Hopf Algebra Actions on Quantum Planes

Brandon Mather

Algebra Seminar November 2023

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x,\ S(y) = y,\ S(z) = z.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[x, y]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[x, y]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_Q[x_1, x_2, x_3, x_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[x,y]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_Q[x_1, x_2, x_3, x_4]$ for

$q_{12} = q_{34}^{-1}, \quad q_{13} = q_{24}^{-1}, \quad q_{14}^2 = 1, \quad q_{23}^2 = -1$ by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_{-1}[u,v]$ by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

16-dim Hopf Algebra

The 16-dimensionl semisimple Hopf algebras have been classified by Kashina. One such example is

$$H_{16} = \left\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ zx = xyz,\ zy = yz \right\rangle$$

with operations:

16-dim Hopf Algebra

The 16-dimensionl semisimple Hopf algebras have been classified by Kashina. One such example is

$$H_{16} = \left\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ zx = xyz,\ zy = yz \right\rangle$$

with operations:

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x^2 + y \otimes 1 - y \otimes x^2)(z \otimes z),$$

16-dim Hopf Algebra

The 16-dimensionl semisimple Hopf algebras have been classified by Kashina. One such example is

$$H_{16} = \left\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ zx = xyz,\ zy = yz \right\rangle$$

with operations:

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x^2 + y \otimes 1 - y \otimes x^2)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

16-dim Hopf Algebra

The 16-dimensionl semisimple Hopf algebras have been classified by Kashina. One such example is

$$H_{16} = \left\langle x, y, z \mid x^4 = y^2 = z^2 = 1,\ xy = yx,\ zx = xyz,\ zy = yz \right\rangle$$

with operations:

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x^2 + y \otimes 1 - y \otimes x^2)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x^3,\ S(y) = y,\ S(z) = \tfrac{1}{2}(1 + x^2 + y - x^2 y)z.$$

Action of H_{16}

H_{16} acts on the algebra

$$\mathbb{C}[t, u, v, w]/(tw + wt, uw + wu)$$

by the representation

$$x \mapsto \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \; y \mapsto \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$z \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$U_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1 \Big\rangle$$

with operations:

Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$U_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \Big\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$U_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

Quantized Universal Enveloping Algebra

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$U_q(\mathfrak{sl}_2) =$$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$
$$KFK^{-1} = q^{-2}F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1},\ S(K^{-1}) = K.$$

Actions of $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x, y]$ with $q^2 = -1$ by the representation

Actions of $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x, y]$ with $q^2 = -1$ by the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Actions of $U_q(\mathfrak{sl}_2)$

$U_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[x, y]$ with $q^2 = -1$ by the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad F \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \qquad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode $S : H \to H$ where the bialgebra operations are

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"'] & H
\end{tikzcd}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$,

$\eta : \mathbb{C} \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"'] & H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$, $\bigtriangleup : H \to H \otimes H$,

$\eta : \mathbb{C} \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"'] & H
\end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd}
H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"'] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\
H \otimes H \arrow[r, "id \otimes \bigtriangleup"'] & H \otimes H \otimes H
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
& H \otimes H \arrow[dd, "\bigtriangledown"] & \\
\mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\
& H &
\end{tikzcd}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$, $\quad\quad \bigtriangleup : H \to H \otimes H$,

$\eta : \mathbb{C} \to H$, $\quad\quad \varepsilon : H \to \mathbb{C}$

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"'] & H \end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"'] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\ H \otimes H \arrow[r, "id \otimes \bigtriangleup"'] & H \otimes H \otimes H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "\eta \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes \eta"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

Counit:

$$\begin{tikzcd} & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\ \mathbb{C} \otimes H & & H \otimes \mathbb{C} \\ & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] & \end{tikzcd}$$

Product and Coproduct compatibility:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes \triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"'] \end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle\otimes\triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & H \arrow[d, "\triangle"] \\ & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] & \end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd} H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle\otimes\triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\ H \otimes H \otimes H \otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & H \arrow[d, "\triangle"] \\ & H \otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] & \end{tikzcd}$$

Antipode:

$$\begin{tikzcd} H \otimes H \arrow[rr, "id\otimes S"] & & H \otimes H \arrow[d, "\nabla"] \\ H \arrow[u, "\triangle"] \arrow[r, "\varepsilon"] \arrow[d, "\triangle"'] & \mathbb{C} \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "S\otimes id"'] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\ H \otimes A \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\ H \otimes A \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\ H \otimes A \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

$$\begin{tikzcd} H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\ H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"'] \end{tikzcd}.$$

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following the diagrams commute:

$$\begin{tikzcd}
H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\
H \otimes A \arrow[r, "\alpha"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\
& A
\end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

$$\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\
H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}.$$

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "\eta"'] & \mathbb{C} \arrow[r, "\eta"] & A \\
H \otimes A \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$