5.5 Newton Raphson Method

Consider the GDP example, assume that the data follows $Gamma(\alpha, \beta)$. What is the MLE of α and β?

From the above derivation, we need to solve the following equations

$$\begin{cases} \frac{1}{n}\sum \log X_i - \psi(\alpha) - \log\beta = 0 \\ \frac{1}{n}\sum x_i - \alpha\beta = 0, \end{cases}$$

How to solve this equations?

The above equations have two parameters. We firstly consider a simpler setting. Solve the equation

$$f(x) - a = 0.$$

1. Choose an initial value x_0;
2. Assume x_k, update x_{k+1} as $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$;
3. Calculate $\Delta = |x_{k+1} - x_k|$

4. If $\Delta > \delta$, go to step 2; otherwise, stop the iteration and use x_{k+1} as the solution of the equation.

Example 5.5.1 (HW revisited)**.**

What if we have multiple variables and multiple functions:

$$\begin{cases} f_1(x_1, \cdots, x_n) = 0, \\ f_2(x_1, \cdots, x_n) = 0, \\ \cdots \\ f_n(x_1, \cdots, x_n) = 0? \end{cases}$$

Let $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \cdots, f_n(\boldsymbol{x}))^T$. Then the update for each step is

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - J(\boldsymbol{x}^i)^{-1}\boldsymbol{f}(\boldsymbol{x}^i).$$

where $J(\boldsymbol{x}^i)$ is the Jacobian matrix given as

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

Example 5.5.2 (MLE of Gamma model)**.**