

# Hopf Module Algebras

Brandon Mather

UNT Master's

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book “Hopf Algebras”
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montgomery writes “Hopf Algebras and Their Actions on Rings”

## Goal

To understand the actions of Hopf algebras on other algebras

Notation:  $\mathbb{C}[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - v_i v_j \rangle$

## Quantum Polynomial Ring

Let  $Q = (q_{ij})$  be an  $n \times n$  matrix of roots of unity where

$$q_{ii} = 1 = q_{ji} q_{ij}.$$

A **quantum polynomial ring** is

$$\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - q_{ij} v_i v_j \rangle.$$

Example:  $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \rangle$

# Motivation

- ① When a grp  $G$  acts on a space  $V$  over  $\mathbb{C}$  linearly, the action can be extended to  $V \otimes V$  by  $g \in G$  acting as

$$g \otimes g = \Delta(g).$$

Then  $\Delta$  defines a coproduct map

$$\Delta : \mathbb{C}G \rightarrow \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,  $\Delta : A \rightarrow A \otimes A$ , we call  $g \in A$  **grouplike** if  $\Delta(g) = g \otimes g$ .

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- ② When a Lie alg  $\mathfrak{g}$  acts on a space  $V$  over  $\mathbb{C}$ , the action can be extended to  $V \otimes V$  by  $x \in \mathfrak{g}$  acting as

$$x \otimes 1 + 1 \otimes x = \Delta(x).$$

Again,  $\Delta$  defines a coproduct map

$$\Delta : \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts,  $\Delta : A \rightarrow A \otimes A$ , we call  $x \in A$  **primitive** if  $\Delta(x) = x \otimes 1 + 1 \otimes x$ .

# Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

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with operations  $\Delta : H_4 \rightarrow H_4 \otimes H_4, \varepsilon : H_4 \rightarrow \mathbb{C}, S : H_4 \rightarrow H_4$

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Primitive

Let  $\tau$  be the 'flip' over the tensor product, so  $\tau(u \otimes v) = v \otimes u$ . Note that  $\tau \circ \Delta(x) \neq \Delta(x)$ , this is called non-cocommutativity.

$H_4$  acts on  $\mathbb{C}_{-1}[v_1, v_2]$  by

$$g \cdot v_1 = v_1, \quad g \cdot v_2 = -v_2, \quad x \cdot v_1 = 0, \quad x \cdot v_2 = v_1.$$

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We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

# Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

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# Actions of Kac-Paljutkin Algebra

$H_8$  acts on  $\mathbb{C}_q[v_1, v_2]$  where  $q^2 = -1$  by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$  for

$q_{12} = q_{34}^{-1}$ ,  $q_{13} = q_{24}^{-1}$ ,  $q_{14}^2 = 1$ ,  $q_{23}^2 = -1$  by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{-1}[v_1, v_2]$  by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

# Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups  $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

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with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

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Note: You can recover  $\mathcal{U}(\mathfrak{sl}_2)$  by limiting  $q \rightarrow 1$ .

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$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

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so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

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Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ 1_H \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes 1_H & \\ \mathbb{C} \otimes H & & H & & H \otimes \mathbb{C} \\ \searrow = & & \nwarrow = & & \end{array}$$

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Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \Delta \downarrow & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}$$

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Counit:

$$\begin{array}{ccccc} & & H & & \\ & \nwarrow = & \downarrow \Delta & \nearrow = & \\ \mathbb{C} \otimes H & & & & H \otimes \mathbb{C} \\ \nwarrow \varepsilon \otimes id & & H \otimes H & & \nearrow id \otimes \varepsilon \end{array}$$

# Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H & & 
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$$\begin{array}{ccc}
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Antipode:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad id \otimes S \quad} & H \otimes H & & \\
 \Delta \uparrow & & \downarrow \nabla & & \\
 H & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{1_H} & H \\
 \Delta \downarrow & & \uparrow \nabla & & \\
 H \otimes H & \xrightarrow{\quad S \otimes id \quad} & H \otimes H & & 
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# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .

Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

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 \qquad
 \begin{array}{ccc}
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 & \searrow = & \downarrow \alpha \\
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 \Delta \otimes id \otimes id \downarrow & & & & & & \uparrow \alpha \otimes \alpha \\
 H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & & & 
 \end{array}$$



# Hopf Algebra Actions

Let  $H$  be a Hopf alg and  $A$  an alg with a map  $\alpha : H \otimes A \rightarrow A$ .  
Then we say  $H$  **acts** on  $A$  by  $\alpha$  if the following diagrams commute:

$$\begin{array}{ccc}
 H \otimes H \otimes A & \xrightarrow{\nabla \otimes id} & H \otimes A \\
 \downarrow id \otimes \alpha & & \downarrow \alpha \\
 H \otimes A & \xrightarrow{\alpha} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{C} \otimes A & \xrightarrow{1_H \otimes id} & H \otimes A \\
 & \searrow = & \downarrow \alpha \\
 & & A
 \end{array}$$

$A$  is called a **module algebra** if the following also commute:

$$\begin{array}{ccccc}
 H \otimes A \otimes A & \xrightarrow{\nabla} & H \otimes A & \xrightarrow{\alpha} & A & \xleftarrow{\nabla} & A \otimes A \\
 \Delta \otimes id \otimes id \downarrow & & & & & & \uparrow \alpha \otimes \alpha \\
 H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & & & 
 \end{array}$$
  

$$\begin{array}{ccccc}
 H & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{1_A} & A \\
 -\otimes 1_A \downarrow & & & \nearrow \alpha & \\
 H \otimes A & & & & 
 \end{array}$$

In words,  $H$  acts on  $A$ , iff  $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And  $A$  is an  $H$ -module alg iff  $H$  acts on  $A$  and  $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_i(a) \cdot h_j(a'), \quad h(1_A) = \varepsilon(h)1_A$$

where  $\Delta(h) = \sum h_i \otimes h_j$ .

# Semidirect Product

Let  $G$  and  $G'$  be groups where  $G'$  acts on  $G$  by automorphisms. Then one can define the semidirect product group,  $G \rtimes G'$ . The action can be extended to the group algebras,  $\mathbb{C}G$  and  $\mathbb{C}G'$ . This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product  $g'g = (g' \cdot g)g'$ .

## Smash Product Algebra

If  $H$  is a Hopf algebra and  $A$  an  $H$ -module algebra, then  $A \# H$  is the smash product algebra defined as  $A \otimes H$  as a vector space and with product

$$ha = \sum_i (g_i \cdot a) k_i$$

where  $a \in A$ ,  $h \in H$  and  $\Delta(h) = \sum_i g_i \otimes k_i$ .

# Smash Product Algebra

## "Group-like" and "Lie-like"

Let  $H$  be a Hopf algebra, define  $G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$  and  $P(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$ .

$G(H)$  is the set of grouplike elements of  $H$  and forms a group under the product.

$P(H)$  is the set of primitive elements of  $H$  and forms a Lie algebra under the commutator bracket.

## Cartier-Kostant-Milnor-Moore Theorem

Let  $H$  be a cocommutative Hopf algebra over  $\mathbb{C}$ , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over  $\mathbb{C}$  is isomorphic to a group algebra.

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- If  $H$  is semisimple and finite-dimensional, and  $A$  is semiprime, is  $A \# H$  semiprime?
- If  $B$  is a Koszul algebra, are there nontrivial PBW deformations of  $B \# \mathcal{U}_q(\mathfrak{sl}_2)$ ?