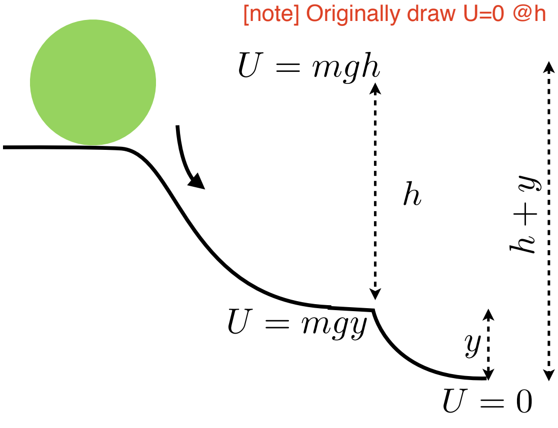
What is Potential?

- Recall from gravity, the change in potential energy of an object above the Earth is: $\Delta U_G = mgh$



- This is a measure of the amount of energy (work) that was required to push the ball up the hill to height h.
- This is also equal to the kinetic energy if the ball rolled back down the hill to h=0
- But what if h does not measure entire height of the hill?

$$\Delta U_G = mg(h+y) - mgy = mgh$$

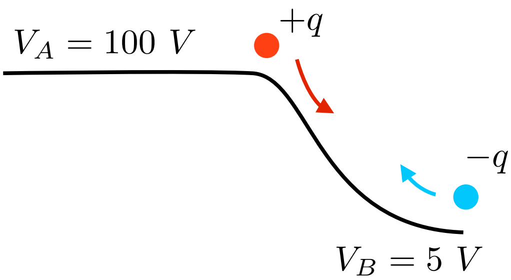
- The change in potential does not care where U=0 is, only the difference matters.
- If $\Delta h = 0$ then the change in potential is zero, for any mass.

- The mass doesn't matter, only the change in height matters, lets divide out the mass:

$$\frac{U_G}{m} = gh = V_G$$

This is the "**Gravitational Potential**": it is a measure of the amount of work required that does not require knowing the mass.

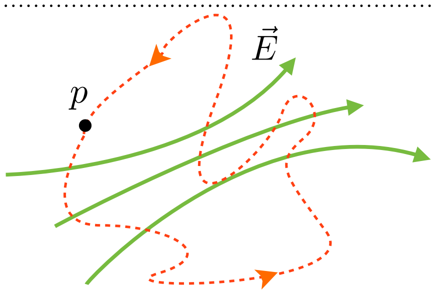
Electric potential works the exact same:



"**Electrostatic Potential**"

$$\frac{U_{el}}{e} = \frac{Q}{4\pi\epsilon_0 r} = V$$

- You can think of (+) charges as rolling down the potential hill to lower V. (-) charges go up the hill to higher potentials.



- Suppose I am given an E-field and a point p
- Now starting at p, I go along an arbitrary path and at the end, go back to point p
- Forces are conservative -> work from p->p is zero

$$V_p - V_p = \oint_A^A \vec{E} \cdot \vec{dl} = 0$$

integrate over closed loop

We now have 4 important equations for electrostatics

Important Electrostatic Equations:

(1) $$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

Gives E-field when charge distribution is known

(2) $$V_p = \int_p^\infty \vec{E}\cdot d\vec{l}$$

Relates potential to E-field along path from $p \to \infty$

(3) $$V_A - V_B = \int_A^B \vec{E}\cdot d\vec{l} \quad \text{or} \quad \Delta V = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$$

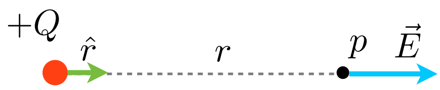
The change in potential from A -> B is related to the amount of E-field along the path from A -> B

(4) $$\oint \vec{E}\cdot d\vec{l} = 0$$

The work done along any closed path is zero.

- If we know the E-field everywhere then **(2)** tells use that we know potential everywhere.
- But given the potential V everywhere, can we find the E-field?
- From **(2)** we see that the potential V is the integral of the E-field.
  - So E-field must be the derivative of the potential
  - Must worry about whether there is a minus sign or not.

- Single point charge:



$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$ vector function

- We have already derived the potential: $$V_p = \frac{Q}{4\pi\epsilon_0 r}$$ scalar function

- Want to show that E is derivative of V:

$$\frac{dV}{dr} = -\frac{Q}{4\pi\epsilon_0 r^2}$$

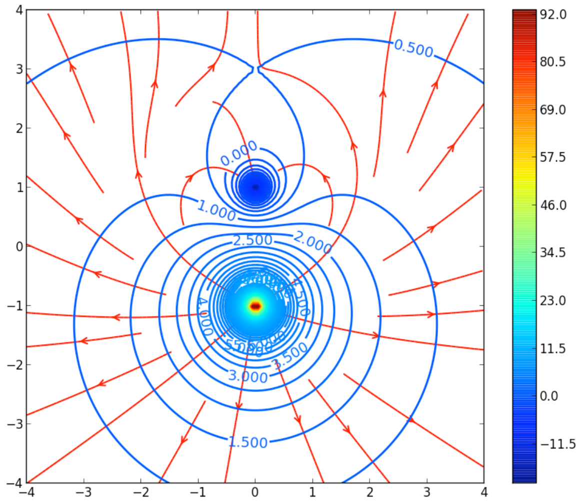
- We want a vector equation, so multiply both sides by $\hat{r}$:

$$\frac{dV}{dr}\hat{r} = -\frac{Q}{4\pi\epsilon_0 r^2}\hat{r} = -\vec{E}$$

$$\boxed{\vec{E} = -\frac{dV}{dr}\hat{r}}$$

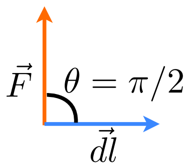
If you know the potential <u>everywhere</u> in space then you can find the E-field.

- Recall: E-field lines are always perpendicular to equipotential surfaces:



- Suppose you have a charge q and you always move perpendicular to E-field:

$$\int_A^B q\vec{E}\cdot d\vec{l} = \int_A^B \vec{F}\cdot d\vec{l} = \int_A^B F\,dl\cos\theta$$

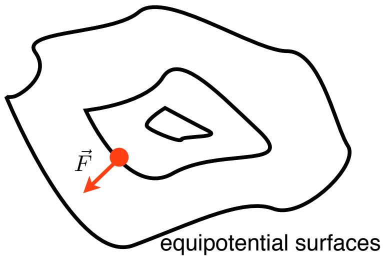


$$\cos\theta = 0$$

$$\int_A^B \vec{F}\cdot d\vec{l} = W_{A\to B} = 0$$

- Work from A->B is zero -> potential is constant

A charge released on an equipotential surface will always move perpendicular to the surface, the the direction of the E-field if (+), or opposite direction if (-).



Deeper Connection Between E-field and V:

- Suppose I am at point P in space with potential Vp and some E-field
- Now I take a step **only** in the x-direction:
  - If potential does not change -> no E-field in the x-direction.
    - E-field (if any) must be perpendicular since V=constant
  - If potential does change, then the electric field in the x-direction can be written as

$$|E_x| = \left|\frac{\Delta V}{\Delta x}\right|_{y,z=\mathrm{const}} \quad \left[\frac{V}{m}\right]$$

Electric field is the change in potential over some distance

- Same in other directions:

$$|E_y| = \left|\frac{\Delta V}{\Delta y}\right|_{x,z=\mathrm{const}} \qquad |E_z| = \left|\frac{\Delta V}{\Delta y}\right|_{x,y=\mathrm{const}}$$

- In x,y,z coordinates, the E-field can be written as: $\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$

$$\boxed{\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right) = -\nabla V}$$

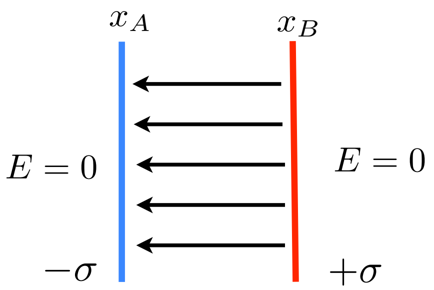
$$\frac{\partial}{\partial x} \to y, z = \mathrm{const}$$

Ex. $V(x) = 10^5 x \quad x \in [0, 0.01 \text{ m}]$ V=0 everywhere else

- What is E?

$$\boxed{\vec{E} = -\frac{\partial V}{\partial x} = -10^5 \hat{x}} \qquad x \in [0, 0.01 \text{ m}] \qquad E_y = E_z = 0$$

- This corresponds to the example of two charged conducting planes



- Lets write out what the E-field is

- Same as previous example $|E| = \dfrac{\sigma}{\epsilon_0} = 10^5 \dfrac{V}{m}$

- What is potential difference?

$$V_A - V_B = \int_{x_A}^{x_B} \vec{E} \cdot \vec{dx} = -10^5 \int_{x_A}^{x_B} \hat{x} \cdot \vec{dx}$$

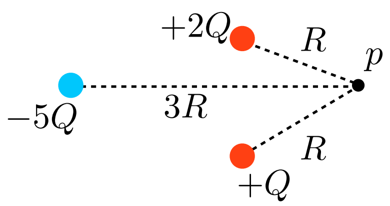
both in same direction

$$V_A - V_B = -10^5 \int_{x_A}^{x_B} dx = -10^5 \left(x_B - x_A\right)$$

- Since distance between A & B is 0.01m: $\boxed{V_A - V_B = -1000 \text{ V}}$

Potential grows linearly from A -> B

Ex. Three point charges:



- What is the potential at point P?

- Recall: $V_p = \dfrac{Q}{4\pi\epsilon_0 r}$

- We can always use superposition to add up individual potential terms

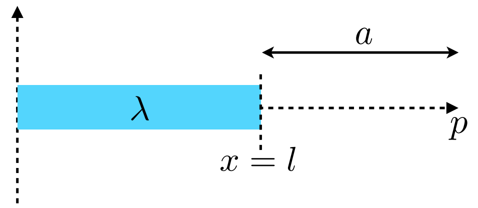
$$V_p = V_p^{+Q} + V_p^{+2Q} + V_p^{-5Q}$$

- Problem is simple now:

$$V_p^{+Q} = \frac{Q}{4\pi\epsilon_0 R} \qquad V_p^{+2Q} = \frac{2Q}{4\pi\epsilon_0 R} \qquad V_p^{-5Q} = \frac{-5Q}{4\pi\epsilon_0 (3R)}$$

$$\boxed{V_p = \frac{4}{3} \frac{Q}{4\pi\epsilon_0 R}}$$

Ex. Charged rod:



- What is the potential at point P from rod with length l and charge density λ.

- We can calculate the E-field and use

$$V_p = \int_p^\infty \vec{E} \cdot \vec{dl}$$

but we are not given E and this is too much work.

- Instead we know that charged rod is just a collection of small individual charges dq.

- So lets add up the potential from each little charge dq to get answer:

$$dV_p = \frac{dq}{4\pi\epsilon_0 d} = \frac{\lambda dx}{4\pi\epsilon_0 \left[(a+l) - x\right]}$$

$$V_p = \frac{\lambda}{4\pi\epsilon_0} \int_0^l \frac{dx}{\left[(a+l) - x\right]} = \frac{\lambda}{4\pi\epsilon_0} \left[\log\left(a+l\right) - \log\left(a\right)\right]$$

$$\boxed{V_p = \frac{\lambda}{4\pi\epsilon_0} \log\left(\frac{a+l}{a}\right)}$$