Combinatorial Analysis

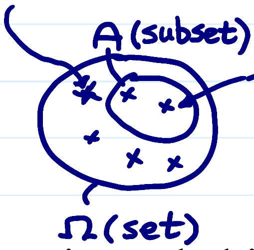
- An example: 古典概型 概率论理论的出现不是数学家钻研出来的
  - ➢ A communication system is to consist of *n seemingly identical* antennas that are to be lined up in a linear order
  - ➢ A resulting system will be functional as long as no two consecutive antennas are defective
  - ➢ If it turns out *m* (=2) of the *n* (=4) antennas are defective, what is the probability that the resulting system will be functional?

前提是原子的概率值相同

1	1	0	0	x
1	0	1	0	v
1	0	0	1	x
0	1	1	0	v
0	1	0	1	v
0	0	1	1	x

$\boxed{\text{Prob.} = \frac{3}{6}}$

an outcome



each outcomes have "equal" chance to happen

$P(A) \equiv \frac{\#A}{\#\Omega}$

$\#\Omega$ 0011 $\binom{4}{2}=6$

$\#A$ 1 1 00 $\binom{3}{2}=3$

- Many problems in probability theory can be solved simply by *counting the number* of different ways that a certain event can occur
- The mathematical theory of counting is formally known as *combinatorial analysis*
- What to Count? (i) Lists, (ii) Permutations, (iii) Combination, (iv) Partition, (v) Number of integer solutions.

- Lists
  - ➢ Definition (n_1 1st, n_2 2nd, ..., n_r r-th)
    - ▪ Ordered Pairs: $(x, y)=(w, z)$ iff $w=x$ and $z=y$.
    - ▪ Ordered Triples: $(x, y, z)=(u, v, w)$ iff $u=x$, $v=y$, and $w=z$.
    - ▪ *List of Length r*: $(x_1, \ldots, x_r)=(y_1, \ldots, y_s)$ iff $s=r$ and $x_i=y_i$ for $i=1, \ldots, r$.

order matter
e.g. (1,2,3) ≠ (3,2,1)

  - ➢ Example (License Plates): A license plate has the form $LMNwxyz$, where (1st 2nd 3rd ... 7th)

    $L, M, N \in \{A, B, \ldots, Z\}$,

    $w, x, y, z \in \{0, 1, \ldots, 9\}$,

    and, so, is a list of length seven.

BAD0001
BAD1000

x □ 1 2 ... m
y □ 1 2 ... n 1 2 ... n ... 1 2 ... n

乘法原则其实是一棵树

  - ➢ The basic principle of counting - multiplication principle
    - ▪ For two: If there are m choices for x and for each choice of x, n choice for y, then there are mn choices for (x, y).
    - ▪ For several: If there are n_i choices for x_i, $i=1, \ldots, r$, then there are

$$n_1 n_2 \cdots n_r$$

choices for $(x_1, \ldots, x_r)$.

- Example:

  As I was going to St. Ives, I met a man with seven wives
  Every wife had seven sacks, Every sack had seven cats
  Every cat had seven kits, Kits, cats, sacks, wives
  How many were going to St. Ives?

  - Ans: none one吧 哈哈
  - However, how many were going the other way?

  By multiplication principle:

  7 Wives, 7×7=49 sacks, 49×7=343 cats, 343×7=2401 kits

  Total=7+49+343+2401=2800 $= 7^1+7^2+7^3+7^4$ (等比級數)

- Example (license plates): A license plate has the form $LMNwxyz$, where

  $L, M, N \in \{A, B, \ldots, Z\}$

  $w, x, y, z \in \{0, 1, \ldots, 9\}$

  There are $26^3 \times 10^4$=175,760,000 license plates. Of these,

  c.f. $(26\times25\times24)\times(10\times9\times8\times7)$ = 78,624,000 ← List + permutation

  of them have distinct letters and digits (no repetition).

- Permutation (r-permutation of n objects, $r \le n$)
  - Definition: For n objects, a *permutation* of length r is a list (order matters) $(x_1, \ldots, x_r)$ with distinct components (no repetition); that is $x_i \ne x_j$ when $i \ne j$
  - Example: (1, 2, 3) is a permutation of three elements; (1, 2, 1) is not a permutation
  - Counting Formulas. From n objects, there are

    $$n^r = n \times \cdots \times n \quad (r \text{ factors})$$

    lists of length r and

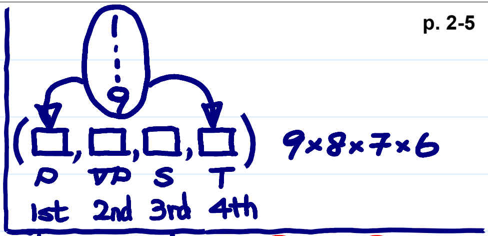
    $$\boxed{(n)_r} \equiv n \times (n-1) \times \cdots \times (n-r+1)$$

    permutations of length r may be formed.

    list: (1st, ..., rth)
    permutation: (1st, ..., rth)
  - Example: There are 10^3=1000 three digit numbers, of which $(10)_3=10\times9\times8=720$ lists with distinct digits.
    all lists
    all permutations
  - Some notations
    - Factorials: For positive integers n and r, when $r=n$, write

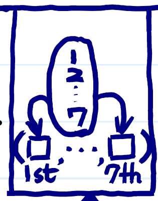
      階乘 $n! \equiv (n)_n = n \times (n-1) \times \cdots \times 2 \times 1$

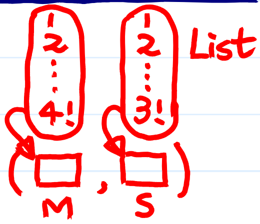
      permutation defined in textbook
    - Conventions: $(n)_0=1$ and $0!=1$



- Some Notes
  - The textbook only consider $r=n$.
  - $(n)_r \equiv 0$, if $r>n$. $\boxed{(n)_r = \frac{n!}{(n-r)!}}$
  - If $r<n$, then $n!=(n)_r(n-r)!$

➢ Example: A group of 9 people may choose officers (P, VP, S, T) in $(9)_4=3024$ ways.





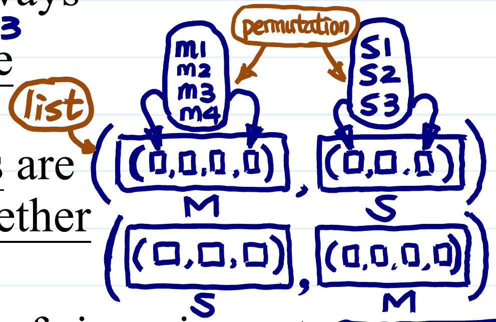
➢ Example: $n=9,\ r=4,\ (9)_4 = 9!/5!$

- 7 books may be arranged in $7!=5040$ ways
- If there are 4 math books ($m_1, \ldots, m_4$) and 3 science books (S_1, S_2, S_3), then there are $2\times(4!\times 3!)=288$ arrangements in which the math books are together and the science books are together

list

$(4! \times 3!)$

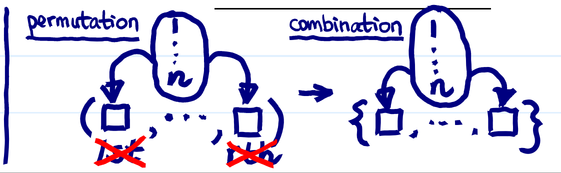
permutation



• Combinations

➢ Definition: For n objects, a *combination* of size r is a set (order ignored) $\{x_1, \ldots, x_r\}$ of r distinct elements. Two combinations equal if they have the same elements, possibly written in different order.

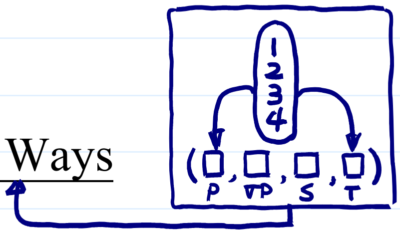
➢ Example: $\{1, 2, 3\}=\{3, 2, 1\}$, $=\{1,3,2\}=\{2,3,1\}=\{2,1,3\}$ $=\{3,1,2\}$
but $(1, 2, 3) \neq (3, 2, 1)$
1st 2nd 3rd



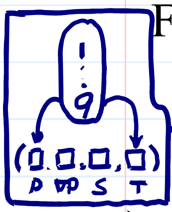
➢ Example: How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

Choose a committee in ?? Ways.

Choose officers from the committee in 4! Ways



From the Basic principle

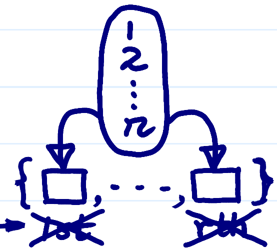


- $(9)_4 = 4! \times ??$
- So, $?? = (9)_4/4! = 126 = \frac{9!}{5!4!}$

➢ Combinations Formula

- From n (≥ 1) objects,

$$\binom{n}{r} = \frac{1}{r!}(n)_r$$

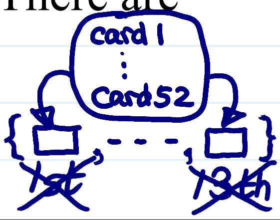


combinations of size $r \leq n$ may be formed

- Example (bridge): A bridge hand is a combination of $r=13$ cards drawn from a standard deck of $n=52$. There are

$$\binom{52}{13} = 635,013,559,600$$

such hands.



- Binomial coefficients
  - Alternatively, $\binom{n}{r} = \boxed{\frac{n!}{r!(n-r)!}} = (n)_r$
  - **The Binomial Theorem**: For all $-\infty < x,\ y < \infty$

$$\boxed{(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.}$$

{ □, ..., □ }
1st ... nth
1, 2, ..., n

- Proof. If

$$(x+y)^n = \underset{1}{(x+y)} \times \cdots \times \underset{n}{(x+y)}.$$

is expanded, then $x^r y^{n-r}$ will appear as often as x can be chosen from r of the n factors; i,e., in $\binom{n}{r}$ ways

- Example. When n=3, $(x+y)^3 = \underset{\binom{3}{3}}{x^3} + \underset{\binom{3}{2}}{3x^2y} + \underset{\binom{3}{1}}{3xy^2} + \underset{\binom{3}{0}}{y^3}$.

  - Binomial identities
    - Setting x=y=1, we get $2^n = \sum_{r=0}^{n} \binom{n}{r}$

$\{1,\dots,n\} \Rightarrow \phi, \{1\}, \{1,2\}, \dots, \{1,\dots,n\}$
$\{2\}, \{1,3\},$
$\vdots \quad \vdots$
$\{n\}, \{n-1, n\},$

(Y/N, Y/N, ..., Y/N)
(□, □, ..., □)
1 2 ... n
List

- Example: how many subsets are there of a set consisting of n elements?

- Letting x=–1 and y=1, we get $0 = \sum_{r=0}^{n} \binom{n}{r} (-1)^r$

9/15

- A useful identity:

$$\boxed{\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}}$$

1, 2, ..., n
{ □ - - - □ }
1st ... rth

contain ①: $\binom{n-1}{r-1}$
not contain ①: $\binom{n-1}{r}$

- Partitions
  - Example: How many distinct arrangements formed from the letters

$$M\ I_1 S_1 S_2 I_2 S_3 S_4 I_3 P_1 P_2 I_4?$$
1 2 3 4 5 6 7 8 9 10 11

permutations
4 "S" treated "different"

partitions
4 "S" treated "identical"

- There are 11 letters which can be arranged in 11! Ways
- But, this leads to double counting. If the 4 "S" are permuted, then nothing is changed. Similarly, for the 4 "I"s and 2 "P"s.
- Each configuration of letters counted

list
permutation

$$4! \times 4! \times 2! \times 1! = 1{,}152$$

times and the answer is $\frac{11!}{4!4!2!1!} = 34,650.$

permutation
S_1, S_2, S_3, S_4
(□ □ □ □)
3 4 6 7

list
1, 2, ..., 4!   1, 2, ..., 4!   1, 2
(□, □, □)
S I P

- Definition: Let Z be a set with n objects. If $r \geq 2$ is an integer, then, an *ordered partition* of Z into r subsets is a list

generalize
combination

e.g. $(\{1,2\},\{3,4\}) \neq (\{3,4\},\{1,2\})$

$$(Z_1, \dots, Z_r)$$

where $Z_1, \dots, Z_r$ are mutually exclusive subsets of Z whose union is Z; i.e.,

combination