# Electromagnetic Waves



**Displacement Current:**

- In the last chapter we have seen that currents and voltages can oscillate in time.

- These oscillations lead to time-changing E-fields and B-fields though Faraday's Law and Ampere's Law:

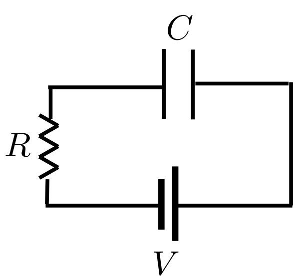
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \qquad \text{(Faraday)}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\rm enc} \qquad \text{(Ampere)}$$

- However, we have a problem, Ampere's law was derived from the Biot-Savart Law where we assumed that the current $I_{\rm enc}$ was constant in time.

- What happens if the current is not constant?

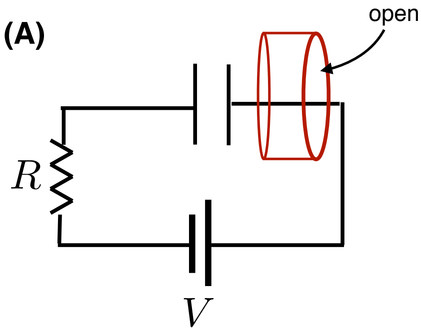
- Let us consider the series RC-circuit driven by a voltage source:




$$q(t) = CV\left[1 - e^{-t/RC}\right]$$

$$I(t) = \frac{V}{R}e^{-t/RC}$$

- Lets calculate the B-field using Ampere's Law: $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$

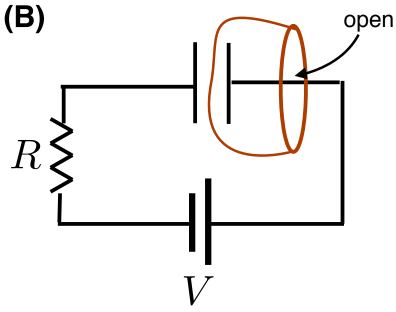



- Recall that we defined the enclosed current as the current going through <u>an open surface</u>.

- If we pick the surface in (A) then the enclosed current is the current in the circuit loop.

- But remember that we are free to pick **any** open surface we want to define the enclosed current.

- Let us pick the surface in (B) where the open surface goes through the middle of the capacitor




- Even if there is current flowing in the wire, there is no current going through the surface in (B)!

- Therefore we always get:

$$\oint_{(B)} \vec{B} \cdot d\vec{l} = 0$$



- We now have a problem, our equations give two different answers depending on how we pick our surface.

- So how do we resolve this problem?

- First note that if $I_{\text{enc}} = 0$ then both sides of Ampere's Law are zero and there is no problem.

- In general, $I(t) = \frac{V}{R} e^{-t/RC}$ because charge is building up on the capacitor

- If the capacitor has area A, then the charge density $\sigma(t)$ on the capacitor is:

$$\sigma(t) = \frac{CV}{A}\left[1 - e^{-t/RC}\right]$$

- Since $E = \sigma/\epsilon_0$ we have a time-changing E-field inside of the capacitor:

$$E(t) = \frac{CV}{\epsilon_0 A}\left[1 - e^{-t/RC}\right]$$

- Expressed in terms of the electric-flux: $\Phi_E(t) = \int \vec{E}(t) \cdot d\vec{A}$

- When the current is not zero, there is a time-changing electric flux:

$$\frac{d\Phi_E(t)}{dt} = \frac{d}{dt} \int \vec{E}(t) \cdot d\vec{A}$$

- Just like Faraday's law: $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$ gives E-field from changing $\Phi_B$

  - We can get a B-field from a time-changing E-flux:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left( \epsilon_0 \frac{d\Phi_E}{dt} \right)$$

**"displacement current"**

- Ampere's Law can now be written as sum of enclosed and displacement currents:

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}}$$

- This is single term is why Maxwell (1861) is famous!



**Maxwell's Equations (In final form):**

$$\int \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$ (Gauss's Law)

$$\int \vec{B} \cdot d\vec{A} = 0$$ (No Magnetic Monopoles)

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ (Faraday's Law)

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$ (Maxwell-Ampere's Law)

## You Now Know Everything About Electromagnetism!

- These equations are what inspired Einstein to formulate special relativity

- The unification of electricity and magnetism is guide for trying to understand how all of the fundamental forces in nature are related

## EM Waves:

- With Maxwell's displacement current, the laws of EM now support traveling waves
    - Light rays, radio waves, microwave (cell phone) signals.

- We will only consider traveling waves in vacuum -> $Q_{\text{enc}} = 0, I_{\text{enc}} = 0$

- Only 2 Maxwell's equations are non-trivial:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$ E-field from changing B-field

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$ B-field from changing E-field

- EM waves can travel even in a vacuum with no charges or currents present!

- One possible solution:

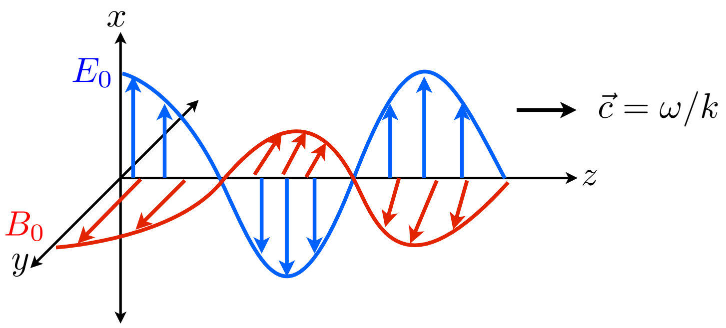
$$k = 2\pi/\lambda \qquad \omega = vk$$

$$\vec{E} = E_0 \hat{x} \cos(kz - \omega t)$$

$$\vec{B} = B_0 \hat{y} \cos(kz - \omega t)$$

Wave moving in +z-direction






- These solutions are called "**Plane Waves**"
    - For a fixed value of z, the amplitudes of the E-field and B-field are the same everywhere in the x-y plane no matter where you are.

- Solutions are valid solutions to Maxwell's equations only if the following are true:

$$B_0 = \frac{E_0}{c} \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

- Maxwell predicted the speed of light:

$$\epsilon_0 = 8.85 \times 10^{-12}$$
$$\mu_0 = 4\pi \times 10^{-7}$$

$$\Rightarrow \quad c \approx 3 \times 10^8 \text{ m/s}$$

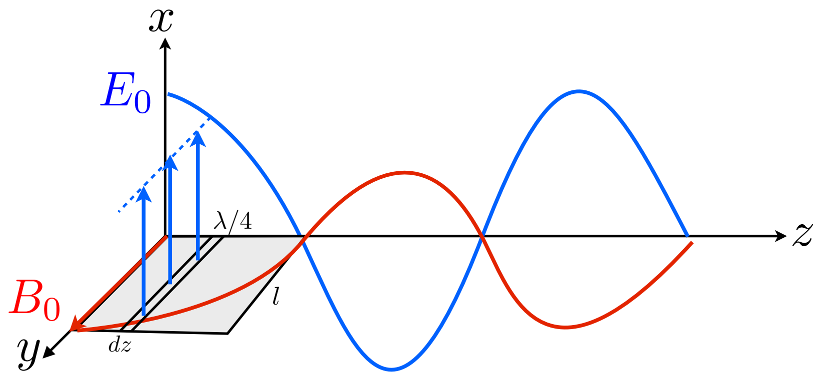
- We need to prove that this conditions are necessary for solutions to be valid

- We will first look at Maxwell-Ampere's Law: $\oint \vec{B}\cdot d\vec{l} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt}$

- We first need to pick a closed loop to define both dl and the area A for the E-flux.

  - We will take t=0, and pick the loop with length $z=\lambda/4$ and height $y=l$

  - Define a small slice of the loop area as ldz




  - E-field changes over loop length, but E-field is constant over slice (plane wave)



- We will pick the area vector for the loop dA as pointing in the same direction as E-field.

- The E-flux is thus: $\Phi_E = \int \vec{E}\cdot d\vec{A} = \int_0^{\lambda/4} l dz E_0 \cos\left(kz-\omega t\right)$

- But we want the change in flux $d\Phi_E/dt$:

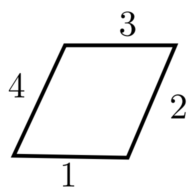
$$\frac{d\Phi_E}{dt} = lE_0\omega \int_0^{\lambda/4} \sin(kz-\overbrace{\omega t}^{=0 \text{ since t=0}})dz$$

- After integration:

$$-\frac{lE_0\omega}{k}\cos(kz)\Big|_0^{\lambda/4} = lE_0\frac{\omega}{k} = lE_0 c$$

- Now we can look at the LHS of Ampere: $\oint \vec{B}\cdot d\vec{l}$

- Divide loop into 4 sides:



- Along 1 & 3, B-field and dl are perpendicular -> $\vec{B} \cdot \vec{dl} = 0$

- Along 2, B-field is everywhere 0 -> $\vec{B} \cdot \vec{dl} = 0$

- Along 4, B-field is everywhere $B_0$ and in same direction as dl -> $\vec{B} \cdot \vec{dl} = B_0 l$

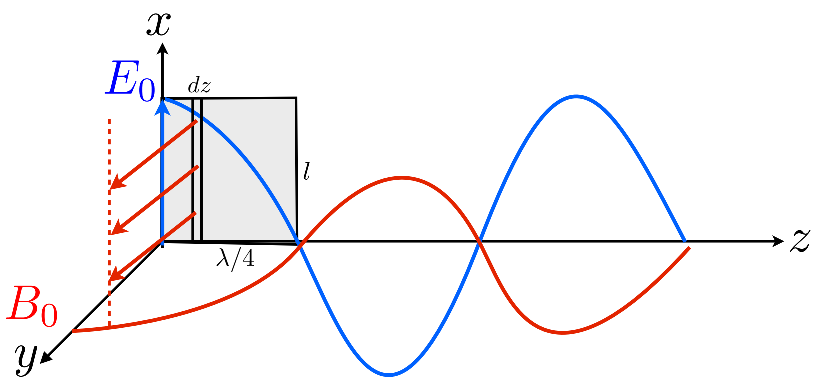
- Therefore, from Maxwell-Ampere we have:

$$B_0 l = \epsilon_0 \mu_0 l E_0 c \quad \Rightarrow \quad \boxed{B_0 = \epsilon_0 \mu_0 E_0 c} \quad \textbf{(\#1)}$$

- We are only half way done.  Now must evaluate Faraday's Law: $\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$

- We will perform the same analysis, but move the integration loop to the x-z plane:






- We will again define dA to be in the same direction as B-field -> dl goes counter clockwise.

- For B-flux through surface we get:

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = \int_0^{\lambda/4} dz l B_0 \cos(kz - \omega t)$$

- The time rate of change of B-field is analogous to E-field from previous calculation:

$$-\frac{d\Phi_B}{dt} = -l B_0 \omega \int_0^{\lambda/4} \sin(kz) dz = -\frac{l B \omega}{k}$$

- In calculating the E-field around the loop, only the branch at z=0 is non-zero:

$$\oint \vec{E} \cdot dl = -E_0 l$$ (minus since E and dl in opposite directions)

- Therefore, Faraday's Law gives us:

$$-E_0 l = -\frac{lB_0\omega}{k} = -lB_0 c \quad \rightarrow \quad \boxed{B_0 = \frac{E_0}{c}} \qquad \textbf{(\#2)}$$

- Plugging #2 into #1 we get:

$$B_0 = \epsilon_0\mu_0 E_0 c \quad \rightarrow \quad \frac{E_0}{c} = \epsilon_0\mu_0 E_0 c \quad \rightarrow \quad \boxed{c = \frac{1}{\sqrt{\epsilon_0\mu_0}}}$$

**General Properties of EM Waves:**

$\vec{E} \perp \vec{v}$

$\vec{B} \perp \vec{v}$

$\vec{E} \perp \vec{B}$

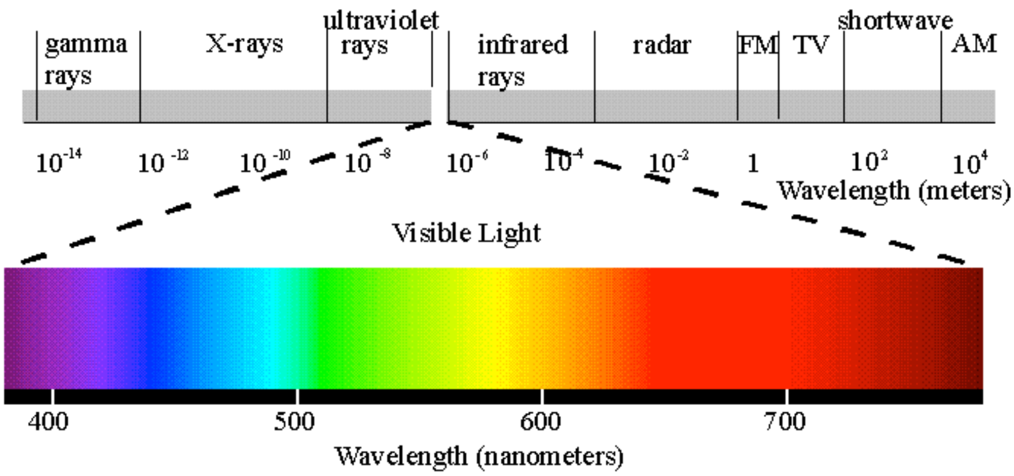
$\vec{E}$ & $\vec{B}$ are in phase (Simultaneously go through zero, or are maximum)

$\vec{E} \times \vec{B} = \vec{v}$

$B_0 = E_0/c$ in vacuum

$c = (\sqrt{\epsilon_0\mu_0})^{-1}$ in vacuum






- Since all EM waves travel at the speed of light, they can be used to calculate distances

$$d = ct$$

- The father the source of light, the farther back in time you are looking.

$t = 1 \times 10^{-9}$ s $\rightarrow$ d $= 30$ cm

$t = 1.2$ s $\rightarrow$ d $= 3.6 \times 10^{8}$ m (distance to the moon)

$t = 499$ s $= 8.3$ min $\rightarrow$ d $= 149.6 \times 10^{9}$ m (distance to the sun)

$t = 7.89 \times 10^{11}$ s $= 25000$ yrs $\rightarrow$ d $= 2.4 \times 10^{20}$ m (distance to nearest galaxy)