8004 Homework 4

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1 Problem 1 In the context of Problem 2 of Homework Assignment 3, use R matrix calculations to do the following in the (non-full-rank) Gauss-Markov normal linear model

(a) Find 90% two-sided confidence limits for σ.

(a).1 Background

The model described in HW3, Problem 2 in $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ matrix form is:

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{42} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 6 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{21} \\ \epsilon_{31} \\ \epsilon_{41} \\ \epsilon_{42} \end{pmatrix}$$

Also, we are given that var(ϵ)= $\mathbf{V}$, for $\mathbf{V}_1$ = diag(1,9,9,1,1,9) and $\mathbf{V_2} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 9 \end{pmatrix}$.

We have $\mathbf{Y} \sim N_n(\mathbf{X}\beta, \sigma^2\mathbf{V})$. To find a suitable estimator for σ^2, first transform the Generalized Least Squares model into an Ordinary Least Squares model by multiplying by $\mathbf{V}^{-1/2}$. This gives $\mathbf{U} + \mathbf{W}\beta = \epsilon^\star$, where $\mathbf{U} = \mathbf{V}^{-1/2}$ $\mathbf{Y}$, $\mathbf{W} = \mathbf{V}^{-1/2}$ $\mathbf{X}$, and $\epsilon^\star = \mathbf{V}^{-1/2}$ ϵ. Note that $\mathbf{U} \sim N_n(\mathbf{W}\beta, \sigma^2\mathbf{I})$.

Now find an estimator for σ^2 for use in construction of the confidence interval using the variance of $\mathbf{U}$. var($\mathbf{U}$) = σ^2 $\mathbf{I}$ = E($\mathbf{U}$ - E($\mathbf{U}$))^2 = E($\mathbf{U}$ - $\mathbf{WB}$)^2. First observe the distribution of $\mathbf{U} - \hat{\mathbf{U}} \sim$ N$_n$($\mathbf{0}$, σ^2 $\mathbf{I}$). Consider

$$\frac{SSE}{\sigma^2} = \frac{(\mathbf{U}-\hat{\mathbf{U}})'(\mathbf{U}-\hat{\mathbf{U}})}{\sigma^2} = \frac{1}{\sigma^2}((\mathbf{I}-\mathbf{P_W})\mathbf{U})'((\mathbf{I}-\mathbf{P_W})\mathbf{U}) = \frac{1}{\sigma^2}\mathbf{U}'(\mathbf{I}-\mathbf{P_W})\mathbf{U}$$

Note that the product of $\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{P_W})$ and $cov(\mathbf{U}) = \sigma^2\mathbf{I}$ is $\mathbf{U}-\hat{\mathbf{U}}$ is $\frac{1}{\sigma^2}(\mathbf{I}-\mathbf{P_W})\sigma^2\mathbf{I} = (\mathbf{I}-\mathbf{P_W})$. The result is a projection matrix orthogonal to C($\mathbf{W}$). It is also idempotent, a property of all projection matrices which can also be shown: ($\mathbf{I}$ - $\mathbf{P}_W$)($\mathbf{I}$ - $\mathbf{P}_W$) = $\mathbf{I}$ - $\mathbf{I}$ $\mathbf{P}_W$ - $\mathbf{P}_W$ $\mathbf{I}$ + $\mathbf{P}_W$ $\mathbf{P}_W$ = $\mathbf{I}$ - $\mathbf{P}_W$. Further rank($\mathbf{I}$-$\mathbf{P}_W$)=n-rank($\mathbf{W}$)

The following theorem applies to the quadratic form $\frac{1}{\sigma^2}\mathbf{U}'(\mathbf{I}-\mathbf{P_W})\mathbf{U}$ and shows that it is distributed $\chi^2((n-rank(*W*))$.

Theorem 1.1. *Let **y** be distributed $N_p(\mu, \Sigma)$, **A** be a symmetric matric of constants, rank(**A**)=r, and define $\lambda = \frac{1}{2}\mu'\mathbf{A}\mu$. Then, **y'Ay** follows $\chi^2(r,\lambda)$ if and only if $\mathbf{A}\Sigma$ is idempotent.*

Here, $\mathbf{y} = \mathbf{U}$, $\mu = \mathbf{W}\beta$, $\Sigma = \sigma^2$ $\mathbf{I}$, $\mathbf{A} = \frac{1}{\sigma^2}(\mathbf{I}-\mathbf{P_W})$, and $\lambda = \frac{1}{2\sigma^2}\beta'\mathbf{W}'(\mathbf{I}-\mathbf{P_W})\mathbf{W}\beta = \mathbf{0}$.
To find two-sided 90% confidence limits for σ^2, we note SSE = $\mathbf{U'(I\text{-}P}_W\mathbf{)U}$ and write:
1 - α = P(lower $\frac{\alpha}{2}$ quantile of χ^2(n-rank($\mathbf{W}$)) < $\frac{SSE}{\sigma^2}$ < upper $\frac{\alpha}{2}$ quantile of χ^2(n-rank($\mathbf{W}$)))
.90 = P(lower .05 quantile of χ^2(n-rank($\mathbf{W}$)) < $\frac{SSE}{\sigma^2}$ < upper .05 quantile of χ^2(n-rank($\mathbf{W}$)))
Solving for an interval for σ^2, we have:
.90 = P($\frac{SSE}{\text{upper .05 quantile of }\chi^2(n\text{−rank}(\mathbf{W}))} < \sigma^2 < \frac{SSE}{\text{lower .05 quantile of }\chi^2(n\text{−rank}(\mathbf{W}))}$))

(a).2 Interval for σ^2 using $\mathbf{V}_1$

```
#Find V^(-1/2)
Vh1 <-solve(V1^(1/2))

#Transform model to OLS
U <- Vh1 %*% Y
W <- Vh1 %*% X

Uhat <- W %*% ginv(t(W) %*% W) %*% t(W) %*% U

SSE <- t(U-Uhat) %*% (U-Uhat)

qr(W)$rank

lowerchi <- qchisq(.05, df=4)
upperchi <- qchisq(.95, df=4)

SSE/lowerchi
SSE/upperchi
```

For the covariance matrix $\mathbf{V}_1$ given in HW3 problem 2, we found an SSE of `0.5` and two-sided 90% confidence limits for σ^2 of $0.0527 < \sigma^2 < 0.7035$.

(a).3 Interval for σ^2 using $\mathbf{V}_2$

```
#Find V^(-1/2) using spectral decompostion
Vh2 <-solve(eigen(V2)$vectors %*% diag(sqrt(eigen(V2)$values)) %*% t(eigen(V2)$vectors))

#Transform model to OLS
U <- Vh2 %*% Y
W <- Vh2 %*% X

Uhat <- W %*% ginv(t(W) %*% W) %*% t(W) %*% U

SSE <- t(U-Uhat) %*% (U-Uhat)

qr(W)$rank

lowerchi <- qchisq(.05, df=4)
upperchi <- qchisq(.95, df=4)
```

For the covariance matrix $\mathbf{V}_2$ given in HW3 problem 2, we found an SSE of `0.458333333333332` and two-sided 90% confidence limits for σ^2 of $0.0483 < \sigma^2 < 0.6449$.

- Find 90% two-sided confidence limits for $\mu + \tau_2$.
- Find 90% two-sided confidence limits for τ_1 - τ_2.
- Find a p-value for testing the null hypothesis $H_0 : \tau_1 - \tau_2 = 0$ vs H_a : not H_0.
- Find 90% two-sided predition limits for the sample mean of /n/=10 future observations from the first set of conditions.

- Find 90% two-sided prediction limints for the difference between a pair of future values, one from the first set of conditions (i.e. with mean $\mu + \tau_1$) and one from the second set of conditions (i.e. with mean $\mu + \tau_2$).

- Find a p-value for testing the following: What is the practical interpretation of this test?

$$H_0 : \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- Find a p-value for testing:

$$H_0 : \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}.$$

2 Problem 2 In the following make use of the data in Problem 4 of Homework Assignment 3. Consider a regression of *y* on $x_1, x_2, \ldots, x_5$. Use R matrix calculations to do the following in a full rank Gauss-Markov normal linear model.

(a) Find 90% two-sided condifence limits for σ.

(b) Find 90% two-sided condifence limits for the mean response under the conditions of data point #1.

(c) Find 90% two-sided condifence limits for the difference in mean responses under the conditions of data points #1 and #2. .

(d) Find a *p*-value for testing the hypothesis that the conditions of data points #1 and #2 produce the same mean response.

(e) Find 90% two-sided prediction limits for an additional response for the set of conditions $x1 = 0.005, x2 = 0.45, x3 = 7, x4 = 45,$ and $x_5 = 6$.

(f) Find a *p*-value for testing the hypothesis that a model including only x_1, x_3, and x_5 is adequare for "explaining" home price. (Hint: write it in the form of $H_0 : C\beta = 0$).

3 Problem 3

(a) In the context of Problem 1, part g), suppose that in fact $\tau_1 = \tau_2$, $\tau_3 = \tau_4 = \tau_1$ - dσ. What is the distribution of the F statistic?

(b) Use R to plot the power of the $\alpha = 0.05$ level test as a function of *d* for *d* ∈ [-5,5], that is plotting *P* (F > the cut-off value) against *d*. The R function pf(q,df1,df2,ncp) will compute cumulative (non-central) F probabilities for you corresponding to the value q, for degrees of freedom df1 and df2 when the noncentrality parameter is ncp.

4 Appendix: Tangled R code

```
library(MASS); library(xtable)
  lvector <- function(x, dig = 2, dsply=rep("f",ncol(x)+1)) {
   x <- xtable(x, align=rep("",ncol(x)+1),display=dsply,digits=dig) # We repeat empty string 6 time
   print(x, floating=FALSE, tabular.environment="pmatrix",
     hline.after=NULL, include.rownames=FALSE, include.colnames=FALSE)
   }

#Variables from Problem 2 of HW3:
  V1 <- diag(c(1,9,9,1,1,9))
  Y <- matrix(c(2, 1, 4, 6, 3, 5), nrow=6, ncol=1)
  X <- matrix(c(rep(1,6),
                1,1,0,0,0,0,
                0,0,1,0,0,0,
                0,0,0,1,0,0,
                0,0,0,0,1,1),nrow = 6,byrow=FALSE)

  V2 <- diag(c(1,9,9,1,1,9))
  V2[1,2] <- 1
  V2[2,1] <- 1
  V2[4,3] <- -1
  V2[3,4] <- -1
  V2[6,5] <- -1
  V2[5,6] <- -1
```