- At the end of last class we saw that, for pure rotational motion, the change in work is proportional to the torque

$$\Delta W = \tau_{\rm net,ext}\Delta\theta$$

- Torque has units of N-m. These are the same units as the Joule. However no one uses Joules for torque, since it is different from an energy.

- In terms of the moment of inertia, the torque can also be written as: $\tau_{\rm net,ext} = I\alpha$

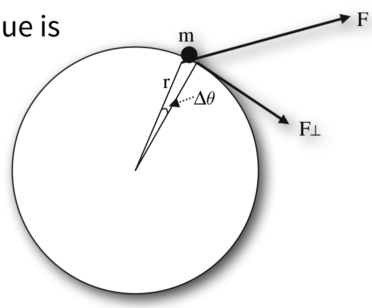
- This is the rotational version of Newton's 2nd law (F=ma).

- Since the torque depends on I, the rotational response of a system depends on how the mass is distributed in the object.

- Lets compare the translational and rotational versions of Newton's 2nd law

- For a particle moving in a circle due to force F, $I = mr^2$, and the torque is

$$\tau = mr^2\alpha$$



- And the torque is also related to the perpendicular component of the applied force $F_\perp$

$$\tau = F_\perp r$$

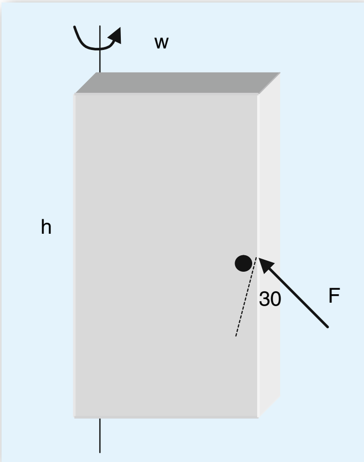
- Equating the two expressions and solving for the tangential force:

$$F_{\perp} = ma_{\rm tang} = m\alpha r$$

- We can equally describe the rotation of a simple particle using tangential accelerations and forces

- However, for anything more complicated than a single particle, you must use torque and angular variables to solve the problem.

<u>Ex.</u> Lets consider the problem of opening a door. Suppose the door has a mass m=10kg and a height h=2.5m and width w=1m. The moment of inertia for the door is $I = (1/3)mw^2$. Supposed the door is pushed with a horizontal force F=5N at and angle of 30deg ($\pi/6$ rad) from perpendicular to the door. Find the angular acceleration of the door, and the time it takes the door to open ($\pi/2$).



<u>Solution:</u>

- A constant torque acts to push the door open, and therefore there is a constant angular acceleration.

- The torque is given by: $\tau = F_{\perp}r = F\cos(\pi/6)w = 5\cos(\pi/6) = 4.3\ \mathrm{N\cdot m}$

- The angular acceleration is constant and given by:

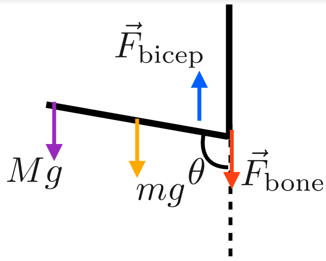
$$\alpha = \frac{\tau}{I} = \frac{4.3}{(1/3)(10)(1)^2} = \boxed{1.3\ \mathrm{rad/s^2}}$$

- Using this constant acceleration we can find the time for the door to rotate by $\pi/2$

$$\theta = \frac{1}{2}\alpha t^2 \quad \text{or} \quad t = \sqrt{\frac{2\theta}{\alpha}} = \sqrt{\frac{2(\pi/2)}{1.3}} = \boxed{1.6\ \mathrm{s}}$$

<u>Ex.</u> Calculate the forces that the biceps and the upper arm bone (Humerus) exert on a persons forearm when supporting a weight as shown without any movement. Here the mass is M=20kg, the length of the persons forearm is 0.4m with mass m=2kg, and the bicep muscles are attached at a distance of d=0.04m from the elbow. The arms make an angle of $2\pi/3$ from the vertical.

<u>Solution:</u>



- In order to solve for the two unknown forces, we must realize that <u>both</u> the net force and the net torque must be zero.

  - There is no translational or rotational motion.

- We have will have two equations for two unknowns

- The equation for the net torque has three components:

$$\tau_{\rm net} = Mg[L\sin(2\pi/3)] + mg[\frac{L}{2}\sin(2\pi/3)] - F_{\rm biceps}[d\sin(2\pi/3)] = 0$$

- We can divide both sides by $\sin(2\pi/3)$ and solve for $F_{\rm biceps}$:

$$F_{\rm biceps} = \left(Mg + \frac{mg}{2}\right)\frac{L}{d} = \frac{[(20)(9.8) + 2(9.8)/2]\,0.4}{0.04} = \boxed{2058\ \rm N}$$

- The equation for the net force is: $Mg + mg + F_{\rm bone} - F_{\rm bicep} = 0$

- We can now solve for the force from the arm bone to find: $\boxed{F_{\rm bone} = 1842\ \rm N}$

- We see that to lift a relatively small weight of 196N requires the muscles and bones in your body to exert a force that is almost 10 times stronger!

<u>Angular Momentum:</u>

- In our last chapter on momentum, we were able to rewrite Newton's second law for a system

$$\vec{F}_{\rm net,ext} = m\vec{a}_{\rm cm}$$

in terms of the CM momentum so that the net external force was equal to the rate of change in the total linear momentum

$$\vec{F}_{\rm net,ext} = \lim_{\Delta t \to 0}\frac{\Delta\vec{P}_{\rm total}}{\Delta t}$$

- If the net force is zero, then this equation shows us that the total momentum of our system is conserved.

- Now we want to introduce another important physical quantity called the **angular momentum**

- The linear momentum is given by $\vec{P} = m\vec{v}$, what do you think the angular momentum formula looks like?

$$\boxed{\vec{L} = I\,\vec{\omega} \qquad \textbf{Angular Momentum}}$$

- To begin, again we look at a single single particle traveling in a circle.

- We know that the moment of inertia is $I = mr^2$ so we can rewrite the angular momentum as

$$L = mr^2\omega = rm(r\omega) = r(mv) = rp$$

- For a system of many particles, where the moments of inertia add together, the angular momentum also adds

$$L_{\text{total}} = \sum_i r_i p_{i,\perp}$$

This has only the momentum perpendicular to the radius, because only the perpendicular forces cause rotations.

- Just like before where we saw that the net external force is related to the rate of change in momentum, we can relate the torque (angular version of force) to the angular momentum

$$\tau_{\text{net,ext}} = \lim_{\Delta t \to 0} \frac{\Delta L_{\text{total}}}{\Delta t}$$

- If there is no net torque on an object, then the total angular momentum of the system remains constant; **The angular momentum is conserved if there is no net torque**.

- For a solid object undergoing pure rotational motion, conservation has a simple form

$$\vec{L} = I\vec{\omega} = \text{constant} \qquad \text{(No external torques)}$$

- The conservation of angular momentum is responsible for your ability to ride a bicycle

- The wheel tries to conserve angular momentum which is a vector quantity

Ex. An ice skater begins a spin by rotating at an angular velocity of 2rad/s with both arms and one leg stretched out as shown. At that time, her moment of inertia is $I = 0.5\ \mathrm{kg}\cdot\mathrm{m}^2$. She then brings her arms over her head and her legs together so that her moment of inertia is $0.2\ \mathrm{kg}\cdot\mathrm{m}^2$. What is her final angular velocity?

Solution:

- Because there is no external torque on the skater, her angular momentum must be conserved.

$$I_{\mathrm{init}}\omega_{\mathrm{init}} = I_{\mathrm{fin}}\omega_{\mathrm{fin}}$$

- In this case her final moment of inertia decreases, so the final angular velocity must increase

$$0.5(2) = 0.3\omega_{\mathrm{fin}} \quad \Rightarrow \quad \boxed{\omega_{\mathrm{fin}} = 3.3\ \mathrm{rad/s}}$$

- The conservation of angular momentum also controls the the rotational motion of a ballerina or diver as they change their moment of inertia with their body.

Worlds fastest ice skating spin, ~5 rev/sec!

Ex. A 5m radius merry-go-round with no friction and a moment of inertia of $2500\ \mathrm{kg}\cdot\mathrm{m}^2$ is turning at 2rpm when the motor is turned off. If there are 10 children, each with mass m=30kg, initially at the outer edge of the ride (r=5m), (a) what is the angular velocity if all of the children move to a radius of 1m? Assume the children are particles. (b) Find the torque required to stop the ride in 10s.

Solution:

- Before the torque is applied to stop the ride, there are no external torques

- So again, we can use conservation of angular momentum

$$L_{\mathrm{init}} = I_{\mathrm{init}}\omega_{\mathrm{init}} = (I_{\mathrm{ride}} + I_{\mathrm{children}})\omega_{\mathrm{init}} = I_{\mathrm{init}}\omega_{\mathrm{init}} = [I_{\mathrm{ride}} + 10(mr^2)]\omega_{\mathrm{init}}$$

$$L_{\mathrm{init}} = (10^4\ \mathrm{kg}\cdot\mathrm{m}^2)(0.2\ \mathrm{rad/s}) = 2000\ \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s} = \mathrm{L}_{\mathrm{final}} = \mathrm{I}_{\mathrm{final}}\omega_{\mathrm{final}}$$

- To get the final angular velocity we need to calculate the moment of inertia with the children moved to r=1m

$$I_{\mathrm{final}} = 2500 + 10(30)1^2 = 2800\ \mathrm{kg}\cdot\mathrm{m}^2 \quad \Rightarrow \quad \omega_{\mathrm{final}} = \frac{2000\ \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}}{2800\ \mathrm{kg}\cdot\mathrm{m}^2} = \boxed{0.71\ \mathrm{rad/s}}$$

- When the torque is applied to slow the ride down, it will produce an angular deceleration

$$\alpha = \frac{\tau}{I}$$

- Since we know the moment of inertia, all we need to find is the angular deceleration

- Given that we know the angular velocity before slowing down from part (a), and we know how long it takes to stop, we can solve for the deceleration.

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{-0.71}{10} = -0.071\ \mathrm{rad/s^2}$$

- Substituting into the top equation we can solve for the torque

$$\tau = I\alpha = 2800(-0.071) = \boxed{-199\ \mathrm{N\cdot m}}$$

Relationship Between Translational and Rotational Motion

Applicability	*Rotational*	*Translational*	*Relations Between Variables*
$\alpha(a)$ = constant	$\omega = \omega_o + \alpha t$	$v = v_o + at$	$s = r\theta$
$\alpha(a)$ = constant	$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o)$	$v^2 = v_o^2 + 2a(x - x_o)$	$v = \omega r$
$\alpha(a)$ = constant	$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2$	$x = x_o + v_o t + \frac{1}{2}at^2$	$a_{\text{tang}} = r\alpha$
General	$\text{KE} = \frac{1}{2}I\omega^2$	$\text{KE} = \frac{1}{2}mv^2$	$I = \sum m_i r_i^2$
General	$\tau_{\text{net,ext}} = I\alpha$	$F_{\text{net,ext}} = ma$	$\tau = rF_{\perp} = r_{\perp}F$
General	$\tau_{\text{net,ext}} = \frac{\Delta L_{\text{total}}}{\Delta t}$	$F_{\text{net,ext}} = \frac{\Delta P_{\text{total}}}{\Delta t}$	$L = I\omega = rp_{\perp}$
General	$\tau_{\text{net,ext}} = 0 \Rightarrow L_{\text{total}}$ = constant	$F_{\text{net,ext}} = 0 \Rightarrow P_{\text{total}}$ = constant	

Rotational Diffusion and Cell Membrane Dynamics:

- We have seen that microscopic objects in a liquid will undergo random translational motion due to random collisions with the fluid molecules due to their thermal motion.

- There will also be random rotational motion around the CM due to random torques acting on the objects.

- In random translational motion there was a friction force proportional to the velocity

$$F_F = -fv$$

- For random rotational motion, there is a frictional torque that is proportional to the angular velocity

$$\tau_F = -f_R\omega$$

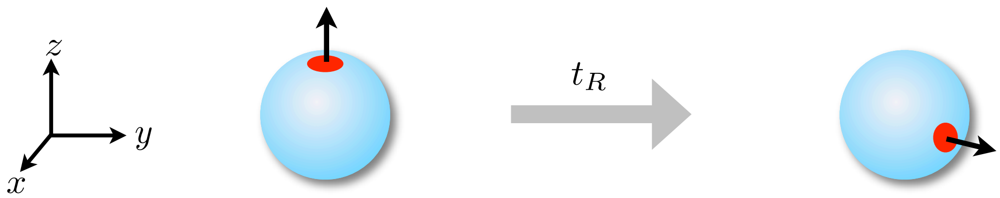
- In both cases, the shape of the object, and the properties of the fluid determine the friction

- For spherical microscopic objects in a fluid, the **rotational friction coefficient** f_R is

$$f_R = 8\pi\eta r^3 \qquad \eta = \text{fluid viscosity}$$

- These random frictional torques will lead to rotational diffusion.

- Rotational diffusion is measured by the amount of time it takes for the orientation of the object to be completely randomized



Object with arrow pointed in specific direction

Object with arrow pointed in random direction

- This time is called the **rotational relaxation time** t_R and it is proportional to the rotational friction coefficient

- For very small objects, this time is very fast: $t_R \approx 10^{-12} - 10^{-9}$ sec

- For macroscopic objects this time can be larger: $t_R \approx 10^{-3}$ or larger

- There is also a number called the **rotational diffusion coefficient** D_R that is proportional to the temperature and inversely proportional to

$$\boxed{D_R = \frac{1}{2t_R} = \frac{k_B T}{f_R}} \qquad \begin{aligned} k_B &= 1.38\times10^{-23}\ \mathrm{kg\cdot m/K\cdot s^2} \\ T &= \text{temperature (Kelvin)} \end{aligned}$$

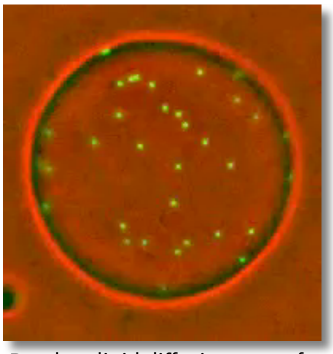
- Both rotational and translational diffusion play an important role in the dynamics of cell membranes.

- A cell membrane is made out of a bunch of lipid (fat) molecules that have a head that is attracted to water (hydrophilic) and a tail that does not like water (hydrophobic)

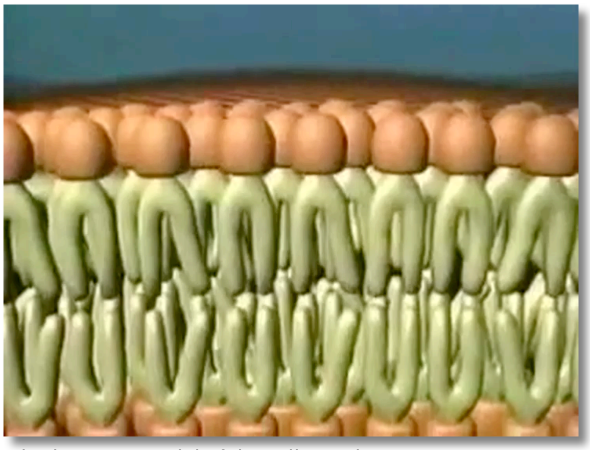
- In addition, there are proteins in the cell membrane that move molecules in and out of the cell.

- Both the lipids and protein are free to move around in the cell membrane. This is called the **fluid-mosaic model**

- The lipids are free to do translational diffusion along the surface



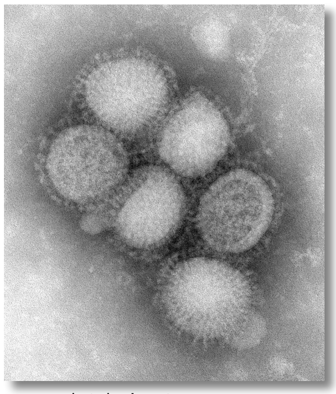
Random lipid diffusion on surface



Fluid-Mosaic model of the cell membrane

- The lipids can also (rarely) flip up and down due to rotational diffusion.

Ex. An almost spherical virus is in water at a temperature of 20C (293K). using laser light, the rotational diffusion time can be calculated to be 0.2ms. Calculate the radius of the virus. For the viscosity of water use the value $\eta_{H_2O} = 0.001\ \mathrm{kg/m \cdot s}$



H1N1 (Pig) Flu Virus

Solution:

- From our previous discussion we know that the rotational diffusion time is related to the rotational diffusion constant

$$t_R = \frac{1}{2D_R}$$

- Using our expression for D_R and the rotational friction coefficient for a spherical object we have

$$t_R = \frac{8\pi\eta r^3}{2k_B T}$$

- We can now rearrange the equation to solve for the radius and plug in our numbers

$$r = \left(\frac{k_\mathrm{B} T t_R}{4\pi\eta}\right)^{1/3} = \left(\frac{1.38\times10^{-23}\cdot 293\cdot 2\times10^{-4}}{4\pi\cdot 0.001}\right)^{1/3} = \boxed{85\ \mathrm{nm}}$$

Static Equilibrium:

- In the special case where both the momentum and and angular momentum of an object are conserved, the object is said to be in **equilibrium**.

$$\vec{P} = \text{constant} \quad \text{and} \quad \vec{L} = \text{constant} \qquad \text{(In equilibrium)}$$

- This obviously includes the special case where $P = L = 0$ and the object is at rest.

- In this case, the object is said to be in **static equilibrium**.

- In terms of both forces and torques, the condition for equilibrium implies that both the net external force and torque are zero

$$F_{x,\mathrm{net}} = 0 \qquad F_{y,\mathrm{net}} = 0 \qquad \tau_{\mathrm{net}} = 0$$

Steps for solving static equilibrium problems:

1) Draw an external force diagram and label all of the forces

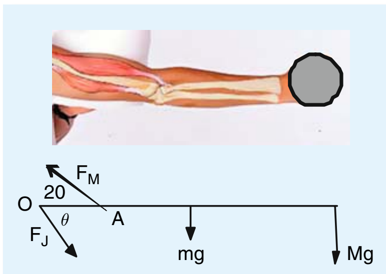
2) Determine which forces you know and what forces you don't know

3) Write the appropriate equations for $F_{\mathrm{net}} = 0$ in every dimension

4) Write the appropriate equations for $\tau_{\mathrm{net}} = 0$ about a single axis of rotation

5) Solve all of the equations for the unknown quantities.

Ex. Consider an arm with a dumbbell held horizontally with length L. The forces are the weight of the dumbbell, the weight of the arm, the pull of the arm (deltoid) muscle at position *A* and an angle of 20deg, and the force of the shoulder joint F_J at some unknown angle θ. If the arm is uniform and weighs 50N, and the dumbbell 75N, find the force from both the arm muscle and the joint. Assume *A*=L/4.



Solution:

1) Draw an external force diagram and label all of the forces

- Why does the should joint have a downward force?

2) Determine which forces you know and what forces you don't know

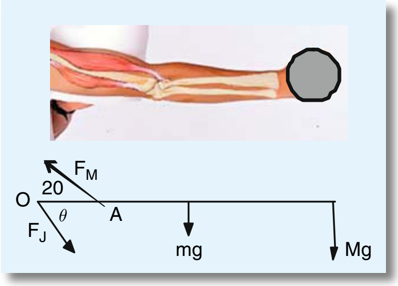
- Three unknowns: F_M, F_j, θ ➡ We need three equations

3) Write the appropriate force equations in every dimension

$$\sum F_{\mathrm{horiz}} = F_J \cos\theta - F_M \cos(20\,\mathrm{deg}) = 0$$

$$\sum F_{\mathrm{vert}} = F_M \sin(20\,\mathrm{deg}) - F_J \sin\theta - mg - Mg = 0$$

4) Write the appropriate equation for torque about a single axis of rotation



- Using *O* as the axis of rotation:

$$\sum \tau_O = mg(L/2) + MgL - F_M(L/4)\sin(20\,\mathrm{deg}) = 0$$

5) Solve all of the equations for the unknown quantities.

- We can solve the torque equation for the muscle force F_M

$$F_M = (mg/2 + Mg)\left(\frac{4}{\sin(20\,\mathrm{deg})}\right) = \boxed{1200\ \mathrm{N}}$$

- Now we use this value in the two force equations:

$$F_J \sin\theta = F_M \sin(20\,\mathrm{deg}) - mg - Mg = 280\ \mathrm{N}$$

$$F_J \cos\theta = F_M \cos(20\,\mathrm{deg}) = 1100\ \mathrm{N}$$

- We can first solve for the angle θ by dividing the first equation by the second

$$\tan\theta = \frac{280\ \mathrm{N}}{1100\ \mathrm{N}} = 0.25 \quad \Rightarrow \quad \boxed{\theta = 14\,\mathrm{deg} \quad \mathrm{or} \quad 0.24\ \mathrm{rad}}$$

- Finally, we can now solve for the joint force

$$\boxed{F_J = 1132\ \mathrm{N}}$$

- Again, we see that the force that the muscles must provide is around 10 times larger than the force of the dumbbell in the persons arm.

- Because your body must produce these large forces, this means that your muscles and joints can be easily injured if you are not careful.