Koszul Resolution of a Polynomial Ring

1 Koszul Resolution

Let $\Bbbk$ be a field, $A = \Bbbk_q[x_1, x_2] = \Bbbk < x_1, x_2 > /(x_2x_1 - qx_1x_2)$ for some $q \in \Bbbk^*$. Let $A_1 = \Bbbk[x_1]$ and $A_2 = \Bbbk[x_2]$ be subalgebras of A so that A is the twisted tensor product $A_1 \otimes^\tau A_2$ where $\tau : \mathbb{Z}^2 \to \Bbbk^*$ is the bicharacter $\tau(m, n) = q^{-mn}$. We consider the Koszul resolutions of A_1 and A_2

$$0 \to A_1^e \xrightarrow{(x_1 \otimes 1 - 1 \otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e \xrightarrow{(x_2 \otimes 1 - 1 \otimes x_2)\cdot} A_2^e \to 0.$$

In order to construct a resolution of A,, the differentials of these resolutions need to be graded maps. To this end, we shift the grading of the homological degree 1 component of both resolutions up by 1:

$$0 \to A_1^e(-1) \xrightarrow{(x_1 \otimes 1 - 1 \otimes x_1)\cdot} A_1^e \to 0$$

$$0 \to A_2^e(-1) \xrightarrow{(x_2 \otimes 1 - 1 \otimes x_2)\cdot} A_2^e \to 0.$$

Then, the differential $(x_1 \otimes 1 - 1 \otimes x_1)\cdot$ maps basis elements as follows

$$(x_1 \otimes 1 - 1 \otimes x_1)(x_1^n \otimes x_1^m) = x_1^{n+1} \otimes x_1^m - x_1^n \otimes x_1^{m+1}.$$

The element on the right has degree $n + m + 1$ in A_1^e and the element $x_1^n \otimes x_1^m$ has shifted degree $n + m + 1$ in $A_1^e(-1)$, so we see the differential is now graded. Similarly, the differential $(x_2 \otimes 1 - 1 \otimes x_2)\cdot : A_2^e(-1) \to A_2^e$ is graded.

Then by a theorem proved by Bergh and Oppermann in "Cohomology of Twisted Tensor Products" (2008), the total complex of the tensor product of these two resolutions is a projective resolution of A as an A^e-module. This resolution is

$$0 \to A_1^e(-1) \otimes A_2^e(-1) \xrightarrow{\partial_2} [A_1^e \otimes A_2^e(-1)] \oplus [A_1^e(-1) \otimes A_2^e] \xrightarrow{\partial_1} A_1^e \otimes A_2^e \to 0$$

where the differentials are given by

$$\partial_2 = \begin{bmatrix} (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id} \\ \mathrm{id} \otimes (1 \otimes x_2 - x_2 \otimes 1)\cdot \end{bmatrix} : x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto \begin{bmatrix} x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d - x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d \\ x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} - x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d \end{bmatrix}$$

$$\partial_1 = \left[\mathrm{id} \otimes (x_2 \otimes 1 - 1 \otimes x_2)\cdot \quad (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \mathrm{id}\right] :$$

$$\begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} \mapsto x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d - x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} + x_1^{r+1} \otimes x_1^s \otimes x_2^u \otimes x_2^v - x_1^r \otimes x_1^{s+1} \otimes x_2^u \otimes x_2^v.$$

Of note is that the last module, $A_1^e \otimes A_2^e$, is isomorphic to A^e as A^e-modules via the map

$$x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto q^{bc} x_1^a x_2^c \otimes x_1^b x_2^d.$$

We will show that this map is A^e-linear. First, if we act and then map, we get

$$\begin{aligned} & x_1^r x_2^s \cdot (x_1^a \otimes x_1^b \otimes x_2^c \otimes c_2^d) \cdot x_1^t x_2^u \\ &= q^{-s(a+b)-st-t(c+d)} (x_1^r \cdot (x_1^a \otimes x_1^b) \cdot x_1^t) \otimes (x_2^s \cdot (x_2^c \otimes x_2^d) \cdot x_2^u) \\ &= q^{-s(a+b)-st-t(c+d)} x_1^{a+r} \otimes x_1^{b+t} \otimes x_2^{c+s} \otimes x_2^{d+u} \\ &\mapsto q^{bc-as-dt} x_1^{a+r} x_2^{c+s} \otimes x_1^{b+t} x_2^{d+u}. \end{aligned}$$

Next if we instead map and then act we get

$$\begin{aligned} & x_1^r x_2^s \cdot (q^{bc} x_1^a x_2^c \otimes x_1^b x_2^d) \cdot x_1^t x_2^u \\ &= q^{bc} x_1^r x_2^s x_1^a x_2^c \otimes x_1^b x_2^d x_1^t x_2^u \\ &= q^{bc} \left(q^{-as} x_1^{a+r} x_2^{c+s}\right) \otimes \left(q^{-dt} x_1^{b+t} \otimes x_2^{d+u}\right) \\ &= q^{bc-as-dt} x_1^{a+r} x_2^{c+s} \otimes x_1^{b+t} x_2^{d+u} \end{aligned}$$

So we see that this map is an A^e-module map. This map is clearly surjective, for any basis element $x_1^a x_2^b \otimes x_1^c x_2^d \in A^e$, the element $q^{-bc} x_1^a \otimes x_1^c \otimes x_2^b \otimes x_2^d \in A_1^e \otimes A_2^e$ is in its pre-image. As well, this defines an inverse of the map, so we conclude that this is in fact an isomorphism of A^e-modules.

For the sake of easier computation of the cohomology, we will perform similar isomorphisms of the other two A^e-modules,

$$\begin{aligned} & A_1^e(-1) \otimes A_2^e(-1) \cong A^e \\ & x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto q^{(b+1)(c+1)} x_1^a x_2^c \otimes x_1^b x_2^d \end{aligned}$$

and

$$\begin{aligned} (A_1^e \otimes A_2^e(-1)) \oplus (A_1^e(-1) \otimes A_2^e) &\cong A^e \oplus A^e \\ \begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} &\mapsto \begin{bmatrix} q^{b(c+1)} x_1^a x_2^c \otimes x_1^b x_2^d \\ q^{(s+1)u} x_1^r x_2^u \otimes x_1^s x_2^v \end{bmatrix} \end{aligned}$$

Hence, we can rewrite the resolution as

$$0 \to A^e \xrightarrow{\partial_2^*} A^e \oplus A^e \xrightarrow{\partial_1^*} A^e \to 0$$

where the differentials are given by

$$\partial_2^* \left(x_1^a x_2^b \otimes x_1^c x_2^d\right) = \begin{bmatrix} q^{-(b+1)} x_1^{a+1} x_2^b \otimes x_1^c x_2^d - x_1^a x_2^b \otimes x_1^{c+1} x_2^d \\ q^{-(c+1)} x_1^a x_2^b \otimes x_1^c x_2^{d+1} - x_1^a x_2^{b+1} \otimes x_1^c x_2^d \end{bmatrix}$$

$$\partial_1^* \left(\begin{bmatrix} x_1^a x_2^b \otimes x_1^c x_2^d \\ x_1^r x_2^s \otimes x_1^u x_2^v \end{bmatrix}\right) = x_1^a x_2^{b+1} \otimes x_1^c x_2^d - q^{-c} x_1^a x_2^b \otimes x_1^c x_2^{d+1} + q^{-s} x_1^{r+1} x_2^s \otimes x_1^u x_2^v - x_1^r x_2^s \otimes x_1^{u+1} x_2^v.$$

2 Computing Cohomology

Apply the functor $\mathrm{Hom}_{A^e}(-, A)$ to this resolution to get the complex

$$0 \to \mathrm{Hom}_{A^e}(A^e, A) \xrightarrow{d_1} \mathrm{Hom}_{A^e}(A^e \oplus A^e, A) \xrightarrow{d_2} \mathrm{Hom}_{A^e}(A^e, A) \to 0.$$

The differentials are given by $d_i(f)(a) = f(\partial_i^*(a))$.

Next, we want to rewrite this complex in a more familiar form. Let $V = \Bbbk x_1 \oplus \Bbbk x_2$, then

$$\begin{aligned} \mathrm{Hom}_{A^e}(A^e, A) &\cong A \otimes \textstyle\bigwedge_q^2(V) \\ f &\mapsto f(1 \otimes 1) \otimes x_1 \wedge x_2 \end{aligned}$$

This map is clearly A^e-linear and surjective. Since every $f \in \mathrm{Hom}_{A^e}(A^e, A)$ is determined by its image on $1 \otimes 1$, it is also injective, and hence, an isomorphism of A^e-modules.

We also have that

$$\begin{aligned} \mathrm{Hom}_{A^e}(A^e \oplus A^e, A) &\cong A \otimes \textstyle\bigwedge_q^1(V) \\ f &\mapsto f(1 \otimes 1, 0) \otimes x_1 + f(0, 1 \otimes 1) \otimes x_2 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}_{A^e}(A^e, A) &\cong A \otimes \textstyle\bigwedge_q^0(V) \\ f &\mapsto f(1 \otimes 1). \end{aligned}$$

Ultimately, we have the complex

$$0 \to A \otimes \textstyle\bigwedge_q^2(V) \xrightarrow{d_1^*} A \otimes \textstyle\bigwedge_q^1(V) \xrightarrow{d_2^*} A \otimes \textstyle\bigwedge_q^0(V) \to 0$$

with differentials given by

$$\begin{aligned} d_1^*(x_1^a x_2^b \otimes x_1 \wedge x_2) &= (q^{-a} - 1) x_1^a x_2^{b+1} + (1 - q^{-b}) x_1^{a+1} x_2^b \\ d_2^*(x_1^a x_2^b \otimes x_1 + x_1^c x_2^d \otimes x_2) &= (1 - q^{-b}) x_1^{a+1} x_2^b + (1 - q^{-c}) x_1^c x_2^{d+1} \end{aligned}$$