# STAT 8003
# Homework 4

Group L

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## 1 Question 1

### (a)

$$
\begin{aligned}
&EX = \theta + 2(1-\theta) = 2 - \theta \\
&m_1 = EX = 2 - \theta \\
&\hat{\theta} = 2 - m_1
\end{aligned}
$$

We have:

$$m_1 = \frac{\sum_{i=1}^{3} x_i}{3} = \frac{1+2+2}{3} = \frac{5}{3}$$

Therefore,

$$\hat{\theta} = \frac{1}{3}$$

### (b)

let $n$ be the total number of independent observations of $X$, $k$ be the number of independent observations of $x_i = 1$, then:

$$
\begin{aligned}
L_n(\theta) &= \prod_{i=1}^{n} f(x_i, \theta) \\
&= \prod_{i=1}^{k} f(1, \theta) \prod_{i=1}^{n-k} f(2, \theta) \\
&= \theta^k (1-\theta)^{n-k}
\end{aligned}
$$

We have $n = 3, k = 1$, therefore

$$L_3(\theta) = \theta(1-\theta)^2$$

### (c)

From (b), we have:

$$\begin{aligned} L_3(\theta) &= \theta(1-\theta)^2 \\ &= \theta^3 - 2\theta^2 + \theta \\ \frac{dL}{d\theta} &= 3\theta^2 - 4\theta + 1 \\ &= (3\theta - 1)(\theta - 1) \end{aligned}$$

When $\frac{dL}{d\theta} = 0$, we will have the MLE of $\theta$. Since $\theta \neq 0$,

$$\hat{\theta} = \frac{1}{3}$$

## 2 Question 2

### (a)

$$\begin{aligned} EX &= \frac{1}{2\sigma}\int_{-\infty}^{+\infty} xe^{-\frac{|x|}{\sigma}}dx \\ &= \frac{1}{2\sigma}[\int_{-\infty}^{0} xe^{\frac{x}{\sigma}}dx + \int_{0}^{+\infty} xe^{-\frac{x}{\sigma}}dx] \\ &= \frac{1}{2\sigma}[\int_{0}^{+\infty} (-1)(-x)e^{-\frac{x}{\sigma}}d(-x) + \int_{0}^{+\infty} xe^{-\frac{x}{\sigma}}dx] \\ &= \frac{1}{2\sigma}[-\int_{0}^{+\infty} xe^{-\frac{x}{\sigma}}dx + \int_{0}^{+\infty} xe^{-\frac{x}{\sigma}}dx] \\ &= 0 \end{aligned}$$

Therefore, we need to calculate $EX^2$ to estimate $\sigma$.

$$\begin{aligned} EX^2 &= \frac{1}{2\sigma}\int_{-\infty}^{+\infty} x^2e^{-\frac{|x|}{\sigma}}dx \\ &= \frac{1}{2\sigma}[\int_{-\infty}^{0} x^2e^{\frac{x}{\sigma}}dx + \int_{0}^{+\infty} x^2e^{-\frac{x}{\sigma}}dx] \\ &= \frac{1}{2\sigma}[\int_{0}^{+\infty} (-1)(-x)^2e^{-\frac{x}{\sigma}}d(-x) + \int_{0}^{+\infty} x^2e^{-\frac{x}{\sigma}}dx] \\ &= \frac{1}{\sigma}\int_{0}^{+\infty} x^2e^{-\frac{x}{\sigma}}dx \\ &= \sigma^2\int_{0}^{+\infty} (\frac{x}{\sigma})^2e^{-\frac{x}{\sigma}}d(\frac{x}{\sigma}) \\ &= \sigma^2\Gamma(3) \\ &= 2\sigma^2 \end{aligned}$$

Therefore, we have

$$m_2 = EX^2 = 2\sigma^2$$

since $\sigma > 0$,

$$\hat{\sigma} = \sqrt{\frac{m_2}{2}} = \sqrt{\frac{\sum_{i=1}^{n} {x_i}^2}{2n}}$$

### (b)

$$L_n(\sigma) = (\frac{1}{2\sigma})^n e^{(-\frac{\sum_{i=1}^{n}|x_i|}{\sigma})}$$

$$l_n(\sigma) = -nlog(2\sigma) - \frac{\sum_{i=1}^{n}|x_i|}{\sigma}$$

$$\frac{dl}{d\sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}|x_i|}{\sigma^2}$$

when $\frac{dl}{d\sigma} = 0$, we will have the MLE of $\sigma$. Therefore,

$$\hat{\sigma} = \frac{\sum_{i=1}^{n}|x_i|}{n}$$

## 3 Question 3

### (a)

$$\begin{aligned}
f(x_i, p_i) &= \binom{2}{x_i}(\frac{e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}})^{x_i}(\frac{1}{(1+e^{\beta_0+\beta_1 t_i})^{2-x_i}}) \\
&= \binom{2}{x_i}\frac{(e^{\beta_0+\beta_1 t_i})^{x_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} \\
L_n(\beta_0, \beta_1) &= \prod_{i=1}^{n}\binom{2}{x_i}\prod_{i=1}^{n}\frac{(e^{\beta_0+\beta_1 t_i})^{x_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} \\
l_n(\beta_0, \beta_1) &= log\prod_{i=1}^{n}\binom{2}{x_i} + \sum_{i=1}^{n}[x_i(\beta_0+\beta_1 t_i) - 2log(1+e^{\beta_0+\beta_1 t_i})]
\end{aligned}$$

**(b)**

$$\frac{\partial l}{\partial \beta_0} = \sum_{i=1}^{n}(x_i - 2\frac{e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}})$$

$$\frac{\partial l}{\partial \beta_1} = \sum_{i=1}^{n}(x_i t_i - 2\frac{t_i e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}})$$

The equations for the maximum likelihood estimator of $\beta_0$ and $\beta_1$ are:

$$\begin{cases} \sum_{i=1}^{n}(x_i - 2\frac{e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) & = 0 \\ \sum_{i=1}^{n}(x_i t_i - 2\frac{t_i e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) & = 0 \end{cases}$$

**(c)**

Step 1:
Set the initial values for $\beta_0, \beta_1$ as 0, 0.

Step 2:
From (b), we have two functions for $\beta_0$ and $\beta_1$:

$$\begin{cases} f_1(\beta_0, \beta_1) & = \sum_{i=1}^{n}(x_i - 2\frac{e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) = 0 \\ f_2(\beta_0, \beta_1) & = \sum_{i=1}^{n}(x_i t_i - 2\frac{t_i e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) = 0 \end{cases}$$

Therefore, we have

$$\frac{\partial f_1}{\partial \beta_0} = -\sum_{i=1}^{n}\frac{2e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2}$$

$$\frac{\partial f_1}{\partial \beta_1} = -\sum_{i=1}^{n}\frac{2t_i e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2}$$

$$\frac{\partial f_2}{\partial \beta_0} = -\sum_{i=1}^{n}\frac{2t_i e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2}$$

$$\frac{\partial f_2}{\partial \beta_1} = -\sum_{i=1}^{n}\frac{2{t_i}^2 e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2}$$

Then the update for each step is:

$$\begin{pmatrix} \beta_0^{i+1} \\ \beta_1^{i+1} \end{pmatrix} = \begin{pmatrix} \beta_0^{i} \\ \beta_1^{i} \end{pmatrix} - \begin{pmatrix} -\sum_{i=1}^{n}\frac{2e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} & -\sum_{i=1}^{n}\frac{2t_i e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} \\ -\sum_{i=1}^{n}\frac{2t_i e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} & -\sum_{i=1}^{n}\frac{2{t_i}^2 e^{\beta_0+\beta_1 t_i}}{(1+e^{\beta_0+\beta_1 t_i})^2} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{n}(x_i - 2\frac{e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) \\ \sum_{i=1}^{n}(x_i t_i - 2\frac{t_i e^{\beta_0+\beta_1 t_i}}{1+e^{\beta_0+\beta_1 t_i}}) \end{pmatrix}$$

Step 3:
calculate $\Delta$ as:

$$\Delta = \sqrt{(\beta_0^{i+1} - \beta_0^i)^2 - (\beta_1^{i+1} - \beta_1^i)^2}$$

Step 4:
In question 3(d), we will set the criteria for $\Delta$ as 0.001. If $\Delta > 0.001$, return to Step 2; If $\Delta < 0.001$, the iteration will be stopped and $\begin{pmatrix} \beta_0^{i+1} \\ \beta_1^{i+1} \end{pmatrix}$ will be used as the solution of the equation.

## (d)

The code of R for the Newton-Raphson algorithm is:

```
> shuttle <- read.csv("http://astro.temple.edu/~zhaozhg/Stat8003/data/shuttle.txt")
>
> beta0 <- 0
> beta1 <- 0
>
> beta.old <- matrix( c(beta0, beta1), 2, 1)
> beta.new <- beta.old
> delta <- 0.001
> Delta <- 1
> itr <- 1
>
> while(Delta > delta)
+ {
+   beta.old <- beta.new
+   a<- -2*exp(beta.old[1]+beta.old[2]*shuttle$temp)/(1+exp(beta.old[1]+beta.old[2]*shuttle$temp))^2
+   b<- -2*shuttle$temp*exp(beta.old[1]+beta.old[2]*shuttle$temp)/(1+exp(beta.old[1]+beta.old[2]*shuttle$temp))^2
+   c <- -2*(shuttle$temp)^2*exp(beta.old[1]+beta.old[2]*shuttle$temp)/(1+exp(beta.old[1]+beta.old[2]*shuttle$temp))^2
+
+   jacobian <- matrix( c(sum(a), sum(b), sum(b), sum( c)), 2, 2)
+
+   d <- shuttle$ndo - 2*exp(beta.old[1]+beta.old[2]*shuttle$temp)/(1+exp(beta.old[1]+beta.old[2]*shuttle$temp))
+   e <- shuttle$ndo*shuttle$temp - 2*shuttle$temp*exp(beta.old[1]+beta.old[2]*shuttle$temp)/(1+exp(beta.old[1]+beta.old[2]*shuttle$temp))
+
+   f.value <- matrix( c(sum(d), sum(e)), 2, 1)
+   f<- solve(jacobian) %*% f.value
+   beta.new <- beta.old - solve(jacobian) %*% f.value
+   Delta <- (((solve(jacobian) %*% f.value)[1])^2+((solve(jacobian) %*% f.value)[2]^2))
+
+   print( paste("iter:", itr,", beta0=", beta.new[1],", beta1=",beta.new[2], sep=" "))
+
+   itr <- itr+1
+ }
[1] "iter: 1 , beta0= 5.39682539682542 , beta1= -0.0950793650793653"
[1] "iter: 2 , beta0= 8.21951434646164 , beta1= -0.141181894622337"
[1] "iter: 3 , beta0= 8.97606878392621 , beta1= -0.15355498556895"
[1] "iter: 4 , beta0= 9.02103253729767 , beta1= -0.154293608953509"
[1] "iter: 5 , beta0= 9.02118524876056 , beta1= -0.154296124762872"
>
>
```

Therefore, $\hat{\beta}_0 \approx 9.021185$; $\hat{\beta}_1 \approx -0.154296$

## (e)

From question (d), we have:

$$\begin{aligned} p &= \frac{e^{\beta_0 + \beta_1 t_i}}{1 + e^{\beta_0 + \beta_1 t_i}} \\ &= \frac{e^{9.021185 - 0.154296 * 31}}{1 + e^{9.021185 - 0.154296 * 31}} \\ &\approx 0.9858 \end{aligned}$$

Therefore, the probability that an o-ring will be damaged is:

$$\begin{aligned} P(X=1) &= \binom{2}{1} p(1-p) \\ &\approx 0.028 \end{aligned}$$

## (f)

The code of R for the probability $p$ against the temperature by letting temperature go from 30 degrees to 90 degrees is:

```
ti= c(30:90)

p_i = exp(beta.new[1]+beta.new[2]*ti)/(1+exp(beta.new[1]+beta.new[2]*ti))

plot(ti,p_i,
     'l',
     ylab = 'o-ring damage probability',
     xlab = 'temperature (degree)',
     main = 'o-ring damage probability v.s. temperature')
```

The plot is: