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Review Paper

Solving the advection diffusion equation in three dimensions in neutral case

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In this paper, the advection diffusion equation is solved in three dimensions space (x, y, z) using separation of variables technique to evaluate pollutant concentration per emission rate, taking eddy diffusivities of pollutants and mean wind speed in neutral case. The used data set was observed from the atmospheric diffusion experiments were made at Indian Institute of Technology (ITT) Delhi. The trace of sulfur hexafluoride (SF_6) was released at a rate q=50 ml min^{-1}. The results show that the predicated Gaussian model Present model, previous model with u*, and slender model with u* are in agreement with the observation at 100m downwind and most cases lie within a factor of two.

Key words: Diffusion, Neutral Case; Eddy diffusivity, Gaussian Model.

INTRODUCTION

Atmospheric dispersion modeling refers to the mathematical description of contaminant transport in the atmosphere. The term dispersion in this context is used to describe the combination of diffusion (due to turbulent eddy motion) and advection (due to the wind) that occurs within the air near the Earth's surface (John- 2011). The analytical solution of the atmospheric diffusion equation in different shapes depends on Gaussian and non-Gaussian solutions. An analytical solution with power law for the wind speed and eddy diffusivity with the realistic assumption was studied by (Demuth-1978). The solution has been implemented in the KAPPA-G model (Tirabassi et al. 1986). (Lin and Hildemann, 1997) extended the solution of (Demuth-1978) under boundary conditions suitable for dry deposition on the ground.

Estimating of crosswind integrated Gaussian and non-Gaussian concentration through different dispersion schemes was studied by (Essa et al. 2011).

In this paper the advection diffusion equation (ADE) is solved in three directions ways to obtain the crosswind integrated ground level concentration using separation variables of technique, taking eddy diffusivities of pollutants and mean wind speed in neutral case. The used data set was observed from the atmospheric diffusion experiments were made in the Indian institute of Technology (ITT) Delhi. The trace SF_6 was released at a rate q=50 ml min^{-1}.

Data and Methodology

The steady state the advection diffusion equation for dispersion of a non reactive contaminates released from continuous source (Seinfeld, 1986).

$$U(z)\frac{\partial C}{\partial x}=\frac{\partial}{\partial y}\left(K_y\frac{dC}{dy}\right)+\frac{\partial}{\partial z}\left(K_z\frac{dC}{dz}\right)\Rightarrow$$

$$U(z)\frac{\partial C}{\partial x}=\left(K_y\frac{d^2C}{dy^2}\right)+\left(K_z\frac{d^2C}{dz^2}+K_z'\frac{dC}{dz}\right)$$

(1)

Where

$$U(z)=\frac{u_*}{k_v}\ln\left(\frac{z}{z_0}\right)$$

$$K_y=\beta U x \qquad (2)$$

$$K_z=k_v w_* z$$

Where β= 3.61*((u*/U) ^2), z_0 is the roughness height (0.78), z is the vertical height at 15 m and Q=50ml min^{-1} (Essa et. al. 2009). Where x, y and z are the coordinates in the along-wind, cross wind and vertical directions respectively. C is the mean concentration of pollutants and U (z) is the mean wind speed in the downwind direction, K_y and K_z are eddy diffusivities of pollutants in the crosswind and vertical directions respectively.

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Where k_v is the von Karman constant (k_v ~0.4), z_s is the stack height (115m) and w* is the convection velocity scale, and β is a parameter depends on the stability condition. u* is frication velocity .

Let the solution of the equation (1) in the form

$$C(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{3}$$

Substituting from equation (3) after differentiation (1) respect to x, y, z and divided on (X(x) Y(y) Z (z)), we get that:-

$$U(z)\,\frac{X'}{X(x)} = \left(K_y\,\frac{Y''}{Y(y)}\right)+\left(K_z\,\frac{Z''}{Z(z)}+K_z'\,\frac{Z'}{Z(z)}\right)=-\lambda^2$$

$$\frac{X'}{X(x)}=\frac{-\lambda^2}{U(z)}\Rightarrow X(x)=C_1 e^{\frac{-\lambda^2}{U(z)}x} \tag{4}$$

$$Y''+\frac{\lambda^2}{K_y}Y(y)=0\Rightarrow Y(y)=C_2\cos\left(\lambda y/\sqrt{K_y}\right)+C_3\sin\left(\lambda y/\sqrt{K_y}\right)$$

$$Z''(z)+\frac{K_z'}{K_z}Z'(z)+\frac{\lambda^2}{K_z}Z(z)=0\Rightarrow Z(z)=AJ_o\left(\lambda/\sqrt{K_z}\right)+BY_0\left(\lambda/\sqrt{K_z}\right)$$

C(x,y,z) is finite at z=0 that B=0 $\therefore\ Z(z)=AJ_o\left(\lambda/\sqrt{K_z}\right)$

Substituting from equation (4) in equation (3), we get that:-

$$C(x,y,z)=e^{\frac{-\lambda^2}{U(z)}x}\,J_o\left(\lambda/\sqrt{K_z}\right)\left(\alpha\cos\left(\lambda y/\sqrt{K_y}\right)+\beta\sin\left(\lambda y/\sqrt{K_y}\right)\right) \tag{5}$$

Let

α= $C_1 C_2$ A , β= $C_1 C_3$ A

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2 4^2}-\ldots\ldots\ldots\ldots$$

where J_0 () is zero Bessel function in first order. (Shamus, 1980).

Equation (5) is subjected to the following boundary condition:-

1-The flux at the top of the mixing layer can be given by:-

$$K_z\,\frac{\partial C}{\partial z}=0 \text{ at z=0, zs} \tag{i}$$

2-The flux at cross wind of the mixing layer can be given by: -

$$\frac{\partial C(x,z)}{\partial y}=0 \text{ at } y=0, y_s \tag{ii}$$

3-the pollutant is released from an elevated point source of strength Q located at the point (0, y_s, z_s) is written in the form:-

$$UC(x, y, z) = Q\,\delta(y-y_s)\,\delta(z-z_s) \quad \text{at } x=0 \tag{iii}$$

where δ is Dirac delta function.
Substituting from equation (ii) in equation (5), we get that:-

$$\beta=0$$
$$\lambda=\frac{n\pi\sqrt{K_y}}{y_s} \tag{6}$$

Substituting from and equation (iii) in equation (3), we get that:-

$$\alpha=\frac{Q}{U(z_s)\,J_0\left(\frac{n\pi}{y_s}\sqrt{\frac{K_{y_s}}{K_{z_s}}}\right)} \tag{7}$$

Substituting from equation (6) and equation (7) in equation (5), we get that:-

$$\frac{C(x,y,z)}{q}=e^{-\left(\frac{n^2\pi^2K_y}{y_s^2U(z)}\right)x}\,J_o\left(\frac{n\pi}{y_s}\sqrt{\frac{K_y}{K_z}}\right)\left(\frac{\cos\left(\frac{n\pi y}{y_s}\right)}{U(z_s)\,J_0\left(\frac{n\pi}{y_s}\sqrt{\frac{K_{y_s}}{K(z_s)}}\right)\cos(n\pi)}\right) \tag{8}$$

Where

$$J_0\left(\frac{n\pi\sqrt{K_y}}{y_s\sqrt{K(z)}}\right)=1-\frac{(n\pi)^2K_y}{2^2{y_s}^2K(z)}+\frac{(n\pi)^4{K_y}^2}{2^24^2{y_s}^4K_{z_s}^2}-\ldots\ldots\ldots\ldots$$

Results and Discussion

Table 1: wind speed and frication velocity in different runs are 0.78m (roughness length) at 15m

run no.	u_* (m/s)	U(m/s) (at 15m)
1	0.34	2.5
2	0.21	1.6
6	0.34	2.5
7	0.37	2.7
8	0.25	1.8
11	0.25	1.8
12	0.35	2.6
13	0.21	1.6

Table 2: peak values of concentration in parts per trillion (ppt) observed and Predicated at 50 downwind of the source

Run No,	Observed	Previous model with u_*	Present Model	Slender plume model with u_*	Gaussian model with u_*
1	832	205	166	426	738
2	1068	173	442	642	1045
6	1101	256	562	426	528
7	248	245	651	392	487
8	1282	669	972	782	1017
11	616	265	547	568	680
12	759	242	368	515	672

13	1060	224	611	971	1200

Table 3: peak values of concentration in parts per trillion (ppt) observed and predicated at 100 downwind of the source

Run No	observed	Previous model with u*	Present model	Slender plume model with u*	Gaussian model With u*
1	832	145	146	345	145
2	1068	195	197	247	195
6	1101	146	147	155	146
7	248	134	136	143	134
8	1282	211	213	287	211
11	616	184	186	194	184
12	759	155	141	186	155
13	1060	193	195	202	193

Table 2 shows peak values of trace concentrations results from the Previous model with u*, present model, the Slender plume and Gaussian models in parts per trillion (ppt) observed and predicated at 50 m downwind of the source. Table (3) show peak values of trace concentrations results from the model with u*, present model, the Slender plume and Gaussian models in parts per trillion (ppt) (Essa et al. (2009) observed and predicated at 100m downwind of the source. Table -4- and Table -5- shows the ratio predicated and observed concentration in parts per trillion (ppt) at 50 m and 100m downwind of the source.

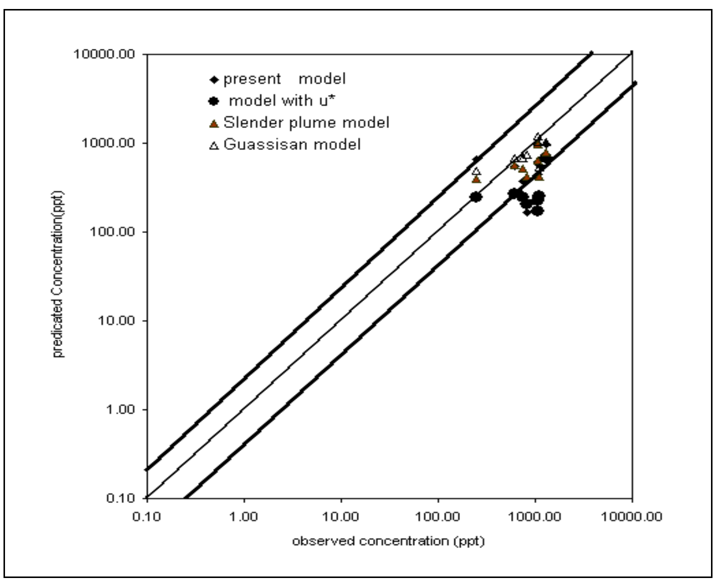

Figure 1: Logarithmic plot of observed at 50 downwind of the source versus predicated concentration (ppt) Middle line is the one to one line where as other represents the factor of two

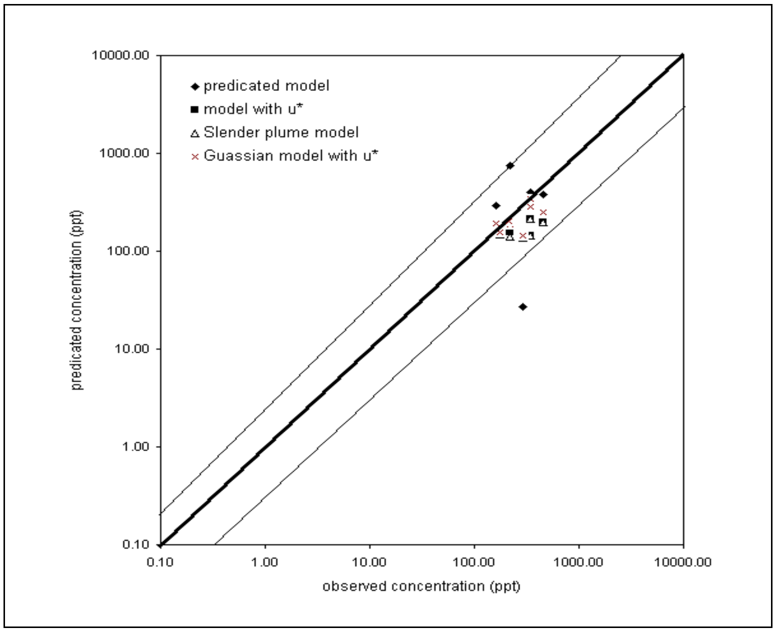

Figure 2: Logarithmic plot of observed at 100 downwind of the source versus predicated concentration (ppt) Middle line is the one to one line where as other represents the factor of two

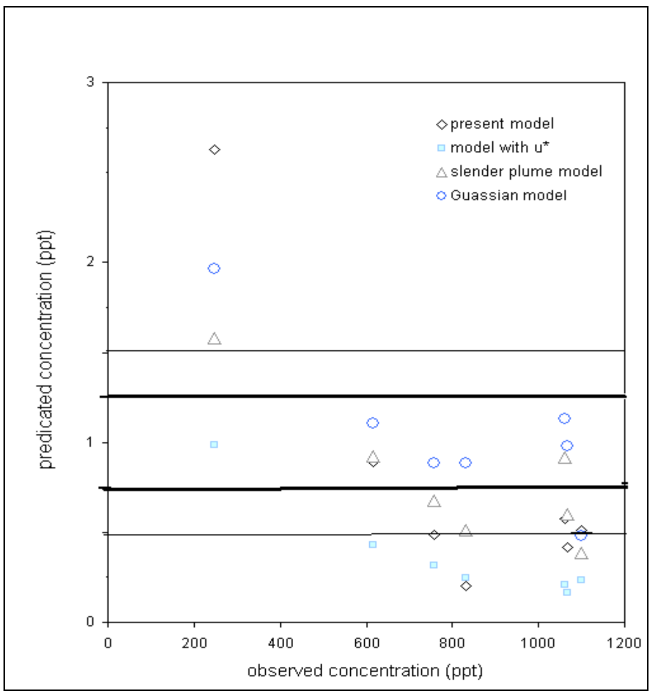

Figure 3: Scatter diagram of the ratio of predicated to observed concentration versus the observed concentration

Fig-1- gives the logarithmic plot for peak values of trace concentrations results from the model with u*, the Slender plume and the Gaussian models .This figure shows that the results from Gaussian model with u*, Present model and model with u* lie within a factor of two with the observation at 50m downwind distance.

Fig-2- gives the logarithmic plot for peak values of trace concentrations results from the model with u*, the Slender plume and the Gaussian models .This figure shows that the results from Gaussian model with u*, Present model and model with u* compares well with the observation at 100m downwind and most cases lie within a factor of two. The used data set was observed from the atmospheric diffusion experiments were made at Indian Institute of Technology (ITT) Delhi. The trace SF6 was released at a rate q=50 ml min^{-1}.

Conclusion

In this paper, the advection diffusion equation is solved in three dimensions space (x,y,z) using separation of variables technique to evaluate crosswind integrated of pollutant concentration per emission rate, taking eddy diffusivities of pollutants and mean wind speed in neutral case. The used data set was observed from the atmospheric diffusion experiments were made at Indian Institute of Technology (ITT) Delhi. The trace SF_6 was released at a rate q=50 ml $min^{-1.}$ Shows that the results from the Gaussian model with u*, Present model and model with u* in agreement with the observation at 100m downwind and most cases lie within a factor of two. Comparison between different models at 50,100 m downwind of the source according to standard statistical performance measure is calculated. We find most cases lie within a factor of two.

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