STAT 8004 – Statistical Methods II
Spring 2015
Homework Assignment 2 – Solutions

1. Write out the following models of elementary/intermediate statistical analysis in the matrix form

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

(a) A one-variable quadratic polynomial regression model

$$y_i = \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 + \varepsilon_i$$

for $(i = 1, 2, \ldots, 5)$.

(b) A two-factor ANCOVA model without interactions

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma(x_{ijk} - \bar{x}) + \varepsilon_{ijk}$$

for $i = 1, 2$, $j = 1, 2$, and $k = 1, 2$.

Solutions:

(a)

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \\ 1 & x_5 & x_5^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & (x_{111} - \bar{x}) \\ 1 & 1 & 0 & 1 & 0 & (x_{112} - \bar{x}) \\ 1 & 1 & 0 & 0 & 1 & (x_{121} - \bar{x}) \\ 1 & 1 & 0 & 0 & 1 & (x_{122} - \bar{x}) \\ 1 & 0 & 1 & 1 & 0 & (x_{211} - \bar{x}) \\ 1 & 0 & 1 & 1 & 0 & (x_{212} - \bar{x}) \\ 1 & 0 & 1 & 0 & 1 & (x_{221} - \bar{x}) \\ 1 & 0 & 1 & 0 & 1 & (x_{222} - \bar{x}) \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma \end{pmatrix} + \begin{pmatrix} \varepsilon_{111} \\ \varepsilon_{112} \\ \varepsilon_{121} \\ \varepsilon_{122} \\ \varepsilon_{211} \\ \varepsilon_{212} \\ \varepsilon_{221} \\ \varepsilon_{222} \end{pmatrix}.$$

2. Use `eigen()` function in R to compute the eigenvalues and eigenvectors of

$$\mathbf{V} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Then use R to find an "inverse square root" of this matrix. That is, find a symmetric matrix $\mathbf{W}$ such that $\mathbf{W}\mathbf{W} = \mathbf{V}^{-1}$.

Solutions:

```
> V <- matrix(c(3,-1,1,-1,5,-1,1,-1,3),3,3,byrow=T)
> EV <- eigen(V)
> W <- EV$vectors%*%diag(1/sqrt(EV$values))%*%t(EV$vectors)
> W
            [,1]       [,2]        [,3]
[1,]  0.61404486 0.05636733 -0.09306192
[2,]  0.05636733 0.46461562  0.05636733
[3,] -0.09306192 0.05636733  0.61404486
> W%*%W
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
> solve(V)
            [,1]       [,2]        [,3]
[1,]  0.38888889 0.05555556 -0.11111111
[2,]  0.05555556 0.22222222  0.05555556
[3,] -0.11111111 0.05555556  0.38888889
```

3. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 4.001 \\ 4.001 & 4.002 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 4 & 4.001 \\ 4.001 & 4.002001 \end{pmatrix}.$$

Obviously, these matrices are nearly identical. Use R and compute the determinants and inverses of these matrices. (Even though the original two matrices are nearly the same, $\mathbf{A}^{-1} \approx -3\mathbf{B}^{-1}$. This shows that small changes in the in the elements of nearly singular matrices can have big effects on some matrix operations.)

<u>Solutions:</u>

```
> library(MASS)
> A <- matrix(c(4,4.001,4.001,4.002),2,2)
> B <- matrix(c(4,4.001,4.001,4.002001),2,2)
> det(A)
[1] -1e-06
> det(B)
[1] 3e-06
> ginv(A)
         [,1]     [,2]
[1,] -4002000  4001000
[2,]  4001000 -4000000
> ginv(B)
```

```
         [,1]     [,2]
[1,]  1334000 -1333667
[2,] -1333667  1333333
> 3*ginv(B)
         [,1]     [,2]
[1,]  4002001 -4001000
[2,] -4001000  4000000
```

4. Write an R function to conduct projection, e.g. with name `project()`, so that the input is the given design matrix $\mathbf{X}$, and the output is the projection matrix $\mathbf{P}_{\mathbf{X}}$ for projecting a vector onto the column space of $\mathbf{X}$.

   Solutions:

```
library(MASS)
project <- function(A) {A%*%ginv(t(A)%*%A)%*%t(A)}
```

5. Consider the (non-full-rank) two-way "effect model" with interactions in the Example (d) in lecture.

   (a) Determine which of the parametric functions below are estimable:

   $$\alpha_1, \alpha_2 - \alpha_1, \mu + \alpha_1 + \beta_1 + \delta_{11}, \delta_{12}, \delta_{12} - \delta_{11} - (\delta_{22} - \delta_{21})$$

   For those that are estimable, find $\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T$, such that $\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T\mathbf{Y}$ produces the estimate of $\mathbf{c}^T\boldsymbol{\beta}$.

   (b) For the parameter vector $\boldsymbol{\beta}$ written in the order used in class, consider the hypothesis H_0 : $\mathbf{C}\boldsymbol{\beta} = \mathbf{0}$ for

   $$\mathbf{C} = \begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

   Is this hypothesis testable? Explain.

   Solutions:

   (a) $\mu + \alpha_1 + \beta_1 + \delta_{11}$ and $\delta_{12} - \delta_{11} - (\delta_{22} - \delta_{21})$ are estimable. Their corresponding $\mathbf{c}^T(\mathbf{X}^T\mathbf{X})^{-}\mathbf{X}^T$ are $(1/2, 1/2, 0, 0, 0, 0, 0, 0)$ and $(-1/2, -1/2, 1/2, 1/2, 1/2, 1/2, -1/2, -1/2)$.

   (b) Though $\text{rank}(C) = 2$, but the first row is not corresponding to an un-estimable item. So it is not testable.