For a constant magnetic field,

$$\frac{d\left(\gamma\vec{\beta}\right)}{dt} = \frac{qB}{m}\vec{\beta} \times \hat{z}.$$

The expression on the left-hand side has units of inverse time. Thus, we have that

$$\left[\frac{qB}{m}\right] = T^{-1},$$

and qB/m is referred to as the magnetic gyrofequency ω_g, or cylcotron frequency, in some contexts. Therefore,

$$\frac{d\left(\gamma\vec{\beta}\right)}{d(\omega_g t)} = \beta_y\hat{x} - \beta_x\hat{y}.$$

In terms of the differential proper time $du{=}dt/\gamma$,

$$\begin{aligned}
\frac{d\left(\gamma\vec{\beta}\right)}{du} &= [\gamma\beta_y\hat{x} - \gamma\beta_x\hat{y}]\omega_g \\
\Rightarrow \frac{d\left(\gamma\vec{\beta}\right)}{du} \cdot \gamma\vec{\beta} &= \frac{d\gamma}{du} \\
&= 0
\end{aligned}$$

which implies that the gamma factor is a constant of motion. Thus,

$$\begin{aligned}
\frac{d\left(\gamma\vec{\beta}\right)}{du} &= \gamma\frac{d\vec{\beta}}{du} \\
&= [\gamma\beta_y\hat{x} - \gamma\beta_x\hat{y}]\omega_g \\
\Rightarrow \frac{d\vec{\beta}}{du} &= \omega_g[\beta_y\hat{x} - \beta_x\hat{y}] \\
\Rightarrow \frac{d\beta_x}{du} &= \omega_g\beta_y, \\
\frac{d\beta_y}{du} &= -\omega_g\beta_x \\
\Rightarrow \frac{d^2\beta_y}{du^2} &= -\omega_g^2\beta_y \\
\Rightarrow \beta_y(u) &= \beta_c\cos(\omega_g u) + \beta_s\sin(\omega_g u), \\
\beta_x(u) &= \beta_c\sin(\omega_g u) - \beta_s\cos(\omega_g u). \qquad (1)
\end{aligned}$$

with constants fixed by boundary conditions. Specifically, we find that

$$\begin{aligned}
\int_0^t dt' &= \int_{u_0}^u \gamma du' \\
t &= \gamma\Delta u.
\end{aligned}$$

Defining $u_0 = u(t=0)$, we have that $t = \gamma(u - u_0)$. Thus, absorbing u_0 into the constants in (1), we obtain

$$\vec{\beta}(t) = \begin{pmatrix} \beta_c\sin(\omega_g t) - \beta_s\cos(\omega_g t) \\ \beta_c\cos(\omega_g t) + \beta_s\sin(\omega_g t) \\ \beta_{z0} \end{pmatrix}$$

where we defined the “relativistic gyrofrequency” $\omega_g' = \omega_g / \gamma = \frac{qB}{\gamma m}$. To specify $\vec{\beta}(0) = \vec{\beta}_0$ we can fix that

$$\begin{aligned}
\begin{pmatrix} \beta_{x0} \\ \beta_{y0} \\ \beta_{z0} \end{pmatrix} &= \begin{pmatrix} -\beta_s \\ \beta_c \\ \beta_{z0} \end{pmatrix} \\
\Rightarrow \vec{\beta}(t) &= \begin{pmatrix} \beta_{y0}\sin(\omega_g t) + \beta_{x0}\cos(\omega_g t) \\ \beta_{y0}\cos(\omega_g t) - \beta_{x0}\sin(\omega_g t) \\ \beta_{z0} \end{pmatrix}
\end{aligned}$$

From this, one can see that $\beta^2(t) = |\vec{\beta}_0|^2$, so that γ is a constant of motion in a particular frame. For the sake of completeness, Thus, the full motion is

$$\vec{x}(t) = \vec{x}_0 + \frac{1}{\omega_g} \begin{pmatrix} \beta_{y_0}(1 - \cos(\omega_g t)) + \beta_{x0}\sin(\omega_g t) \\ \beta_{y0}\sin(\omega_g t) + \beta_{x0}(\cos(\omega_g t) - 1) \\ \beta_{z0}\omega_g t \end{pmatrix}.$$