Quantum Group Actions and Hopf Algebras

Brandon Mather

University of North Texas Master's Defense
Department of Mathematics
February 28th 2024

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation: $\mathbb{C}[v_1, \dots, v_n] = \mathbb{C}\langle v_1, \dots, v_n \mid v_j v_i - v_i v_j \rangle$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where

$$q_{ii} = 1 = q_{ji} q_{ij}.$$

A **quantum polynomial ring** is

$$\mathbb{C}_Q[v_1, \dots, v_n] = \mathbb{C}\langle v_1, \dots, v_n \mid v_j v_i - q_{ij} v_i v_j \rangle .$$

Example: $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C}\langle v_1, v_2 \mid v_1 v_2 + v_2 v_1 \rangle$

Motivation

1. When a grp G acts on a space V over $\mathbb{C}$ linearly, the action can be extended to $V \otimes V$ by $g \in G$ acting as

$$g \otimes g = \triangle(g).$$

Then $\triangle$ defines a coproduct map

$$\triangle : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts,
$\triangle : A \to A \otimes A$,
we call $g \in A$ **grouplike** if
$\triangle(g) = g \otimes g$.

Motivation

1. When a grp G acts on a space V over $\mathbb{C}$ linearly, the action can be extended to $V \otimes V$ by $g \in G$ acting as
$$g \otimes g = \triangle(g).$$
Then $\triangle$ defines a coproduct map
$$\triangle : \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$
For arbitrary coproducts, $\triangle : A \to A \otimes A$, we call $g \in A$ **grouplike** if $\triangle(g) = g \otimes g$.

2. When a Lie alg $\mathfrak{g}$ acts on a space V over $\mathbb{C}$, the action can be extended to $V \otimes V$ by $x \in \mathfrak{g}$ acting as
$$x \otimes 1 + 1 \otimes x = \triangle(x).$$
Then $\triangle$ defines a coproduct map
$$\triangle : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$
For arbitrary coproducts, $\triangle : A \to A \otimes A$, we call $x \in A$ **primitive** if $\triangle(x) = x \otimes 1 + 1 \otimes x$.

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$$\triangle(g) = g \otimes g, \qquad \triangle(x) = x \otimes 1 + g \otimes x$$

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$$\begin{aligned} \triangle(g) &= g \otimes g, & \triangle(x) &= x \otimes 1 + g \otimes x \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0 \end{aligned}$$

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$$\begin{aligned} \triangle(g) &= g \otimes g, & \triangle(x) &= x \otimes 1 + g \otimes x \\ \varepsilon(g) &= 1, & \varepsilon(x) &= 0 \\ S(g) &= g^{-1}, & S(x) &= -x \end{aligned}$$

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$$\begin{aligned} \triangle(g) &= g \otimes g, \\ \varepsilon(g) &= 1, \\ S(g) &= g^{-1}, \end{aligned} \qquad\qquad \begin{aligned} \triangle(x) &= x \otimes 1 + g \otimes x \\ \varepsilon(x) &= 0 \\ S(x) &= -x \end{aligned}$$

Group-like

$(1, g)$ – Primitive

Sweedler's Algebra

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

$$H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$$

with operations $\triangle : H_4 \to H_4 \otimes H_4, \varepsilon : H_4 \to \mathbb{C}, S : H_4 \to H_4$

$\triangle(g) = g \otimes g,$	$\triangle(x) = x \otimes 1 + g \otimes x$
$\varepsilon(g) = 1,$	$\varepsilon(x) = 0$
$S(g) = g^{-1},$	$S(x) = -x$
Group-like	$(1, g) -$ Primitive

Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(x) \neq \triangle(x)$, so H is non-cocommutative.

Actions of Sweedler's Algebra

H_4 acts on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$g \cdot v_1 = v_1,\ g \cdot v_2 = -v_2,\ x \cdot v_1 = 0,\ x \cdot v_2 = v_1.$$

Actions of Sweedler's Algebra

H_4 acts on $\mathbb{C}_{-1}[v_1, v_2]$ by

$$g \cdot v_1 = v_1,\ g \cdot v_2 = -v_2,\ x \cdot v_1 = 0,\ x \cdot v_2 = v_1.$$

We can express this action on the generators v_1, v_2 as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy)\right\rangle$$

with operations

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x, \ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

Kac-Paljutkin Algebra

The unique 8-dim'l non-comm., non-cocomm. Hopf alg given by G. Kac and V. Paljutkin in "Finite Ring Groups" (1966):

$H_8 =$

$$\left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \tfrac{1}{2}(1+x+y-xy) \right\rangle$$

with operations

$$\triangle(x) = x \otimes x,\ \triangle(y) = y \otimes y,$$

$$\triangle(z) = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),$$

$$\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1,$$

$$S(x) = x,\ S(y) = y,\ S(z) = z.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Actions of Kac-Paljutkin Algebra

H_8 acts on $\mathbb{C}_q[v_1, v_2]$ where $q^2 = -1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$q_{12} = q_{34}^{-1}, \;\; q_{13} = q_{24}^{-1}, \;\; q_{14}^2 = 1, \;\; q_{23}^2 = -1$ via the rep

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

And on $\mathbb{C}_{-1}[v_1, v_2]$ via the rep

$$x \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups:
$\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups: $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \Big\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups: $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E, \right.$$

$$\left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups: $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\Big\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E,$$

$$KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1\Big\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1},\ S(K^{-1}) = K.$$

Quantized Universal Enveloping Algebra

Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups: $\mathcal{U}_q(\mathfrak{sl}_2) =$

$$\left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1}\left(K - K^{-1}\right), KEK^{-1} = q^2 E, \right.$$

$$\left. KFK^{-1} = q^{-2} F, KK^{-1} = K^{-1}K = 1 \right\rangle$$

with operations:

$$\triangle(E) = E \otimes 1 + K \otimes E,\ \triangle(F) = F \otimes K^{-1} + 1 \otimes F,$$

$$\triangle(K) = K \otimes K,\ \triangle(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$\varepsilon(E) = \varepsilon(F) = 0,\ \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$

$$S(E) = -K^{-1}E,\ S(F) = -FK,\ S(K) = K^{-1},\ S(K^{-1}) = K.$$

Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \to 1$.

$\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[v_1, v_2]$ via the representation

$\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[v_1, v_2]$ via the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad\qquad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Actions of $\mathcal{U}_q(\mathfrak{sl}_2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\mathbb{C}_q[v_1, v_2]$ via the representation

$$E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$K \mapsto \begin{bmatrix} q & 0 \\ 0 & q^{-1} \end{bmatrix} \qquad K^{-1} \mapsto \begin{bmatrix} q^{-1} & 0 \\ 0 & q \end{bmatrix}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode $S : H \to H$ where the bialgebra operations are

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"'] & H \end{tikzcd}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$,

$1_H : \mathbb{C} \to H$,

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\ H \otimes H \arrow[r, "\bigtriangledown"'] & H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} & H \otimes H \arrow[dd, "\bigtriangledown"] & \\ \mathbb{C} \otimes H \arrow[ur, "1_H \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes 1_H"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

Hopf Algebra

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\nabla : H \otimes H \to H,$ $\triangle : H \to H \otimes H,$

$1_H : \mathbb{C} \to H,$

so that the following commute:

Associativity:

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"'] & H \otimes H \arrow[d, "\nabla"] \\ H \otimes H \arrow[r, "\nabla"'] & H \end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd} H \arrow[r, "\triangle"] \arrow[d, "\triangle"'] & H \otimes H \arrow[d, "\triangle \otimes id"] \\ H \otimes H \arrow[r, "id \otimes \triangle"'] & H \otimes H \otimes H \end{tikzcd}$$

Unit:

$$\begin{tikzcd} & H \otimes H \arrow[d, "\nabla"] & \\ \mathbb{C} \otimes H \arrow[ur, "1_H \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes 1_H"'] \arrow[dl, "="] \\ & H & \end{tikzcd}$$

A **Hopf algebra** is a bialgebra H over a field with an antipode

$S : H \to H$ where the bialgebra operations are

$\bigtriangledown : H \otimes H \to H$, $\bigtriangleup : H \to H \otimes H$,

$1_H : \mathbb{C} \to H$, $\varepsilon : H \to \mathbb{C}$

so that the following commute:

Associativity:

$$\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\bigtriangledown \otimes id"] \arrow[d, "id \otimes \bigtriangledown"'] & H \otimes H \arrow[d, "\bigtriangledown"] \\
H \otimes H \arrow[r, "\bigtriangledown"'] & H
\end{tikzcd}$$

Coassociativity:

$$\begin{tikzcd}
H \arrow[r, "\bigtriangleup"] \arrow[d, "\bigtriangleup"'] & H \otimes H \arrow[d, "\bigtriangleup \otimes id"] \\
H \otimes H \arrow[r, "id \otimes \bigtriangleup"'] & H \otimes H \otimes H
\end{tikzcd}$$

Unit:

$$\begin{tikzcd}
 & H \otimes H \arrow[dd, "\bigtriangledown"] & \\
\mathbb{C} \otimes H \arrow[ur, "1_H \otimes id"] \arrow[dr, "="'] & & H \otimes \mathbb{C} \arrow[ul, "id \otimes 1_H"'] \arrow[dl, "="] \\
 & H &
\end{tikzcd}$$

Counit:

$$\begin{tikzcd}
 & H \arrow[dl, "="'] \arrow[dr, "="] \arrow[dd, "\bigtriangleup"] & \\
\mathbb{C} \otimes H & & H \otimes \mathbb{C} \\
 & H \otimes H \arrow[ul, "\varepsilon \otimes id"] \arrow[ur, "id \otimes \varepsilon"'] &
\end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd}
H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes \triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"']
\end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd}
H \otimes H \arrow[r, "\nabla"] \arrow[d, "\triangle \otimes \triangle"'] & H \arrow[r, "\triangle"] & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "id \otimes \tau \otimes id"'] & & H \otimes H \otimes H \otimes H \arrow[u, "\nabla \otimes \nabla"']
\end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "1_H"] \arrow[dr, "1_H \otimes 1_H"'] & H \arrow[d, "\triangle"] \\
& H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\varepsilon"] & \mathbb{C} \\
H \otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon \otimes \varepsilon"'] &
\end{tikzcd}$$

Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{tikzcd} H\otimes H \arrow[r, "\nabla"] \arrow[d, "\Delta\otimes\Delta"'] & H \arrow[r, "\Delta"] & H\otimes H \\ H\otimes H\otimes H\otimes H \arrow[rr, "id\otimes\tau\otimes id"'] & & H\otimes H\otimes H\otimes H \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

Unit and Counit compatibility:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1_H"] \arrow[dr, "1_H\otimes 1_H"'] & H \arrow[d, "\Delta"] \\ & H\otimes H \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{C} \\ H\otimes H \arrow[u, "\nabla"] \arrow[ur, "\varepsilon\otimes\varepsilon"'] & \end{tikzcd}$$

Antipode:

$$\begin{tikzcd} H\otimes H \arrow[rr, "id\otimes S"] & & H\otimes H \arrow[d, "\nabla"] \\ H \arrow[u, "\Delta"] \arrow[d, "\Delta"'] \arrow[r, "\varepsilon"] & \mathbb{C} \arrow[r, "1_H"] & H \\ H\otimes H \arrow[rr, "S\otimes id"] & & H\otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Quantum Group

Common philosophy: A **quantum group** is a Hopf alg H with a bijective antipode, and some invertible $R \in H \otimes H$ witnessing how close H is to being cocommutative and satisfying:

$$(1) R\left(\sum h_{(1)} \otimes h_{(2)}\right) R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$
$$(2) \triangle \otimes id(R) = R_{1,3}R_{2,3}$$
$$(3) id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where $R = \sum R_{(1)} \otimes R_{(2)}$, $\quad R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$. Hence, R is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}.$$

Quantum Group Examples

The three Hopf algs examples above are all qunatum groups. Sweedler's algebra, $H_4 = \left\langle g, x \mid g^2 = 1, x^2 = 0, gx = -xg \right\rangle$, has the universal R-matrix

$$R = 1 \otimes 1 - 2\tfrac{1-g}{2} \otimes \tfrac{1-g}{2} + x \otimes x + 2x\tfrac{1-g}{2} \otimes x\tfrac{1-g}{2} - 2x \otimes x\tfrac{1-g}{2}.$$

The Kac-Paljutkin algebra, $H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \right\rangle$, has 6 non-isomorphic quasitriangular structures.

And the ambiguity of what is a quantum group can be best seen with $\mathcal{U}_q(\mathfrak{sl}_2)$. This is undoubtedly a quantum group as it was the inspiration for the concept, but its universal R-matrix exists only in a completion of the tensor product $\mathcal{U}_q(\mathfrak{sl}2) \otimes \mathcal{U}_q(\mathfrak{sl}2)$, and so is a formal power series of pure tensors.

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$.
Then we say H **acts** on A by α if the following diagrams commute:

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$.
Then we say H **acts** on A by α if the following diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\ H \otimes A \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "1_H \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$.
Then we say H **acts** on A by α if the following diagrams commute:

$$\begin{tikzcd}
H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\
H \otimes A \arrow[r, "\alpha"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \otimes A \arrow[r, "1_H \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\
 & A
\end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following diagrams commute:

$$\begin{tikzcd} H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\ H \otimes A \arrow[r, "\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C} \otimes A \arrow[r, "1_H \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\ & A \end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

$$\begin{tikzcd} H \otimes A \otimes A \arrow[r, "id \otimes \nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\ H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"'] \end{tikzcd}.$$

Hopf Algebra Actions

Let H be a Hopf alg and A an alg with a map $\alpha : H \otimes A \to A$. Then we say H **acts** on A by α if the following diagrams commute:

$$\begin{tikzcd}
H \otimes H \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \alpha"] & H \otimes A \arrow[d, "\alpha"] \\
H \otimes A \arrow[r, "\alpha"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{C} \otimes A \arrow[r, "1_H \otimes id"] \arrow[dr, "="'] & H \otimes A \arrow[d, "\alpha"] \\
& A
\end{tikzcd}$$

A is called an H-**module algebra** if the following also commute:

$$\begin{tikzcd}
H \otimes A \otimes A \arrow[r, "id \otimes \nabla"] \arrow[d, "\triangle \otimes id \otimes id"'] & H \otimes A \arrow[r, "\alpha"] & A & A \otimes A \arrow[l, "\nabla"'] \\
H \otimes H \otimes A \otimes A \arrow[rrr, "id \otimes \tau \otimes id"] & & & H \otimes A \otimes H \otimes A \arrow[u, "\alpha \otimes \alpha"']
\end{tikzcd}.$$

$$\begin{tikzcd}
H \arrow[r, "\varepsilon"] \arrow[d, "id \otimes 1_A"'] & \mathbb{C} \arrow[r, "1_A"] & A \\
H \otimes A \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

Hopf Algebra Actions

In words, H **acts on** A iff you can multiply in H and then act on A or act on A consecutively, $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-**module alg** iff H acts on A and $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a), \quad h(1_A) = \varepsilon(h)1_A$$

where $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms, giving the semidirect product group, $G \rtimes G'$.
The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product $g'g = g'(g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then $A\#H$ is defined as $A \otimes H$ as a vector space and with product

$$(a' \otimes h)(a \otimes h') = \sum_i a' h_{(1)}(a) \otimes h_{(2)} h'$$

where $a, a' \in A$, $h, h' \in H$ and $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.

Smash Product Algebra

"Group-like" and "Lie-like"

For Hopf alg H, define

$$G(H) = \{h \in H \mid \triangle(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over $\mathbb{C}$, then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dimpt'l Hopf alg over $\mathbb{C}$ is iso to a grp alg.

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".

Research Directions

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?

Research Directions

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?

Research Directions

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is $A\#H$ semiprime?

Research Directions

- Which actions on algebras by Hopf algebras factor through a group action? If the action does not factor through a group action, it is said to have "quantum symmetry".
- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is $A\#H$ semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B\#\mathcal{U}_q(\mathfrak{sl}_2)$?