

Hopf Module Algebras

Brandon Mather

UNT Master's

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book “Hopf Algebras”
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montgomery writes “Hopf Algebras and Their Actions on Rings”

Goal

To understand the actions of Hopf algebras on other algebras

Notation: $\mathbb{C}[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - v_i v_j \rangle$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ji} q_{ij}$.

$\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C} \langle v_1, \dots, v_n \mid v_j v_i - q_{ij} v_i v_j \rangle$ is called a **quantum polynomial ring**.

Example: $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \rangle$

Motivation

- 1 When a grp G acts on a space V by automorphism, the action can be extended to $V \otimes V$ by $g \in G$ acting as $g \otimes g = \Delta(g)$, this is called group-like

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These motivating examples generalize to a Hopf alg, H , a structure that is both an alg and a coalg with an antipode $S : H \rightarrow H$. We also require the alg and coalg structures of H to be compatible, so Δ and ε are alg homomorphisms. As well, S will be forced to be an alg antihomomorphism.

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \rangle$$

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Group-like

Primitive

Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \Delta(g) = \Delta(g)$ but $\tau \circ \Delta(x) \neq \Delta(x)$, this is what is called non-cocommutativity.

H_4 acts on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$g \cdot v_1 = v_1, \quad g \cdot v_2 = -v_2, \quad x \cdot v_1 = 0, \quad x \cdot v_2 = v_1.$$

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We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

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$$S(x) = x, S(y) = y, S(z) = z.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ by

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}$, $q_{13} = q_{24}^{-1}$, $q_{14}^2 = 1$, $q_{23}^2 = -1$ by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in “Quantum linear problem...” (1983), leading Vladimir Drinfeld to quantum groups $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} (K - K^{-1}), KEK^{-1} = q^2 E, \right. \\ \left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \rightarrow 1$.

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with operations:

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$

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$$\varepsilon(E) = \varepsilon(F) = 0, \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E, S(F) = -FK, S(K) = K^{-1}, S(K^{-1}) = K.$$

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$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix}$$

$$F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

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$$\nabla : H \otimes H \rightarrow H,$$

so that the following commute:

Associativity:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\ id \otimes \nabla \downarrow & & \downarrow \nabla \\ H \otimes H & \xrightarrow{\nabla} & H \end{array}$$

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Unit:

$$\begin{array}{ccccc} & & H \otimes H & & \\ \eta \otimes id \nearrow & & \downarrow \nabla & \nwarrow id \otimes \eta & \\ \mathbb{C} \otimes H & & & & H \otimes \mathbb{C} \\ \searrow = & & \downarrow & & \swarrow = \\ & & H & & \end{array}$$

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Coassociativity:

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \Delta \downarrow & & \downarrow \Delta \otimes id \\ H \otimes H & \xrightarrow{id \otimes \Delta} & H \otimes H \otimes H \end{array}$$

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Counit:

$$\begin{array}{ccccc} & & H & & \\ & \nwarrow = & \downarrow \Delta & \nearrow = & \\ \mathbb{C} \otimes H & & & & H \otimes \mathbb{C} \\ \varepsilon \otimes id \nwarrow & & & \nearrow id \otimes \varepsilon & \\ & & H \otimes H & & \end{array}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccc}
 H \otimes H & \xrightarrow{\quad \nabla \quad} & H & \xrightarrow{\quad \Delta \quad} & H \otimes H \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \nabla \otimes \nabla \\
 H \otimes H \otimes H \otimes H & \xrightarrow{\quad id \otimes \tau \otimes id \quad} & H \otimes H \otimes H \otimes H
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 \end{array}$$

Antipode:

$$\begin{array}{ccccc}
 H \otimes H & \xrightarrow{\quad id \otimes S \quad} & H \otimes H & & \\
 \Delta \uparrow & & & & \downarrow \nabla \\
 H & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{\quad \eta \quad} & H \\
 \Delta \downarrow & & & & \uparrow \nabla \\
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Hopf Algebra Actions

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 H \otimes A \otimes A & \xrightarrow{\nabla} & H \otimes A & \xrightarrow{\alpha} & A & \xleftarrow{\nabla} & A \otimes A \\
 \Delta \otimes id \otimes id \downarrow & & & & & & \uparrow \alpha \otimes \alpha \\
 H \otimes H \otimes A \otimes A & \xrightarrow{id \otimes \tau \otimes id} & H \otimes A \otimes H \otimes A & & & &
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 \end{array}$$

$$\begin{array}{ccccc}
 H & \xrightarrow{\varepsilon} & \mathbb{C} & \xrightarrow{\eta} & A \\
 \eta \downarrow & & & \nearrow \alpha & \\
 H \otimes A & & & &
 \end{array}$$

In words, H acts on A , iff

$$(hh')(a) = h(h'(a)), \quad 1_h(a) = a.$$

And A is an H -module alg iff H acts on A and

$$h(aa') = \sum h_i(a) \cdot h_j(a), \quad h(1_A) = \varepsilon(h)1_A$$

where $\Delta(h) = \sum h_i \otimes h_j$.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group, $G \rtimes G'$. The action can be extended to the group algebras, $\mathbb{C}G$ and $\mathbb{C}G'$. This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product $g'g = (g' \cdot g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H -module algebra, then $A \# H$ is the smash product algebra defined as $A \otimes H$ as a vector space and with product

$$ha = \sum_i (g_i \cdot a) k_i$$

where $a \in A$, $h \in H$ and $\Delta(h) = \sum_i g_i \otimes k_i$.

Smash Product Algebra

"Group-like" and "Lie-like"

Let H be a Hopf algebra, define $G(H) = \{h \in H \mid \Delta(h) = h \otimes h\}$ and $P(H) = \{h \in H \mid \Delta(h) = h \otimes 1 + 1 \otimes h\}$.

$G(H)$ is the set of grouplike elements of H and forms a group under the product.

$P(H)$ is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over \mathbb{C} , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over \mathbb{C} is isomorphic to a group algebra.

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- If H is semisimple and finite-dimensional, and A is semiprime, is $A \# H$ semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B \# \mathcal{U}_q(\mathfrak{sl}_2)$?