Solution

Problem 5. The incidence of a rare disease seems to be decreasing. In successive years, the number of new cases is $y_1, \ldots, y_n$. We assume that $y_1, \ldots, y_n$ are independent random variables from Poisson distributions with means $\theta, \theta^2, \ldots, \theta^n$ respectively.

a) Forumlate a likelihood ratio test for testing $\mathrm{H}_0: \theta = 1$ versus $\mathrm{H}_a: \theta < 1$. For $(y_1, y_2) = (2, 0)$, would such test with size 0.20 test accept or reject H_0?

b) Describe a procedure for forming a level 0.95 one-sided confidence interval of the form $(0, \theta_u)$ [you do not need to come up with a closed form expression and can express that you would need to calculate the quantiles of certain distributions and do a numerical search to form the confidence interval]. Use your procedure to find (approximately) a realized confidence interval of the form $(0, \theta_u)$ for the sample $(y_1, y_2) = (2, 0)$ (you may want to write a computer program for this).

Solution.

a) The likelihood function of (y_1, y_2) is

$$L(\theta) = \frac{\theta^{y_1+2y_2} \exp(-\theta - \theta^2)}{y_1! y_2!}.$$

For $\theta_0 = 1$, and any $\theta_1 < 1$, the likelihood ratio is

$$\frac{L(\theta_1)}{L(\theta_0)} = \left(\frac{\theta_1}{\theta_0}\right)^{y_1+2y_2} \exp(-\theta_1 - \theta_1^2 + \theta_0 + \theta_0^2),$$

which is a strictly decreasing function of $y_1 + 2y_2$. We reject H_0 if $L(\theta_1)/L(\theta_0)$ is large, or equivalently when $y_1 + 2y_2 \leq y$.

Let $Y = Y_1 + 2Y_2$. Under H_0, $Y_1, Y_2 \sim \text{Poisson}(1)$ *i.i.d.*. We have

$$\begin{aligned}
\mathtt{P}(Y = 0) &= \mathtt{P}(Y_1 = 0, Y_2 = 0) = 0.1353 \\
\mathtt{P}(Y = 1) &= \mathtt{P}(Y_1 = 1, Y_2 = 0) = 0.1353
\end{aligned}$$

Therefore,

$$\mathtt{P}(Y \le 0 \mid \mathrm{H}_0) = 0.1353 < 0.2, \quad \mathtt{P}(Y \le 1 \mid \mathrm{H}_0) = 0.2706 > 0.2.$$

The size 0.20 test will reject H_0 when $y = 0$, *i.e.* $y_1 = y_2 = 0$. When $(y_1, y_2) = (2, 0)$, we do not reject H_0.

b) To find the 95% upper bound for θ, we first do hypothesis test $\mathrm{H}_0 : \theta = \theta_0$ *vs.* $\mathrm{H}_1 : \theta = \theta_1 < \theta_0$, and find the largest θ_0 so that H_0 is not rejected at the level 0.05 with the oberservation $(y_1, y_2) = (2, 0)$.

By a), we know that we will reject H_0 only if $Y = Y_1 + 2Y_2$ is small. When $\theta = \theta_0$,

$$\begin{aligned}
&\mathtt{P}(Y = 0) = \mathtt{P}(Y_1 = 0, Y_2 = 0) = \exp(-\theta_0 - \theta_0^2) \\
&\mathtt{P}(Y = 1) = \mathtt{P}(Y_1 = 1, Y_2 = 0) = \theta_0 \exp(-\theta_0 - \theta_0^2) \\
&\mathtt{P}(Y = 2) = \mathtt{P}(Y_1 = 2, Y_2 = 0) + \mathtt{P}(Y_1 = 0, Y_2 = 1) = 3\theta_0^2 \exp(-\theta_0 - \theta_0^2)/2
\end{aligned}$$

We need to find the largest θ_0 so that $\mathtt{P}(Y \le 2) \le 0.05$. Let

$$f(\theta_0) = (1 + \theta_0 + 3\theta_0/2) \exp(-\theta_0 - \theta_0^2)$$

It is easy to see that $f'(\theta_0) < 0$. Therefore, $f(\theta_0)$ is a monotone decreasing function. Equivalently, we set $f(\theta_0) = 0.05$ and find the root $\theta_0 = 1.80$ (by R function *uniroot*).

Therefore, the 0.95 one-sided CI of θ is $(0, 1.80)$.

1 The Duarity between Confidence Interval and Hypothesis Testing

Example. Suppose $X_1, \ldots, X_n$ *i.i.d.* $\sim N(\mu, 1)$. We would like to test

$$\mathrm{H}_0 : \mu = \mu_0, \quad vs. \quad \mathrm{H}_1 : \mu \ne \mu_0$$

with type I error α.

In last semester, we learnt that we can use the following test:

$$T = \sqrt{n}(\bar{X} - \mu_0)$$

and reject H_0 when $|T| > z_{1-\alpha/2}$, where $z_{1-\alpha/2}$ is the $1 - \alpha/2$-th quantile of the standard normal density.

On the other hand, after we observe $X_1, \ldots, X_n$, we can build up a $(1-\alpha)$ confidence interval of μ:

$$(\bar{X} - z_{1-\alpha/2}/\sqrt{n},\ \bar{X} + z_{1-\alpha/2}/\sqrt{n}).$$

Now suppose $\bar{X} = 1$, $n = 10$, and $\alpha = 0.05$. The 95% CI of μ is $(0.38, 1.62)$. How does this link to hypothesis testing?

Now consider $\mathrm{H}_0:\ \mu_0 = 0.38$ and $\mathrm{H}_1:\ \mu_1 \neq 0.38$. Then $T = 1.96$. It is on the rejection cutoff boundary. If $\mathrm{H}_0:\ \mu_0 = 0.39$, then $T = 1.93 < 1.96$, we do not reject H_0. And if $\mathrm{H}_0:\ \mu = 0.37$, then $T = 1.99 > 1.96$, and we reject H_0. This is to say that $\mu_0 = 0.38$ is the smallest μ_0 so that $\mathrm{H}_0:\ \mu = \mu_0$ is not rejected at the level of 0.05. Similarly, we can show that $\mu_0 = 1.61$ is the largest μ_0 so that H_0 is not rejected at the same level.

This builds up a link between hypothesis testing and confidence interval. To find $1 - \alpha$ two sideded confidence interval of μ, we only need to find the smallest and largest μ_0 such that

$$\mathrm{H}_0:\ \mu = \mu_0, \quad vs. \quad \mathrm{H}_1:\ \mu \neq \mu_0$$

is not rejected under the level of α.

Example. How about one-sided confidence interval? Following the above example, we know that the one-sided lower confidence interval for μ is

$$(-\infty,\ \bar{X} + z_{1-\alpha}/\sqrt{n}) = (-\infty, 1.52)$$

Consider

$$\mathrm{H}_0:\ \mu = \mu_0, \quad vs. \quad \mathrm{H}_1:\ \mu < \mu_0$$

To control type I error at level α, we reject H_0 when $T < -1.64$.

Now let $\mu_0 = 1.52$, then $T = -1.64$, which is on the rejection boundary. When $\mu_0 = 1.51$, then $T = -1.61$, and therefore we do not reject H_0. And when $\mu_0 = 1.53$, then $T = -1.67$, and therefore we reject H_0. Therefore, $\mu_0 = 1.52$ is the largest μ_0 such that the one sided test

$$\mathrm{H}_0:\ \mu = \mu_0, \quad vs. \quad \mathrm{H}_1:\ \mu < \mu_0$$

is not rejected.