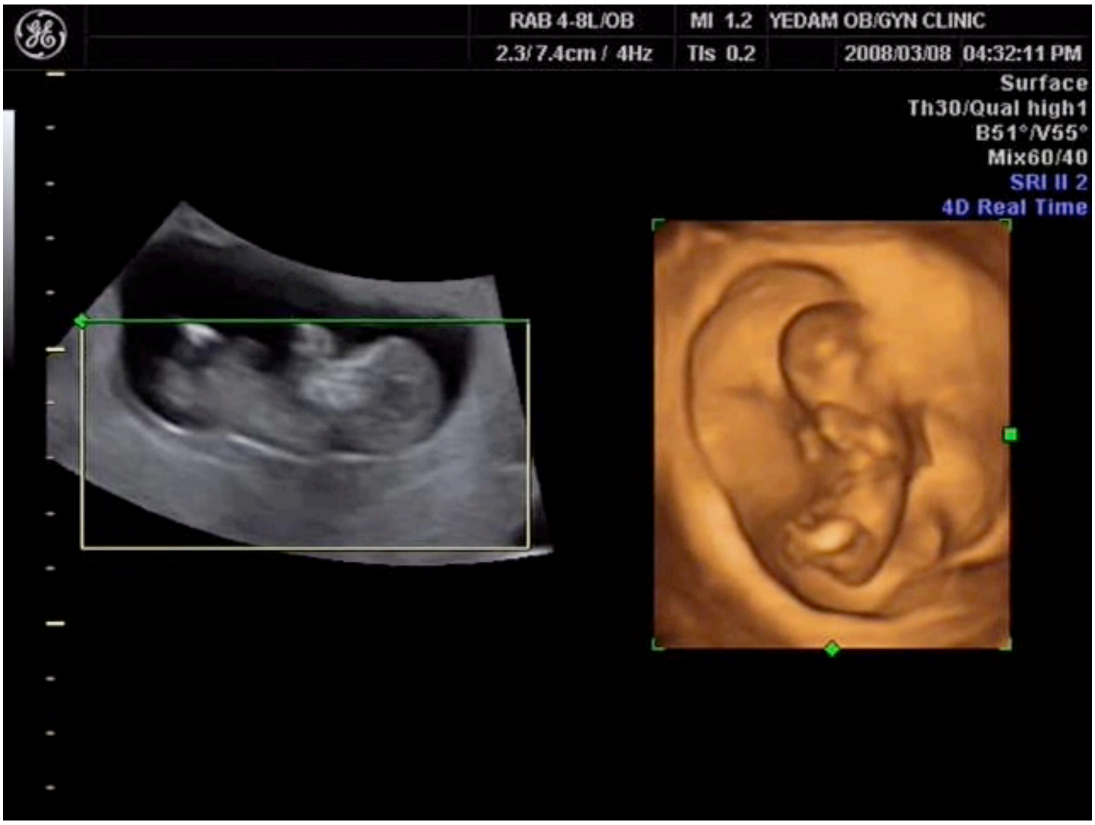


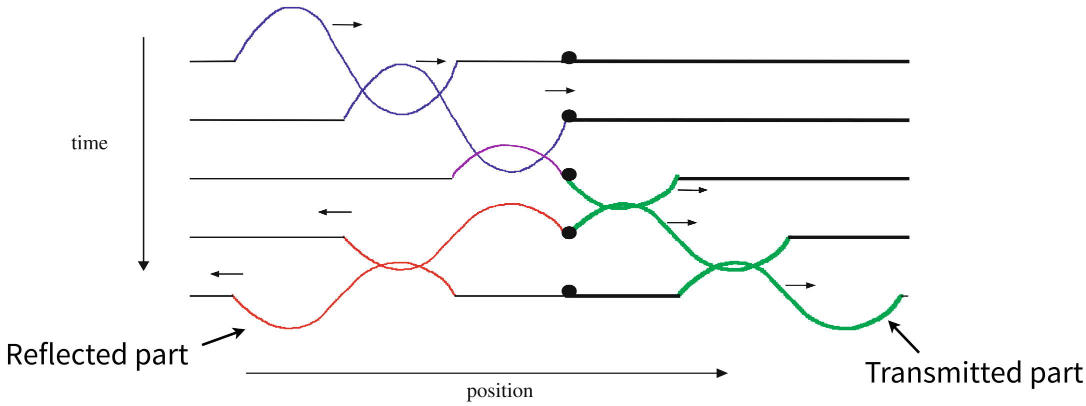
Wave interference:

- When waves encounter a boundary formed by two different materials several different things can occur.

- Part of the wave will continue to travel in the same direction. This is called the **transmitted wave**.

- Part of the wave will be reflected and travel in the opposite direction. This is called the **reflected wave**.

- For example, if a wave traveling along a string hits a boundary formed by a string with different material properties, there will be reflection and transmission.



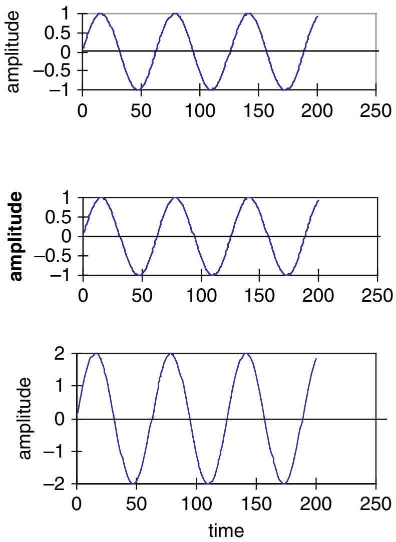
- The reflected part of the wave interacts with the initial wave in a process called **interference**.

- Interference is an example of the **superposition principle**: The total wave amplitude is obtained by adding the amplitudes of the individual waves.

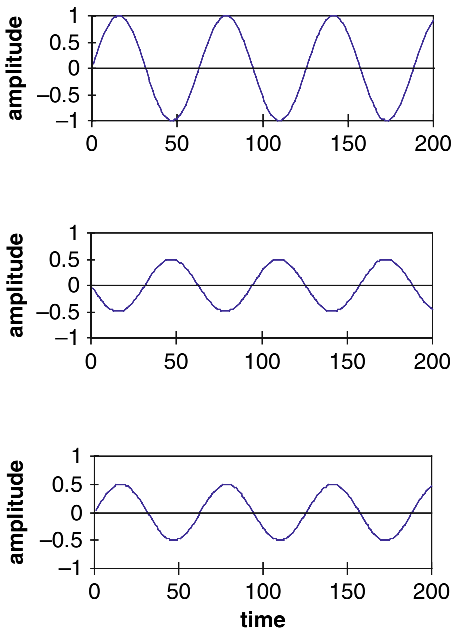
- Consider the two waves:

$$y_1(x,t) = A\sin(kx - \omega t) \qquad y_2(x,t) = B\sin(kx - \omega t + \phi)$$

- If $\phi = 0$ then the total wave amplitude is A+B and the waves are said to **interfere constructively**.



- If $\phi = \pi$ then the total wave amplitude is reduced to |A-B| and the waves are said to **interfere destructively**.



- Assuming A=B, then the amplitude is given by:

$$y = y_1 + y_2 = A\sin(kx - \omega t) + A\sin(kx - \omega t + \phi)$$

- Using trigonometry we can simply this equation:

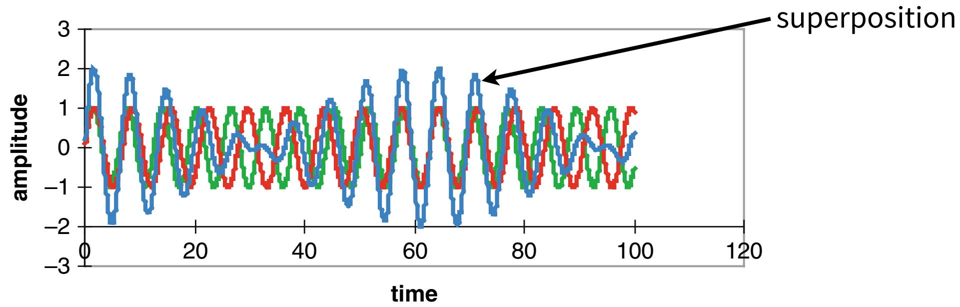
$$y = [2A\cos(\phi/2)]\sin(kx - \omega t + \phi/2)$$

- Wave still oscillates at same frequency, but with phase shift $\phi/2$

- We see that if $\phi = n\pi, n = 1, 3, 5, \ldots$ the amplitude is zero.

- if $\phi = n\pi, n = 0, 2, 4, \ldots$ the amplitude is maximum 2A.

- We can also consider the case where the frequencies are slightly different.



- Here the red and green waves have same amplitude but differ in frequency by about 10%. The blue curve is the superposition.

- The superposition wave will oscillate at the average frequency $(\omega_1 + \omega_2)/2$, but the amplitude of the resultant wave will vary at a frequency that is the difference $(\omega_1 - \omega_2)/2$

- The frequency $\omega_{\text{beat}} = \omega_1 - \omega_2$ is called the **beat frequency**.

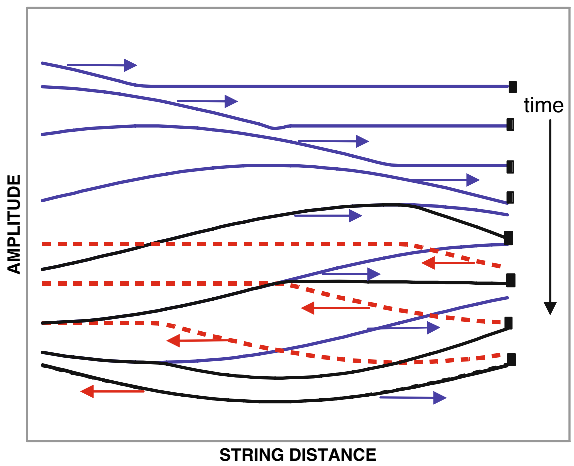
Standing Waves and Resonance:

- We have seen that when you shake one end of a string you can create waves along the string.

- If the other end of the string is fixed so that $y(x_{\text{end}}, t) = 0$ then all of the energy is reflected by to the other end.

- The amount of time it takes for the wave to come back to where it started is $t = 2L/v_{\text{wave}}$ where L is the length of the string.

- If this amount of time is equal to the period of oscillation $t = T = 1/f$, then the incoming and reflected waves will interfere constructively and will produce a standing wave.



- These standing waves can be seen by adding up two waves traveling in opposite directions

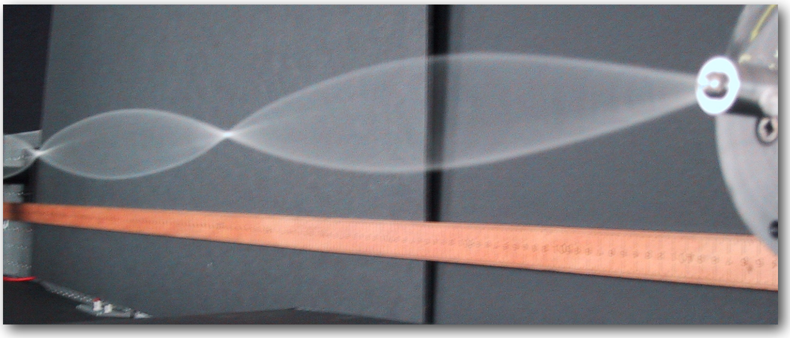
$$y_1 = A\sin(kx - \omega t)$$
$$y_2 = A\sin(kx + \omega t)$$
$$\rightarrow \quad y = y_1 + y_2 = 2A\sin(kx)\cos(\omega t)$$

- The sum of these two traveling waves is **<u>not a traveling wave</u>**!

- Each position x oscillates with frequency ω but no wave travels down the string.

- To see this we can see that whenever $kx = n\pi, n = 0, 1, 2, \ldots$ the amplitude is always zero.

- These points are called **nodes**.

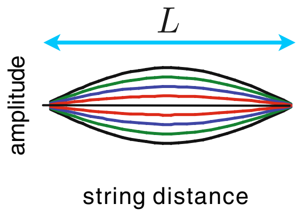


- Because standing waves are only generated when the wave traveling time is equal to the period of oscillation, only special frequencies are able to create standing waves.

- These frequencies are called **resonant frequencies**.

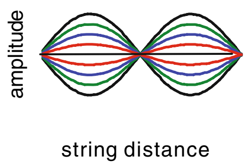
- The lowest possible frequency is called the **fundamental frequency**, and corresponds to the inverse travel time.

$$f_1 = \frac{v_{\text{wave}}}{2L} \xrightarrow{v_{\text{wave}} = f\lambda} \lambda_1 = 2L$$



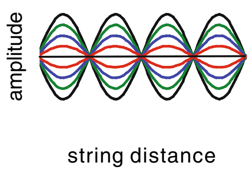
- Above the fundamental frequency there are a collection of **harmonics** given by

$$f_n = \frac{v_{\text{wave}}}{\lambda_n} \qquad \lambda_n = \frac{2L}{n}, n = 1, 2, \ldots$$



$\lambda_2 = L$

- Each harmonic adds one node to the string



$\lambda_4 = \frac{L}{2}$

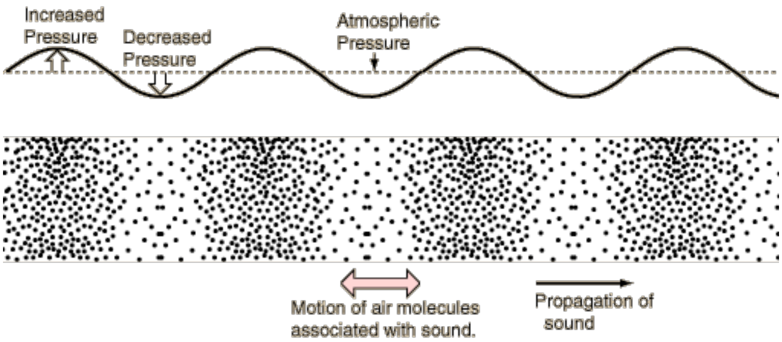
Sound Waves:

- Sound waves represent one of the most important waves in the world around us.

- Sound waves are longitudinal waves generated by varying the pressure in the air.

- This change in pressure, with respect to the normal atmospheric pressure, is given by

$$\Delta P = \Delta P_{\text{max}} \sin(kx - \omega t) \qquad \Delta P = P - P_{\text{atm}}$$



- The maximum difference in pressure ΔP_{max} is usually very small, around $\Delta P_{\text{max}} < 0.03 P_{\text{atm}}$

- The velocity of sound waves depends on two properties of the material they are traveling in:

$$v_{\text{sound}} = \sqrt{\frac{B}{\rho}}$$

$B = \text{bulk modulus}$ ← "How much your material can stretch"

$\rho = \text{density of material}$

Material (20°C Unless Noted)	*Density (kg/m³)*	*Speed (m/s)*
Air	1.20	343
Water	998	1,482
Seawater	1,025	1,522
Body tissue (37°C)	1,047	1,570
Glass (pyrex)	2,320	5,170

- Because the density of gases, and some fluids, changes with temperature, the velocity of sound changes with temperature.

- If the temperature of a gas goes up then the density goes down, and the velocity goes up.

Intensity of Sound:

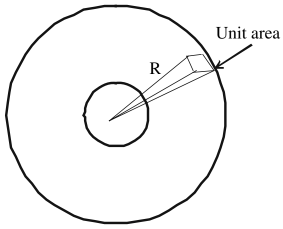
- The **intensity** *I* of a wave represents the energy per time (power) passing through a unit of surface area.

- For a wave moving in one-dimension, the amplitude of the wave remains the same, and the energy per time (power) traveling with the wave is a constant.

- In three-dimensions, a wave will spread out in all directions.

- If the sound started at a single point, then the energy passes through a spherical area that depends on the distance.

$$A = 4\pi r^2$$



Spherical area for sound from a single source in 3D.

- Because the area increases with distance, the intensity will decrease with distance.

$$I = \frac{P}{A} = \frac{P}{4\pi r^2}$$

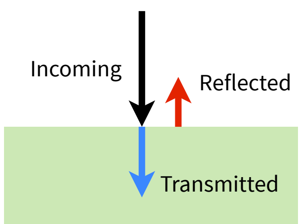
- Assuming P is a constant, then we have the important relation: $I \propto 1/r^2$

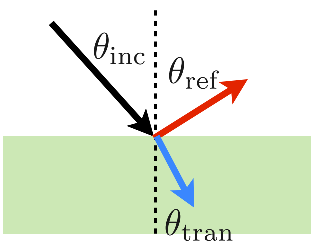
- In terms of the pressure difference, the intensity can be written as: $I_{\mathrm{max}} = \dfrac{\Delta P_{\mathrm{max}}^2}{2\rho v_{\mathrm{wave}}}$

Intensity is proportional to the amplitude squared

Superposition of Sound Waves:

- Consider sound waves moving in more than one-dimension that encounter a boundary formed by two different materials.





- If the wave is perpendicular to the surface, then the reflected and transmitted waves are along the same direction (like 1D string)

- If the wave comes in at an angle θ_{inc} then the reflected and transmitted waves travel in different directions.

- This process is extremely important for all imaging processes, including ultrasounds, x-rays,...

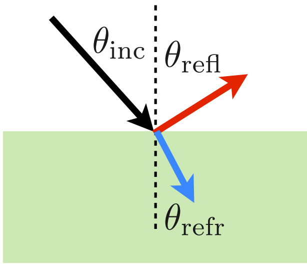
- The wave that enters the new material does not travel in the same direction as the incoming wave. The wave is said to be **refracted** ("bent") because the velocity of sound in the new material is not the same.

- In the new material, the wavelength of the light changes, the frequency stays the same.

- When going from one material to another at an angle, the transmitted wave angle is called the **angle of refraction**.

- The incoming angle and refracted angles can be related to the two different sound velocities via the **law of refraction**:

$$\boxed{\frac{\sin\theta_{\rm inc}}{\sin\theta_{\rm refr}} = \frac{v_{\rm inc}}{v_{\rm refr}}}$$



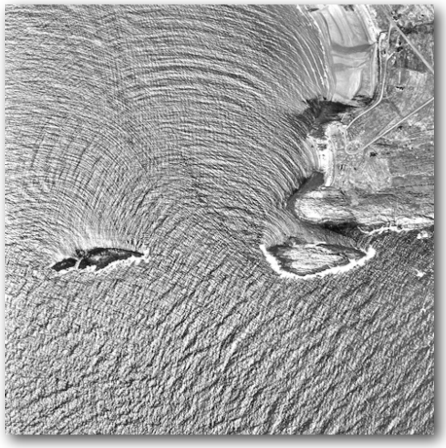
The angle in the new material is called the angle of refraction.

- Another key property of waves is **diffraction**.

- Diffraction is the ability for waves to spread out (bend) behind objects.

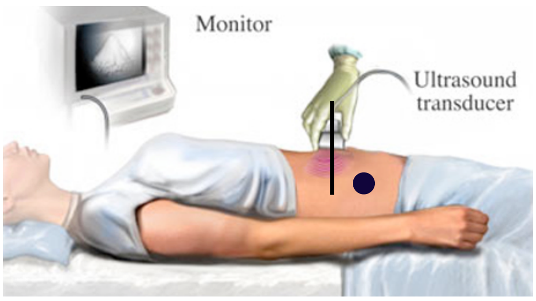
- How much diffraction depends on the size of the object and the wavelength of the wave.

- If the object is much larger than the wavelength then there will be little diffraction.



Diffraction of water waves around rocks.

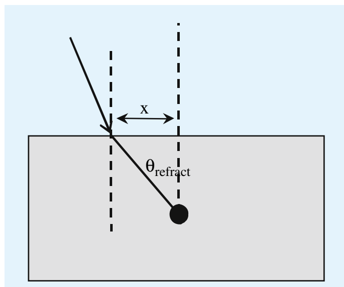
Ex. An ultrasound wave is incoming onto a persons abdomen at an angle of $\theta_{\rm inc} = 20\,{\rm deg}$. Where should the ultrasound be placed so the wave will hit a kidney stone located 7cm below the surface as shown? The ultrasound waves are emitted from a gel with velocity $v_{\rm gel} = 1400m/s$, into the persons tissue with velocity $v_{\rm tiss} = 1570m/s$.



Ultrasound of abdomen.

Solution:

- The wave entering the persons body will refract at the boundary at an angle of refraction given by the law of refraction



$$\sin\theta_{\rm refr} = \sin\theta_{\rm inc}\left(\frac{v_{\rm tiss}}{v_{\rm gel}}\right) = \sin(20)\left(\frac{1570}{1400}\right) = 0.38$$

- Therefore we have: $\theta_{\rm refr} = 22.6\,{\rm deg}$

- Since we know the depth of the kidney stone, we can use trigonometry to figure out what *x* should be

$$\tan\theta_{\rm refr} = x/y = x/7\ {\rm cm} \rightarrow \boxed{{\rm x} = 2.9\ {\rm cm}}$$

Spatial Superpositions of Sound Waves:

- We have already examined the superposition of two waves with different frequencies.

- Now we will look the the superposition of two waves that are created at two different locations but added at a single position.

- In this special case, we can write the two waves in one-dimension as:

$$y_1 = A\sin(kx - \omega t + \phi_1) \qquad y_2 = A\sin(kx - \omega t + \phi_2)$$

depends on location x_1 depends on location x_2

- The initial positions for y1 and y2 are represented in the two phase variables ϕ_1 and ϕ_2

- The net wave is found from the superposition of y1 and y2 to be:

$$y_{\mathrm{net}} = y_1 + y_2 = 2A\left[\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\right]\sin(kx - \omega t)$$

- We see that the amplitude of the wave depends on the phase difference, which depends on the locations x1 and x2 of the initial waves.

$$\mathrm{Amplitude} = 2A\left[\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\right]$$

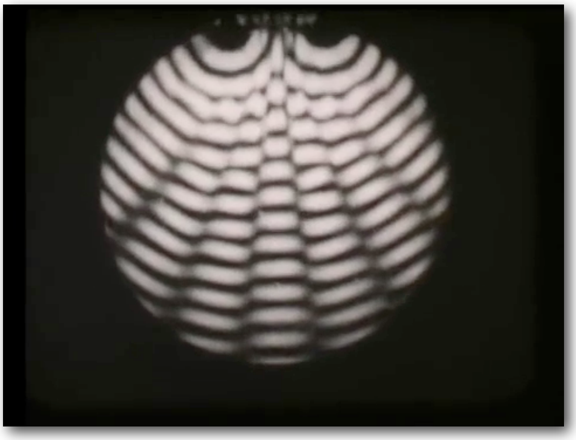
- If the phase difference is $\phi_1 - \phi_2 = 2n\pi, n = 0, 1, 2, \ldots$

$\Rightarrow$ $\mathrm{Amplitude} = 2A$ $\Rightarrow$ $I = 4A^2$ (Constructive interference)

- If the phase difference is $\phi_1 - \phi_2 = n\pi, n = 1, 3, \ldots$

$\Rightarrow$ $\mathrm{Amplitude} = 0$ $\Rightarrow$ $I = 0$ (Destructive interference)

- In more than one-dimension the results are the same, but more complicated mathematics



Ultrasound:

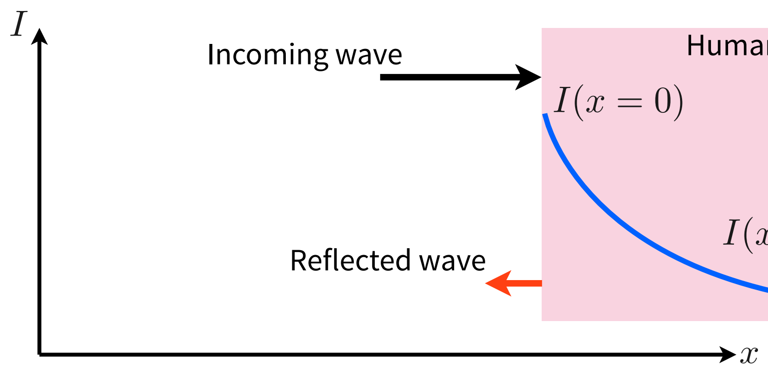
- The normal frequencies that a human can hear ranges from $20 - 20000\ \mathrm{Hz}$

- Frequencies above 20000 Hz are called ultrasound

- The only difference between ultrasound and normal sound waves is the higher frequency, and shorter wavelength

- The wavelength for ultrasound waves is about 1mm. Therefore, the smallest object that you can detect with ultrasound is around 1mm in size.

- In ultrasound imaging, sound is traveling through water or biological tissue where the velocity of sound is larger than in air (~1480m/s).



- As the wave travels through a material, some part of the wave gets absorbed.

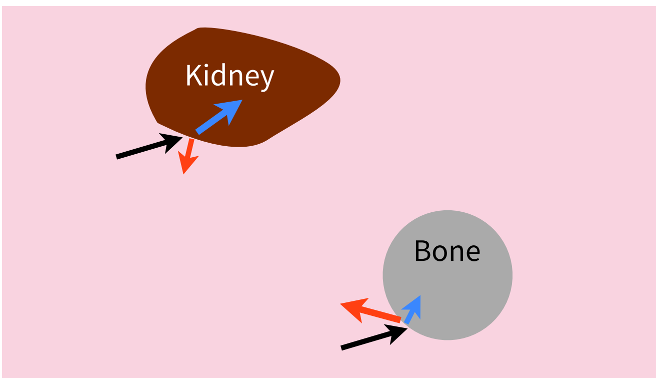
$$I(x) = I(0)e^{-\alpha x}$$

- This absorption is determined by the **absorption constant** α

- If the larger the absorption constant, the less depth the wave can penetrate.

- The absorption constant depends on the material, and on the frequency of the wave.

- We have seen that some part of a wave gets reflected when it encounters a boundary. These boundaries also occur when you have two different materials inside of your body (muscle and bone).



- Ultrasound measures <u>reflected intensity</u>

- Most of the wave is transmitted through kidney

- Most of the wave is reflected from a bone.

- How much of wave is reflected/ transmitted depends on materials

- The amount of reflected wave is determined by the **acoustic** ("sound") **impedance** *z* of the two materials at the boundary.

$$z_1 = \rho_1 v_{\mathrm{wave},1} \qquad z_2 = \rho_2 v_{\mathrm{wave},2}$$

- The amplitude of the reflected wave can be calculated from:

$$\frac{I_{\mathrm{refl}}}{I_{\mathrm{inc}}} = \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2}$$

- If the impedances of the two materials are the same, then there is no reflected intensity

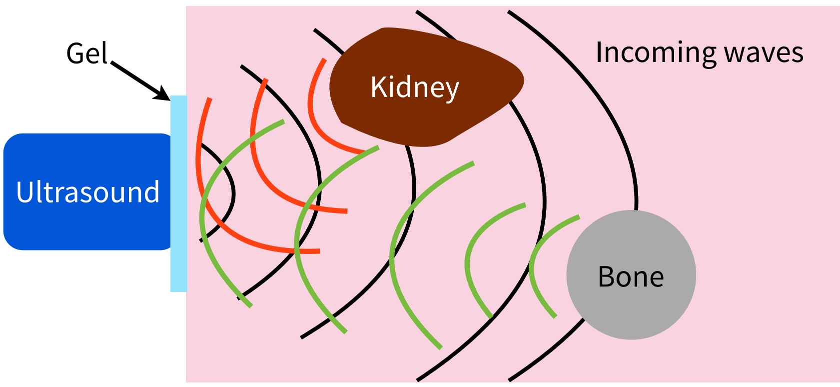
- If the impedances of the two materials are almost the same, then the reflected intensity is small

  - It is difficult to image objects that have an impedance near the impedance of the surrounding material (in the body this is water).

Material	*Acoustic Impedance (kg/m²s)*
Air	430 ←
Water	1.48×10^6
Fat	1.33×10^6
Muscle	1.64×10^6
Bone	6.27×10^6

Very different from water, so ultrasound sends waves through a gel

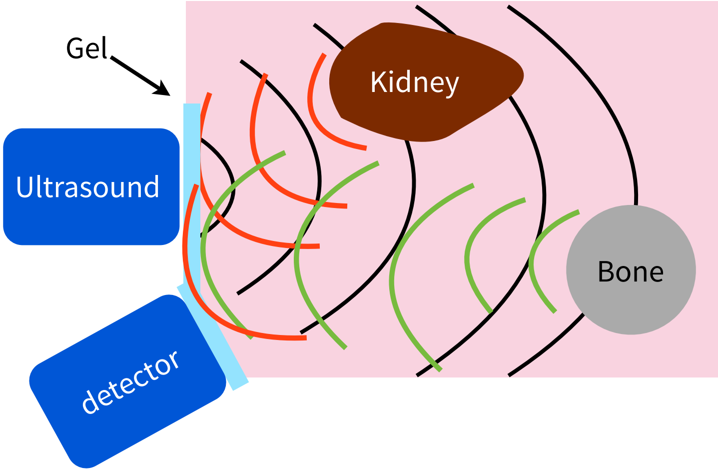
- Ultrasounds must be extremely sensitive to measure the weak reflected intensities
  - Reflected intensities are also reduced from absorption when leaving the body.



- The reflected signals arrive back at the ultrasound at different times depending on the depth of the object

$$t = \frac{2d}{v_{\mathrm{tissue}}} \rightarrow \boxed{d = \frac{v_{\mathrm{tissue}} t}{2} = \frac{(1570\ \mathrm{m/s})\mathrm{t}}{2}}$$

- This will tell us how deep the objects are, but not where they are in the body.



- Using more than one detector, that is at a different angle from the first allows one to get position information on the objects.

→ Can do 3D imaging

3D ultrasound image

Ultrasound can show you what a baby looks like before it comes out