Quantum Symmetry of Hopf Actions

Brandon Mather

Algebra Seminar, November 2023

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

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Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[u,v]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ by

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And on $\mathbb{C}_{-1}[u,v]$ by

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Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

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$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \qquad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

Hopf Algebra

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so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"'] & H \end{tikzcd}$$

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Unit:

$$\begin{tikzcd} & H \otimes H \arrow[d, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

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Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & H \otimes H \arrow[d, "\triangle\otimes id"] \\ H \otimes H \arrow[r, "id\otimes\triangle"'] & H \otimes H \otimes H \end{tikzcd}$$

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 & H &
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
 & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\triangle"] & \\
\mathbb{C} \otimes H & & H \otimes \mathbb{C} \\
 & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] &
\end{tikzcd}$$

Product and Coproduct compatibility:

$$\begin{tikzcd}
H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes \triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"']
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Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\Delta\otimes\Delta"'] & H \arrow[r, "\Delta"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & H \arrow[d, "\Delta"] \\ & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] & \end{tikzcd}$$

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Antipode:

$$\begin{tikzcd} H \otimes H \arrow[rr, "id\otimes S"] & & H \otimes H \arrow[d, "\nabla"] \\ H \arrow[u, "\Delta"] \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & \mathbb{C} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes id"] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

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Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

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$$\begin{tikzcd} H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\ H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"'] \end{tikzcd} .$$

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