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STAT 8003, Homework 2

Group # 8
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Problem 1. Suppose that X is a discrete random variable with P(X = 0) = :25, P(X = 1) = :125, P(X = 2) = :125, and P(X = 3) = :5. Graph the cdf of X.

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set.seed(42)
xvls <- c(0,1,2,3) #possible values of our discrete random variable.
pvls <- c(.25, .125, .125, .5)
X <- sample(xvls, size = 100000, replace=TRUE,  prob=pvls)
Fn <- ecdf(X)
plot(Fn)
```

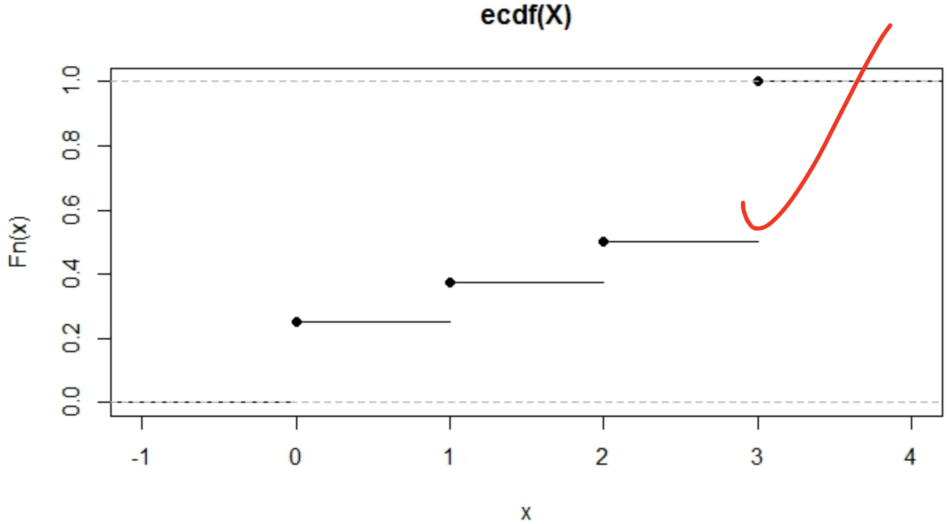

Figure 1: cdf of X

Problem 2. A light bulb manufacturer claims his light bulbs will last 500 hours on the average. The lifetime of a light bulb is assumed to follow an exponential distribution.

Given in the problem that expected failure rate, λ, is 500.We will use cdf of the exponential distribution to calculate probabilities:

$$F(X = x) = 1 - \mathrm{e}^{-x/\lambda}, x \geq 0$$

(a). What is the probability that the light bulb will have to replaced within 500 hours?

$$P(X \leq 500) = 1 - \mathrm{e}^{-500/500}$$

Using R, we calculate cdf with the following statement: pexp(500,1/500) to get probability of 0.6321206, or 63%.

Solution: The probability that the light bulb will have to be replaced within 500 hours is 63%.

(b). What is the probability that the light bulb will last more than 1000 hours?

$$\begin{aligned} P(X \geq 1000) &= 1 - (1 - \mathrm{e}^{-1000/500}) \\ &= \mathrm{e}^{-1000/500} \end{aligned}$$

Using R, we calculate cdf with the following statement: pexp(1000,1/500, lower.tail = 0) to get probability of 0.1353353, or 14%.

Solution: The probability that the light bulb will last more than 1000 hours is 14%.

(c). Suppose a light bulb has already been working for 300 hours. What is the probability that it can work at least 300 hours more?

Taking advantage of the memoryless property of exponential function:

$$\begin{aligned} P(X \geq 600 | X \geq 300) &= P(X \geq 300) \\ &= 1 - (1 - \mathrm{e}^{-300/500}) \\ &= \mathrm{e}^{-300/500} \end{aligned}$$

Using R, we calculate cdf with the following statement:pexp(300,1/500,lower.tail = 0) to get probability of 0.5488116 or 55%.

Solution: The probability that the light bulb can work at least 300 hours more provided it has worked 300 hours is 55%.

Problem 3. A pipe smoker carries one box of matches in his left pocket and one box in his right. Initially, each box contains n matches. If he needs a match, the smoker is equally likely to choose either pocket. What is the cdf for the number of matches in the other box when he first discovers that one box is empty?

Solution: We will use negative binomial distribution to solve this problem. Let X_l=number of matches left in the box in the left pocket when he first discovers the box in his right pocket is empty, and X_r=number of matches left in the box in the right pocket when he first discovers the box in his left pocket is empty.
Let's assume that the smoker first discovers the left pocket is empty. k is the number of matches left in the right box when the left pocket is discoverd to be empty. In that case, the smoker must have picked the left pocket for $n+1$ times, and picked the right pocket for $n-k$ times,and the last time the smoker picked the left pocket.

Hence,

$$p(X_r = k) = \binom{2n-k}{n} \frac{1}{2}^{n+1} \frac{1}{2}^{n-k} \quad (0 \leq K \leq n)$$

It is obvious that

$$X_l = k \bigcap X_r = k = \varnothing$$

And as the probability of each time the smoker picks the right pocket or the left pocket is equal to $\frac{1}{2}$, thus

$$p(X_l = k) = p(X_r = k)$$

Then,

$$\begin{aligned} p(X = k) &= p((X_l = k) \bigcup (X_r = k)) \\ &= p(X_l = k) + p(X_r = k) \\ &= 2\binom{2n-k}{n} \frac{1}{2}^{n+1} \frac{1}{2}^{n-k} \\ &= \binom{2n-k}{n} \frac{1}{2}^{2n-k}; \quad (0 \leq K \leq n) \end{aligned}$$

Hence, the *cdf* denoted as $F(k)$ is

$$F(k) = \sum_{m=0}^{k} \binom{2n-m}{n} \frac{1}{2}^{2n-m}; \quad (0 \leq K \leq n)$$

Problem 4. Suppose there is a continuous random variable X with cdf F(x). Let Y = F(X). What is the distribution of Y?

According to the instructions, we are going to find the *pdf* and *cdf* of the random variable Y. Let's denote the *pdf* of Y as $f(Y)$ and the *cdf* of Y as $F(Y)$, respectively. Then we have,

$$\begin{aligned} F(y;Y) &= Pr(Y \leq y) \\ &= Pr(F(X) \leq y) \end{aligned}$$

Since $F(X)$ is the *cdf* of the ranom variable X, thus $F(X)$ must be non-decreasing, thus, $F(X) \leq y$ could be replaced by $X \leq F^{-1}(y)$
Then we have

$$\begin{aligned} F(y;Y) &= Pr(X \leq F^{-1}(y)) \\ &= F(F^{-1}(y)) \\ &= y; \quad (0 \leq y \leq 1) \end{aligned}$$

$$f(y;Y) = F^{'}(y;Y) = 1 \quad (0 \leq y \leq 1)$$

Conclusion: the *cdf* of Y is $F(Y) = y; \quad (0 \leq y \leq 1)$, and the *pdf* of Y is $f(Y) = 1; \quad (0 \leq y \leq 1)$.

What distribution?