oblig 1 16/09/19

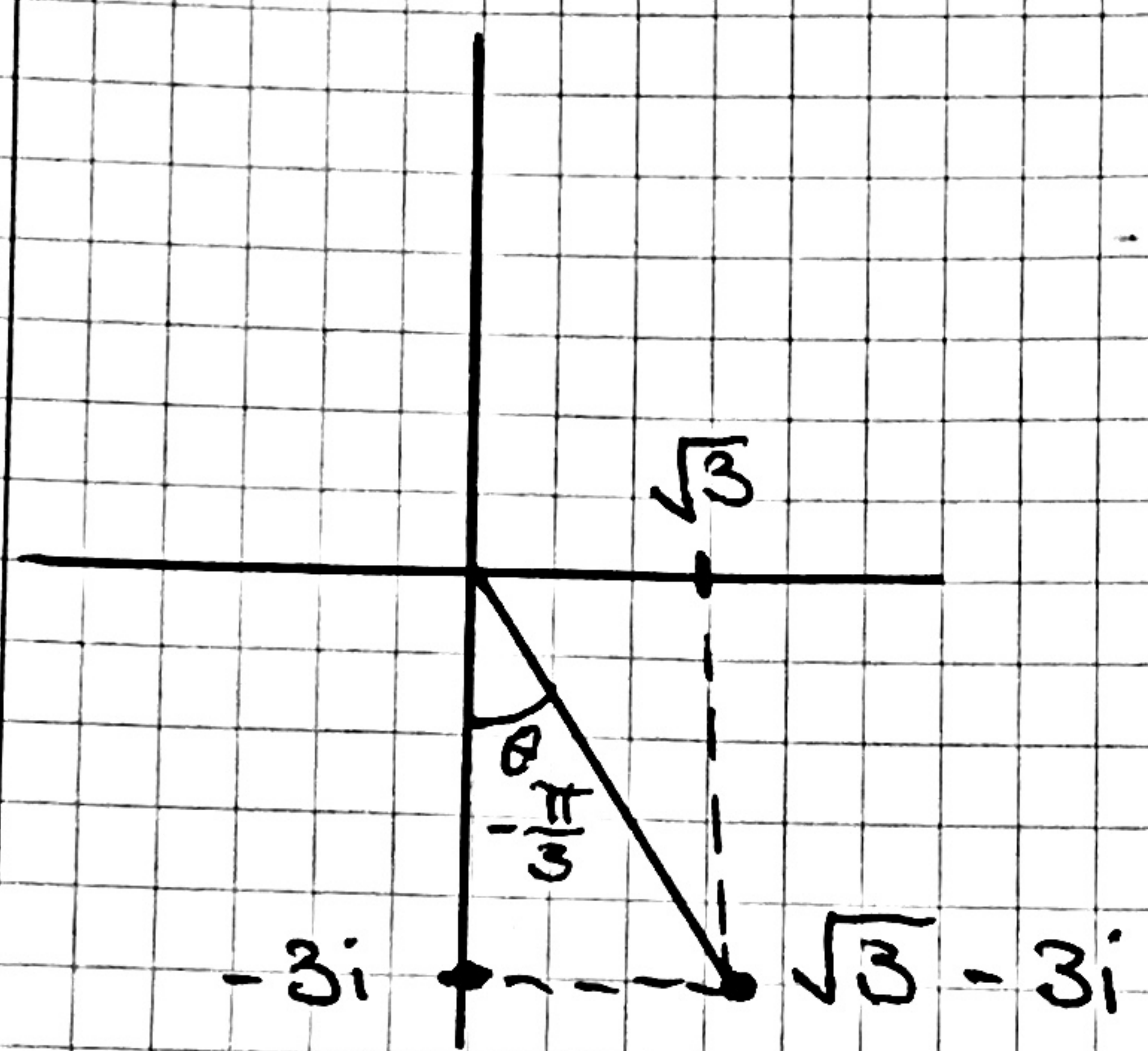
Noora Lindeflaten

oppgåve 1

a) $w = \sqrt{3} - 3i$

$|w| = \sqrt{(\sqrt{3})^2 + (-3)^2}$

$= \underline{2\sqrt{3}}$



$w = r\left(\cos\theta + i\sin\theta\right)$

$\cos\theta = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2}$

$\theta = -\dfrac{\pi}{3}$

polarform

$$\underline{\underline{w = 2\sqrt{3}\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)}}$$

$$w^7 = (2\sqrt{3})^7 \left(\cos -\frac{\pi}{3} + i \sin -\frac{\pi}{3}\right)^7$$

$$w^7 = (2\sqrt{3})^7 \left(\cos 7 \cdot -\frac{\pi}{3} + i \sin 7 \cdot -\frac{\pi}{3}\right)$$

$$w^7 = (2\sqrt{3})^7 \left(\cos -\frac{7\pi}{3} + i \sin -\frac{7\pi}{3}\right)$$

$$w^7 = (2\sqrt{3})^7 \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right)$$

$$w^7 = 2^7 \cdot 3^3 \cdot \sqrt{3} \left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$$

$$\underline{\underline{w^7 = 2993 - 5184i}}$$

b)

i) $(3+5i) \cdot \overline{(2+3i)}$

$= (3+5i)(2-3i)$

$= 6 + 10i - 15i^2 - 9i$

$= 6 + i - 15i^2$

$= 6 + i + 15$

$\underline{\underline{= 21 + i}}$

ii) $\dfrac{25i}{3+4i} - 3i = \dfrac{25i}{(3+4i)} - \dfrac{3i(3+4i)}{(3+4i)}$

$= \dfrac{16i + 12}{(3+4i)} = \dfrac{4\cancel{(3+4i)}}{\cancel{(3+4i)}} = \underline{\underline{4}}$

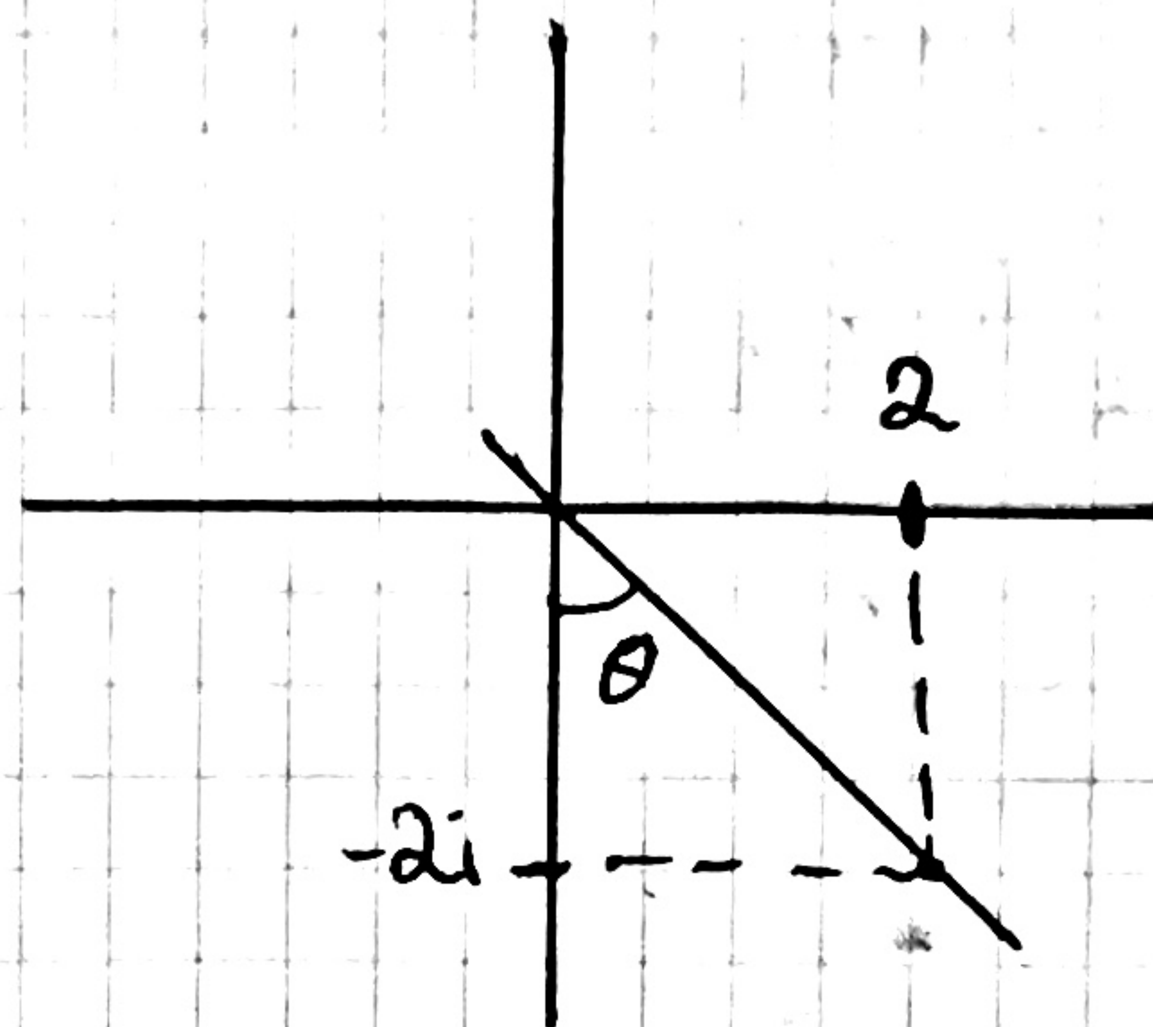
c) $z^7 - 2 + 2i = 0$

$z^7 = 2 - 2i$

Vi skal finne alle røtene til z, alle komplekse løysinger.

$r^2 = 2^2 + (-2)^2$

$r = \underline{2\sqrt{2}}$



$\theta = \tan \frac{b}{a} = -\frac{2}{2} = -1$

$\theta = -\frac{\pi}{4}$

- Jeg har kommet frem til en generell løysing slik:

$$z_k = r e^{i \cdot \frac{\pi}{4} + 2\pi \cdot k}$$

$$z_k = (2\sqrt{2})^{1/7} e^{\frac{k \cdot 2\pi - \frac{\pi}{4}}{7} \cdot i}$$

$$= 2^{\frac{1}{7}} \cdot 2^{\frac{1}{14}} \cdot e^{\frac{k \cdot 8\pi - \pi}{28} \cdot i}$$

$$= \underline{\sqrt[14]{2^3} \cdot e^{\frac{k 8\pi - \pi}{28} i}}$$

$k = 0, 1, 2, 3, 4, 5, 6$

$$z_0 = \underline{\underline{\sqrt[14]{2^3}\ e^{-\frac{\pi}{28} i}}}$$

$$z_1 = \sqrt[14]{2^3} \cdot e^{\frac{\pi}{4} \cdot i}$$

$$z_2 = \sqrt[14]{2^3} \cdot e^{i\frac{15\pi}{28}}$$

$$z_3 = \sqrt[14]{2^3}\, e^{i\frac{23\pi}{28}}$$

$$z_4 = \sqrt[14]{2^3}\, e^{i\frac{31\pi}{28}}$$

$$z_5 = \sqrt[14]{2^3}\, e^{i\frac{39\pi}{28}}$$

$$z_6 = \sqrt[14]{2^3}\, e^{i\frac{47\pi}{28}}$$

d) $P(z) = z^3 - 3z^2 + z - 3$

• sett inn $z=3$

$P(3) = 3^3 - 3 \cdot 3^2 + 3 - 3 = 0$

Vi ser at $x=3$ er eit nullpunkt og $(x-3)$ er derfor ein faktor i uttrykket, det gjer at vi kan bruke polynomdivisjon for å faktorisere hele uttrykket

$$
\begin{array}{l}
z^3 - 3z^2 + z - 3 : (z-3) = \underline{z^2 + 1} \\
\underline{-(z^3 - 3z^2)} \\
\quad\quad 0 + z - 3 \\
\quad\quad\ \ \underline{-(z-3)} \\
\quad\quad\quad\quad 0
\end{array}
$$

Ved hjelp av abc-formelen og kunnskap om komplekse tal ser vi at

faktorisering over $\underline{\underline{\mathbb{C} = (z-3)(z-i)(z+i)}}$

faktorisering over $\underline{\underline{\mathbb{R}. = (z-3)(z^2+1)}}$

oppgåve 2

a) $f(x) = \ln x$

- skal ved induksjon vise at den niendederiverte til f i $x=1$ er $f^9 = 40320$ og $f^{15} = 87178291200$

- rekner nokre verdiar for å sjå eit mønster

$$f^{(1)} = \frac{1}{x} \qquad f^{(2)} = -\frac{1}{x^2}$$

$$f^{(3)} = \frac{2}{x^3} \qquad f^{(4)} = -\frac{6}{x^4}$$

- får formelen $f^{(n)}(x) = -\dfrac{(n-1)!}{(-x)^n}$

- Sjekker om det stemmer for 1

$$f^{(1)}(x) = -\frac{(1-1)!}{(-x)^1} = \frac{1}{x}$$ Ja!

- Sjekker om det stemmer for k og $k+1$

$$f^{k}(x) = -\frac{(k-1)!}{(-x)^{k}} \Rightarrow f^{k+1}(x) = -\frac{((k+1)-1)!}{(-x)^{k+1}}$$

- Deriverer k

$$f^{k'}(x) = -\frac{(k-1)!}{k(-x)^{k+1}}$$

- mellomregning

$$k!\,(k-1) = (k-1)!$$

$$\frac{(k-1)!}{k!} = (k-1)$$

$$k! = \frac{(k-1)}{(k-1)}$$

$$\underline{\underline{f^{k'}(x) = -\frac{k!}{(-x)^{k+1}}}}$$

$$f^{9}(1) = -\frac{(9-1)!}{(-1)^{9}} = \underline{40\ 320}$$

$$f^{15}(1) = -\frac{(15-1)!}{(-1)^{15}} = \underline{87\ 178291200}$$

Q.E.D

b) $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

$n \in \mathbb{N}$

- Vise at det stemmer for 1

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\Rightarrow \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2} \quad \text{ok!}$$

- går utifrå at dersom k stemmer, stemmer k+1

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$$

$$\frac{(k+2)}{(k+2)}\;\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

• finn felles nemnar på venstre side

$$\Rightarrow \frac{k^2+2k}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

$$\Rightarrow \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

1. kvadratsetning

$$\frac{(k+1)\cancel{(k+1)}}{(k+2)\cancel{(k+1)}} = \frac{k+1}{k+2}$$

$$\underline{\underline{\frac{k+1}{k+2} = \frac{k+1}{k+2}}}$$

Q.E.D

Oppgave 8

a) $$\lim_{x \to 2} \frac{x^2 - x - 2}{\sqrt{x^2 - 3x + 2}}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}$$

$$= \lim_{x \to 2} \frac{\sqrt{(x-2)}\,(x+1)}{\sqrt{(x-1)}}$$

$$= \frac{\sqrt{(2-2)} \cdot (2+1)}{\sqrt{(2-1)}} = \frac{0}{1} = \underline{\underline{0}}$$

b) $$\lim_{x \to -1} \frac{x^2 - x - 2}{\sqrt{x^2 - 3x + 2}} = \frac{(-1)^2 - (-1) - 2}{\sqrt{(-1)^2 - 3(-1) + 2}} = \frac{0}{\sqrt{6}}$$

$$= \underline{\underline{0}}$$

c) $$\lim_{x \to 2} \cos\left(\frac{x^2 - x - 2}{\sqrt{x^2 + 3x + 2}}\right)$$

$$= \lim_{x \to 2} \cos\left(\frac{(x-2)(x+1)}{(x-2)^{\frac{1}{2}}(x-1)^{\frac{1}{2}}}\right)$$

$$= \cos\left(\frac{0}{1}\right) = \underline{\underline{1}}$$

d) $\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2+3}}$

$= \lim_{x\to\infty} \frac{2x+3}{(x^2+3)^{\frac{1}{2}}} = \lim_{x\to\infty} \frac{2x+3}{x+3^{\frac{1}{2}}}$

· Deler på høgaste grad av x

$= \lim_{x\to\infty} \frac{\frac{2x}{x} + \cancel{\frac{3}{x}}}{\frac{x}{x} + \cancel{\frac{\sqrt{3}}{x}}} = \frac{2}{1} = \underline{\underline{2}}$

går mot 0

e) $\lim_{x\to 1} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)$

bruk skviseteoremet

$\left|\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)\right| \leq 1$ for alle $x \neq 0$

og dermed er

$\left|(x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right)\right| \leq |x|$ for alle $x \neq 0$

som gjør at

$-|x| \leq (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) \leq |x|$ for alle $x \neq 0$

siden $\lim_{x\to 1} |(x-1)| = 0$ gjer det at

$\lim_{x\to 1} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) = \underline{\underline{0}}$

funksjonen er kontinuerlig i 0, fordi den er definert.

oppg. 4 a

$$\lim_{x \to 1} (x^2 + x + 1) = 3$$

gitt $\varepsilon > 0$ ønsker å finne

en $\delta > 0$ slik at

$$0 < |x-1| < \delta \Rightarrow 0 < |(x^2 + x + 1) - 3| < \varepsilon$$

$$0 < |(x^2 + x + 1) - 3| < \varepsilon$$

$$\Downarrow$$

$$|x+2| \cdot |x-1|$$

antar at $\delta \leq 1$

$$|x-1| \leq 1$$

$$\Downarrow$$

$$0 < x < 2 \quad | +2$$

$$\Downarrow$$

$$2 < x+2 < 4$$

$$\Downarrow$$

$$|x+2| < 2$$

$|x+2| \cdot |x-1| < 4|x-1|$ viss

$|x-1| < \delta \leq 1$ men $4|x-1| < \varepsilon$ viss

$|x-1| < \varepsilon/4$

derfor viss vi tar

$\delta = \min \{1, \varepsilon/4\}$

så er

$|f(x) - 3| < 4\,|x-1| < 4 \cdot \varepsilon/4 = \varepsilon$

viss $|x-1| < \delta$

som beviser at

$\lim_{x \to 1} f(x) = 3$

4b $f(a) = g(a) = 0 \qquad g'(a) \neq 0$

begrunn at $\dfrac{f'(a)}{g'(a)} = \lim_{x \to a} \dfrac{f(x)}{g(x)}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$\frac{f'(a)}{g'(a)} = \lim_{h \to 0} \frac{\frac{f(a+h)}{h}}{\frac{g(a+h)}{h}}$$

$$\frac{f'(a)}{g'(a)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} \Rightarrow \lim_{x \to a} \frac{f(a)}{g(a)}$$

c) bruker konklusjonen i b og får

$$\lim_{x \to 0} \frac{x}{e^x - 1} = \lim_{x \to 0} \frac{1}{e^x} = \frac{1}{e^0} = \underline{\underline{1}}$$

oppgåve 5

a) $f(x) = 2x^3 + x^2 - 4x - 2 + \cos x$

$f'(x) = 6x^2 + 2x - 4 - \sin x$

$f'(0) = -4$

$y - (-1) = -4(x - 0)$

$y = -4x - 1$

$$\text{normalen} = \frac{-1}{\text{stig. talet til tangenten}}$$

$$= \underline{\underline{\frac{-1}{-4x} = \boxed{\frac{1}{4}x - 1}}}$$ ← ligningen til normalen til grafen i punktet $(0, -1)$

b) $f(x) = 2x^3 + x^2 - 4x - 2 + \cos x$

La f være en funksjon som er deriverbar i a så er f kontinuerlig i a

$$0 = \left(\lim_{x \to a} f(x)\right) - f(a) = \lim_{x \to a} \left(f(x) - f(a)\right)$$

$$= \lim_{x \to a} \underbrace{(x-a)}_{\downarrow\, 0} \cdot \underbrace{\frac{f(x)-f(a)}{x-a}}_{f'(a)}$$

$$= \lim_{x \to a} (x-a) \cdot \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = 0 \cdot f'(a) = \underline{\underline{0}}$$

Q. E. D

c) $f(x) = 0$ har minst 3 løysingar

$$f(-2) = 2(-2)^3 + (-2)^2 - 4(-2) + \cos(-2)$$

$$= -16 + 4 + 8 + \cos(-2)$$

$$= -6 + \cos(-2)$$

$$f(-2) \in [-7, -5]$$

$$f(-1) = 2(-1)^3 + (-1)^2 - 4(-1)^{-2} + \cos(-1)$$

$$= -2 + 1 + 4 + \pi$$

$$= \underline{\underline{1 + \pi}}$$

$$f(1) = 2(1)^3 + (1)^2 - 4(1) - 2 + \cos(1)$$

$$= 2 + 1 - 4 - 2 + \cos 1$$

$$= \underline{\underline{-3}}$$

oppgåve 6

a)
$$f(x) = \begin{cases} (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) & \text{når } x \neq 1 \\ 0 & \text{når } x = 1 \end{cases}$$

avgjøre om f er kontinuerlig i $x = 1$

$$0 = f(x) = \lim_{x \to 1} (x-1)\left(1 + \sin\left(\frac{1}{x^2-1}\right)\right)$$

$$-1 \leq \sin\left(\frac{1}{x^2-1}\right) \leq 1$$

$$0 \leq 1 + \sin\left(\frac{1}{x^2-1}\right) \leq 2$$

$$0 \leq (x-1)\left(1+\sin\left(\frac{1}{x^2-1}\right)\right) \leq 2(x-1)$$

$$\lim_{x \to 1} 2(x-1) = 0$$

f er kontinuerlig for $x=1$

siden $\lim_{x \to 1} f(x) = f(1) = 0$.

b)

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \left((1+h) - 1 \right) \left(1 + \sin \left(\frac{1}{(1+h)^2 - 1} \right) \right) - 0$$

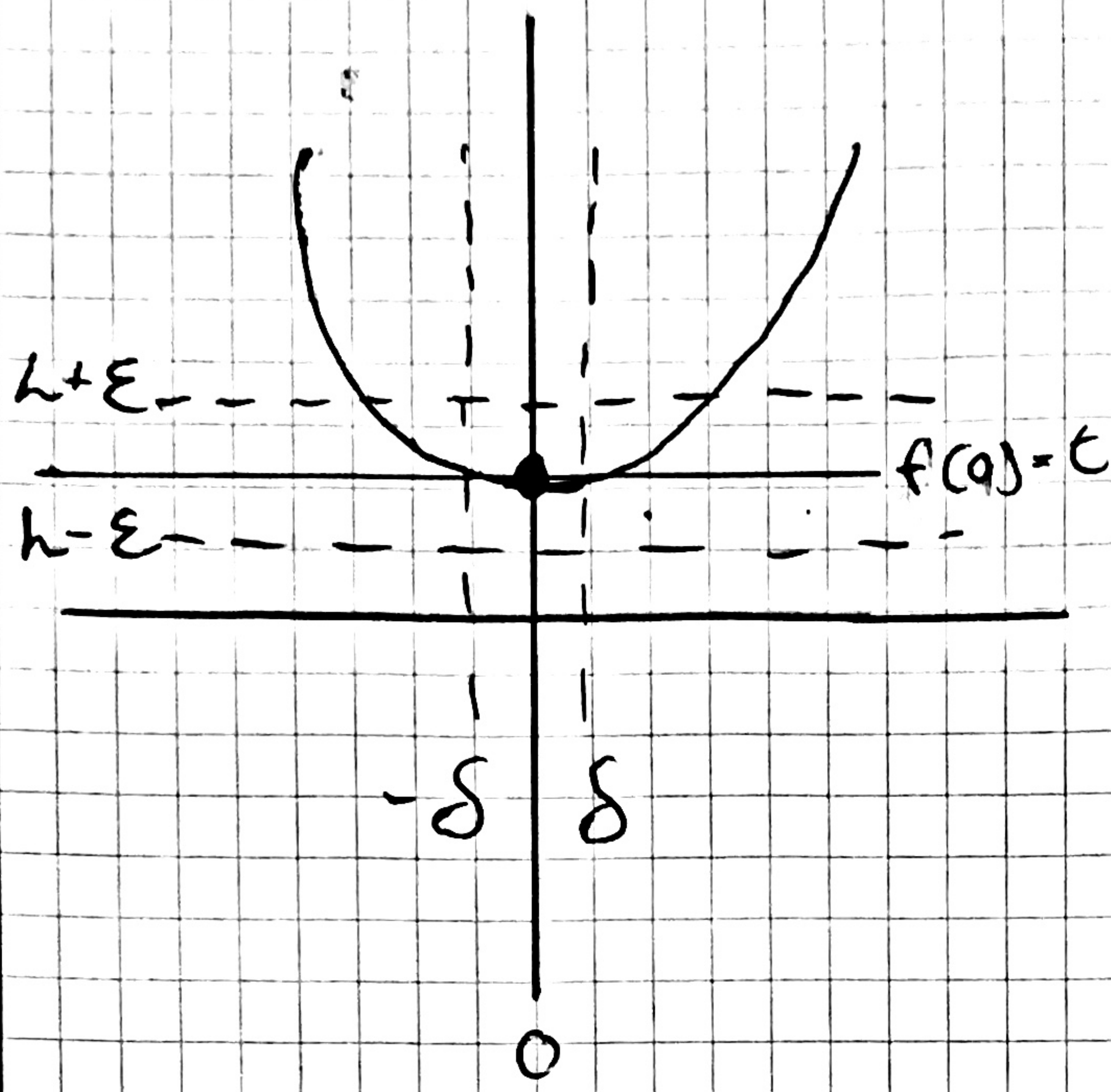
$$= \lim_{h \to 0} \frac{h \left(1 + \sin \left(\frac{1}{2h + h^2} \right) \right)}{h}$$

$$= \lim_{h \to 0} \underbrace{1}_{1} + \sin \left(\frac{1}{2h + h^2} \right)$$

denne vil hoppe mellom 1 og -1

funksjonen er derfor ikkje deriverbar i $x=1$

oppgåve 7



Definisjonen av grenseverdi

$|f(x)-L| < \varepsilon$ $\quad \varepsilon > 0$

for alle x slik at

$0 < |x-a| < \delta$ $\quad \delta > 0$

når $x \to a$

Definisjonen av kontinuitet

$f(a) = \lim\limits_{x \to a} f(x)$

Når $f(x) \geq 0$ for alle $x \neq 0$

og $f(x)$ er kontinuerlig i

$x = 0$ da må $f(x) = 0 \geq 0$