Review of Calculus and Matrix

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

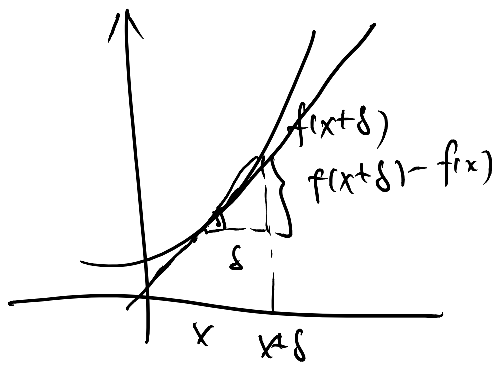
$$\frac{1}{x}$$

(ϵ, δ) - definition : f be a function

$$\lim_{x \to c} f(x) = L$$

$\forall \epsilon > 0$, there $\exists \delta$ such that $|x - c| < \delta$, then

$$|f(x) - L| < \epsilon$$



Derivative: $y = f(x)$

$$y' = \frac{dy}{dx} = (f(x))' = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

$$(x^n)' = n x^{n-1}$$

$$(\sin x)' = \cos x$$

$$(c)' = 0$$

100%

1 $\qquad$ $1 + 1$

1 $\qquad$ $(1+0.5)$ $\qquad$ $(1+0.5)^2 = 2.25$

$(c)' = 0$ $\qquad 1$

$(e^x)' = e^x$ $\qquad 1 \ (1+0.25)\ (1+0.25)^5 \ \cdots (1+0.25)^4$

$\vdots$

$\Downarrow$

$e = 2.71$

3. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

$$\left(\frac{x}{e^x}\right)' = \frac{1 \cdot e^x - x e^x}{(e^x)^2} = \frac{1-x}{e^x}$$

5. Chain rule.

$$f(g(x)) = f'(g(x))\,g'(x)$$

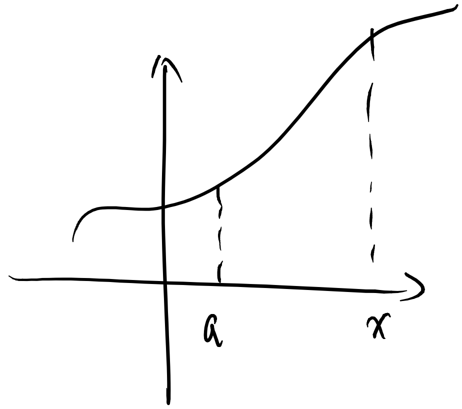
$$(\sin(e^x))' = \cos(e^x)e^x$$

6. If $f(x)$ is inverse function of $g(x)$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$f(g(x)) = x.$$

$$f'(g(x))\, g'(x) = 1$$



Integral

$$F_a(x) = \int_a^x f(x)\,dx$$

$$\boxed{F'(x) = f(x)}$$

Integration by Parts

$$\int_0^\infty f(x)\, g'(x)\,dx = f(x)\, g(x) - \int f'(x)\, g(x)\,dx$$

$$\int_0^\infty x e^{-x}\,dx = x\left(-e^{-x}\right)\Big|_0^\infty - \int_0^\infty 1 \cdot \left(-e^{-x}\right)dx$$

$f(x) = x$

$g(x) = -e^{-x}$

$$= 0 + \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1.$$

L'Hospital Rule

$$\frac{f(x)}{\ }\qquad \frac{f'(x)}{\ }$$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = 1$$

$$\lim_{x \to 0} \frac{e^x}{x} = \lim_{x \to 0} \frac{e^x}{1} \cancel{=} 1$$

Taylor Expansion

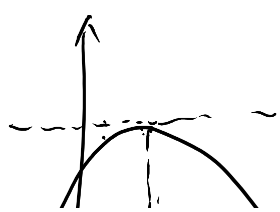
$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

$$+ \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

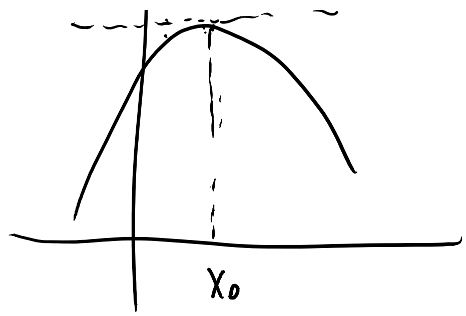
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\frac{e^x - 1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots$$

Local Extrema



Local Extrema



$f(x)$

$f'(x) = 0$
$f''(x) < 0$ } local Maxima

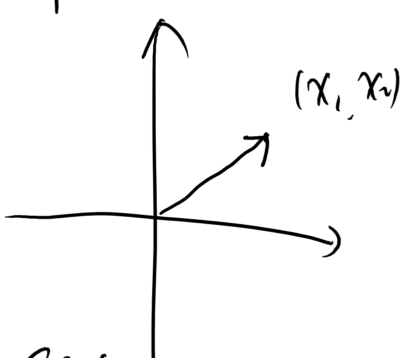
$f(x) = x^3$ $\quad f'(x) = 3x^2$ $\quad$ when $x = 0$ $\quad f'(0) = 0$

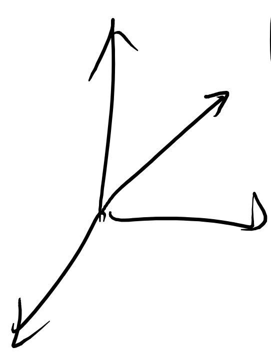
$f'(x) = 0$
$f''(x) > 0$ } $\rightarrow$ local minimum

Veetors:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}$$ p-dimensional column vector

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$





$$cV = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \end{pmatrix}$$

$$cX = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_p \end{pmatrix}$$

$$\underline{X + Y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_p + y_p \end{pmatrix}$$

<u>Matrices,</u>

$$X = \begin{pmatrix} 5 & 8 & 2 \\ -1 & 0 & 7 \end{pmatrix}_{2\times 3} \qquad X_{m\times n}$$

∪|

squard matrix $\quad m = n$

∪|

Symmetric Matrix $\quad x_{ij} = x_{ji} \quad \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$

∪|

diagonal Matrix $\quad x_{ij} = 0, \ \underline{i \neq j} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \checkmark$$

∪|

Identity Matrix $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow$ "$\underline{1}$"

Matrix Operation

"+" $\quad \underline{A} + \underline{B}$

Scalar Product $\quad c \cdot A$

Matrix Product $\quad \underset{m \times n}{A} \cdot \underset{n \times s}{B} = \underset{m \times s}{C}$

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

i-th in A $\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix}$ j-th column B $\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}$$

$2 \times 3 \quad 3 \times 2$

1 $\quad A + B = B + A$

2. $(cA)B = c(AB)$

3. $c(A+B) = cA + cB$

4. $(AB)C = A(BC)$

5. $A(B+C) = AB + AC$

$$AB \cancel{=} BA$$

6. $AI = A \qquad BI = B$

Matrix Inverse. $\qquad 2 \cdot \frac{1}{2} = 1$

A is a squard matrix, if there exists B such that $\underline{A \cdot B} = I$,

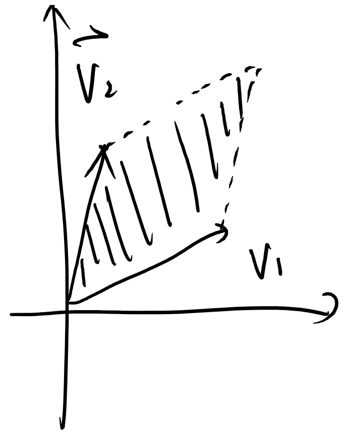
$A^{-1} \qquad A \cdot A^{-1} = A^{-1} A = I$

Call A invertible.

Determint: $\underline{\det(A)} \qquad |A|$

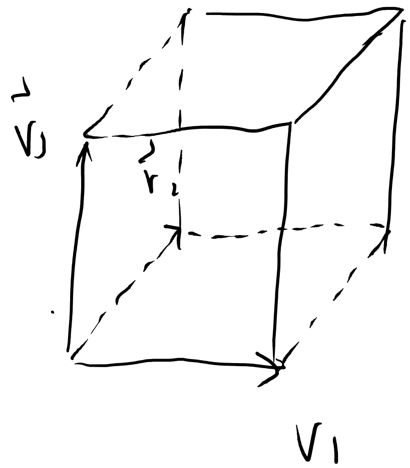
$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{array}{cc} \shortparallel & \shortparallel \\ \vec{v}_1 & \vec{v}_2 \end{array}$$



$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$



① A is invertible iff $\det(A) \neq 0$.

② $\det(AB) = \det(A) \cdot \det(B)$

③ $\det(A^{-1}) = (\det A)^{-1}$

Special Matrices

Orthogonal Matrix

A is orthogonal iff $A^TA = AA^T = I_n$

A is orthogonal iff $A^T A = A A^T = I_n$

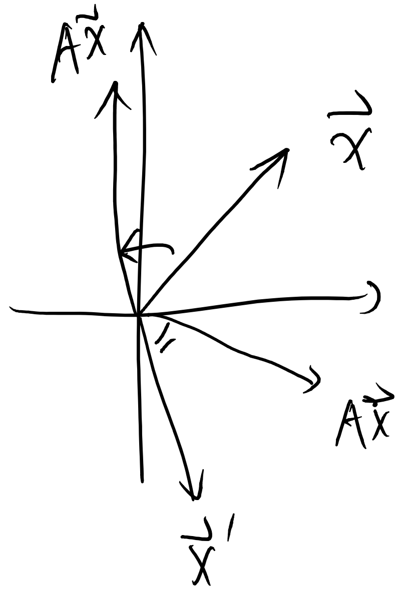
$$A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad A^T A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix

$$\underset{n \times n}{A} : \underset{\text{n-dimensil vector}}{R^n} \longmapsto \underset{\text{n-dimensl}}{R^n}$$

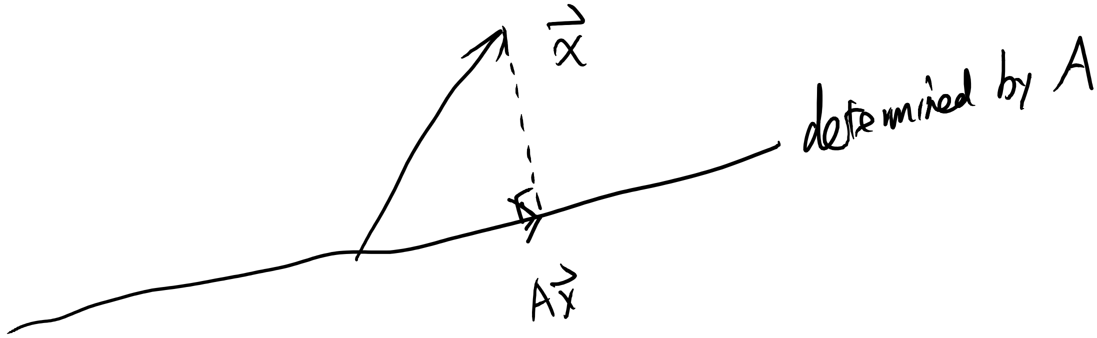
$$\vec{x} \longmapsto A\vec{x}$$



② Idempotent : A iff. $AA = A^2 = A$

$$\Downarrow$$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad A^2 = A \quad A^n = A$



$$\underline{A^2 \vec{x} = A\vec{x}} \Leftrightarrow A^2 = A$$

Positive Definite Matrix. $\quad x^T a x > 0$

A is positive definite if $\quad (A > 0)$

$$\underset{1\times n}{x^T} \underset{n\times n}{A} \underset{n\times 1}{x} > 0 \text{ for any nonzero } x.$$

$(A \geq 0)$

$x^T A x \geq 0 \rightarrow$ Semi-positive definite

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$x^T A x = (x_1 \ \ x_2)\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2x_1^2 - 2x_1x_2 + x_2^2$$

$$= x_1^2 + (x_1 - x_2)^2 > 0 \quad \text{if } \vec{x} \neq 0$$

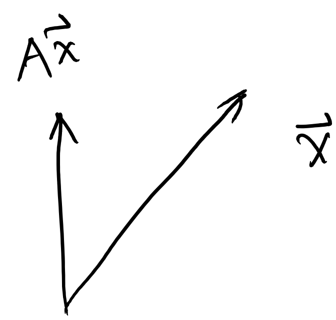
$$A > 0$$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

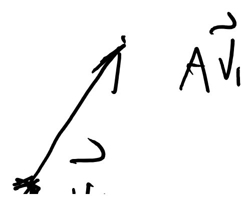
$$\vec{x} B x = -2$$

Eigen Value / vector.

$A \to$ squard matrix.

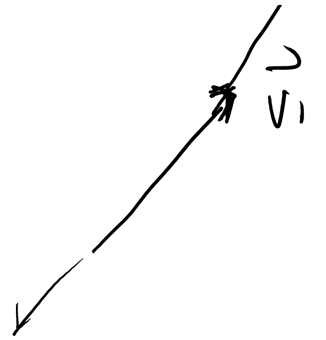


$$\underline{A}\vec{v}_1 = \lambda_1 \vec{v}_1$$



$\underline{\underline{A}} v_1 = \lambda_1 v_1$

eigenvalue   eigenvector



$$\underline{\det(A - \lambda I) = 0}$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0$$

$$\underline{\lambda = 1} \text{ or } 3$$

$$A\vec{v}_1 = \lambda_1 \vec{v}_1 \qquad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda = 3$

$$A\vec{v}_2 = 3\vec{v}_2 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$