8004 Homework 1

Nooreen Dabbish
January 28, 2015

1 Write out the following models of elementary/intermediate statistical analysis in the matrix form:

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

(a) A one-variable quadratic polynomial regression model

$$y_i = \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \epsilon_i \text{ for } (i = 1, 2, \ldots, 5)$$

.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_1^2 \end{pmatrix}, \beta = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix}, \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix}$$

$\mathbf{y} = \mathbf{X}\beta + \epsilon$ in this model is therefore:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_1^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \end{pmatrix}$$

(b) A two-factor ANCOVA model without interactions

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma(x_{ijk} - \bar{x}) + \epsilon_{ijk} \text{ for } i = 1, 2, \ j = 1, 2, \text{ and } k = 1, 2.$$

This model describes an 8-dimensional data space:

$$\mathbf{y} = \begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{pmatrix}, \epsilon = \begin{pmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \end{pmatrix}$$

The vector of centered x-values may be calculated as

$$\boldsymbol{x}_c = (\boldsymbol{I} - \frac{1}{n}\boldsymbol{J}) \begin{pmatrix} x_{111} \\ x_{112} \\ x_{121} \\ x_{122} \\ x_{211} \\ x_{212} \\ x_{221} \\ x_{222} \end{pmatrix} = \begin{pmatrix} x_{111} - \bar{x} \\ x_{112} - \bar{x} \\ x_{121} - \bar{x} \\ x_{122} - \bar{x} \\ x_{211} - \bar{x} \\ x_{212} - \bar{x} \\ x_{221} - \bar{x} \\ x_{222} - \bar{x} \end{pmatrix}$$

The design matrix $\mathbf{X}$ and regression coefficient vector β are given by:

$$\mathbf{X} = \begin{pmatrix} 1.00 & 1.00 & 0.00 & 1.00 & x_{111}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 1.00 & x_{112}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 0.00 & x_{121}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 0.00 & x_{122}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 1.00 & x_{211}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 1.00 & x_{212}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 0.00 & x_{221}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 0.00 & x_{222}-\bar{x} \end{pmatrix}, \beta = \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma \end{pmatrix}$$

Putting these together gives the model:

$$\begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{pmatrix} = \begin{pmatrix} 1.00 & 1.00 & 0.00 & 1.00 & x_{111}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 1.00 & x_{112}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 0.00 & x_{121}-\bar{x} \\ 1.00 & 1.00 & 0.00 & 0.00 & x_{122}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 1.00 & x_{211}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 1.00 & x_{212}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 0.00 & x_{221}-\bar{x} \\ 1.00 & 0.00 & 1.00 & 0.00 & x_{222}-\bar{x} \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma \end{pmatrix} + \begin{pmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \end{pmatrix}$$

2 * Use `eigen()` function in R to compute the eigenvalues and eigenvectors of

eigen() tt function in R to compute the eigenvalues and eigenvectors of

$$\mathbf{V} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

Then use R to find and “inverse square root” of this matrix. That is, find a symmetric matrix $\mathbf{W}$ such that $\mathbf{WW} = \mathbf{V}^{-1}$.

(a) Eigenvalues and Eigenvectors

(a) Eigenvalues and Eigenvectors

The eigenvalues are $\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$.

With corresponding eigenvectors:

$$\mathbf{e_1} = \begin{pmatrix} 0.41 \\ -0.82 \\ 0.41 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0.58 \\ 0.58 \\ 0.58 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0.71 \\ 0.00 \\ -0.71 \end{pmatrix}.$$

(b) Inverse Square Root

(b) Inverse Square Root

$$\mathbf{W} = \begin{pmatrix} 0.3889 & 0.0556 & -0.1111 \\ 0.0556 & 0.2222 & 0.0556 \\ -0.1111 & 0.0556 & 0.3889 \end{pmatrix}$$

3 Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 4.00 & 4.00 \\ 4.00 & 4.00 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4.00 & 4.00 \\ 4.00 & 4.00 \end{pmatrix}.$$

Obviously, these matrices are nearly identical. Use R and compute the determinants and inverses of these matrices. (Even though the original two matrices are nearly the same, $\mathbf{A}^{-1} \approx -3\mathbf{B}^{-1}$. This shos that small changes in the elements of nearly singular matrices can have big eggects on some matrix operations.)

(a) Determinants and Inverses.

The determinant of A is -1.00000000513756e-06 and the determinant of B is SRC_R[:session **hw1** :results raw]{det(B)}.

##

$$\mathbf{A}^{-1} = \begin{pmatrix} -4001999.98 & 4000999.98 \\ 4000999.98 & -3999999.98 \end{pmatrix} \text{ and } \mathbf{B}^{-1} = \begin{pmatrix} 1334000.33 & -1333666.67 \\ -1333666.67 & 1333333.33 \end{pmatrix}$$

#

$$-3\mathbf{B}^{-1} = \begin{pmatrix} -4002001.00 & 4001000.00 \\ 4001000.00 & -4000000.00 \end{pmatrix}$$

4 Write an R function to conduct projection, e.g. with the name `project()` .

project() tt.

The input is the given design matrix $\mathbf{X}$, and the output is the projection matrix $\mathbf{P_X}$ for projecting a vector onto the column space of $\mathbf{X}$.

```
project <- function (X) {X%*%(solve(t(X)%*%X))%*%t(X)}
```

5 Consider the (non-full-rank) two-way "effect model" with interactions in the Example (d) in lecture.

(a) Determine which of the parametric functions below are estimable:

α_1,α_2 - α_a, μ + α_1+ β_1+δ_{11}, δ_{12}, δ_{12} - δ_{11} - (δ_{22} - δ_{21})

For those that are estimable, find $\mathbf{C}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T$, such that $\mathbf{C}^T(\mathbf{X}^T\mathbf{X})^-\mathbf{X}^T\mathbf{Y}$ procudes the extimate os $\mathbf{C}^T\beta$.

(b) For thte parameter vector β written in the order used in class, consider the hypothesis H_0 : \$$\mathbf{C}\beta = \mathbf{0} for$

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

Is this hypothesis testable? Explain.