# Quantum Symmetry of Hopf Actions

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Algebra Seminar, November 2023

The unique 8-dim'l non-commutative, non-cocommutative Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$$

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## Actions of Kac-Paljutkin Algebra

$$H_8$$
 acts on  $\mathbb{C}_q[u,v]$  where  $q^2=-1$  by 
$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on  $\mathbb{C}_{\mathcal{O}}[v_1, v_2, v_3, v_4]$  for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
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And on  $\mathbb{C}_{-1}[u,v]$  by

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Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left( K - K^{-1} \right), KEK^{-1} = q^2 E, 
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with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E, \ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

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$$E\mapsto egin{bmatrix} 0 & 1 \ 0 & 0 \end{bmatrix} & F\mapsto egin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix} \\ \mathcal{K}\mapsto egin{bmatrix} q & 0 \ 0 & q^{-1} \end{bmatrix} & \mathcal{K}^{-1}\mapsto egin{bmatrix} q^{-1} & 0 \ 0 & q \end{bmatrix}$$

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$$\nabla: H \otimes H \to H$$
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so that the following commute:

#### Associativity:

$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
\downarrow^{id} \otimes \nabla \downarrow & & \downarrow \nabla \\
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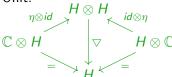
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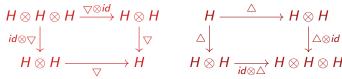
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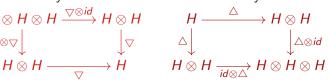
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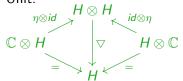
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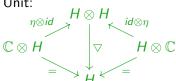
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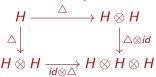
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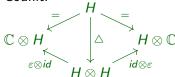
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### Hopf Algebra Diagrams

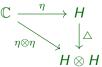
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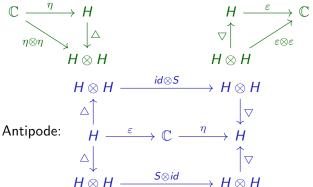


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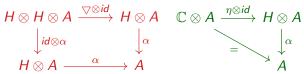
$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} H & \stackrel{\triangle}{\longrightarrow} H \otimes H \\ & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} H \otimes H \otimes H \otimes H \end{array}$$

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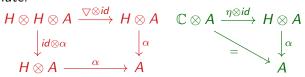


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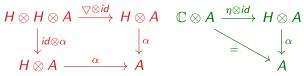


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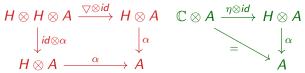
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