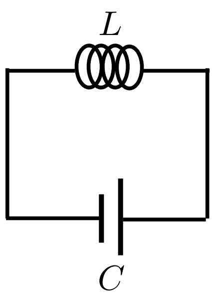
Electromagnetic Oscillations

- So far we have only discussed circuits with currents that flow only in one direction

- In this chapter we look at circuits where the current and voltage oscillate sinusoidally in time with a given frequency

- The simplest example is the **LC-Oscillator:**



- Assume capacitor is initially charged

- The energy stored in each component is given via:

$$U_C = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$
$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2$$

- We could use Faraday to get circuit equation of motion, or we can use the fact that the total energy $U = U_C + U_L$ is conserved $dU/dt = 0$:

$$\frac{dU}{dt} = \frac{d}{dt}\left[\frac{q^2}{2C} + \frac{1}{2}L\left(\frac{dq}{dt}\right)^2\right] = \frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} = 0$$

- Divide both sides by dq/dt

- This ignores the trivial solution where there is no initial charge on capacitor

$$\boxed{\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = 0}$$

- Charge on capacitor obeys the equation of motion for a harmonic oscillator.

- Solution is given by: $q(t) = q_{max} \cos(\omega_0 t - \phi)$ $\quad \omega_0 = \dfrac{1}{\sqrt{LC}}$

Phase-angle determined by initial conditions

- Current can be found by differentiating: $I(t) = \dfrac{dq}{dt} = \underbrace{-q_{max}\omega_0}_{\text{I\_max}} \sin(\omega_0 t - \phi)$

$$\Rightarrow I(t) = -I_{\max} \sin(\omega_0 t - \phi)$$

- What about the energy stored in capacitor and inductor?

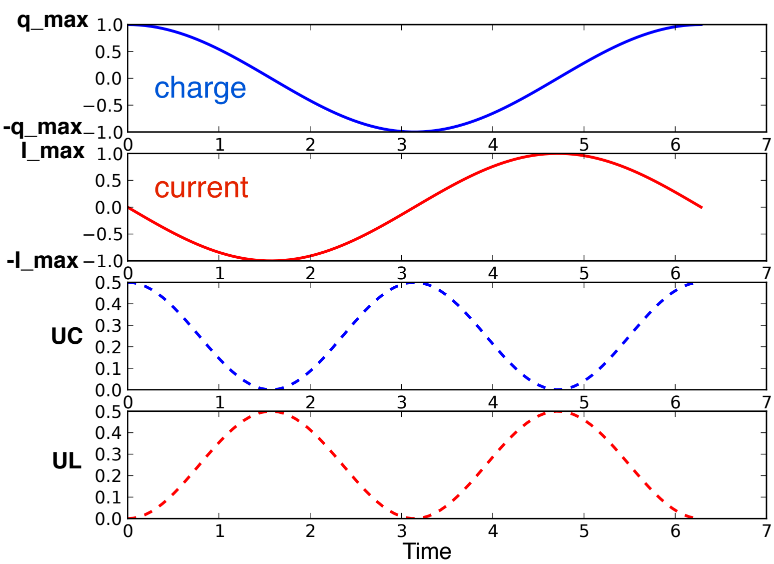
$$U_C = \frac{q^2}{2C} = \frac{q_{max}^2}{2C} \cos^2(\omega t - \phi)$$

$$U_L = \frac{1}{2}LI^2 = \frac{1}{2}LI_{\max}^2 \sin^2(\omega t - \phi)$$

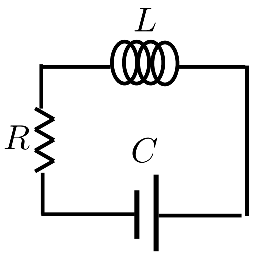
- but, $I_{\max}^2 = q_{\max}^2\omega_0^2 = \dfrac{q_{\max}^2}{LC}$ $\Rightarrow$ $\dfrac{1}{2}LI_{\max}^2 = \dfrac{q_{\max}^2}{2C}$

$U_L = \dfrac{q_{\max}^2}{2C} \sin^2(\omega t - \phi)$ -U_L is maximum when U_C is minimum

<u>Ex</u>. L=C=1 -> $\omega = 1$ & $\phi = 0$



LRC-Circuit



- Now energy is no longer conserved, but is dissipated in resistor.
- Energy lost must be equal to power dissipated by resistor:

$$\frac{dU}{dt} = -I^2 R$$

- Equation of motion for charge is now:

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} = -I^2 R \quad \Rightarrow \quad \frac{q}{C}\frac{dq}{dt} + L\frac{dq}{dt}\frac{d^2q}{dt^2} + \left(\frac{dq}{dt}\right)^2 R = 0$$

- We will again divide by dq/dt and divide by L:

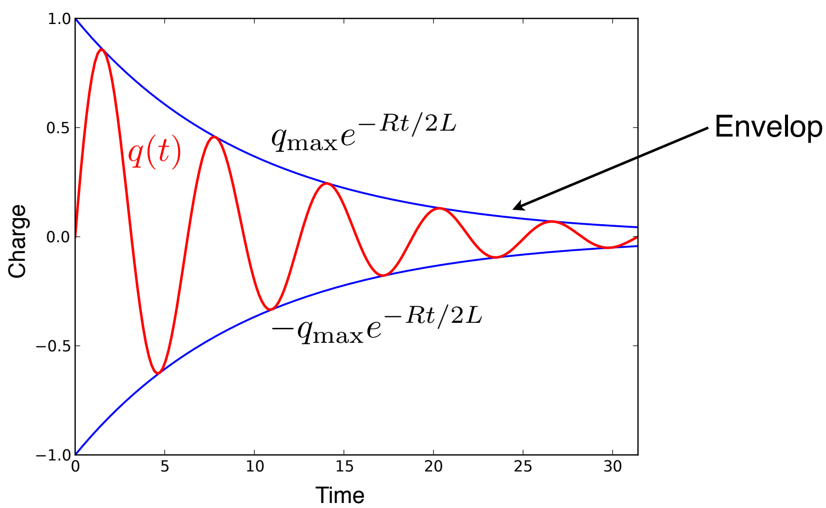
$$\boxed{\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0}$$

Damped Harmonic Oscillator Equation

- If resistance is small: $q(t) = q_{\max} e^{-Rt/2L} \cos(\omega t - \phi)$

not the same frequency!

- Frequency is shifted: $w = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$ $\qquad \omega_0 = 1/\sqrt{LC}$



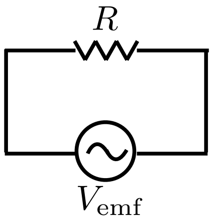
AC-Circuits:

- We will now look at circuits driven by sinusoidally varying voltage sources:

$$V = V_{\max} \sin(\omega t) \quad \Rightarrow \quad I = I_{\max} \sin(\omega t - \phi)$$

- Current may or may not be in phase with the driving voltage

Resistor only circuit:



- Using Kirchoff's voltage law:

$$V_R - V_{\text{emf}} = 0$$

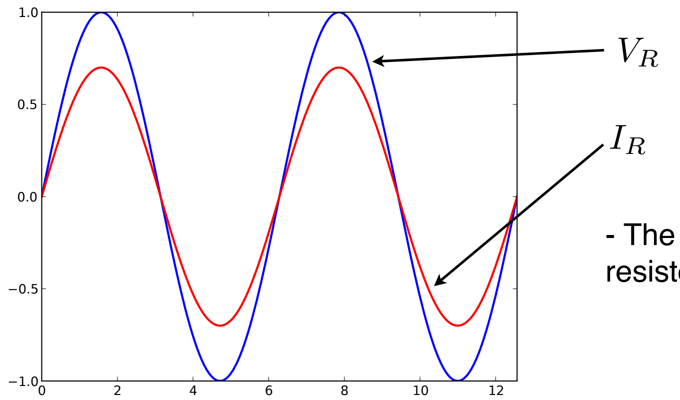
- Therefore the voltage across the resistor is:

$$V_R = V_{\text{emf}}^{\max}\sin(\omega t) = V_R^{\max}\sin(\omega t)$$

- The voltage and current through the resistor are given by the standard relation: $V_R = I_R R$

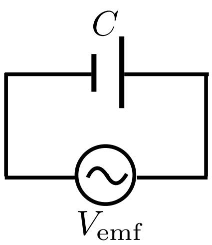
- The current through the resistor is given by Ohm's law:

$$I_R = \frac{V_R}{R} = \frac{V_R^{\max}}{R}\sin(\omega t) = I_R^{\max}\sin(\omega t)$$



- The current and voltage through the resistor are in-phase -> $\phi = 0$

Capacitor only circuit:



- Again, using Kirchoff's voltage law: $V_C - V_{\text{emf}} = 0$

$$V_C = V_{\text{emf}}^{\max}\sin(\omega t) = V_C^{\max}\sin(\omega t)$$

- To get expression for current we must use $q = CV$

$$q(t) = CV_C^{\max}\sin(\omega t) \Rightarrow I(t) = \frac{dq}{dt} = CV_C^{\max}\omega\cos(\omega t)$$

- We can define the **reactance** of a capacitor: $X_C = \dfrac{1}{\omega C}$

$$\Rightarrow I_C(t) = \frac{V_C^{\max}}{X_C}\cos(\omega t) = I_C^{\max}\cos(\omega t)$$

- The amplitude of the current and voltage are related by: $V_C = I_C X_C$

  - We see that the reactance behaves like a resistor, but no energy is lostω

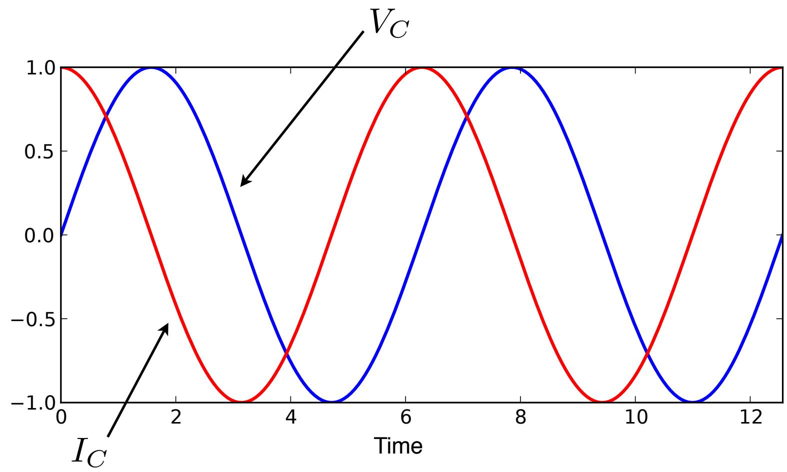
  - A large reactance can be made by a large capacitor or if the frequency is very high.

- We have $I_C(t) = I_C^{\max}\cos(\omega t)$ but $\cos(\omega t) = \sin(\omega t + \pi/2)$

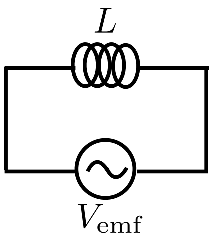
- Therefore, the current can be expressed as: $I_C(t) = I_C^{\max}\sin(\omega t + \underbrace{\pi/2}_{\text{phase}})$

- The current is pi/2 out of phase with the voltage.

- Since $\phi = -\pi/2$ we say that "**the current leads the voltage by pi/2**"



<u>Inductor only circuit:</u>



- Can no longer use Kirchoff's voltage law, must use Faraday

$$-V_{\text{emf}} = -L\frac{dI}{dt} \qquad V_{\text{emf}} = L\frac{dI}{dt}$$

$$L\frac{dI}{dt} = V_{\text{emf}}^{\max}\sin(\omega t) = \mathcal{E}_L^{\max}\sin(\omega t) \;\Rightarrow\; \frac{dI}{dt} = \frac{\mathcal{E}_L^{\max}}{L}\sin(\omega t)$$

- But we want current I, not dI/dt: $I_L(t) = \int \frac{dI_L}{dt}dt = \int \frac{\mathcal{E}_L^{\max}}{L}\sin(\omega t)dt$

$$\Rightarrow\; I_L(t) = -\frac{\mathcal{E}_L^{\max}}{\omega L}\cos(\omega t)$$

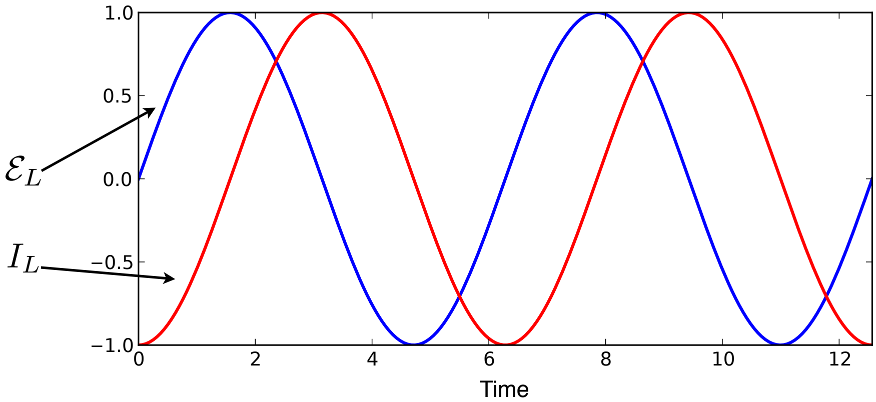
- Now define the reactance of an inductor: $X_L = \omega L$

$$\Rightarrow\; I_L(t) = -\frac{\mathcal{E}_L^{\max}}{X_L}\cos(\omega t) = -I_L^{\max}\cos(\omega t)$$

- We now use the relation: $-\cos(\omega t) = \sin(\omega t - \pi/2)$

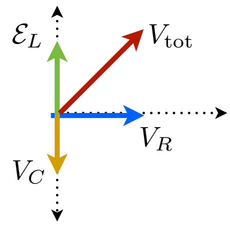
$$\Rightarrow \quad I_L(t) = I_L^{\max} \sin(\omega t - \pi/2)$$

- For circuit with only an inductor $\phi = +\pi/2$ so "**the current is behind voltage by pi/2**".



- The EMF of the inductor can be expressed as: $\mathcal{E}_L = I_L X_L$

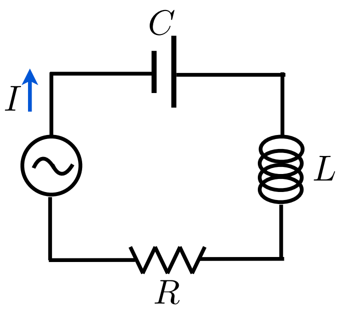
$$V_{\text{tot}}^2 = V_R^2 + (\mathcal{E}_L - V_C)^2$$



Summary

Resistor:	$V_R = I_R R$	Current and voltage in phase	$\phi = 0$
Capacitor:	$V_C = I_C X_C$	Current leads the voltage	$\phi = -\pi/2$
Inductor:	$\mathcal{E}_L = I_L X_L$	Current is behind the voltage	$\phi = +\pi/2$

Series LRC Circuit:



- Lets assume the driving voltage is given by: $V(t) = V_0 \cos(\omega t)$

$$\oint \vec{E} \cdot d\vec{l} \neq 0$$ Must use Faraday

- Go around loop in same direction as the current

$$V_C + 0 + IR - V_0 \cos(\omega t) = -L\frac{dI}{dt}$$

inductor

- Using $I = dq/dt$ and $V_c = q/C$:

$$\boxed{L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \cos(\omega t)}$$

Damped-Driven Harmonic Oscillator Equation

- Second-order ordinary differential equation
- We do not know how to solve this differential equation
- We will just analyze the answer without deriving it.

- The equation for the current in the circuit is given by:

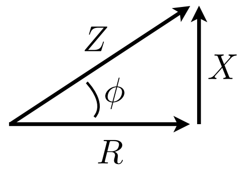
$$I(t) = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \cos(\omega t - \phi) = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \cos(\omega t - \phi)$$

- Let us define the total reactance: $X = X_L - X_C$

- Denominator of current equation is now: $\sqrt{R^2 + X^2} \equiv Z$ **Impedance**

$$\Rightarrow \quad I(t) = \frac{V_0}{Z} \cos(\omega t - \phi)$$

- The resistance R, reactance X, and impedance Z form triangle
(remember vector sum of voltages)



$$\tan\phi = \frac{X}{R}$$

- The impedance Z behaves as an effective resistance for the circuit

- This is the steady-state solution.. only valid after long times.

- Current can be delayed if $\phi > 0$ (from inductor)

- Current can lead voltage if $\phi < 0$ (from capacitor)

- There is only one frequency at which the current in the circuit reaches its maximum value. This is called the **resonance frequency**.

$$I_{\max} = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + X^2}} \longleftarrow \left(\omega L - \frac{1}{\omega C}\right)^2$$

- Resistance of circuit is fixed, therefore maximum value of current through circuit is given by the relation

$$X = 0 \rightarrow \omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}$$ **Resonance frequency**

not the same as driving frequency

- On resonance $Z = R$ and therefore $I_{\max} = \frac{V_0}{R}$