# STAT 8004 Lecture 8

Thursday, March 26, 2015 5:19 PM

Review: Linear mixed (effect) model.

$$\underset{n\times1}{Y} = \underset{k\times1}{X\beta} + \underset{g\times1}{Zu} + \varepsilon$$

$$\begin{pmatrix} u \\ \varepsilon \end{pmatrix} \sim N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} G & 0 \\ 0 & R \end{pmatrix} \right)$$

$E(Y) = X\beta$

$Var(Y) = ZGZ^T + R \leftarrow$

- Key: what is the random effect?

- Estimation - REML for parameters in $\Sigma$

REML / ML:
- $\boxed{\text{– Likelihood framework for its inference} \quad \text{– estimator \& the associated variance}}$
- – GLS framework for fixed effect

  $\boxed{\widehat{c^T\beta} = c^T(X^T\hat{\Sigma}^{-1}X)^{-}X^T\hat{\Sigma}^{-1}Y}$

  its estimated variance, distribution inferences.
- – Prediction for $s^Tu$

  based on the joint distribution of $\begin{pmatrix} u \\ Y \end{pmatrix}$

  $\widehat{s^Tu} = s^T\hat{E}(u|Y)\,(\hat{\Sigma})$

  its associated variance
- $\hat{\ell} = \widehat{c^T\beta} + s^T\hat{u}$

  its associated variance

Remark: - This is an asymptotic framework ($n\to\infty$)
- Consider the robustness when the normality assumption is violated
- # parameters in $R$ & $G$. $\leftarrow$

$Y = X\beta + Zu + \varepsilon$

- By all means, one may pretend that $u$ is also a fixed effect, then using $Z$ similar to $X$, one is able to fit a linear model.
- This procedure provides outputs.
- How to make use of this kind of analysis?

- Fact: to some (not full) extent, it is informative and it is very useful in practice in conjunction with the ANOVA.

( Chapter 25 onwards, Applied Linear Statistical Models by Kutner, Nachtsheim, Neter & Li )

- ANOVA type analysis for LME

$\rightarrow$
$$\boxed{\begin{gathered} y_{ijk} = \mu + \underbrace{\alpha_i + \delta_{ij}}_{\text{random effect}} + \varepsilon_{ijk} \qquad \text{for } i,j,k = 1,2 \quad n=8 \\ \alpha_i \sim N(0,\sigma_\alpha^2) \quad \delta_{ij} \sim N(0,\sigma_\delta^2) \\ \varepsilon_{ijk} \sim N(0,\sigma_\varepsilon^2) \end{gathered}}$$

$$Y = \begin{pmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}\mu + \begin{pmatrix} 1 & & 1 & & & \\ & & & 1 & & \\ & 1 & & & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \delta_{11} \\ \delta_{12} \\ \delta_{21} \\ \delta_{22} \end{pmatrix} + \varepsilon$$

let $X_1 = \begin{pmatrix} 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\left(y_{222}\right)$

let $X_1 = \begin{pmatrix} 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 1 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$C(X_1) \subset C(X_2) \subset C(X_3)$

$P_{X_j} = X_j (X_j^T X_j)^- X_j$

ANOVA table

| | | d.f |
|---|---|---|
| SSA | $Y^T(P_{X_2} - P_{X_1})Y$ | $rank(X_2) - rank(X_1) = 2-1 = 1$ |
| SSB(A) | $Y^T(P_{X_3} - P_{X_2})Y$ | $rank(X_3) - rank(X_2) = 4-2 = 2$ |
| SSE | $Y^T(I - P_{X_3})Y$ | $rank(I) - rank(X_3) = 4$ |
| | $Y^T(I - P_1)Y$ | 7 |

$$\begin{cases} E(MSE) = (E\frac{SSE}{4}) = \sigma_\varepsilon^2 \\ E(MSB(A)) = E\left(\frac{SSB(A)}{2}\right) = \sigma_\varepsilon^2 + 2\sigma_\delta^2 \\ E(MSA) = E\left(\frac{SSA}{1}\right) = \sigma_\varepsilon^2 + 2\sigma_\delta^2 + 4\sigma_\alpha^2 \end{cases}$$

under the mixed model.

$$\left( E(Y^T A Y) = tr(A\Sigma) + (E(Y))^T A (E(Y)) \right.$$

- Method of moment estimation for $\sigma_\varepsilon^2, \sigma_\alpha^2, \sigma_\delta^2$

$$\rightarrow \begin{cases} \hat{\sigma}_\varepsilon^2 = MSE \leftarrow \\ \hat{\sigma}_\delta^2 = \boxed{\frac{MSB(A) - MSE}{2}} \leftarrow \\ \hat{\sigma}_\alpha^2 \ \ - - - \end{cases}$$

C.I for

$\sigma_\varepsilon^2, \sigma_\delta^2, \sigma_\alpha^2$
$\uparrow \quad \uparrow \quad \uparrow$

- How about their associated variance? distribution

let us consider a more general situation:

Suppose $MS_1, MS_2, \dots, MS_\ell$ are independent r.vs

and $\underline{(df_i)\frac{MS_i}{E(MS_i)}} \sim \chi^2_{df_i}$

Then for a linear combination

$$\boxed{S^2 = a_1 MS_1 + a_2 MS_2 + \dots + a_\ell MS_\ell}$$

Then $E(S^2) = a_1 E(MS_1) + \dots + a_\ell E(MS_\ell)$

$var(S^2) = \sum a_i^2 Var(MS_i)$ (How to calculate it?)

$= \sum a_i^2 Var\left(\frac{E(MS_i)}{df_i} \underbrace{\frac{df_i (MS_i)}{E(MS_i)}}\right)$

$= \sum a_i^2 \left(\frac{E(MS_i)}{df_i}\right)^2 var(a\ \chi^2_{df_i}\ r.v.)$

$= \sum a_i^2 \left(\frac{E(MS_i)}{df_i}\right)^2 2\, df_i$

$= \sum a_i^2 \left[ \frac{(E(MS_i))^2}{\dots} \right.$

$$= 2\sum a_i^2 \left(\frac{\quad}{df_i}\right) \quad \leq a_i^{\ldots}$$

$$= 2\sum a_i^2 \boxed{\frac{(E(MS_i))^2}{df_i}}$$

then $\widehat{Var}(S^2) = 2\sum a_i^2 \boxed{\frac{(MS_i)^2}{df_i}}$

Only need to calculate $MS_i$, nice!

An alternative way

$$E(MS_i)^2 = Var(MS_i) + (E(MS_i))^2$$

$$= \frac{2(E(MS_i))^2}{df_i} + (E(MS_i))^2$$

$$= (E(MS_i))^2 \quad \frac{2+df_i}{df_i}$$

$$(E(MS_i))^2 = \boxed{\frac{df_i}{df_i+2} \; E(MS_i)^2}$$

Then, an alternative formula is

$$\widehat{Var}(S^2) = 2\sum a_i^2 \frac{(MS_i)^2}{df_i+2}$$

This gives smaller variance estimation.

- Can do normal approximation for confidence intervals of $E(S^2)$

—x—

- The Cochran-Satterthwaite approximation

using $\frac{v(S^2)}{E(S^2)} \sim \chi^2_v$ approximately

If $v$ is known

$$P\left(\chi^2_{v,lower} < \frac{v(S^2)}{E(S^2)} < \chi^2_{v,upper}\right) = 1-\alpha$$

$$\parallel$$

$$P\left(\frac{vS^2}{\chi^2_{v,upper}} < \underline{\underline{E(S^2)}} < \frac{vS^2}{\chi^2_{v,lower}}\right)$$

$v$?

By solving $\begin{cases} E\left(\frac{v(S^2)}{E(S^2)}\right) = \underline{\underline{v}} \\ Var\left(\frac{v(S^2)}{E(S^2)}\right) = 2\underline{\underline{v}} \end{cases}$ for $v$

got $\underline{\underline{v}} = \frac{(ES^2)^2}{\sum \frac{(a_i E(MS_i))^2}{df_i}}$

$$\hat{v} = \frac{(S^2)^2}{\sum (a_i MS_i)^2}$$

$$\boxed{\hat{\nu} = \frac{(S^2)^2}{\sum \frac{(a_i MS_{i})^2}{df_i}}}$$

replacing $\nu$ with $\hat{\nu}$ to obtain CI for $E(S^2)$.

—x—

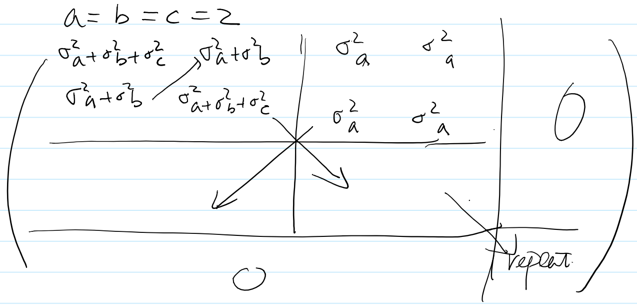
- Hierarchical model (nested)

In general, for A with total $a$ levels, there are a total $b$ levels of B nested within each level of A, $c$ levels of C nested within each level of B

The variance

$$\Sigma = \sigma_A^2 I_{a\times a} \otimes J_{bc\times bc} + \sigma_B^2 I_{ab\times ab} \otimes J_{c\times c} + \sigma_C^2 I_{abc\times abc}$$

$a = b = c = 2$




- ANOVA analysis

$$E(MSA) = \sigma_c^2 + c\sigma_b^2 + (bc)\sigma_a^2$$

$$E(MSB(A)) = \sigma_c^2 + c\sigma_b^2$$

$$E(MSE) = \sigma_c^2$$

$$\hat{\sigma}_c^2 = MSE$$

$$\begin{cases} \hat{\sigma}_b^2 = \frac{1}{c}(MSB(A) - MSE) \\ \hat{\sigma}_a^2 = \frac{1}{bc}(MSA - MSB(A)) \end{cases}$$

All take the form of $S^2$

$$\left( \frac{\hat{\nu} S^2}{\chi^2_{\hat{\nu}, upper}} \quad \frac{\hat{\nu} S^2}{\chi^2_{\hat{\nu}, lower}} \right) \text{ as C.I.}$$

ANOVA table

| SS | df |
|---|---|
| SSA | $a-1$ |
| SSB(A) | $a(b-1)$ |
| SSE | $ab(c-1)$ |

| | |
|---|---|
| SSB(A) | $a(b-1)$ |
| SSE | $ab(c-1)$ |

- Fixed effect:

$$y_{ijk} = \mu + \alpha_i + \delta_{ij} + \varepsilon_{ijk}$$

...

$\mu$ is the only fixed effect

$$\bar{y}_{\cdot\cdot\cdot} = \mu + \bar{\alpha} + \bar{\delta}_{\cdot\cdot} + \bar{\varepsilon}_{\cdot\cdot\cdot}$$

$$\bar{\alpha} = \sum_i \sum_j \sum_k \alpha_i \Big/ abc = \frac{\sum \alpha_i}{a}$$

...

$$\mathrm{var}(\bar{y}_{\cdot\cdot\cdot}) = \frac{\sigma_\alpha^2}{a} + \frac{\sigma_\delta^2}{ab} + \frac{\sigma_\varepsilon^2}{abc}$$

$$= \frac{1}{abc}\left\{ bc\,\sigma_\alpha^2 + c\,\sigma_\delta^2 + \sigma_\varepsilon^2 \right\}$$

$$= \frac{1}{abc} E(MSA)$$

$$\widehat{\mathrm{var}}(\bar{y}_{\cdot\cdot\cdot}) = \frac{MSA}{abc} \qquad \left( \frac{\sum_{ijk}(y_{ijk} - \bar{y}_{\cdot})^2}{a-1} \right) \text{ `` } MSA$$

- $$\boxed{\bar{y}_{\cdot\cdot\cdot} \pm \underset{df = a-1}{t} \sqrt{\frac{MSA}{abc}}}$$ for $\mu$.

- Random blocking (Penicillin example)
  producing process
  blocked by batch

Factor A: fixed effect $a$ levels

Factor B: random "blocks" $b$ levels
( one observation in each cell)

$$\boxed{y_{ij} = \underbrace{\boxed{\mu + \alpha_i}}_{\text{fixed effect}} + \underbrace{\beta_j}_{\text{random}} + \varepsilon_{ij}}$$

$i = 1, \cdots, a \quad j = 1, \cdots b$

$$Y = \begin{pmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{a1} \\ \vdots \\ Y_{1b} \\ Y_{2b} \\ \vdots \\ Y_{ab} \end{pmatrix} = \begin{pmatrix} 1 & I_{a\times a} \\ & I_{a\times a} \\ & \vdots \\ & I_{a\times a} \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \vdots \\ \alpha_a \end{pmatrix} +$$

$ab\times 1$ $\quad 1_{a\times 1} \quad \beta_1$

$Y_{ab}$

$ab \times 1$

$$\begin{pmatrix} 1_{a\times1} & & \\ & 1_{a\times1} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_b \end{pmatrix} + \varepsilon$$

$$G = \mathrm{Var}\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_b \end{pmatrix} = \sigma_b^2 I_{b\times b}$$

$$R = \mathrm{var}(\varepsilon) = \sigma_\varepsilon^2 I_{ab\times ab}$$

$$\Sigma = ZGZ^T + R = \sigma_b^2 I_{b\times b} \otimes J_{a\times a} + \sigma_\varepsilon^2 I_{ab\times ab}$$

$$J_{a\times a} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{a\times a}$$

ANOVA with no interaction

| Source | SS | df | MS |
|---|---|---|---|
| A | SSA | $a-1$ | $\frac{SSA}{a-1}$ |
| B | SSB | $b-1$ | $\frac{SSB}{b-1}$ |
| error | SSE | $(b-1)(a-1)$ | MSE |
| | | $ab-1$ | |

$$\begin{cases} E(MSE) = \sigma_\varepsilon^2 \\ E(MSB) = E\left(\frac{SSB}{b-1}\right) = \sigma_\varepsilon^2 + a\sigma_b^2 \end{cases}$$

$$\Rightarrow \hat{\sigma}_\varepsilon^2 = MSE$$

$$\hat{\sigma}_b^2 = \frac{1}{a}(MSB - MSE)$$

- $E(MSA)$ involves fixed effect.

Now $\left( \frac{(a-1)(b-1)\,MSE}{\chi^2_{(a-1)(b-1),\,upper}} \quad \frac{(a-1)(b-1)\,MSE}{\chi^2_{(a-1)(b-1),\,lower}} \right)$

CI for $\sigma_\varepsilon^2$.

for $\sigma_b^2$, need to use Cochran-Satterthwaite approximation)

- for fixed effect

$$\widehat{\mu+\alpha_i} = \frac{1}{b}\sum_j y_{ij} = \bar{y}_{i\cdot}$$

$$= \mu + \alpha_i + \bar{\beta}_\cdot + \bar{\varepsilon}_{\cdot\cdot}$$

$$S^2_{\bar{y}_{i\cdot}} = \widehat{Var}(\bar{y}_{i\cdot}) = \frac{1}{b}\sigma^2_\varepsilon + \frac{1}{b}\sigma^2_b$$

$$= \frac{1}{ba}\left(E(MSB - MSE)\right) + \frac{1}{b}E(MSE)$$

$$= \frac{1}{ab}E(MSB) + \frac{a-1}{ab}E(MSE)$$

$$\boxed{\frac{\bar{y}_{i\cdot} - (\mu + \alpha_i)}{\sqrt{S^2_{\bar{y}_{i\cdot}}}} \sim \boxed{t_{\hat{\nu}}}}$$

from CS approximation

- Called Cochran – Satterthwaite approximation based confidence interval for fixed effect.

— x —

- $\bar{y}_{i\cdot} \pm t_{\hat{\nu}}\sqrt{S^2_{\bar{y}_{i\cdot}}}$ as an approximate C.I. for $\mu + \alpha_i$

- How about $(\alpha_i - \alpha_{i'})$

$$\bar{y}_{i\cdot} - \bar{y}_{i'\cdot} = (\mu + \alpha_i + \bar{\beta}_{\cdot} + \bar{\varepsilon}_{i\cdot})$$
$$- (\mu + \alpha_{i'} + \bar{\beta}_{\cdot} + \bar{\varepsilon}_{i'\cdot})$$

$$= (\alpha_i - \alpha_{i'}) + (\bar{\varepsilon}_{i\cdot} - \bar{\varepsilon}_{i'\cdot})$$

$$E(\bar{y}_{i\cdot} - \bar{y}_{i'\cdot}) = \alpha_i - \alpha_{i'}$$

$$Var(\qquad) = 2\frac{\sigma^2_\varepsilon}{b}$$

$$\Rightarrow \boxed{\bar{y}_{i\cdot} - \bar{y}_{i'\cdot} \pm t\sqrt{2\frac{MSE}{b}}}$$

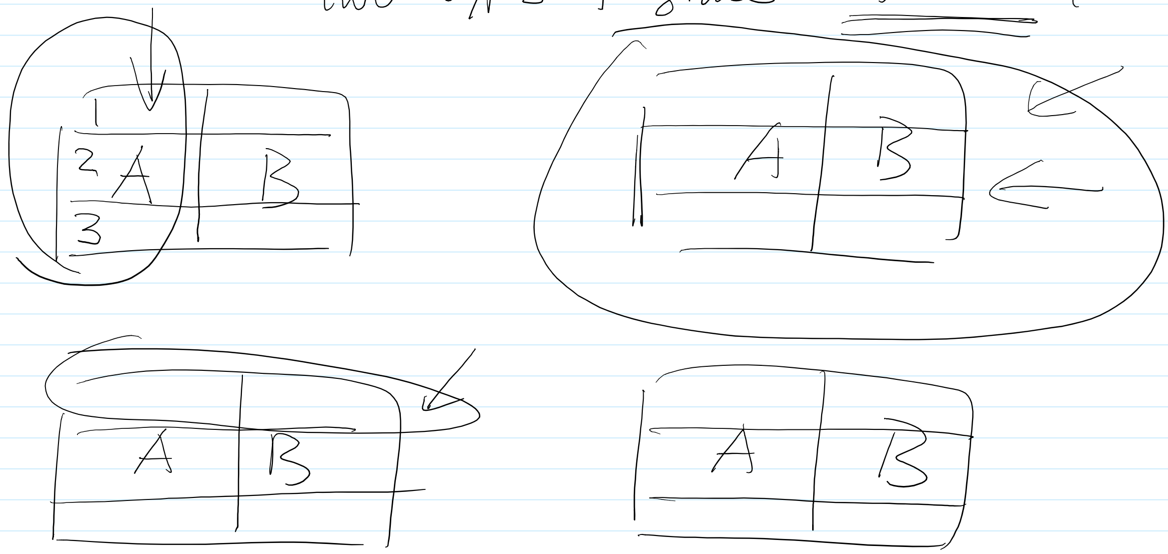
$t$: df of MSE $(a-1)(b-1)$.

— x —

- "Split-plot"

example : - four fields for growing grasses
- two types of grass A & B (wholeplot)

- two types of grass <u>A & B</u> (whole plot)




- 3 baterial innoculation ← (subplot)

- Main effect: $\begin{cases} \text{- grass effect} \\ \text{- baterial innoculation} \end{cases}$
  fixed

outcome $Y$ is the <u>Yield</u>

$$y_{ijk} = \boxed{\mu + \alpha_i + \beta_j + \delta_{ij}}$$

$i$th grass with $j$th type of innoculation on $k$th field.

fixed

$$+ \underline{\gamma_k} + \underline{\eta_{ik}} + \varepsilon_{ijk}$$

$\gamma_k$: random effect associated with field

$\eta_{ik}$: random effect associated with the same grass for the three innoculation

- Reason for split-plot:

  practical constraints!

$$\gamma_k \sim N(0, \sigma_\gamma^2)$$

$$\eta_{ik} \sim N(0, \sigma_\eta^2)$$

$$\varepsilon_{ijk} \sim N(0, \sigma_{ijk}^2)$$

```
> summary(grass.aov <- aov(Yield ~ Culti*Innoc +
+           Error(Block/Culti),data=grass))

Error: Block
          Df Sum Sq Mean Sq F value Pr(>F)
Residuals  3  25.32    8.44

Error: Block:Culti
          Df Sum Sq Mean Sq F value Pr(>F)
Culti      1  2.407   2.407   0.762  0.447
Residuals  3  9.480   3.160

Error: Within
            Df Sum Sq Mean Sq F value    Pr(>F)
Innoc        2 118.18   59.09  83.763 8.92e-08 ***
Culti:Innoc  2   1.83    0.91   1.294     0.31
Residuals   12   8.46    0.71
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

**ANOVA table**:

| Source of Variation | df | SS |
|---|---|---|
| Blocks | $r-1=3$ | 25.32 |
| Cultivars | $a-1=1$ | 2.41 |
| Whole Plot error (Block × cultivar interaction) | (r-1)(a-1)=3 | 9.48 |
| Innoculants | $b-1=2$ | 118.18 |
| Cult. × Innoc. | $(a-1)(b-1)=2$ | 1.83 |
| Sub-plot error | 12 | 8.465 |
| Corrected total | 23 | 165.673 |

```
> o1=lm(Yield~Culti*Innoc+Block*Culti,data=grass)
> anova(o1)
Analysis of Variance Table

Response: Yield
            Df  Sum Sq Mean Sq F value    Pr(>F)
Culti        1   2.407   2.407  3.4117 0.0895283 .
Innoc        2 118.176  59.088 83.7631 8.919e-08 ***
Block        3  25.320   8.440 11.9646 0.0006428 ***
Culti:Innoc  2   1.826   0.913  1.2942 0.3097837
Culti:Block  3   9.480   3.160  4.4796 0.0249095 *
Residuals   12   8.465   0.705
---
```

```
Culti:Innoc   2   1.826   0.913  1.2942 0.3097837
Culti:Block   3   9.480   3.160  4.4796 0.0249095 *
Residuals    12   8.465   0.705
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

**ANOVA table:**

| Source of Variation | df | MS | $E(MS)$ |
|---|---|---|---|
| Blocks | 3 | 8.44 | $\sigma_e^2 + b\sigma_w^2 + ba\sigma_\beta^2$ |
| Cultivars | 1 | 2.41 | $\sigma_e^2 + b\sigma_w^2 + (i)$ |
| Whole Plot error (Block $\times$ cult. interaction) | 3 | 3.16 | $\sigma_e^2 + b\sigma_w^2$ |
| Innoculants | 2 | 59.09 | $\sigma_e^2 + (ii)$ |
| Cult. $\times$ Innoc. | 2 | 0.91 | $\sigma_e^2 + (iii)$ |
| Sub-plot error | 12 | 0.705 | $\sigma_e^2$ |
| Corrected total | 23 | | |

**(i)** $\dfrac{br\Sigma_{i=1}^{a}(\alpha_i+\bar{\delta}_{i.}-\bar{\alpha}_{.}-\bar{\delta}_{..})^2}{a-1}$

**(ii)** $\dfrac{ar\Sigma_{k=1}^{b}(\gamma_k+\bar{\delta}_{.k}-\bar{\gamma}_{.}-\bar{\delta}_{..})^2}{b-1}$

**(iii)** $\dfrac{r\Sigma_i\Sigma_k(\delta_{ik}-\bar{\delta}_{i.}-\bar{\delta}_{.k}+\bar{\delta}_{..})^2}{(a-1)(b-1)}$

```
> o3=lme(Yield~Culti*Innoc,random=~1|Block/Culti,data=grass)
> anova(o3)
            numDF denDF   F-value p-value
(Intercept)     1    12 2630.8256  <.0001
Culti           1     3    0.7616  0.4471
Innoc           2    12   83.7631  <.0001
Culti:Innoc     2    12    1.2942  0.3098
```

fixed effect testing

- Example "repeated measurements"
"longitudinal data"
Chapter 9 of Faraway (2006)

Example:
- Exercise therapy study, subjects are assigned three weight lifting program (1. 2. 3)

- measurements of the strengths on days
  2, 4, 6, 8, 10, 12, 14

- Dependence / correlation due to the
  measurements on the same subject.

$$y_{ijk} = \mu + \overbrace{\alpha_i + \tau_k + \delta_{ik}}^{\text{fixed}}$$

$y_{ijk}$: kth strength measurement of the jth subject in the ith program

$\alpha_i$: program, $\tau_k$: time

$$+ \gamma_{ij} + \varepsilon_{ijk}$$

$\gamma_{ij}$: random effect associated with subject

$$Y = X\beta + Zu + \varepsilon$$

$$\mathrm{cov}(y_{ijk}, y_{ijk'}) = ?$$

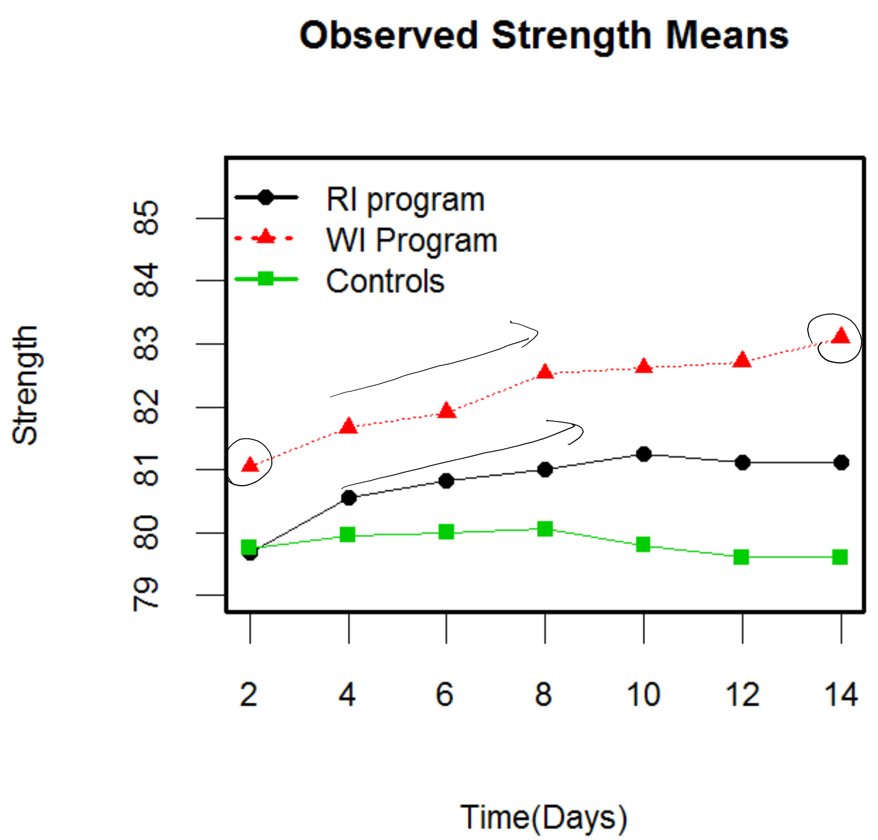
$$\mathrm{corr}(y_{ijk}, y_{ijk'}) = ?$$

$$\mathrm{cov}(y_{ijk}, y_{ij'k'}) = ?$$

$$\gamma_{ij} \overset{iid}{\sim} N(0, \sigma_\gamma^2) \quad ?$$

$$\varepsilon_{ijk} \sim N(0, \sigma_\varepsilon^2)$$

- compound symmetric correlation structure !




Code $g$(time) into the model

```
> m1=lmer(Strength~Program*Timef+(1|Subj),data=weight)
> summary(m1)
Linear mixed model fit by REML ['lmerMod']
Formula: Strength ~ Program * Timef + (1 | Subj)
   Data: weight

REML criterion at convergence: 1420.8

Scaled residuals:
     Min       1Q   Median       3Q      Max
-3.03413 -0.63265  0.00009  0.56698  3.15030

Random effects:
 Groups   Name        Variance Std.Dev.
 Subj     (Intercept) 9.603    3.099
 Residual             1.197    1.094
Number of obs: 399, groups:  Subj, 57
```

```
Fixed effects:
                       Estimate Std. Error t value
(Intercept)           79.687500   0.821592   96.99
ProgramWI              1.360119   1.090554    1.25
ProgramXCONT           0.062500   1.102281    0.06
Timef4                 0.875000   0.386793    2.26
Timef6                 1.125000   0.386793    2.91
Timef8                 1.312500   0.386793    3.39
Timef10                1.562500   0.386793    4.04
Timef12                1.437500   0.386793    3.72
Timef14                1.437500   0.386793    3.72
ProgramWI:Timef4      -0.255952   0.513417   -0.50
ProgramXCONT:Timef4   -0.675000   0.518938   -1.30
ProgramWI:Timef6      -0.267857   0.513417   -0.52
ProgramXCONT:Timef6   -0.875000   0.518938   -1.69
ProgramWI:Timef8       0.163690   0.513417    0.32
ProgramXCONT:Timef8   -1.012500   0.518938   -1.95
ProgramWI:Timef10      0.008929   0.513417    0.02
ProgramXCONT:Timef10  -1.512500   0.518938   -2.91
ProgramWI:Timef12      0.229167   0.513417    0.45
ProgramXCONT:Timef12  -1.587500   0.518938   -3.06
```

p-value

confint

- $Y = X\beta + Zu + \varepsilon$

$$\begin{pmatrix} u \\ \varepsilon \end{pmatrix} \sim N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} G & 0 \\ 0 & R \end{pmatrix} \right)$$

- Generalized linear models

$Y$
$\downarrow$
broaden