樣本空間 Sample Space and Events

can chosen to be larger than all possible outcomes

- *Sample Space* Ω: the set of all possible outcomes in a random phenomenon. Examples:

known

final outcome is not 100% predictable

discrete → countable

1. Sex of a newborn child: Ω = {girl, boy}
2. The order of finish in a race among the 7 horses 1, 2, …, 7:

   finite set

   Ω = { all 7! Permutations of (1, 2, 3, 4, 5, 6, 7)} — a list
3. Flipping two coins: Ω = {(H, H), (H, T), (T, H), (T, T)}

   infinite set
4. Number of phone calls received in a year: Ω = {0, 1, 2, 3, …}

   countably infinite
5. Lifetime (in hours) of a transistor: $\Omega = [0, \infty)$

   uncountably infinite

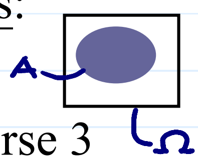
continuous → uncountable

might be difficult to identify in some cases

- *Event*: Any (measurable) subset of Ω is an event. Examples:

事件

1. A={girl}: the event - child is a girl.
2. A={all outcomes in Ω starting with a 3}: the event - horse 3 wins the race.



- A={(H, H), (H, T)}: the event - head appears on the 1st coin.
- A={0, 1, …, 500}: the event - no more than 500 calls received
- A=[0, 5]: the event - transistor does not last longer than 5 hours.

power set → 2^Ω: collection of all subsets in Ω | $\Omega=\{\omega_1, \ldots, \omega_n\}$ (1 or 0 … 1 or 0)

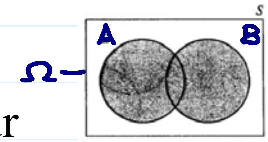
- ➢ an event occurs ⇔ outcome ∊ the event (subset)
- ➢ **Q**: How many different events if $\#\Omega=n<\infty$?

$2^\Omega=\{\phi, \{\omega_1\}, \ldots, \{\omega_n\}, \{\omega_1,\omega_2\}, \ldots, \Omega\}$

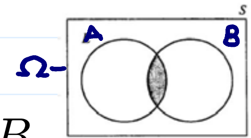
Ans. $\#2^\Omega = 2^n$

- Set Operations of Events
  - ➢ Union. $C = A \cup B \Rightarrow C$: either A or B occurs
  - ➢ Intersection. $C = A \cap B \Rightarrow C$: both A and B occur
  - ➢ Complement. $C = A^c \Rightarrow C$: A does not occur
  - ➢ Mutually exclusive (disjoint). $A \cap B = \emptyset \Rightarrow A$ and B have no outcomes in common.

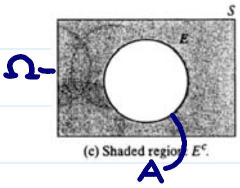
    including countably infinite many
  - ➢ Definitions of union and intersection for more than 2 events can be defined in a similar manner



(a) Shaded region: $E \cup F$.

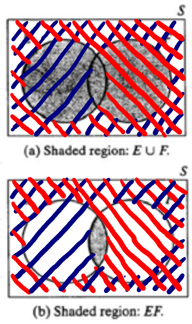


(b) Shaded region: EF.



(c) Shaded region: E^c.

- Some Simple Rules of Set Operations
  - Commutative Laws. $A \cup B = B \cup A$ and $A \cap B = B \cap A$
  - Associative Laws. $(A \cup B) \cup C = A \cup (B \cup C)$
    $(A \cap B) \cap C = A \cap (B \cap C)$.
  - Distributive Laws. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
    $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
  - DeMorgan's Laws.


(a) Shaded region: $E \cup F$.
(b) Shaded region: EF.

required, not optional

$$(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c \quad \text{and} \quad (\cap_{i=1}^{n} A_i)^c = \cup_{i=1}^{n} A_i^c.$$

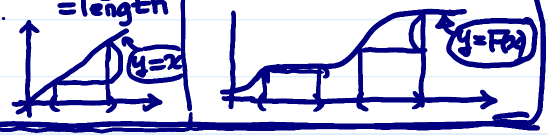
by induction (exercise)

Reading: textbook, Sec 2.2

機率測度 → Probability Measure → a function: $2^{\Omega} \to [0,1]$

- The Classical Approach
  - Sample Space Ω is a *finite* set
  - Probability: For an event A,

测度论： $\int g(x)dx$ weight = length | $\int g(x)dF(x)$ non-decreasing



probability space $(\Omega, \mathcal{F}, P)$ collection of events

$$P(A) = \frac{\#A}{\#\Omega}$$

This explains why combinatorial Thm plays an important role in probability.

- Example (Roulette): 哈哈 绿色为了不让红色超过一半
  - Ω = {0, 00, 1, 2, 3, 4, …, 35, 36}
  - P({Red Outcome}) = 18/38 = 9/19.

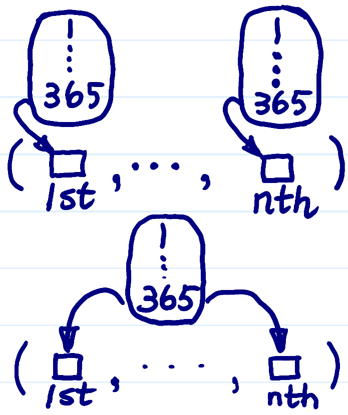
#A #Ω

assume birthday is uniform in 365 days

- Example (Birthday Problem): n people gather at a party. What is the probability that they all have different birthdays?
  - Ω = lists of n from {1, 2, 3, …, 365}
  - A= {all permutations}
  - $P_n(A) = (365)_n / 365^n$ (#A, #Ω) 哈哈 绝对赚钱！！

n	8	16	22	23	32	40
$P_n(A)$	.926	.716	.524	.492	.247	.109

smaller



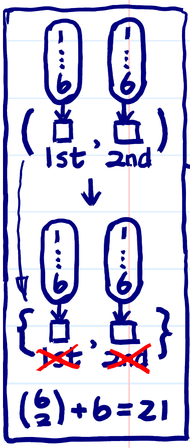
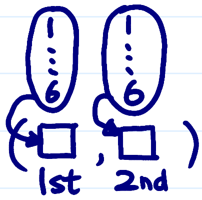
- Inadequacy of the Classical Approach
  - It requires: $P(A) = \dfrac{\#A}{\#\Omega}$
    - Finite Ω
    - Symmetric Outcomes

i.e., all outcomes in Ω are equally likely to occur, for $\omega \in \Omega$ $P(\{\omega\}) = \dfrac{1}{\#\Omega}$

➢ Example (Sum of Two Dice Being 6)

symmetric outcomes

- Ω_1 = {(1,1), (1,2), (2,1), (1,3), (3,1), …, (6,6)}, $\#\Omega_1=36$,
  ↑{lists} 1/36
  A={(1,5), (2,4), (3,3), (4,2), (5,1)}, $P(A)$=5/36.

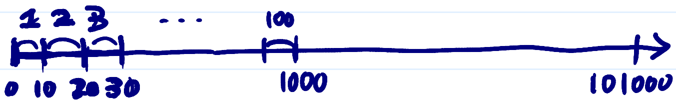
- Ω_2 = {{1,1}, {1,2}, {1,3}, …, {6,6}}, $\#\Omega_2=21$,
  ↑{sets} 1/36 2/36 2/36
  A={{1,5}, {2,4}, {3,3}}, $P(A)$=~~3/21~~.
- Ω_3 = {2, 3, 4, …, 12}, $\#\Omega_3=11$,
  ↑{sums} 1/36 2/36 3/36
  A={6}, $P(A)$=~~1/11~~.

non equally-likely outcomes

$\binom{6}{2}+6=21$

➢ Example (Sampling Proportional to Size): 正确的古典概型都建立不出来

N=101
n=4
{10,21,39,85}




- N invoices.
- Sample $n < N$.
- Pick large ones with higher probability.
- Note: Finite Ω, but *non equally-likely* outcomes.

each dollar should have equal chance of being selected, not each invoice

$\#\Omega=\binom{N}{n}$