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STAT 8003, Homework 3

Group # 8

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Problem 1. Problem 1. (20 points) X and Y are independent random variables with exponential distributions with expectations and, respectively. Sometimes it is impossible to obtain direct observations of X and Y . Instead, we observe the random variables Z and W, where

$Z = \min(X;Y)$ and $W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$

(This is a situation that arises, in particular, in medical experiments. The X and Y variables are censored).

a) Find the joint distribution of Z and W.

For W=0:

$$\begin{aligned} P(Z \leq z, W = 0) &= P(\min(X,Y) \leq z,\, Y \leq X) \\ &= P(Y \leq z,\, Y \leq X) \\ &= \int_0^z \int_y^\infty \frac{1}{\lambda} e^{\frac{-x}{\lambda}} \frac{1}{\mu} e^{\frac{-y}{\mu}} \,\mathrm{d}x \,\mathrm{d}y \\ &= \frac{1}{\cancel{\mu}} \left(\frac{-\cancel{\mu}\lambda}{\lambda+\mu} \right) e^{-y(\frac{1}{\lambda}+\frac{1}{\mu})} \Big|_{y=0}^{z} \\ &= \frac{\lambda}{\lambda+\mu} \left(1 - e^{-\left(\frac{1}{\lambda}+\frac{1}{\mu}\right) z} \right). \end{aligned}$$

For W=1:

$$\begin{aligned}
P(Z \le z, W=1) &= P(\min(X,Y) \le z,\ X \le Y) \\
&= P(Y \le z,\ X \le Y) \\
&= \int_0^z \int_x^\infty \frac{1}{\lambda} e^{\frac{-x}{\lambda}} \frac{1}{\mu} e^{\frac{-y}{\mu}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^z \frac{1}{\lambda} e^{\frac{-x(\lambda+\mu)}{\lambda\mu}}\,\mathrm{d}x \\
&= \frac{\mu}{\lambda+\mu}(1 - e^{(\frac{-1}{\lambda}+\frac{-1}{\mu})z}).
\end{aligned}$$

b) Prove that Z and W are independent. (Hint: show that $P(Z \le z|W=w) = P(Z \le z)$ for $w = 0$ or 1.)

$$\begin{aligned}
P(W=0) &= P(Y \le X) \\
&= \int_0^\infty \int_x^\infty \frac{1}{\lambda} e^{\frac{-x}{\lambda}} \frac{1}{\mu} e^{\frac{-y}{\mu}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{\lambda}{\lambda+\mu}.
\end{aligned}$$

$$\begin{aligned}
P(W=1) &= 1 - P(W=0) \\
&= \frac{\mu}{\lambda+\mu}.
\end{aligned}$$

$$\begin{aligned}
P(Z \le z) &= P(Z \le z, W=0) + P(Z \le z, W=1) \\
&= 1 - e^{(\frac{-1}{\lambda}+\frac{-1}{\mu})z}.
\end{aligned}$$

$$P(Z \le z, W=i) = P(Z \le z)P(W=i) \quad \text{for } i = 1, 0, \quad z > 0.$$

Z and W are independent.

Problem 2. (20 points) Let X and Y have the joint density function

$$f(x;y) = k(xy); 0 \leq y \leq x \leq 1$$

and 0 elsewhere.

a). Sketch the region over which the density is positive and use it in determining limits of integration to answer the following questions.

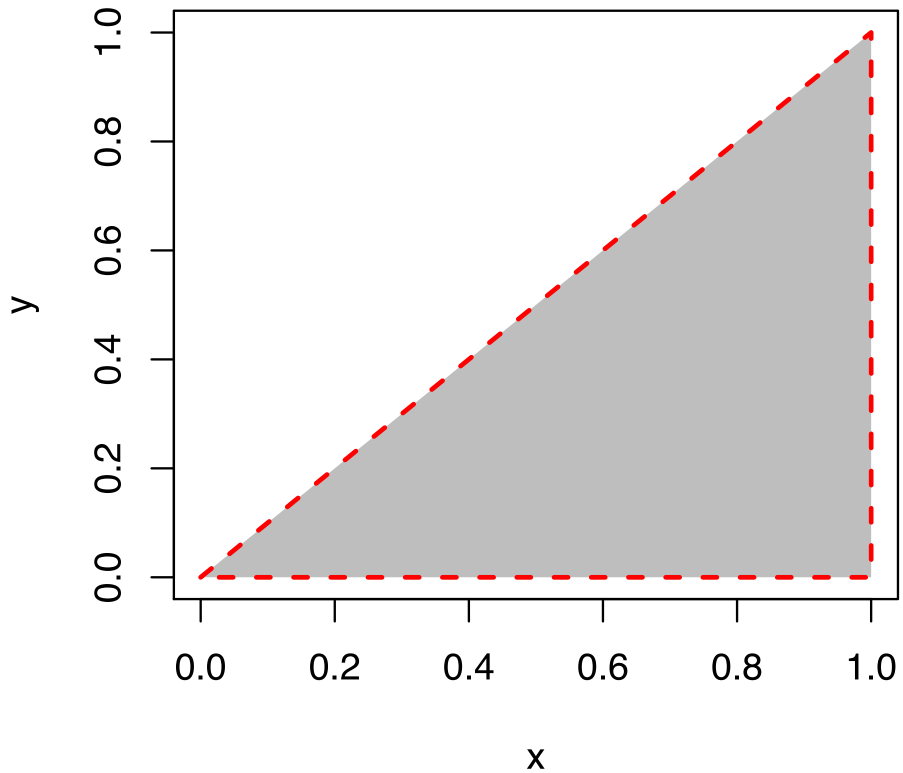

Figure 1: Region over which the density is positive

b). Find k.

We want to chose a k such that the pdf sums to 1 over the area:

$$
\begin{aligned}
1 &= k\int_0^1\int_0^x (x-y)\,\mathrm{d}y\mathrm{d}x \\
&= k\int_0^1 \left[xy - \frac{y^2}{2}\Big|_{y=0}^{y=x} \right] \mathrm{d}x \\
&= k\int_0^1 \left[\frac{x^2}{2}\right] \mathrm{d}x \\
&= k\frac{x^3}{6}\Big|_0^1 \\
&= \frac{k}{6} \\
k &= 6
\end{aligned}
$$

c). Find the marginal densities of X and Y .

$$
\begin{aligned}
f_X(x) &= 6\int_0^x (x-y)\mathrm{d}y \\
&= 6\left(xy - \frac{y^2}{2}\right)\Big|_{y=0}^{x} \\
&= 3x^2 \quad (0 \le x \le 1)
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= 6\int_y^1 (x-y)\mathrm{d}x \\
&= 6\left(\frac{x^2}{2} - xy\right)\Big|_{x=y}^{1} \\
&= k\left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right) \\
&= \frac{6}{2}(y^2 - 2y + 1) \\
&= 3(y-1)^2 \quad (0 \le y \le 1)
\end{aligned}
$$

d). Find the conditional densities of Y given X and X given Y .

$$
\boxed{\text{Conditional density } f(y|X=x) = \frac{\text{joint density } f_{X,Y}(x,y)}{\text{marginal } f_X(x)}}
$$

$$f(y|X=x) = \frac{\cancel{6}(x-y)}{\frac{\cancel{6}x^2}{2}} = \frac{2(x-y)}{x^2} \quad (0 \leq y \leq x \leq 1)$$
$$f(x|Y=y) = \frac{\cancel{6}(x-y)}{\frac{\cancel{6}(y-1)^2}{2}} = \frac{2(x-y)}{(y-1)^2} \quad (0 \leq y \leq x \leq 1)$$

Problem 3. (10 points) A couple decides to continue to have children until a daughter is born. What is expected number of children of this couple?

Let X be the number of children until a daughter is born. Then, X would follow geometric distribution. Pmf of X is:

$$p(X,x) = (1-p)^{x-1}p$$

Expectation of geometric random variable is $E(X) = 1/p$. It can be derived in the following way:

Let q=1-p. Then,

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x(q)^{x-1}p \\
&= p\sum_{x=1}^{\infty} xq^{x-1} \\
&= p\frac{\mathrm{d}}{\mathrm{d}p}\sum_{x=1}^{\infty} q^x \\
&= p\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{q}{1-q}\right) \\
&= \frac{p}{(1-q)^2} \\
&= \frac{1}{p}
\end{aligned}$$

Then, $E(X) = \frac{1}{1/2} = 2$. Expected number of children is 2.

Problem 4 . (20 points) Let X have pdf

$$f(x) = \frac{1}{2}(1+x); 1 < x < 1$$

a). Find the pdf of $Y = X^2$.

$$g(x) = Y = X^2 \qquad\qquad g^{-1}(y) = X = \pm\sqrt{y}$$

Define support sets for x: $A_0 = \{0\}, A_2 = (-1, 0),\ A_1 = (0, 1)$

$$g'(x) = 2x \qquad\qquad g'(x) = \begin{cases} 2x & > 0 \quad \text{increasing on } A_1 \\ 2x & < 0 \quad \text{decreasing on } A_2 \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) = \begin{cases} \frac{1}{2} y^{-\frac{1}{2}} & 0 < y < 1 \quad \text{gives } A_1 \\ -\frac{1}{2} y^{-\frac{1}{2}} & 0 < y < 1 \quad \text{gives } A_2 \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) & g(x) \text{ increasing} \\ -f_X(g^{-1}(y)) \frac{\mathrm{d}}{\mathrm{d}y} g^{-1}(y) & g(x) \text{ decreasing} \end{cases}$$

$$f_Y(y) = \underbrace{\frac{1}{2}(1 + y^{\frac{1}{2}}) \frac{1}{2} y^{-\frac{1}{2}}}_{\text{Contribution of increasing part, } A_1} + \underbrace{\frac{1}{2}(1 - y^{\frac{1}{2}}) \frac{1}{2} y^{-\frac{1}{2}}}_{\text{Contribution of decreasing part, } A_2}$$

$$= \frac{1}{2\sqrt{y}}$$

b). Find $E(Y)$ and $Var(Y)$.

You find $E(Y)$ directly from f_Y above or using $g(x)$ and f_X.

$$E(Y) = \int y f_y \mathrm{d}y$$

$$= \int_0^1 \frac{y \mathrm{d}y}{2\sqrt{y}} = \frac{1}{2} \int_0^1 \sqrt{y} \mathrm{d}y = \frac{1}{\cancel{2}} \frac{\cancel{2}}{3} y^{\frac{3}{2}} \Big|_{y=0}^{1} = \frac{1}{3}$$

$$
\begin{aligned}
E(Y) = E(g(x)) &= \int g(x) f_X \mathrm{d}x \\
&= \int_{-1}^{1} x^2 \frac{1}{2}(1+x)\mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} \left(x^2 + x^3\right) \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^4}{4}\right]\Bigg|_{-1}^{1} = \frac{1}{2}\left[\frac{1}{3} + \cancel{\frac{1}{4}} + \frac{1}{3} - \cancel{\frac{1}{4}}\right] \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y) &= E(Y^2) - (E(Y))^2 \\
&= E((X^2)^2) - \left(\frac{1}{3}\right)^2 \\
&= E(X^4) - \frac{1}{9} \\
&= \int x^4 f(x)\mathrm{d}x - \frac{1}{9} = \frac{1}{2}\int_{-1}^{1}(x^4 + x^5)\mathrm{d}x - \frac{1}{9} \\
&= \frac{1}{2}\left(\frac{x^5}{5} + \frac{x^6}{6}\right)\Bigg|_{x=-1}^{1} - \frac{1}{9} = \frac{1}{2}\left[\frac{1}{5} + \cancel{\frac{1}{6}} + \frac{1}{5} - \cancel{\frac{1}{6}}\right] - \frac{1}{9} = \frac{1}{5} - \frac{1}{9} \\
Var(Y) &= \frac{4}{45}
\end{aligned}
$$

Problem 5. (20 points) Suppose that the random variable Y has a binomial distribution with n trials and success probability X, where n is a given constant and X is a Unif(0; 1) random variable.

a) Find $E(Y)$ and $Var(Y)$.

$$
\begin{aligned}
E(Y) &= E[E(Y|X=x)] \\
&= E[nX] \\
&= nE[X] = \frac{1}{2}n
\end{aligned}
$$

To find $Var(Y)$ we used $Var(Y) = E[Var(Y|X = x)] + Var[E(Y|X = x)]$

$$\begin{aligned} E[Var(Y|X = x)] &= E[nX(1 - X)] \\ &= nE[X(1 - X)] \\ &= n\int_0^1 x(1 - x)\mathrm{d}x \\ &= \frac{1}{6}n \end{aligned}$$

$$\begin{aligned} Var[E(Y|X = x)] &= Var(nX) \\ &= n^2 Var(X) \\ &= n^2\int_0^1 (x - \frac{1}{2})^2\mathrm{d}x \\ &= \frac{1}{12}n^2 \end{aligned}$$

Plugging to solve for $Var(Y)$:

$$\begin{aligned} Var(Y) &= E[Var(Y|X = x)] + Var[E(Y|X = x)] \\ &= \frac{1}{6}n + \frac{1}{12}n^2 \end{aligned}$$

b) Find the joint distribution of X and Y.

$$P(X, Y) = P(Y|X = x)f(X) = \binom{n}{y}x^y(1 - x)^{n-y} \quad (y = 0, 1, 2 \cdots n, x \in (0, 1))$$

c) Find the marginal distribution of Y .

$$P(Y = y) = \binom{n}{y}\int_0^1 x^y(1 - x)^{n-y}\mathrm{d}x$$

Using definition of $\Gamma y = (y - 1)!$

$$\binom{n}{y} = \frac{n!}{y!(n - y)!} = \frac{\Gamma(n + 1)}{\Gamma(y + 1)\Gamma(n - y + 1)}$$

Beta density pdf is $g(u) = \frac{\Gamma(\alpha+\beta)}{\Gamma\alpha\Gamma\beta}u^{\alpha-1}(1 - u)^{\beta-1}$, $0 \le u \le 1$

Beta density integrates to 1 with leading $\frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)}$

$$\int_0^1 u^{\alpha-1}(1 - u)^{\beta-1}\mathrm{d}u = \frac{\Gamma(\alpha + \beta)}{\Gamma\alpha\Gamma\beta}$$

Do substitution $u = x; (\alpha - 1) = y; (\beta - 1) = (n - y)$

$$\begin{aligned}
P(Y = y) &= \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \int_0^1 x^y (1-x)^{n-y} \mathrm{d}x \\
&= \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \\
&= \frac{1}{n+1}
\end{aligned}$$

This means that each outcome of Y is equally likely if $X \sim$ Uniform distribution.

Problem 6. (10 points) Let X, Y and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2 and σ_Z^2, respectively. Let

$$\begin{aligned}
U &= Z + X \\
V &= Z + Y
\end{aligned}$$

Find $Cov(U, V)$ and $\rho(U, V)$. Note that $\rho(U, V)$ is defined as

$$\rho(U, V) = \frac{Cov(U; V)}{\sqrt{Var(U)Var(V)}}$$

$$\begin{aligned}
E(UV) &= E[(Z + X)(Z + Y)] \\
&= E(Z^2 + XZ + XY + ZY) \\
&= E(Z^2) + E(XZ) + E(XY) + E(ZY)
\end{aligned}$$

Since X,Y,Z are uncorrelated,

$$\begin{aligned}
E(XZ) &= E(X)E(Z) \\
E(XY) &= E(X)E(Y) \\
E(ZY) &= E(Z)E(Y)
\end{aligned}$$

Then,

$$E(UV) = E(Z^2) + E(X)E(Z) + E(X)E(Y) + E(Z)E(Y)$$

$$
\begin{aligned}
E(U)E(V) &= E(Z+X)E(Z+Y) \\
&= [E(Z)+E(X)][E(Z)+E(Y)] \\
&= E^2(Z)+E(X)E(Z)+E(X)E(Y)+E(Z)E(Y)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Cov(U,V) &= E(UV)-E(U)E(V) \\
&= E(Z^2)-E^2(Z) \\
&= Var(Z) \\
&= \sigma_Z^2
\end{aligned}
$$

$$
\begin{aligned}
\rho(U,V) &= \frac{Cov(U,V)}{\sqrt{Var(U)Var(V)}} \\
&= \frac{Cov(U,V)}{\sqrt{Var(Z+X)Var(Z+Y)}}
\end{aligned}
$$

Since X,Y,Z are uncorrelated,

$$
\begin{aligned}
Var(Z+X) &= Var(Z)+Var(X) \\
Var(Z+Y) &= Var(Z)+Var(Y)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\rho(U,V) &= \frac{Cov(U,V)}{\sqrt{[Var(Z)+Var(X)][Var(Z)+Var(Y)]}} \\
&= \frac{\sigma_z^2}{\sqrt{[(\sigma_Z^2+\sigma_X^2)(\sigma_Z^2+\sigma_Y^2)]}}
\end{aligned}
$$