Koszul Resolution of a Polynomial Ring

1 Koszul Resolution

Let $\Bbbk$ be a field, $A = \Bbbk[x_1, \ldots, x_n]$, and $V = \Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n$. Fix a basis of $A^e = \Bbbk[x_1, \ldots, x_n] \otimes \Bbbk[x_1, \ldots, x_n]$ as a $\Bbbk$-vector space

$$\{x_1^{i_1} \ldots x_n^{i_n} \otimes x_1^{j_1} \ldots x_n^{j_n} = \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \mid \vec{i}, \vec{j} \in \mathbb{N}^n\}.$$

Also fix a basis of $(A^e)^n$ as an A^e-module

$$\left\{ e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \;\middle|\; 0 \leq i \leq n, \text{ 1 in ith row} \right\}.$$

Finally, fix a basis of V as $\Bbbk$-vector space

$$\{x_1, \ldots, x_n\}.$$

The sequence $(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n)$ is clearly regular in A^e, so we get a free resolution from the Koszul complex

$$0 \to A^e \to \bigwedge^{n-1} (A^e)^n \to \cdots \to \bigwedge^{2} (A^e)^n \to (A^e)^n \to A^e \to 0$$

of $A^e/(x_1 \otimes 1 - 1 \otimes x_1, \ldots, x_n \otimes 1 - 1 \otimes x_n) \cong A$. The differentials are given by

$$\begin{aligned} \partial_k : \bigwedge^{k} (A^e)^k &\to \bigwedge^{k-1} (A^e)^n \\ \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} e_{m_0} \wedge \cdots \wedge e_{m_{k-1}} &\mapsto \sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) e_{m_0} \wedge \cdots \wedge \widehat{e_{m_t}} \wedge \cdots \wedge e_{m_{k-1}} \end{aligned}$$

where $\widehat{e_{m_t}}$ means ommitting that term. Note: $\bigwedge^k (A^e)^n$ is being taken over A^e, and so we are identifying $\bigwedge^n (A^e)^n \cong A^e$ and $\wedge^1 (A^e)^n \cong (A^e)^n$.

2 Isomorphic Resolution

There is an isomorphism of A^e-modules

$$\bigwedge^k (A^e)^n \cong A^e \otimes \bigwedge^k V$$
$$\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} e_{m_0} \wedge \cdots \wedge e_{m_{k-1}} \mapsto \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_1} \wedge \cdots \wedge x_{m_{k-1}}.$$

Note that $\bigwedge^k V$ is being taken over $\Bbbk$. This isomorphism induces a chain map isomorphism from the Koszul resolution to the resolution

$$0 \to A^e \to A^e \otimes \bigwedge^{n-1} V \to \cdots \to A^e \otimes \bigwedge^2 V \to A^e \otimes V \to A^e \to 0$$

where we identify $A^e \otimes \bigwedge^n V \cong A^e$, $A^e \otimes \bigwedge^1 V \cong A^e \otimes V$. The induced differentials of this resolution are given by

$$\partial'_k : A^e \otimes \bigwedge^k V \to A^e \otimes \bigwedge^{k-1} V$$
$$\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto \sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}}.$$

Henceforth, this resolution will be identified as the Koszul resolution of A.

3 Calculating Cohomology

Recall that as A is a $\Bbbk$-vector space, and so a free $\Bbbk$-module, $HH^k(A) = Ext^k_{A^e}(A, A)$. This can be computed as the k-th homology group of the complex given by from applying the functor $Hom_{A^e}(-, A)$ to the Koszul resolution. This complex is

$$0 \to Hom_{A^e}(A^e, A) \to Hom_{A^e}(A^e \otimes V, A) \to \cdots \to Hom_{A^e}(A^e \otimes \bigwedge^{n-1} V, A) \to Hom_{A^e}(A^e, A) \to 0$$

with differentials

$$d_k : Hom_{A^e}(A^e \otimes \bigwedge^k V, A) \to Hom_{A^e}(A^e \bigwedge^{k-1} V, A)$$

$$d_k(g)(\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_k}) = \sum_{t=0}^{k} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) g(1 \otimes 1 \otimes x_{m_1} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_k}).$$

Note that we are using the fact that g is A^e-linear to factor out $\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t})$. As well, $g(1 \otimes 1 \otimes x_{m_1} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_k}) \in A = \Bbbk[x_1, \ldots, x_n]$, so we can write it as $p(x_1, \ldots, x_n)$. But

$$(x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \cdot p(x_1, \ldots, x_n) = x_{m_t} p(x_1, \ldots, x_n) - p(x_1, \ldots, x_n) x_{m_t} = 0$$

for all t and any $p(x_1, \ldots, x_n) \in A$. Hence, $d_k(g) = 0$ for all $g \in Hom_{A^e}(A^e \otimes \bigwedge^k V, A)$ and for all k, so every differential is the 0 map. Then, the homology groups are $Ext^k_{A^e}(A, A) = Hom_{A^e}(A^e \otimes \bigwedge^k V, A)$ for all $n \leq k \leq 0$.

There is an isomorphism of A^e-modules

$$A \otimes \bigwedge^k V \cong Hom_{A^e}(A^e \otimes \bigwedge^k V, A)$$
$$\vec{x}^{\vec{i}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto (1 \otimes 1 \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \mapsto \vec{x}^{\vec{i}}).$$

Therefore, $HH^k(\Bbbk[x_1, \ldots, x_n]) = \Bbbk[x_1, \ldots, x_n] \otimes \bigwedge^k (\Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n)$.

4 Calculating Homology

As with the cohomology, since A is a free $\Bbbk$-module, $HH_k(A) = Tor_k^{A^e}(A, A)$. This can be calculated as the k-th homology group of the complex given by applying the functor $- \otimes_{A^e} A$ to the Koszul resolution. This complex is

$$0 \to A^e \otimes_{A^e} A \to A^e \otimes \bigwedge^{n-1} V \otimes_{A^e} A \to \cdots \to A^e \otimes V \otimes_{A^e} A \to A^e \otimes_{A^e} A \to 0$$

$$d'_k : A^e \otimes \bigwedge^k V \otimes_{A^e} A \to A^e \otimes \bigwedge^{k-1} V \otimes_{A^e} A$$

$$d'_k(\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^{\vec{l}}) =$$
$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^{\vec{l}}.$$

But as the functor $- \otimes_{A^e} A$ is tensoring over A^e, we can rewrite this as

$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^{\vec{l}} =$$
$$\sum_{t=0}^{k-1} (-1)^t \vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge \widehat{x_{m_t}} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} (x_{m_t} \otimes 1 - 1 \otimes x_{m_t}) \vec{x}^{\vec{l}} =$$
$$0.$$

So again, all of the differentials are 0, and hence $HH_k(A) = A^e \otimes \bigwedge^k V \otimes_{A^e} A$.

As with the cohomology, there is an isomorphism of A^e-modules

$$A^e \otimes \bigwedge^k V \otimes_{A^e} A \cong A \otimes \bigwedge^k V$$
$$\vec{x}^{\vec{i}} \otimes \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}} \otimes_{A^e} \vec{x}^{\vec{l}} \mapsto \vec{x}^{\vec{i}} \vec{x}^{\vec{l}} \vec{x}^{\vec{j}} \otimes x_{m_0} \wedge \cdots \wedge x_{m_{k-1}}.$$

Therefore, $HH_k(\Bbbk[x_1, \ldots, x_n]) = \Bbbk[x_1, \ldots, x_n] \otimes \bigwedge^k (\Bbbk x_1 \oplus \cdots \oplus \Bbbk x_n)$.