

# Koszul Resolution of a Polynomial Ring

## 1 Koszul Resolution

Let  $\mathbb{k}$  be a field,  $A = \mathbb{k}_q[x_1, x_2] = \mathbb{k} \langle x_1, x_2 \rangle / (x_2x_1 - qx_1x_2)$  for some  $q \in \mathbb{k}^*$ . Let  $A_1 = \mathbb{k}[x_1]$  and  $A_2 = \mathbb{k}[x_2]$  be subalgebras of  $A$  so that  $A$  is the twisted tensor product  $A_1 \otimes^\tau A_2$  where  $\tau : \mathbb{Z}^2 \rightarrow \mathbb{k}^*$  is the bicharacter  $\tau(m, n) = q^{-mn}$ . We consider the Koszul resolutions of  $A_1$  and  $A_2$

$$\begin{aligned} 0 \rightarrow A_1^e &\xrightarrow{(x_1 \otimes 1 - 1 \otimes x_1) \cdot} A_1^e \rightarrow 0 \\ 0 \rightarrow A_2^e &\xrightarrow{(x_2 \otimes 1 - 1 \otimes x_2) \cdot} A_2^e \rightarrow 0. \end{aligned}$$

In order to construct a resolution of  $A$ , the differentials of these resolutions need to be graded maps. To this end, we shift the grading of the homological degree 1 component of both resolutions up by 1:

$$\begin{aligned} 0 \rightarrow A_1^e(-1) &\xrightarrow{(x_1 \otimes 1 - 1 \otimes x_1) \cdot} A_1^e \rightarrow 0 \\ 0 \rightarrow A_2^e(-1) &\xrightarrow{(x_2 \otimes 1 - 1 \otimes x_2) \cdot} A_2^e \rightarrow 0. \end{aligned}$$

Then, the differential  $(x_1 \otimes 1 - 1 \otimes x_1) \cdot$  maps basis elements as follows

$$(x_1 \otimes 1 - 1 \otimes x_1)(x_1^n \otimes x_1^m) = x_1^{n+1} \otimes x_1^m - x_1^n \otimes x_1^{m+1}.$$

The element on the right has degree  $n + m + 1$  in  $A_1^e$  and the element  $x_1^n \otimes x_1^m$  has shifted degree  $n + m + 1$  in  $A_1^e(-1)$ , so we see the differential is now graded. Similarly, the differential  $(x_2 \otimes 1 - 1 \otimes x_2) \cdot : A_2^e(-1) \rightarrow A_2^e$  is graded.

Then by a theorem proved by Bergh and Oppermann in "Cohomology of Twisted Tensor Products" (2008), the total complex of the tensor product of these two resolutions is a projective resolution of  $A$  as an  $A^e$ -module. This resolution is

$$0 \rightarrow A_1^e(-1) \otimes A_2^e(-1) \xrightarrow{\partial_2} [A_1^e \otimes A_2^e(-1)] \oplus [A_1^e(-1) \otimes A_2^e] \xrightarrow{\partial_1} A_1^e \otimes A_2^e \rightarrow 0$$

where the differentials are given by

$$\partial_2 = \begin{bmatrix} (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \text{id} \\ \text{id} \otimes (1 \otimes x_2 - x_2 \otimes 1) \cdot \end{bmatrix} : x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto \begin{bmatrix} x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d - x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d \\ x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} - x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d \end{bmatrix}$$

$$\partial_1 = [\text{id} \otimes (x_2 \otimes 1 - 1 \otimes x_2) \cdot (x_1 \otimes 1 - 1 \otimes x_1) \cdot \otimes \text{id}] :$$

$$\begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} \mapsto x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d - x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1} + x_1^{r+1} \otimes x_1^s \otimes x_2^u \otimes x_2^v - x_1^r \otimes x_1^{s+1} \otimes x_2^u \otimes x_2^v.$$

Of note is that the last module,  $A_1^e \otimes A_2^e$ , is isomorphic to  $A^e$  as  $A^e$ -modules via the map

$$x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \mapsto q^{bc} x_1^a x_2^c \otimes x_1^b x_2^d.$$

We will show that this map is  $A^e$ -linear. First, if we act and then map, we get

$$\begin{aligned} & x_1^r x_2^s \cdot (x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d) \cdot x_1^t x_2^u \\ &= q^{-s(a+b)-st-t(c+d)} (x_1^r \cdot (x_1^a \otimes x_1^b) \cdot x_1^t) \otimes (x_2^s \cdot (x_2^c \otimes x_2^d) \cdot x_2^u) \\ &= q^{-s(a+b)-st-t(c+d)} x_1^{a+r} \otimes x_1^{b+t} \otimes x_2^{c+s} \otimes x_2^{d+u} \\ &\mapsto q^{bc-as-dt} x_1^{a+r} x_2^{c+s} \otimes x_1^{b+t} x_2^{d+u}. \end{aligned}$$

Next if we instead map and then act we get

$$\begin{aligned} & x_1^r x_2^s \cdot (q^{bc} x_1^a x_2^c \otimes x_1^b x_2^d) \cdot x_1^t x_2^u \\ &= q^{bc} x_1^r x_2^s x_1^a x_2^c \otimes x_1^b x_2^d x_1^t x_2^u \\ &= q^{bc} (q^{-as} x_1^{a+r} x_2^{c+s}) \otimes (q^{-dt} x_1^{b+t} x_2^{d+u}) \\ &= q^{bc-as-dt} x_1^{a+r} x_2^{c+s} \otimes x_1^{b+t} x_2^{d+u} \end{aligned}$$

So we see that this map is an  $A^e$ -module map. This map is clearly surjective, for any basis element  $x_1^a x_2^b \otimes x_1^c x_2^d \in A^e$ , the element  $q^{-bc} x_1^a \otimes x_1^c \otimes x_2^b \otimes x_2^d \in A_1^e \otimes A_2^e$  is in its pre-image. As well, this defines an inverse of the map, so we conclude that this is in fact an isomorphism of  $A^e$ -modules.

## 2 Computing Cohomology

Apply the functor  $\text{Hom}_{A^e}(-, A)$  to this resolution to get the complex

$$0 \rightarrow \text{Hom}_{A^e}(A^e, A) \xrightarrow{d_1} \text{Hom}_{A^e}(A_1^e \otimes A_2^e(-1) \oplus A_1^e(-1) \otimes A_2^e, A) \xrightarrow{d_2} \text{Hom}_{A^e}(A_1^e(-1) \otimes A_2^e(-1), A) \rightarrow 0.$$

The differentials are given by

$$d_1(f) \left( \begin{bmatrix} x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d \\ x_1^r \otimes x_1^s \otimes x_2^u \otimes x_2^v \end{bmatrix} \right) = q^{b(c+1)} f(x_1^a x_2^{c+1} \otimes x_1^b x_2^d) - q^{bc} f(x_1^a x_2^c \otimes x_1^b x_2^{d+1}) \\ + q^{su} f(x_1^{r+1} x_2^u \otimes x_1^s x_2^v) - q^{s(u+1)} f(x_1^r x_2^u \otimes x_1^{s+1} x_2^v)$$

$$d_2(f)(x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^d) = f(x_1^{a+1} \otimes x_1^b \otimes x_2^c \otimes x_2^d, x_1^a \otimes x_1^b \otimes x_2^c \otimes x_2^{d+1}) \\ - f(x_1^a \otimes x_1^{b+1} \otimes x_2^c \otimes x_2^d, x_1^a \otimes x_1^b \otimes x_2^{c+1} \otimes x_2^d)$$

Next, we want to rewrite this resolution in a more familiar form. Let  $V = \mathbb{k}x_1 \oplus \mathbb{k}x_2$ , then

$$\text{Hom}_{A^e}(A^e, A) \cong A \otimes \bigwedge_q^2(V) \\ f \mapsto f(1 \otimes 1) \otimes x_1 \wedge x_2$$

This map is clearly  $A^e$ -linear and surjective. Since every  $f \in \text{Hom}_{A^e}(A^e, A)$  is determined by its image on  $1 \otimes 1$ , it is also injective, and hence, an isomorphism of  $A^e$ -modules.

We also have that

$$\text{Hom}_{A^e}((A_1^e \otimes A_2^e(-1)) \oplus (A_1^e(-1) \otimes A_2^e)) \cong A \otimes \bigwedge_q^1(V)$$