1 Relativistic Lorentz Force

The first equation of motion is Newton's second law:

$$\frac{d\vec{p}}{dt} = \vec{F}(\vec{r}, \vec{p}).$$

Non-relativistically,

$$m\frac{d\dot{\vec{x}}}{dt} + \dot{\vec{x}}\frac{dm}{dt} = \vec{F}(\vec{r}, \vec{p}).$$

Now, assume constant mass particles, then, in full relativity, (for something like the lorentz force)

$$\begin{aligned} \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r}, \vec{p}) \\ = m\left(\frac{d(\gamma\dot{\vec{x}})}{dt}\right) &= \vec{F}(\vec{r}, \vec{p}) \\ = m\dot{\gamma}\dot{\vec{x}} + m\gamma\ddot{\vec{x}} &= \vec{F}(\vec{r}, \vec{p}) \end{aligned}$$

where the first term is a relativistic drag term. We have that

$$\begin{aligned} \dot{\gamma} &= \frac{d}{dt}\left(\frac{1}{\sqrt{1 - |\dot{\vec{x}}|^2/c^2}}\right) \\ &= -\frac{-\frac{1}{c^2}\frac{d}{dt}(|\dot{\vec{x}}|^2)}{2(1 - |\dot{\vec{x}}|^2/c^2)^{3/2}} \\ &= \frac{1}{2(1-\beta^2)^{3/2}}\frac{1}{c^2}\frac{d}{dt}(|\dot{\vec{x}}|^2) \end{aligned}$$

We have that

$$\begin{aligned} \frac{d}{dt}(v^2) &= \frac{d}{dt}\left(\sum_i v_i v_i\right) \\ &= 2\sum_i \dot{v_i} v_i \\ &= 2\vec{a}\cdot\vec{v}. \end{aligned}$$

So,

$$\begin{aligned} \dot{\gamma} &= \frac{\gamma}{2c^2(1-\beta^2)}2(\vec{a}\cdot\vec{v}) \\ &= \frac{\gamma}{c^2(1-\beta^2)}(\vec{a}\cdot\vec{v}) \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\gamma}{c^2(1-\beta^2)}(\ddot{\vec{x}}\cdot\dot{\vec{x}})\dot{\vec{x}} + \gamma\ddot{\vec{x}} &= \frac{1}{m}\vec{F}(\vec{r}, \vec{p}) \\ \Rightarrow \gamma\left(I + \frac{1}{c^2(1-\beta^2)}\dot{\vec{x}}\otimes\dot{\vec{x}}\right)\cdot\ddot{\vec{x}} &= \frac{1}{m}\vec{F}(\vec{r}, \vec{p}) \end{aligned}$$

where I is the identity tensor. Now, for the lorentz force,

$$\gamma\ddot{\vec{x}}\cdot\left(I + \frac{\dot{\vec{x}}\otimes\dot{\vec{x}}}{c^2 - v^2}\right) = \frac{e}{m}\left[\vec{E}(\vec{r}, t) + \frac{\vec{v}}{c}\times\vec{B}\right].$$

Given a time with $\vec{x}(t) = \sum_i x_i(t)\hat{x}_i$ and $\vec{p}(t) = m\gamma(t)\vec{v} = m\gamma(t)\sum_i v_i(t)\hat{x}_i$, we have in matrix notation,

$$\begin{aligned}
\begin{pmatrix} 1+\frac{v_x^2}{c^2-v^2} & \frac{v_xv_y}{c^2-v^2} & \frac{v_xv_z}{c^2-v^2} \\ \frac{v_xv_y}{c^2-v^2} & 1+\frac{v_y^2}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} \\ \frac{v_xv_z}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} & 1+\frac{v_z^2}{c^2-v^2} \end{pmatrix}\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} &= \frac{e\sqrt{1-v^2/c^2}}{m}\left[\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + \frac{1}{c}\begin{pmatrix} v_yB_z - v_zB_y \\ v_zB_x - v_xB_z \\ v_xB_y - v_yB_x \end{pmatrix}\right] \\
\Rightarrow \Gamma\vec{a} &= \frac{e}{m\gamma}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \\
\Rightarrow \vec{a} &= \frac{\Gamma^{-1}}{\gamma}\frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right).
\end{aligned}$$

We see that the this is the old lorentz force in a rest frame with a matrix correction

$$\frac{1}{\gamma}\begin{pmatrix} 1+\frac{v_x^2}{c^2-v^2} & \frac{v_xv_y}{c^2-v^2} & \frac{v_xv_z}{c^2-v^2} \\ \frac{v_xv_y}{c^2-v^2} & 1+\frac{v_y^2}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} \\ \frac{v_xv_z}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} & 1+\frac{v_z^2}{c^2-v^2} \end{pmatrix}^{-1}.$$

Note that Γ can be written

$$\begin{aligned}
& \begin{pmatrix} 1+\frac{v_x^2}{c^2-v^2} & \frac{v_xv_y}{c^2-v^2} & \frac{v_xv_z}{c^2-v^2} \\ \frac{v_xv_y}{c^2-v^2} & 1+\frac{v_y^2}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} \\ \frac{v_xv_z}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} & 1+\frac{v_z^2}{c^2-v^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{c^2-v^2+v_x^2}{c^2-v^2} & \frac{v_xv_y}{c^2-v^2} & \frac{v_xv_z}{c^2-v^2} \\ \frac{v_xv_y}{c^2-v^2} & \frac{c^2-v^2+v_y^2}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} \\ \frac{v_xv_z}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} & \frac{c^2-v^2+v_z^2}{c^2-v^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{c^2-v_y^2-v_z^2}{c^2-v^2} & \frac{v_xv_y}{c^2-v^2} & \frac{v_xv_z}{c^2-v^2} \\ \frac{v_xv_y}{c^2-v^2} & \frac{c^2-v_x^2-v_z^2}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} \\ \frac{v_xv_z}{c^2-v^2} & \frac{v_yv_z}{c^2-v^2} & \frac{c^2-v_x^2-v_y^2}{c^2-v^2} \end{pmatrix} \\
&= \frac{1}{c^2-v^2}\begin{pmatrix} c^2-v_y^2-v_z^2 & v_xv_y & v_xv_z \\ v_xv_y & c^2-v_x^2-v_z^2 & v_yv_z \\ v_xv_z & v_yv_z & c^2-v_x^2-v_y^2 \end{pmatrix} \\
&= \frac{1}{c^2}\gamma^2\begin{pmatrix} c^2-v_y^2-v_z^2 & v_xv_y & v_xv_z \\ v_xv_y & c^2-v_x^2-v_z^2 & v_yv_z \\ v_xv_z & v_yv_z & c^2-v_x^2-v_y^2 \end{pmatrix} \\
\Rightarrow \Gamma^{-1} &= \frac{c^2}{\gamma^2}\left[\begin{pmatrix} c^2-v_y^2-v_z^2 & v_xv_y & v_xv_z \\ v_xv_y & c^2-v_x^2-v_z^2 & v_yv_z \\ v_xv_z & v_yv_z & c^2-v_x^2-v_y^2 \end{pmatrix}\right]^{-1}
\end{aligned}$$

Therefore, we have that

$$\vec{a} = \frac{c^2}{\gamma^3}\begin{pmatrix} c^2-v_y^2-v_z^2 & v_xv_y & v_xv_z \\ v_xv_y & c^2-v_x^2-v_z^2 & v_yv_z \\ v_xv_z & v_yv_z & c^2-v_x^2-v_y^2 \end{pmatrix}^{-1}\frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right).$$