Hopf Algebra Actions on Quantum Planes

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History

- (1939) Heinz Hopf works on homology of a compact Lie group leading to Hopf algs.
- (1969) Moss Sweedler writes seminal book "Hopf Algebras".
- (1986) Vladimir Drinfeld gives an address at ICM popularizing quantum groups.
- (1992) Susan Montogomery writes seminal book "Hopf Algebras and Their Actions on Rings".
- (Modern Day) Researchers work towards classification problems and actions of Hopf algs.

Goal

Understand actions of Hopf algs on noncommutative algs.

Quantum Polynomial Ring

Definition

Let $Q = (q_{ij})$ be an $n \times n$ matrix, each entry q_{ij} a root of unity and $q_{ii} = q_{ij}q_{ji} = 1$. A **quantum polynomial ring** is a ring

$$\mathbb{C}_Q[x_1,\ldots,x_n]=\mathbb{C}[x_1,\ldots,x_n]/\langle x_jx_i-q_{ij}x_ix_j\rangle.$$

For example, $\mathbb{C}_{-1}[u,v] = \mathbb{C}[u,v]/\langle uv + vu \rangle$

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, xg = -gx \right\rangle$$
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Actions of Sweedler's Algebra

 H_4 acts on the quantum plane $\mathbb{C}_{-1}[u,v]$ by

$$g \cdot u = u$$
, $g \cdot v = -v$, $x \cdot u = 0$, $x \cdot v = u$

giving the representation

$$g\mapsto egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix}, \qquad x\mapsto egin{bmatrix} 0 & 1 \ 0 & 0 \end{bmatrix}.$$

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$$

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Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on $\mathbb{C}_q[x,y]$ where $q^2=-1$ by
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{\mathcal{O}}[x_1, x_2, x_3, x_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
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Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$egin{aligned} U_q(\mathfrak{sl}_2) = \ & \left\langle {\it E}, {\it F}, {\it K}, {\it K}^{-1} \mid {\it EF-FE} = (q - q^{-1})^{-1} \left({\it K} - {\it K}^{-1}
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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$
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$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

Actions of $U_q(\mathfrak{sl}_2)$

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so that the following commute:

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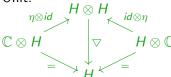
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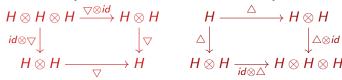
$$\nabla: H \otimes H \to H$$
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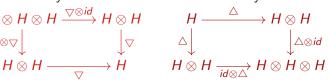
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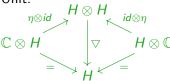
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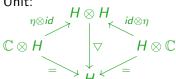
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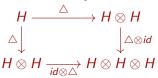
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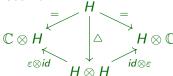
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Hopf Algebra Diagrams

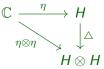
Product and Coproduct compatibility:

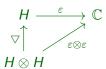
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$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} & H & \stackrel{\triangle}{\longrightarrow} & H \otimes H \\ & & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} & H \otimes H \otimes H \otimes H \end{array}$$

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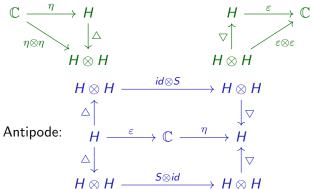


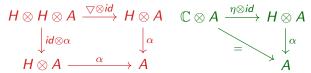
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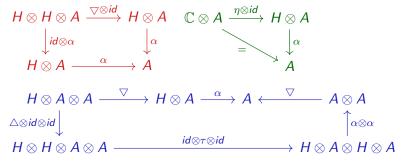
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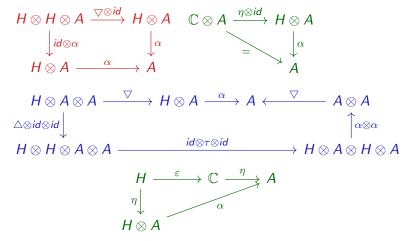
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Unit and Counit compatibility:









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- When are the invariant subrings from Hopf alg actions Artin-Schelter Gorenstein?
- If H is semisimple and finite dimensional and A a semiprime alg, is A#H semiprime?
- If B is a Koszul alg, are there nontrivial PBW deformations of $B\#U_q(\mathfrak{sl}_\mathfrak{n})$?

Thank You!

References I

- [1] G. I. Kats. "Finite Ring Groups". In: <u>Dokl. Akad. Nauk SSSR</u> 147 (), pp. 21–24.
- [2] Kenneth Chan et al. "Quantum binary polyhedral groups and their actions on quantum planes". en. In: arXiv:1303.7203 (July 2014). arXiv:1303.7203 [math]. URL: http://arxiv.org/abs/1303.7203.
- [3] Miriam Cohen and Davida Fishman. "Hopf algebra actions". en. In: <u>Journal of Algebra</u> 100.2 (May 1986), pp. 363–379. ISSN: 00218693. DOI: 10.1016/0021-8693(86)90082-7.
- [4] Juan Cuadra, Pavel Etingof, and Chelsea Walton. "Finite dimensional Hopf actions on Weyl algebras". en. In: arXiv:1509.01165 (July 2016). arXiv:1509.01165 [math]. URL: http://arxiv.org/abs/1509.01165.

References II

- [5] E. Kirkman, J. Kuzmanovich, and J.J. Zhang. "Gorenstein subrings of invariants under Hopf algebra actions". en. In:

 <u>Journal of Algebra</u> 322.10 (Nov. 2009), pp. 3640–3669. ISSN:

 00218693. DOI: 10.1016/j.jalgebra.2009.08.018.
- [6] Susan Montgomery. Hopf algebras and their actions on rings. Vol. 82. Providence, R.I.; 4: Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, 1993. ISBN: 0-8218-0738-2.
- [7] Chelsea Walton and Sarah Witherspoon.

 "Poincaré–Birkhoff–Witt deformations of smash product algebras from Hopf actions on Koszul algebras". en. In:

 Algebra & Number Theory 8.7 (Oct. 2014), pp. 1701–1731.

 ISSN: 1944-7833, 1937-0652. DOI:

 10.2140/ant.2014.8.1701.