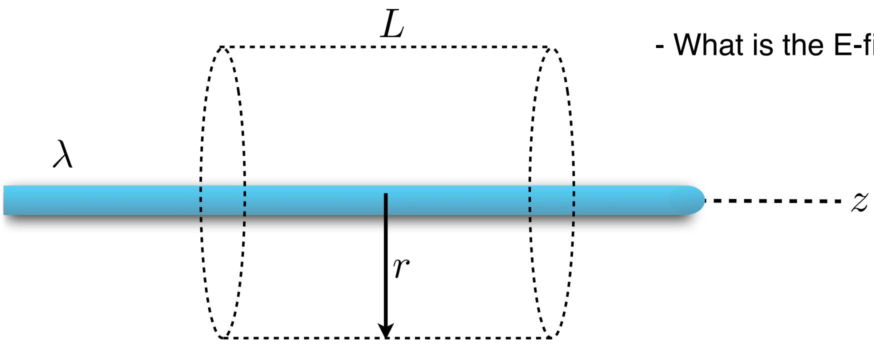
Ex. Cylindrical Symmetry:



- Consider an infinitely long rod with charge density λ

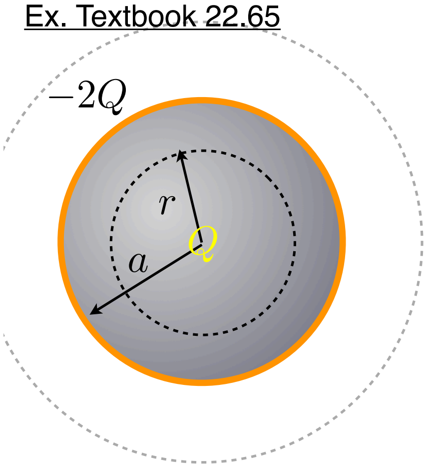
- What is the E-field everywhere outside?

- Choose gaussian surface:
    - Rod is cylindrical -> choose cylinder.
    - Now use symmetries
    - Can rotate about the z-axis and problem does not change -> E-field not in θ direction
    - Can move up and down z-axis and nothing changes -> E-field not in z-direction
    - E-field only in radial direction!

$$\phi = 2\pi r L E = \frac{Q_{\rm enc}}{\epsilon_0} = \lambda L \quad \Rightarrow \quad \boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}}$$

Electric Fields & Gauss' Law Sample Problems

Ex. Textbook 22.65



An insulating sphere of radius “a” has a total charge Q and is covered by a thin conducting shell of charge -2Q. What is the E-field inside and outside? Draw E-field as function of r.

Inside:

- We are given only total charge Q, convert to charge density:

$$\rho = \frac{Q}{V} = \frac{Q}{4/3\pi a^3}$$

- Recall that E-field is only in radial direction, as is dA and E-field constant over any spherical surface -> flux integral is just E*A
- Draw spherical gaussian surface at distance r
- E-field at distance r from center is then:

$$\phi = 4\pi r^2 E = \frac{Q_{\text{enc}}}{\epsilon_0} = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3$$

- plug back in charge density:

$$\boxed{\vec{E}_{\text{inside}} = \frac{Qr}{4\pi\epsilon_0 a^3}\hat{r}}$$

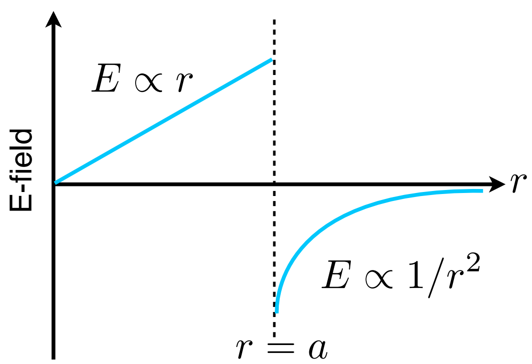
Outside:

- Remember, the E-field of spherical charges looks just like point charges!
- Draw spherical gaussian surface over entire sphere.
- What is net enclosed charge? $Q_{\text{net}} = Q + (-2Q) = -Q$

- Answer is simple point charge with charge -Q:

$$\boxed{\vec{E}_{\text{outside}} = \frac{-Q}{4\pi\epsilon_0 r^2}\hat{r}}$$

- Draw E-field as function of radius:



Ex. Textbook 22.66

- Suppose a nonconducting sphere of radius R has charge density: $\rho(r) = \frac{\beta}{r}\sin(\pi r/2R)$

- What is the total charge, and what is the E-field inside and outside?

Total Charge:

- Recall, $dQ = \rho dv \rightarrow Q = \int_V \rho dV$ $\qquad dV = 4\pi r^2 dr$ (for sphere)

$$Q = \beta \int_0^R \sin(\pi r/2R)\frac{4\pi r^2 dr}{r} = 4\pi\beta \int_0^R r\sin(\pi r/2R)\,dr$$

Integrate by parts: $\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)dx$

$$u = r \;\; ; \;\; v' = \sin(\pi r/2R) \rightarrow v = -\frac{2R}{\pi}\cos(\pi r/2R)$$

$$Q = 4\pi\beta\left[-\frac{2Rr}{\pi}\cos(\pi r/2R)\Big|_0^R - \int_0^R -\frac{2R}{\pi}\cos(\pi r/2R)\right]$$

$$Q = 4\pi\beta\left[-\frac{2Rr}{\pi}\cos(\pi r/2R) + \frac{4R^2}{\pi^2}\sin(\pi r/2R)\right]\Bigg|_0^R$$

- Boundaries are simple to evaluate:

@R:

$$\cos(\pi r/2R) \rightarrow 0$$
$$\sin(\pi r/2R) \rightarrow 1$$

@0:

$$\sin(\pi r/2R) \rightarrow 0$$
$$r\cos(\pi r/2R) \rightarrow 0$$

- Total charge is thus:

$$\boxed{Q = 4\pi\beta\frac{4R^2}{\pi^2} = \frac{16\beta R^2}{\pi}}$$

E-field outside:

- Again, E-field outside is easy, just like a point charge!!

$$\boxed{\vec{E}_{\mathrm{outside}} = \frac{Q_{\mathrm{enc}}}{4\pi\epsilon_0 r^2}\hat{r} = \frac{16\beta R^2}{\pi}\frac{1}{4\pi\epsilon_0 r^2}\hat{r} = \frac{4\beta}{\pi^2\epsilon_0}\frac{R^2}{r^2}\hat{r}}$$

E-field Inside:

- The E-field inside is also like that of a point charge, except we need to find the charge enclosed by a sphere of radius r.

- Fortunately, we have already calculated the charge as a function of r:

$$Q = 4\pi\beta\left[-\frac{2Rr}{\pi}\cos\left(\pi r/2R\right) + \frac{4R^2}{\pi^2}\sin\left(\pi r/2R\right)\right]$$

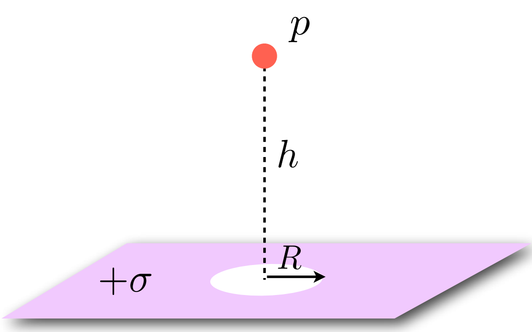
$$\vec{E}_{\text{inside}} = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 r^2}\hat{r} = \frac{4\pi\beta}{4\pi\epsilon_0 r^2}\hat{r}\left[-\frac{2Rr}{\pi}\cos\left(\pi r/2R\right) + \frac{4R^2}{\pi^2}\sin\left(\pi r/2R\right)\right]$$

$$\boxed{\vec{E}_{\text{inside}} = \frac{\beta}{\epsilon_0 r^2}\hat{r}\left[-\frac{2Rr}{\pi}\cos\left(\pi r/2R\right) + \frac{4R^2}{\pi^2}\sin\left(\pi r/2R\right)\right]}$$

At r=R:

$$\vec{E}_{\text{outside}} = \frac{4\beta}{\pi^2\epsilon_0}\hat{r} \qquad\qquad \vec{E}_{\text{inside}} = \frac{4\beta}{\pi^2\epsilon_0}\hat{r}$$

Ex. Textbook 22.68



Find the electric field at point P above an infinite charged plane with charge density $+\sigma$ and a hole cut or radius R cut out directly below P.

IMPORTANT TIP

Any time a problem has a hole in it, use the superposition principle:

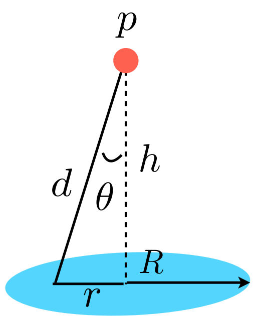
$$(+\sigma) + (-\sigma) = 0 = \text{Hole}$$

- Problem has two parts: 1) E-field from plane. 2) E-field from disc with density $\sigma' = -\sigma$

1): E-field from plane we already know: $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z}$

2): Now need E-field from disk, similar to charged disc problem.

$$dE = \frac{\sigma' dA}{4\pi\epsilon_0 d^2}$$



- Only need z-component due to symmetry

$$dE = \frac{\sigma' dA}{4\pi\epsilon_0 d^2}\cos\theta = \frac{\sigma' dA}{4\pi\epsilon_0 d^2}\frac{h}{d} \qquad d = \sqrt{r^2 + h^2}$$

- Need to integrate over disc area $dA = 2\pi r dr$

$$E = \frac{\sigma' h}{2\epsilon_0}\int_0^R \frac{r}{(r^2+h^2)^{3/2}}dr$$

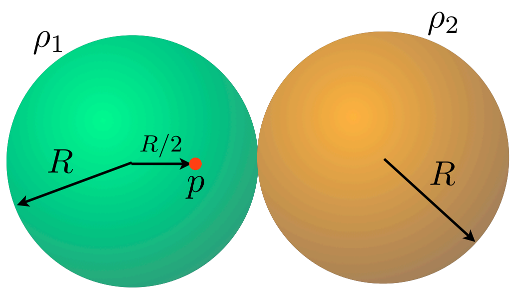
- Recall that this can be solved with a substitution: $u = r^2 + h^2$

$$\vec{E} = -\frac{\sigma'}{2\epsilon_0}\left[\frac{h}{\sqrt{R^2+h^2}} - 1\right]\hat{z} = -\frac{\sigma}{2\epsilon_0}\left[1 - \frac{h}{\sqrt{R^2+h^2}}\right]\hat{z}$$

- Combine the two E-field terms:

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\frac{h}{\sqrt{R^2+h^2}} = \frac{\sigma}{2\epsilon_0}\cos\theta \rightarrow \frac{\sigma}{2\epsilon_0} \qquad h \gg R}$$

Ex. Finding Charge Densities



Two spheres of radius R with charge densities ρ_1 and ρ_2 are touching. If the E-field at point P is zero, what is the ratio ρ_1/ρ_2?

- Problem is spherical.

- Remember that outside, spherical charge E-field looks like point charge

Sphere #1:

- Draw spherical gauss surface through P at R/2
- Only enclosed charge matters.
- Calculate total charge inside sphere of radius R/2

$$Q_1^{\text{enc}} = \frac{4}{3}\rho_1\pi\left(R/2\right)^3 = \frac{\rho_1\pi R^3}{6}$$

- E-field like point charge $E = \dfrac{Q^{\text{enc}}}{4\pi\epsilon_0 r^2}$

$$\Rightarrow \boxed{E_1 = \frac{\rho_1\pi R^3}{6}\frac{1}{4\pi\epsilon_0(R/2)^2} = \frac{\rho_1 R}{6\epsilon_0}}$$

Sphere #2:

- Same thing as sphere #1, but must add up charge over entire sphere #2.

$$Q_2^{\text{enc}} = \frac{4}{3}\rho_2 \pi R^3$$

- E-field at P like point charge a distance 3R/2 away.

$$\boxed{E = \frac{4}{3}\rho_2 \pi R^3 \frac{1}{4\pi\epsilon_0 (3R/2)^2} = \frac{4\rho_2 R}{27\epsilon_0}}$$

- To find ratio, set E1=E2 and solve:

$$\frac{\rho_1 R}{6\epsilon_0} = \frac{4\rho_2 R}{27\epsilon_0} \Rightarrow \boxed{\frac{\rho_1}{\rho_2} = \frac{8}{9}}$$