PHYS-183 : Day #10

Center of Mass:

- Here we begin to study more complicated systems consisting of two or more particles, and even objects with finite sizes such as a person.

- For any system, there is a well defined point called the **center of mass** (CM) at which the entire mass of the object can be considered to be for translational motion

- For example consider two particles with masses m_1 and m_2 attached by a (massless) rod of length L.

- If we threw this system into the air, it would rotate and translate before it lands.

- The CM of the system would move just like a single particle with mass $(m_1 + m_2)$

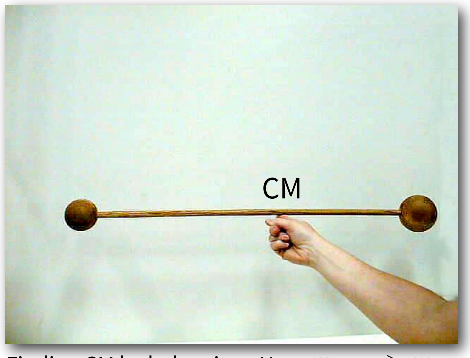
- How do we find the center of mass of a system?

- You can think of the CM as the “balance point” of the system

- If the masses are the same $m_1 = m_2$ then obviously the CM should be in the middle of the masses.

$$x_{\mathrm{CM}} = L/2$$

- If $m_1 > m_2$ then the CM should move closer to mass m1, but how much closer?



Finding CM by balancing. Here $m_{\mathrm{right}} > m_{\mathrm{left}}$

- Find the CM in one direction is like finding the average position of the mass in that direction:

I) Define the origin of your coordinate system

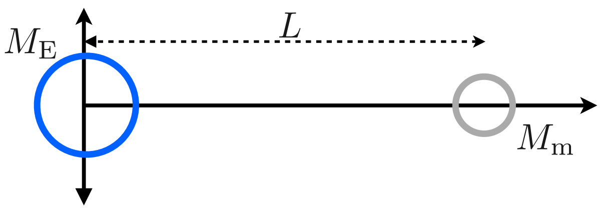
II) Calculated the average position, weighted by the masses

$$x_{\mathrm{CM}} = \left(\frac{m_1}{m_1 + m_2}\right) x_1 + \left(\frac{m_2}{m_1 + m_2}\right) x_2$$

here *x1* and *x2* are the locations of *m1* and *m2* in our coordinate system.

Ex. Find the center of mass of the Earth and moon given that the radius of the Earth is 6.37×10^6 m , the radius of the moon is 1.74×10^6 m, the distance between the Earth and moon is 3.82×10^8 m, and that the Earth is 81.5 times heavier than the moon.

- To make things simple, set origin of coordinates at center of one of the objects.



- Since the separation between Earth and moon is larger than the Earth and moon radius, we can treat each of them as particles.

$$x_{\mathrm{CM}} = \frac{M_{\mathrm{E}}(0) + M_{\mathrm{m}}(L)}{M_{\mathrm{E}} + M_{\mathrm{m}}} = \frac{1}{1 + M_{\mathrm{E}}/M_{\mathrm{m}}} L = 0.012L = 4.63 \times 10^6 \text{ m}$$

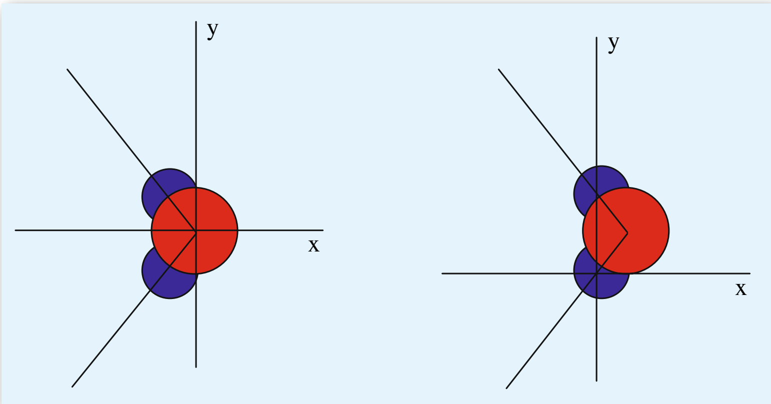
- In this case, the CM of the entire system is still inside the Earth.

- We can generalize how to find the CM to many particles and directions:

$$x_{\mathrm{cm}} = \sum_{i=1}^{N} \left(\frac{m_i}{M}\right) x_i \qquad y_{\mathrm{cm}} = \sum_{i=1}^{N} \left(\frac{m_i}{M}\right) y_i \qquad z_{\mathrm{cm}} = \sum_{i=1}^{N} \left(\frac{m_i}{M}\right) z_i$$

where *M* is the total mass of the system and *N* is the total number of particles.

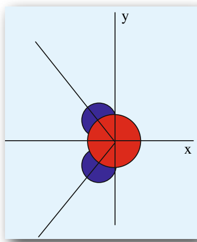
Ex. Find the center of mass of a water molecule using the following information: radius of O= 0.14nm, radius of H=0.12nm, bond length of O-H bond = 0.097nm, and the H-H angle is 104.5 deg.



- We will solve this problem using the Oxygen as center, and then a Hydrogen as center of coordinate system.

- Using Oxygen as the center of our coordinates (0,0) the Hydrogen are located at

$$(-0.097\cos 52.3\,\mathrm{deg}, \pm 0.097\sin 52.3\,\mathrm{deg})\ \mathrm{nm}$$



Coordinates #1

- Taking the mass of Oxygen to be 16amu and the mass of Hydrogen to be 1amu:

$$x_{\mathrm{CM}}^{(1)} = \frac{16(0) + 1(-0.059) + 1(-0.059)}{16+1+1} = \boxed{-0.0066\ \mathrm{nm}}$$

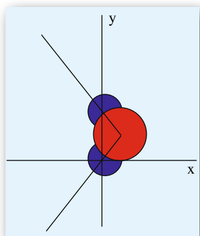
$$y_{\mathrm{CM}}^{(1)} = \frac{16(0) + 1(0.077) + 1(-0.077)}{18} = \boxed{0\mathrm{nm}}$$

We could have guessed this since the molecule looks the same above and below the x-axis.

- Using the lower hydrogen as the center (0,0):

$$(0.097\cos 52.3\,\mathrm{deg}, 0.097\sin 52.3\,\mathrm{deg})\ \mathrm{nm} \quad \text{(Oxygen)}$$

$$(0, 2\cdot 0.097\sin 52.3\,\mathrm{deg})\ \mathrm{nm} \quad \text{(Upper Hydrogen)}$$



Coordinates #2

$$x_{\mathrm{CM}}^{(2)} = \frac{16(0.059) + 1(0) + 1(0)}{18} = \boxed{0.053\ \mathrm{nm}}$$

$$y_{\mathrm{CM}}^{(2)} = \frac{16(0.077) + 1(0) + 1(0.15)}{18} = \boxed{0.077\ \mathrm{nm}}$$

- These answers are equivalent, just written in different coordinate systems.

  - In first coordinate system, the origin of the second is at (-0.059,-0.077)

$$x^{(2)}_{\mathrm{CM}} + (-0.059) = -0.0066 = x^{(1)}_{\mathrm{CM}}$$

$$y^{(2)}_{\mathrm{CM}} + (-0.077) = 0 = y^{(1)}_{\mathrm{CM}}$$

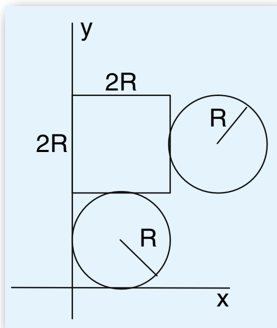
- In the case of large, solid objects, if they uniform throughout and have simple shapes, it is easy to solve for the CM.

density of material

- For uniform simple objects, the mass of the object is: $M = \rho V$ volume of object

- If the object is made from many simple objects, then you just calculate the CM for each simple object and add them together.

Ex. Find the CM of the object on the right if the objects all are made out of the same material, and have thickness t and R=0.1m.



Example object

- Here the volume of each object is $V_i = tA_i$ where *t* is the thickness.

- Since all the thicknesses and densities are the same, they will all cancel

- We only need to know the object areas and the location of the CM for each object.

square lower circle upper circle

$$x_{cm} = \frac{4R^2(R) + \pi R^2 R + \pi R^2(3R)}{4R^2 + \pi R^2 + \pi R^2} = \boxed{0.16\ \mathrm{m}}$$

$$y_{cm} = \frac{4R^2(3R) + \pi R^2 R + \pi R^2(3R)}{4R^2 + \pi R^2 + \pi R^2} = \boxed{0.24\ \mathrm{m}}$$

- For more complicated objects, you need to do an experiment to find the CM by balancing in x, y, and z directions.

Finding CM of difficult object by balancing.

Center of Mass Motion and Newton's 2nd Law:

- In the last section we have seen how to find in the CM of, in principle, any object we want.

- Here we will show that the translational motion of a system of many particles is completely described by its center of mass motion.

- The derivation is given in the box on the right

- The generalized form for Newton's 2nd law is:

$$\boxed{\vec{F}_{\mathrm{net,ext}}=\sum \vec{F}_{\mathrm{net}}=M\vec{a}_{\mathrm{CM}}} \quad M=\text{total mass}$$

- As usual, only external forces cause accelerations

- Any object can be described as a collection of particles

- For all the particles that make up the object, the particle-particle forces cancel because of Newton's 3rd law.

Translational motion of an object can be completely described by replacing the object with a particle with mass *M*, located at the objects CM location $\vec{r}_{\mathrm{CM}}$

For just the x-axis we have:

$$M\vec{x}_{\mathrm{CM}}=\sum m_i\vec{x}_i$$

Taking the time derivative of both side:

$$M\vec{v}_{\mathrm{CM}}=\sum m_i\vec{v}_i$$

or

$$\vec{p}_{\mathrm{CM}}=\sum \vec{p}_i$$

where we assume mass is fixed. We see that the CM momentum is equal to the total momentum of all the particles. Taking another time derivative

$$M\vec{a}_{\mathrm{CM}}=\frac{d\vec{p}_{\mathrm{CM}}}{dt}$$
$$=\sum\frac{d\vec{p}_i}{dt}=\sum\vec{F}_{i,\mathrm{net}}$$

Only external forces are counted on the right hand side, since internal forces cancel via Newton's 3rd law.

- As shown in the box, we also can show that the CM momentum is equal to the combined momentum of all the individual particles.

$$\vec{p}_{\mathrm{CM}}=M\vec{v}_{\mathrm{CM}}=\sum\vec{p}_i$$

- Therefore, we can also express the CM motion in terms of the change in CM momentum

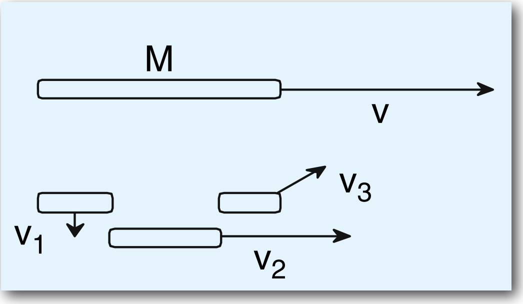
$$\vec{F}_{\mathrm{net,ext}}=\lim_{\Delta t\to 0}\frac{\Delta\vec{p}_{\mathrm{CM}}}{\Delta t}$$

Key Idea: The CM moves as if all the mass of the system is located there, and all of the external forces are acting on it.

- In using the CM momentum, we can also extend the validity of the conservation of momentum:

If there are no external forces acting on an object, the CM momentum, or total momentum of the system is conserved.

Ex. A rocket of mass *M* explodes into 3 pieces at the top of its trajectory where it was traveling horizontally at a speed of 10m/s at the moment of the explosion. If one piece with mass 0.25*M* falls vertically down at a speed of 1.2m/s, a second piece with mass 0.5*M* continues in the original direction, and the third piece moves in the forward direction at an angle of $\pi/4$. Find the final velocities of the second and third pieces. The forces of the explosion are so strong, you can ignore the force of gravity.



- Since there are no external forces in the problem, the total momentum must be conserved.

  - The initial momentum is: $\vec{P}_{\text{init}} = M\vec{v}$ (in +x-direction)

- Lets write the conservation of momentum in the x and y directions separately:

$$Mv = \frac{1}{2}Mv_2 + \frac{1}{4}Mv_3\cos(\pi/4) \quad \text{(in x-direction)}$$

$$0 = \frac{1}{4}Mv_1 - \frac{1}{4}Mv_3\sin(\pi/4) \quad \text{(in y-direction)}$$

- Since we know *v1*=1.2m/s we can solve for *v3* in the second equation:

$$v_3 = \frac{1.2\ \text{m/s}}{\sin(\pi/4)} = \boxed{1.7\ \text{m/s}} \quad \text{(at an angle of } \pi/4\text{)}$$

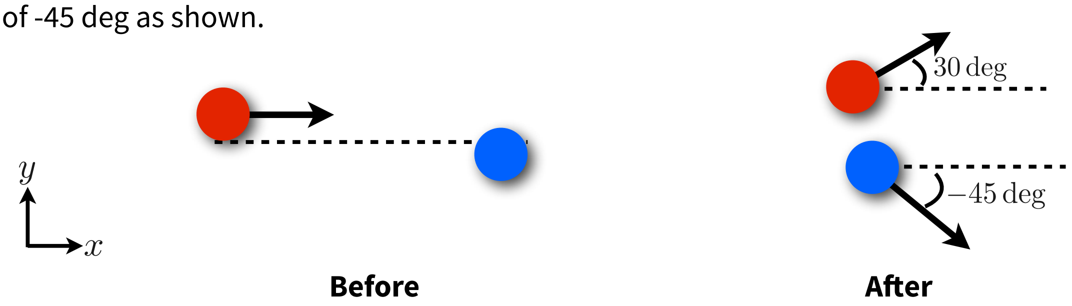
- Now using *v3* in the first equation, we can solve for *v2*:

$$10 = 0.5v_2 + 0.25\cdot 1.7\cos(\pi/4)$$

$$\boxed{v_2 = 19\ \text{m/s}} \quad \text{(in the +x-direction)}$$

-Note that kinetic energy is not conserved in this example.

Ex. A red hockey puck with mass 0.5kg is traveling at a speed of 5m/s and then hits a blue hockey puck at rest and with mass 0.5kg off-center. If after the collision, the red puck moves at an angle of 30 deg and with a velocity of 3m/s, find the final velocity of the blue puck if it moves at an angle of -45 deg as shown.



- Again, there are no external horizontal forces acting on our system, so the momentum in the horizontal directions is conserved (gravity does not matter here).

- The initial momentum is only in the x-direction: $P_x^{(\mathrm{init})} = mv_0 = (0.5)5 = 2.5\ \mathrm{kg}\cdot\mathrm{m/s}$

- After the collision, the x-components of the red and blue puck momentum must add up to the initial momentum.

$$2.5\ \mathrm{kg}\cdot\mathrm{m/s} = (0.5)(3\cos(30)) + (0.5)(\mathrm{v}_{\mathrm{blue}}\cos(45))$$

- We can now solve for the velocity of the blue puck: $\boxed{v_{\mathrm{blue}} = 3.4\ \mathrm{m/s}}$