Brandon Mather

Algebra Seminar, November 2023

History

- (1939) Heinz Hopf works on homology of sphere groups
- (1969) Moss Sweedler writes seminal book "Hopf Algebras"
- (1986) Vladimir Drinfeld gives ICM address on quantum groups
- (1992) Susan Montogomery writes "Hopf Algebras and Their Actions on Rings"

Goal

To understand the actions of Hopf algebras on other algebras

Quantum Plane

Notation:
$$\mathbb{C}[x_1,\ldots,x_n] = \mathbb{C}\langle x_1,\ldots,x_n\rangle/(x_jx_i-x_ix_j)$$

Quantum Polynomial Ring

Let $Q = (q_{ij})$ be an $n \times n$ matrix of roots of unity where $q_{ii} = 1 = q_{ii}q_{ii}$.

 $\mathbb{C}_Q[x_1,\ldots,x_n] = \mathbb{C}\langle x_1,\ldots,x_n\rangle/(x_jx_i-q_{ij}x_ix_j)$ is called a **quantum polynomial ring**.

Example: $\mathbb{C}_{-1}[u,v] = \mathbb{C}\langle u,v\rangle/(uv+vu)$

The unique 4-dim'l non-commutative, non-cocommutative Hopf alg given by Moss Sweedler (1969):

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 $\triangle(x) = x \otimes 1 + 1 \otimes x$ $\varepsilon(g) = 1,$ $\varepsilon(x) = 0$ $S(g) = g^{-1},$ $S(x) = -x$ "Group-like" "Lie-like"

Let τ be the 'flip' over the tensor product, so $\tau(u \otimes v) = v \otimes u$. Note that $\tau \circ \triangle(g) = \triangle(g)$ but $\tau \circ \triangle(x) \neq \triangle(x)$, this is what is called non-cocommutativity.

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Actions of Sweedler's Algebra

$$\mathcal{H}_4$$
 acts on $\mathbb{C}_{-1}[u,v]$ by
$$g\cdot u=u,\ g\cdot v=-v,\ x\cdot u=0,\ x\cdot v=u.$$

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$$g \cdot u = u$$
, $g \cdot v = -v$, $x \cdot u = 0$, $x \cdot v = u$.

Giving the representation on the vector space $\mathbb{C}_{-1}[u,v]$

$$g \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ x \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

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$$H_8 = \left\langle x, y, z \mid x^2 = y^2 = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{2}(1+x+y-xy) \right\rangle$$

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Actions of Kac-Paljutkin Algebra

$$\mathcal{H}_8$$
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$$x\mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \ y\mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ z\mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_Q[v_1, v_2, v_3, v_4]$ for

$$q_{12}=q_{34}^{-1}, \;\; q_{13}=q_{24}^{-1}, \;\; q_{14}^2=1, \;\; q_{23}^2=-1$$
 by

$$x \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad y \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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And on $\mathbb{C}_{-1}[u,v]$ by

$$x \mapsto egin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & y \mapsto egin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} & z \mapsto egin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Described by Piotr Kulish and Nicolai Reshetikhin in "Quantum linear problem for the sine-Gordon equation and highest weight representations" (1983), leading Vladimir Drinfeld to quantum groups

$$\mathcal{U}_q(\mathfrak{sl}_2) = \ \left\langle E, F, K, K^{-1} \mid EF - FE = (q - q^{-1})^{-1} \left(K - K^{-1} \right), KEK^{-1} = q^2 E,
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$$S(E) = -K^{-1}E, \ S(F) = -FK, \ S(K) = K^{-1}, \ S(K^{-1}) = K.$$

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so that the following commute:

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$$\begin{array}{ccc}
H \otimes H \otimes H & \xrightarrow{\nabla \otimes id} & H \otimes H \\
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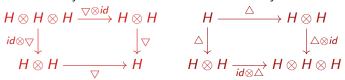
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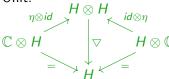
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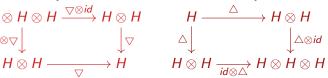
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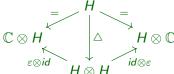
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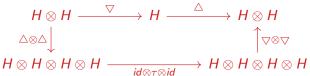


Counit:



Hopf Algebra Diagrams

Product and Coproduct compatibility:



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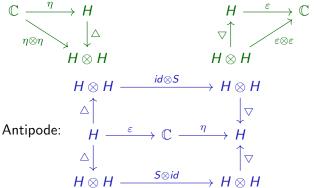


Hopf Algebra Diagrams

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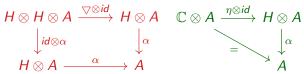
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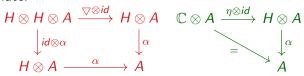


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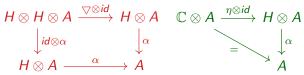


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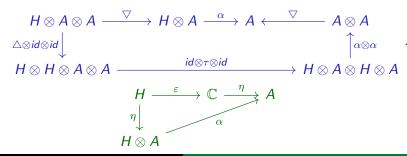
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Semidirect Product

Let G and G' be groups where G' acts on G by automorphisms. Then one can define the semidirect product group, $G \rtimes G'$. The action can be extended to the group algebras, $\mathbb{C}G$ and $\mathbb{C}G'$. This will give the group algebra

$$\mathbb{C}(G \rtimes G') = \mathbb{C}G\#\mathbb{C}G'$$

with product $g'g = (g' \cdot g)g'$.

Smash Product Algebra

If H is a Hopf algebra and A an H-module algebra, then A#H is the smash product algebra defined as $A\otimes H$ as a vector space and with product

$$ha = \sum_{i} (g_i \cdot a) k_i$$

where $a \in A$, $h \in H$ and $\triangle(h) = \sum_i g_i \otimes k_i$.

Smash Product Algebra

"Group-like" and "Lie-like"

Let H be a Hopf algebra, define $G(H) = \{h \in H \mid \triangle(h) = h \otimes h\}$ and $P(H) = \{h \in H \mid \triangle(h) = h \otimes 1 + 1 \otimes h\}$.

G(H) is the set of grouplike elements of H and forms a group under the product.

P(H) is the set of primitive elements of H and forms a Lie algebra under the commutator bracket.

Cartier-Kostant-Milnor-Moore Theorem

Let H be a cocommutative Hopf algebra over \mathbb{C} , then

$$H \cong \mathcal{U}(P(H)) \# \mathbb{C}G(H)$$

as Hopf algebras.

As a corollary, any finite-dimensional Hopf algebra over $\mathbb C$ is isomorpihe to a group algebra.

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- Which Hopf Algebras act on AS-regular algebras?
- When are the invariant subrings from Hopf actions AS-Gorenstein?
- If H is semisimple and finite-dimensional, and A is semiprime, is A#H semiprime?
- If B is a Koszul algebra, are there nontrivial PBW deformations of $B\#\mathcal{U}_q(\mathfrak{sl}_2)$?