STAT 8004 – Statistical Methods II
Spring 2015
Homework Assignment 6 – Solutions

1. A meat scientist is studying the effect of storage temperature on meat quality. The temperatures of interest are 34, 40, and 46 degrees Fahrenheit. Twelve coolers are available for the study. The three temperatures are randomly assigned to the twelve coolers using a balanced and completely randomized design. Two large cuts of fresh beef are stored in each cooler. After three days, each member of a team of experts independently assigns a quality score to each cut of beef. The experts are not told about the storage conditions of each cut. The scores assigned by the team to each cut of beef are averaged to produce an overall quality score for each cut.

   (a) Write down a model for the overall quality score data. Define your notation thoroughly.

   Solution:

   A model for the overall quality score is the following:

$$y_{ijk} = \mu + \alpha_i + u_{ij} + \varepsilon_{ijk}$$

   where α_i $(i = 1, 2, 3)$ is the fixed effect of the respective temperature, $u_{ij} \overset{iid}{\sim} N(0, \sigma_u^2)$ is the random effect of jth cooler $(j = 1, 2, 3, 4)$ at ith temperature, $\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma_e^2)$ is the random error for $k = 1, 2$. Here, the source of dependence is that two pieces of beef stored in the same cooler.

2. Let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma^2/2 & 0 \\ \sigma^2/2 & \sigma^2 & \sigma^2/2 \\ 0 & \sigma^2/2 & \sigma^2 \end{pmatrix} \right)$$

   where μ_1, μ_2 and σ^2 are unknown parameters. Find the REML of σ^2. Please start with writing it as $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, and then try to find $\mathbf{M}$ for calculating the REML.

   Solution:

   Write the model as $\mathbf{Y} = \mathbf{X}\boldsymbol{\mu} + \boldsymbol{\varepsilon}$ with

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \text{ and } \boldsymbol{\varepsilon} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix}$$

   with the covariance matrix

$$\boldsymbol{\Sigma} = \sigma^2 \begin{pmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}.$$

   Since $\text{rank}(\mathbf{X}) = 2$, $n - \text{rank}(\mathbf{X}) = 3 - 2 = 1$.

It is easy to see that $\mathbf{B} = (1, -1, 0)$ that satisfies $\mathbf{BX} = 0$, with $\mathrm{rank}(\mathbf{B}) = 1$: Then it is straightforward to verify that $\mathbf{BY} = (1, -1, 0)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_1 - y_2$ is normally distributed with mean 0, and variance σ^2. Thus it follows easily that the REML $\hat{\sigma}^2 = (y_1 - y_2)^2$.