proof. Prove by induction.

$n=2$, it holds ← by ④

$n=3$, $P(A_1\cup A_2\cup A_3) \overset{\text{by ④}}{=} P(A_1\cup A_2)+P(A_3)-P((A_1\cup A_2)\cap A_3)$

$P(A_1\cup A_2) = P(A_1)+P(A_2)-P(A_1\cap A_2)$ (by ④)

$(A_1\cap A_3)\cup(A_2\cap A_3) = (A_1\cup A_2)\cap A_3$

$P((A_1\cup A_2)\cap A_3) = P(A_1\cap A_3)+P(A_2\cap A_3)-P(A_1\cap A_2\cap A_3)$ (by ④)

$= P(A_1)+P(A_2)+P(A_3)-P(A_1\cap A_2)-P(A_1\cap A_3)-P(A_2\cap A_3)$
$+P(A_1\cap A_2\cap A_3)$

For $n>3$, by mathematical induction (exercise) → $\underline{A_1\cup\cdots\cup A_{n-1}}\cup \underline{A_n}$

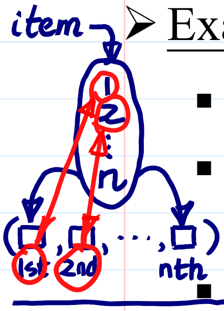
➢ Notes:

- There are $\binom{n}{k}$ summands in σ_k
- In symmetric examples, (e.g. $k=2$, $P(A_1\cap A_2)=P(A_1\cap A_3) = \cdots = P(A_{n-1}\cap A_n)$)

$$\sigma_k = \binom{n}{k} P(A_1\cap\cdots\cap A_k)$$

- It can be shown that (3rd proposition in LNp.3-11) (For proof, see textbook)

$$\begin{aligned} P(A_1\cup\cdots\cup A_n) &\le \sigma_1 \quad \text{加太多} \\ P(A_1\cup\cdots\cup A_n) &\ge \sigma_1-\sigma_2 \quad \text{減太多} \\ P(A_1\cup\cdots\cup A_n) &\le \sigma_1-\sigma_2+\sigma_3 \quad \text{加太多} \\ \cdots \quad & \cdots \quad \cdots \end{aligned}$$

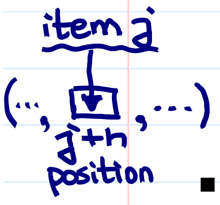


➢ Example (The Matching Problem).

- Applications: (a) Taste Testing. (b) Gift Exchange. (symmetric outcomes)
- Let Ω be all permutations $\omega = (i_1, \ldots, i_n)$ of $1, 2, \ldots, n$. Thus, $\#\Omega = n!$. (e.g. $(3, 1, 5, \cdots)$ 1st 2nd 3rd)
- Let

$$A_j = \{\omega: i_j = j\} \quad \text{and} \quad A = \cup_{i=1}^n A_i,$$

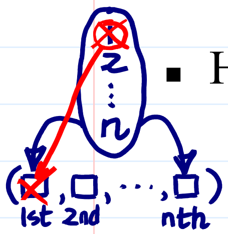
(item; A^c: none of them get his/her own gift; A: at least one person get his/her own gift)



Q: $P(A)$=? (What would you expect when n is large?) ($P(A)$↑? to 1? $P(A)$↓? to 0? or others?)

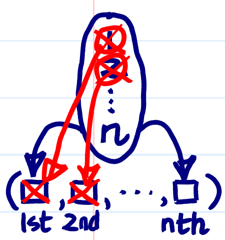
- By symmetry,

$$\sigma_k = \binom{n}{k} P(A_1\cap\cdots\cap A_k),$$



- Here,

$$\begin{aligned} P(A_1) &= \frac{1\times(n-1)!}{n!} = \frac{1}{n}, \\ P(A_1\cap A_2) &= \frac{(n-2)!}{n!} = \frac{1}{(n)_2}, \\ \cdots &= \cdots, \\ P(A_1\cap\cdots\cap A_k) &= \frac{(n-k)!}{n!} = \frac{1}{(n)_k}, \cdots \end{aligned}$$



for $k = 1, \ldots, n$.

- So, $\sigma_k = \binom{n}{k}\frac{1}{(n)_k} = \frac{1}{k!}$, $= \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{n!}$

$\boxed{e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k}$

$$P(A) = \sigma_1 - \sigma_2 + \cdots + (-1)^{n+1}\sigma_n = \sum_{\boxed{k=1}}^{n}(-1)^{k+1}\frac{1}{k!},$$

$\boxed{\text{c.f.}}$ $$P(A) = 1 - \sum_{\boxed{k=0}}^{n}(-1)^{k}\frac{1}{k!} \approx 1 - \frac{1}{e} = 0.632 \Rightarrow P(A^c) \approx e^{-1} = 0.368$$

$\boxed{\text{when } n \to \infty}$

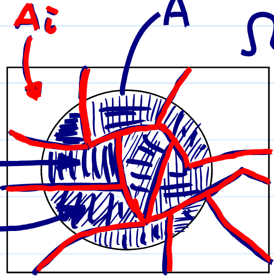
- Note: approximation accurate to 3 decimal places if $n \geq 6$.

countably infinite many or $\boxed{\text{finite}}$ $\boxed{\text{Let } A_{n+1} = A_{n+2} = \cdots = \phi}$

- Proposition: If $A_1, A_2, \ldots$, is a *partition* of Ω, i.e.,
  1. $\cup_{i=1}^{\infty} A_i = \Omega$,
  2. $A_1, A_2, \ldots$, are mutually exclusive,

  then, for any event $A \subset \Omega$,

$$P(A) = \sum_{i=1}^{\infty} P(A \cap A_i).$$



proof. $A = A \cap \Omega = A \cap (\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty}(A \cap A_i)$ $\boxed{\text{mutually exclusive}}$

$P(A) = P(\bigcup_{i=1}^{\infty}(A \cap A_i)) = \sum_{i=1}^{\infty} P(A \cap A_i)$

❖ **Reading**: textbook, Sec 2.4 & 2.5

Probability Measure for Continuous Sample Space

⊙ Q: How to define probability in a continuous sample space?

c.f. How to define P. M. for discrete Ω — check Example in LNp.3-1 ~ 3-6

- Monotone Sequences of sets $\boxed{\text{(LNp.3-7)}}$

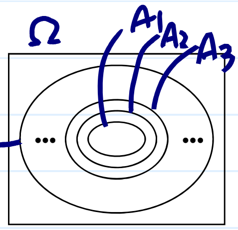
  ➢ Definition: A sequence of events $A_1, A_2, \ldots$, is called *increasing* if

  countably infinite many

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \subset \Omega$$

  and *decreasing* if

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots \supset \emptyset$$

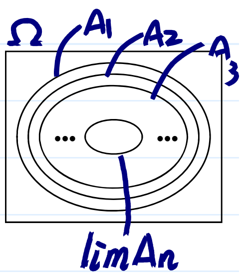


  The limit of an increasing sequence is defined as

$\boxed{A_1 \cup \cdots \cup A_n}$ $$\lim_{n\to\infty} A_n = \cup_{i=1}^{\infty} A_i$$ 不一定是 Ω

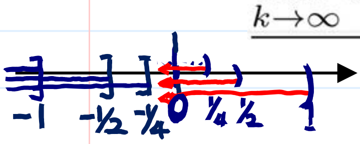
  and the limit of an decreasing sequence is $\boxed{\text{countably infinite}}$

$\boxed{A_1 \cap \cdots \cap A_n}$ $$\lim_{n\to\infty} A_n = \cap_{i=1}^{\infty} A_i$$ 不一定是 $\emptyset$



  ➢ Example: If $\Omega = \mathbb{R}$ and $A_k = (-\infty, 1/k)$, then A_k's are decreasing and

$$\lim_{k\to\infty} A_k = \{\omega : \omega < 1/k \text{ for all } k \in \mathbb{Z}_+\} = (-\infty, 0]$$



(exercise) $A_k = (-\infty, -\frac{1}{k}] \uparrow$, $\lim_{k\to\infty} A_k = (-\infty, 0)$

- **Proposition**: If $A_1, A_2, \dots$, is increasing or decreasing, then

cf LNp. 3-3 DeMorgan's Law

$$\left(\lim_{n\to\infty} A_n\right)^c = \lim_{n\to\infty} A_n^c$$

$\cup$ $\uparrow$ ⟷ $\cap$ $\downarrow$

$\cap$ $\downarrow$ ⟷ $\cup$ $\uparrow$

$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c = \lim_{n\to\infty} A_n^c$ (A_n ↑, A_n^c ↓)

$\left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c = \lim_{n\to\infty} A_n^c$ (A_n ↓, A_n^c ↑)

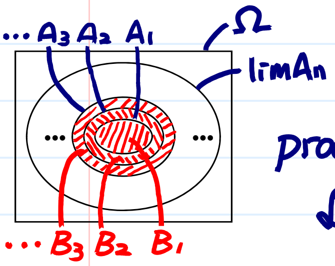
- **Proposition**: If $A_1, A_2, \dots$, is increasing or decreasing, then

$$\lim_{n\to\infty} P(A_n) = P\left(\lim_{n\to\infty} A_n\right). \quad \text{—(5)}$$

a P.M.

$\boxed{\text{lim \& } P \text{ exchangable}}$

连续的情况



proof. (a) A_n increasing.

Let $B_1 = A_1,\ B_2 = A_2 \cap A_1^c, \dots, B_n = A_n \cap A_{n-1}^c, \dots$

$\Rightarrow \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n,\quad \bigcup_{n=1}^{k} B_n = \bigcup_{n=1}^{k} A_n,\quad B_i \cap B_j = \phi$ for $i \neq j$. (mutually exclusive)

$P(\lim_{n\to\infty} A_n) = P(\bigcup_{n=1}^{\infty} A_n) = P(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} P(B_n) = \lim_{k\to\infty} \sum_{n=1}^{k} P(B_n)$

$= \lim_{k\to\infty} P(\bigcup_{n=1}^{k} B_n) = \lim_{k\to\infty} P(\bigcup_{n=1}^{k} A_n) = \lim_{k\to\infty} P(A_k)$

by Ax3 Additivity

$\boxed{\text{countable sums}}$

(b) A_n decreasing $\Rightarrow$ A_n^c increasing

$1 - P(\lim_{n\to\infty} A_n) = P((\lim_{n\to\infty} A_n)^c)$

$= P(\lim_{n\to\infty} A_n^c) = \lim_{n\to\infty} P(A_n^c) = \lim_{n\to\infty} 1 - P(A_n) = 1 - \lim_{n\to\infty} P(A_n)$

by (a)