# Brachistochrone Tunnel

## 1. Equation of motion:

To calculate the fastest path between two points on the surface, we have to find the function minimizing the functional

$$T = \int_A^B \frac{\mathrm{d}s}{v(r)}\,, \tag{1}$$

where $\mathrm{d}s$ is the infinitesimal arclength element along the path:

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2} \tag{2}$$

And the speed $v(s)$ can be obtained from the conversation of mechanical energy:

$$E = \frac{1}{2} m\, v(r)^2 + U(r) \tag{3}$$

Inside a uniform-density sphere, the force is linear:

$$F(r) \;=\; -\frac{GM_{\mathrm{eff}}\, m}{r^2} \;=\; -\frac{GMm}{R^3}\, r \tag{4}$$

where $M_{\mathrm{eff}} = 4/3\pi r^3 \rho = Mr^3/R^3$ is the effective mass contained inside a sphere of radius $r$. Hence the potential energy is

$$U(r) \;=\; \int F(r)\,\mathrm{d}r \;=\; \frac{GMm}{2R^3}\, r^2 + C\,. \tag{5}$$

where the integration constant can safely assumed to be $C = 0$. Assuming $v(R) = 0$ at the surface, (3) can be written as

$$\frac{GMm}{2R} \;=\; \frac{1}{2} m\, v(r)^2 + \frac{GMm}{2R^3}\, r^2 \tag{6}$$

from which we get

$$v(r) \;=\; \sqrt{\frac{GM}{R}\left(1 - \frac{r^2}{R^2}\right)} \;=\; \sqrt{gR\left(1 - \frac{r^2}{R^2}\right)}\,, \tag{7}$$

where $g = GM/R^2$. Therefore, the travel time becomes

$$T \;=\; \sqrt{\frac{R}{g}} \int_A^B \sqrt{\frac{(\mathrm{d}r/\mathrm{d}\theta)^2 + r^2}{R^2 - r^2}}\,\mathrm{d}\theta \;=\; \sqrt{\frac{R}{g}} \int_A^B \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}}\,\mathrm{d}\theta\,, \tag{8}$$

with $r' = \mathrm{d}r/\mathrm{d}\theta$.

Since the integrand is of the form $L(r, r')$, with no explicit $\theta$-dependence, we can use the Beltrami identity (special case of the Euler-Lagrange equation) to find the $r$ minimizing $T$:

$$L \;-\; r'\,\frac{\partial L}{\partial r'} \;=\; c\,, \tag{9}$$

for some constant $c$. Substituting

$$L(r, r') = \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}}\,, \tag{10}$$

we get

$$\frac{r^2}{\sqrt{r'^2 + r^2}\sqrt{R^2 - r^2}}\,, \tag{11}$$

from which we can isolate $r' = \mathrm{d}r/\mathrm{d}\theta$:

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} \;=\; \frac{r}{c}\sqrt{\frac{(1+c^2)\,r^2 - c^2R^2}{R^2 - r^2}} \tag{12}$$

Equation (12) is a separable ODE, and can be rewritten as:

$$\int \frac{c}{r}\sqrt{\frac{R^2 - r^2}{(1+c^2)r^2 - c^2R^2}}\,\mathrm{d}r \;=\; \theta\,. \tag{13}$$

Introducing the turning radius $r_0 = \sqrt{\frac{c^2R^2}{1+c^2}}$, where $\mathrm{d}r/\mathrm{d}\theta = 0$, we get

$$\int \frac{c}{r}\sqrt{\frac{R^2 - r^2}{(1+c^2)r^2 - c^2R^2}}\,\mathrm{d}r \;=\; \int \frac{r_0}{rR}\sqrt{\frac{R^2 - r^2}{r^2 - r_0^2}}\,\mathrm{d}r. \tag{14}$$

With the change of variables

$$u = \sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}, \qquad r = \sqrt{\frac{u^2R^2 + r_0^2}{1+u^2}}, \qquad \mathrm{d}r = \frac{u}{r}\,\frac{R^2 - r_0^2}{(1+u^2)^2}\,\mathrm{d}u, \tag{15}$$

the integral becomes

$$\int \frac{r_0(R^2 - r_0^2)}{R}\,\frac{\mathrm{d}u}{(u^2R^2 + r_0^2)(1+u^2)} \;=\; \frac{1}{R}\int \frac{r_0R^2}{u^2R^2 + r_0^2}\,\mathrm{d}u \;-\; \frac{1}{R}\int \frac{r_0}{1+u^2}\,\mathrm{d}u\,, \tag{16}$$

and the antiderivative is

$$\theta(r) \;=\; \arctan\!\Big(\frac{R}{r_0}\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}\Big) \;-\; \frac{r_0}{R}\,\arctan\!\Big(\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}\Big) \;+\; C_0. \tag{17}$$

The constant $C_0$ is an overall rotation. This is the equation of a hypocycloid, the path of a point of a circle, rolling inside larger circle

## 2. Travel time:

As established, the time functional is

$$T \;=\; \sqrt{\frac{R}{g}} \int\limits_{\theta_A}^{\theta_B} \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}}\, \mathrm{d}\theta\,. \tag{18}$$

Applying a change of variables with $\theta' = \mathrm{d}\theta/\mathrm{d}r$, we get:

$$T \;=\; 2\sqrt{\frac{R}{g}} \int\limits_{r_0}^{R} \sqrt{\frac{1 + (r\,\theta')^2}{R^2 - r^2}}\, \mathrm{d}r. \tag{19}$$

where we take the integral from $R$ (surface radius) to $r_0$ (turning radius), which covers only half of the path, hence the factor of 2 in the front. From (12), we can calculate $r\theta'$ by considering that $\mathrm{d}r/\mathrm{d}\theta = 1/\theta'$:

$$r\theta' = \frac{r_0}{\sqrt{R^2 - r_0^2}} \sqrt{\frac{R^2 - r^2}{(1 + c^2)r^2 - c^2 R^2}} \tag{20}$$

Substituting this into (19), and utilizing $c = r_0/\sqrt{R^2 - r_0^2}$, we get:

$$T = 2\sqrt{\frac{R^2 - r_0^2}{R\,g}} \int\limits_{R}^{r_0} \frac{r}{\sqrt{(R^2 - r^2)(r^2 - r_0^2)}} \mathrm{d}r \tag{21}$$

Which is admittedly a tricky integral, but can be calculated using the substitution

$$r^2 = r_0^2 \cos^2 u + R^2 \sin^2 u\,, \qquad\qquad u \in [0, \tfrac{\pi}{2}]\,, \tag{22}$$

making:

$$r\frac{\mathrm{d}r}{\mathrm{d}u} = (R^2 - r_0^2) \sin u \cos u \qquad \rightarrow \qquad r\mathrm{d}r = (R^2 - r_0^2) \sin u \cos u \mathrm{d}u \tag{23}$$

$$R^2 - r^2 = (R^2 - r_0^2) \cos^2 u \tag{24}$$

$$r^2 - r_0^2 = (R^2 - r_0^2) \sin^2 u \tag{25}$$

Substituting these back into (21), the integrand reduces to 1, and:

$$\begin{aligned} T = 2\sqrt{\frac{R^2 - r_0^2}{R\,g}} \int\limits_{0}^{\pi/2} \frac{(R^2 - r_0^2) \sin u \cos u}{\sqrt{(R^2 - r_0^2) \cos^2 u \; (R^2 - r_0^2) \sin^2 u}} \mathrm{d}u = 2\sqrt{\frac{R^2 - r_0^2}{R\,g}} \int\limits_{0}^{\pi/2} \mathrm{d}u = 2\sqrt{\frac{R^2 - r_0^2}{R\,g}} \frac{\pi}{2} = \\ = \pi\sqrt{\frac{R^2 - r_0^2}{R\,g}} \end{aligned} \tag{26}$$

or, using $g = GM/R^2$:

$$T = \pi\sqrt{\frac{R(R^2 - r_0^2)}{GM}} \tag{27}$$

## Note:

To parametrize the curves more naturally with the angle $\theta_{\mathrm{AB}}$ between the start and end points, we can substitute

$$r_0 = R\left(1 - \frac{\theta_{\mathrm{AB}}}{\pi}\right) \tag{28}$$

in all the above calculations.