Quantum Group Actions and Hopf Algebras

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Department of Mathematics

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation: $\mathbb{C}[v_1,\ldots,v_n] = \mathbb{C}\langle v_1,\ldots,v_n \mid v_j v_i - v_i v_j \rangle$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where

$$q_{ii}=1=q_{ji}q_{ij}.$$

A quantum polynomial ring is

$$\mathbb{C}_{Q}[v_{1},\ldots,v_{n}] = \mathbb{C}\left\langle v_{1},\ldots,v_{n} \mid v_{j}v_{i} - q_{ij}v_{i}v_{j}\right\rangle.$$

Example: $\mathbb{C}_{-1}[v_1, v_2] = \mathbb{C} \langle v_1, v_2 | v_1 v_2 + v_2 v_1 \rangle$

Motivation

• When a grp G acts on a space V over $\mathbb C$ linearly, the action can be extended to $V\otimes V$ by $g\in G$ acting as

$$g \otimes g = \triangle(g)$$
.

Then \triangle defines a coproduct map

$$\triangle: \mathbb{C}G \to \mathbb{C}G \otimes \mathbb{C}G.$$

For arbitrary coproducts, $\triangle: A \rightarrow A \otimes A$.

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When a Lie alg $\mathfrak g$ acts on a space V over $\mathbb C$, the action can be extended to $V\otimes V$ by $x\in \mathfrak g$ acting as

$$x \otimes 1 + 1 \otimes x = \triangle(x)$$
.

Then \triangle defines a coproduct map

$$\triangle: \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

For arbitrary coproducts, $\triangle: A \rightarrow A \otimes A$, we call $x \in A$ **primitive** if $\triangle(x) = x \otimes 1 + 1 \otimes x$.

Hopf algebras combine the notions of Algebras with products and Coalgebras with coproducts.

We are looking for actions of Hopf algebras. In particular on non-commutative algebras like Quantum Polynomial Rings.

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by M. Sweedler (1969):

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Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(x) \neq \triangle(x)$, so H is non-cocommutative.

Actions of Sweedler's Algebra

$$\mathit{H}_4$$
 acts on $\mathbb{C}_{-1}[\mathit{v}_1,\mathit{v}_2]$ by

$$g \cdot v_1 = v_1, \ g \cdot v_2 = -v_2, \ x \cdot v_1 = 0, \ x \cdot v_2 = v_1.$$

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We can express this action on the generators as

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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$$H_8 =$$

$$\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \rangle$$

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Actions of Kac-Paljutkin Algebra

 H_8 acts on $\mathbb{C}_q[v_1,v_2]$ where $q^2=-1$ via the representation

$$x \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{\mathcal{O}}[v_1, v_2, v_3, v_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 via the rep

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Described by P. Kulish and N. Reshetikhin in "Quantum linear problem..." (1983), leading Vladimir Drinfeld to quantum groups $\mathcal{U}_q(\mathfrak{sl}_2)=$

$$\left\langle {\it E},{\it F},{\it K},{\it K}^{-1}\mid {\it EF}-{\it FE}=(q-q^{-1})^{-1}\left({\it K}-{\it K}^{-1}
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with operations:

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Note: You can recover $\mathcal{U}(\mathfrak{sl}_2)$ by limiting $q \to 1$.

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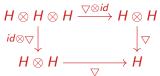
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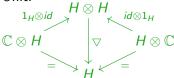
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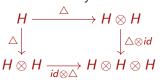
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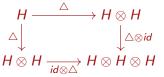
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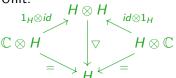
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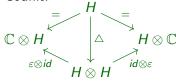
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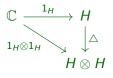
Hopf Algebra Diagrams

Product and Coproduct compatibility:

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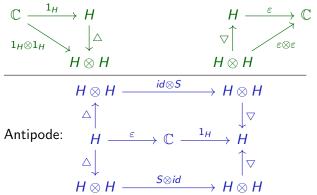


Hopf Algebra Diagrams

Product and Coproduct compatibility:

$$\begin{array}{ccc} H \otimes H & \stackrel{\nabla}{\longrightarrow} & H & \stackrel{\triangle}{\longrightarrow} & H \otimes H \\ & & & & \uparrow_{\nabla \otimes \nabla} \\ H \otimes H \otimes H \otimes H & \stackrel{id \otimes \tau \otimes id}{\longrightarrow} & H \otimes H \otimes H \otimes H \end{array}$$

Unit and Counit compatibility:



Quantum Group

Common philosophy holds that a **quantum group** might be defined as a Hopf alg with a bijective antipode, S, and an invertible element $R \in H \otimes H$ satisfying

$$(1)R\left(\sum h_{(1)} \otimes h_{(2)}\right)R^{-1} = \sum h_{(2)} \otimes h_{(1)}$$

$$(2)\triangle \otimes id(R) = R_{1,3}R_{2,3}$$

$$(3)id \otimes \triangle(R) = R_{1,3}R_{1,2}$$

where $R = \sum R_{(1)} \otimes R_{(2)}$, $R_{1,2} = \sum R_{(1)} \otimes R_{(2)} \otimes 1$, $R_{1,3} = \sum R_{(1)} \otimes 1 \otimes R_{(2)}$, and $R_{2,3} = \sum 1 \otimes R_{(1)} \otimes R_{(2)}$. The element R witnesses how close being cocommutative the quantum group is. As well, one can show that

$$R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}.$$

Hence, *R* is a solution to the quantum Yang-Baxter equation, and so is often called a **universal R-matrix**.

Quantum Group Examples

The three Hopf algs we have already discussed are all quantum groups.

Sweedler's algebra, $H_4=\left\langle g,x\mid g^2=1,x^2=0,gx=-xg\right\rangle$, has the universal R-matrix

$$R = 1 \otimes 1 - 2\frac{1-g}{2} \otimes \frac{1-g}{2} + x \otimes x + 2x\frac{1-g}{2} \otimes x\frac{1-g}{2} - 2x \otimes x\frac{1-g}{2}$$

The Kac-Paljutkin algebra,

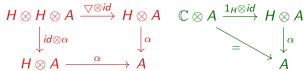
$$H_8 = \langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1 + x + y - xy) \rangle$$
 has 6 non-isomorphic quasitriangular

 $\frac{1}{2}(1+x+y-xy)$, has 6 non-isomorphic quasitriangular structures.

And the ambiguity of what is a quantum group can be best seen with $\mathcal{U}_q(\mathfrak{sl}_2)$. This is undoubtedly a quantum group as it was the inspiration for the concept, but its universal R-matrix exists only in a completion of the tensor product $\mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathcal{U}_q(\mathfrak{sl}_2)$, and so is a formal power series of pure tensors.

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$$H \otimes H \otimes A \xrightarrow{\nabla \otimes id} H \otimes A \quad \mathbb{C} \otimes A \xrightarrow{1_H \otimes id} H \otimes A$$

$$\downarrow^{id \otimes \alpha} \qquad \downarrow^{\alpha} \qquad \downarrow^{\alpha}$$

$$H \otimes A \xrightarrow{\alpha} A$$

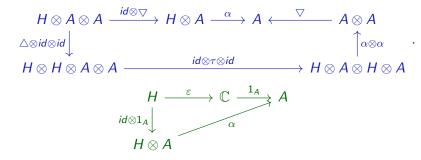
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In words, H acts on A iff you can multiply in H and then act on A or act on A consecutively, $\forall h, h' \in H, \forall a \in A$

$$(hh')(a) = h(h'(a)), \quad 1_H(a) = a.$$

And A is an H-module alg iff H acts on A and $\forall h \in H, \forall a, a' \in A$

$$h(aa') = \sum h_{(1)}(a) \cdot h_{(1)}(a), \quad h(1_A) = \varepsilon(h)1_A$$

where
$$\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$$
.

Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms, giving the semidirect product group, $G \rtimes G'$.

The action can be extended to the group algebras:

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G \# \mathbb{C}G'$$

with product g'g = g'(g)g'.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is defined as $A\otimes H$ as a vector space and with product

$$(a'\otimes h)(a\otimes h')=\sum_i a'h_{(1)}(a)\otimes h_{(2)}h'$$

where $a, a' \in A$, $h, h' \in H$ and $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$.

Smash Product Algebra

"Group-like" and "Lie-like"

For Hopf alg H, define

$$G(H) = \{h \in H \mid \triangle(h) = h \otimes h\} = \text{grouplike elements}$$

which forms a group, and define

$$P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\} = \text{primitive elements}$$

which forms a Lie alg.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over \mathbb{C} , then as Hopf alg,

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

Cor: Any cocomm, finite-dimpt'l Hopf alg over $\mathbb C$ is iso to a grp alg.

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- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of B#U_q(sl₂)?