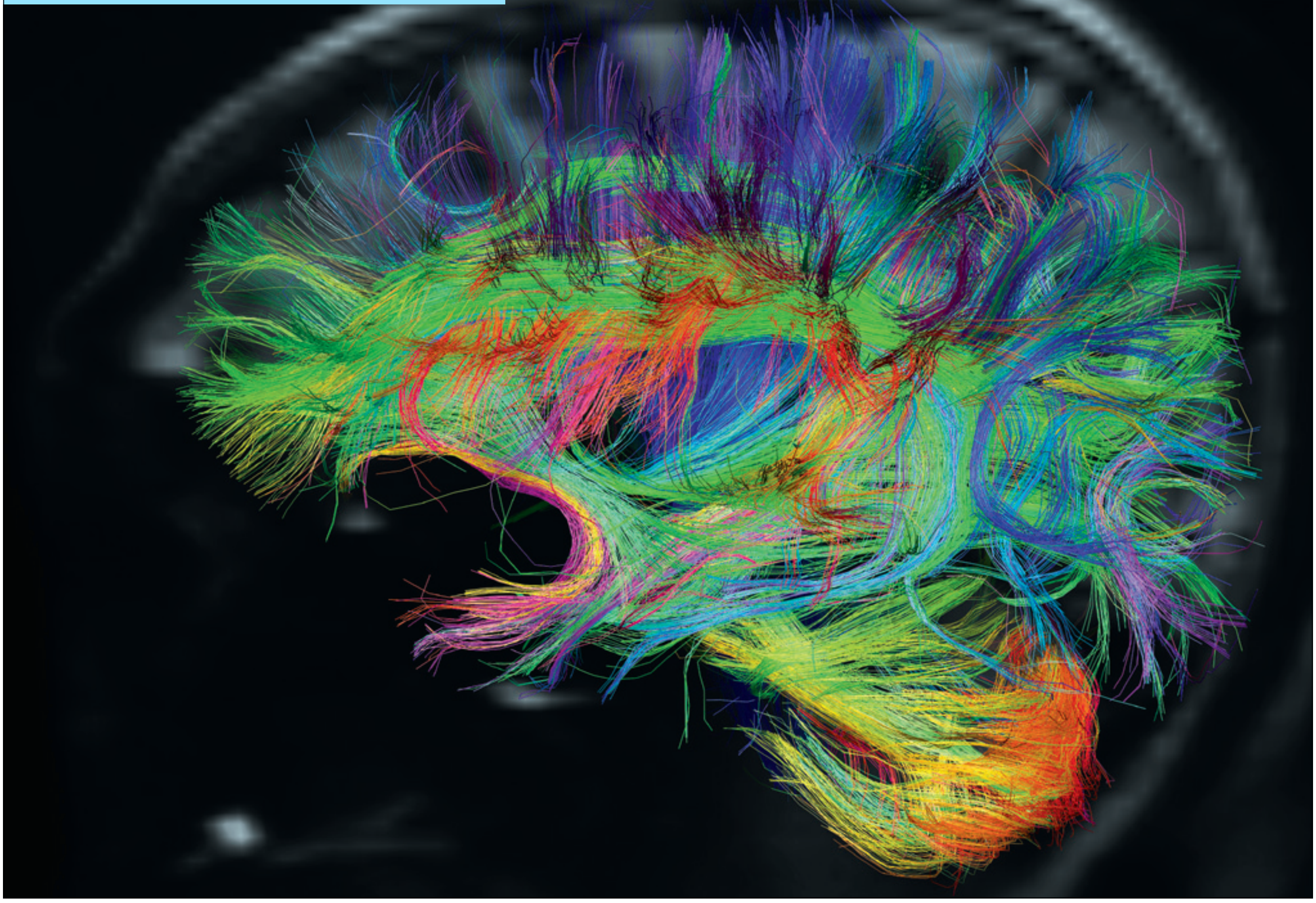
PHYS-183 : Day #26



Applications of Ohm's Law - Electrical Circuits:

- Now that we have learned about electrical current and resistance, we are now prepared to analyze some basic circuits.

- Our goal is to obtain equations for how the current through our circuit behaves as a function of time given that we know the batteries voltage and the resistance of the circuit.

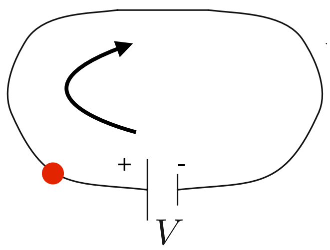
- To analyze any circuit device we must use the rules known as **Kirchoff's Laws**

- The first rule is called **Kirchoff's voltage law** (KVL)

The drop in electrical potential around any closed loop in the circuit must be equal to zero.

- The change in voltage must be zero because you start and stop at the same place

$$\Rightarrow \sum_i V_i = 0$$ along any closed loop



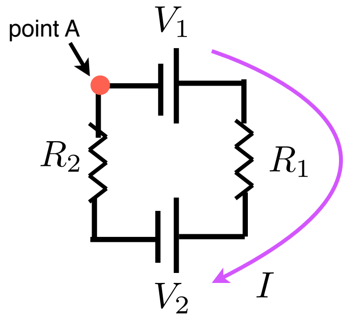
- Some potential terms (voltages) must be zero.

- Changes in potential occur only across the circuit elements (i.e battery resistors, ...)

- In order to use KVL, we must first decide the direction of the circuit.

- For simple circuits, we can guess the direction, for more complicated circuits we may not know.

Always assume that the current moves in the clockwise direction around a circuit loop



A complicated circuit where the current direction is not known.

- We must also decide what the sign of the potential across each device will be:

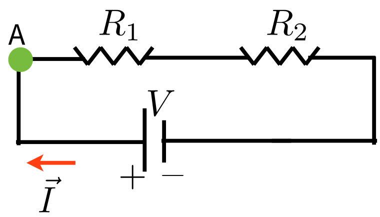
If going from low -> high potential: voltage is positive (+)

If going from high -> low potential: voltage is negative (-)

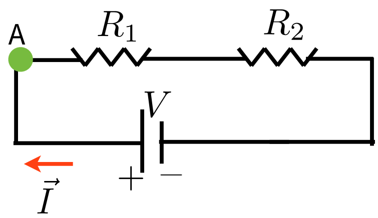
- Finally, go around each loop in the circuit in the same direction as the current goes (clockwise)

- Lets now use these rules to analyze a circuit with a battery and two **resistors in series**.

-As always, we assume that the current goes in the clockwise direction, and starting at point A, we go around the loop adding voltages in the same direction as the current



- Before we do do this we use charge conservation to note that the current through each of the circuit elements must be the same.



- Writing KVL for this circuit in the clockwise direction:

$$-V_{R_1} - V_{R_2} + V = 0$$

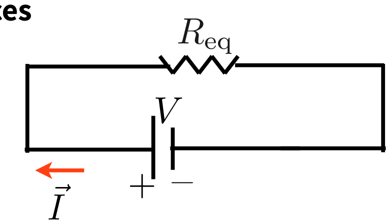
- Using Ohm's law and the fact that the current is the same we have

$$-IR_1 - IR_2 + V = 0$$

$$V = IR_1 + IR_2 = I(R_1 + R_2) \equiv IR_{eq}$$

- **If we have two or more resistors <u>in series</u>, we can simplify them into a single resistor where the total resistance is the sum of the individual resistances**

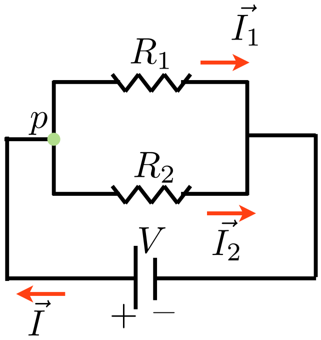
$$R_{eq} = \sum_i R_i$$



- In this case the current $I = V/R_{eq}$ is positive because we picked the correct direction for the current (clockwise)

- If you get a negative current at the end of you calculation, then the current is actually going in the opposite direction (counter-clockwise)

- In addition to having two resistors in series, we can have two **resistors in parallel**.

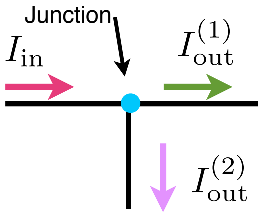


- At point P, the current splits into two components I1 and I2.

- By conservation of electric charge we must have

$$I = I_1 + I_2$$

- This can be generalized as Kirchoff's Current Law (KCL):

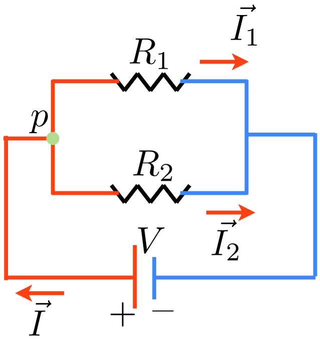


The current flowing into any junction in the circuit must be equal to the total current exiting the junction

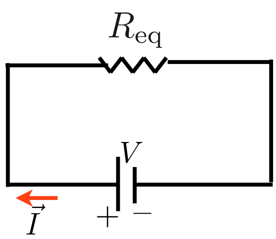
$$I_{\mathrm{in}} = \sum_k I_{\mathrm{out}}^{(k)}$$

- It is also important to note that for two resistors in parallel, the voltage drop across both resistors is the same

- This is easy to see if you draw the circuit using a different color for each value of the potential.



- Using KCL and the fact that the voltage across the two resistors is the same, solving for the total current we find



$$I = I_1 + I_2 = \frac{V}{R_1} + \frac{V}{R_2} = V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \equiv \frac{V}{R_{\mathrm{eq}}}$$

- **For two resistors <u>in parallel</u>, the resistors can be combined into a single equivalent resistor with the total resistance given by**

$$\frac{1}{R_{\mathrm{eq}}} = \sum_i \frac{1}{R_i}$$

<u>To summarize:</u>

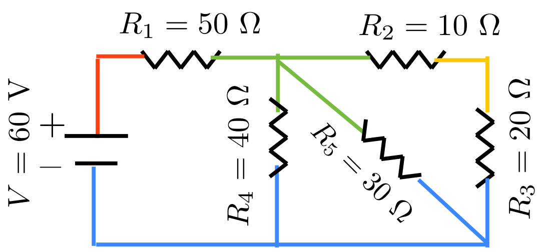
KVL : The voltage drop along any closed circuit loop is zero $\quad \sum_i V_i = 0$

KCL : The current entering any part of the circuit must equal the current leaving that part $\quad I_{\mathrm{in}} = \sum_k I_{\mathrm{out}}^{(k)}$

For **resistors in series**: $\quad R_{\mathrm{eq}} = \sum_i R_i$

For **resistors in parallel**: $\quad \frac{1}{R_{\mathrm{eq}}} = \sum_i \frac{1}{R_i}$

Ex. Simplifying resistors:



- What is the equivalent resistance of the 5 resistors?

- Use color for potentials (as before)

-> R4 & R5 in parallel

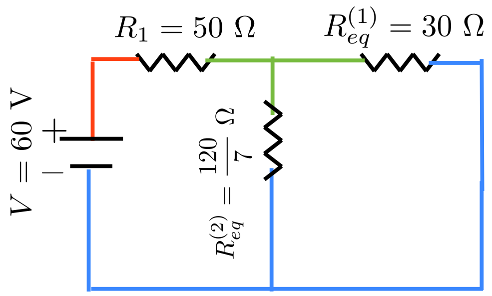
-> R2 & R3 in series

- R5 is connected in the middle of R1,R2 & R4 -> we cannot combine R1 and R4, or R1 and R2 yet

- Start with R2 and R3 (or R4 and R5): $R_{eq}^{\text{series}} = R_2 + R_3 = 30\ \Omega$

- Now do R4 and R5: $\dfrac{1}{R_{eq}^{\text{para}}} = \dfrac{1}{R_4} + \dfrac{1}{R_5} = \dfrac{7}{120} \quad \rightarrow \quad R_{eq}^{\text{para}} = \dfrac{120}{7}\ \Omega$

- Redraw the circuit:



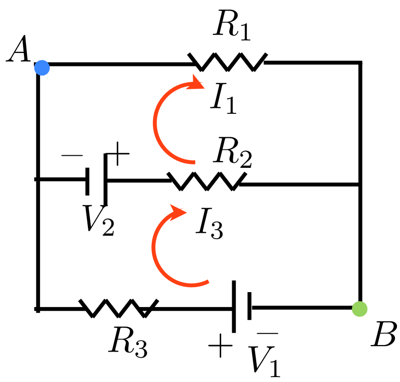
- Now $R_{eq}^{(1)}$ and $R_{eq}^{(2)}$ are in parallel, do them next:

$$\frac{1}{R_{eq}^{(3)}} = \frac{1}{R_{eq}^{(1)}} + \frac{1}{R_{eq}^{(2)}} = \frac{11}{120} \quad \rightarrow \quad R_{eq}^{(3)} = \frac{120}{11}\ \Omega$$

- We are left with R1 in series with $R_{eq}^{(3)}$

$$R_{eq}^{\text{total}} = R_1 + R_{eq}^{(3)} = 50\ \Omega + \frac{120}{11}\ \Omega = \boxed{\frac{670}{11}\ \Omega}$$

Ex. Solving circuits:



$$R_1 = 10\ \Omega \qquad R_2 = 20\ \Omega \qquad R_3 = 30\ \Omega$$

$$V_1 = 15\ \mathrm{V} \qquad V_2 = 9\ \mathrm{V}$$

Question: What is the power through each of the resistors if the resistors obey Ohm's Law?

- For Ohmic resistors we know: $P = I^2 R$

- Must find the currents flowing through each resistor.

Step #1: Pick the directions for the currents I_1 and I_3

- Current through resistor #2 (I_2) will obviously be a combination of I_1 and I_3

Step #2: Pick a starting point, and go around each loop in the direction of the currents from step #1.

For current #1 (starting at A): $-I_1R_1 - I_1R_2 + I_3R_2 - V_2 = 0$

For current #3 (starting at B): $V_1 - I_3R_3 + V_2 - I_3R_2 + I_1R_2 = 0$

- We are left with two equations and two unknown variables (I1 & I3)

-We need to solve the equation for current #1 to get I1 in terms of I3 and other given variables, and then plug into the equation for current #3.

$$\boxed{I_1 = \frac{3}{110}\ \mathrm{A}} \qquad \boxed{I_3 = \frac{54}{110}\ \mathrm{A}}$$

- Since I3 >I1, the current I2 is simply I3-I1:

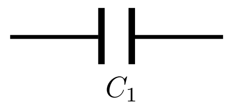
$$\boxed{I_2 = I_3 - I_1 = \frac{54}{110} - \frac{3}{110} = \frac{51}{110}\ \mathrm{A}}$$

- Having found the currents, the power is easy to evaluate:

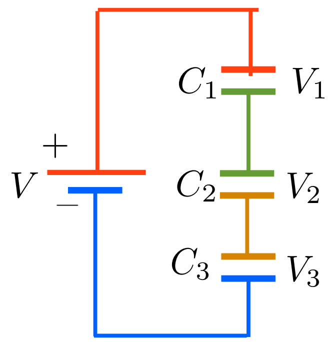
$$\boxed{\begin{aligned} P_1 &= I_1^2 R_1 = 0.0074\ \mathrm{W} \\ P_2 &= I_2^2 R_2 = 4.30\ \mathrm{W} \\ P_3 &= I_3^2 R_3 = 7.23\ \mathrm{W} \end{aligned}}$$

-We can also consider simple circuits consisting of capacitors and a battery.

Capacitor:

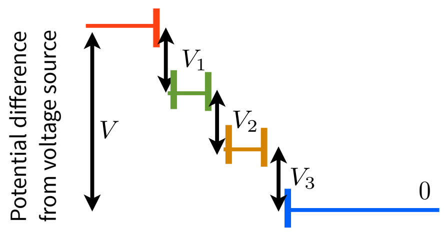


- First let use consider a circuit with three **capacitors in series**:



- No longer have just two different voltages.

- Now have three separate voltage drops

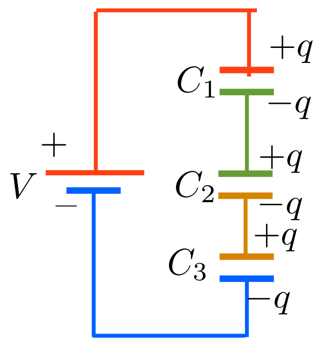


- Remember that only voltages across circuit elements matter

- What is the charge on each capacitor?

  - If there is no voltage source initially, all parts of the circuit are neutral

  - Once voltage is connected, a charge builds up on the capacitors



- The voltage causes a charge +q on the plate of C1 at voltage V

- An induced charge of -q is then created on the opposite plate.

- Because the circuit was originally neutral, and not net charge can be created, this induced -q causes +q to move to the opposite plate on C2.

- This happens for <u>every</u> capacitor in the series.

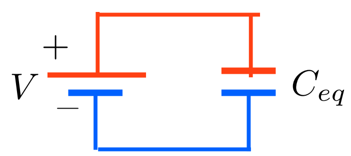
- Since we know $V = V_1 + V_2 + V_3$, each capacitor has charge q and $V = q/C$

$$V = V_1 + V_2 + V_3 = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) = \frac{q}{C_{eq}}$$

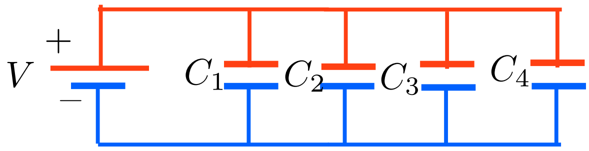
- We see that **a collection of capacitors in series is again equivalent to a single capacitor with capacitance:**

$$\boxed{\frac{1}{C_{eq}^{\text{series}}} = \sum \frac{1}{C_i^{\text{series}}}}$$

Example above becomes



- We can also have **capacitors in parallel** (side by side)



- Each capacitor has the same voltage drop across it

- Capacitance of each element can be different

- The charge on each capacitor is thus:

$$Q_1 = C_1 V \qquad Q_2 = C_2 V \qquad Q_3 = C_3 V \qquad Q_4 = C_4 V$$

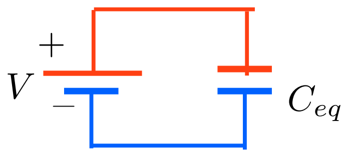
- Since the voltage V is the same for all, lets combine to find the total charge:

$$Q_{\text{total}} = C_1 V + C_2 V + C_3 V + C_4 V = (C_1 + C_2 + C_3 + C_4)\, V = C_{\text{eq}} V$$

- **Capacitors in parallel are equivalent (eq) to a single capacitor with capacitance:**

$$\boxed{C_{eq}^{\text{parallel}} = \sum C_i^{\text{parallel}}}$$

Example above becomes

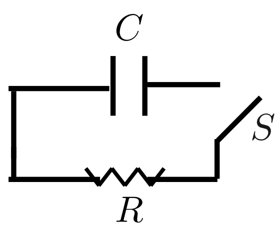


<u>Membrane Electrical Currents:</u>

- Last week we looked at cell membranes as simple capacitors

- This is a very good approximation for a pure phospholipid bilayer that has a very high resistivity of about $10^{15}\ \Omega \cdot cm$

- However, we have seen that the cell membrane is covered with membrane channels that let currents of ions flow across the membrane.



Simple RC-series circuit; a resistor and capacitor in series with a switch *S*

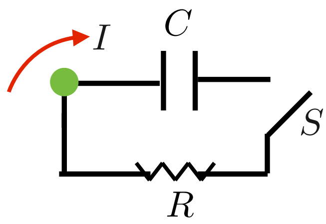
- The simplest model, also called equivalent circuit, for the membrane is an **RC-series circuit**.

- Here we will assume that initially the capacitor is fully charged with a voltage $V_0 = Q_0/C$ and the switch is initially open so no current is flowing

- At time t=0, the switch is closed so that current is allowed to flow. This is equivalent to the membrane channel opening and letting charged ions through

- The capacitor will not lose all of its charge at once, but varies with time in a manner that is determined by the resistance and capacitance of the circuit.

- To find this time-dependence, we must use KVL to find the equation for the circuit



$$-\frac{Q(t)}{C} - I(t)R = 0$$

- Since the current is the rate at which the charges are moving $I = dQ/dt$

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{RC} = 0$$

- This is a differential equation for the amount of charge on the capacitor as a function of time

- In this case, you can verify that $q(t) = Ae^{-t/RC}$ is a solution to the equation

- The charge on the capacitor is given by: $\boxed{Q(t) = Q_0 e^{-\frac{t}{RC}}}$

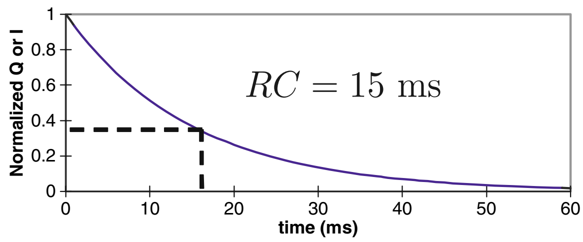
- And we can take the time derivative to get the current: $\boxed{I = I_0 e^{-\frac{t}{RC}}}$ $\quad I_0 = \frac{Q_0}{RC}$

- And since $V = Q/C$: $V(t) = \frac{Q_0}{C} e^{-\frac{t}{RC}} = \boxed{V_0 e^{-\frac{t}{RC}}}$

- The key parameter is the product *RC*, in units of time, called the **RC time constant** $\tau = RC$

- This constant determines the rate at which all the values go to zero.

- Numerically, this time constant determines the amount of time it takes from the initial value of charge or current to decrease to $1/e \sim 0.37$ of its initial value

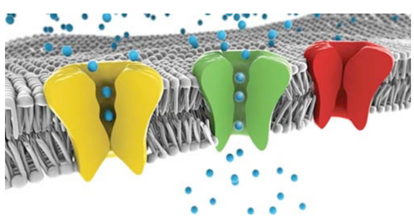


- Lets now apply these ideas to membrane channels to see how many ions go through each channel when it is opened for 1ms

- In the resting state , the membrane RC times are in the range of $10\ \mu\text{s} \rightarrow 1\ \text{s}$

- Typically people use the capacitance per unit area, $C/A \sim 1\ \mu\text{F}/\text{cm}^2$ and the resistance times area $RA = \rho L \sim 10 \rightarrow 10^6\ \Omega \cdot \text{cm}^2$

- These values vary a lot because the number of membrane channels per unit area, and the time that these channels are open, is different for different cell types.



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- Using the resting potential of 0.1V, the typical charge density on the cell surface is

$$\sigma = \frac{Q_0}{A} \sim 0.1\ \mu\mathrm{C/cm^2}$$

- We can get an approximate value for the average current per area through all channels by dividing by the RC time constant.

$$\frac{I}{A} = \frac{Q_0}{A\tau} \sim 100\ \mu\mathrm{A/cm^2} \qquad \tau = 1\ \mathrm{ms}$$

- Using an experimentally measured value of 10^9 channels/cm^2, the current flowing through a single channel is

$$\frac{I}{\mathrm{channel}} = \frac{I}{A}\frac{1}{10^9} = \frac{10^{-4}}{10^9} = 10^{-13}\ \mathrm{A}$$

- Again using a value of 1ms as the time the channel is open, this corresponds to

$$\boxed{600000\ \mathrm{ions/s} = 600\ \mathrm{ions/ms}}$$