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ANALYSIS OF THE THREE-DIMENSIONAL MOTION OF FLEXIBLE CYLINDERS UNDER VORTEX-INDUCED VIBRATIONS USING INVARIANT MANIFOLDS

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Abstract. *The present work presents a methodology for obtaining a minimal model to represent the phenomenon of vortex-induced vibrations on flexible cylinder with the usage of wake-oscillators. The continuous model is discretized via a Galerkin scheme to obtain a reduced-order model. The resulting set of equations is then further reduced by the proposed methodology of obtaining nonlinear modes of vibration to represent the wake-variable as function of the cross-wise motion. The results show that the use of the nonlinear mode retains the main qualitative aspects of the phenomenon while also presenting sufficient quantitative agreement with the complete system. In addition, the use of nonlinear modes allow for a significantly smaller computational time and a fast and precise identification of the lock-in region.*

Keywords: *Vortex-Induced Vibrations, Nonlinear Modes, Reduced-Order Modelling, Flexible Structures*

1. INTRODUCTION

A classical problem of fluid-structure interaction in many engineering applications is that of vortex-induced vibrations (VIV). It is a self-sustained and self-limited phenomenon activated by a near-resonant condition between the vortex shedding around a structure and its modes of vibration, rendering it a significant problem for structural analysis, especially concerning fatigue evaluation, and more recently, in low-power energy harvesters. The literature on the topic is rich in many aspects, with a detailed description of the phenomenon for its simplest case, an elastic-mounted rigid cylinder oscillating in a single direction, being presented in Khalak and Williamson (1999). The extension for the case of two degrees-of-freedom (DOF) is also well-documented. In this condition, the presence of the inline displacement interacting with the wake and the structure generates higher amplitudes of response in the cross-wise direction, as well as the phenomenon of lock-in also in the inline direction for the appropriate values of reduced velocities. Detailed explanations of the phenomenon together with experimental results for support and illustration can be found in Dahl *et al.* (2007), Jauvtis and Williamson (2004), Blevins and Coughran (2009), Franzini *et al.* (2012), and Franzini *et al.* (2013).

Concerning flexible structures, significant changes happen to the phenomenon, and many techniques can be employed to investigate it. The modal amplitudes of response are usually larger than in the rigid case, and since the structure is flexible, it often happens a jump of the lock-in from one mode to another rather than the classical desynchronization that appears in the rigid cylinder case. In Chaplin *et al.* (2005b) and Chaplin *et al.* (2005a), experimental results show that reponses dominated by a single mode, multi-modal responses, or travelling wave conditions may appear in the flexible cylinder depending on the reduced velocity. In Fujarra *et al.* (2001) and later in Defensor Filho *et al.* (2022), the influence of orthotropic stiffness of the structure is shown in different experimental results, including the appearing of a new response branch of high amplitudes and oscillating frequencies. In Franzini *et al.* (2016), it is shown by experimental results that some reduced velocities lead to the synchronization of inline and cross-wise vibrations in the flexible structure.

Regarding the modelling of VIV, the problem can be tackled from two broad strategies. One is the usage of numerical schemes for the whole problem, modelling the structure as a flexible continuum, with the dimensionality required by the shape of the structure (rod, shell, plate, or solid body), and the fluid modelled by classical fluid mechanics equations. However, this leads to a high computational load for implementing the finite element method (FEM) for the structure and computational fluid dynamics (CFD) for the fluid, which can prove unfeasible, or demand excessive amounts of time in certain scenarios, like the one of very flexible risers used in the offshore industry. An alternative is the use of the phenomenological models. Such models are empirically based, in which oscillators are used to represent a wake-variable that is taken as an idealization of the effects of the flow over the structure. The oscillator adopted is chosen by its qualitative properties, such as its capacity of generating limit cycles. One of the most commonly found oscillators in the literature is the Van der Pol oscillator, which presents the necessary features to model VIV. Actually, in Aranha (2004), a solution based on perturbation techniques is used to show that for the case of fixed cylinders, the dynamics of the wake actually follows a Ginzburg-Landau equation up to the first order, which in the abscence of the diffusion term along the length of the cylinder becomes the Van der Pol oscillator, giving then a mathematical foundation for the use of such a model.

Many wake-oscillator models are available in the literature, such as the ones presented in Facchinetti *et al.* (2004), Ogink and Metrikine (2010), and Qu and Metrikine (2020). The usual application is to discretize the structural equations of motion using a FEM decomposition or by means of Galerkin projections. Some works then attach the wake oscillators directly to the discretized equations, while others consider the wake variable to be spatially distributed and then apply an *ad-hoc* shape for its projection. In any case, the number of equations to integrate can be large, leading to significant computational costs, especially if a large number of scenarios need to be evaluated. Because of this, the idea of further reducing the size of the dynamical system becomes appealing. One of the few works in the literature to deal with that idea is presented in Keber and Wiercigroch (2008) in which nonlinear modes of vibration are used to account for the nonlinear contribution of higher structural modes over the single one that is simulated together with an attached wake-oscillator.

Nonlinear modes can be a very useful and systematic way of reducing further the size of dynamical systems, since they are able to retain the most important aspects of the dynamical response without recquiring the integration of the full system. However, its applications in the literature do not go further than what is presented in Keber and Wiercigroch (2008) when VIV is concerned. Some initial attempts with some degree of success were presented in Vernizzi and Franzini (2019) for a rigidly mounted cylinder, and Vernizzi *et al.* (2020) for a flexible cylinder considering only the cross-wise vibrations. In the present work, a complete scheme of discretizing the continuum problem using a Galerkin discretization followed by a further reduction of the system using nonlinear modes is presented for the three-dimensional vibrations of a flexible cylinder. The dynamics of the wake-oscillator are retained by means of nonlinear modes representing the wake-variable as a function of the cross-wise motion, leading to a system that recquires fewer equations to be integrated.

2. BRIEF DESCRIPTION OF NONLINEAR MODES AND THEIR EVALUATION

An usually adopted method for investigating the dynamics of linear systems is by means of a modal decomposition using the modes of vibration of the system. In the particular case of ROMs representing flexible structures, obtained by discretization methods such as the Galerkin projection, two possibilities can occur. The first one is that the exact modal shapes of the original problem were used as projection functions in the Galerkin method, and the resulting system is already in the form of its modal equations. The other scenario is that the projection functions are not the modal shapes of the original problem, which may occur due to the lack of an expression or approximation for the modal shapes, or even due to mathematical difficulties arising when using the actual modal shapes as projection functions. This condition leads to a discretized system that is not in the form of its modal equation, but can be transformed into its modal form by an appropriate coordinate transform using the eigenvectors of the system of ordinary differential equations obtained.

For linear systems, there are important advantages to working with the modal dynamics. The dynamics can be solved for each of the modal components in a rather simple way, and then the resulting response for the entire system can be evaluated by a simple linear superposition of the individual responses. Furthermore, given initial conditions related to one particular mode will result in responses contained in that mode. Also, forcing terms applied to the system that are resonant or nearly resonant to a specific mode will cause a response that has most of its content contained in such a mode.

In order to expand such concepts and advantages to nonlinear systems, attempts were made to address the existence and definition of what would be nonlinear modes of vibration, retaining similar properties from its linear counterpart. An early successfull description was made by Rosenberg (1966), describing nonlinear modes as an invariant set as function of the degrees of freedom of the system and using an energy conservation relation to express the velocities. The definition of the nonlinear modes as an invariant set recalls to one of the properties of linear modes, that is, if an initial condition is given that lies over the invariant set, then the motion will remain bounded to that set for all its development.

A general extension of such concept was then proposed by Shaw and Pierre (1993), defining the nonlinear modes as a manifold that represent the response of the generalized coordinates of a dynamical system as a function of a chosen group of master coordinates. In the first proposition of such idea, the authors used two generalized coordinates, being a displacement/velocity pair, as master coordinates. Using a geometrical explanation, each of the coordinates is then designated by a spatial curve, the manifold, parametrized by the master coordinates. The coordinate then assumes only values that lies on such manifold if the dynamical response is contained in such mode. It is straightforward to conclude then that the difference in such representation for linear or nonlinear modes is that in the linear case, such curves are always a plane, while in the nonlinear case, it can be any curve defined by a function of the master coordinates.

Since then, the literature on nonlinear modes became extensive in terms of computation techniques, mathematical definitions, and problems to which the concept is applied. Classical examples of applications can be found in Nayfeh *et al.* (1996), King and Vakakis (1996), Pesheck *et al.* (2002), and Jiang *et al.* (2005). An approach to structures discretized via the Finite Element Method can be found in Soares and Mazzilli (2000), Mazzilli and Baracho Neto (2002), and Baracho Neto and Mazzilli (2005). Investigations concerning the application over the original continuous model are presented in Nayfeh and Nayfeh (1994) and Nayfeh *et al.* (1999). Finally, more detailed reviews, including methods for the computation of the nonlinear modes are found in Kerschen *et al.* (2009), Peeters *et al.* (2009), and Mazzilli *et al.* (2022).

Resuming to the effective computation of the shapes of the nonlinear modes, consider the 2N-dimensional dynamical system representing an N-DOF mechanical system given as

$$\dot{x}_i = y_i \qquad i = 1, ..., N, \tag{1}$$

$$\dot{y}_i = f_i\left(\mathbf{x}; \mathbf{y}\right) \qquad i = 1, ..., N, \tag{2}$$

with overdots representing a time derivative as commonly practiced, $\mathbf{x}$ is the vector of generalized positions, $\mathbf{y}$ is the vector of generalized velocities and $f_i\left(\mathbf{x}; \mathbf{y}\right)$ are the generalized forces normalized by the inertial term from each equation of motion. Assume then that exists at least one motion such that the generalized coordinates and velocities can be functionally related to a single associated pair of generalized position/velocity. The master pair can then be defined as (x_1, y_1) without loss of generality, since the indexing of the equations does not impact the results. Now it is necessary to seek relations of the form $x_i = X_i\left(x_1, y_1\right)$ and $y_i = Y_i\left(x_1, y_1\right)$ for $i = 2, ..., N$, observing that $X_1 = x_1$ and $Y_1 = y_1$.

The manifolds sought are then 2-dimensional, as a natural implication of chosing two master coordinates. It is possible to obtain manifolds of larger dimensions by using more master coordinates, which may be required depending on the system under study. Such approach, however, does not change the general mathematical steps needed for obtaining the manifolds. Using now the same techniques of center manifold theory (see for example Nayfeh and Balachandran (1995)) to eliminate the temporal dependence of the equations, it is possible to arrive to

$$Y_i = \frac{\partial X_i}{\partial x_1} y_1 + \frac{\partial X_i}{\partial y_1} f_1\left(x_1, X_2, ..., X_N; y_1, Y_2, ..., Y_N\right) \qquad i = 1, ..., N, \tag{3}$$

$$f_i\left(x_1, X_1, ..., X_N; y_1, Y_1, ..., Y_N\right) = \frac{\partial Y_i}{\partial x_1} y_1 + \frac{\partial Y_i}{\partial y_1} f_1\left(x_1, X_2, ..., X_N; y_1, Y_2, ..., Y_N\right) \qquad i = 1, ..., N. \tag{4}$$

The problem is that Eqs. 3 and 4 usually are at least as difficult to solve as the initial problem. However, it is possible to use appropriate coordinate changes and series expansions to find an approximated geometry of the manifolds near an equilibrium solution. The main aspects of the dynamical response of the system are then captured by the geometry of the manifolds, and then there is only need to find a temporal solution or numerically integrate the modal equations given by

$$\dot{x}_1 = y_1, \tag{5}$$

$$\dot{y}_1 = f_1\left(x_1, X_2, ..., X_N; y_1, Y_2, ..., Y_N\right). \tag{6}$$

Finally, the complete solution is simply obtained by substituting the temporal series of $x_1(t)$ and $y_1(t)$ into the functions that define the geometry of the obtained manifolds.

3. MATHEMATICAL MODEL

The mathematical model is developed first for the structure and then for the fluid interaction with the structure in a step by step manner, with the coupled model being presented at the end of this section.

3.1 Structural model

The model herein considered is that of a vertical beam immersed in fluid, pinned at both ends, with circular cross-section, and subjected to the free stream of velocity U_∞, considered constant along the length of the beam. This scenario is very common in offshore engineering and of high practical interest. The structure is modelled as an Euler-Bernoulli beam allowed to vibrate in both transversal directions, and the nonlinear structural terms are disregarded due to the small amplitude of vibrations typical of VIV, leaving all the nonlinearities to the wake-oscillator model. The incorporation of such terms in the manifold scheme is straightforward and does not impose additional complexity in the methodology as well. The effect of an initial tensile force and also the weight of the structure are both considered in the model since they have a significant contribution to the linear dynamics of the structure.

The complete development of the equations of motion for this problem are found in the literature. Using the simmetry of the cross-section and the model available in Vernizzi *et al.* (2019), the equations of motion of the structure read

$$\left(\mu + \mu_a\right)\ddot{U} + EIU'''' - \left(\left(T_b + \gamma_s Z\right)U'\right)' = f_x, \tag{7}$$

$$(\mu + \mu_a)\ddot{V} + EIV'''' - ((T_b + \gamma_s Z)V')' = f_y, \tag{8}$$

where U and V are the inline and cross-wise displacements of the beam respectively, with respect to the free-stream direction. The term γ_s is the 'submerged weight' of the structure, resulting from the combination of the weigth and buoyancy forces, EI is the flexural stiffness of the beam, μ is the mass per unit length, Z is the axial coordinate, considered zero at the lowest end of the vertical beam, μ_a is the potential added mass per unit length and T_b is the tensile force at the bottom cross-section of the beam. The forcing terms that appear in the equations, f_x and f_y, are the composition of all external forces that may act on the beam in the indicated directions.

3.2 Fluid interaction model

In order to obtain the forcing terms over the structure due to the fluid-structure interaction, let us start by considering the vectorial scheme shown in Fig. 1, which includes the vectorial evaluation of the relative velocity.

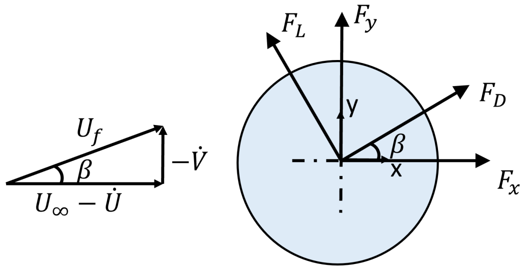

Figure 1. Relative velocity composition

Analysing Fig. 1, it is a simple task to write the forcing terms over the structure in the x and y directions as functions of the developed drag and lift forces, F_D and F_L respectively. The relation for the relative velocity between the structure and the fluid is also easily obtained.The resulting forces in the x and y directions are simply

$$F_x = \frac{1}{2}\rho U_f^2 D\left(-C_L \sin\beta + C_D \cos\beta\right) = \frac{1}{2}\rho D\sqrt{\left(U_\infty - \dot{U}\right)^2 + \dot{V}^2}\left(C_L\dot{V} + C_D\left(U_\infty - \dot{U}\right)\right), \tag{9}$$

$$F_y = \frac{1}{2}\rho U_f^2 D\left(C_L \cos\beta + C_D \sin\beta\right) = \frac{1}{2}\rho D\sqrt{\left(U_\infty - \dot{U}\right)^2 + \dot{V}^2}\left(C_L\left(U_\infty - \dot{U}\right) - C_D\dot{V}\right), \tag{10}$$

with ρ being the specific mass of the surrounding fluid, D is the diameter of the beam, C_L and C_D are instantaneous lift and draf coefficients respectively. Now, both the lift and drag coefficients are written in terms of a wake-variable, tied to a wake-oscillator model based on a Van der Pol equation. Many different models following this mathematical structure are available in the literature. Since the task itself of obtaining nonlinear modes for the wake-variable is still on its initial steps in the literature, this work uses a simplified version of an existing model, namely, the one presented in Qu and Metrikine (2020). In such case, the coefficients are given as

$$C_L = \frac{1}{2}qC_L^0, \tag{11}$$

$$C_D = \overline{C}_D + \alpha\left(\frac{qC_L^0}{2}\right)^2, \tag{12}$$

with C_L^0 being the amplitude of the fluctuation of the lift coefficient for a stationary cylinder, $\overline{C}_D$ being the mean drag coefficient of a stationary cylinder and α is a calibration constant that can be adjusted with results from experimental studies. The variable q is the wake-variable, and in this work will obey the equation

$$\ddot{q} + \varepsilon\omega_s\left(q^2 - 1\right)\dot{q} + \omega_s^2 q = \frac{A}{D}\ddot{V}, \tag{13}$$

were ω_s is the vortex shedding frequency, while ε and A are calibration coefficients. Notice that there is no effect of the inline displacement over the wake-oscillator equation as in Qu and Metrikine (2020). This simplification is herein made due to the complications posed by this term in the present attempt, being left for future investigations.

3.3 Coupled model

Considering that no other external forces act over the beam, the final model can be obtained by simply using $f_x = F_x$ and $f_y = F_y$ in Eqs. 7 and 8, together with adding Eq. 13 in the set of equations to be solved. Now, another simplification is adopted due to the presence of the square roots in the forcing terms. The problem is that they generate an obstacle in obtaining ROMs via the Galerkin method, since the necessary projection integrals would be dependent on the state of the system, thus the reduction of the continuum equations to a discretized set would require an evaluation of projection integrals for each time instant. That said, the square roots are expanded in polynomial series. Additionally, only terms that are linear on the variables U, V are retained, while nonlinear terms in q related to the drag force fluctuation are kept. This actually renders a model similar to that presented in Facchinetti *et al.* (2004), which can be obtained by applying the same mathematical procedures over the hydrodynamic forces decomposition, but preserving the qualitative aspect of a fluctuating drag force as well during the vortex shedding. Another reason for keeping a higher order term in q alone is because this variable usually achieves amplitudes of 5 to 10 times those of the variables U and V in such wake-oscillator models. Using the proposed expansion, the model becomes

$$(\mu + \mu_a)\ddot{U} + EIU'''' - ((T_b + \gamma_s Z)U')' = \frac{\rho D \overline{C}_D U_\infty^2}{2} + \frac{\rho D \alpha \left(\frac{qC_L^0}{2}\right)^2 U_\infty^2}{2} - \rho D \overline{C}_D U_\infty \dot{U}, \tag{14}$$

$$(\mu + \mu_a)\ddot{V} + EIV'''' - ((T_b + \gamma Z)V')' = \frac{\rho D C_L^0 U_\infty^2 q}{4} - \frac{\rho D \overline{C}_D U_\infty \dot{V}}{2}, \tag{15}$$

$$\ddot{q} + \varepsilon\omega_s\left(q^2 - 1\right)\dot{q} + \omega_s^2 q = \frac{A}{D}\ddot{V}. \tag{16}$$

Defining dimensionless variables $\xi = Z/\ell$, $\tau = \omega_n t$, $v = V/D$ and $u = U/D$, the equations of motion read

$$\ddot{u} + \frac{\rho D^2 \overline{C}_D U_r}{2\pi(\mu+\mu_a)}\dot{u} + \frac{EIu''''}{(\mu+\mu_a)\omega_n^2\ell^4} - \frac{(T_b + \gamma_s \ell\xi)u''}{(\mu+\mu_a)\omega_n^2\ell^2} - \frac{\gamma_s u'}{(\mu+\mu_a)\omega_n^2\ell} = \frac{\rho D^2 \overline{C}_D U_r^2}{8\pi^2(\mu+\mu_a)} + \frac{\rho D^2\left(qC_L^0\right)^2 U_r^2}{32\pi^2(\mu+\mu_a)}, \tag{17}$$

$$\ddot{v} + \frac{\rho D^2 \overline{C}_D U_r}{4\pi(\mu+\mu_a)}\dot{v} + \frac{EIv''''}{(\mu+\mu_a)\omega_n^2\ell^4} - \frac{(T_b + \gamma_s \ell\xi)v''}{(\mu+\mu_a)\omega_n^2\ell^2} - \frac{\gamma_s v'}{(\mu+\mu_a)\omega_n^2\ell} = \frac{\rho C_L^0 U_r^2 D^2}{16\pi^2(\mu+\mu_a)}q, \tag{18}$$

$$\ddot{q} + \varepsilon U_r S_t\left(q^2 - 1\right)\dot{q} + (U_r S_t)^2 q = A\ddot{v}, \tag{19}$$

where U_r is the reduced velocity and S_t is the Strouhal number, commonly found in VIV analysis. The final step of the modelling is the Galerkin projection. Since the objective of this work is focused on the nonlinear mode for the wake-variable, a single mode will be used in each direction of motion, namely the first mode of vibration. For the variable q, it is assumed that its spatial distribution will follow the modal shape of the structure, corresponding to assume that the motion of the structure dominates the vortex shedding synchrony along the length. This is made considering the lack of a better discussion on this matter in the literature to furnish a better alternative, up to the best of the authors' knowledge. Since the beam is under the effect of a varying tensile force along its length, the modal shape is not a trigonometric function. Thus, a Bessel-like function presented in Mazzilli *et al.* (2014) is used. The application of such function in a Galerkin scheme for a similar problem can be found in Vernizzi *et al.* (2019). The final model can be put in the generic form

$$\ddot{u} + 2\beta_1\dot{u} + \beta_2 u + \beta_4 q^2 = \beta_5 \tag{20}$$

$$\ddot{v} + \beta_1\dot{v} + \beta_2 v + \beta_3 q = 0 \tag{21}$$

$$\ddot{q} + \theta_1 q^2\dot{q} + \theta_2\dot{q} + \theta_3 q + \theta_4\ddot{v} = 0, \tag{22}$$

with the coefficients being evaluated by the Galerkin projection. The same symbol for the continuous displacements are herein used for the discrete ones without loss of understanding since the expansion is made in a single projection function for each coordinate. Notice also that the coefficients are function of the reduced velocity.

3.4 Nonlinear mode scheme

The nonlinear modes are calculated to set q and $\dot{q}$ as functions of the pair $(v, \dot{v})$, being named X_q and Y_q. Variable u is not included since it does not influence the wake-oscillator in the present model. One could argue that nonlinear modes could be conceived to represent u as well, since it will respond as a function of q. This possibility however will not be taken herein, since a future expansion including the inline influece over the wake-oscillator may remove this possibility.

In order to compute the desired nonlinear modes, a coordinate change is necessary, since a classical polynomial expansion in the pair $(v, \dot{v})$ is not able to predict the actual response in the typical limit cycle amplitudes of VIV. To that end, the variables may be written as $v = r\cos\phi$ and $\dot{v} = r\sin\phi$. The equations of motion for the new variables are easily obtained by substituting the transformation in Eq. 21 together with the compatibility equation resulting from the transformation itself given by

$$\dot{r}\cos\phi - r\dot{\phi}\sin\phi = r\sin\phi. \tag{23}$$

Now, the manifolds are assumed to be an expansion in these new variables as

$$X_q = a_1 r\cos\phi + a_2 r\sin\phi + a_3 r^3\cos\phi + a_4 r^3\sin\phi + a_5 r\cos 3\phi + a_6 r\sin 3\phi, \tag{24}$$

$$Y_q = b_1 r\cos\phi + b_2 r\sin\phi + b_3 r^3\cos\phi + b_4 r^3\sin\phi + b_5 r\cos 3\phi + b_6 r\sin 3\phi. \tag{25}$$

Notice that quadratic terms in $r\cos\phi$ and $r\sin\phi$ are not present in the expansions. This is done since such terms break the quadrant simmetry of the Van der Pol oscillator, thus not being part of a nonlinear mode representing this type of oscillator. The manifolds geometries are then substituted in Eqs. 3 and 4, together with the definitions of f_i for this problem that are obtained from Eqs. 21 and 22, rendering partial differential equations in both r and θ. The solution of the resulting equations is made using another Galerkin projection, this time using the functions that multiply each of the coefficients in the manifolds expressions. This allow for a set of 12 nonlinear algebraic equations to be solved. Since the nonlinear mode must be tangent to the linear mode at the equilibrium point, it is possible to solve the terms corresponding to the linear representation of X_q and Y_q before solving the remaining terms. This leads to a separation of the system in 4 equations involving a_1, a_2, b_1 and b_2, and another system involving the remaining coefficients. All these tasks were performed with the aid of symbolic computation using the Wolfram Mathematica. The nonlinear modes were obtained for U_r varying from 1 to 15 in increments of 0.01 in a brute force continuation scheme, in which the coefficients of a previous solution are used as initial guess for the iterative solution of the following problem.

4. RESULTS AND DISCUSSION

In order to illustrate the capability of the nonlinear modes to represent the wake-variable and be used to obtain the responses on the inline and cross-wise directions, a numerical example is developed. The numerical data adopted is shown in Tab. 1, adapted from Sparks (2002), Ogink and Metrikine (2010) and Qu and Metrikine (2020).

Table 1. Numerical data for the illustrative model.

Parameter	Value	Parameter	Value
$(\mu + \mu_a)$	1200kg/m	ρ	1025kg/m^3
EI	318.6MNm^2	C_L^0	0.3842
γ	3433.5N	$\overline{C}_D$	1.1856
ℓ	2000m	α	2.2
T_b	633kN	D	0.5588m

The parameter ε is taken as 0.05 for $U_r < 6$ and 0.3 otherwise, while A is taken as 4 or 12 in the same intervals of reduced velocity. Two different scenarios were then used for the numerical integration of the model. In the first one, the complete set of ordinary differential equations before the obtaining of the nonlinear mode were integrated for reduced velocities ranging from 1 to 12, with a discretization of 0.1. Each integration was carried up to 4000 units of dimensionless time in order to guarantee that a limit cycle was achieved. The second approach involves a numerical integration of only the equations in r and ϕ for the same reduced velocities and amount of time. Then, the resulting time-series are used in the expressions for X_q and Y_q to obtain the time-series for q and its derivative. Finally, equation 20 can be numerically integrated for the resulting q. The results for the dimensionless amplitude of the cross-wise motion are shown in Fig. 2.

The first thing to notice is that the results using the nonlinear mode approach are in good qualitative agreement with the reference system, and not far off in quantitative terms. One aspect to be noticed is that the nonlinear mode does not recover any response outside the lock-in region. This is actually expected, since the basic principle for the nonlinear mode to work is that there is a mutual dependence of the oscillations in q and v. Outside the lock-in region, variable q receives an insignificant influence of the variable v, becoming an almost isolated oscillator which then acts as a forcing term in the equation of motion for v. This simply means that the obtained mode is not adequate to represent the system outside the lock-in region. This causes no issues for practical terms, since the lock-in region is the region of interest in VIV analysis. To better illustrate that the break of representation capability is related to the behaviour of the variable q, the steady-state amplitudes of response of this variable are shown in Fig. 3

From observing the response amplitudes of q it is then clear that this variable has its own behaviour outside the lock-in range. Notice also that the limit-cycle amplitude in this case is not 2 for the q, the commonly develope value for a Van

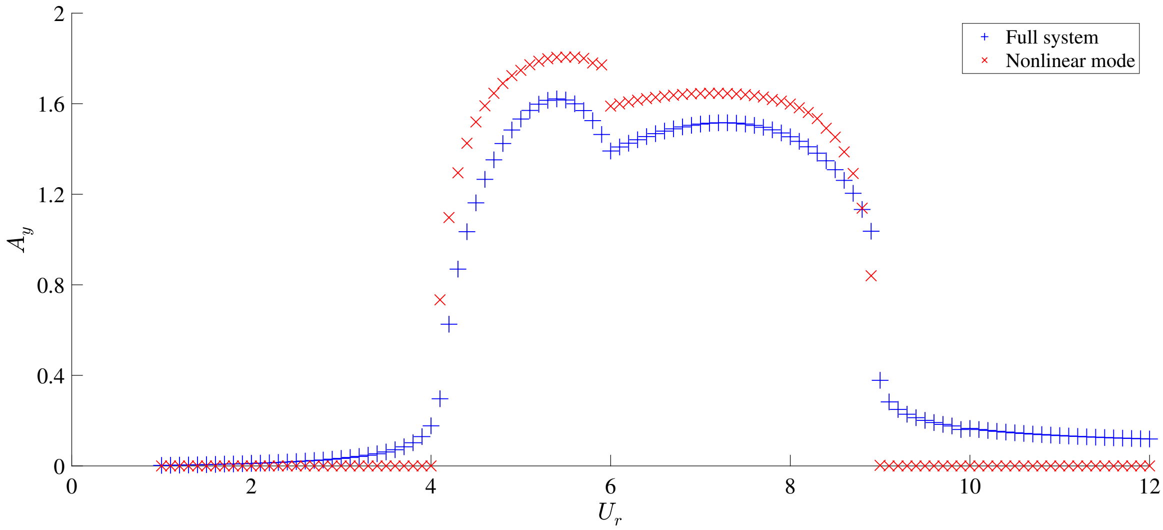

Figure 2. Dimensionless amplitude of the cross-wise motion (A_y) in steady-state regime as function of the reduced velocity U_r. Blue crosses (+) indicate the result of integrating the full system of equations while red xs (x) indicate the results of integrating the resulting modal equation using the nonlinear mode approach.

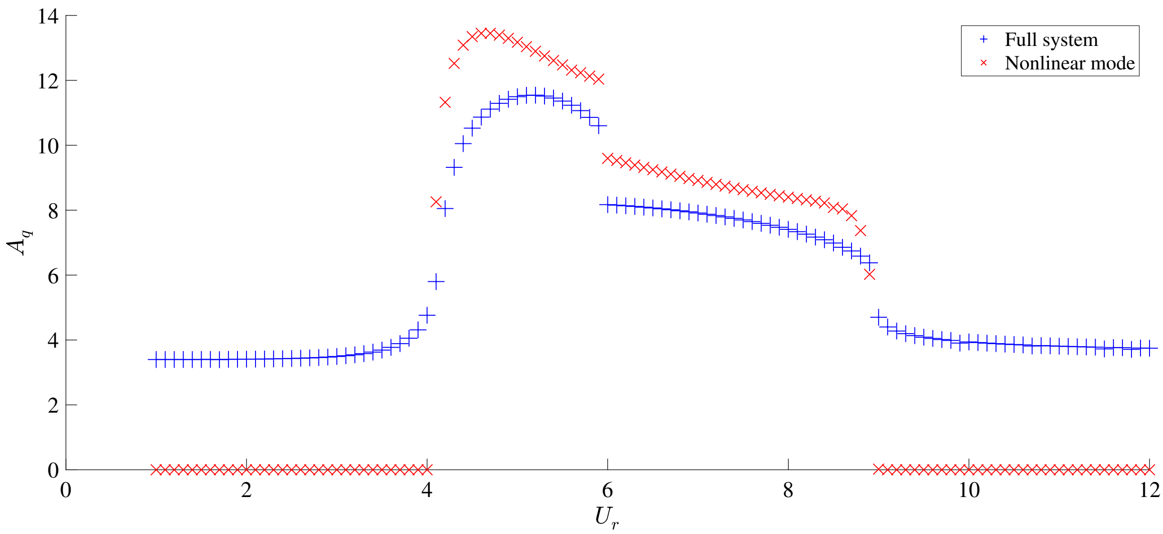

Figure 3. Amplitude of the wake-variable response(A_q) in steady-state regime as function of the reduced velocity U_r. Blue crosses (+) indicate the result of integrating the full system of equations while red xs (x) indicate the results of integrating the resulting modal equation using the nonlinear mode approach.

der Pol oscillator. This happens because due to the Galerkin projection, the coefficients θ_1 and θ_2 are not equal in Eq. 22, while they would be in the classical Van der Pol oscillator. It is possible to apply a Harmonic Balance approximation to obtain the amplitudes in this case, which shows the dependence on θ_1 and θ_2, which is not made here for the sake of the length of this paper and to not divert the focus. Nonetheless, the qualitative behaviour of q in the lock-in range is noticeably agreeable with the simulations of the complete system, with the differences being of quantitative order. Since there is a difference in the values of the response of q, it is expected an impact on the response of u as well. This is shown by means of the average dimensionless value of u in Fig. 4 and the amplitude of the oscillating component of u in Fig. 5.

The effects of the overestimation of the response of q in the lock-in region and its underestimation outside of it are clearly visible in both the mean and oscillatory portions of the behaviour in the inline direction. Again, most of the difference is quantitative rather than qualitative. The only exception being the resonance of the inline motion that occurs for reduced velocities around 3. This is simply a result of the nonlinear mode not being adequate for representation in that range of reduced velocities.

Considering the exposed results, one could question the advantage of using the nonlinear mode. The first simple advantage is that of computational effort. Obtaining the results using the nonlinear mode took 2/3 of the time that the integration of the full system took on the same computer, a 10th-gen Intel i7 with 16 GB of RAM, both in a single-core setup. Considering a practical scenario where some nonlinearities can cause interactions with higher modes of the

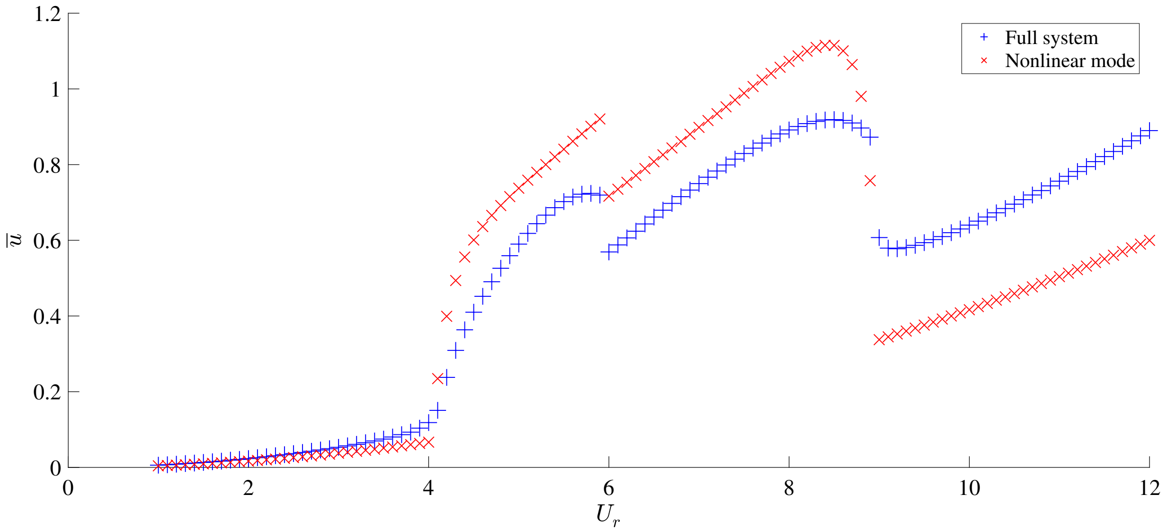

Figure 4. Dimensionless mean value of the inline motion ($\overline{u}$) in steady-state regime as function of the reduced velocity U_r. Blue crosses (+) indicate the result of integrating the full system of equations while red xs (x) indicate the results of integrating the resulting modal equation using the nonlinear mode approach.

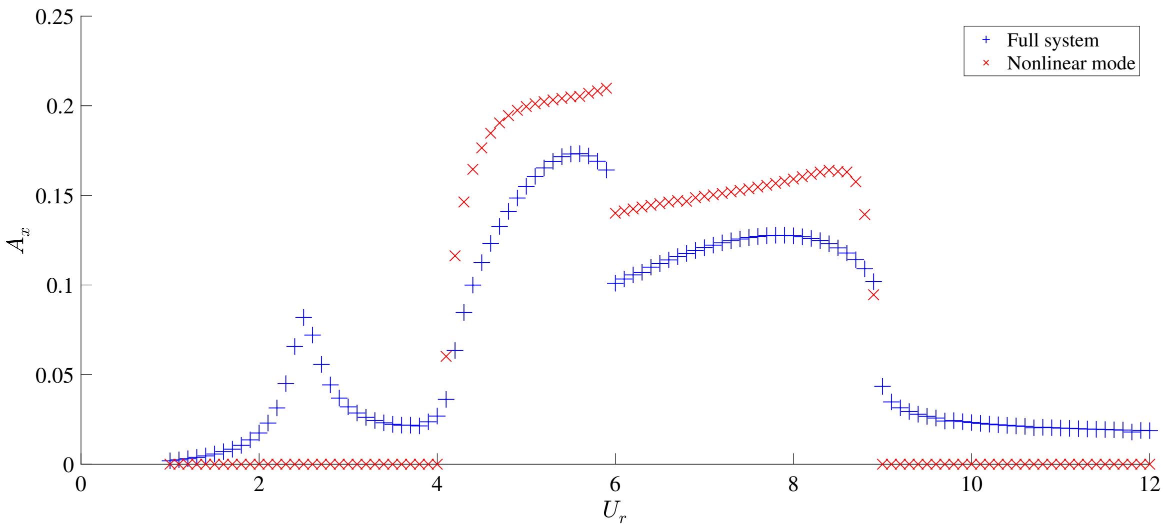

Figure 5. Dimensionless amplitude of the inline motion (A_x) in steady-state regime as function of the reduced velocity U_r. Blue crosses (+) indicate the result of integrating the full system of equations while red xs (x) indicate the results of integrating the resulting modal equation using the nonlinear mode approach.

structure, this advantage can be potentiated by accounting for the effects of the higher modes over the locked one using nonlinear modes, as is done in Keber and Wiercigroch (2008), while integrating the full system would involve integrating the equations of motion for all the spatial modes.

Another advantage that can be explored is related to the geometry of the manifolds, which can be inferred by means of the coefficients that define it. There are clear qualitative changes in the geometry of the manifolds when transitioning between regions of different behaviour, like entering or exiting the lock-in region. Although this is not explored due to the limited length of this work. A detailed assessment of this type of relationship between the geometry of the manifold and the behaviour of the system can be used to infer what type of response the system will present without having to actually integrating it. This can be particularly useful, for example, as an early stage of verification of new phenomenological models, by inspecting if the manifolds present the desired geometrical aspects to cause the expected type of response of VIV.

5. CONCLUSIONS

A working methodology is proposed in order to obtain reduced-order models (ROM) of flexible structures under VIV using a Galerkin scheme, followed by a further order reduction by means of the usage of nonlinear modes of vibration to represent the wake dynamics. The main steps necessary to obtain the discretized ROM and for the subsequente calculation of the shapes of the manifolds that describe the nonlinear modes are discussed step by step, without showing the minor algebraic steps that can be performed with the aid of symbolic computation.

One numerical example is developed in order to compare results of the original discretized system with those obtained by the use of the nonlinear mode. The results show a noticeable qualitative agreement between the full system integration and the results with the nonlinear mode. In terms of quantitative results, the integration with the nonlinear mode always furnished higher results, with up to 15% larger values in the cross-wise ampplitude of motion. The nonlinear mode integration possess the advantage of being faster in obtaining the response of the system. It is also able to hint some aspects of the response prior to the integration as function of the unfolding of the geometry of the manifolds, such as the points of synchronization and desynchrronization between the wake and the structural response.

This type of analysis involving a nonlinear mode to describe the wake dynamics in terms of the structural response is still lacking in the literature, with a lot of possibilities of investigation to be made. Further works include a deeper investigation of the relationship between the manifolds geometries and the expected response of the system. Other possibilities are the application to models discretized with various modal shapes and then reducing the obtained model to a minimum by writing the nonlinear mode as function of the generalized coordinates expected to dominate the response of the complete system. The application to more complex phenomenological models is also yet to be seen, which will likely involve the use of an inline pair of generalized displacement/velocity together with a cross-wise one in order to better represente the phenomenon.

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