8004 Homework 9

Nooreen Dabbish
April 9, 2015

1 Data is generated from the exponetial distribution with density

$$f(y) = \lambda \exp(-\lambda y), \text{ where } \lambda, y > 0.$$

(a) Show that it belongs to the exponential family distributions be indentifyting θ, b(θ), ϕ, a(ϕ) and c(y;ϕ).

An exponential family distribution can be written in the form

$$\exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)\right\}.$$

We write:

$$f(y|\lambda) = \exp(-\lambda y + \log \lambda)$$

and equate θ = -λ, b(θ) = -log λ =-log(-θ), (note that λ >0, so $\theta < 0$), ϕ = 1, a(ϕ) = 1, c(y;ϕ) = 1.

(b) What is the canonical link and variance functions for a GLM with the response following the exponential distribution?

The link function connects the linear predictor μ to the parameter θ in the exponential family distribution definition above. To find the canonical link, we want μ = E(Y) = b'(θ). we find the first moment of y:

$$EY = \int_0^\infty \lambda y e^{-y} dy = \frac{1}{\lambda}$$

$$\text{Note } b'(\theta) = \frac{-1}{\theta} = \frac{1}{\lambda} = \mu \text{ and } b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2} = \mu^2$$

We write θ as a funcion of μ

$$\theta(\mu) = -\frac{1}{\mu}$$

$$b'^{-1}(\cdot) = \text{negative inverse function.}$$

Since var(Y) = b”(θ)a(ϕ), and a(ϕ) = 1, var(Y) = b”(θ) = μ^2.

(c) Is there any practical difficulty for using the canonical link in practice?

Especially in small samples, canoncial links have desirable properties. However, they may not be the best fit for a model (McCullagh and Nelder pg 32).

Note in this case that the exponential mean is restricted to positive values. However our μ is a linear combination of predictors. This does not guarantee a positive restriction on our estimates of the mean.

(d) Express the deviance as a function of y$_i$ and fitted mean μ_i (i = 1, . . . , n).

2 Data is generated from the exponetial distribution with density

$$f(y) = \lambda \exp(-\lambda y), \text{ where } \lambda, y > 0.$$

(a) Show that it belongs to the exponential family distributions be indentifyting θ, b(θ), ϕ, a(ϕ) and c(y;ϕ).

An exponential family distribution can be written in the form

$$\exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)\right\}.$$

We write:

$$f(y|\lambda) = \exp(-\lambda y + \log\lambda)$$

and equate θ = -λ, b(θ) = -log λ =-log(-θ), (note that λ >0, so θ < 0), ϕ = 1, a(ϕ) = 1, c(y;ϕ) = 1.

(b) What is the canonical link and variance functions for a GLM with the response following the exponential distribution?

The link function connects the linear predictor μ to the parameter θ in the exponential family distribution definition above. To find the canonical link, we want μ = E(Y) = b'(θ). we find the first moment of y:

$$EY = \int_0^\infty \lambda y e^{-y} dy = \frac{1}{\lambda}$$

$$\text{Note } b'(\theta) = \frac{-1}{\theta} = \frac{1}{\lambda} = \mu \text{ and } b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2} = \mu^2$$

We write θ as a funcion of μ

$$\theta(\mu) = -\frac{1}{\mu}$$

$$b'^{-1}(\cdot) = \text{negative inverse function.}$$

Since var(Y) = b"(θ)a(ϕ), and a(ϕ) = 1, var(Y) = b"(θ) = μ^2.

(c) Is there any practical difficulty for using the canonical link in practice?

Especially in small samples, canoncial links have desirable properties. However, they may not be the best fit for a model (McCullagh and Nelder pg 32).

Note in this case that the exponential mean is restricted to positive values. However our μ is a linear combination of predictors. This does not guarantee a positive restriction on our estimates of the mean.

(d) Express the deviance as a function of y_i and fitted mean μ_i (i = 1, ..., n).

3 Data is generated from the exponetial distribution with density

$$f(y) = \lambda\exp(-\lambda y), \text{ where } \lambda, y > 0.$$

(a) Show that it belongs to the exponential family distributions be indentifyting θ, b(θ), ϕ, a(ϕ) and c(y;ϕ).

An exponential family distribution can be written in the form

$$\exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)\right\}.$$

We write:

$$f(y|\lambda) = \exp(-\lambda y + \log\lambda)$$

and equate θ = -λ, b(θ) = -log λ =-log(-θ), (note that λ >0, so θ < 0), ϕ = 1, a(ϕ) = 1, c(y;ϕ) = 1.

(b) What is the canonical link and variance functions for a GLM with the response following the exponential distribution?

The link function connects the linear predictor μ to the parameter θ in the exponential family distribution definition above. To find the canonical link, we want μ = E(Y) = b'(θ). we find the first moment of y:

$$EY = \int_0^\infty \lambda y e^{-y} dy = \frac{1}{\lambda}$$

$$\text{Note } b'(\theta) = \frac{-1}{\theta} = \frac{1}{\lambda} = \mu \text{ and } b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2} = \mu^2$$

We write θ as a funcion of μ

$$\theta(\mu) = -\frac{1}{\mu}$$

$$b'^{-1}(\cdot) = \text{negative inverse function.}$$

Since var(Y) = b"(θ)a(ϕ), and a(ϕ) = 1, var(Y) = b"(θ) = μ^2.

(c) Is there any practical difficulty for using the canonical link in practice?

Especially in small samples, canoncial links have desirable properties. However, they may not be the best fit for a model (McCullagh and Nelder pg 32).

Note in this case that the exponential mean is restricted to positive values. However our μ is a linear combination of predictors. This does not guarantee a positive restriction on our estimates of the mean.

(d) Express the deviance as a function of y$_i$ and fitted mean μ_i (i = 1, . . . , n).

4 Data is generated from the exponetial distribution with density

$$f(y) = \lambda \exp(-\lambda y), \text{ where } \lambda, y > 0.$$

(a) Show that it belongs to the exponential family distributions be indentifyting θ, b(θ), ϕ, a(ϕ) and c(y;ϕ).

An exponential family distribution can be written in the form

$$\exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)\right\}.$$

We write:

$$f(y|\lambda) = \exp(-\lambda y + \log \lambda)$$

and equate θ = -λ, b(θ) = -log λ =-log(-θ), (note that λ >0, so $\theta < 0$), ϕ = 1, a(ϕ) = 1, c(y;ϕ) = 1.

(b) What is the canonical link and variance functions for a GLM with the response following the exponential distribution?

The link function connects the linear predictor μ to the parameter θ in the exponential family distribution definition above. To find the canonical link, we want μ = E(Y) = b'(θ). we find the first moment of y:

$$EY = \int_0^\infty \lambda y e^{-y} dy = \frac{1}{\lambda}$$

$$\text{Note } b'(\theta) = \frac{-1}{\theta} = \frac{1}{\lambda} = \mu \text{ and } b''(\theta) = \frac{1}{\theta^2} = \frac{1}{\lambda^2} = \mu^2$$

We write θ as a funcion of μ

$$\theta(\mu) = -\frac{1}{\mu}$$

$$b'^{-1}(\cdot) = \text{negative inverse function.}$$

Since var(Y) = b"(θ)a(ϕ), and a(ϕ) = 1, var(Y) = b"(θ) = μ^2.

(c) Is there any practical difficulty for using the canonical link in practice?

Especially in small samples, canoncial links have desirable properties. However, they may not be the best fit for a model (McCullagh and Nelder pg 32).

Note in this case that the exponential mean is restricted to positive values. However our μ is a linear combination of predictors. This does not guarantee a positive restriction on our estimates of the mean.

(d) Express the deviance as a function of y$_i$ and fitted mean μ_i (i = 1, . . . , n).

We have scaled deviance given by

$$\frac{D(y;\hat{\mu})}{\phi} = 2\sum \frac{w_i}{\phi}\{y_i(\widetilde{\theta}_i - \hat{\theta}_i) - b(\widetilde{\theta}_i) + b(\hat{\theta}_i)\}$$

with a(ϕ) = ϕ/w, $\tilde{\theta} = \theta(y)$ denoting the full model (n parameter) estimate of θ, and $\hat{\theta} = \theta(\hat{\mu})$ denoting the null model (one parameter) estimate of θ.

Evaluating for b(θ) = - log (- θ) and ϕ = 1, w$_i$ = 1 gives

$$D(y;\hat{\mu}) = 2\sum \left(y_i(\widetilde{\theta}_i - \hat{\theta}_i) + \log\left(\frac{\widetilde{\theta}_i}{\hat{\theta}_i}\right)\right)$$

From above, we have that $\theta(\mu) = -\frac{1}{\mu}$. Evaluating for $\hat{\theta}_i = 1/\hat{m}u_i)$ and $\tilde{\theta}_i = y_i$ gives

$$D(y;\hat{\mu}) = 2\sum \left(y_i(y_i - \hat{\mu}_i) + \log\left(\frac{\hat{\mu}_i}{y_i}\right)\right)$$