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STAT 8003, Homework 6

Group # 8
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Problem 1. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is 1/2 versus the alternative that the probability is not 1/2. The test rejects if either 0 or 10 heads are observed.

a). What is the significance level of the test?

X is random variable denoting the number of heads.
$X \sim$ binomial (10, p), where p denote the probability of heads.
$H_0 : p = 0.5$
$H_A : p \neq 0.5$

$$\begin{aligned}\alpha &= P(\text{Reject } H_0 | H_0 \text{ is true}) \\ &= P(X = 0 | p = 0.5) + P(X = 10 | p = 0.5) \\ &= \binom{10}{0} 0.5^0 (1 - 0.5)^{10-0} + \binom{10}{10} 0.5^{10} (1 - 0.5)^{10-10}\end{aligned}$$

Calculate in R, we have $\alpha = 0.002$.

b). If in fact the probability of heads is .1, what is the power of the test?

Since the probability of heads is 0.1, we change our hypothesis to
$H_0 : p = 0.5$
$H_A : p = 0.1$

$$
\begin{aligned}
1-\beta &= P(\text{Reject } H_0 | H_A \text{ is true}) \\
&= P(X=0|p=0.1) + P(X=10|p=0.1) \\
&= \binom{10}{0} 0.1^0 (1-0.1)^{10-0} + \binom{10}{10} 0.1^{10} (1-0.1)^{10-10}
\end{aligned}
$$

Calculate in R, we have $1-\beta = 0.3487$

Problem 2. Suppose that $X \sim Bin(100; p)$. Consider the test that rejects $H_0 : p = .5$ in favor of $H_A : p \neq .5$ for $|X - 50| > 10$. Use the normal approximation to the binomial distribution to answer the following:

a). What is α?

$H_0 : p = 0.5$
$H_A : p \neq 0.5$

Since a binomial random variable is the sum of independent Bernoulli random variables, its distribution can be approximated by a normal distribution.

$$
E(X) = np = 100 * 0.5 = 50
$$

$$
Var(X) = np(1-p) = 100 * 0.5 * 0.5 = 25
$$

$$
\begin{aligned}
\alpha &= P(\text{Reject } H_0 | H_0 \text{ is true}) \\
&= P((|X-50| > 10)|p=0.5) \\
&= P(\frac{|X-50|}{5} > \frac{10}{5})|p=0.5) \\
&\approx \Phi(-2) + 1 - \Phi(2)
\end{aligned}
$$

Calculate in R, we have $\alpha = 0.0455$.

b). Graph the power as a function of p.

When H_A is true,

$$\frac{10(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{D} N(0,1)$$

then,

$$\begin{aligned}
\beta &= p(\text{accept } H_0|H_A) \\
&= p(|X-50| \leq 10|H_A) \\
&= p(0.4 \leq \bar{X} \leq 0.6|H_A) \\
&= p\left(\frac{10(0.4-p)}{\sqrt{p(1-p)}} \leq \frac{10(\bar{X}-p)}{\sqrt{p(1-p)}} \leq \frac{10(0.6-p)}{\sqrt{p(1-p)}}\right) \\
&\approx \Phi\left(\frac{10(0.6-p)}{\sqrt{p(1-p)}}\right) - \Phi\left(\frac{10(0.4-p)}{\sqrt{p(1-p)}}\right)
\end{aligned}$$

thus,

$$\begin{aligned}
\text{power} &= 1-\beta \\
&\approx 1-\Phi\left(\frac{10(0.6-p)}{\sqrt{p(1-p)}}\right) + \Phi\left(\frac{10(0.4-p)}{\sqrt{p(1-p)}}\right)
\end{aligned}$$

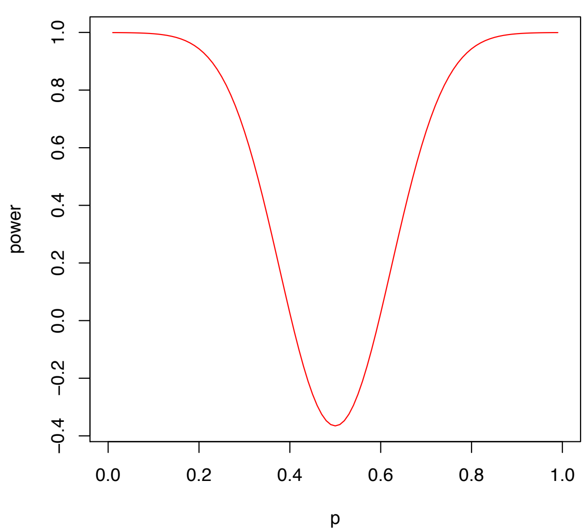

Figure 1: Power as a function of p

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R code:
p=seq(0,1,by=0.01)
power=1-pnorm(10*(0.6-p))/sqrt(p*(1-p))+pnorm(10*(0.4-p))/sqrt(p*(1-p))
plot(p,power,type="l",col="red",xlab="p",ylab="power")
```

Problem 3. Suppose that a single observation X is taken from a uniform density on $[0;\theta]$, and consider testing $H_0 : \theta = 1$ versus $H_1 : \theta = 2$.

a). Find a test that has significance level $\alpha = 0$. What is its power?

$$\begin{aligned} P(\text{Reject } H_0|H_0 \text{ is true}) &= 0 \\ P(c < X < 2|H_0) &= 0 \\ 1 - P(0 < X < c|H_0) &= 0 \\ 1 - \int_0^c \frac{1}{\theta_0}\mathrm{d}x &= 0 \\ 1 - \frac{c}{\theta_0} + 0 &= 0 \\ 1 - \frac{c}{1} &= 0 \\ c &= 1 \end{aligned}$$

Then, the test is to reject X when $X \in (1, 2)$, $\alpha = 0$.

We will use this to find power of the test:

$$\begin{aligned} 1 - \beta &= P(\text{Reject } H_0|H_1 \text{ is true}) \\ &= P(1 < X < 2|H_1) \\ &= \frac{1}{2} \end{aligned}$$

b). For $0 < \alpha < 1$, consider the test that rejects when $X \in [0; \alpha]$. What is its significance level and power?

Find significance:

$$\begin{aligned} \alpha &= P((0 < X < \alpha)|H_0) \\ &= \int_0^\alpha \frac{1}{\theta_0} \mathrm{d}x \\ &= \frac{\alpha}{1} - 0 \\ &= \alpha \end{aligned}$$

Find power:

$$\begin{aligned} (1-\beta) &= P((0 < X < \alpha)|H_1) \\ &= \int_0^\alpha \frac{1}{\theta_1} \mathrm{d}x \\ &= \frac{\alpha}{2} - 0 \\ &= \frac{\alpha}{2} \end{aligned}$$

c). What is the significance level and power of the test that rejects when $X \in [1-\alpha; 1]$?

Find significance:

$$\begin{aligned} \alpha &= P(1-\alpha < X < 1)|H_0) \\ &= \int_{1-\alpha}^1 \frac{1}{\theta_0} \mathrm{d}x \\ &= 1 - 1 + \alpha \\ &= \alpha \end{aligned}$$

Find power:

$$
\begin{aligned}
(1-\beta) &= P(((1-\alpha) < X < 1)|H_1) \\
&= \int_{1-\alpha}^{1} \frac{1}{\theta_1} \mathrm{d}x \\
&= \frac{1-1+\alpha}{2} \\
&= \frac{\alpha}{2}
\end{aligned}
$$

d). Find another test that has the same significance level and power as the previous one.

Let the test be rejecting when $X \in (a, b)$, where $a \geq 0$ and $b \leq 1$.

then we'd like

$$
\begin{aligned}
p(X \in (a,b)|H_0) &= \int_a^b \mathrm{d}x \\
&= b - a \\
&= \alpha \\
1-\beta &= 1 - (p(x \in (0,a)|H_1) + p(x \in (b,2)|H_1) \\
&= 1 - (\int_0^a \frac{1}{2}\mathrm{d}x + \int_b^2 \frac{1}{2}\mathrm{d}x) \\
&= \frac{1}{2}(b-a) \\
&= \frac{\alpha}{2}
\end{aligned}
$$

Hence, as long as the test is rejecting X when $X \in (a, b)$, where $a \geq 0$, $b \leq 1$ and $b - a = \alpha$, it will have the same significance level and power, for example, reject X when $X \in [(1-\alpha)/2, (1+\alpha)/2]$.

e). Does the likelihood ratio test determine a unique rejection region?

From results in sections b and c, we anticipate that the likelihood ratio test would not determine a unique rejection region.

Calculate likelihood ratio:

$$f_0(X) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_1(X) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$LR = \frac{lik(\theta_0)}{lik(\theta_1)} = \frac{f_0(X)}{f_1(X)} = \begin{cases} 2 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

According to the Neyman Pearson Lemma, the most powerful test rejects when LR takes small value, then we can choose to reject when $X \in (1, 2)$, but in this case α would be 0. If $\alpha > 0$, then when $0 < x < 1$, we will have the same LR. Thus, the rejection region should fall into (0,1). And From the above, we know that as long as the width of the interval within $(0, 1)$ is equal to α, the test will have the same significance level and power. Hence, the LR can not determine a unique rejection region.

f). What happens if the null and the alternative hypothesis are interchanged - $H_0 : \theta = 2$ versus $H_1 : \theta = 1$?

If the null and the alternative hypothesis are interchanged, then, all of the significant level and power of the above questions will be changed.

i. $\alpha = 0$

$$\begin{aligned} P(\text{Reject } H_0 | H_0 \text{ is true}) &= 0 \\ &= P(c < X < 2 | H_0) \\ &= 1 - \frac{c}{\theta_0} \\ &= 1 - \frac{c}{2} \\ c &= 2 \end{aligned}$$

Then, when $\alpha = 0$, the test of rejecting X is when $2 < x < 2$, which means we never reject X.

We will use this to find power of the test:

$$\begin{aligned} 1 - \beta &= P(\text{Reject } H_0 | H_1 \text{ is true}) \\ &= 0 \end{aligned}$$

ii. For $0 < \alpha < 1$, consider the test that rejects when $X \in [0; \alpha]$. What is its significance level and power?

Find significance:

$$\begin{aligned}\alpha &= P((0 < X < \alpha)|H_0)\\ &= \int_0^\alpha \frac{1}{\theta_0}\mathrm{d}x\\ &= \frac{\alpha}{2} - 0\\ &= \frac{\alpha}{2}\end{aligned}$$

Find power:

$$\begin{aligned}(1-\beta) &= P((0 < X < \alpha)|H_1)\\ &= \int_0^\alpha \frac{1}{\theta_1}\mathrm{d}x\\ &= \frac{\alpha}{1} - 0\\ &= \alpha\end{aligned}$$

iii. What is the significance level and power of the test that rejects when $X \in [1-\alpha; 1]$?

Find significance:

$$\begin{aligned}\alpha &= P(1-\alpha < X < 1)|H_0)\\ &= \int_{1-\alpha}^1 \frac{1}{\theta_0}\mathrm{d}x\\ &= \frac{1}{2} - \frac{(1-\alpha)}{2}\\ &= \frac{\alpha}{2}\end{aligned}$$

Find power:

$$
\begin{aligned}
(1-\beta) &= P(((1-\alpha) < X < 1)|H_1) \\
&= \int_{1-\alpha}^{1} \frac{1}{\theta_1} \mathrm{d}x \\
&= 1 - 1 + \alpha \\
&= \alpha
\end{aligned}
$$

v. Find another test that has the same significance level and power as the previous one.

Let the test be rejecting when $X \in (a, b)$, where $a \geq 0$ and $b \leq 1$.

then we'd like

$$
\begin{aligned}
p(X \in (a,b)|H_0) &= \int_a^b \frac{1}{2} \mathrm{d}x \\
&= \frac{b-a}{2} \\
&= \frac{\alpha}{2} \\
1-\beta &= 1 - (p(x \in (0,a)|H_1) + p(x \in (b,1)|H_1) \\
&= 1 - (\int_0^a \mathrm{d}x + \int_b^1 \mathrm{d}x) \\
&= b - a \\
&= \alpha
\end{aligned}
$$

Hence, as long as the test is rejecting X when $X \in (a, b)$, where $a \geq 0$, $b \leq 1$ and $b - a = \alpha$, it will have the same significance level and power, for example, reject X when $X \in [(1-\alpha)/2, (1+\alpha)/2]$.

v). Does the likelihood ratio test determine a unique rejection region?

Calculate likelihood ratio:

$$
f_0(X) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}
$$

$$f_1(X) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$LR = \frac{lik(\theta_1)}{lik(\theta_0)} = \frac{f_1(X)}{f_0(X)} = \begin{cases} 2 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

According to the Neyman Pearson Lemma, the most powerful test rejects when LR takes large value, then we can choose to reject when X fall into the region of $(0, 1)$. And from the above, we know that as long as the width of the interval within $(0, 1)$ is equal to α, the test will have the same significance level and power. Hence, the LR can not determine a unique rejection region.