8004 Homework 4

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1 Problem 1 In the context of Problem 2 of Homework Assignment 3, use R matrix calculations to do the following in the (non-full-rank) Gauss-Markov normal linear model

(a) Find 90% two-sided confidence limits for σ.

The model described in HW3, Problem 2 in $\mathbf{Y} = \mathbf{X}\beta + \epsilon$ matrix form is:

$$\begin{pmatrix} y_{11} \\ y_{12} \\ y_{21} \\ y_{31} \\ y_{41} \\ y_{42} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \\ 6 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \epsilon_{21} \\ \epsilon_{31} \\ \epsilon_{41} \\ \epsilon_{42} \end{pmatrix}$$

Because the problem statement says this is a Gauss-Markov normal linear model, we know that $\mathbf{Y} \sim \mathrm{N}(\mathbf{X}\beta, \sigma^2\, \mathbf{I})$.

Using the hand-written function `sigmacalc`, included in the appendix. The following two-sided 90% confidence limits for σ were obtained: $0.646 < \sigma < 4.9366$.

(b) Find 90% two-sided confidence limits for $\mu + \tau_2$.

Using the t-distribution describing the distribution of estimable function c'β, the handwritten R function `cbetacalc` included in the appendix, was used to caluclate confidence limits for this entity, where c' = (1, 0, 1, 0 , 0).

$0.7354 < \mu + \tau_2 < 7.2646$

(c) Find 90% two-sided confidence limits for τ_1 - τ_2.

Proceeding as in part b, here $\tau_1 - \tau_2 = \mathbf{c}'\beta = (0, 1, -1, 0, 0) \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix}$. The function `cbetacalc` was used once again with **c** above.

$-6.4984 < \tau_1 - \tau_2 < 1.4984$

(d) Find a *p*-value for testing the null hypothesis $H_0 : \tau_1 - \tau_2 = 0$ vs H_a : not H_0.

(d).1 General Linear Hypothesis Test

The general linear hypothesis test is the following F test for $H_0 : \mathbf{C}\beta = \mathbf{0}$ verus $H_1 : \mathbf{C}\beta \neq \mathbf{0}$, given $\mathbf{y} \sim \mathrm{N}_n(\mathbf{X}\beta, \sigma^2\, \mathbf{I})$, $\mathbf{C}$ q x (/k/+1), rank($\mathbf{C}$) = q, with SSH = the sum of squares due to the hypothesis or due to $\mathbf{C}\beta$. Note that

$\frac{SSH}{\sigma^2} = \frac{(\mathbf{C}\hat{\beta})'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}']^{-1}\mathbf{C}\hat{\beta}}{\sigma^2} \sim \chi^2(q, \frac{(\mathbf{C}\beta)'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}']^{-1}\mathbf{C}\beta}{2\sigma^2})$

and

$\frac{SSE}{\sigma^2} = \frac{\mathbf{y}'[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}']\mathbf{y}}{\sigma^2} \sim \chi^2(n - rank(X))$.

Taking the ratio gives us our test statistic:

$$F = \frac{SSH/q}{SSE/(n - rank(X))}$$

- If $H_0 : \mathbf{C}\beta = \mathbf{0}$ is false, F ~ F(q,n-rank(X),λ), where $\lambda = \frac{(\mathbf{C}\beta)'[\mathbf{C}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{C}']^{-1}\mathbf{C}\beta}{2\sigma^2}$).
- Notice that if $\mathbf{C}\beta = \mathbf{0}$ is true, λ defined above = 0, giving F ~ F(q, n-rank(X)).

(d).2 *p*-value from the F statistic

We need to find the F statistic described above. Here **C** is **a**' from above, **a**'=(0,1,-1,0,0), and **C** is 1 x 5, rank 1.

We used the handwritten function `Cbetahatd` throughout for General Linear Hypothesis Testing. It is included in the appendix for your reference.

The *p*-value obtained was `0.209430584957905`.

(e) Find 90% two-sided predition limits for the sample mean of $n = 10$ future observations from the first set of conditions.

(e).1 A t statistic for prediction

Consider future observation y_0, $y_0 = \mathbf{x}_0' \beta + \epsilon_0$ with $\hat{y}_0 = \mathbf{x_0'}\hat{\beta}$, where $\hat{y}_0$ is computed from n observations and y_0 is obtained independently. We find that $E(y_0 - \hat{y}_0) = 0$ and

$var(y_0 - \hat{y}_0) = var(\epsilon_0) + var(\mathbf{x_0'}\hat{\beta}) = \sigma^2[1 + \mathbf{x_0'}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_0}]$, where $\widehat{var(y - \hat{y})} = s^2 2[1 + \mathbf{x_0'}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_0}]$. Because of the independence of s^2 and y_0 and $\hat{y}_0$, we have the following t statistic:

$$t = \frac{y_0 - \hat{y}_0 - 0}{s\sqrt{1 + \mathbf{x_0'}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_0}} \sim t(n-k-1)}$$

Therefore,

$$P = \left[-t_{\alpha/2,n-k-1} \leq \frac{y_0 - \hat{y}_0 - 0}{s\sqrt{1 + \mathbf{x_0'}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_0}}} \leq t_{alpha/2,n-k-a} \right] = 1 - \alpha$$

Re-arranging in terms of $\mathbf{x_0'}\hat{\beta} = \hat{y}_0$ gives:

$$\mathbf{x_0'}\hat{\beta} \pm t_{\alpha/2,n-k-1} s\sqrt{1 + \mathbf{x_0'}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x_0}}.$$

(e).2 Predicitions for n observations from $\mu + \tau_1$

Using the preceeding theory and the handwritten R function, `predict`, which is included in the appendix. I ran a prediction fo n=10 from the first condition $\mathbf{x}_0$ = (1,0,1,0,0).

The 90% confidence limits obtained for the mean were `0.576` to `7.424`.

(f) Find 90% two-sided prediction limits for the difference between a pair of future values, one from the first set of conditions (i.e. with mean $\mu + \tau_1$) and one from the second set of conditions (i.e. with mean $\mu + \tau_2$).

Similar to part (e) above, here I used my `predict` function again, except an n of 2 and a $\mathbf{x}_0$ vector of the difference of the first two conditions:

(1,1,0,0,0) - (1,0,1,0,0) = (0,1,-1,0,0).

This gave 90 % prediction limits for the difference as follows: `-7.1169` to `2.1169`.

(g) Find a *p*-value for testing the following: What is the practical interpretation of this test?

$$H_0: \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The practical interpretation of this test is to ask if all of the parameters are equal. I performed the test using the General Linear Hypothesis Testing function described above, `Cbetahatd`.

```
G <- t(matrix(c(0,1,-1,0,0,
                      0,1,0,-1,0,
                      0,1,0,0,-1),nrow=3,ncol=5, byrow=TRUE))
    Cbetahatd(Y1,X1,G,c(0,0,0))
```

I obtained a p value of 0.20643991448067, indicating that it is unlikely that all of the parameters are equal.

(h) Find a *p*-value for testing:

$$H_0: \begin{pmatrix} 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}.$$

I tested this hypotheis as in question 1g), using the General Linear Hypothesis and the F-test implemented in my function `Cbetahatd`, note that the vector (10,0) was entered for the **d** vector.

```
H <- t(matrix(c(0, 1, -1, 0, 0, 0, 0, 1, -1, 0), nrow=2, ncol=5, byrow=T))

   Cbetahatd(Y1,X1,H,c(10,0))
```

A significant *p*-value of 0.0134 was obtained, suggesting that this hypothesis is acceptable.

2 Problem 2 In the following make use of the data in Problem 4 of Homework Assignment 3. Consider a regression of *y* on $x_1, x_2, \ldots, x_5$. Use R matrix calculations to do the following in a full rank Gauss-Markov normal linear model.

(a) Find 90% two-sided condifence limits for σ.

Calling our `sigmacalc` function on the Boston data set, we find 90% confidence limits for sigma of `5.6106` $< \sigma <$ `6.2263`.

(b) Find 90% two-sided confidence limits for the mean response under the conditions of data point #1.

To find these 90% confidence limits, we will use the t-distribution of $, where c' is the first row of our data set (data point #1).

Using the `cbetacalc` function to do this, as `cbetacalc(YB,XB, .1, XB[1,])` we find a 90% confidence interval of `25.2114` < mean response under the conditions of data point #1 < `26.1973`.

(c) Find 90% two-sided condifence limits for the difference in mean responses under the conditions of data points #1 and #2. .

To find these 90% confidence limits, we will use the t-distribution of $, where c' is the difference beteen the first row of our data set and the second row (data points #1 and #2).

Using the `cbetacalc` function to do this, as `cbetacalc(YB,XB, .1, (XB[1,]-XB[2,]))` we find `1.2025` to `2.6125` is a 90% confidence interval for the difference in mean responses under conditions 1 and 2.

(d) Find a *p*-value for testing the hypothesis that the conditions of data points #1 and #2 produce the same mean response.

An F-test was used to test the hypothesis that the product between the vector describing the differences between conditions 1 and 2 and beta is **0**. That is H_0 : c'β = **0**, where c' = XB[1,] - XB[2,]. This was done using my general linear hypothesis testing function: `Cbetahatd(YB,XB, (XB[1,]-XB[2,]))`. The *p*-value obtained was `1.01975837067947e-05`.

(e) Find 90% two-sided prediction limits for an additional response for the set of conditions $\mathbf{x}_1$ = 0.005, $\mathbf{x}_2$ = 0.45, $\mathbf{x}_3$ = 7, $\mathbf{x}_4$ = 45, and $\mathbf{x}_5$ = 6.

90 % prediction limits for an additional response from these conditions were obtained using the conditions as our c-vector in the `predict` function: `predict(YB,XB, .1, c(0,0.005,0.45,7,45,6), 1)`. The limits obtained were `24` to `47.7985`.

(f) Find a *p*-value for testing the hypothesis that a model including only x_1, x_3, and x_5 is adequare for "explaining" home price.

Using an F-test on the hypothesis that c'β = $\beta_2 + \beta_4 = 0$, we find a *p*-value of `6.73025042030595e-06` for this model.

3 Problem 3

(a) In the context of Problem 1, part g), suppose that in fact $\tau_1 = \tau_2$, $\tau_3 = \tau_4 = \tau_1$ - dσ. What is the distribution of the F statistic?

The F statistic for Problem 1, part g is given by $F = \frac{Q/s}{SSE/N-\text{rank}(X)} \sim F(s, N-\text{rank}(X), \lambda)$.

Where $Q = (\widehat{C'\beta} - d)'(C'(X'X)^- C)^{-1}(\widehat{C'\beta} - d)$ and $\lambda = \frac{1}{2\sigma^2}(C'\beta - d)'(C'(X'X)^- C)^{-1}(C'\beta - d)$.

Therefore, if $\tau_1 = \tau_2$, and τ_3=τ_4=τ_1 - dσ, our non-centrality parameter will equal

$$\lambda = \frac{1}{2\sigma^2}(0, d\sigma, d\sigma)(C'(X'X)^-)C)^{-1}\begin{pmatrix} 0 \\ d\sigma \\ d\sigma \end{pmatrix}.$$

Evaluating for (C'(X'X)$^-$C)$^{-1}$ in R, we find:

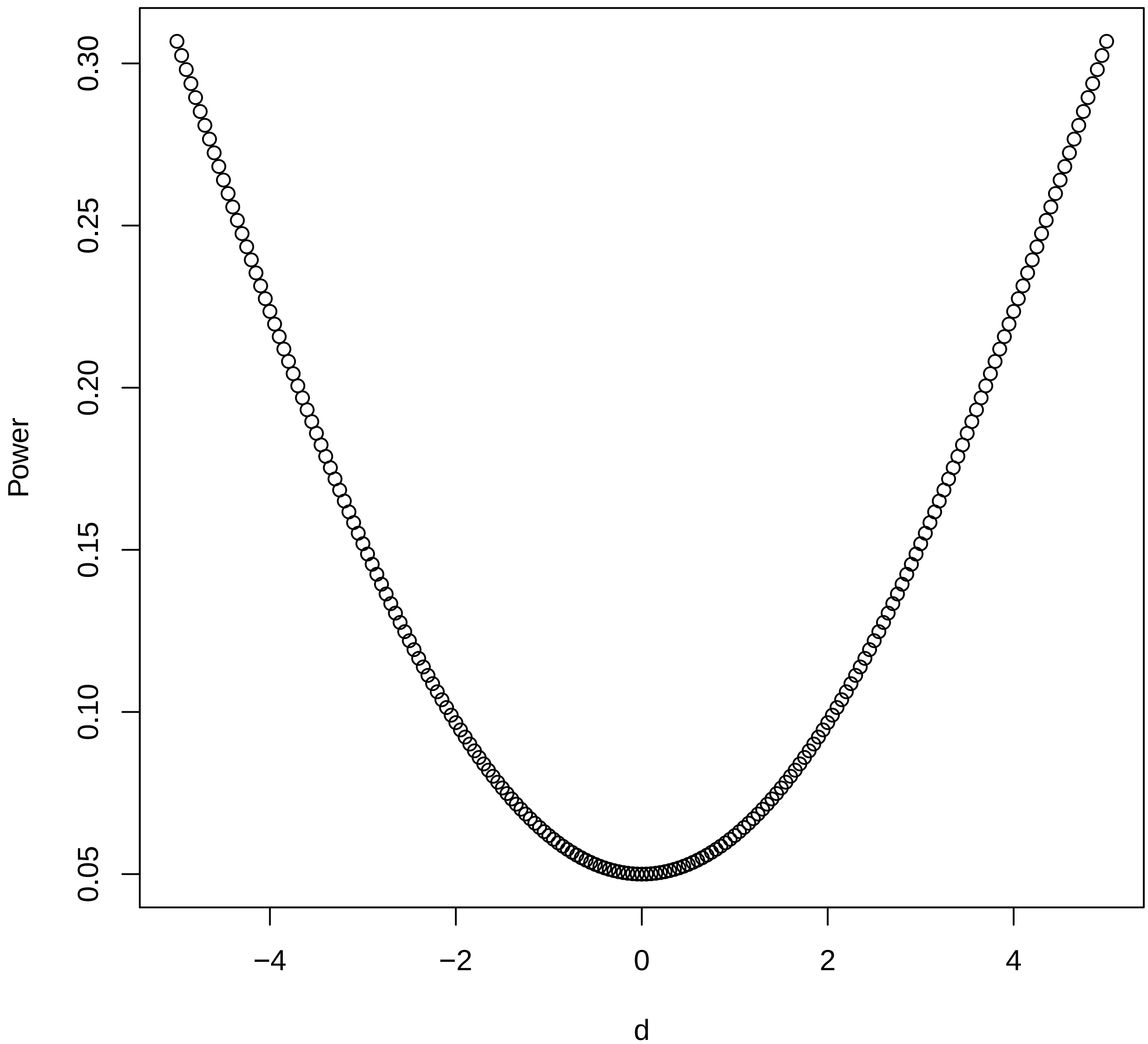
```
fractions(ginv(t(C1g)%*%ginv(t(X1)%*%X1)%*%C1g))
```

$$(C'(X'X)C)^{-1} = \begin{pmatrix} 5/6 & -1/6 & -1/3 \\ -1/6 & 5/6 & -1/3 \\ -1/3 & -1/3 & 4/3 \end{pmatrix}$$

Giving $\lambda = \frac{3}{4}d^2$ so the final distribution of the F statistic is F(3, 2, $\frac{3}{4}d^2$).

(b) Use R to plot the power of the α = 0.05 level test as a function of *d* for *d* $\in$ [-5,5], that is plotting *P* (F > the cut-off value) against *d*. The R function pf(q,df1,df2,ncp) will compute cumulative (non-central) F probabilities for you corresponding to the value q, for degrees of freedom df1 and df2 when the noncentrality parameter is ncp.

```
d <- seq(-5,5,by=.05)
Power <- 1-pf(qf(0.95,3,2),3,2,.75*d^2)
plot(d, Power)
```



r0.4 :

Figure 1: Power of an $\alpha = 0.05$ level test as a function of d.

4 Appendix: Tangled R code

```
library(MASS); library(xtable)
  lvector <- function(x, dig = 2, dsply=rep("f",ncol(x)+1)) {
   x <- xtable(x, align=rep("",ncol(x)+1),display=dsply,digits=dig) # We repeat empty string 6 time
   print(x, floating=FALSE, tabular.environment="pmatrix",
     hline.after=NULL, include.rownames=FALSE, include.colnames=FALSE)
   }

#Variables from Problem 2 of HW3:
  V1 <- diag(c(1,9,9,1,1,9))
  Y <- matrix(c(2, 1, 4, 6, 3, 5), nrow=6, ncol=1)
  X <- matrix(c(rep(1,6),
                1,1,0,0,0,0,
                0,0,1,0,0,0,
                0,0,0,1,0,0,
                0,0,0,0,1,1),nrow = 6,byrow=FALSE)

  V2 <- diag(c(1,9,9,1,1,9))
  V2[1,2] <- 1
  V2[2,1] <- 1
  V2[4,3] <- -1
  V2[3,4] <- -1
  V2[6,5] <- -1
  V2[5,6] <- -1


#Variables from Problem 4 of HW3:
data(Boston)
Y_B = as.matrix(Boston$medv)
X_B = as.matrix(Boston[,c('crim','nox','rm','age','dis')])
X_B = cbind(rep(1,dim(Boston)[1]),X_B)
bhat_B <- ginv(t(X_B)%*%X_B) %*% t(X_B) %*% Y_B
Yhat_B <- X_B %*% bhat_B
err_B <- Y_B - Yhat_B
sigsqhat_B <- t(err_B) %*% err_B / (dim(X_B)[1] - qr(X_B)$rank)

#Find V^(-1/2)
Vh1 <-solve(V1^(1/2))

#Transform model to OLS
U <- Vh1 %*% Y
W <- Vh1 %*% X

Uhat <- W %*% ginv(t(W) %*% W) %*% t(W) %*% U

SSE <- t(U-Uhat) %*% (U-Uhat)

qr(W)$rank
```

```
lowerchi <- qchisq(.05, df=(length(U) - qr(W)$rank))
upperchi <- qchisq(.95, df=(length(U) - qr(W)$rank))

SSE/lowerchi
SSE/upperchi

#Find V^(-1/2) using spectral decompostion
Vh2 <-solve(eigen(V2)$vectors %*% diag(sqrt(eigen(V2)$values)) %*% t(eigen(V2)$vectors))

#Transform model to OLS
U <- Vh2 %*% Y
W <- Vh2 %*% X

Uhat <- W %*% ginv(t(W) %*% W) %*% t(W) %*% U

SSE <- t(U-Uhat) %*% (U-Uhat)

qr(W)$rank

lowerchi <- qchisq(.05, df=(length(U) - qr(W)$rank))
upperchi <- qchisq(.95, df=(length(U) - qr(W)$rank))

Yhat <- X %*% ginv(t(X) %*% X) %*% t(X) %*% Y

SSE <- t(Y-Yhat) %*% (Y-Yhat)

lowerchi <- qchisq(.05, df=(length(Y) - qr(X)$rank))
upperchi <- qchisq(.95, df=(length(Y) -qr(X)$rank))

#Find the t distribution quantile
t_1b <- qt(.05, (length(Y) - qr(W)$rank - 1) )

a_1b = matrix(c(1,0,1,0,0))
s_1b <- sqrt(SSE/(length(Y) - qr(W)$rank - 1))
Bhat_1b <- ginv(t(W) %*% W) %*% t(W) %*% U
quad_1b <- sqrt(t(a_1b) %*% ginv(t(W)%*%W) %*% a_1b)
upper1b <- t(a_1b) %*% Bhat_1b - t_1b * s_1b * quad_1b
lower1b <- t(a_1b) %*% Bhat_1b + t_1b * s_1b * quad_1b

a_1c = matrix(c(0,1,-1,0,0))

quad_1c <- sqrt(t(a_1c) %*% ginv(t(W)%*%W) %*% a_1c)
upper1c <- t(a_1c) %*% Bhat_1b - t_1b * s_1b * quad_1c
lower1c <- t(a_1c) %*% Bhat_1b + t_1b * s_1b * quad_1c

SSH <- t(t(a_1c) %*% Bhat_1b) %*% ginv(t(a_1c)%*%ginv(t(W)%*%W)%*%a_1c)%*%t(a_1c)%*%Bhat_1b

p_1d <- pf(SSH/SSE, 1, 1, lower.tail=FALSE)
```

```
#Find SSR in the full model.
SSR_Bf <- t(bhat_B) %*% t(X_B) %*% Y_B - (length(Y_B)*(mean(Y_B))^2)

#create reduced model design matric and X1_B and estimator bhat1_B
X1_B <- X_B[,-c(3,5)]
bhat1_B <- ginv(t(X1_B)%*%X1_B) %*% t(X1_B) %*% Y_B
SSR_Br <- t(bhat1_B) %*% t(X1_B) %*% Y_B - (length(Y_B)*(mean(Y_B))^2)

SSE_B <- t(Y_B)%*%Y_B - t(bhat_B)%*%t(X_B)%*%Y_B

F_2f <- ((SSR_Bf - SSR_Br)/2)/(SSE_B/(length(Y_B) - qr(X_B)$rank))

pf_2f <- pf(F_2f, 2, (length(Y_B)-(qr(X_B)$rank)), lower.tail=F)
pf_2f
```

5 Appendix: Additional Notes

(a) Useful Theorems

Theorem 5.1. *Suppose* $\mathbf{Y} \sim MVN_n(\mu, \textbf{Sigma})$*,* Σ *positive definite. Also suppose* $\mathbf{A}_{n\times n}$ *symmetric and rank(*$\mathbf{A}$*) =* k*. If* $\mathbf{A}\Sigma$ *idempotent,* $\mathbf{Y}'\mathbf{A}\mathbf{Y} \sim \chi^2_k(\mu'\mathbf{A}\mu)$*.*

Theorem 5.2. *Suppose* $\mathbf{Y} \sim MVN_n(\mu, \sigma^2\mathbf{I})$*. And the product* $\mathbf{BA} = \mathbf{0}$*, with A and B of appropriate size. Then,*

[(a)]If $\mathbf{A}$ *symmetric,* $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *and* $\mathbf{BY}$ *are independent. If both* $\mathbf{B}$ *and* $\mathbf{A}$ *symmetric,* $\mathbf{Y}'\mathbf{A}\mathbf{Y}$ *and* $\mathbf{Y}'\mathbf{B}\mathbf{Y}$ *are independent.*

(b) Distributions of interests

(b).1 SSE/σ^2

Using theorem 5.1 above, we can show:

$$\frac{SSE}{\sigma^2} = \frac{(\mathbf{Y}-\hat{\mathbf{Y}})'(\mathbf{Y}-\hat{\mathbf{Y}})}{\sigma^2} \sim \chi^2_{n-\text{rank}(X)}$$

Rearranging to find confidence limits for σ gives:

$$P\left(\sqrt{\frac{SSE}{\text{upper}\,\alpha/2\,\text{quantile of}\,\chi^2_{\text{n-rank(X)}}}} < \sigma < \sqrt{\frac{SSE}{\text{upper}\,\alpha/2\,\text{quantile of}\,\chi^2_{\text{n-rank(X)}}}}\right) = 1-\alpha$$

(b).2 Estimable functions c'β

For an estimable **c'**β, we have:

$$\frac{\widehat{\mathbf{c}'\beta} - \mathbf{c}'\beta}{\sqrt{MSE}\sqrt{\mathbf{C}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}}} \sim t_{n-\text{rank}(X)}$$

Note that $MSE = \frac{SSE}{n-\text{rank}(X)}$. Rearranging to find 1 - α confidence limits for **c'**β, denoting $\text{t}^{\star}$ = the upper $\alpha/2$ quantile of $\text{t}_{\text{n-rank(X)}}$, we have:

$$P\left(\widehat{\mathbf{c}'\beta} - t^{\star}\sqrt{MSE}\sqrt{\mathbf{C}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}} < \mathbf{c}'\beta < \widehat{\mathbf{c}'\beta} + t^{\star}\sqrt{MSE}\sqrt{\mathbf{C}'(\mathbf{X}'\mathbf{X})^{-}\mathbf{C}}\right) = 1-\alpha$$