PHYS-183 : Day #7

<u>Vector Algebra:</u>

- We are now ready to look at motion, forces, and energy is more than one dimension.

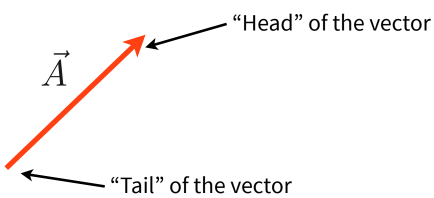
- We have seen that there are mathematical quantities such as position, velocity, acceleration, and forces, that have both a magnitude and a direction. We called these objects **vectors** .

- We have also looked at objects that <u>do not</u> have a direction. These objects, such as speed, time, energy, power, work, are called **scalar** objects.

- In one-dimension, vectors are very simple. In two or more dimensions, vector mathematics becomes much more complicated.

- In physics, we always draw vectors to help us understand what is going on.

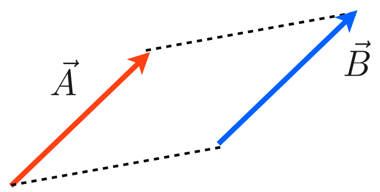
- A vector $\vec{A}$ is drawn as:



- The length of the vector is proportional to the vectors magnitude.

- The end of the vector with the arrowhead shows the direction of the vector.

- Two vectors $\vec{A}$ and $\vec{B}$ are said to be equal, $\vec{A} = \vec{B}$, if they have the same magnitude and point in the same direction

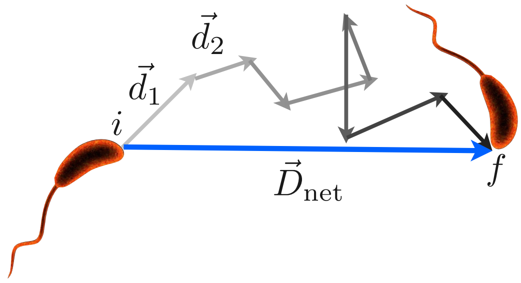


- Two vectors are the same if you can move one on top of another without rotating the vectors or changing their magnitudes

- We also need to be able to add vectors together.

- Consider a series of displacements ($\vec{d}_i$) of a bacteria

- Starting from the initial location, each new vector starts at the head of the previous vector



- The net displacement $\vec{D}_{\rm net}$, is the vector starting from the tail of the first vector to the head of the final vector

- Remember, you can not add vectors that correspond to different variables

$$\cancel{\vec{d} + \vec{F}}$$

- To make adding vectors easier, we can also define a coordinate system such as the Cartesian coordinate system (x,y).

- Given a coordinate system, a vector can be expressed by an ordered list of numbers:

$$\vec{A} = (A_x, A_y) \qquad \text{(In two-dimensions)}$$

- Here, $A_x$is called the x-component of the vector, and A_y is the y-components

- The x and y-components of a vector are independent, and therefore we can treat the components separately.

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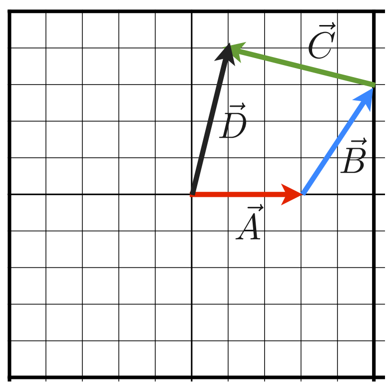
<u>Ex. Vector Addition:</u>

Add the following three vectors

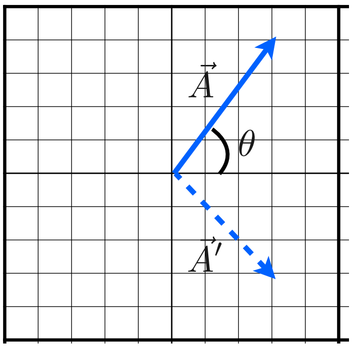
$$\vec{A} = (3,0) \quad \vec{B} = (2,3) \quad \vec{C} = (-4,1)$$
$$\vec{D} = \vec{A} + \vec{B} + \vec{C}$$
$$\vec{D} = (3+2-4, 0+3+1) = (1,4)$$



- Because the components of a vector are independent, we can define the **magnitude of the vector** $|\vec{A}|$ in terms of the individual components:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

- The direction of the vector can be found with trigonometry:



- Let θ be the angle of the vector with respect to the x-axis

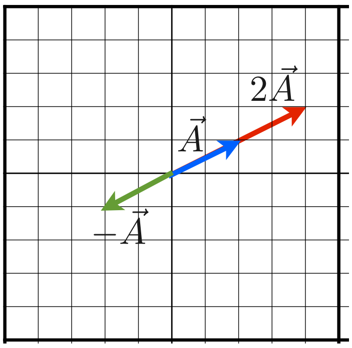
$$\cos(\theta) = \frac{A_x}{|\vec{A}|}$$

- The cosine function can have the same value for more than one angle, so you must draw a picture to get the right direction

- Both $\vec{A}$ and $\vec{A}'$ have the same $\cos(\theta)$ but the angles are not the same

- We also can define vector multiplication using vector components:

$$\vec{A} = (2, 1)$$
$$2\vec{A} = (2 \cdot 2, 2 \cdot 1) = (4, 2)$$
$$-\vec{A} = (-1 \cdot 2, -1 \cdot 1) = (-2, -1)$$



- For any vector: $\vec{A} - \vec{A} = \vec{0}$ (zero vector)

- The zero-vector is still a vector, but with zero magnitude.

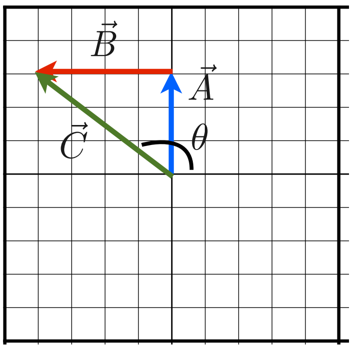
<u>Ex. Vector Addition:</u>

- Calculate the sum of the two vectors $\vec{A} = (0, 3)$ and $\vec{B} = (-4, 0)$ with the magnitude and direction

$$\vec{C} = (0, 3) + (-4, 0) = (0 - 4, 3 + 0) = (-4, 3)$$

$$|\vec{C}| = \sqrt{4^2 + 3^2} = 5$$

$$\cos(\theta) = \frac{C_x}{|\vec{C}|} = \frac{-4}{5} \quad \Rightarrow \quad \theta = 143\,\mathrm{deg}$$



Steps in Vector Addition /Subtraction:

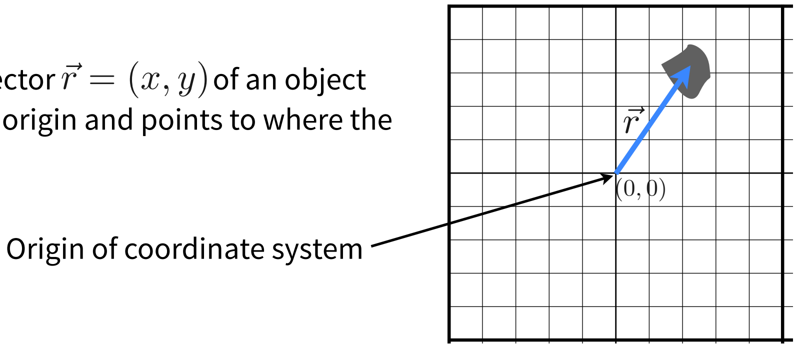
1) Make a drawing of the vectors, if not given.
2) Find the components of each vector, if not given.
3) Perform the addition or subtraction for each component.
4) If needed, combine all components to get final vector, the magnitude, and direction.

Kinematics in 2D:

- Now that we know how to deal with vectors, we can study the motion of objects in more than one dimension.

- We start by identifying a single point in our coordinate system called the **origin**, that defines the (x,y)=(0,0) location.

- We can define the position vector $\vec{r} = (x, y)$ of an object as the vector that starts at the origin and points to where the objects center of mass is.



- As the object moves, the objects (x,y) location changes in time. This change in position can be used to define the velocity vector

$$\vec{v} = (v_x, v_y) \qquad v_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \qquad v_y = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

- If the objects velocity also changes in time, then we can define the acceleration vector in a similar way:

$$\vec{a} = (a_x, a_y) \qquad a_x = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} \qquad a_y = \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t}$$

- Like displacement vectors, the individual components of any vector are independent of the others.

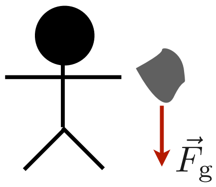
- Keep in mind that the position, velocity, and acceleration vectors can point in different directions.

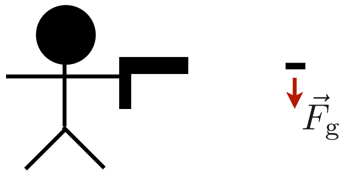
Case #1: Constant force, free fall, projectile motion.

- We have already discussed the force of gravity in one-dimension
- Gravity always acts vertically, resulting in a constant acceleration $g_{\mathrm{E}} = 9.8\ \mathrm{m/s}^2$
- Since vector components are independent, a vertical acceleration can only change the vertical velocity of an object.

➡ - When the only force is gravity, the horizontal and vertical components of an objects motion are completely independent.

Ex. Suppose a rock is dropped from 2m high, and a bullet is shot from the same height with a horizontal velocity of 500m/s. How long does it take for the rock and bullet to hit the ground?





$$y = \frac{1}{2} g_{\mathrm{E}} t^2 \qquad\qquad y = \frac{1}{2} g_{\mathrm{E}} t^2$$

$$t = \sqrt{\frac{2y}{g_{\mathrm{E}}}} \qquad\qquad t = \sqrt{\frac{2y}{g_{\mathrm{E}}}}$$

Both objects hit the ground at the same time!

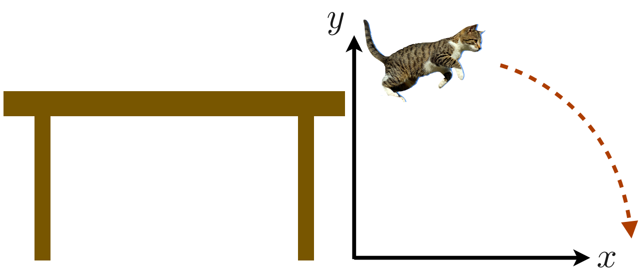
- In general, for an object with only the force of gravity acting on it, we have the following vector components if we define +y to be in the up direction:

$$\vec{r} = \left([x_0 + v_{0x} t], [y_0 + v_{0y} t - \frac{1}{2} g_{\mathrm{E}} t^2]\right)$$

$$\vec{v} = \left(v_{0x}, [v_{0y} - g_{\mathrm{E}} t]\right)$$

$$\vec{a} = (0, -g_{\mathrm{E}})$$

Ex. Suppose a cat initially running horizontally with a velocity of 1.6m/s runs off a table 0.8m high. (a) How long is the cat in the air? (b) How far does it travel horizontally before it lands? (c) What is its velocity just before it hits the ground?



Part (a)

- Once the cat leaves the table, the only acceleration on the cat is from gravity

$$y(t) = y(0) + v_{0y} t + \frac{1}{2} g t^2$$

$$0 = 0.8 + 0 - \frac{1}{2} g_{\mathrm{E}} t^2 \quad \Rightarrow \quad t = \sqrt{\frac{2 \cdot 0.8\ \mathrm{m}}{9.8\ \mathrm{m/s}^2}} = \boxed{0.4\ \mathrm{s}}$$

Part (b)

- Now that we know how long the cat is in the air, we can use that time to find how far in the +x-direction the cat moved.

$$x(t) = x(0) + v_{0x}t = v_{0x}t = 1.6 \text{ m/s}(0.4 \text{ s}) = \boxed{0.64 \text{ m}}$$

Part (c)

- The velocity of the cat, just before landing has two components. The x-component is given in the problem $v_x = v_{0x} = 1.6 \text{ m/s}$

- The y-component can be calculated from the gravitational acceleration and time

$$v_y = -g_{\text{E}}t = -9.8 \text{ m/s}^2(0.4 \text{ s}) = -3.9 \text{ m/s}$$

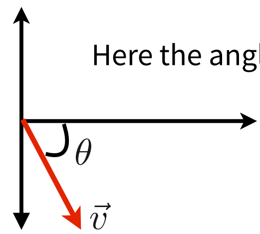
- Therefore, the final velocity vector can be written as $\vec{v} = (1.6, -3.9)(\text{m/s})$

or in terms of magnitude and direction:

$$|\vec{v}| = \sqrt{1.6^2 + 3.9^2} = 4.2 \text{ m/s}$$

$$\theta = \cos^{-1}\left(\frac{1.6}{4.2}\right) = 68 \text{ deg}$$

Here the angle is below the x-axis



Ex. A football is kicked with a speed of 40m/s at an angle of 40 degrees above the x-axis. (a) Find the velocity after 1s. (b) Find the time for the ball to hit the ground.

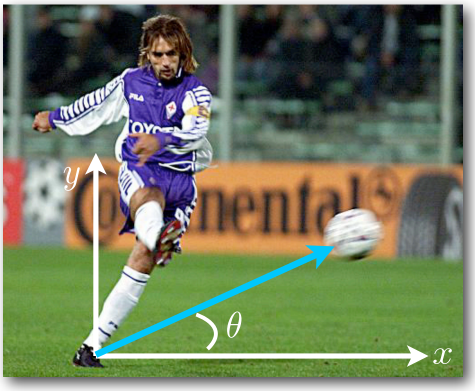
Part (a)

- Since the initial velocity is not pointed along the x or y axis, we need to find the components of the velocity

$$v_{0x} = |\vec{v}|\cos\theta = 40\cos(40 \text{ deg}) = 30.64 \text{ m/s}$$

$$v_{0y} = |\vec{v}|\sin\theta = 40\sin(40 \text{ deg}) = 25.71 \text{ m/s}$$

$$\vec{v}(0) = (30.64, 25.71)(\text{m/s})$$



- Again, the only force is due to gravity pointing in the -y-direction. Therefore v_x is constant

- In the y-direction: $v_y(t) = v_y(0) - \frac{1}{2}g_{\text{E}}t = 25.71 - 9.8(1^2) \text{ m/s} = 15.91 \text{ m/s}$

- Therefore, at t=1s, the velocity is: $\boxed{\vec{v}(1) = (30.64, 15.91)(\text{m/s})}$

Part (b)

- Remember that the x and y components are independent

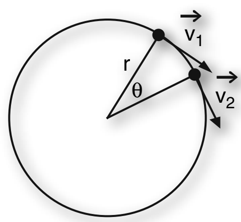
- Therefore, the time it takes to fall, depends only on the initial velocity,acceleration, and height in y-direction only.

- Since the ball starts and stops on the ground (y=0) the time it takes for the ball to hit the ground is

$$0 = v_{0y}t - \frac{1}{2}g_{\mathrm{E}}t^2 \quad \Rightarrow \quad t = \frac{2v_{0y}}{g_{\mathrm{E}}} = \frac{2(25.71\ \mathrm{m/s})}{9.8\ \mathrm{m/s^2}} = \boxed{5.2\ \mathrm{s}}$$

Case #2: Circular Motion:

- Having moved into more than one-dimension, we can now look at circular motion.



Object undergoing circular motion

- In circular motion, the direction of the velocity vector is always changing as a function of time.

- Since the velocity is changing, there must be a nonzero acceleration.

- Here we will consider **uniform circular motion**:

$$|\vec{v}_1| = |\vec{v}_2| = v \qquad \text{(constant magnitude)}$$

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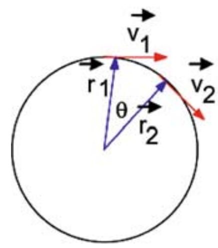
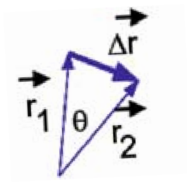


- Consider the object at two different times t_1 and t_2 separated by $\Delta t = t_2 - t_1$

- In the time Δt, the object has traveled a distance equal to $v\Delta t$ and has gone through an angle θ

- During this same time, the velocity vector has also rotated by the angle θ



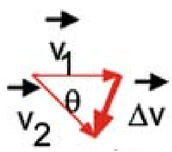
Change in position

- Because the angles are the same, we can write the relation

$$\frac{|\Delta\vec{v}|}{|\vec{v}_1|} = \frac{|\Delta\vec{r}|}{|\vec{r}_1|}$$

- Since the radius and magnitude of velocity are constant:

$$|\Delta\vec{v}| = \frac{v}{r}|\Delta\vec{r}|$$

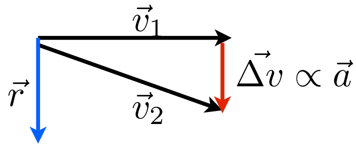


Change in velocity

- Now divide both sides by Δt and let $\Delta t \to 0$. Then $|\Delta\vec{r}|/\Delta t \to v$ and

$$\boxed{a_{\mathrm{cent}} = \lim_{\Delta t \to} \frac{|\Delta\vec{v}|}{\Delta t} = \frac{v^2}{r}}$$

- As $\Delta t \to 0$, the <u>acceleration points toward the center of the circle</u>



- Therefore, the acceleration in uniform circular motion is called **radial acceleration**, or **centripetal acceleration** (centripetal = “points to center of circle”)

- This acceleration has constant magnitude, but the direction is <u>always</u> changing because the direction of the velocity is always changing.

- We can not use our formulas for constant acceleration to solve circular motion problems.

<u>Ex.</u> A protein molecule is spinning in ultracentrifuge at 80,000 rpm at a fixed distance of 5cm from the axis of rotation. Find the acceleration of the protein, and express the acceleration in terms of g_{E}

The radius is 0.05m, therefore we just need to convert rpm to m/s

$$v = 8\times 10^4 \frac{\text{rev}}{\text{min}}\cdot\frac{1\ \text{min}}{60\ \text{s}}\cdot\frac{2\pi r}{\text{rev}} = 420\ \text{m/s}$$

Then the acceleration is

$$a_{\text{cent}} = v^2/r = \boxed{3.5\times 10^6\ \text{m/s}^2}$$

- In terms of the gravitational acceleration

$$\frac{a_{\text{cent}}}{g_{\mathrm{E}}} = \boxed{360000}$$

The protein feels an acceleration that is 360,000 times stronger than gravity

<u>Dynamics in 2D:</u>

- With the aid of vector mathematics, it is easy to extend Newton’s laws to work in two or more dimensions.

- Newton’s 1st and 3rd laws do not require any change

    - Objects will still move at a constant velocity $\vec{v}$ unless there is a force acting

    - For every external force on an object, the object pushes back with a force that is equal in magnitude and in the opposite direction.

- Newton’s 2nd law is a vector equation: $\dfrac{\vec{F}_{\text{net}}}{m} = \vec{a}$

- Now that we know how vectors work, it is easy to solve Newton’s 2nd law

    - Add all force vectors to get net force and divide by the mass to get acceleration.

Steps to solving problems in two or more dimensions:

(1) First, make a drawing of the problem, if there is not one, and identify the object whose motion you want to study.

(2) Identify all of the forces on the object (only that object), by drawing a force diagram.

(3) Write down the equations of motion; write down the components of Newtons 2nd law. Make sure the x and y components are separate

(4) Once the equations of motion are found. Solve for the variables that you do not know.

(5) If possible, check your results.

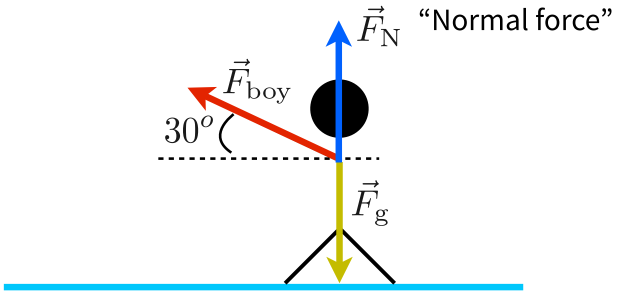
Ex. Suppose there is a boy and girl on ice skates. The boy pulls the girl at an angle of 30 deg with a a force of 30N. What is the girls acceleration if the girls mass is 40kg?

(1) Draw the picture, focus on the girl.

(2) Draw all the forces on the girl.

The reaction force on an object that opposes the force of gravity is called the "**Normal Force**"

"Normal" = "Perpendicular"



(3) Find the y and x components for the forces.

$$F_y = F_N + F_{\rm boy}\sin(30\,{\rm deg}) - F_{\rm g} = 0$$ (In the y-direction)

$$F_x = F_{\rm boy}\cos(30\,{\rm deg})$$ (In the x-direction)

(4) Solve for the acceleration in the x-direction (there is no acceleration in the y-direction because the girl does not come of the ice

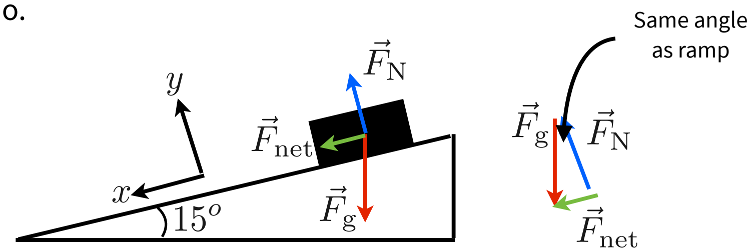
$$a_x = \frac{F_{\rm boy}\cos(30\,{\rm deg})}{m} = \frac{26\ {\rm N}}{40\ {\rm kg}} = \boxed{0.65\ {\rm m/s^2}}$$ (to the left)

Ex. A piano of mass 100kg slides down a frictionless ramp that is 5m long and at an angle of 15 deg. If the piano starts from rest (v=0), what this the speed of the piano at the bottom.

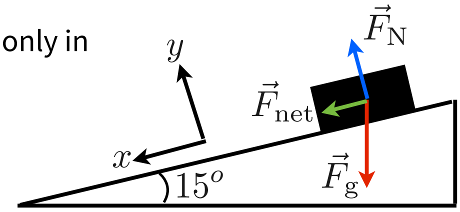
(1) Draw the picture, focus on the piano.

(2) Draw all the forces on the piano.

- Pick good coordinate system



(3) Write down the equations of motion. Here we are interested only in the x-direction



$$F_x = F_{\mathrm{g}} \sin(15\,\mathrm{deg})$$

$$a_x = \frac{F_{\mathrm{g}} \sin(15\,\mathrm{deg})}{m} = g_{\mathrm{E}} \sin(15\,\mathrm{deg}) = 9.8 * (0.26)\ \mathrm{m/s^2} = 2.54\ \mathrm{m/s^2}$$

- Since the initial velocity is zero, and we know how far the piano moves (x=5m) we can use

$$v^2(t) = 2a_x x = 2(2.54\ \mathrm{m/s^2})(5\ \mathrm{m}) \quad \Rightarrow \quad \boxed{v = 5.0\ \mathrm{m/s}}$$

Work and Energy:

- Work and energy are scalar quantities, so we might think that we do not need to work about vectors.

- However, work is the product of a force and a displacement, both of which are vectors.

- In this section we generalize our definition of work to work in more than one-dimension.