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Exploring Boundedness and Oscillatory Behavior in Impulsive Stochastic Fractional Differential Equations: Theoretical Insights and Practical Applications

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Exploring Boundedness and Oscillatory Behavior in Impulsive Stochastic Fractional Differential Equations: Theoretical Insights and Practical Applications

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1 Abstract

This study investigates the impulsive stochastic fractional differential equation (ISFDE) of order $\alpha \in (0, 1)$, integrating fractional calculus, impulsivity, and stochasticity. The ISFDE formulation encapsulates continuous fractional dynamics, sudden changes from impulses, and stochastic components, characterizing systems with memory, abrupt alterations, and uncertainty. The research establishes the existence and uniqueness of solutions to the ISFDE, demonstrating their boundedness, and most important oscillatory behaviour under suitable conditions. The findings contribute to advancing the theoretical understanding and practical applicability of ISFDEs, underscoring their significance in modeling diverse dynamic systems while ensuring the stability and well-behaved nature of solutions.

Keywords: Impulsive Stochastic equations; Caputo fractional derivative; Mittag-Leffler functions; Existence; Uniqueness; Boundedness; Oscillatory nature; Example.
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2 Introduction

Fractional calculus is a branch of mathematics that deals with derivatives and integrals of non-integer orders. Instead of the traditional integer order derivatives and integrals you're likely familiar with (like first, second, or third derivatives), fractional calculus extends these concepts to include non-integer or fractional orders [1]. In essence, it involves generalizing the notions of differentiation and integration to non-integer orders, allowing for a deeper understanding and analysis of various phenomena in science, engineering, physics, and other fields.The key elements in fractional calculus include fractional derivatives and fractional integrals [2]. Fractional derivatives generalize the concept of derivatives to non-integer orders, while fractional integrals are their inverse operations.These concepts have found applications in various fields such as signal processing, physics (for instance, in describing phenomena like diffusion and viscoelasticity), finance (in modeling complex systems), and engineering (for analyzing systems with memory or long-range interactions)[3]. The mathematics behind fractional calculus is intricate and often involves complex analysis, special functions, and advanced mathematical techniques. But its applications have proven to be valuable in modeling real-world phenomena that cannot be accurately described by classical integer-order calculus [4]. Differential equations with fractional-order derivatives have gained importance due to their various applications in science and engineering such as rheology, dynamical processes in self-similar and porous structures, heat conduction, control theory, electroanalytical chemistry, chemical physics, and economics, [5] and [6]. Impulsive fractional differential equations (IFDEs) combine two fundamental mathematical concepts: impulsivity and fractional calculus. These equations model systems where the dynamics involve both fractional derivatives and instantaneous changes or impulses at specific time points. Fractional derivatives represent the non-integer order derivatives that describe the behavior of the system with memory and non-local effects.The fractional derivatives capture the past history of the system up to a fractional order. It represents sudden, instantaneous changes or events that occur at specific time points. These impulses can alter the system's state or behavior abruptly.An impulsive fractional differential equation typically combines fractional derivatives with Dirac delta functions or other impulse functions. The equation incorporates both the fractional derivative terms describing the continuous evolution of the system and terms accounting for the instantaneous changes due to impulses. IFDEs can be expressed in various forms, such as Caputo or Riemann-Liouville fractional derivatives, coupled with delta functions or other types of impulse functions. Solutions to impulsive fractional differential equations often exhibit intricate behaviors due to the combined effects of fractional derivatives and sudden changes induced by impulses. The impulses can lead to discontinuities or abrupt changes in the system's state or derivatives, adding complexity to the analysis of solutions. IFDEs find applications in control theory, where sudden changes or inputs (impulses) affect the system's behavior, and fractional derivatives capture the system's inherent memory. Modeling biological phenomena like neuronal firing patterns or physiological processes that involve both continuous dynamics and sudden events. Systems exhibiting intermittent dynamics, like systems with intermittent control actions or sudden external disturbances, can be described using IFDEs. Analyzing solutions to impulsive fractional differential equations requires a solid understanding of both fractional calculus and the theory of impulses. Computing solutions to IFDEs can be challenging due to the combined complexity of fractional derivatives and discontinuities caused by impulses. Interpreting the behavior of solutions to IFDEs can be non-intuitive due to the interaction between continuous fractional dynamics and sudden impulses. Impulsive fractional differential equations provide a mathematical framework to describe systems that undergo continuous fractional evolution while being influenced by sudden, instantaneous changes or events. They are a powerful tool in modeling real-world phenomena where both memory effects and abrupt alterations play crucial roles in system behavior [7]. Impulsive fractional stochastic equations merge three fundamental concepts in mathematics and modeling: fractional calculus, impulsivity, and stochasticity. These equations represent systems that involve fractional derivatives, instantaneous changes or impulses, and randomness or uncertainty. Represent sudden, instantaneous changes or events occurring at specific times [8]. Involves randomness or uncertainty, usually modeled using stochastic processes such as Brownian motion or random noise [9]. These equations combine fractional derivatives with impulses and stochastic terms, expressing the continuous evolution of the system, sudden changes due to impulses, and random fluctuations. Expressions of impulsive fractional stochastic equations involve fractional derivatives, impulse functions, and stochastic integrals or differential equations, incorporating randomness into the system's dynamics. Solutions to these equations are stochastic processes exhibiting both continuous fractional dynamics and abrupt changes caused by impulses, along with random fluctuations. Modeling financial markets where sudden events (like news releases) cause instant market changes, while fractional calculus captures long-term memory effects. Stochasticity accounts for random fluctuations in prices. Modeling biological phenomena such as neuron firing patterns or physiological processes that involve both continuous fractional evolution and stochastic fluctuations. Describing systems where sudden unpredictable events (impulses) affect the system's dynamics, coupled with inherent stochastic disturbances [10]. Solving impulsive fractional stochastic equations often requires advanced mathematical techniques due to the complexity arising from fractional derivatives, impulses, and stochastic elements. Simulating solutions to these equations accurately can be computationally challenging due to the combination of fractional calculus and stochastic processes. Interpreting the behavior of solutions to these equations can be intricate because of the interaction between continuous fractional dynamics, sudden impulses, and stochastic fluctuations. Impulsive fractional stochastic equations offer a powerful framework to model systems undergoing continuous fractional evolution, sudden changes, and random fluctuations. They find applications in diverse fields where both memory effects, abrupt alterations, and uncertainty play significant roles in system behavior [11]. This particular issue aims to provide a platform for distinguished scholars across several sectors of engineering, especially applied mathematicians, to showcase its innovative research. our main focus is on recent developments in differential and integral equations of arbitrary order that originate in physical networks, as well as analytical and numerical techniques using state-of-the-art mathematical modeling.[12].

$$\begin{cases} {}_{t_k}D^{\alpha}\Big(p(t)[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)]\Big)+q(t)x(t)=0, \qquad t>t_0, \quad t\neq t_k, \quad \alpha\in(0,1), \\ x(t_k^+)=a_k x(t_k^-), \qquad {}_{t_k}D^{\alpha}x(t_k^+)=b_{t_k-1}D^{\alpha}x(t_k^-), \qquad k=1,2.... \end{cases}$$

$$D^{\beta}_{+,t}u(x,t)+a(t)D^{\beta-1}_{+,t}u(x,t)=b(t)\Delta u(x,t)+\sum_{k=1}^{m}c_k(t)\Delta u(x,t-\tau_k)-F(x,t),$$

under the impulsive condition

$$D^{\beta-1}_{+,t}u(x,t_j^+) - D^{\beta-1}_{+,t}u(x,t_j^-) = \sigma(x,t_j)D^{\beta-1}_{+,t}u(x,t_j), \quad j = 1,2....,(x,t) \in \Omega \times \mathbb{R}_+.$$

with two kinds of boundary conditions

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial N} + f(x,t)u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \quad t \neq t_j \\
&\textit{and} \\
u(x,t) &= 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \quad t \neq t_j,
\end{aligned}$$

$$\begin{cases} {}^{Hc}D^{\alpha}_{t_k}y(t) \in F(t,y(t)), \quad t \in J = (t_k, t_k+1), \\ y(t_k^+) = I_k\big(y(t_k^-)\big), \quad k = 1,2...., \\ y(1) = y_*. \end{cases}$$

The existence theorems of oscillatory and non-oscillatory solutions of the equation mentioned previously were derived by the researchers by applying the fixed point theorem and the concepts of upper and lower solutions. We examine the oscillatory behavior of solutions to the following fractional impulsive differential equation, inspired from the earlier publications [11].

$$\begin{cases} {}^{c}D^{\delta}_{a}x(t) = A(t)x(t) + f(t,x(t)) + \sigma(t,x_t)\dfrac{dB(t)}{dt} \\ x(t) = \phi(t); \qquad -\omega \leq t < 0 \\ \Delta x(t_k) = y_k; \qquad k = 1,2... \end{cases} \tag{1}$$

${}^{c}D^{\delta}_{a}$ is Caputo Fractional derivative.

$$x_t = \{x(t+\theta) : \omega \leq \theta \leq 0\}$$

where $\omega \in [0,+\infty)$ can be regarded as $PC([\omega,0];\mathfrak{R}^s) - value$ stochastic process, where $\delta : [t_o,T] \times PC([\omega,0];\mathfrak{R}^s) \to \mathfrak{R}^s$ and $\sigma : [t_o,T] \times PC([\omega,0];\mathfrak{R}^s) \to \mathfrak{R}^{s\times n}$. P(t) is $n-dimentional$ standard Brownian motion. Brownian motion paths are continuous but nowhere differentiable. Thus, discussing the boundedness of its derivative in the classical sense is not applicable because the derivative does not exist in the usual calculus framework for Brownian motion. The following represents the setting up value:

$$x_{t_o} = \phi = \{\phi(\theta) : -\omega \leq \theta \leq 0\}$$

is an F_{t_o} measurable. Further $PC([-\omega,0];\mathfrak{R}^s)$-valued random variable such that

$$\phi \in M^2([-\omega,0];\mathfrak{R}^s) \;\; where \;\; M^2([-\omega,0];\mathfrak{R}^s)$$

denotes the family of process $\{\phi(t)\}_{t_o \leq 0} \in \mathfrak{L}^p([-\omega,0];\mathfrak{R}^s)$ such that $\mathfrak{E}\int_{-\omega}^{0}|\phi(t)|^2 dt < \infty$. However, in a more generalized sense or in certain contexts, derivatives of stochastic processes like Brownian motion can be understood using stochastic calculus, where concepts like the integral and stochastic derivatives are employed. In this framework, stochastic calculus allows handling certain types of derivatives for stochastic processes, including Brownian motion, leading to quantities like stochastic integrals or stochastic differentials. Within stochastic calculus, the derivative of Brownian motion with respect to time $dB(t)/dt$ can be represented as a stochastic process known as the stochastic differential, typically denoted as $dB(t) or dW(t)$. However, this stochastic differential is not a classical derivative in the deterministic calculus sense, and its behavior doesn't necessarily relate to boundedness in the traditional sense.The behavior of the stochastic differential $dB(t) or dW(t)$ involves properties different from those of classical derivatives. It is often used in the context of stochastic differential equations, where it represents the randomness or the infinitesimal change in a stochastic process. In summary, Brownian motion does not possess a classical derivative due to its non-differentiable paths, but within the framework of stochastic calculus, it has a stochastic differential that is not bounded in the traditional sense. The expression $\sigma(t, x_t)$ typically represents the volatility or the diffusion term in stochastic differential equations (SDEs), where x_t is stochastic process and $\sigma(t, x_t)$ represents the volatility coefficient. The boundedness of $\sigma(t, x_t)$ depends on the specific properties of the function σ and the behavior of the stochastic process x_t it's associated with. Here are some considerations:

- **Properties of** σ: The boundedness of $\sigma(t, x_t)$ may rely on the nature of the function σ if σ is a bounded function, meaning its values are limited within a certain range for all t and X_t then $\sigma(t, x_t)$ would also be bounded.

- **Properties of** x_t: The behavior of the stochastic process x_t can influence the boundedness of $\sigma(t, x_t)$. Here x_t is a bounded process itself (for instance, a process confined to a finite interval or a compact set), it might contribute to the boundedness of $\sigma(t, x_t)$.

- **Stochastic Process Behavior**:The behaviour of x_t might involve moments like variance or higher-order moments that could influence the boundedness of $\sigma(t, x_t)$.

3 preliminaries

Definition 3.1. $\mathfrak{R}^s$-value stochastic process x(t) defined on $t_o - \omega < t < T$ is called a solution of (1) with initial value $x_{t_o} = \phi = \{\phi(\theta) : -\omega \leq \theta \leq 0\}$ if x(t) satisfies the following properties:
(i) x(t) is continuous and $\{x(t)\}_{t_o \leq t \leq T}$ is F_t adapted.
(ii) $\sigma(t, x_t) \in \mathfrak{L}^2([t_o, T]; \mathfrak{R}^{s\times n})$;
(iii) $x_{t_o} = \phi$, $\forall\ t_o \leq t \leq T$,

$$x(t) = \begin{cases} \phi_o + \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} \Big[A(s)x(s) + f(s, x(s)) + \sigma(t, x_t) \frac{dB(s)}{ds} \Big], & \forall t \in (t_0, t_1) \\ \phi_o + \sum_{i=1}^{k} y_i + \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} \Big[A(s)x(s) + f(s, x(s)) + \sigma(t, x_t) \frac{dB(s)}{ds} \Big], & \forall t \in (t_0, t_1) \end{cases}$$

This is the answer to equation (1) above. It is a special remedy. In the event that $\tilde{x}(t)$ is not different from x(t), then

$$P\{x(t) = \tilde{x}(t) \ for \ any \ t_o - \omega < t \leq T\} = 1$$

Lemma 3.1 ([16]). *Let ϵ, β and p be positive constants such that $[P(\epsilon-1)+1] > 0, [P(\beta-1)+1] > 0$. Then*

$$\begin{aligned} \int_0^t (t-s)^{p(\epsilon-1)} s^{p(\beta-1)} ds &= t^\mu \mathcal{B}\big(P(\beta-1)+1, P(\epsilon-1)+1\big), \quad t \geq 0, \\ \mathcal{B}(\xi,\eta) &= \int_0^1 s^{\xi-1}(1-s)^{\eta-1} ds, \\ \mu > 0, \xi > 0 \ \ and \ \ \mu &= P(\epsilon+\beta-1)+1 \end{aligned}$$

Definition 3.2 ([15]). Let $\delta \in (0,1]$ and $\nu : [0,\infty) \to X$. The fractional integral of order δ for a function ν with a lower limit of zero is defined as

$$I_t^\delta \nu(t) = \int_0^t g_\delta(t-s)\nu(s) ds, \quad t > 0,$$

Assuming that right hand-side is point wise defined on $[0,\infty)$, where g_a denote Riemann Liouville (RL) kernel.

$$g_\delta(t) = \frac{t^{\delta-1}}{\Gamma(\delta)}, \quad t > 0.$$

Definition 3.3 ([15]). Furthermore, the Caputo fractional derivative operator of order β is represented by ${}^C D_t^\delta$, it is defined by

$${}^C D_t^\delta \nu(t) = \frac{d}{dt}[I_t^{1-\delta}(\nu(t)-\nu(0))] = \frac{d}{dt}\bigg(\int_0^t g_{1-\delta}(t-s)(\nu(t)-\nu(0)) ds\bigg), \quad t > 0.$$

More generally, for $u : [0,\infty] \times \mathbb{R}^n \to \mathbb{R}^n$. Caputo fractional derivative with respect to time of function w can be written as

$$\partial_t^\delta w(t,x) = \partial_t\bigg(\int_0^t g_{1-\delta}(t-s)(w(t,x)-w(0,x)) ds\bigg), \quad t > 0.$$

Definition 3.4. Let's look into Mittag-Leffler special functions in general:

$$\begin{aligned} E_\delta(-t^\delta A) &= \int_0^\infty \mathfrak{M}_\delta(s) e^{-st^\delta A} ds, \\ E_{\delta,\delta}(-t^\delta A) &= \int_0^\infty \delta s \mathfrak{M}_\delta(s) e^{-st^\delta A} ds, \end{aligned}$$

where $\mathfrak{M}_\delta(\lambda)$ denotes the Mainardi Wright type function, which is defined by

$$\mathfrak{M}_\delta(\lambda) = \sum_{n=0}^\infty \frac{\lambda^n}{n!\Gamma(1-\delta(1+n))}.$$

Proposition 3.2.

$$\begin{aligned}(i)\quad E_{\delta,\delta}(-t^{\beta}A) &= \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\delta,\delta}(-t^{\delta}\mu)(\mu I+A)^{-1}d\mu;\\ (ii)\quad A^{\delta}E_{\delta,\delta}(-t^{\delta}A) &= \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^{\delta}E_{\delta,\delta}(-t^{\delta}\mu)(\mu I+A)^{-1}d\mu.\end{aligned}$$

proof. *(i)In the view of* $\int_0^\infty \delta s\mathfrak{M}_\delta(s)e^{-st}ds = E_{\delta,\delta}(-t)$ *and Fubini theorem, we have*

$$\begin{aligned}E_{\delta,\delta}(-t^{\delta}A) &= \int_0^\infty \delta s\mathfrak{M}_\delta(s)e^{-st^{\delta}A}ds\\ &= \frac{1}{2\pi i}\int_0^\infty \delta s\mathfrak{M}_\delta(s)\int_{\Gamma\theta} e^{-\mu st^{\delta}}(\mu I+A)^{-1}d\mu ds\\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} E_{\delta,\delta}(-t^{\delta}\mu)(\mu I+A)^{-1}d\mu,\end{aligned}$$

here Γ_θ *is appropriate integral route.*

(ii) Similarly

$$\begin{aligned}A^{\alpha}E_{\delta,\delta}(-t^{\delta}A) &= \int_0^\infty \delta s\mathfrak{M}_\delta(s)A^{\alpha}e^{-st^{\delta}A}ds\\ &= \frac{1}{2\pi i}\int_0^\infty \delta s\mathfrak{M}_\delta(s)\int_{\Gamma\theta} \mu^{\alpha}e^{-\mu st^{\delta}}(\mu I+A)^{-1}d\mu ds\\ &= \frac{1}{2\pi i}\int_{\Gamma\theta} \mu^{\alpha}E_{\delta,\delta}(-\mu t^{\delta})(\mu I+A)^{-1}d\mu.\end{aligned}$$

The results immediately arise generally equivalent.

Lemma 3.3. *([14]).The operators* $E_\delta(-t^\delta A)$ *and* $E_{\delta,\delta}(-t^\delta A)$ *are the continuous for* $t>0$*, in uniform operator topology. Further, the uniformly continuous on* $[r,\infty)$ *for* $r>0$*.*

Lemma 3.4. *([15])*

. *if* $0<\delta<1$ *it gives*

(i) $\forall\ u\in X,\ \lim_{t\to 0^+} E_\delta(-t^\delta A)u = u;$

(ii) $\forall\ u\in D(A) and\ t>0,\ {}^{C}D_t^{\delta}E_\delta(-t^\delta A)u{=}{-}AE_\delta(-t^\delta A)u;$

(iii) $\forall\ u\in X,\ E_\delta^{'}(-t^\delta A)u{=}{-}t^{\delta-1}AE_{\delta,\delta}(-t^\delta A)u;$

(iv) $\forall\ t>0,\ E_\delta(-t^\delta A)u{=}I_t^{1-\delta}\left(t^{\delta-1}E_{\delta,\delta}(-t^\delta A)u\right).$

4 Existence

Lemma 4.1. *Let $\delta \in (0,1)$ and $\left\{A(t)x(t) + f(t,x(t)) + \sigma(t,x_t)\frac{dB(t)}{dt}\right\} : J \to \mathbb{R}$ is mapping which is well defined and continuous. An operation x as provided by*

$$\begin{cases} {}^cD_a^\delta x(t) = A(t)x(t) + f(t,x(t)) + \sigma(t,x_t)\frac{dB(t)}{dt} \\ x(t) = \phi(t); \qquad -\omega \le t < 0 \\ \Delta x(t_k) = y_k; \qquad k = 1,2... \end{cases} \tag{2}$$

By applying RL integral on both side of equation and put conditions we get following results

$$x(t) = \begin{cases} \phi_o + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\left[A(s)x(s) + f(s,x(s)) + \sigma(t,x_t)\frac{dB(s)}{ds}\right], \quad \forall t \in (t_0,t_1) \\ \phi_o + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\left[A(s)x(s) + f(s,x(s)) + \sigma(t,x_t)\frac{dB(s)}{ds}\right], \quad \forall t \in (t_0,t_1) \end{cases} \tag{3}$$

is the solution of above equation with $x(0) = \phi(0) = \phi_o$ It's a special kind of answer. While $\tilde{x}(t)$ is not different from x(t) then the function x, provided by

$$P\{x(t) = \tilde{x}(t) \ for \ any \ t_o - \omega < t \le T\} = 1$$

This is expected, a solution is considered oscillating whenever it exhibits neither positive nor negative situations. If not, it is referred to be non-oscillating. A solution is considered oscillatory if, for every $\{\xi_k\}k \in \mathbb{N} \subset [\theta_0, \infty)$ there's an increasing divergent sequence, That is to say $x(\xi_k^+)x(\xi_k^-) \le 0 \ \forall \ k \in \mathbb{N}$.

Lemma 4.2.

$$\begin{cases} {}^cD_a^\delta x(t) = A(t)x(t) + f(t,x(t)) + \sigma(t,x_t)\dfrac{dB(t)}{dt} \\ x(t) = \phi(t); \qquad -\omega \le t < 0 \\ \Delta x(t_k) = y_k; \qquad k = 1,2... \end{cases}$$

satisfying solution

$$x(t) = \begin{cases} \phi_0 + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\left[A(s)x(s) + f(s,x(s)) + \sigma(t,x_t)\frac{dB(s)}{ds}\right], \quad \forall t \in (t_0,t_1) \\ \phi_0 + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\left[A(s)x(s) + f(s,x(s)) + \sigma(t,x_t)\frac{dB(s)}{ds}\right], \quad \forall t \in (t_0,t_1) \end{cases}$$

$$\begin{aligned} x(t) &= \phi_o + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)x(s)ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\sigma(t,x_t)dB(s) \end{aligned}$$

Taking Laplace on both sides

$$x(\lambda) = \frac{\phi_o}{\lambda} + \frac{1}{\lambda^\delta}[A(\lambda)x(\lambda)] + \frac{1}{\lambda^\delta}F(\lambda,u(\lambda)) + \frac{1}{\lambda^\delta}\sigma(\lambda,x_\lambda)dB(\lambda)$$

$$
\begin{aligned}
\lambda^\delta x(\lambda) &= \phi_o\lambda^{\delta-1} + [A(\lambda)x(\lambda)] + F(\lambda, x(\lambda)) + \sigma(\lambda, x_\lambda)dB(\lambda)\\
(\lambda^\delta + A)x(\lambda) &= \phi_o\lambda^{\delta-1} + F(\lambda, x(\lambda)) + \sigma(\lambda, x_\lambda)dB(\lambda)\\
x(\lambda) &= (\lambda^\delta + A)^{-1}\phi_o\lambda^{\delta-1} + (\lambda^\delta + A)^{-1}F(\lambda, x(\lambda))\\
&+ (\lambda^\delta + A)^{-1}\sigma(\lambda, X_\lambda)dB(\lambda)
\end{aligned}
$$

Considering the inverse of Laplace on each side

$$
\begin{aligned}
x(t) &= \phi_o\int_a^t E_\delta(-(t-s)^\delta A)ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^\delta A)F(\lambda, x(\lambda))ds\\
&+ \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^\delta A)\sigma(\lambda, x_\lambda)dB(\lambda).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
x(t) &= \left(\phi_o + \sum_{i=1}^{k} y_i\right)\int_a^t E_\delta(-(t-s)^\delta A)ds + \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^\delta A)F(\lambda, x(\lambda))ds\\
&+ \int_a^t (t-s)^{\delta-1}E_{\delta,\delta}(-(t-s)^\delta A)\sigma(\lambda, x_\lambda)dB(\lambda).
\end{aligned}
$$

5 Bounded Behaviour

Theorem 5.1. *Assume the* $0 < \delta < 1, P > 1, \beta > 0, P(\delta-1)+1 > 0, P(\beta-1)+1 > 0, q = \frac{p}{p-1}$ *with continuous mapping* $\left\{A(t)x(t) + f(t, x(t)) + \sigma(t, x_t)\frac{dB(t)}{dt}\right\} : J \to \mathbb{R}$ *exists such that*

$$
\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds \le d \tag{4}
$$

is bounded $\forall t \ge a$. *Given function f(t,x) holds the following conditions.*

- (P_1) : *Continuity of f(t,x) exists in domain of interval* $D = \{(t, x) : t \in j, x \in \mathbb{R}\}$
- (P_2) : *Both g and h are continuously and non-negative functions exists such that* $g, h : \mathbb{R}^+ := [a, \infty) \to \mathbb{R}$ *also g is increasing function. Take* $0 < \beta \le 2 - \delta - \frac{1}{P}$

$$
|f(t, x(t))| \le t^{\beta-1}h(t)g(\frac{|x|}{t}), \quad t > a, \quad (t, x) \in D, \tag{5}
$$

and

$$
\int_a^\infty s^{\frac{\mu q}{p}} h^q(s)ds < \infty, \tag{6}
$$

where $\mu := P(\delta + \beta - 2) + 1 \le 0$.

- (P_3) :

$$
\int_a^\infty \frac{d\eta}{g^q(\eta)} \to \infty. \tag{7}
$$

The impulsive values satisfy the subsequent prerequisite.

- (P_4) : *The value* $\bar{\mathcal{M}}$ *exists in order to*

$$\left|\sum_{i=1}^{k} y_i\right| < \bar{\mathcal{M}}, \quad k = 1, 2.... \tag{8}$$

Should x(t) be the answer to (1), subsequently

$$\lim_{t\to\infty} Sup \frac{|x(t)|}{t} < \infty. \tag{9}$$

- (P_5) *From the above conditions of boundedness we can find bounded value of stochastic function value*

$$\left|\sigma(t, x_t)\frac{dB(t)}{dt}\right| \leq Mt^{\beta-1}h^*(t)g^*(\frac{|x_t|}{t}), \quad t > a, \quad (t, x) \in D \tag{10}$$

$$\int_a^\infty s^{\frac{\mu q}{p}} h^{*q}(s)ds < \infty$$

where $\mu := P(\delta + \beta - 2) + 1 \leq 0.$

$$\int_a^\infty \frac{d\eta}{g^{*q}(\eta)} \to \infty.$$

$$\lim_{t\to\infty} Sup \frac{|x_t|}{t} < \infty.$$

$$\begin{aligned}
|x(t)| &\leq |\phi_o| + |\sum_{i=1}^{k} y_i| + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|f(s, x(s))|ds \\
&+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|\sigma(s, x_s)dB(s)|, \quad t \in (t_k, t_{k+1}].
\end{aligned}$$

Apply (5), Then obtain

$$\begin{aligned}
|x(t)| &\leq |\phi_o| + |\sum_{i=1}^{k} y_i| + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds \\
&+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h(s)g(\frac{|x(s)|}{s})ds \\
&+ \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h^*(s)g^*\left(\frac{|x_s|}{s}\right)ds, \quad t \in (t_k, t_{k+1}].
\end{aligned}$$

From (4) we find $\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds \leq d \quad \forall t \geq a,$ *Suppose*

$$\begin{aligned}
\mathfrak{D}(k) &= |x_0| + |\sum_{i=1}^{k} y_i| + \frac{d}{\Gamma(\delta)} \\
|x(t)| &\leq \mathfrak{D}(k)t + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h(s)g\left(\frac{|x(s)|}{s}\right)ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}s^{\beta-1}h^*(s)g^*\left(\frac{|x_s|}{s}\right)ds \\
&\leq \mathfrak{D}(k)t + \frac{1}{\Gamma(\delta)}(t-a)\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g\left(\frac{|x(s)|}{s}\right)ds \\
&+ \frac{M}{\Gamma(\delta)}(t-a)\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*\left(\frac{|x_s|}{s}\right)ds
\end{aligned}$$

That results from inequity,

$$\begin{aligned}
\frac{|x(t)|}{t} &\leq \mathfrak{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g\left(\frac{|x(s)|}{s}\right)ds \\
&+ \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*\left(\frac{|x_s|}{s}\right)ds
\end{aligned}$$

put

$$\begin{aligned}
\mathcal{S}(t) &= \mathfrak{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h(s)g\left(\frac{|x(s)|}{s}\right)ds \\
\mathcal{S}^*(t) &= \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-2}s^{\beta-1}h^*(s)g^*\left(\frac{|x_s|}{s}\right)ds
\end{aligned}$$

$$\frac{|x(t)|}{t} \leq \mathcal{S}(t,k), \qquad t \in (t_k, t_{k+1}]. \tag{11}$$

$$\frac{|x(t)|}{t} \leq \mathcal{S}^*(t,k), \qquad t \in (t_k, t_{k+1}]. \tag{12}$$

here both g^* *and* g *are increasing functions*

$$\begin{aligned}
g\left(\frac{|x(t)|}{t}\right) &\leq g(\mathcal{S}(t,k)), \qquad t \in (t_k, t_{k+1}]. \\
g^*\left(\frac{|x(t)|}{t}\right) &\leq g^*(\mathcal{S}^*(t,k)), \qquad t \in (t_k, t_{k+1}].
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}(t,k) \leq 1 \quad &+ \quad \mathcal{D}(k) + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\epsilon-1}s^{\gamma-1}h(s)g(\mathcal{S}(s,k))ds \\
\mathcal{S}^*(t,k) \quad &\leq \quad \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\beta-1}s^{\gamma-1}h^*(s)g^*(\mathcal{S}^*(s,k))ds, \qquad t \in (t_k, t_{k+1}].
\end{aligned} \tag{13}$$

Take $\mu < \epsilon = \delta - 1 < 1$

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\epsilon-1}s^{\beta-1}h(s)g(\mathcal{S}(s,k))ds + \frac{M}{\Gamma(\delta)}\int_a^t (t-s)^{\epsilon-1}s^{\beta-1}h^*(s)g^*(\mathcal{S}^*(s,k))ds \\
&\le \frac{1}{\Gamma(\delta)}\bigg(\int_a^t (t-s)^{p(\epsilon-1)}s^{p(\beta-1)}ds\bigg)^{\frac{1}{p}}\bigg(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds\bigg)^{\frac{1}{q}} \\
&+\frac{M}{\Gamma(\delta)}\bigg(\int_a^t (t-s)^{p(\epsilon-1)}s^{p(\beta-1)}ds\bigg)^{\frac{1}{p}}\bigg(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds\bigg)^{\frac{1}{q}} \\
&\le \frac{1}{\Gamma(\delta)}\bigg(\mathcal{B}t^{\mu}\bigg)^{\frac{1}{p}}\bigg(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds\bigg)^{\frac{1}{q}} + \frac{M}{\Gamma(\delta)}\bigg(\mathcal{B}t^{\mu}\bigg)^{\frac{1}{p}}\bigg(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds\bigg)^{\frac{1}{q}} \\
&= \frac{1}{\Gamma(\delta)}\bigg(\mathcal{B}t^{\mu}\bigg)^{\frac{1}{p}}\bigg[\bigg(\int_a^t h^q(s)g^q(\mathcal{S}(s,k))ds\bigg)^{\frac{1}{q}} + M\bigg(\int_a^t h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds\bigg)^{\frac{1}{q}}\bigg]. \quad t\in(t_k,t_k+1],
\end{aligned}$$

where $\mathcal{B} := \mathcal{B}\bigg(P(\beta-1)+1, P(\epsilon-1)+1\bigg)$, $\mu = P(\delta+\beta-2)+1 \le 0$.
By applying $\mu \le 0$ *with* $t > s \ge a$, *it gives,*

$$\int_a^t (t-s)^{\epsilon-1}s^{\beta-1}h(s)g(\mathcal{S}(s,k))ds.$$

$$\le \frac{1}{\Gamma(\delta)}\mathcal{B}^{\frac{1}{p}}\bigg[\bigg(\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds\bigg)^{\frac{1}{q}} + M\bigg(\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds\bigg)^{\frac{1}{q}}\bigg], \qquad \forall t\in(t_k,t_{k+1}].$$

using (14 we have,

$$(x+y)^q \le 2^{q-1}(x^q+y^q), \quad x,y\ge 0, \quad q>1.$$

Fort $\in (t_k, t_{k+1}]$,*we obtain from (13) that*

$$\begin{aligned}
\mathcal{S}^q(t,k) \quad &\le 2^{q-1}\bigg((1+\mathfrak{D}(k)^q) + (\mathcal{B}^{\frac{1}{p}}\tfrac{1}{\Gamma(\delta)})^q\bigg[\bigg(\textstyle\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds\bigg)^{\frac{1}{q}} \\
&+M\bigg(\textstyle\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds\bigg)^{\frac{1}{q}}\bigg]\bigg).
\end{aligned}$$

Setting $\mathfrak{P}_1(k) = 2^{q-1}[(1+\mathfrak{D}(k)^q)]$, $\mathfrak{Q}_1 = 2^{q-1}(\mathcal{B}^{\frac{1}{p}}\frac{1}{\Gamma(\delta)})^q$, $\mathfrak{R}_1 = 2^{q-1}M$. *Then*

$$\mathcal{S}^q(t,k) \le \mathfrak{P}_1(k) + \mathfrak{Q}_1\int_a^t s^{\frac{\mu q}{p}}h^q(s)g^q(\mathcal{S}(s,k))ds + \mathfrak{R}_1\int_a^t s^{\frac{\mu q}{p}}h^{*q}(s)g^{*q}(\mathcal{S}^*(s,k))ds,$$

Denote

$$\psi(\eta) = g^q(\eta),$$
$$\Psi(\xi) = \int_{s_k}^{\xi}\frac{d\eta}{\psi(\eta)}, \quad s_k = s(t_k^+,k). \tag{14}$$

$$\psi^*(\eta) = g^{*q}(\eta),$$

$$\Psi^*(\xi) = \int_{s_k}^{\xi} \frac{d\eta}{\psi^*(\eta)}, \quad s_k = s(t_k^+, k). \tag{15}$$

As we know that

$$\Psi(\mathcal{S}(t,k)) = \int_{s_k}^{s(t,k)} \frac{d\eta}{\psi(\eta)}$$

$$\Psi^*(\mathcal{S}(t,k)) = \int_{s_k}^{s(t,k)} \frac{d\eta}{\psi^*(\eta)},$$

P_3 implies that

$$\lim_{\mathcal{S}(t,k))\to\infty} \Psi(\mathcal{S}(t,k)) = \infty \quad and \quad \lim_{\mathcal{S}(t,k))\to\infty} \Psi^*(\mathcal{S}(t,k)) = \infty$$

According to Bihari Lemma ([17])

$$\mathcal{S}^q(t,k) \le L(k) := \Psi^{-1}\left(\Psi\mathfrak{P}_1(k) + \mathfrak{Q}_1 \int_a^t s^{\frac{\mu q}{p}} h^q(s)ds\right)$$

$$\mathcal{S}^q(t,k) \le L^*(k) := \Psi^{*-1}\left(\mathfrak{R}_1 \int_a^t s^{\frac{\mu q}{p}} h^{*q}(s)\right)ds$$

From axiom (P_4) and boundedness of $\mathfrak{P}_1(k)$. And from P_2 and (14) with (15) we find result $L(k)$, k=1,2,3... is bounded. Then

$$\mathcal{S}^q(t,k) \le L = \sup_{k\ge1} L(k)$$
$$\mathcal{S}(t,k) \le L^{\frac{1}{q}}$$
$$\mathcal{S}(t,k) \le L^{*\frac{1}{q}}$$

From (11) and (12), we have

$$\frac{|x(t)|}{t} \le L^{\frac{1}{q}},$$
$$\frac{|x_t|}{t} \le L^{*\frac{1}{q}}.$$

It draw the conclusion with

$$\lim_{t\to\infty} Sup \frac{|x(t)|}{t} < \infty,$$
$$\lim_{t\to\infty} Sup \frac{|x_t|}{t} < \infty.$$

and these concludes the demonstration.

6 Oscillatory Behaviour

.

Theorem 6.1. *let's suppose δ, p, q, β are constants and μ be defined in Theorem (5.1), and conditions are fulfil from $(P_1) - (P_5)$. If for any constant $\bar{m} \in \Bigg(M\Gamma(\delta), 1 + M\Gamma(\delta)\Bigg)$, then (1) refers oscillatory behaviour if following condition meets.*

$$\lim_{t\to\infty} Inf\Bigg[\bar{m}t + \int_a^t (t-s)^{\delta-1}A(s)x(s)ds\Bigg] = -\infty \tag{16}$$

$$\lim_{t\to\infty} Sup\Bigg[\bar{m}t + \int_a^t (t-s)^{\delta-1}A(s)x(s)ds\Bigg] = \infty \tag{17}$$

***Proof*.**

$$\lim_{t\to\infty}\int_t^\infty s^{\frac{\mu q}{p}} h^q(s) g^q\Bigg(\frac{x(s)}{s}\Bigg)ds = 0,$$
$$\lim_{t\to\infty}\int_t^\infty s^{\frac{\mu q}{p}} h^{*q}(s) g^{*q}\Bigg(\frac{x_s}{s}\Bigg)ds = 0.$$

So, there is $d_1 \geq d_o$ that satisfies

$$0 < \int_{d_1}^\infty s^{\frac{\mu q}{p}} h^q(s) g^q\Bigg(\frac{x(s)}{s}\Bigg)ds < 1, \tag{18}$$

$$0 < \int_{d_1}^\infty s^{\frac{\mu q}{p}} h^{*q}(s) g^{*q}\Bigg(\frac{x_s}{s}\Bigg)ds < 1. \tag{19}$$

One can deduce while losing generalization as $a \leq d_0 \leq d_1 < t_1$.
Following the same steps as in the Theorem (5.1) evidence that one get to

$$\begin{aligned} x(t) &\leq \phi_o + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)x(s)ds + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}\sigma(s,x_s)dB(s), \quad t \in (t_k, t_{k+1}], \quad k \geq 1. \end{aligned}$$

$$\begin{aligned} x(t) &\leq \phi_o + \sum_{i=1}^k y_i + \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)x(s)ds + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1}f(s,x(s))ds \\ &+ \frac{1}{\Gamma(\delta)}\int_{d_1}^t (t-s)^{\delta-1}s^{\beta-1}h^q(s)g^q\Bigg(\frac{x(s)}{s}\Bigg)ds + \frac{1}{\Gamma(\delta)}\int_a^{d_1} (t-s)^{\delta-1}\sigma(s,x_s)dB(s) \\ &+ M\frac{1}{\Gamma(\delta)}\int_{d_1}^t (t-s)^{\delta-1}s^{\beta-1}h^{*q}(s)g^{*q}\Bigg(\frac{x_s}{s}\Bigg)ds \end{aligned}$$

$$
\begin{aligned}
x(t) &\leq \phi_0+\sum_{i=1}^{k} y_i+\frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)x(s)ds+\frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}f(s,x(s))ds\\
&+ \frac{1}{\Gamma(\delta)}t\int_{d_1}^t (t-s)^{\delta-2}s^{\beta-1}h^q(s)g^q\left(\frac{x(s)}{s}\right)ds+\frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}\sigma(s,x_s)dB(s)\\
&+ M\frac{1}{\Gamma(\delta)}t\int_{d_1}^t (t-s)^{\delta-2}s^{\beta-1}h^{*q}(s)g^{*q}\left(\frac{x_s}{s}\right)ds, \quad t\in(t_k,t_{k+1}], \quad k\geq 1
\end{aligned}
$$

$$
\begin{aligned}
x(t) &\leq \phi_0+\bar{\mathcal{M}}+\frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}A(s)x(s)ds+\frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}f(s,x(s))ds\\
&+ \frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}\sigma(s,x_s)dB(s)\\
&+ \frac{1}{\Gamma(\delta)}t\mathcal{B}^{\frac{1}{p}}\left[\left(\int_{d_1}^t s^{\frac{\theta q}{p}}h^q(s)g^q(\frac{x(s)}{s})ds\right)^{\frac{1}{q}}+M\left(\int_{d_1}^t s^{\frac{\theta q}{p}}h^{*q}(s)g^{*q}(\frac{x_s}{s})ds\right)^{\frac{1}{q}}\right]
\end{aligned}
\tag{20}
$$

$$
x(t)\leq m+\frac{1}{\Gamma(\delta)}\left(\bar{m}t+\int_a^t (t-s)^{\delta-1}A(s)x(s)ds\right)t \tag{21}
$$

where

$$
m=\phi_o+\bar{M}+\frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}f(s,x(s))ds+\frac{1}{\Gamma(\delta)}\int_a^{d_1}(t-s)^{\delta-1}\sigma(s,x_s)dB(s)
$$

and

$$
\bar{m}:=\bar{\mathcal{M}}\Gamma(\delta)+\Gamma(\delta)\mathcal{B}^{\frac{1}{p}}\left[\left(\int_{d_1}^t s^{\frac{\theta q}{p}}h^q(s)g^q(\frac{x(s)}{s})ds\right)^{\frac{1}{q}}+M\left(\int_{d_1}^t s^{\frac{\theta q}{p}}h^{*q}(s)g^{*q}(\frac{x_s}{s})ds\right)^{\frac{1}{q}}\right]
$$

Here m and $\bar{m}$ are constants. From (18) and (19), we have $\bar{\mathcal{M}}\Gamma(\delta)<\bar{m}<1+\bar{\mathcal{M}}\Gamma(\delta)$
Take limit $t\to\infty$ to (21) and utilizing (16) runs opposed to the claim that x(t) is going to end up positive. In the event that $x(t)<0$, by taking y=-x. it is simply observed, y fulfills (1) by substituting -A(t)x(t) for x(t) and -f(t,-y) for f(t,x). Since the current case's proof is identical to that of the previous one, it is excluded. That whole demonstration.

7 Example

Here we discuss example on Impulsive Stochastic Fractional Differential Equation.

Example 7.1. Consider (ISFDE) with $\frac{1}{2}$th-order

$$
\begin{cases}
{}^cD_a^\delta x(t)=A(t)x(t)+t^{\beta-1}h(t)g(\frac{|x|}{t})+Mt^{\beta-1}h^*(t)g^*(\frac{|x_t|}{t}), \quad J:=[3,\infty),\delta=\frac{1}{2}\\
x(3)=\phi_o; \qquad -\omega\leq t<0\\
\Delta x(t_k)=\frac{1}{k(k+1)}; \qquad k=1,2...
\end{cases}
\tag{22}
$$

here $\phi_o=1$, $h(t)=(t+2)^{\frac{-4}{3}}$, $g(\eta)=\frac{1}{\eta^{1/q}}$, $A(t)x(t)=-t^{\frac{-3}{2}}$, $s\geq 3$, $p=\frac{3}{2}$, $\beta=\frac{5}{6}$, M=1, a=3, from these values our satisfied results are $p(\delta-1)+1=\frac{1}{4}>0$, $p(\beta-1)+1=\frac{3}{4}>0$, $q=\frac{p}{p-1}=3$, and

$\mu = p(\delta+\beta-2)+1 = 0$. So that results will be,

$$\begin{aligned}\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds = & \quad \frac{1}{t}\int_3^t (t-s)^{\frac{-1}{2}}|s^{\frac{-1}{2}}|ds \\ = & \quad \frac{1}{t}\int_3^t t^{-1/2}(1-\frac{s}{t})^{-1/2}s^{\frac{-1}{2}}ds \end{aligned} \tag{23}$$

By setting $\theta := \frac{s}{t}$ so above equation (23) becomes

$$\begin{aligned}\frac{1}{t}\int_a^t (t-s)^{\delta-1}|A(s)x(s)|ds \quad & = \frac{1}{t}\int_{\frac{3}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}t^{\frac{1}{2}}t^{\frac{-1}{2}}td\theta \\ & \int_{\frac{3}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta \\ & \leq \int_0^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta = \mathcal{B}(\frac{1}{2},\frac{3}{2}), t \geq 3.\end{aligned}$$

$$\int_a^\infty s^{\frac{\mu q}{p}}h^q(s)ds = \int_3^\infty (s+2)^{-4}ds = -\frac{1}{64}.$$

$$\int_a^\infty \frac{d\eta}{g^q(\eta)} = \int_3^\infty \eta d\eta \to \infty.$$

$$\left|\sum_{i=1}^k y_i\right| < 1, \quad k = 1,2,3....$$

$$\lim_{t\to\infty} Sup\frac{|x(t)|}{t} < \infty.$$

Similarly for stochastic term selection is $h^*(t) = (2t+3)^{\frac{-2}{3}}$, $g^*(\eta) = \frac{1}{(2\eta)^{1/q}}$ the result is

$$\begin{aligned}\int_3^\infty s^{\frac{\mu q}{p}}h^{*q}(s)ds &= -\frac{1}{18}, \\ \int_a^\infty \frac{d\eta}{g^{*q}(\eta)} &= \int_3^\infty 2\eta d\eta \to \infty \\ \lim_{t\to\infty} Sup\frac{|x_t|}{t} &< \infty.\end{aligned}$$

$$\bar{m}t + \int_a^t (t-s)^{\delta-1}A(s)x(s)ds = \bar{m}t - t\int_{\frac{2}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta = t\left(\bar{m} - \int_{\frac{2}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta\right)$$

$$\lim_{t\to\infty}\int_{\frac{2}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta = \mathcal{B}(\frac{1}{2},\frac{3}{2}).$$

From above results $M\Gamma(\delta) < \bar{m} < 1 + M\Gamma(\delta)$, we have

$$\bar{m} < \mathcal{B}(\frac{1}{2},\frac{3}{2}).$$

$$\bar{m} - \int_{\frac{2}{t}}^1 \theta^{-1/2}(1-\theta)^{-1/2}d\theta < 0$$

for sufficiently larger t.

So,

$$\lim_{t\to\infty}\left(\bar{m}t+\int_a^t (t-s)^{\delta-1}A(s)x(s)ds\right)\to -\infty.$$

Hence it is concluded that all necessary conditions $(P_1)-(P_5)$ with equation (16) are satisfied by equation (22). Hence all solution is oscillatory.

8 Declaration of Competing Interest

The authors state that they are clear of any financial conflicts of interest or close personal connections that might have seemed to have an effect on the research presented in this study.

Data availability statement:

No new data were created this study.

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