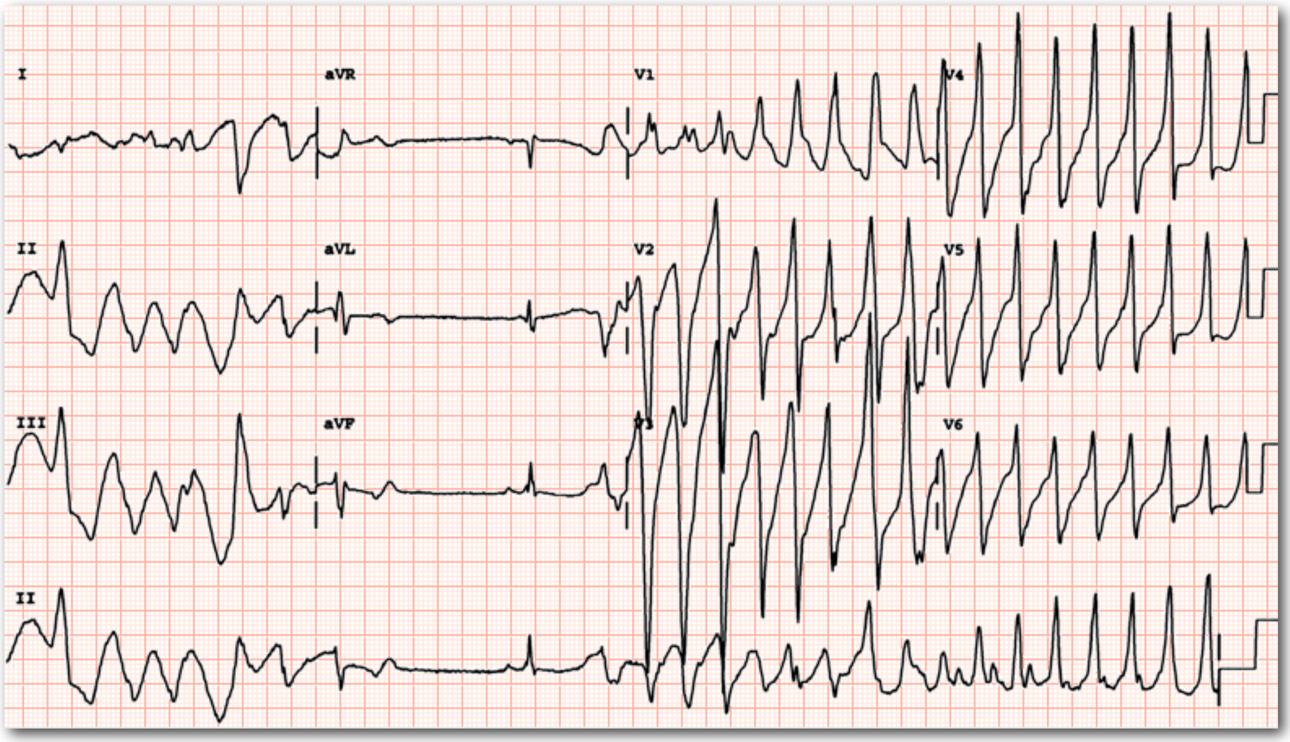
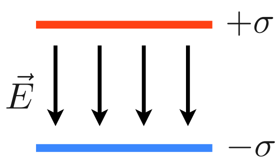
PHYS-183 : Day #24



<u>Capacitors and Cell Membranes:</u>

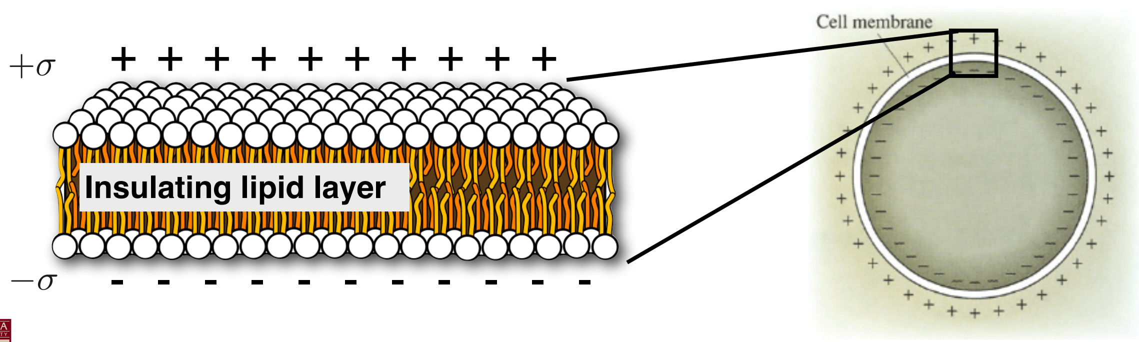
- At the end of last time, we looked at an example where the electric field was constant in between two charged plates.



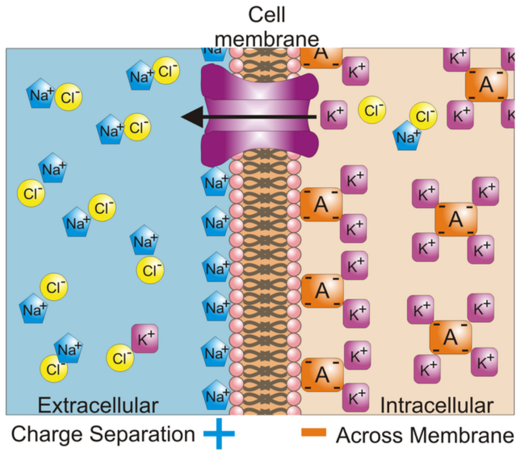
- This example highlights a particular configuration of charge called a **capacitor**

- Capacitors are used in <u>every</u> electronic device and are also used in living cells.

- The lipid bilayer the forms the cell membrane can be modeled as consisting of two conducting sheets with an insulator in between them



- The has a net negative charge inside because it uses come energy to pump sodium (Na +) and potassium (K+) ions, both with charge +q, out of the cell.



- Typical, the cell moves enough charges to the outside that the electric field that the charges produce generate an electrostatic potential called the **resting potential** of about

$$V_{\rm rest} = -100\ {\rm mV}$$

- Since only potential differences matter, we have set the outside voltage to be at zero Volts.

- To move charges in and out of the cell, the cell membrane has many proteins that act as doorways that let the charges move in an out.

- If the cell lets positive charges back in, then the cells potential becomes $V_{\rm cell} > V_{\rm rest}$ and the cell is said to be **depolarized**

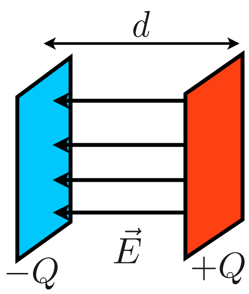
-If the cell potential becomes more negative, it is called **hyperpolarized**.

- To understand the charge properties of the cell membrane, we must first try to understand the basic properties of a capacitor.

- We have seen that the work done in bringing charges together results in a non-zero electrostatic potential energy.

- Therefore, if we can make a device that stores electric charges, this device will also store energy.

- The simplest example of such a device is the parallel plate capacitor



- Suppose there equal and opposite charges on the two plates

- There will then be an electric field generated between the plates

- This in turn corresponds to a potential between the plates given by

$$\boxed{V = Ed} \quad \text{(since E is constant inside)}$$

- In general, the charge on the plates and the voltage between them are linearly proportional with a proportionality constant called the **capacitance** *C*:

$$\boxed{Q = CV}$$

- The capacitance measures how much charge the plates will have for a given amount of voltage

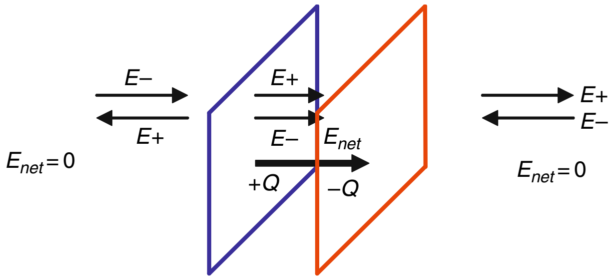
$$C = \frac{Q}{V}$$

- To find the capacitance, lets find the value for the E-field between the plates.

- The electric field from a single charge plane was given previously as

$$E = 2\pi k\sigma = 2\pi k\frac{Q}{A}$$ ← The area of the plate

- Again, this value does not depend on the distance from the plate.



- The E-field to the left and right of the plates is zero.

- Since the E-field from the two plates points in the same direction in between the plates, the total E-field inside is

$$\boxed{E_{\text{inside}} = 4\pi k\frac{Q}{A}}$$

- The electrostatic potential between the plates can now be calculated:

$$V = Ed = 4\pi k\frac{Q}{A}d$$

- We can now solve for the capacitance of the plates using $C = Q/V$:

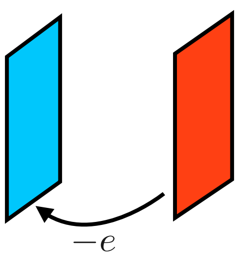
$$\boxed{C = \frac{1}{4\pi k}\frac{A}{d} \quad \text{[F]}}$$

Key Idea: The capacitance of the plates, or any capacitor, is determined purely by the geometry of the device

- Capacitance is measured in units called **Farads** [F]

- The plates store not only charge, but also energy

- Imagine the plates are initially uncharged, and we transfer one electron at a time from one plate to another.



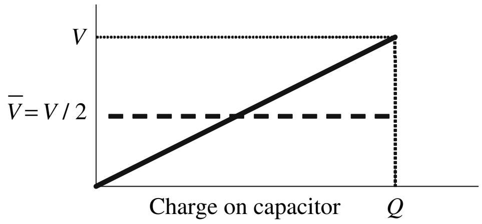
- After transferring the first electron, we must do work to transfer the second since there is now a potential V' and a Coulomb force

$$W = qV'$$

- Since the potential depends on the plate charge, the third electron will take even more work to transfer.

- Since the amount of work changes with the charge, to get the total energy we can not simply multiple the final charge by the final potential to get the correct answer.

- However, we can image transferring all of the charges at the same time in a potential that is 1/2 of the maximum potential: $W = Q(V/2)$



The total work is the area under the line. This area is the same as that under the dashed line representing 1/2 of the total potential.

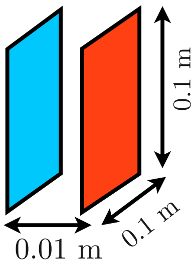
- Since energy is conserved, the work (energy) used to move the charges is now stored as potential energy in the capacitor.

$$U_c = \frac{1}{2}QV$$

- We can use the capacitance *C* to get an equation with a single variable only:

$$\boxed{U_c = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}}$$

<u>Ex.</u> What is the charge on two 0.1x0.1m plates separated by 0.01m if the potential across the plates is 12V?



- The charge on the plates and the voltage are related via the capacitance.

- First, need to find the capacitance using the geometry of the plates

$$C = \frac{1}{4\pi k}\frac{0.1^2}{0.01} = 8.8 \times 10^{-12}\ \mathrm{F}$$

- The charge on the plates is simply the capacitance times the electrostatic potential.

$$Q = CV = (8.8 \times 10^{-12})(12) = \boxed{1.1 \times 10^{-10}\ \mathrm{C}}$$

- What does it mean when we say that the work done by moving charges on to the capacitor is stored as potential energy?

- We will show that the energy is stored in the electric field between the plates.

$$U_{\mathrm{C}} = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{1}{4\pi k}\frac{A}{d}\right)(Ed)^2 = \frac{1}{4\pi k}\frac{E^2}{2}(Ad)$$

- We see that the potential energy is proportional to the E-field squared and the volume between the plates *Ad*

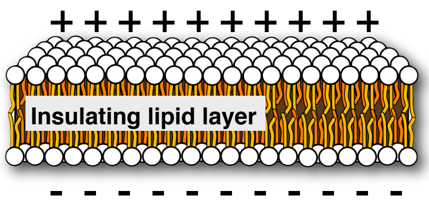
- Therefore, if we look at the potential energy per volume inside the plates we have

$$\boxed{\frac{U_{\mathrm{C}}}{(\mathrm{Vol})} = \frac{1}{4\pi k}\frac{E^2}{2}}$$

- This is a fundamental relationship for the energy stored in the E-field.

Key Idea: There is energy stored in the E-field and the magnitude of the energy is proportional to the amplitude of the E-field squared.

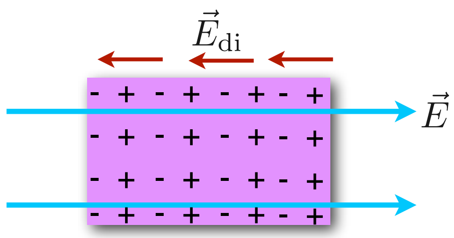
- In the cell membrane we saw that there was an insulator made out of lipids in between the two plates of charge.



- How do the properties of a capacitor change when there is an insulator placed in between the plates?

- Charges are not free to move in an insulator.

- However, for in some insulators called **dielectrics** the charge can be moved by a small amount when an E-field goes through them



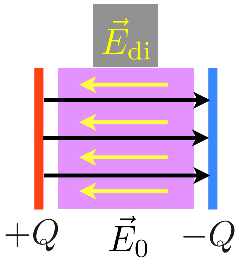
- Neutral dielectric (insulator)

- Dielectric that is **polarized**

- Negative charges move slightly in the direction of -$\vec{E}$

- Therefore, when there is an E-field applied to a dielectric, the dielectric generates its own E-field that points in the opposite direction $\vec{E}_{\mathrm{di}}$

- What happens if we put a dielectric in between two capacitor plates?



- Total E-field is the super position of the two E-fields that point in opposite directions

A dielectric in a capacitor reduces the E-field inside the capacitor.

- The factor that determines how much smaller the E-field is is called the **dielectric constant** κ

$$\boxed{E = \frac{E_0}{\kappa}}$$

Material	*Dielectric Constant*
Air	1.00054
Paper	~4
Pyrex glass	4.7
Rubber (Neoprene)	~7
Ethanol	25
Water	80

Dielectric constants of some materials

- Since V~E and C~1/V we also have:

$$\boxed{V = \frac{V_0}{\kappa}} \qquad \boxed{C = C_0\kappa}$$

- A dielectric will decrease the voltage and E-field in a capacitor allowing more charges to be stored and therefore increases the capacitance.

Ex. A $0.1\ \mu\mathrm{F}$ parallel plate capacitor has a 12V electrostatic potential across it. A dielectric with constant $\kappa = 4$ in placed inside. Find the charge on the plates and the voltage after the dielectric has been inserted.

Solution:

- The voltage on the plates has been reduced according to

$$V = \frac{V_0}{\kappa} = \frac{12}{4} = \boxed{3\ \mathrm{V}}$$

- Before the dielectric is placed inside, the charge on the plates is

$$Q = CV = (10^{-7})(12) = 1.2 \times 10^{-6}\ \mathrm{C}$$

- The charge after the dielectric is inserted is the same as before. This can be seen via:

$$Q = CV = (C_0\kappa)\left(\frac{V_0}{\kappa}\right) = C_0V_0$$

Electric Properties of Cell Membranes:

- The capacitance per unit area for a cell membrane was first determined in the 1920's to have a value of

$$\frac{C_{\rm cell}}{\rm cm^2} = 1\ \mu{\rm F/cm^2}$$

- Modeling the membrane as a parallel plate capacitor, we can calculate the thickness of the membrane since capacitance is determined by geometry.

- Using a dielectric constant of $\kappa = 3$ for lipids:

$$\frac{C_{\rm cell}}{A} = \frac{\kappa}{4\pi k}\frac{1}{d_{\rm mem}} \quad \Rightarrow \quad \boxed{d_{\rm mem} \approx 3\times 10^{-9}\ {\rm m} = 3\ {\rm nm}}$$

- This simple model is close to the actual value of $d_{\rm mem} \approx 7.5\ {\rm nm}$

- This length scale indicates that the cell membrane is made out macromolecules since they are also a few nanometers in length.

- Assuming that the cell membrane is a spherical conductor with a uniform charge distribution, we can find the total amount of charge on the membrane

$$Q = CV \quad \xrightarrow{\text{dividing both sides by the area}} \quad \frac{Q}{A} = \frac{C}{A}V$$

- Again, the magnitude of the potential across the cell membrane is $V_{\rm cell} = 0.1\ V$

- Using our capacitance per unit area $C_{\rm cell}/{\rm cm^2} = 1\ \mu{\rm F/cm^2}$

$$\frac{Q}{A} = 1\ \mu{\rm F/cm^2}\cdot(0.1{\rm V}) = \boxed{0.1\ \mu{\rm C/cm^2}}$$

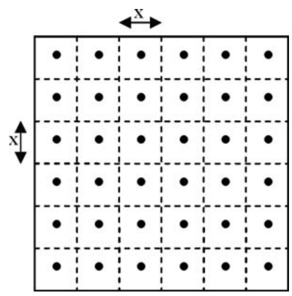
- We can get some idea for this charge density by calculating the average space between each charge on the cell membrane.

$$\frac{1\ \rm charge}{x^2\ \rm cm^2} = \frac{0.1\times 10^{-6}\ {\rm C/cm^2}}{1.6\times 10^{-19}\ {\rm C}} = 6.25\times 10^{11}\ {\rm charges/cm^2}$$

- Solving for x, and converting to nanometers we find one charge per 13nm.

- Having found the potential across the cell membrane and with width of the cell membrane we can quickly calculate the E-field across it

$$E = \frac{V_{\mathrm{cell}}}{\kappa d_{\mathrm{mem}}} = \frac{0.1\ \mathrm{V}}{3 \cdot 3 \times 10^{-9}\ \mathrm{m}} = \boxed{1.1 \times 10^{5}\ \mathrm{V/m}}$$



Uniform surface charge model of the cell membrane. Each box has a height and width of ~13nm.

- This is an extremely high E-field value!

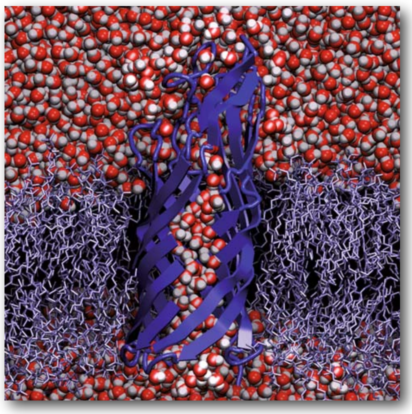
- The largest E-field that you can create in dry air is only 30% of this value.

- This large E-field indicates that there are relatively large forces on the molecules that are inside the membrane.

- Since the E-field also contains energy. This large value tells us that this E-field can release a sizable amount of energy if the cell is triggered to do so.

<u>Charge Transport Through Cell Membranes:</u>

- To create a potential across the cell membrane, cells contain many **membrane channels** that are designed to transport ions, water, and some larger macromolecules though the cell membrane


Molecular model of a membrane channel formed by a protein (blue)

- These channels are formed from proteins, sugars, and fatty acids.

- Each channel is designed to transport a single type of molecule (ex. Na+ and K+ ions)

- To get a resting potential, the cell uses energy to actively transport Na+ ions out of the cell and K+ ions inside.

  - 3 Na+ go out, 2K+ go in

- When the channels open and close, the cell can become depolarized or hyperpolarized depending on which channels are active.

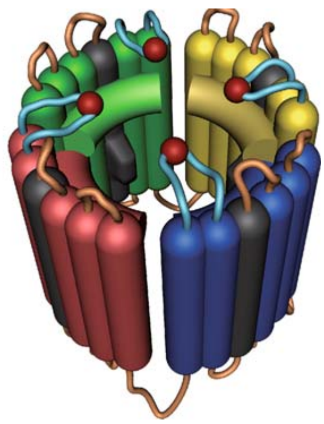
- There are two ways that the channels can be activated:

  - The channel can be activated by the membrane potential -> **Voltage-gated channels**

  - The channel can be activated by a chemical signal -> **Ligand-gated channels**

- The Na+ (sodium ion) channel is a very important voltage-gated channel.

- The channel is formed by about 2000 amino acids, with some extra sugars and fatty acids.



Molecular model of the Na+ channel

- In a muscle, there are about 50 to 500 Na+ channels per $\mu\mathrm{m}^2$

- These channels are usually closed, but become activated by a change in the cell membrane potential.

- When opened, the channels let about 10^3 Na+ ions into the cell in about 1ms

- Because these ions are positively charged, they cause the cell to become depolarized.

- If the cell is a muscle cell, then this depolarization causes the muscle to contract.

- This depolarization is what causes your heart to beat and your muscles to move.

- Because depolarization involves charges and E-fields, this process can actually be monitored by a machine.

- The most important example is an Electrocardiograph or EKG or ECG measuring the potential change in your heart.