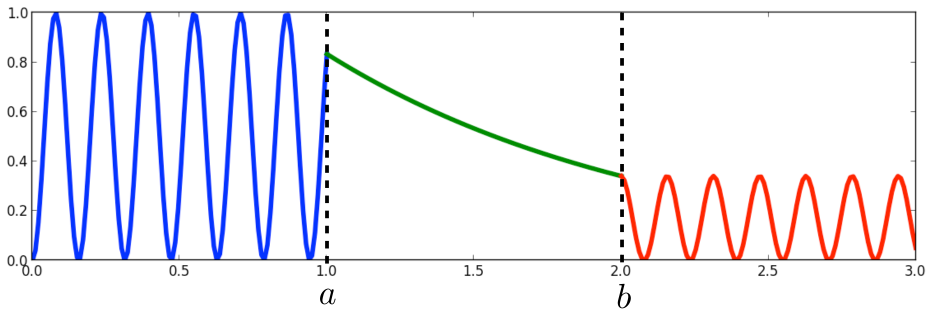


- If the amplitude of the wave function does not decay to zero inside the forbidden region (E<U), then the amplitude for x>b must also be non-zero since the boundary conditions must match at x=b.

→ - There is a non-zero probability of the particle being found on the other side of the barrier x>b

- This is called **Tunneling,** and it is impossible in classical physics

- The probability that a particle on the left will appear on the right of the barrier is called the **transmission amplitude** and it is given by:

$$\boxed{T = \frac{|\Psi(b)|^2}{|\Psi(a)|^2} = \frac{|Fe^{-\gamma b}|^2}{|Fe^{-\gamma a}|^2} = \left|e^{-\gamma(b-a)}\right|^2 = e^{-2\gamma(b-a)}}$$

- Transmission decays exponentially with barrier width.

<u>Harmonic Oscillator:</u>

- The most important 1D example is that of the harmonic oscillator potential

$$U(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega_0^2 x^2$$

- The kinetic energy is given by $K(x) = E - U(x)$

  - The points at which K(x)=0 are called the classical **Turning Points**, since the oscillator “must turn around here to avoid K(x)<0.

  - However we saw that in quantum mechanics, the wave function can actually extend into this forbidden region.

- The Schrödinger equation for the harmonic oscillator reads:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega_0^2 x^2\Psi(x) = E\Psi(x)$$

$$\Downarrow$$

$$\frac{d^2\Psi(x)}{dx^2} = -\left(\frac{2mE}{\hbar^2} - \frac{m^2\omega_0^2}{\hbar^2}x^2\right)\Psi(x)$$

- Like the potential well with E<U, the energies have discrete values:

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots$$

- The corresponding wave functions are the product of gaussian functions and special functions called Hermite polynomials:

$$\Psi_n(x) = \frac{1}{\sqrt{\sigma}\pi^{1/4}} \frac{1}{\sqrt{n!\,2^n}} H_n\left(\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}$$

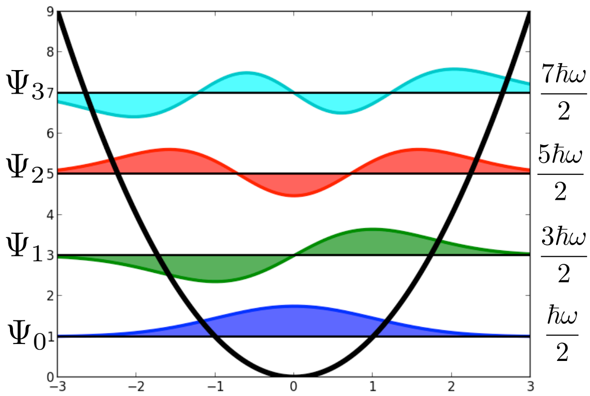
- Here σ is the half-width of the gaussian: $\sigma = \sqrt{\hbar/m\omega_0}$

- The first few states are given below:

$$\Psi_0(x) = \frac{1}{\sqrt{\sigma}\pi^{1/4}} e^{-x^2/2\sigma^2}$$

$$\Psi_1(x) = \frac{1}{\sqrt{\sigma}\pi^{1/4}} \frac{1}{\sqrt{2}} \left(2\frac{x}{\sigma}\right) e^{-x^2/2\sigma^2}$$

$$\Psi_2(x) = \frac{1}{\sqrt{\sigma}\pi^{1/4}} \frac{1}{\sqrt{8}} \left(4\frac{x^2}{\sigma^2} - 2\right) e^{-x^2/2\sigma^2}$$



Measurements in Quantum Mechanics:

- Having found the wave function $\Psi(x)$, how do we measure physical quantities?

- Recall that things like momentum, kinetic energy,... (observables) all have an associated operator such that (using momentum as an example):

$$\hat{p}\Psi(x) = p\Psi(x)$$

operator corresponding observable

- We can obtain the average value for the observable we want to measure using the conservation of probability

$$\int_{-\infty}^{\infty} \Psi^*(x)\Psi(x)dx = \int_{-\infty}^{\infty} P(x)dx = 1$$

- For a given observable A we can get the **expectation value** (average value) as:

$$\boxed{\langle A \rangle = \int_{-\infty}^{\infty} \Psi^*(x)\hat{A}\Psi(x)dx}$$

Ex. uncertainty of HO in the ground state:

- Here we will find the uncertainty in the position and momentum of a harmonic oscillator in its n=0 (“ground state”).

- Recall that the uncertainty is given in terms of the operator variances:

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$$

$$(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2$$

- The ground state wave function is $\Psi_0(x) = \dfrac{1}{\sqrt{\sigma}\pi^{1/4}} e^{-x^2/2\sigma^2}$

- Solving for $\langle x\rangle$:

Odd function

$$\langle x\rangle = \frac{1}{\sigma\sqrt{\pi}}\int_{\infty}^{\infty} e^{-x^2/2\sigma^2} x e^{-x^2/2\sigma^2} dx = 0$$

Odd function

- Solving for $\langle p\rangle$:

$$\langle p\rangle = \frac{-i\hbar}{\sigma\sqrt{\pi}}\int_{\infty}^{\infty} e^{-x^2/2\sigma^2}\frac{d}{dx}e^{-x^2/2\sigma^2}dx = \frac{i\hbar}{\sigma^3\sqrt{\pi}}\int_{\infty}^{\infty} e^{-x^2/2\sigma^2} x e^{-x^2/2\sigma^2} = 0$$

- Solving for $\langle x^2\rangle$:

$$\langle x^2\rangle = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^2 e^{-x^2/2\sigma^2} dx = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2/\sigma^2} dx$$

$$\rightarrow \quad \langle x^2\rangle = \frac{\sigma^2}{2} \quad \rightarrow \quad \boxed{(\Delta x)^2 = \frac{\sigma^2}{2}}$$

- Solving for $\langle p^2\rangle$:

$$\langle p^2\rangle = \frac{-\hbar^2}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2}\frac{d^2}{dx^2}e^{-x^2/2\sigma^2}dx$$

$$\rightarrow \quad \langle p^2\rangle = \frac{\hbar^2}{2\sigma^2} \quad \rightarrow \quad \boxed{(\Delta p)^2 = \frac{\hbar^2}{2\sigma^2}}$$

- Since the Heisenberg uncertainty relation is given as the square root of the product of the variances

$$\boxed{\Delta x\Delta p = \frac{\hbar}{2}}$$

- The ground state of the HO gives the minimal value for the Heisenberg uncertainty relation