8004 Homework 2

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1 Suppose that we are working under the Gauss-Markov model

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

where $E(\epsilon) = \mathbf{0}$ and $var(\epsilon) = \sigma^2\mathbf{I}$. Let $\hat{\mathbf{Y}}$ be the ordinary least square estimator of $\mathbf{Y}$.

(a) Show that $\hat{\mathbf{Y}}$ and $\mathbf{Y} - \hat{\mathbf{Y}}$ are uncorrelated.

Let $\hat{\mathbf{Y}} = P_X Y = X(X'X)^- X'Y$.

First, note that $\hat{\mathbf{Y}}$ and $\mathbf{Y} - \hat{\mathbf{Y}}$ are orthogonal.

$$\hat{\mathbf{Y}}'(\mathbf{Y} - \hat{\mathbf{Y}}) = (\mathbf{P_X Y})'(\mathbf{Y} - \mathbf{P_x Y}) = \mathbf{Y}'\mathbf{P_X}'(\mathbf{Y} - \mathbf{P_x Y}) = \mathbf{Y}'\mathbf{P_X Y} - \mathbf{Y}'\mathbf{P_X}'\mathbf{P_X Y} = \mathbf{Y}'\mathbf{P_X Y} - \mathbf{Y}'\mathbf{P_X Y} = \mathbf{0}$$

Therefore, the expectation

$$E(\hat{\mathbf{Y}}'(\mathbf{Y} - \hat{\mathbf{Y}})) = E(\mathbf{0}) = \mathbf{0}$$

$$E(\hat{\mathbf{Y}}) = E(\mathbf{X}\hat{\beta}) = XE(\hat{\beta}) = X\beta$$

$$E(\mathbf{Y} - \hat{\mathbf{Y}}) = \mathbf{X}\beta - \mathbf{X}\beta = \mathbf{0}$$

This shows $\hat{\mathbf{Y}}$ and $\mathbf{Y} - \hat{\mathbf{Y}}$ are uncorrelated because $E(\hat{\mathbf{Y}}(\mathbf{Y} - \hat{\mathbf{Y}}) - E(\hat{\mathbf{Y}})E(\mathbf{Y} - \hat{\mathbf{Y}})$ is zero.

(b) Show that

$$E\{(\mathbf{Y} - \hat{\mathbf{Y}})^T(\mathbf{Y} - \hat{\mathbf{Y}})\} = \sigma^2\{n - \text{rank}(\mathbf{X})\}.$$

You may use Theorem 5.2a of R&S.

Theorem 5.2a states: If $\mathbf{y}$ is a random vector with mean μ and covariance matrix Σ and if $\mathbf{A}$ is a symmetric matrix of contants, then

$$E(\mathbf{y}'\mathbf{A}\mathbf{y}) = tr(\mathbf{A}\Sigma) + \mu'\mathbf{A}\mu.$$

We can apply Theorem 5.2a with $\mathbf{A} = \mathbf{I}$ and $\mathbf{y} = \mathbf{Y} - \hat{\mathbf{Y}}$. From part a) above, our mean μ is $\mathbf{0}$.

$$Var(\mathbf{Y} - \hat{\mathbf{Y}}) = Var((\mathbf{I} - \mathbf{P_X})Y) = \sigma^2(\mathbf{I} - \mathbf{P_X})$$

The trace of an nxn identity matrix $\mathbf{I}$ is n, and the trace a projection matrix is the rank of target space, $tr(P_X) = rank(X)$.

This gives the desired result:

$$E\{(\mathbf{Y} - \hat{\mathbf{Y}})^T(\mathbf{Y} - \hat{\mathbf{Y}})\} = \sigma^2\{n - \text{rank}(\mathbf{X})\}.$$

2 Consider the one-way ANOVA model $y_{ij} = \mu + \tau_i + \epsilon_{ij}$ for the jth individual of the ith group.

Suppose there are 4 treatments (groups) and the sample sizes are respectively 2,1,1,2 for treatments. Now suppose that $\mathbf{Y} = (y_{11}, y_{12}, y_{21}, y_{31}, y_{41}, y_{42})^T = (2, 1, 4, 6, 3, 5)^T$ contains the observations.

Use R and weighted generalized least squares to find an appropriate estimate for

$$E(\mathbf{Y}) \text{ and } \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \beta$$

in the Aiken model with $\text{var}(\epsilon) = \mathbf{V}$ for two cases where

(a) $\mathbf{V} = \mathbf{V}_1 = \text{diag}(1, 9, 9, 1, 1, 9)$

The full model described in this question in $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ matrix form is:

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{42} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 6 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \delta_{11} \\ \delta_{12} \\ \delta_{21} \\ \delta_{31} \\ \delta_{41} \\ \delta_{42} \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{21} \\ \epsilon_{31} \\ \epsilon_{41} \\ \epsilon_{42} \end{pmatrix}$$

We have $var(\epsilon) = \sigma^2\mathbf{V}$, so we must re-write the model in terms of $\mathbf{U} = \mathbf{V}^{-1/2}\mathbf{Y}$ as follows:

$$\begin{aligned} \mathbf{V} &= \mathbf{V}^{1/2}\mathbf{V}^{1/2}, \text{ V is a diagonal matrix} \\ \text{Let } \mathbf{U} &= \mathbf{V}^{-1/2}Y \\ E(\mathbf{U}) &= \mathbf{V}^{-1/2}EY = \mathbf{V}^{-1/2}\mathbf{X}\beta \\ &= \mathbf{W}\beta \\ Var(\mathbf{U}) &= \mathbf{V}^{-1/2}Var(\mathbf{Y})\mathbf{V}^{-1/2} \\ &= \sigma^2\mathbf{V}^{-1/2}\mathbf{V}\mathbf{V}^{-1/2} \\ &= \sigma^2\mathbf{I} \\ \epsilon^\star &= \mathbf{V}^{-1/2}\epsilon \end{aligned}$$

This gives us $\mathbf{U} = \mathbf{W}\beta + \epsilon^\star$, where the Gauss-Markov assumptions hold for $\mathbf{U}$.
We first calculate $\mathbf{V}^{-1/2}$:

$$\begin{aligned} \mathbf{V} = \mathbf{V}_1 &= \text{diag}(1, 9, 9, 1, 1, 9) \\ \text{So, } \mathbf{V}_1^{1/2} &= \text{diag}(1, 3, 3, 1, 1, 3) \\ \mathbf{V}_1^{-1/2} &= \text{diag}(1, 1/3, 1/3, 1, 1, 1/3) \end{aligned}$$

We can check this in R,

```
V <- diag(c(1,9,9,1,1,9))
Vhi <- solve(V^(1/2))
lvector(Vhi)
```

$$\mathbf{V}^{-1/2} = \begin{pmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.33 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.33 \end{pmatrix}$$

$$\mathbf{U} = \mathbf{V}^{-1/2}\mathbf{Y} = \begin{pmatrix} y_{11} \\ \frac{1}{3}y_{12} \\ \frac{1}{3}y_{21} \\ y_{31} \\ y_{41} \\ \frac{1}{3}y_{42} \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{3} \\ \frac{4}{3} \\ 6 \\ 3 \\ \frac{5}{3} \end{pmatrix}$$

Checking **U** in R gives:

```
Y <- matrix(c(2, 1, 4, 6, 3, 5), nrow=6, ncol=1)
U <- Vhi %*% Y
lvector(U)
```

$$\mathbf{U} = \begin{pmatrix} 2.00 \\ 0.33 \\ 1.33 \\ 6.00 \\ 3.00 \\ 1.67 \end{pmatrix}$$

$$\begin{aligned} \mathbf{W} &= \mathbf{V}^{-1/2}\mathbf{X} \\ &= \text{diag}(1, 1/3, 1/3, 1, 1, 1/3) \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \end{aligned}$$

Checking **W** in R gives:

```
X <- matrix(c(rep(1,6),
              1,1,0,0,0,0,
              0,0,1,0,0,0,
              0,0,0,1,0,0,
              0,0,0,0,1,1,
```

```
               rep(c(1,rep(0,6)),5),1
               ),nrow = 6,byrow=FALSE)
W <- Vhi %*% X
lvector(W)
```

$$\mathbf{W} = \begin{pmatrix} 1.00 & 1.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.33 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.33 & 0.00 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 & 0.33 & 0.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.33 & 0.00 & 0.00 & 0.00 & 0.33 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.33 \end{pmatrix}$$

(a).1 Solving $\mathbf{U} = \mathbf{W}\beta + \epsilon$ for $\hat{\mathbf{U}}$

$$\hat{\mathbf{U}} = \mathbf{W}(\mathbf{W}'\mathbf{W})^{-}\mathbf{W}'\mathbf{Y}$$

```
Uhat <- W %*% ginv(t(W) %*% W) %*% t(W) %*% U
Uhat
W %*% ginv(t(W) %*% W) %*% t(W)
```

(a).2

(b) $\mathbf{V}_2$

$$\mathbf{V} = \mathbf{V}_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 9 \end{pmatrix}$$