohomology, since  $A$  is a free  $\mathbb{k}$ -module,  $HH_k(A) = \text{Tor}_k^{A^e}(A, A)$ . This can be calculated as the  $k$ -th homology group of the complex given by applying the functor  $- \otimes_{A^e} A$  to the Koszul resolution. This complex is

$$0 \rightarrow A^e \otimes_{A^e} A \rightarrow A^e \otimes \bigwedge^{n-1} V \otimes_{A^e} A \rightarrow \cdots \rightarrow A^e \otimes V \otimes_{A^e} A \rightarrow A^e \otimes_{A^e} A \rightarrow 0$$

$$d'_k : A^e \otimes \bigwedge^k V \otimes_{A^e} A \rightarrow A^e \otimes \bigwedge^{k-1} V \otimes_{A^e} A$$

$$d'_k(\vec{x}^i \otimes \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^l) =$$

$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^i \otimes \vec{x}^j (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^l.$$

But as the functor  $- \otimes_{A^e} A$  is tensoring over  $A^e$ , we can rewrite this as

$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^i \otimes \vec{x}^j (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^l =$$

$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^i \otimes \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \vec{x}^l =$$

$$0.$$

So again, all of the differentials are 0, and hence  $HH_k(A) = A^e \otimes \bigwedge^k V \otimes_{A^e} A$ .

As with the cohomology, there is an isomorphism of  $A^e$ -modules

$$A^e \otimes \bigwedge^k V \otimes_{A^e} A \cong A \otimes \bigwedge^k V$$

$$\vec{x}^i \otimes \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^l \mapsto \vec{x}^i \vec{x}^l \vec{x}^j \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}}.$$

Therefore,  $HH_k(\mathbb{k}[x_1, \dots, x_n]) = \mathbb{k}[x_1, \dots, x_n] \otimes \bigwedge^k (\mathbb{k}x_1 \oplus \cdots \oplus \mathbb{k}x_n)$ .