STAT 8003
Homework 6

Group L

October 9, 2014

1 Question 1

1.a

Let Y be the the total number of heads.
According to the question, $Y \sim Bin(n, \tau)$, where $n = 10$.

$H_o : \tau = \frac{1}{2}, \qquad H_1 : \tau \neq \frac{1}{2}$

The rejection rule is $R = \{Y : Y = 0, Y = 10\}$.

$$\begin{aligned}\alpha &= P(R|H_0)\\ &= P(Y = 0, Y = 10|\tau = \frac{1}{2})\\ &= P(Y = 0|\tau = \frac{1}{2}) + P(Y = 10|\tau = \frac{1}{2})\\ &= (1 - \frac{1}{2})^{10} + (\frac{1}{2})^{10}\\ &\approx 0.002\end{aligned}$$

Thus, the significant level of the test is about 0.002.

1.b

$$\begin{aligned}power &= 1 - \beta\\ &= 1 - P(R^c|H_1)\\ &= 1 - P(Y = 1, 2, ..., 9|\tau = 0.1)\\ &= P(Y = 0, Y = 10|\tau = 0.1)\\ &= P(Y = 0|\tau = 0.1) + P(Y = 10|\tau = 0.1)\\ &= (1 - 0.1)^{10} + 0.1^{10}\\ &\approx 0.3487\end{aligned}$$

The power of the test is about 0.3487.

2 Question 2

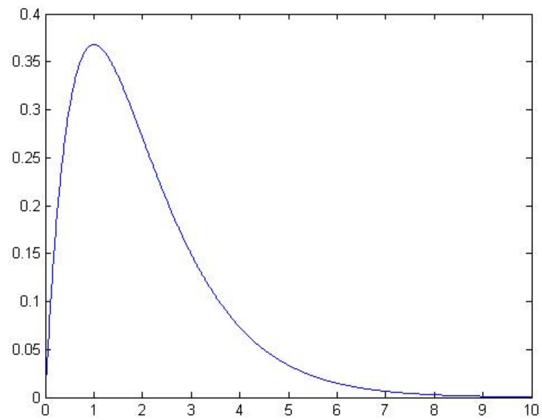
2.a

According to the question, $X_1, X_2, ..., X_n \sim Exp(\theta)$,
thus, the likelihood function is $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \theta^n exp(-\sum_{i=1}^{n} \theta x_i)$
According to MLE, the maximum $\hat{\theta} = \frac{1}{\bar{X}}$

$$\begin{aligned}
\wedge &= \frac{max_{\theta \in \Theta_0 \cup \Theta_1} L(\theta)}{max_{\theta \in \Theta_0} L(\theta)} \\
&= \frac{L(\frac{1}{\bar{X}})}{L(1)} \\
&= \frac{(\frac{1}{\bar{X}})^n exp(-\frac{1}{\bar{X}} \sum_{i=1}^{n} x_i)}{exp(-\sum_{i=1}^{n} x_i)} \\
&= (\frac{1}{\bar{X}})^n exp(\sum_{i=1}^{n} X_i(1 - \frac{1}{\bar{X}})) \\
&= (\frac{1}{\bar{X}})^n exp(n\bar{X}(1 - \frac{1}{\bar{X}})) \\
&= (\frac{1}{\bar{X}})^n exp(n\bar{X}) exp(-n) \geq k^* \\
&\Rightarrow (\frac{1}{\bar{X}})^n exp(n\bar{X}) \geq k^* exp(n) \\
&\Rightarrow (\bar{X})^n exp^n(-\bar{X}) \leq \frac{1}{k^*} exp(-n) \\
&\Rightarrow \bar{X} exp(-\bar{X}) \leq (k^*)^{\frac{-1}{n}} exp(-1) = c
\end{aligned}$$

Thus, the rejection region is of the form $R = \{\bar{X} exp(-\bar{X}) \leq c\}$, where $c = (k^*)^{\frac{-1}{n}} exp(-1)$.

2.b



The plot of $\bar{X}exp(-\bar{X})$ shows above.
Let $g(x) = \bar{X}exp(-\bar{X})$. According to the plot, we know that when $\bar{X}$ at a certain value, $g(x)$ reaches maximum.
Take a derivative of $g(x)$

$$g(x)' = -\bar{X}exp(-\bar{X}) + exp(\bar{X}) = \begin{cases} \geq 0, & \bar{X} \leq 1 \\ < 0, & \bar{X} > 1 \end{cases}$$
$$maxg(x) = exp(-1)$$

$g(x)$ is monotonic increasing when $\bar{X} < 1$, and monotonic decreasing when $\bar{X} > 1$.

Also,

$$n \geq 0, k^* \geq 1$$
$$(k^*)^{\frac{-1}{n}} \leq 1$$
$$c = (k^*)^{\frac{-1}{n}} exp(-1) \leq exp(-1) = maxg(x)$$

Thus, there exsit and only exist two x values, x_0 and x_1. Reject regoin is $R = \{\bar{X} \leq x_0\} \cup \{\bar{X} \geq x_1\}$, where $x_0 exp(-x_0) = x_1 exp(-x_1) = c$.

2.c

Since $\theta = 1$, $\sum_{i=1}^{n} x_i \sim$ Gamma$(n, 1)$, we know that $\bar{X} \sim$ Gamma$(n, \frac{1}{n})$.
Let F be the cdf of Gamma$(n, \frac{1}{n})$,we can get x_0 and x_1 values by solving the following equations:

$$F(x_0) + 1 - F(x_1) = \alpha = 0.05 \tag{1}$$
$$x_0 exp(-x_0) = x_1 exp(-x_1) \tag{2}$$

Take a logarithm of equation (2), we have

$$logx_0 - x_0 = logx_1 - x_1 \tag{3}$$

Solving the equation (1) and (3), we can get x_0 and x_1 value.
Once we have x_0 and x_1, we can get c value with $c = x_0 exp(-x_0) = x_1 exp(-x_1)$.

3 Question 3

3.a

According to the sample data, we can calculate sample mean:

$$\bar{X} = \frac{\sum_{i=1}^{n} x_i}{n} \approx 1.595$$

Because $\bar{X} \leq 1$, let $\bar{X} = x_1$,

$$\begin{aligned}
x_1 &= \bar{X} = 1.595 \\
c &= 1.595exp(-1.595) = 0.324 \\
c &= x_0 exp(-x_0) = 0.324 \\
\Rightarrow x_0 &= 0.577
\end{aligned}$$

$$\begin{aligned}
p - value &= P(\bar{X} \leq 0.577) + P(\bar{X} \geq 1.595) \\
&= F(0.577) + 1 - F(1.599) \\
&= 0.113
\end{aligned}$$

Because $p - value = 0.113 > \alpha = 0.05$, we fail to reject H_0.

3.b

$$\begin{aligned}
2log\wedge &= 2log[(\frac{1}{\bar{X}})^n exp(n\bar{X})exp(-n)] \\
&= 2[(-n)log(\bar{X}) + n\bar{X} - n] \\
&= 2[(-10) \times log(1.595) + 10 \times 1.595 - 10] \\
&\approx 2.5625
\end{aligned}$$

Under H_o, $2log\wedge = \chi^2_1$

$$\begin{aligned}
p - value &= P(\chi^2_1 > 2.5625) \\
&\approx 0.109
\end{aligned}$$

Because $p - value = 0.109 > \alpha = 0.05$, we fail to reject H_0.