Structural reliability analysis based on the dynamic integrity of an attractor. Part I

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Abstract. This paper addresses the reliability analysis of a dynamical system attractor, provided its dynamic-integrity measure has been previously assessed in terms of a meaningful parameter for which the probability density function is known. The probability that the dynamic-integrity measure should be equal or larger than a prescribed safe reference value, for the attractor to be considered "reliable", is sought. Although the ideas addressed here are applicable to any dynamical system, the structural stability case is focused herewith for the sake of an illustration, aiming at characterizing the safe loading threshold in an archetypal model liable to elastic buckling. By assuming that the buckling strength is an input Gaussian random variable, the basic statistical properties of the output variable, namely the dynamic-integrity measure, are estimated, from which a simple procedure to obtain a reliability appraisal is proposed. Another procedure is also addressed, considering a more formal way to obtain the output statistical properties. It is expected that both the simplified and the more formal procedures may help the assimilation of reliability analysis within current structural engineering design practices.

Keywords: Structural Stability, Dynamic Integrity, Reliability Analysis.

1 Introduction

Stability, dynamic integrity, and reliability are essential properties of a dynamical system's attractor, for the purpose of a safe design in different branches of engineering and technology. The ideas and procedures discussed herewith are applicable to any dynamical system but will be illustrated within the context of structural stability, benefitting from the rich historical developments of classical elastic buckling analysis.

1.1 Determination of the Buckling Strength of a Structure: a Historical Perspective

The foundations of the elastic stability theory were established by Euler [1], by evaluating the buckling load of compressed rods (let's call it Euler's load) in the ideal scenario of 'perfect' straight axis, and in the absence of either lateral loading or axial load offsets. Extension to other slender structural systems, such as plates and shells followed, as reported for instance by Timoshenko and Gere [2].

Nevertheless, it was noticed a long time ago that the theoretical estimates of the buckling strength could not always be confirmed experimentally, sometimes leading to values of the order of only 30% of the results evaluated for the so-called 'perfect' system. That was the case of the experimental studies of von Kármán [3] with shells. Although the suspicion that small imperfections would be the cause for this unexpected result (how such minor deviations from the 'perfect' model could explain such a large decrease in the buckling strength?), it was necessary to wait for the advancements of nonlinear beam and shell theories, as proposed by Koiter [4], and Donnell and Wan [5], to fully understand it. As a matter of fact, although remaining unknown for a long time, the studies by Koiter, originally written in Dutch and only translated into English two decades later, are widely recognized as those that clearly established the concept of imperfection sensitivity of buckling loads (let's call them Koiter's load), showing its relationship with the patterns of unstable symmetric bifurcations (as in the case of cylindrical and spherical shells) and of asymmetric bifurcations (as in the case of framed asymmetric structures).

Thompson [6] and Roorda [7] anticipated that it was also important to consider the statistical variation of the buckling strengths as a consequence of the imperfections being a random variable, but perhaps Perry and Chilver [8] were the first to introduce the reliability analysis into the study of structural stability. They departed from the relation between the buckling strength and the imperfection parameter, as obtained from a deterministic model, yet taking into account the statistical properties of the imperfection (input variable), to evaluate the corresponding statistical properties of the buckling strength (output variable), thus characterizing its probability density function. Finally, they confronted the buckling strength's probability density function with that of the applied load, determining the probability of failure in the case the latter would overcome the former. It was indeed a pioneering work linking stability and reliability analysis.

A further important advancement came a few years later when Soliman and Thompson [9] introduced the concept of dynamic integrity of a dynamical system's attractor, expressing the need to somehow 'measure' the compact and non-eroded part of its basin of attraction. In fact, it is not enough to know that a solution is stable (in other words it is an attractor), if its basin of attraction is either too short or fractal, as it might happen in the case of a load just below the Koiter's load. Of course, to assess a measure of dynamic integrity, it was an absolute must to model the problem of structural buckling as a truly dynamical system, not just as a statical one, as usually done. Besides the dynamic-integrity measures proposed by Soliman and Thompson, namely the *GIM* (global integrity measure), based on the 'area' of the basin of attraction, and the *LIM* (local integrity measure), based on the largest radius of a hypersphere of the phase space centered in the attractor and tangent to the basin of attraction boundaries, Rega and Lenci [10] proposed still another measure called *IF* (integrity factor), similar to *LIM*, yet with the hypersphere not necessarily centered in the attractor, it henceforth being a property of the basin, as opposed to *LIM*, which is a property of the attractor.

Lenci and Rega [11] are among the first to connect the concepts of stability and dynamic integrity. They did that in a study of the buckling strength of an archetypal model. In this problem they introduced what they called Thompson's load, as opposed to Euler's load (for the 'perfect' model) and Koiter's load (for the 'imperfect' model), by imposing a minimum 'safe' *GIM*.

Yet, a reliability measure for Thompson's load was still missing, since the integrity measure may be affected by the statistical variation of Koiter's load, considered as a random variable, depending, on its turn, on the imperfection randomness. Hence, what is aimed in future works is something in the sequel, very similar to what Perry and Chilver have done with respect to Koiter's load, yet this time with Thompson's load. The probability that the integrity measure would be at least a safely chosen value is of course associated with the load not overcoming the associated 'safely chosen' Thompson's load, which would be the buckling strength considering simultaneously structural stability, dynamic integrity, and reliability.

1.2 Integrity Measures and Erosion Profiles

As mentioned before, the usual dynamic-integrity measures are *GIM*, *LIM* and *IF*. To calculate them it is necessary to obtain the basin of attraction of the attractor of interest, which is still a computationally demanding task for systems with several degrees of freedom, and consequently with high-dimensional phase spaces.

It is supposed that some tool, such as the in-house code Poli BoA [12], is available for the purposes of both obtaining the basins of attraction and the associated dynamic-integrity measure chosen, generically referred to as I. This procedure is repeated for different values of a convenient system parameter A, within a convenient range. Typically, the graph $I(A)$ will indicate a decrease in the dynamic-integrity measure I as the parameter A increases, even with a steep gradient, thus justifying the nomenclature of an 'erosion profile'. Thompson refers to it as a 'Dover Cliff' profile to the resemblance to the well-known English coastal topography (see Fig. 1).

2 Statistical Properties and Probability Density Functions of Input and Output

Supposing that the 'input' parameter A is a random variable, its statistical properties associated to its probability density function $f(A)$ can be communicated to the 'output' dynamic-integrity measure's statistical properties and its associated probability density function $f(I)$.

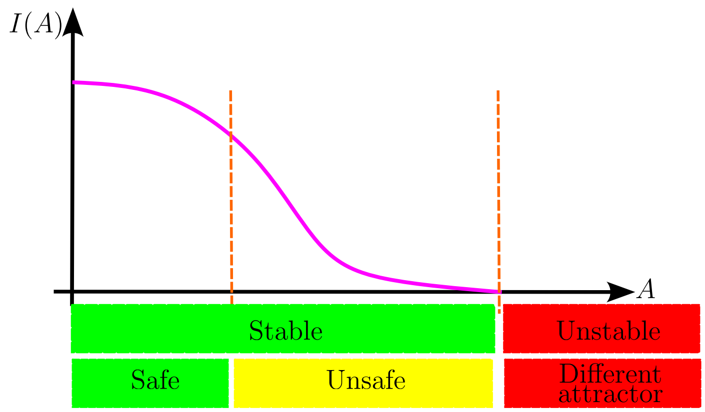

Fig. 1. Sketch of an erosion profile, adapted from [13].

From the fact that in usual applications the randomness of parameter A comes from multiple and independent factors, it is a reasonable assumption that an adequate probability density function is a normal distribution, resorting to the central limit theorem, as mentioned by Roorda [7]. Hence, for $f(A)$ it is written

$$f(A) = \frac{1}{\sigma_A \sqrt{2\pi}} e^{-\frac{(A-\bar{A})^2}{2\sigma_A^2}} \tag{1}$$

where $\bar{A}$ is the expected value and σ_A is the standard deviation of parameter A.

For $f(I)$ it is also reasonable to assume that it is a normal distribution, so that

$$f(I) = \frac{1}{\sigma_I \sqrt{2\pi}} e^{-\frac{(I-\bar{I})^2}{2\sigma_I^2}} \tag{2}$$

where $\bar{I}$ is the expected value and σ_I is the standard deviation of the dynamic-integrity measure I.

2.1 A Simplified Procedure to Evaluate the Dynamic Integrity Statistical Properties

A simplified methodology is proposed considering that the erosion profile $I(A)$ is known, so that $\tan\alpha = -dI/dA$ is the absolute value of its local slope. It is postulated that the output integrity measure standard deviation is given by $\sigma_I = \sigma_A \tan\alpha$ about the expected value $\bar{I}$. This assumption will be later discussed.

Hence, for every point $(\bar{A}, \bar{I})$ of the erosion profile, it can be defined the cut-off region for which the integrity measure complies with $I \geq I_{ref}$, provided $A \leq A_{ref}$, leading to the probability assigned to safety as a direct reliability measure.

For an illustrative example the reader is referred to Sec. 3.

2.2 A Formal Procedure to Evaluate the Dynamic Integrity Statistical Properties

A more formal procedure for determining the statistical properties of the output distribution, using the moment-generating function [14], is now concisely presented. For such a procedure, auxiliary variables β and $\rho(\beta)$ are introduced such as

$$\beta = A - \bar{A} \tag{3}$$

$$\rho(\beta) = I(A) - I(\bar{A}) = I(\bar{A} + \beta) - I(\bar{A}) \tag{4}$$

For $\rho(\beta)$ a power series expansion can be written as

$$\rho(\beta) = \sum_{n=1}^{\infty} \frac{1}{n!} a_n \beta^n \tag{5}$$

Using the notation $E[x]$ for the expected value of a generic variable x, and t for an auxiliary dummy variable, the moment-generating function can be concisely written as

$$M_\rho(t) = E[e^{\rho t}] = E\left[e^{t \sum\limits_{n=1}^{\infty} \frac{1}{n!} a_n \beta^n}\right] = E\left[\prod_{n=1}^{\infty} e^{\frac{1}{n!} a_n \beta^n t}\right] \tag{6}$$

Equation 6 can be expanded in power series of the dummy variable t, so that

$$\begin{aligned} M_\rho(t) &= 1 + t \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n + \\ &+ \frac{t^2}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n!^2} a_n^2 \mu_{2n} + 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} \frac{1}{n! m!} a_n a_m \mu_{n+m} \right) + \dots \end{aligned} \tag{7}$$

where the central moments μ_n of the input random variable are defined by

$$\mu_n = E[\beta^n] \tag{8}$$

Notice that $\mu_1 = 0$ from Eq. 3. Now, the moment-generating function can be used to obtain the expected value $\bar{\rho}$ (first central moment η_1) and the other central moments η_r of the output distribution, as follows:

$$\bar{\rho} = \bar{I} - I(\bar{A}) = \left[\frac{\partial M_\rho}{\partial t}\right]_{t=0} = \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n \tag{9}$$

$$\eta_r = \left[\frac{\partial^r M_\rho}{\partial t^r}\right]_{t=0}, r = 2, 3, \dots \tag{10}$$

From Eq. 9 the expected value of the output distribution comes out

$$\bar{I} = I(\bar{A}) + \bar{\rho} = I(\bar{A}) + \sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n \tag{11}$$

From Eq. 10 the variance (second central moment) also comes out

$$\eta_2 = \sum_{n=1}^{\infty} \frac{1}{n!^2} a_n^2 + \mu_{2n} + 2 \sum_{n=1}^{\infty} \sum_{m=n+1}^{\infty} a_n a_m \mu_{n+m} - \left(\sum_{n=2}^{\infty} \frac{1}{n!} a_n \mu_n \right)^2 \tag{12}$$

Now, the procedure described in Sec. 2.1 can be recast, using an improved estimation for $\bar{I}$, as in Eq. 11, and for σ_I, taking the square root of η_2 , as defined in Eq. 12.

3 Application to an Archetypal Model

The archetypal model discussed by Lenci and Rega [11] is addressed herewith (see Fig. 2). It is a rigid bar hinged at the bottom and constrained laterally by an inclined linear elastic spring. The buckling strength of the system under axial load (p) is sought in presence of an imperfection simulated by a small lateral load (q)[1].

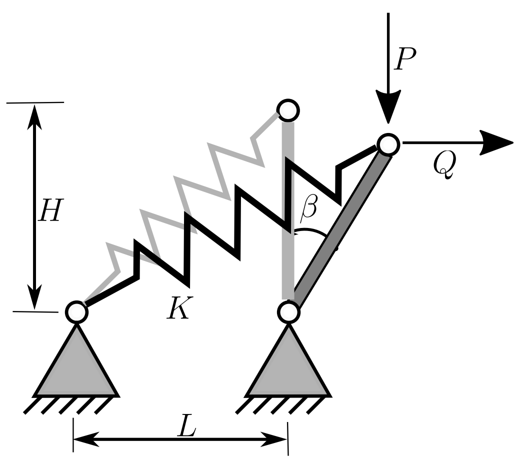

Fig. 2. Archetypal model, adapted from [11].

Figure 3 depicts the erosion profile of the integrity measure $I = GIM$ in function of the axial load $A = p$, as found in [11], indicating a suggestion of Thompson's load to be $\bar{A} = p_T \approx 0.175$ and the corresponding $I(\bar{A}) = GIM_T \approx 0.100$ (this will be referred to as Scenario 2 in Tab. 1).

[1] Upper case parameters/variables are dimensional; lower case parameters/variables are dimensionless.

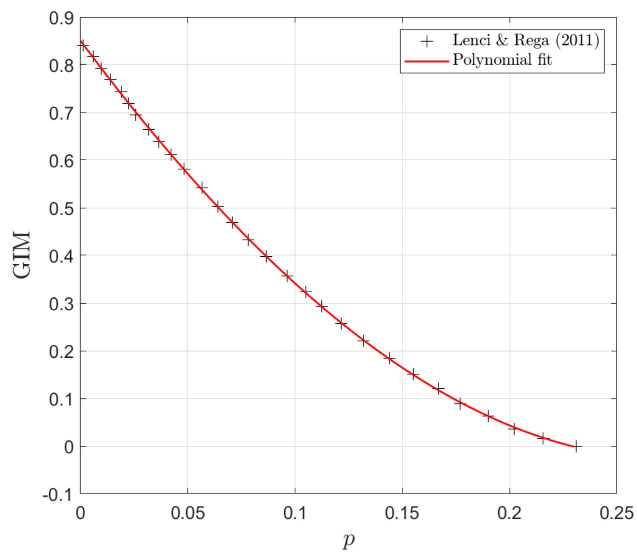

Fig. 3. Polynomial fitting for the erosion profile for $q = 0.03$, extracted from [11].

A cubic polynomial fitting for the curve $I(A) = GIM(p)$ is given by:

$$I(A) = b_0 + b_1 A + b_2 A^2 + b_3 A^3 \tag{13}$$

where $b_0 = 0.8500$, $b_1 = -6.0556$, $b_2 = 9.2479$ and $b_3 = 4.2930$. As for the local erosion profile slope in Scenario 2, from Eq. 13 it is found $\tan\alpha = -(dI/dA)(0.175) \approx 2.4244$. The simplified analysis is then carried out following the procedure presented in Sec. 2.1. Assuming, for the sake of an example, a standard deviation $\sigma_A = 0.020$, the estimated output standard deviation would be $\sigma_I = 0.049$, leading to a probability of 31.7% for GIM to be at least $GIM_T + \sigma_I = 0.149$ provided p is not larger than $p_T + \sigma_A = 0.195$; a probability of 50% for GIM to be at least $GIM_T = 0.100$ provided p is not larger than $p_T = 0.175$; and a probability of 68.3% for GIM to be at least $GIM_T - \sigma_I = 0.051$ provided p is not larger than $p_T - \sigma_A = 0.155$. Table 1 displays other simulations for the probabilities of both lower bounds of GIM and upper bounds of p.

Notice that for $p \leq 0.175$, there is a 31.73% probability that $GIM \geq 0.208$ (Scenario 3) and 50% probability that $GIM \geq 0.100$ (Scenario 2), but a 31.73% probability that GIM would not be acceptable, since $GIM_T - \sigma_I$ is almost null (Scenario 1)! Alternatively, for $p \leq 0.155$, there is a 31.73% probability that $GIM \geq 0.279$ (Scenario 4), 50% probability that $GIM \geq 0.150$ (Scenario 3), and a 68.27% probability that $GIM \geq 0.051$ (Scenario 2). Finally, for $p \leq 0.135$, there is a 31.73% probability that GIM would be (anti-economically?) large (Scenario 5), a 50% probability that $GIM \geq 0.212$ (Scenario 4) and a 68.27% probability that $GIM \geq 0.092$. These results could be used to decide whether

Table 1. Lower bounds of GIM and upper bounds of p for $\sigma_A = 0.040$.

Prob. [%]	Scenario 1 $(\bar{p}; \overline{GIM})$ $(0.215; 0.018)$		Scenario 2 $(\bar{p}; \overline{GIM})$ $(0.175; 0.100)$		Scenario 3 $(\bar{p}; \overline{GIM})$ $(0.135; 0.208)$		Scenario 4 $(\bar{p}; \overline{GIM})$ $(0.095; 0.362)$		Scenario 5 $(\bar{p}; \overline{GIM})$ $(0.055; 0.546)$	
	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$	$GIM \geq$	$p \leq$
31.73	0.092	0.215	0.149	0.195	0.208	0.175	0.279	0.155	0.357	0.135
50	0.053	0.195	0.100	0.175	0.150	0.155	0.212	0.135	0.282	0.115
68.27	0.004	0.175	0.051	0.155	0.092	0.135	0.145	0.115	0.207	0.095

the choice of $p_T = 0.155$ or $p_T = 0.135$ would not be a better choice than $p_T = 0.175$ for a safe engineering design.

Next, the procedure presented in Sec. 2.2 is followed for the sake of a comparison with the simplified analysis just carried out. Only Scenario 2 is focused on what follows. Assuming a normal distribution for the input variable, due to its symmetry it is clear that $\mu_3 = 0$. Hence, for the cubic polynomial fitting, the only non-null central moment to be considered is $\mu_2 = \sigma_A^2 = 0.0004$ for the example at hand. From Eq. 4, considering the cubic polynomial fitting of Eq. 13, it is straightforward to arrive at

$$\rho(\beta) = a_1\beta + a_2\beta^2 + a_3\beta^3 \tag{14}$$

where, for $\bar{A} = 0.175$, $a_1 = b_1 + 2b_2\bar{A} + 3b_3\bar{A}^2 = -2.4244$, $a_2 = b_2 + 3b_3\bar{A} = 11.5017$ and $a_3 = b_3 = 4.2930$, which is a particular case of Eq. 5. Hence, it follows from Eqs. 9 and 12 that $\bar{\rho} = (1/2)a_2\mu_2 \approx 0.002$ and $\eta_2 = a_1^2\mu_2 - (1/4)a_2^2\mu_2^2 = 0.002$. Finally, for the output variable $I = GIM$, the following statistical properties are found: $\bar{I} = I(\bar{A}) + \bar{\rho} \approx 0.102$, to be compared with $GIM_T \approx 0.100$, and $\sqrt{\eta_2} \approx 0.048$, to be compared with $\sigma_I = 0.049$, indicating a good agreement with the simplified method. Of course, if σ_A would be larger (that is, a relative standard deviation larger than 11%), the simplified method might deviate too much from the formal procedure, and this latter would be more adequate to use in the analysis.

4 Concluding remarks

The paper proposes a simplified method to carry out a reliability analysis based on the dynamical integrity of a system, provided the statistical properties of the parameter used to define the erosion curve of the chosen integrity measure are known. A first try is simply based on the slope of the erosion curve to evaluate the standard deviation of the integrity measure, giving the probability that a minimum reference value is attained, provided a threshold of the input system parameter is not surpassed. A more systematic procedure is then discussed, by which the statistical properties of the output variable (in this case the dynamic integrity measure) are evaluated from those of the input variable (in this case the system parameter used to obtain the erosion curve), as derivable from the moment-generating function. Both procedures are illustrated for an archetypal model in which a safe threshold is searched for the buckling load so as not to undergo a critical reduction (or fractalization) of the basin of attraction of the desired equilibrium configuration. A critical reasoning is raised about the choice of a safe, yet still economical, choice of Thompson's load for the problem at hand. The easiness of application of both the simpler and the more formal procedures, which closely agree with each other, gives hope to their adoption in the engineering design practice, although one should not underestimate the determination of the erosion curve stage, which still poses some difficulties for systems with large number of degrees of freedom, thus emphasizing the importance of reduced-order models.

Acknowledgements

Both authors acknowledge the support from the São Paulo Research Foundation (FAPESP) through project 2022/00770-0 and the Brazilian Council for Scientifical and Technological Development (CNPq) for the grants 301050/2018-0 and 305945/2020-3. The second author acknowledges the support of BEPE scholarship 19/15046-2 from FAPESP .

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