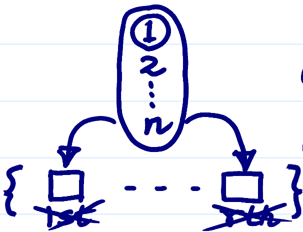
9/15

➢ A useful identity:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$



contain ①: $\binom{n-1}{r-1}$

not contain ①: $\binom{n-1}{r}$

- Partitions

➢ Example: How many distinct arrangements formed from the letters

$$\underset{1}{M}\ \underset{2}{I_1}\underset{3}{S_1}\underset{4}{S_2}\underset{5}{I_2}\underset{6}{S_3}\underset{7}{S_4}\underset{8}{I_3}\underset{9}{P_1}\underset{10}{P_2}\underset{11}{I_4}?$$

permutations

4 "S" treated "different"

partitions

4 "S" treated "identical"

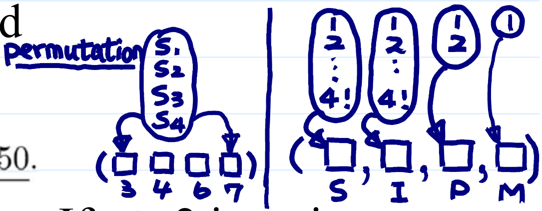
- There are 11 letters which can be arranged in 11! Ways
- But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s. List
- Each configuration of letters counted

list

permutation

$$4! \times 4! \times 2! \times 1! = 1{,}152$$

times and the answer is $\frac{11!}{4!4!2!1!} = 34,650.$



➢ Definition: Let Z be a set with n objects. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

$$(Z_1, \ldots, Z_r)$$

where $Z_1, \ldots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e.,

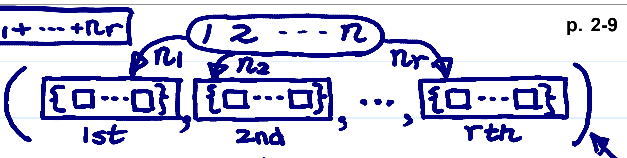
gener-alize

combination (r=2)

combination

$(\{1,2\},\{3,4\},\{5,6\})$
$\neq (\{3,4\},\{5,6\},\{1,2\})$
$(\{1,2\},\{3,4\},\{5,6\})$
$= (\{2,1\},\{4,3\},\{6,5\})$

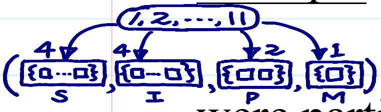
- $Z_i \cap Z_j = \emptyset$, if $i \neq j$, and
- $Z_1 \cup \cdots \cup Z_r = Z$.



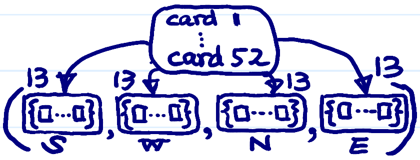
➢ Let $n_i = \#Z_i$, the number of elements in Z_i. Then, $n_1, \ldots, n_r \geq 0$, and $n_1 + \cdots + n_r = n$.

c.f. math, sci books example (LNp.2-5)

- Example: In the "MISSISSIPPI" example, 11 positions,

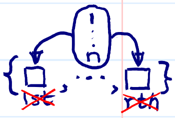


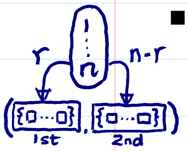
$$Z=\{1, 2, \ldots, 11\}$$



were partitioned into four groups of size

n_1=4 "I"s, n_2=1 "M"s, n_3=2 "P"s, n_4=4 "S"s





- In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

if r=2 ⇒ combination

➢ The Partitions Formula. Let n, $r \geq 1$, and $n_1, \ldots, n_r \geq 0$ be integers s.t. $n_1 + \cdots + n_r = n$. If Z is a set of n objects, then there are

combination

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots \times \binom{n_r}{n_r} = \binom{n}{n_1, \cdots, n_r} \overset{\text{EX in LNp2-8}}{\equiv} \frac{n!}{n_1! \times \cdots \times n_r!}$$

list

C.f. binomial coefficient (LNp.2-7), r=2

(called *multinomial coefficients*) ways to partition Z into r subsets $(Z_1, \ldots, Z_r)$ for which $\#Z_i = n_i$ for $i=1, \ldots, r$.

➢The multinomial theorem

binomial Theorem (LNp.2-7) c.f.

$$(x_1+\cdots+x_r)^n = \sum_{n_1+\cdots+n_r=n}\binom{n}{n_1,\cdots,n_r}x_1^{n_1}\cdots x_r^{n_r}$$

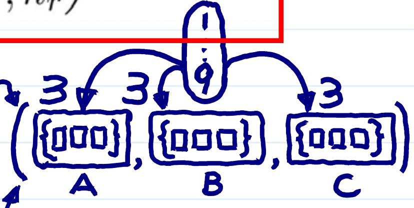
➢Examples:

diff.

- 9 children divided into A, B, C 3 teams of 3 each. How many different divisions? Ans: $\binom{9}{3\,3\,3}=\frac{9!}{3!3!3!}$

order matters

Note. ordered partition



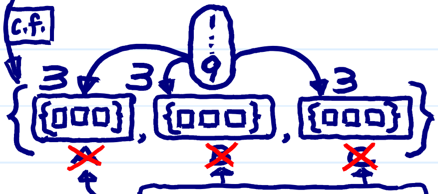
A	1	4
	2	5
	3	6
B	4	7
	5	8
	6	9
C	7	1
	8	2
	9	3

same

- 9 children divided into 3 groups of 3 each, to play a game. How many different divisions? Ans. $\binom{9}{3\,3\,3}/3!$

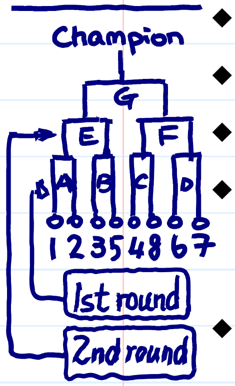
order ignored

c.f.



Note. order can be ignored only when $n_1=n_2=\cdots=n_r$

- a knockout tournament involving $n=2^m$ players
  - ◆ n players divided into $n/2$ pairs
  - ◆ losers of each pair eliminated; winner go next round
  - ◆ the process repeated until a single player remains
  - ◆ Q: How many possible outcomes for the 1st round?



~~1~~ beat 2
~~3~~ : 5
~~8~~ : 4
~~6~~ : 7

① # of different pairing (order matters) $\frac{n!}{2!\cdots2!}=\frac{n!}{2^{n/2}}$

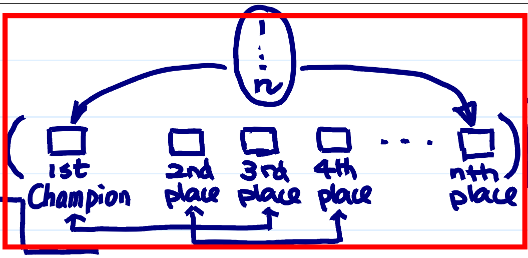
② $\frac{n!}{2^{n/2}}\times\frac{1}{(n/2)!}\times\underbrace{(2\times\cdots\times2)}_{n/2}=\frac{n!}{(n/2)!}$

order ignored

- ◆ Q: How many possible outcomes of the tournament?

1st round $\frac{n!}{(n/2)!}\times\frac{(n/2)!}{(n/4)!}\times\cdots\times\frac{2!}{1!}=n!$ 2nd round

Alternative argument:



n! permutation

Q: How many terms in multinomial Thm (LNp.2-10)?

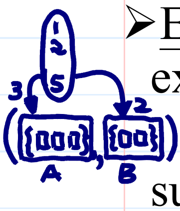
• The Number of Integer Solutions

➢If n and r are positive integers, how many integer solutions are there to the equations: $n_1,\ldots,n_r\geq 0$ and $n_1+\cdots+n_r=n$?

➢Example: How many arrangements from a A's and b B's, for example, ABAAB? There are

e.g. $a=3$, $b=2$

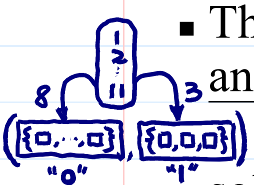
$$\binom{a+b}{a}=\binom{a+b}{b}$$



such arrangements, since an arrangement is determined by the a places occupied by A.

➢Example: Suppose n=8 and r=4. Represent solutions by "o" and "+" by "|".

- For example, $ooo|oo||ooo$ means n_1=3, n_2=2, n_3=0, n_4=3.
- Note: only r−1 (=3) "|"s are needed.
- There are as many solutions as there are ways to arrange "o" and "|". By the last example, there are

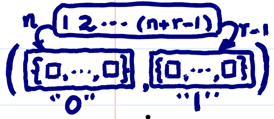


$a=n$
$b=r-1$

$$\binom{\overset{n}{8}+\underset{r-1}{3}}{3}=\binom{11}{3}=165$$

solutions.

➢A general formula. For positive integers n and r, there are



ⓐ ⓑ

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

special case of combination or partition of 2 subsets

integer solutions to $n_1,\ldots,n_r \ge 0$ and $n_1+\cdots+n_r = n$.

➢If $n \ge r$, then there are $\binom{n-1}{r-1}$ solutions with $n_i \ge 1$, for i=1, …, r.

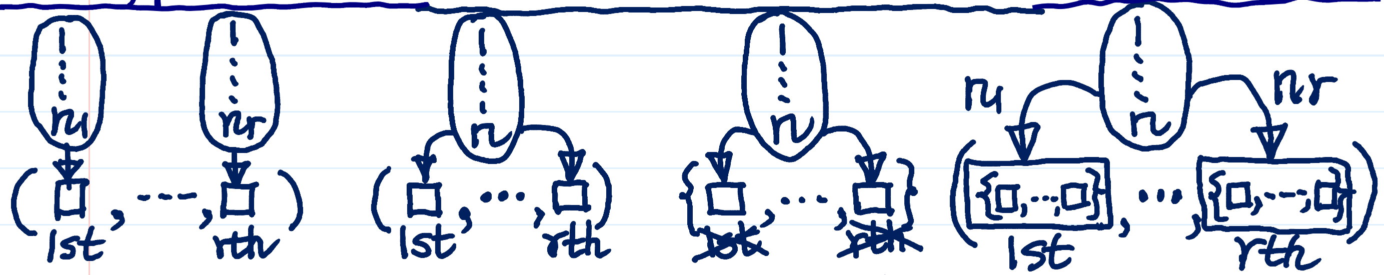
$n_i' = n_i - 1$

$n_i' \ge 0$

$n_1'+n_2'+\cdots+n_r' = n_1+\cdots+n_r-r = n-r$

$\Rightarrow n' = n-r$

Summary



allow repetition	no repetition	no repetition	no repetition
order matters	order matters	order ignored	[order matter between [order not matter within
List	Permutation	Combination ⟶	Partition
$n_1\times\cdots\times n_r$	$(n)_r = \frac{n!}{(n-r)!}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$	$\binom{n}{n_1\ n_2\cdots n_r}$

Integer Solution

2 groups ⟶ r groups

required, not optional

$n!$ ← $r=n$

❖ **Reading**: textbook, Chapter 1