

Brachistochrone Tunnel

1. Equation of motion:

To calculate the fastest path between two points on the surface, we have to find the function minimizing the functional

$$T = \int_A^B \frac{ds}{v(r)}, \quad (1)$$

where ds is the infinitesimal arclength element along the path:

$$ds = \sqrt{dr^2 + r^2 d\theta^2} \quad (2)$$

And the speed $v(s)$ can be obtained from the conversation of mechanical energy:

$$E = \frac{1}{2}m v(r)^2 + U(r) \quad (3)$$

Inside a uniform-density sphere, the force is linear:

$$F(r) = -\frac{GM_{\text{eff}} m}{r^2} = -\frac{GMm}{R^3} r \quad (4)$$

where $M_{\text{eff}} = 4/3\pi r^3 \rho = Mr^3/R^3$ is the effective mass contained inside a sphere of radius r . Hence the potential energy is

$$U(r) = \int F(r) dr = \frac{GMm}{2R^3} r^2 + C. \quad (5)$$

where the integration constant can safely assumed to be $C = 0$. Assuming $v(R) = 0$ at the surface, (3) can be written as

$$\frac{GMm}{2R} = \frac{1}{2}m v(r)^2 + \frac{GMm}{2R^3} r^2 \quad (6)$$

from which we get

$$v(r) = \sqrt{\frac{GM}{R} \left(1 - \frac{r^2}{R^2}\right)} = \sqrt{gR \left(1 - \frac{r^2}{R^2}\right)}, \quad (7)$$

where $g = GM/R^2$. Therefore, the travel time becomes

$$T = \sqrt{\frac{R}{g}} \int_A^B \sqrt{\frac{(dr/d\theta)^2 + r^2}{R^2 - r^2}} d\theta = \sqrt{\frac{R}{g}} \int_A^B \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}} d\theta, \quad (8)$$

with $r' = dr/d\theta$.

Since the integrand is of the form $L(r, r')$, with no explicit θ -dependence, we can use the Beltrami identity (special case of the Euler-Lagrange equation) to find the r minimizing T :

$$L - r' \frac{\partial L}{\partial r'} = c, \quad (9)$$

for some constant c . Substituting

$$L(r, r') = \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}}, \quad (10)$$

we get

$$\frac{r^2}{\sqrt{r'^2 + r^2} \sqrt{R^2 - r^2}}, \quad (11)$$

from which we can isolate $r' = dr/d\theta$:

$$\frac{dr}{d\theta} = \frac{r}{c} \sqrt{\frac{(1 + c^2) r^2 - c^2 R^2}{R^2 - r^2}} \quad (12)$$

Equation (12) is a separable ODE, and can be rewritten as:

$$\int \frac{c}{r} \sqrt{\frac{R^2 - r^2}{(1 + c^2)r^2 - c^2 R^2}} dr = \theta. \quad (13)$$

Introducing the turning radius $r_0 = \sqrt{\frac{c^2 R^2}{1+c^2}}$, where $dr/d\theta = 0$, we get

$$\int \frac{c}{r} \sqrt{\frac{R^2 - r^2}{(1 + c^2)r^2 - c^2 R^2}} dr = \int \frac{r_0}{rR} \sqrt{\frac{R^2 - r^2}{r^2 - r_0^2}} dr. \quad (14)$$

With the change of variables

$$u = \sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}, \quad r = \sqrt{\frac{u^2 R^2 + r_0^2}{1 + u^2}}, \quad dr = \frac{u}{r} \frac{R^2 - r_0^2}{(1 + u^2)^2} du, \quad (15)$$

the integral becomes

$$\int \frac{r_0(R^2 - r_0^2)}{R} \frac{du}{(u^2 R^2 + r_0^2)(1 + u^2)} = \frac{1}{R} \int \frac{r_0 R^2}{u^2 R^2 + r_0^2} du - \frac{1}{R} \int \frac{r_0}{1 + u^2} du, \quad (16)$$

and the antiderivative is

$$\theta(r) = \arctan\left(\frac{R}{r_0} \sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}\right) - \frac{r_0}{R} \arctan\left(\sqrt{\frac{r^2 - r_0^2}{R^2 - r^2}}\right) + C_0. \quad (17)$$

The constant C_0 is an overall rotation. This is the equation of a hypocycloid, the path of a point of a circle, rolling inside larger circle

2. Travel time:

As established, the time functional is

$$T = \sqrt{\frac{R}{g}} \int_{\theta_A}^{\theta_B} \sqrt{\frac{r'^2 + r^2}{R^2 - r^2}} d\theta. \quad (18)$$

Applying a change of variables with $\theta' = d\theta/dr$, we get:

$$T = 2 \sqrt{\frac{R}{g}} \int_{r_0}^R \sqrt{\frac{1 + (r\theta')^2}{R^2 - r^2}} dr. \quad (19)$$

where we take the integral from R (surface radius) to r_0 (turning radius), which covers only half of the path, hence the factor of 2 in the front. From (12), we can calculate $r\theta'$ by considering that $dr/d\theta = 1/\theta'$:

$$r\theta' = \frac{r_0}{\sqrt{R^2 - r_0^2}} \sqrt{\frac{R^2 - r^2}{(1 + c^2)r^2 - c^2 R^2}} \quad (20)$$

Substituting this into (19), and utilizing $c = r_0/\sqrt{R^2 - r_0^2}$, we get:

$$T = 2 \sqrt{\frac{R^2 - r_0^2}{Rg}} \int_R^{r_0} \frac{r}{\sqrt{(R^2 - r^2)(r^2 - r_0^2)}} dr \quad (21)$$

Which is admittedly a tricky integral, but can be calculated using the substitution

$$r^2 = r_0^2 \cos^2 u + R^2 \sin^2 u, \quad u \in [0, \frac{\pi}{2}], \quad (22)$$

making:

$$r \frac{dr}{du} = (R^2 - r_0^2) \sin u \cos u \quad \rightarrow \quad r dr = (R^2 - r_0^2) \sin u \cos u du \quad (23)$$

$$R^2 - r^2 = (R^2 - r_0^2) \cos^2 u \quad (24)$$

$$r^2 - r_0^2 = (R^2 - r_0^2) \sin^2 u \quad (25)$$

Substituting these back into (21), the integrand reduces to 1, and:

$$\begin{aligned} T &= 2 \sqrt{\frac{R^2 - r_0^2}{Rg}} \int_0^{\pi/2} \frac{(R^2 - r_0^2) \sin u \cos u}{\sqrt{(R^2 - r_0^2) \cos^2 u (R^2 - r_0^2) \sin^2 u}} du = 2 \sqrt{\frac{R^2 - r_0^2}{Rg}} \int_0^{\pi/2} du = 2 \sqrt{\frac{R^2 - r_0^2}{Rg}} \frac{\pi}{2} = \\ &= \pi \sqrt{\frac{R^2 - r_0^2}{Rg}} \end{aligned} \quad (26)$$

or, using $g = GM/R^2$:

$$T = \pi \sqrt{\frac{R(R^2 - r_0^2)}{GM}} \quad (27)$$

Note:

To parametrize the curves more naturally with the angle θ_{AB} between the start and end points, we can substitute

$$r_0 = 1 - \frac{\theta_{AB} R}{\pi} \quad (28)$$

in all the above calculations.