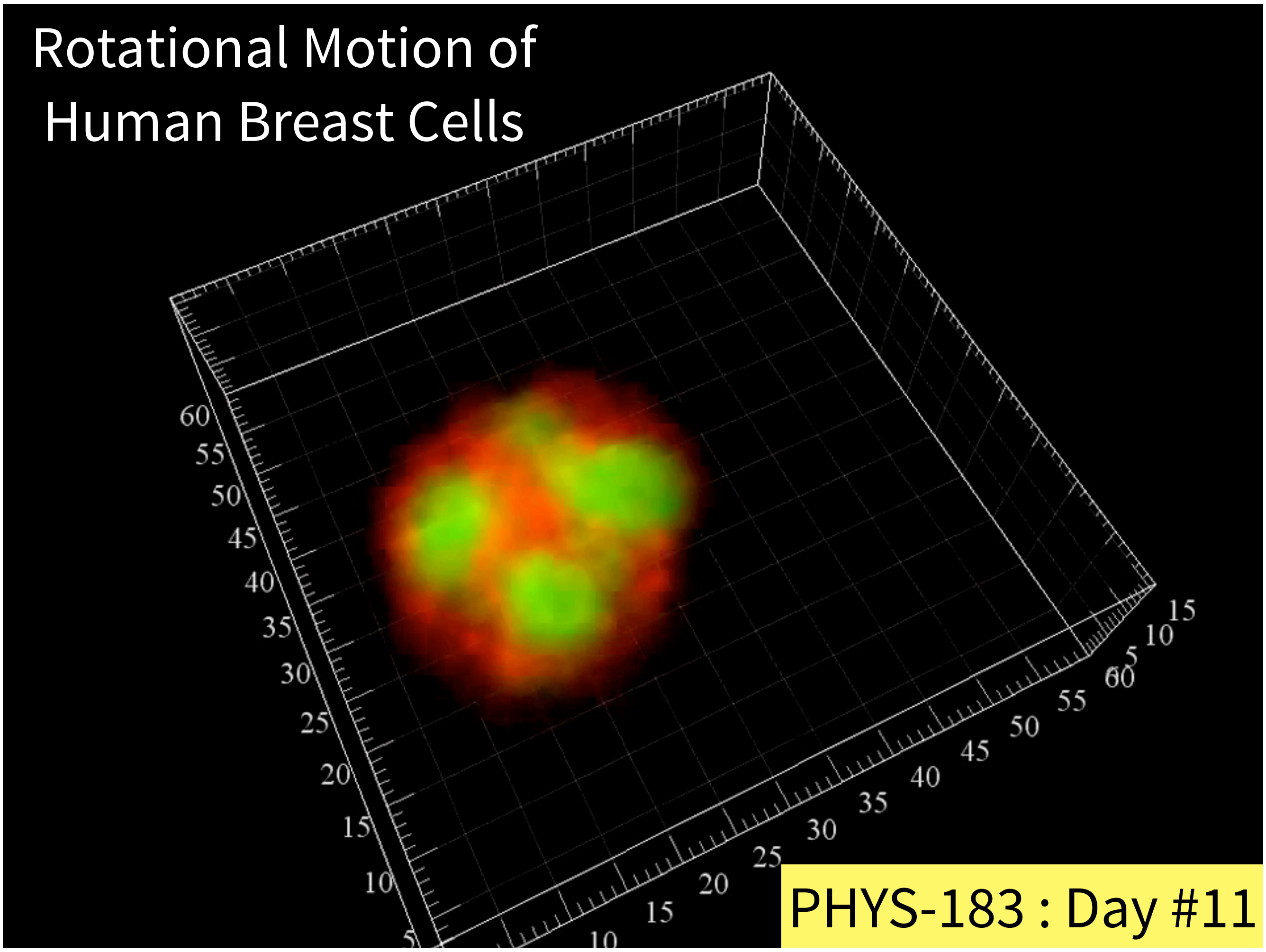


Rotational Motion:

- Once translational motion is accounted for, all other motion of an object can be described in a reference frame where the center of mass (CM) is fixed at the origin.

Motion of birds in a reference frame where the CM is (approximately) fixed.

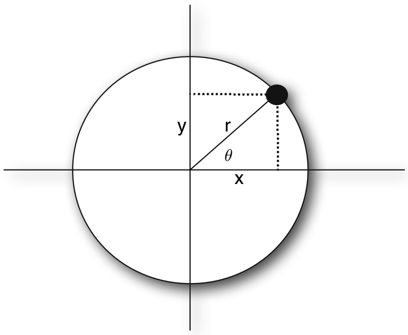
- Here we look at motion about the center of mass viewed from a one of these reference frames

- For a solid object (can not change shape), any motion can be written as a sum of translational and rotational motion about the CM.

- We begin by looking at pure rotational motion about a fixed **axis of rotation**.

- A solid object has motions that are limited to pure translations and rotations about its CM.

- We want to describe pure rotational motion about the CM like the motion of a particle moving on a circle.



Motion of particle on a circle.

- We could use x- and y-coordinates, but it is easier to use the radius of the circle (r) and the angle from the x-axis (θ).

- Since the radius is constant, using these polar coordinates (r,θ) there is really only one variable θ.

- The velocity of the θ variable is known as the **angular velocity**

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t}$$

- The angular velocity measures the **angular displacement** $\Delta\theta$ as a function of time.

- The angular momentum must be measured in radians (radians have no units).

- If the particle travels at a constant speed, doing uniform circular motion, then the instantaneous value of ω is equal to the average value $\bar{\omega}$.

- The angular velocity can be positive or negative, just like 1D velocity.
  - For can pick the +/- sign for whatever direction you want, but you must keep your choice.

- If the speed of the particle changes in time, then the angular velocity will vary, and we need to define the **instantaneous angular velocity**

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

- The distance traveled by the particle along the circumference of the circle Δs is proportional to the angular displacement $\Delta\theta$ from the relation $s = r\theta$. Therefore, the velocity of the particle is

$$v = \frac{\Delta s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r\omega$$

- To get the motion of the particle, we also need to define the angular acceleration. Just like the acceleration measures change in velocity, the angular acceleration measures changes in angular velocity.

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t}, \quad \alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

- We can also connect the linear acceleration to the angular acceleration using the radius of the particles motion.

$$a_{\text{tang}} = r\alpha \qquad \text{(tangential direction)}$$

- The linear acceleration is called the **tangential acceleration** since it is parallel to the tangential velocity.

- The tangential acceleration is responsible for the change in the particles velocity.

- The tangential acceleration is not the same as the centripetal acceleration

$$a_{\text{cent}} = \frac{v^2}{r} \qquad \text{(radial direction)}$$

- The tangential acceleration is zero in uniform circular motion, but $a_{\text{cent}} \neq 0$.

- It is useful to rewrite the centripetal acceleration using the angular frequency.

- For extended objects, such as a car wheel, the velocity is different for different parts of the wheel depending on the distance from the axis of rotation.

- In these cases, it is better to write the centripetal acceleration as

$$a_{\text{cent}} = \frac{v^2}{r} = \omega^2 r$$

- Now that we have the three angular variables θ, ω, α , needed to describe angular motion, we can derive the angular equations of motion for a constant α

- Since we know how θ, ω, α are related to r, v, a, we can just divide our original equations by the radius to get

$$\boxed{\begin{aligned} \omega(t) &= \omega(0) + \alpha t \\ \omega^2(t) &= \omega^2(0) + 2a\left[\theta(t) - \theta(0)\right] \\ \theta(t) &= \theta(0) + \omega(0)t + \frac{1}{2}\alpha t^2 \end{aligned}}$$

(Only in the case of constant angular acceleration)

Bacteria driven by rotation of their flagella.

Ex. Helical bacterial flagella drive E. Coli bacteria at a constant speed when they rotate counterclockwise at a uniform angular velocity. Occasionally, the flagella switch rotational directions, causing the bacteria to stop, before they again move in the counterclockwise direction. If the motor rotates at 4Hz in both directions and takes 5ms to switch directions, what is the average angular acceleration during this time?

Solution:

- The initial angular velocity is $\omega(0) = -2\pi(4)\ \mathrm{rad/s}$ (set counterclockwise to be negative)

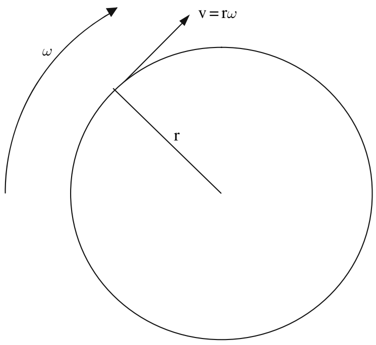
- The final angular velocity is $\omega(t) = +2\pi(4)\ \mathrm{rad/s}$

- Therefore, the average angular acceleration is:

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega(t)-\omega(0)}{\Delta t} = \frac{2\pi(4)-2\pi(-4)}{5\times 10^{-3}\ \mathrm{s}} = \boxed{1.0\times 10^{4}\ \mathrm{rad/s^2}}$$

Rotational Energy:

- We are now in a position to ask what is the kinetic energy of an object undergoing rotational motion about a fixed axis of rotation?



- In this case all parts of the object rotate about this axis

- Let us consider a single particle moving in a circle

- Using our equation for kinetic energy $T = (1/2)mv^2$, and the fact that the velocity can be written in terms of angular velocity $v = \omega r$

$$T = \frac{1}{2}m(\omega r)^2$$

- Or defining the moment of inertia $I = mr^2$ for a single particle we have

$$T = \frac{1}{2}I\omega^2$$ (**Rotational Kinetic Energy**)

- There is a relationship between $m \leftrightarrow I$ and $v \leftrightarrow \omega$.

- If the particle travels in uniform circular motion then kinetic energy is fixed, otherwise not.

Ex. A 25kg girl rides on the outer edge of a merry-go-round with a 10m diameter and has a rotational kinetic energy of 20J. What is the girls moment of inertia relative to the axis of rotation (assume the girl can be modeled as a single particle)? How many revolutions does the ride make per minute?

Solution:

- The girls moment of inertia is $I = mr^2 = 25(5)^2 = \boxed{625\ \mathrm{kg}\cdot\mathrm{m}^2}$

- To get the number of rotations per minute, we need to use the rotational kinetic energy

$$T = \frac{1}{2}I\omega^2 \rightarrow \omega = \sqrt{\frac{2T}{I}}$$

- Therefore, $\omega = \sqrt{2(20)/625} = 0.25\ \mathrm{rad/s}$

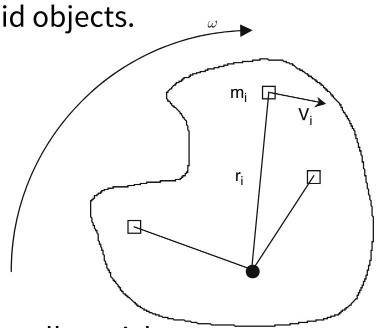
- In one minute the girl goes around an angle of $\theta = \omega(60) = 0.25(60) = 15\ \mathrm{rad}$

- Finally, the revolutions per minute is: $15\ \mathrm{rad}/2\pi = \boxed{2.7\ \mathrm{rev/min}}$

- In the previous example, we modeled the girl as a single particle.
    - Obviously this is not a good approximation.

- Next we want to generalize our discussion to included extended solid objects.

- As a simple model, consider a collection of particles, each with mass m_i attached together by massless rods.



- If the whole system rotates about the same axis, then each particle also has a radius r_i.

- The total rotational kinetic energy of our system is then the sum over all particles.

$$T = \sum_i T_i = \frac{1}{2}\sum_i m_i v_i^2$$

and because $v_i = r_i\omega$ we have

$$T = \frac{1}{2}\sum_i (m_i r_i^2)\omega^2$$

- We can define the moment of inertia for the entire system as $I = \sum_i (m_i r_i^2)$

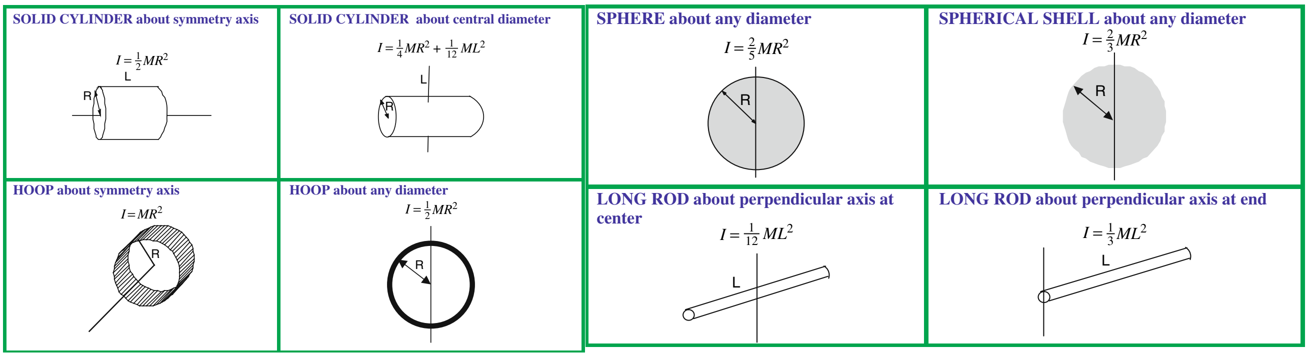
- Therefore, the kinetic energy keeps its simple form $T = (1/2)I\omega^2$

- For any solid object, we can break up the object into little pieces and find the moment of inertia

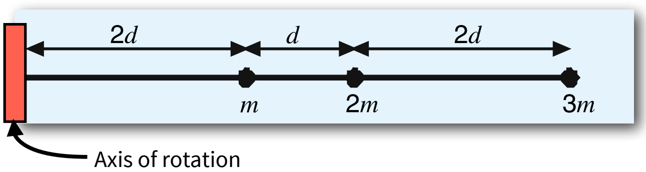
- In the limit where the size of the pieces becomes really small: $I = \int r^2 dm$

- Or, if the object has a constant density ρ : $I = \rho \int r^2 dV \leftarrow$ Integration over the volume of object

Table of common moments of inertia:



Ex. Calculate the moment of inertia for the system shown below. The masses are attached by a massless rod and rotate about the left end of the rod. Use m=1.5kg and d=0.2m. Also calculate the moment of inertia if the object rotates about the center



Solution:

- Using our equation for the moment of inertia, we can simply add each masses contribution

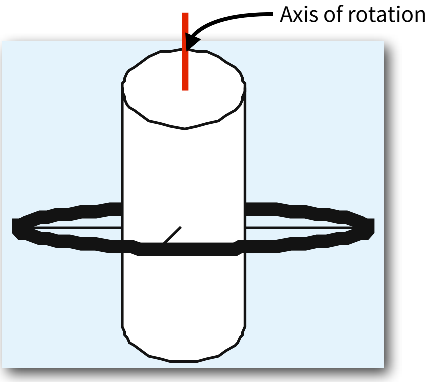
$$I = \sum_i m_i r_i^2 = m(2d)^2 + 2m(3d)^2 + 3m(5d)^2 = 97md^2 = \boxed{5.8\ \mathrm{kg}\cdot\mathrm{m}^2}$$

- Switching the axis of rotation to the middle of the rod, the moment of inertia becomes

$$I = m\left(\frac{d}{2}\right)^2 + 2m\left(\frac{d}{2}\right)^2 + 3m(2.5d)^2 = 19.5md^2 = \boxed{1.2\ \mathrm{kg}\cdot\mathrm{m}^2}$$

- This makes sense as the masses in the second part are traveling in smaller radii circles.

Ex. Find the moment of inertia for the object at right. The cylinder has a mass M, radius r, and length L, whereas the hoop has a mass M/10 and radius 3r. Use M=0.1kg, r=0.05m, and L=0.25.



Solution:

- Since all of the mass of the hoop is at the same radius, the moment of inertia easy to calculate

$$I_{\text{hoop}} = \frac{M}{10}(3r)^2$$

- The moment of inertia for the cylinder must be looked up from the table

$$I_{\text{cylinder}} = \frac{1}{2}Mr^2$$

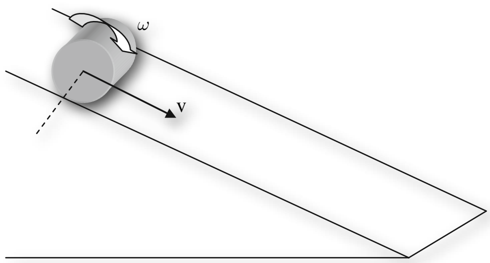
- The total moment for the system is just the sum of the two moments:

$$I = \frac{1}{2}Mr^2 + \frac{M}{10}(3r)^2 = 1.4Mr^2 = \boxed{3.5\times10^{-4}\ \text{kg}\cdot\text{m}^2}$$

Conservation of Energy:

- We can now look at the conservation of energy in objects that have both translational and rotational motion.

- A solid object that rolls has two components for its kinetic energy



$$T = \underbrace{\frac{1}{2}mv^2}_{\text{translational kinetic energy}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{rotational kinetic energy}}$$

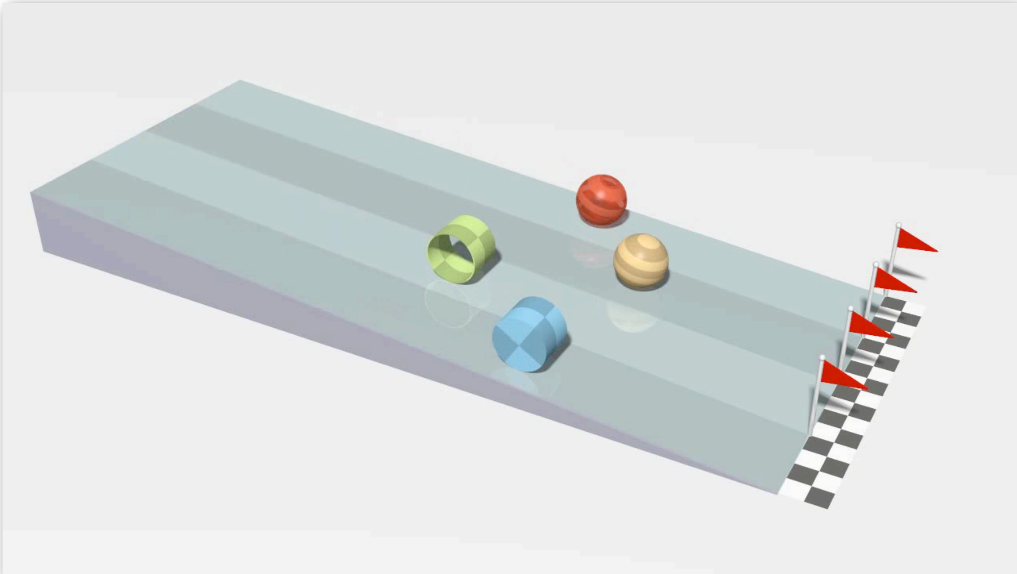
- If all of the forces action on the object are conservative forces (no friction), then the conservation of energy can be written as

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + U = E = \text{constant}$$ (**Conservation of mechanical energy**)

Key Idea: If the object begins to rotate, then the rotational energy increases, and the translational kinetic energy is reduced (if mass is fixed, velocity goes down).

- The amount of rotational kinetic energy depends on the objects moment of inertia and angular velocity.

- What if we have objects with different moments of inertia?



- The solid sphere wins because it has the smallest moment of inertia

→ The sphere has more mass located closer to the axis of rotation.

Ex. Suppose that the hoop and cylinder have the same radius r and mass, both roll down the inclined plane, with angle θ, from rest at a height H. What is the velocity of both objects at the bottom?

Solution:

- We can use conservation of energy to solve this problem

- All the initial energy is potential energy, so the conservation of energy for each object can be written as

$$mgH = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

- The angular velocity of each object is also related to the objects velocity $v = \omega r$

- Solving for the velocity in the energy conservation equation ($\omega = v/r$)

$$v^2 = \frac{2mgH}{(m + I/r^2)}$$

- To get the velocity for each object, we need to look up the moment of inertia for each

$$I_{\rm hoop} = mr^2 \qquad\qquad I_{\rm cylinder} = \frac{1}{2}mr^2$$

- Now we can solve for the final velocity of each object:

$$v_{\rm hoop} = \sqrt{\frac{2mgH}{(m+m)}} = \sqrt{gH} \qquad v_{\rm cylinder} = \sqrt{\frac{2mgH}{(m+m/2)}} = \sqrt{\frac{4gH}{3}}$$

- The final velocities <u>do not</u> depend on the mass, radius, or angle θ!

- Obviously the cylinder, with the larger final velocity, reaches the bottom first.

- Note that if the objects did not spin, then both objects would reach the bottom at the same time, with the same final velocity.

<u>Ex.</u> An empty bucket of m=1kg mass is in a well, attached by a massless rope over a pulley is released from rest. If the pulley is a 0.15m uniform cylinder of M=10kg mass, find the speed of the bucket as it hits the water 12m below.

Well and bucket

<u>Solution:</u>

- The total initial energy is equal to the potential energy of the bucket

$$U = mgh$$

- Since there is no friction, the total mechanical energy is constant

$$mgh = \frac{1}{2}m_{\rm b}v^2 + \frac{1}{2}I\omega^2$$

where v is the velocity of the bucket as it hits the water, ω is the pulleys angular velocity, and

$$I = (1/2)Mr^2 \qquad \text{(for a cylinder)}$$

- Since the bucket moves the pulley, we know that the velocity of the bucket is related to the angular velocity of the pulley as $v = \omega r$

- Then we can rewrite the energy conservation as

$$mgh = \frac{1}{2}mv_{\rm b}^2 + \frac{1}{2}\left(\frac{1}{2}Mr^2\right)\omega^2 = \frac{1}{2}\left(m + \frac{1}{2}M\right)v^2$$

- Solving for the buckets velocity we have $v = \sqrt{\dfrac{2mgh}{(m+\frac{1}{2}M)}}$

- Plugging in the corresponding numbers $v = \sqrt{\dfrac{2\cdot 9.8\cdot 12}{(1+0.5\cdot 10)}} = \boxed{6.3\ \mathrm{m/s}}$

- If the pulley had no mass (no rotational kinetic energy) then the buckets velocity would have been

$$v = \sqrt{2gh} = 15.3\ \mathrm{m/s}$$

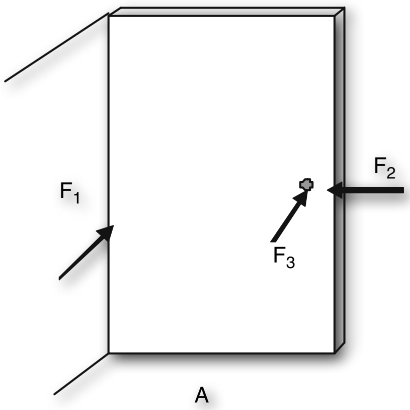
- We see that the velocity of the bucket has been reduced because some of the total energy has gone into rotating the pulley.

Torque and Rotational Dynamics:

- In this section, we will now explain how rotational motion is produced.

- To produce translational motion, we require a net external force to act on the object.

- However a force, no matter how large, does not necessarily result in the rotation of the object



Only F3 results in rotation of the door

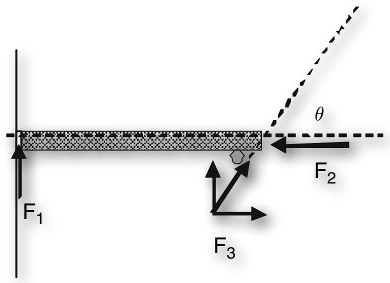
- Pushing on the side of a door (F2) does not result in rotation.

- Pushing on the part of the door near the axis of rotation (F1) also does nothing to rotate the door.

- Only pushing on the section of the door near the door handle (F3) results in the door opening.

- So only some forces give rise to rotations of an object.

- To understand what kind of forces result in rotations, lets look at the door from the top.



- Suppose that force F3 acts at an angle θ from the horizontal.

- Breaking up F3 into components, we see that only the component that is perpendicular to door causes rotations

- We also know that if this component of force is closer to the axis of rotation, then the force is less effective at rotating the door

- From the work-energy theorem, we know that change in kinetic energy is equal to the net external work.

- If the object only rotates, then the change in kinetic energy is all rotational

$$W_{\text{net,ext}} = \Delta T_{\text{rot}} = \Delta \frac{1}{2} I \omega^2$$

- Suppose that for a short time Δt, a net force causes an amount of work ΔW that changes the angular velocity $\omega \rightarrow \omega + \Delta\omega$. Then the work-energy theorem is:

$$\Delta W_{\text{net,ext}} = \frac{1}{2} I (\omega + \Delta\omega)^2 - \frac{1}{2} I \omega^2 = I\omega\Delta\omega + \frac{1}{2} I (\Delta\omega)^2$$

- We are interested in taking the limit as $\Delta t \rightarrow 0$. But in this case $\Delta\omega \rightarrow 0$ as well.

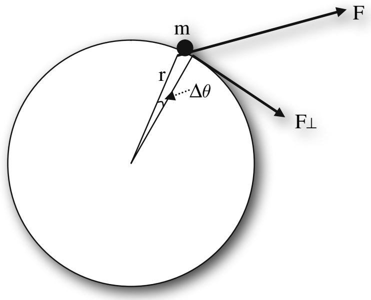
- The second term on the right goes to zero faster, so we will ignore it.

$$\Delta W_{\text{net,ext}} = I\omega \frac{\Delta\omega}{\Delta t} \Delta t$$

- Now writing $\omega \Delta t = \Delta\theta$ and $\Delta\omega / \Delta t = \alpha$ that are valid when $\Delta t \rightarrow 0$:

$$\Delta W_{\text{net,ext}} = I\alpha\Delta\theta$$

- In our case of pure rotational motion, all points of the object move on circles



- From the general definition of work $\Delta W = F_{\mathrm{net,ext}}\Delta x$, we know that net external force for a particle on a circle does the work

$$\Delta W = F_{\perp} r \Delta\theta$$

where $F_{\perp}$ is the force perpendicular to the radial vector, and $\Delta x = r\Delta\theta$ is the distance over which the force acts.

- We can now define the rotational version of the force called the **torque** to be:

$$\tau = F_{\perp} r$$

so that the net work done for pure rotational motion is

$$\boxed{\Delta W = \tau_{\mathrm{net,ext}}\Delta\theta}$$

- Compare this to the translational work: $\Delta W = F_{\mathrm{net,ext}}\Delta x$