

# STAT 8003, Homework 7



Group # 1

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Due at 5:30pm on class on Thu., Oct. 31. Please submit one and only one pdf file for your group via blackboard. Each sub-problem is 10 points (Total 80 points).

**Problem 1.** Suppose that  $X_1, \dots, X_{25}$  form a random sample from a normal distribution having a variance of 80. Graph the power of the likelihood ratio test of  $H_0 : \mu = \mu_0$  versus  $H_A : \mu = \mu_1$  as a function of  $\mu_1$ , at significance levels 0.10 and 0.05. Do the same for a sample size of 100. Compare the graphs and explain what you see.

First, calculate the likelihood ratio as follows:

$$\begin{aligned} LR &= \frac{L_n(\mu_1)}{L_n(\mu_0)} \\ &= \frac{\prod_{i=1}^{25} \frac{1}{\sqrt{2\pi 80}} \exp\left(-\frac{(x_i - \mu_1)^2}{80}\right)}{\prod_{i=1}^{25} \frac{1}{\sqrt{2\pi 80}} \exp\left(-\frac{(x_i - \mu_0)^2}{80}\right)} \\ &= \exp\left(\sum_{i=1}^{25} \frac{(x_i - \mu_0)^2}{80} - \sum_{i=1}^{25} \frac{(x_i - \mu_1)^2}{80}\right) \\ &= \exp\left(\frac{50\bar{X}_{25}(\mu_1 - \mu_0) + 25(\mu_0^2 - \mu_1^2)}{80}\right) \end{aligned}$$

When  $\mu_1 > \mu_0$ , the likelihood ratio will be large when  $\bar{X}_{25}$  is large, so reject  $H_0$  for some  $c$  s.t.

$$P(\bar{X}_{25} > c | \bar{X}_{25} \stackrel{H_0}{\sim} N(\mu_0, 80/25)) = \alpha$$

with corresponding power:

$$P(\bar{X}_{25} > c | \bar{X}_{25} \stackrel{H_1}{\sim} N(\mu_1, 80/25))$$

When  $\mu_0 > \mu_1$ , the likelihood ratio will be large when  $\bar{X}_{25}$  is small, so reject  $H_0$  for some  $c$  s.t.

$$P(\bar{X}_{25} < c | \bar{X}_{25} \stackrel{H_0}{\sim} N(\mu_0, 80/25)) = \alpha$$

with corresponding power:

$$P(\bar{X}_{25} < c | \bar{X}_{25} \stackrel{H_1}{\sim} N(\mu_1, 80/25))$$

Since the only difference between the null and alternative hypotheses is the location parameter, we can represent the power as a function of the difference between  $\mu_0$  and  $\mu_1$  ( $\delta = \mu_1 - \mu_0$ ):

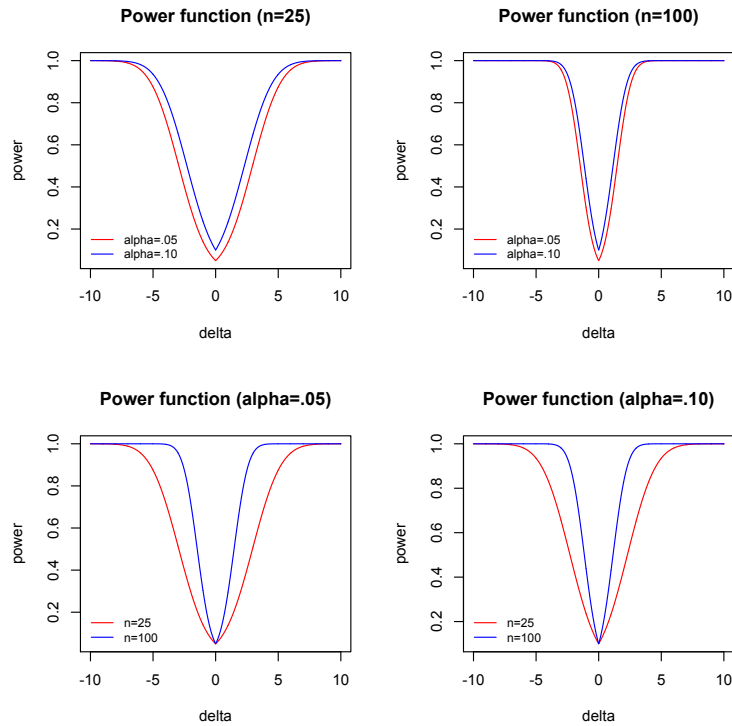
$$\delta > 0, 1 - \beta = 1 - F(z_{1-\alpha} * \sqrt{80/n} - \delta)$$

$$\delta < 0, 1 - \beta = F(z_{\alpha} * \sqrt{80/n} - \delta)$$

$$\delta = 0, 1 - \beta = \alpha$$

Where  $F$  is the cdf function for a  $N(0, 80/n)$  random variable.

Here are the resulting graphs for  $\alpha = 0.05$  and  $0.10$  and  $n = 25$  and  $100$ :



Power function for alpha=0.05 and 0.10 and n=25 and 100

From these graphs you can see two important trends. As  $n$  increases, the power increases for the same difference between  $\mu_1$  and  $\mu_0$ . This makes sense because the variance of the distribution of the test statistic ( $\text{Var}(\bar{X}_n) = 80/n$ ) decreases as  $n$  increases. This means we can more confidently evaluate tests where the hypothesized means are closer together when we have larger sample sizes.

Second, the power of a test with a higher  $\alpha$  is higher holding  $n$  constant. This also is intuitive since there is an inherent tradeoff between Type I and Type II errors for such tests. If you allow the possibility of Type I error to be larger, the rejection region becomes larger thus increasing the power and decreasing the likelihood of Type II error.

**Problem 2.** Let  $X_1, \dots, X_n$  be a random sample from an exponential distribution with the density function  $f(x|\theta) = \theta \exp(-\theta x)$ . Set the desired Type I error rate at 5 %.

a). Derive a generalized likelihood ratio test and show that the rejection region is of the form  $\bar{X} \exp(-\theta_0 \bar{X}) \leq c$

Assume the test is as follows:  $H_0 : \theta = \theta_0$   $H_1 : \theta \neq \theta_0$ . We will calculate  $\Lambda$  as follows and reject  $H_0$  when  $\Lambda$  large.

$$\Lambda = \frac{\max_{\theta > 0} L(\theta)}{\max_{\theta = \theta_0} L(\theta)}$$

The MLE estimator of  $\theta$  will maximize the numerator and we must plug in  $\theta_0$  in the likelihood function on the denominator. So we get:

$$\Lambda = \frac{L(\hat{\theta}_{mle})}{L(\theta_0)}$$

So first we need to find  $\hat{\theta}_{mle}$ :

$$\begin{aligned}
 L_n(\theta) &= \prod_{i=1}^n \theta e^{-\theta x_i} \\
 l_n(\theta) &= n \log \theta - \theta \sum_{i=1}^n x_i \\
 l'_n(\theta) &= \frac{n}{\theta} - \sum_{i=1}^n x_i \\
 &\rightarrow \hat{\theta} = \frac{1}{\bar{X}} \\
 l''_n(\theta) &= -\frac{n}{\theta^2} < 0 \\
 &\rightarrow \hat{\theta} = \frac{1}{\bar{X}} = \hat{\theta}_{mle}
 \end{aligned}$$

The we now calculate  $\Lambda$  as:

$$\begin{aligned}
 \Lambda &= \frac{\prod_{i=1}^n \bar{X}^{-1} \exp(-x_i/\bar{X})}{\prod_{i=1}^n \theta_0 \exp(-\theta_0 x_i)} \\
 &= \left( \frac{1}{\bar{X} \theta_0} \right)^n \exp \left( \theta_0 \sum_{i=1}^n x_i - n \right) \\
 &= \left[ \left( \frac{1}{\theta_0 \bar{X}} \right) \exp(\theta_0 \bar{X} - 1) \right]^n
 \end{aligned}$$

And we will reject  $H_0$  when  $\Lambda \geq c_1$  for a specified  $c_1$ . Rearranging the inequality we get the desired result:

$$\begin{aligned}
 \left[ \left( \frac{1}{\theta_0 \bar{X}} \right) \exp(\theta_0 \bar{X} - 1) \right]^n &\geq c_1 \\
 (\bar{X}) \exp(1 - \theta_0 \bar{X}) &\leq \frac{1}{\theta_0 (c_1)^{1/n}} \\
 (\bar{X}) \exp(\theta_0 \bar{X}) &\leq \frac{1}{e \theta_0 (c_1)^{1/n}}
 \end{aligned}$$

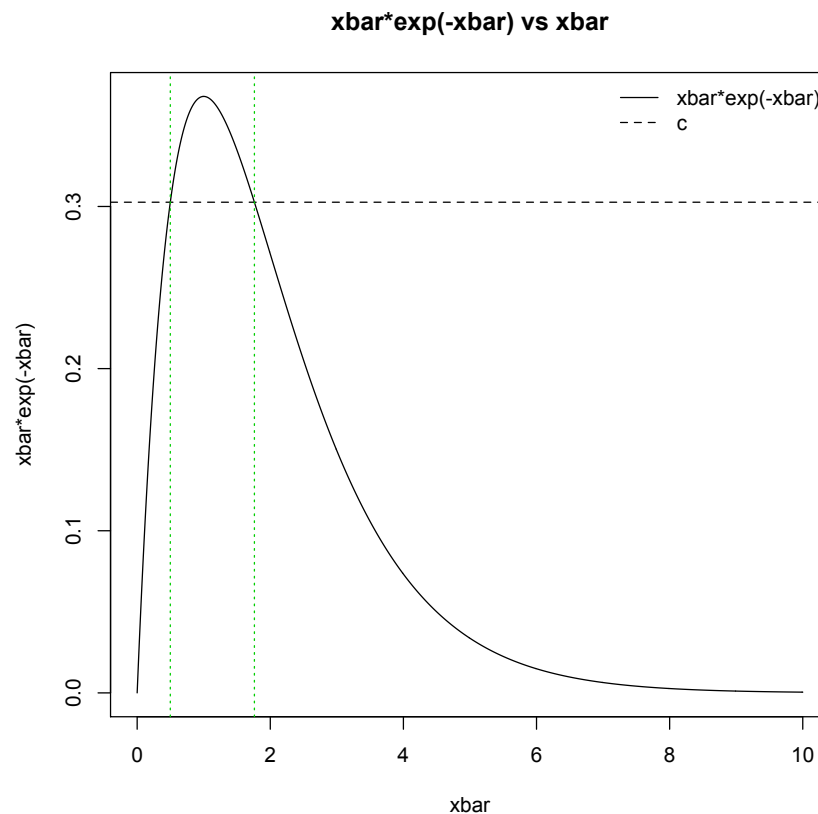
Where  $c_1$  chosen s.t.  $P(\Lambda \geq c_1 | H_0) = \alpha = 0.05$ . So our rejection region is of the form  $\bar{X} \exp(-\theta_0 \bar{X}) \leq c$  where  $c = 1/[e \theta_0 (c_1)^{1/n}]$ .

b). Suppose  $\theta_0 = 1, n = 10$ . Show that the rejection region in a). is of the form  $\{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$ , where  $x_0$  and  $x_1$  are determined by  $c$ .

Since  $\theta_0 = 1, n = 10$ , our rejection region is based on

$$\begin{aligned} f(\bar{X}) &= \bar{X}e^{-\bar{X}} \\ f'(\bar{X}) &= (1 - \bar{X})e^{-\bar{X}} \\ f''(\bar{X}) &= \bar{X}e^{-\bar{X}} - 2e^{-\bar{X}} < 0 \end{aligned}$$

Setting  $f'(\bar{X}) = 0$  gives us the maximum of this function at  $\bar{X} = 1$  since the second derivative is negative. We note that the likelihood ratio function  $f(\bar{X})$  increasing on  $(0, 1)$  and decreasing on  $(1, \infty)$ . Because of the nature of this function, there are two ranges of  $\bar{X}$  s.t.  $f(\bar{X}) < c$  and can be best illustrated graphically



The range of  $\bar{X}$  to the left and right of the green dashed lines will ensure  $f(\bar{X}) < c$ . So the rejection region is of the form  $\{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$ , where  $x_0$  and  $x_1$  are determined by the left and right green lines respectively in the graph above.

c). Explain why  $\sum_{i=1}^n X_i$  and hence  $\bar{X}$  follow gamma distributions when  $\theta_0 = 1$ . How could this knowledge be used to choose  $c$ ?

We know  $X_1, X_2, \dots, X_n$  iid  $\exp(1)$  under  $H_0$ . This is equivalent to  $\text{gamma}(1,1)$ . To find the distributions of  $\sum_{i=1}^n X_i$  and  $\bar{X}$  we can look at their moment generating functions.

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_0^\infty e^{tx} e^{-x} dx \\ &= \frac{1}{t-1} e^{x(t-1)} \Big|_{x=0}^\infty \quad t < 1 \\ &= \frac{1}{1-t} \quad t < 1 \end{aligned}$$

Since  $X_i$  are i.i.d.,

$$\begin{aligned} M_{\sum X_i}(t) &= E[e^{t \sum X_i}] = (E[e^{tx}])^n \\ &= \left( \frac{1}{1-t} \right)^n \quad t < 1 \end{aligned}$$

Which is the same as the mgf for a  $\text{gamma}(n,1)$  distribution.

$$\begin{aligned} M_{\bar{X}}(t) &= E[e^{(t/n) \sum X_i}] = (E[e^{(t/n)x}])^n \\ &= \left( \frac{1}{1-(t/n)} \right)^n \quad t < n \end{aligned}$$

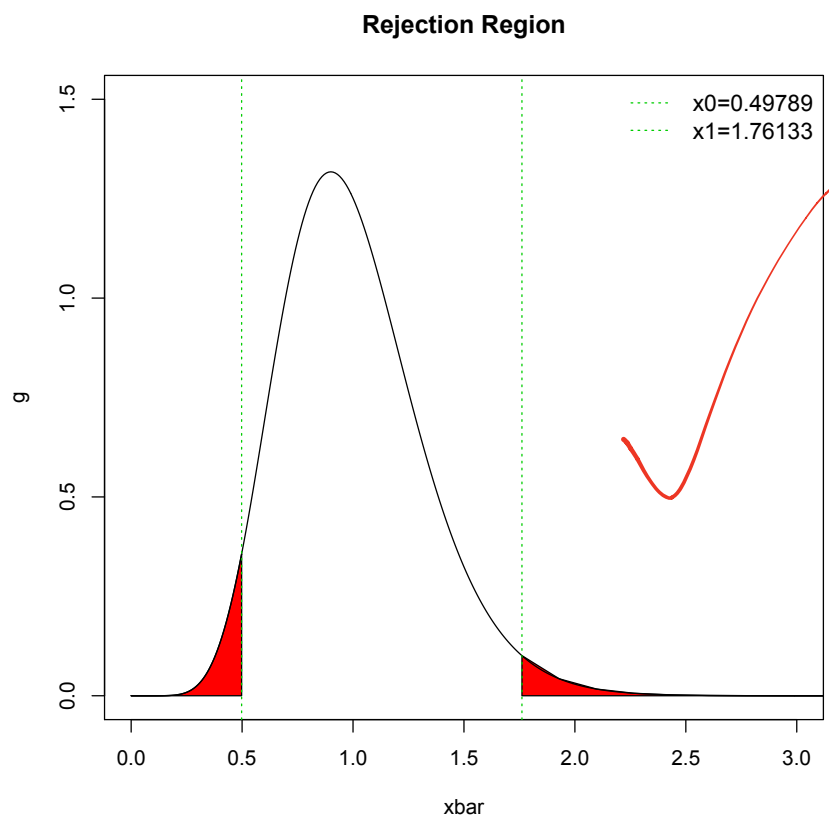
Which is the same as the mgf for a  $\text{gamma}(n,1/n)$  distribution.

In summary, under  $H_0$ ,  $X_i \sim \exp(1)$ ,  $\bar{X} \sim \text{gamma}(n,1/n)$ . This is extremely useful in choosing  $c$  since we've already determined that the rejection region is determined by  $\bar{X}$  and we know the exact distribution of  $\bar{X}$  under the null hypothesis. To find  $c$ , we need to satisfy two conditions: 1)  $P(\{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\} | H_0) = \alpha$  and 2)  $x_0 e^{-x_0} = x_1 e^{-x_1} = c$ . Since we know the exact distribution of  $\bar{X}$  we can identify  $x_0, x_1$ , and  $c$  to satisfy these conditions.

Start with  $x_0$  and  $x_1$  such that they each have equal probability  $\alpha/2$  in the tails. Since gamma is not symmetric, it will not satisfy the second condition, so we can move both points up or down using trial and error until it satisfies the second condition (see R code in appendix for details).

Don't need to list so many digits, round your solution to 0.01

We get  $x_0 = 0.4978931$ ,  $x_1 = 1.7613349$ ,  $c = 0.3026244$  as our final answer also illustrated graphically below:



**Problem 3.** Suppose, to be specific, that in Problem 2, the observed data are the following:

1.07	0.88	0.66	0.55	1.15	0.65	3.45	3.55	3.51	0.48
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a). Based the result in Problem 2, will you reject  $H_0$ , What's your  $p$ -value?

$\bar{X}$  calculated from the data above is 1.595. In part c of problem 2, we found  $x_0 = 0.4978931$ ,  $x_1 = 1.7613349$  for  $\alpha = 0.05$ . Since 1.595 is between  $x_0$  and  $x_1$ . We fail to reject  $H_0$  for this data and  $\alpha=0.05$ . In order to calculate the  $p$ -value, we must consider extreme values on both sides of the distribution since  $H_1 : \theta \neq \theta_0$ . We can get  $P(\bar{X} > 1.595)$  from R:

```
1-pgamma(1.595,shape=10,scale=0.1)
[1] 0.04437559
```

However we cannot simply multiply this probability by 2 to get the  $p$ -value because

the distribution of  $\bar{X}$  is not symmetric. However, we can find the other value of  $\bar{X}$  s.t.  $1.595e^{-1.595} = \bar{X}_2 e^{-\bar{X}_2}$  since we must satisfy that constraint as in the previous determination of the rejection range of  $\bar{X}$ . We can also calculate in R by trial and error:

```
1.595*exp(-1.595)
[1] 0.3236391
qgamma(.04437559,shape=10,scale=0.1)
[1] 0.5307017 #not same as above, keep going...
qgamma(.0679997,shape=10,scale=0.1)*
exp(-qgamma(.0679997,shape=10,scale=0.1))
[1] 0.3236391
qgamma(.0679997,shape=10,scale=0.1)
[1] 0.5753507
```

So the other value of  $\bar{X}$  is 0.5753507 and the CDF at this point is 0.0679997. That means our p-value is

$$P(\bar{X} \geq 1.595) + P(\bar{X} \leq 0.5753507) = 0.04437559 + 0.0679997 = 0.112375$$

b). If we start from generalized likelihood ratio test, and use the asymptotical distribution of  $2\log \text{GLR}$ , will you reject  $H_0$ ? What's your  $p$ -value? From Problem 2, we get the generalized likelihood ratio model is

$$\Lambda = \left[ \frac{1}{\bar{X}\theta_0} e^{\theta_0 \bar{X} - 1} \right]^n$$

We will reject null hypothesis if  $\Lambda \geq c$ . In this case under  $H_0$  we know

$$2\log \Lambda \sim \chi^2(1)$$

$$2\log \Lambda = 2[n(\theta_0 \bar{X} - 1) - n\log(\bar{X}\theta_0)] = 2.5625$$

Since we know  $n = 10$   $\theta_0 = 1$  and from the data observed  $\bar{X} = 1.595$

```
qchisq(.95,1); [1] 3.841459
```

$2\log \Lambda < q_{0.95}$  so we fail to reject  $H_0$

P-value =  $1 - F_{\chi^2(1)}(2.5625)$

```
1-pchisq(2.5625,1); [1] 0.1094254
```

So P-value=0.10943



c). Derive a Wald test of  $H_0 : \theta = 1$  versus  $H_A : \theta \neq 1$ . Do you reject  $H_0$ ? What's your  $p$ -value?

For  $H_0 : \theta = \theta_0$  vs  $H_A : \theta \neq \theta_0$

$$W = \frac{|\hat{\theta}_{mle} - \theta_0|}{\sqrt{\text{Var}(\hat{\theta}_{mle})}}$$

$$l(\theta) = \log(\theta) - \theta \sum_{i=1}^n X_i$$

$$l'(\theta) = \frac{n}{\theta} - \sum_{i=1}^n X_i = 0$$

$$l''(\theta) = -\frac{n}{\theta^2}$$

$$-E(l''(\theta)) = -E\left(-\frac{n}{\theta^2}\right) = \frac{n}{\theta^2}$$

$$\text{Var}(\hat{\theta}_{mle}) = (n\bar{X}^2)^{-1}$$

$$W = \frac{|\hat{\theta}_{mle} - \theta_0|}{\sqrt{\text{Var}(\hat{\theta}_{mle})}} = \frac{1/1.595 - 1}{(1/1.595)/\sqrt{10}} = 1.88$$

$$q_{0.975} = 1.96$$

$$|w| < 1.96 \quad \text{so we fail to reject } H_0$$

$$P\text{-value} = 2(1 - F_{N(0,1)}(1.88)) = 0.0601$$

d). Derive a score test of  $H_0 : \theta = 1$  versus  $H_A : \theta \neq 1$ . Do you reject  $H_0$ ? What's your  $p$ -value?

$$U(\theta_0) = U(1) = \frac{dl(\theta_0)}{d\theta_0} = \frac{n}{\theta_0} - \sum_{i=1}^n X_i = n - \sum_{i=1}^n X_i$$

$$I(\theta_0) = I(1) = -E(l''(\theta_0)) = \frac{n}{\theta_0^2} = n$$

$$U = \frac{n - \sum_{i=1}^n X_i}{\sqrt{n}} = \frac{1 - \bar{X}}{\sqrt{n}} \sim N(0, 1)$$

Plugging in data  $\bar{X} = 1.595, n = 10$

$$U = -1.88$$

$$q_{critical} = \pm 1.96$$

$$|U| < 1.96$$



So we do not reject  $H_0$

$$P\text{-value} = 2 * F_{N(0,1)}(-1.88) = 0.0601$$

## Appendix

R code

```
#####  
# Problem 1 #  
#####  
#Create positive and negative delta vectors  
d.pos<-seq(0.01,10,by=0.01)  
d.neg<-c(-rev(d.pos),0)  
d<-c(d.neg,d.pos)  
  
#Calculate the corresponding power functions on the positive and negative sides  
p.25.05.pos<-1-pnorm(qnorm(0.95,mean=0,  
sd=sqrt(80/25))-d.pos,mean=0,sd=sqrt(80/25))  
p.100.05.pos<-1-pnorm(qnorm(0.95,mean=0,  
sd=sqrt(80/100))-d.pos,mean=0,sd=sqrt(80/100))  
p.25.10.pos<-1-pnorm(qnorm(0.9,mean=0,  
sd=sqrt(80/25))-d.pos,mean=0,sd=sqrt(80/25))  
p.100.10.pos<-1-pnorm(qnorm(0.9,mean=0,  
sd=sqrt(80/100))-d.pos,mean=0,sd=sqrt(80/100))  
p.25.05.neg<-pnorm(qnorm(0.05,mean=0,  
sd=sqrt(80/25))-d.neg,mean=0,sd=sqrt(80/25))  
p.100.05.neg<-pnorm(qnorm(0.05,mean=0,  
sd=sqrt(80/100))-d.neg,mean=0,sd=sqrt(80/100))  
p.25.10.neg<-pnorm(qnorm(0.1,mean=0,  
sd=sqrt(80/25))-d.neg,mean=0,sd=sqrt(80/25))  
p.100.10.neg<-pnorm(qnorm(0.1,mean=0,  
sd=sqrt(80/100))-d.neg,mean=0,sd=sqrt(80/100))  
  
#Combine the positive and negative vectors for plotting
```

```

p.25.05<-c(p.25.05.neg,p.25.05.pos)
p.25.10<-c(p.25.10.neg,p.25.10.pos)
p.100.05<-c(p.100.05.neg,p.100.05.pos)
p.100.10<-c(p.100.10.neg,p.100.10.pos)

#Create 4 plots to compare across alpha and n:
par(mfrow=c(2,2))
plot(d,p.25.05,type="l",col="red",xlab="delta",
ylab="power",main="Power function (n=25)");
lines(d,p.25.10,col="blue");
legend("bottomleft",legend=c("alpha=.05",
"alpha=.10"),col=c("red","blue"),lty=c(1,1),box.lty=0,cex=.8);

plot(d,p.100.05,type="l",col="red",xlab="delta",
ylab="power", main="Power function (n=100)");
lines(d,p.100.10,col="blue");
legend("bottomleft",legend=c("alpha=.05",
"alpha=.10"),col=c("red","blue"),lty=c(1,1),box.lty=0,cex=.8);

plot(d,p.25.05,type="l",col="red",xlab="delta",
ylab="power", main="Power function (alpha=.05)");
lines(d,p.100.05,col="blue");
legend("bottomleft",legend=c("n=25","n=100"),
col=c("red","blue"),lty=c(1,1),box.lty=0,cex=.8);

plot(d,p.25.10,type="l",col="red",xlab="delta",
ylab="power", main="Power function (alpha=.10)");
lines(d,p.100.10,col="blue");
legend("bottomleft",legend=c("n=25","n=100"),
col=c("red","blue"),lty=c(1,1),box.lty=0,cex=.8);

#####
# Problem 2 #
#####
#Function to shade area under curve
polyCurve <- function(x, y, from, to, n = 50, miny, col = "red", border = col)
{drawPoly <- function(fun, from, to, n = 50, miny, col, border)
{Sq <- seq(from = from, to = to, length = n)
polygon(x = c(Sq[1], Sq, Sq[n]),
y = c(miny, fun(Sq), miny),col = col, border = border)}

```

```

lf <- length(from) stopifnot(identical(lf, length(to)))
if(length(col) != lf) col <- rep(col, length.out = lf)
if(length(border) != lf) border <- rep(border, length.out = lf)
if(missing(miny)) miny <- min(y) interp <- approxfun(x = x, y = y)
mapply(drawPoly, from = from, to = to, col = col, border = border,
MoreArgs = list(fun = interp, n = n, miny = miny)) invisible()
}

```

```

from <- c(0, 1.7613349)
to <- c(0.4978931, 10)
cols <- c("red", "red")

```

```

#Plot of how to determine c
xbar<-seq(0,10,by=0.01);
y<-xbar*exp(-xbar)
plot(xbar,y,type="l",
main="xbar*exp(-xbar) vs xbar", xlab="xbar",ylab="xbar*exp(-xbar)")
abline(a=0.3026244,b=0, lty=2)
abline(v=0.4978931,col=3,lty=3)
abline(v=1.7613349,col=3,lty=3)
legend("topright",legend=c("xbar*exp(-xbar)","c"),lty=c(1,2),box.lty=0);

```

```

#Code to determine c
#start with alpha/2 and 1-alpha/2 quantiles for gamma of xbar
x<-c(qgamma(.025,shape=10,scale=0.1),qgamma(.975,shape=10,scale=0.1))
x*exp(-x) # 0.2968676 0.3094756
#not the same, so we can move the points up to keep the probability in the tails
#as alpha but move closer to satisfy the condition that x*exp(-x) same for both
d<-0.006
x<-c(qgamma(.025+d,shape=10,scale=0.1),qgamma(.975+d,shape=10,scale=0.1))
x*exp(-x) #0.3025640 0.3027156
#much closer! keep going...only kept last solution to save space
d<-0.0060704
x<-c(qgamma(.025+d,shape=10,scale=0.1),qgamma(.975+d,shape=10,scale=0.1))
x*exp(-x) #0.3026244 0.3026244
x #0.4978931 1.7613349

```

```

#Plot of rejection region based on xbar

```

```

g<-dgamma(xbar,shape=10,scale=0.1)
plot(xbar, g, type="l", xlim=c(0,3),ylim=c(0,1.5),
main="Rejection Region",panel.first =polyCurve(xbar, g, from = from,
to = to,
col = cols, border = "black"))
abline(v=0.4978931,col=3,lty=3);
abline(v=1.7613349,col=3,lty=3);
legend("topright",legend=c("x0=0.49789","x1=1.76133"),
col=c(3,3),lty=c(3,3),box.lty=0,cex=1.1);

```