# Hopf Algebra Actions on Quantum Planes

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# History

- (1939) Heinz Hopf works on homology of a compact Lie group leading to Hopf algs.
- (1969) Moss Sweedler writes seminal book "Hopf Algebras".
- (1986) Vladimir Drinfeld gives an address at ICM popularizing quantum groups.
- (1992) Susan Montogomery writes seminal book "Hopf Algebras and Their Actions on Rings".
- (Modern Day) Researchers work towards classification problems and actions of Hopf algs.

#### Goal

Understand actions of Hopf algs on noncommutative algs.

# Quantum Polynomial Ring

#### Definition

Let  $Q = (q_{ij})$  be an  $n \times n$  matrix, each entry  $q_{ij}$  a root of unity and  $q_{ii} = q_{ij}q_{ji} = 1$ . A **quantum polynomial ring** is a ring

$$\mathbb{C}_Q[x_1,\ldots,x_n]=\mathbb{C}[x_1,\ldots,x_n]/\langle x_jx_i-q_{ij}x_ix_j\rangle.$$

For example,  $\mathbb{C}_{-1}[u,v] = \mathbb{C}[u,v]/\langle uv + vu \rangle$ 

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx \right\rangle$$
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## Actions of Sweedler's Algebra

 $H_4$  acts on the quantum plane  $\mathbb{C}_{-1}[u,v]$  by

$$g \cdot u = u$$
,  $g \cdot v = -v$ ,  $x \cdot u = 0$ ,  $x \cdot v = v$ 

giving the representation

$$g\mapsto egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix}, \qquad x\mapsto egin{bmatrix} 0 & 1 \ 0 & 0 \end{bmatrix}.$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$$

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## Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on  $\mathbb{C}_q[x,y]$  where  $q^2=-1$  by 
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{\mathcal{O}}[x_1, x_2, x_3, x_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{-1}[u,v]$  by

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Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$egin{aligned} U_q(\mathfrak{sl}_2) = \ & \left\langle {\it E}, {\it F}, {\it K}, {\it K}^{-1} \mid {\it EF-FE} = (q - q^{-1})^{-1} \left( {\it K} - {\it K}^{-1} 
ight), {\it KEK}^{-1} = q^2 {\it E}, \ & {\it KFK}^{-1} = q^{-2} {\it F}, {\it KK}^{-1} = {\it K}^{-1} {\it K} = 1 
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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$
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$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

# Actions of $U_q(\mathfrak{sl}_2)$

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 $K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \quad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$ 

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so that the following commute:

#### Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
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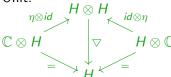
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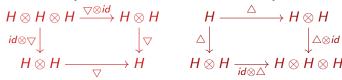
$$\nabla: H \otimes H \to H$$
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$$\triangle: H \to H \otimes H$$
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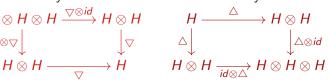
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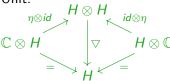
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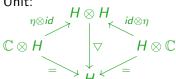
$$\varepsilon: H \to \mathbb{C}.$$

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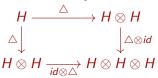
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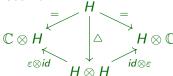
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### Hopf Algebra Diagrams

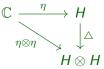
Product and Coproduct compatibility:

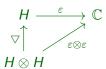
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Product and Coproduct compatibility:

$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} & H & \stackrel{\triangle}{\longrightarrow} & H \otimes H \\ & & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} & H \otimes H \otimes H \otimes H \end{array}$$

Unit and Counit compatibility:



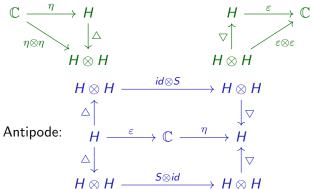


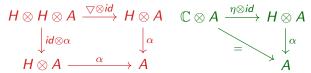
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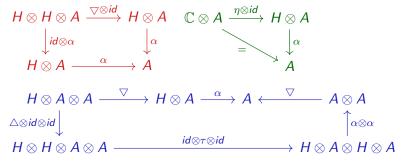
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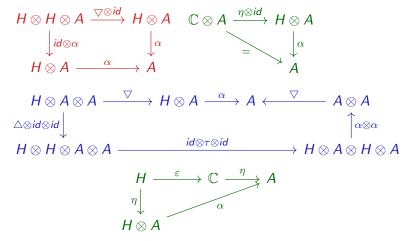
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Unit and Counit compatibility:









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- If H is semisimple and finite dimensional and A a semiprime alg, is A#H semiprime?
- If B is a Koszul alg, are there nontrivial PBW deformations of  $B\#U_q(\mathfrak{sl}_\mathfrak{n})$ ?

# Thank You!

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