(i) Find the derivation of the following formula:

$f(z) = \log_e(1+z)$, where $z = x^T x$, $x \in \mathbb{R}^d$

Solⁿ:

Given, $f(z) = \log_e(1+z)$

By chain rule:

$$\frac{d}{dx} f(z) = \frac{d}{dz} f(z) \cdot \frac{dz}{dx}$$

$$\Rightarrow \frac{d}{dx} f(z) = \frac{d}{dz} \log_e(1+z) \cdot \frac{d}{dx}(x^T x)$$

$$= \frac{1}{1+z} \cdot 2 \cdot x^T$$

$$\boxed{\frac{dz}{dx} = \frac{d}{dx} x^T x = 2x^T \text{ by metrics derivative rule}}$$

$$\therefore \frac{d}{dx} f(z) = \frac{2x^T}{1+x^T x}$$

(ii) Find the derivation of the following formula:

$$f(z) = e^{-z/2}, \text{ where } z = g(y),$$
$$g(y) = y^T S^{-1} y$$
$$y = h(x), \quad h(x) = x - u$$

Sol^n:

Given composite functions $f(z), g(y), h(x)$

So, According to chain rule:

$$\frac{d}{dx} f(z) = \frac{d\,f(z)}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Now,

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left(e^{-z/2}\right) = -\frac{1}{2} e^{-z/2}$$

$$\frac{dz}{dy} = \frac{d}{dy}\left(y^T S^{-1} y\right) = \frac{d}{dy}\left(y^T \cdot y \cdot S^{-1}\right)$$

$$= S^{-1} \frac{d}{dy}\left(y^T y\right)$$

$$= 2 y^T \cdot S^{-1}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x - \mu) \qquad \Big| \ \mu = \text{constant}$$

$$= 1$$

Now, after combining all

$$\frac{d}{dx} f(z) = \frac{d}{dz} f(z) \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

$$= \left\{ -\frac{1}{2} e^{-\frac{z}{2}} \right\} \cdot \left\{ 2y^T S^{-1} \right\} \cdot 1$$

$$\frac{d}{dx} f(z) = -e^{-\frac{z}{2}} \cdot y^T S^{-1}$$

$$\therefore \frac{d}{dx} f(z) = e^{-\frac{(y^T S^{-1} y)}{2}} \cdot (x - \mu)^T \cdot S^{-1}$$