In the name of God,
the merciful, the compassionate

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

HOMEWORK 7
(OBSERVER DESIGN)

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Problem 1 Description

An observer for the inverted pendulum on a motor-driven cart is to be designed using the measurement of the displacement of the cart ($y = x_1$). Determine the observer gain matrix for which the observer poles lie in a fourth-order Butterworth pattern of radius 5, i.e., the characteristic equation is to be: (Problem 7.2 Friedland)

$$\left(\frac{s}{5}\right)^4 + 2.613\left(\frac{s}{5}\right)^3 + (2+\sqrt{2})\left(\frac{s}{5}\right)^2 + 2.613\left(\frac{s}{5}\right) + 1 = 0$$

Solution

State Space representation of this system is,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{k^2}{MRr^2} & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{k^2}{MlRr^2} & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k}{MRr} \\ 0 \\ -\dfrac{k}{MlRr} \end{bmatrix} [e]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

where,

$$m = 0.1\ \text{kg}, M = 1.0\ \text{kg}, l = 0.1\ \text{m}, g = 9.8\ \text{m.s}^{-2}, k = 1\ \text{V.s}, R = 100\ \Omega, r = 0.02\ \text{m}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -25 & -0.98 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 25 & 10.78 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \end{bmatrix}$$

First and foremost, the observability matrix should be evaluated,

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -0.25 & -0.98 & 0 \\ 0 & 625 & 24 & -0.98 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -0.25 & -0.98 & 0 \\ 0 & 625 & 24 & -0.98 \end{vmatrix} = 0.9604 \neq 0$$

It is observable.

$$|sI - A| = s^4 + 25s^3 - 10.78s^2 - 245s = 0$$

$$\rightarrow a_0 = 0$$

$$a_1 = -245$$

$$a_2 = -10.78$$

$$a_3 = 25$$

New characteristics equation,

$$\left(\frac{S}{5}\right)^4 + 2.613\left(\frac{S}{5}\right)^3 + \left(2+\sqrt{2}\right)\left(\frac{S}{5}\right)^2 + 2.613\left(\frac{S}{5}\right) + 1 = 0$$

$$s^4 + 13.065s^3 + 25\left(2+\sqrt{2}\right)s^2 + 326.625s + 625 = 0$$

$$\alpha_0 = 625, \alpha_1 = 326.625, \alpha_2 = 25\left(2+\sqrt{2}\right), \alpha_3 = 13.065$$

$$\Psi = \begin{vmatrix} 1 & 25 & -10.78 & -245 \\ 0 & 1 & 25 & -10.78 \\ 0 & 0 & 1 & 25 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\boldsymbol{\alpha} = [\alpha_3 \quad \alpha_2 \quad \alpha_1 \quad \alpha_0]^T, \boldsymbol{a} = [a_3 \quad a_2 \quad a_1 \quad a_0]^T$$

$$L = [Q\Psi]^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}) = [-11.935 \quad 394.5103 \quad -452.0058 \quad -1993.6186]^T$$

Problem 2 Description

In the system below, consider that state T_3 is measured with a sensor with sufficient accuracy.

$$\frac{d}{dt}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -7 & 2 \\ 2 & 1 & -6 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} 0 & 3.54 \\ 0 & 0.026 \\ 144 & 0.026 \end{bmatrix}\begin{bmatrix} u \\ T_{d\_out} \end{bmatrix}$$

$$Y = T_3$$

Design a reduced-order observer in order to place the poles of the error dynamic in -10, -15.

Solution

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -7 & 2 \\ 2 & 1 & -6 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} 0 & 3.54 \\ 0 & 0.026 \\ 144 & 0.026 \end{bmatrix}\begin{bmatrix} u \\ T_{d_{out}} \end{bmatrix}$$

$$Y = [0 \quad 0 \quad 1]\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$A_{ee} = \begin{bmatrix} -7 & 2 \\ 1 & -5 \end{bmatrix}$$

$$C_s A_{se} = [1 \quad 2]$$

Checking the observability,

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & -8 \end{bmatrix} \rightarrow full\ rank$$

IT IS OBSERVABLE.

$$|sI - (A - LC)|$$

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -7 & 2 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 2]\right)\right| = \begin{vmatrix} s + 7 + l_1 & 2l_1 - 2 \\ l_2 - 1 & s + 5 + 2l_1 \end{vmatrix} = 0$$

$$s^2 + (12 + l_1 + 2l_2)s - (l_2 - 1)(2l_1 - 2) + (7 + l_1)(5 + 2l_2) = 0$$

Considering $s_1 = -10, s_2 = -15$

Therefore,

$$l_1 = -13$$
$$l_2 = 13$$

Problem 3 Description

Consider the system below:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]x$$

a. Check the observability of the system.

b. Find full state observer gains with Direct method in order to place the error system poles at $-5, 2 \pm 3.464$.

c. Design the same observer using Ackerman and Bass-Gura methods.

Solution

a) The observability matrix is as follows,

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \rightarrow full\ rank$$

It is observable.

b) Direct method for observer gains,

$$k = [k_1 \quad k_2 \quad k_3]^T$$

$$|sI - A + kC| = \begin{vmatrix} s-k_1 & -1 & 0 \\ k_2 & s & -1 \\ k_3+6 & 1 & s+6 \end{vmatrix}$$
$$= s^3 + (k_1+6)s^2 + (6k_1+k_2+11)s + 6k_2 + k_3 + 6$$

Desired poles are as follows,

$$p_d = (s+5)(s+2+3.464i)(s+2-3.464i) = s^3 + 9s^2 + 36s + 80$$

Comparing the obtained equations,

$$k_1 = 3, k_2 = 7, k_3 = -1$$

which demonstrates the feedback gain of the system.

$$p_d(A) = A^3 + 9A^2 + 36A + 80I = \begin{bmatrix} 74 & 25 & 3 \\ -18 & 41 & 7 \\ -42 & -95 & -1 \end{bmatrix}$$

c) According the Ackermann formulation,

$$G = p_d Q^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 74 & 25 & 3 \\ -18 & 41 & 7 \\ -42 & -95 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = [3 \quad 7 \quad -1]^T$$

d) According the Bass-Gurra formulation,

$$|sI - A| = s^3 + 6s^2 + 11s + 6$$

$$\Psi = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\alpha} = [80 \quad 36 \quad 9]$$

$$\boldsymbol{a} = [6 \quad 11 \quad 6]$$

$$L = [Q\Psi]^{-1}(\boldsymbol{\alpha} - \boldsymbol{a}) = [3 \quad 7 \quad -1]^T$$

Problem 4 Description

For the system with transfer function $G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s(s+1)}$, and with state variables $x_1 = y, x_2 = \dot{x}_1$:

a. Design a feedback control so that the closed loop system has natural frequency $\omega_n = 3$ and damping ration $\xi = 0.5$.

b. Design a state estimator so that the error dynamic has $\omega_{n1} = 15$ and $\xi = 0.5$.

Solution

The transfer function can be obtained as,

$$G(s) = \frac{10}{s^2 + s} = \frac{Y}{U} \to 10U = s^2Y + sY \to \ddot{x} + \dot{x} = 10u \to$$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

or,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

To design a controller, first, the controllability matrix should be evaluated,

$$\boldsymbol{Q}_c = [\boldsymbol{B} \quad \boldsymbol{AB}] = \begin{bmatrix} 0 & 10 \\ 10 & -10 \end{bmatrix} \to |\boldsymbol{Q}_c| = -100 \neq 0 \to full\ rank$$

The system is controllable.

$$\zeta = 0.5 \to \sin^{-1} 0.5 = 30^o$$

$$\omega_n = 3\frac{\text{rad}}{\text{s}}, \sigma = \zeta\omega = (0.5)(3) = 1.5$$

$$\frac{1.5}{\tan 60} = 2.5981$$

a) Direct Method

$$\boldsymbol{A_c} = \boldsymbol{A} - \boldsymbol{BK} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 10 \end{bmatrix} [K_1 \quad K_2] = \begin{bmatrix} 0 & 1 \\ -10K_1 & -1 - 10K_2 \end{bmatrix}$$

$$|\boldsymbol{sI} - \boldsymbol{A}_c| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -10K_1 & -1 - 10K_2 \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ 10K_1 & s + 1 + 10K_2 \end{vmatrix}$$

$$= s(s + 1 + 10K_2) + 10K_1$$

$$= s^2 + (10K_2 + 1)s + 10K_1 = (s + 1.5 + 2.5981j)(s + 1.5 - 2.5981j)$$

$$= s^2 + 3s + 9 \to$$

$$1 + 10K_2 = 3 \to K_2 = 0.2$$

$$10K_1 = 9 \to K_1 = 0.9$$

$$K = [0.9 \quad 0.2]$$

b) State Estimator

The observability matrix is as follows,

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow full\ rank$$

The system is observable.

$$\zeta = 0.5 \rightarrow \sin^{-1} 0.5 = 30^o$$

$$\omega_n = 15 \frac{\text{rad}}{\text{s}}, \sigma = \zeta\omega = (0.5)(3) = 7.5$$

$$\frac{7.5}{\tan 60} = 12.9904$$

Error:

$$\dot{e} = A_e e$$

where,

$$A_e = A - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} C \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -l_1 & 1 \\ -l_2 & -1 \end{bmatrix}$$

$$|sI - A_e| = \begin{bmatrix} s + l_1 & -1 \\ l_2 & s+1 \end{bmatrix} = (s + l_1)(s + 1) + l_2 = s^2 + (1 + l_1)s + l_1 + l_2$$

$$= (s + 7.5 - 12.9904i)(s + 7.5 + 12.9904i) = s^2 + 15s + 225$$

$$1 + l_1 = 15 \rightarrow l_1 = 14$$

$$l_1 + l_2 = 225 \rightarrow l_2 = 211$$

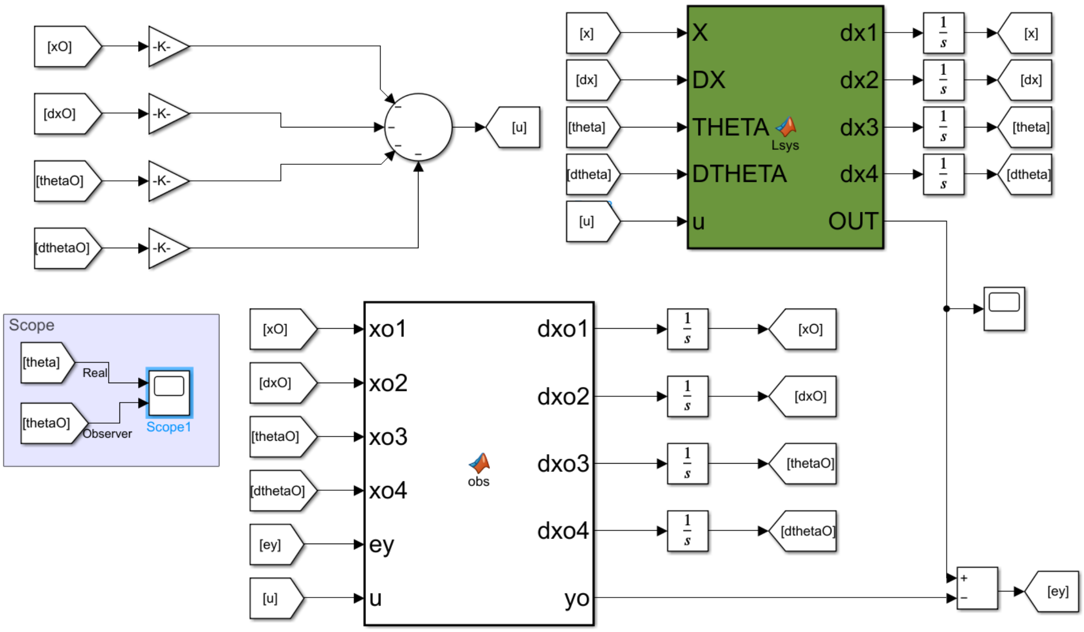
$$L = [14 \quad 211]^T$$

Problem 2.1 Description

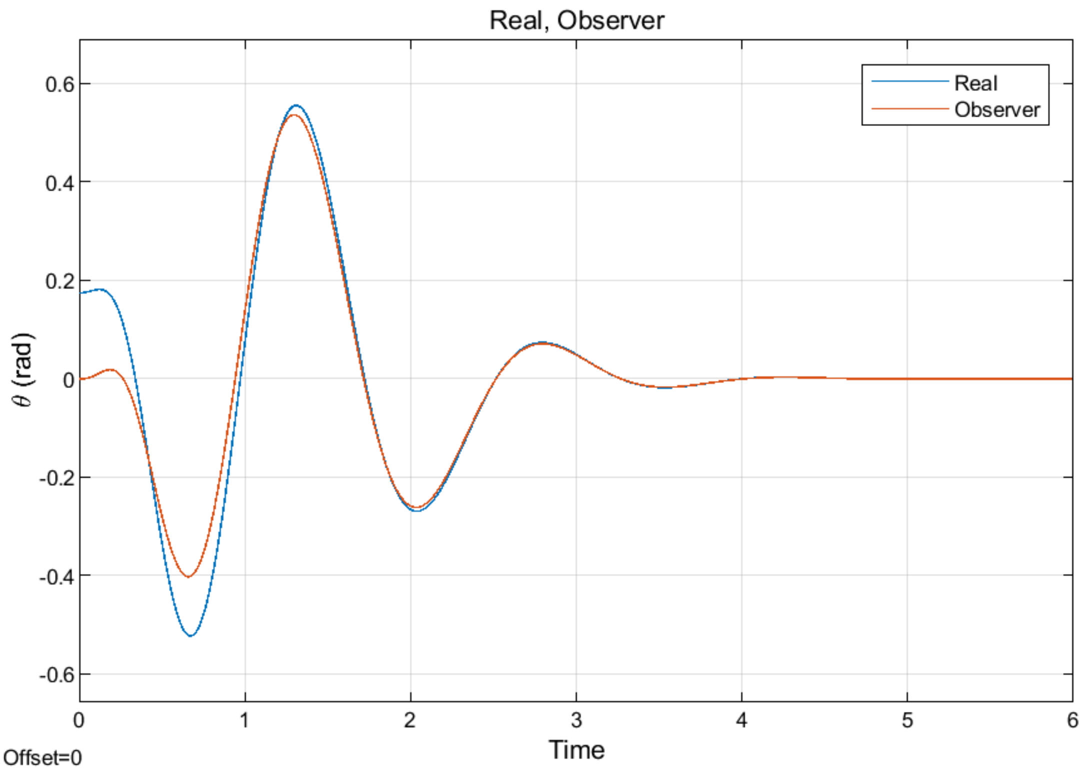
Implement Part1.Q1 system and observer in Simulink. Consider initial conditions to be $[0, 0, 10°, 0]^T$. Compare the estimated and real state responses on the same plots and explain your results.

Solution

The system is implemented in MATLAB Simulink and the results are plotted in this part. Furthermore, feedback controller from previous homework is also used to stabilized the results.



It can be observed that the observer can estimated the results after a couple of time.

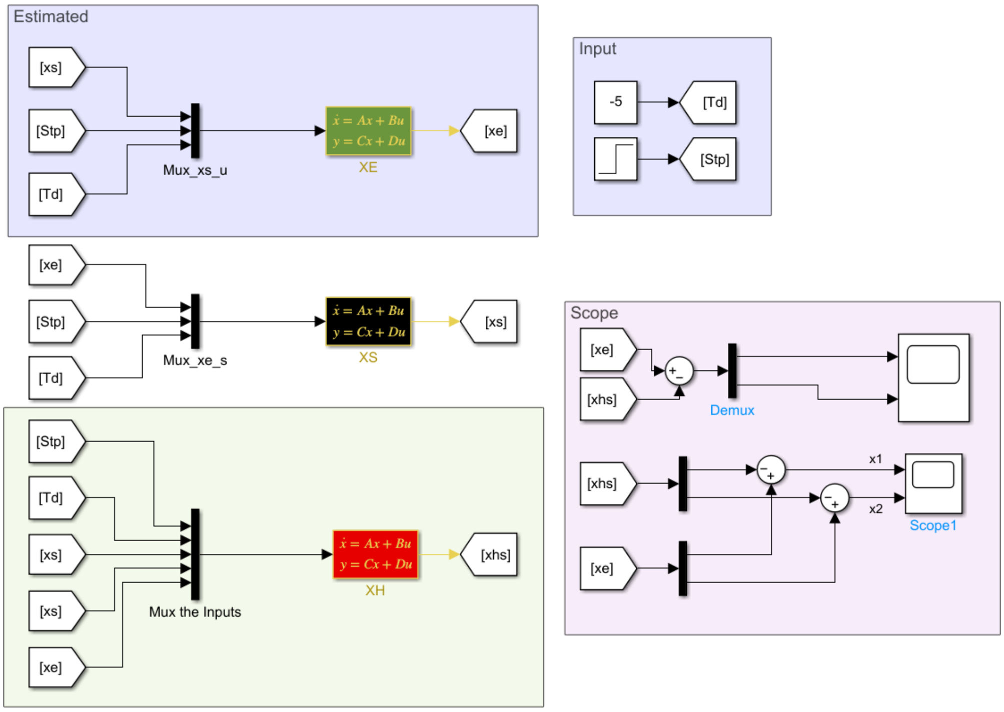


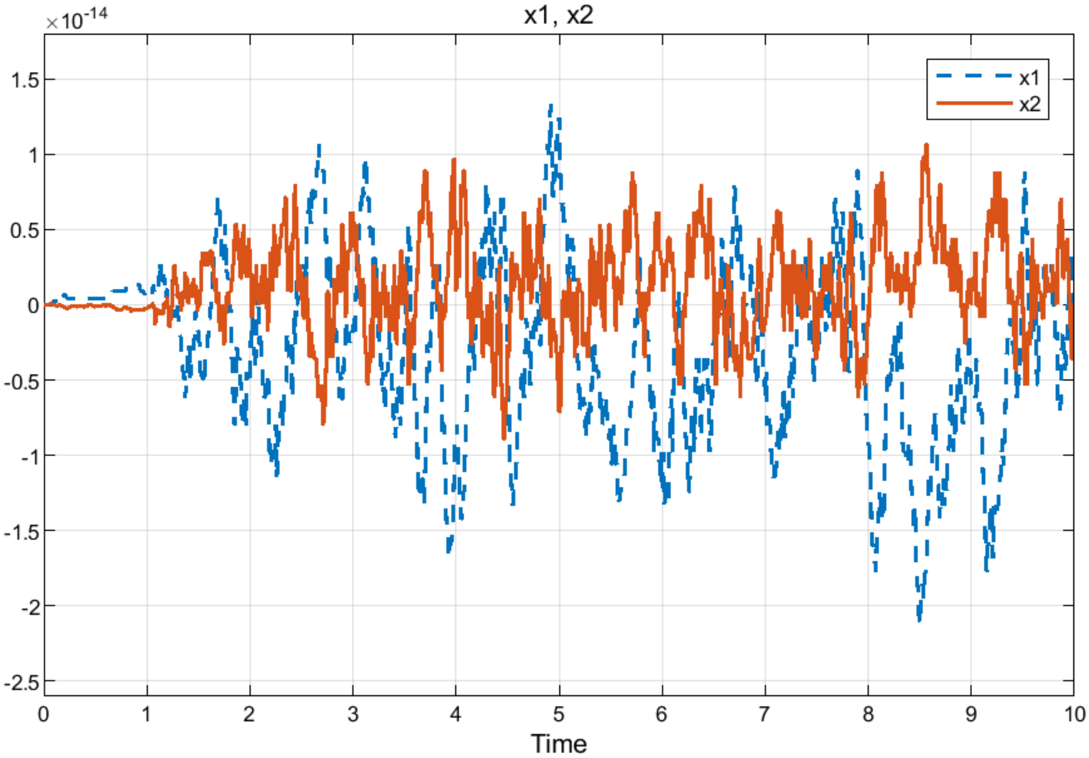
Problem 2.2 Description

Implement Part1.Q2 system and observer in Simulink. Then compare the real states and estimated states with a step control input and $T_{d_{out}} = -5$. (Consider zero initial conditions.)

Solution

The system has been implemented in MATLAB Simulink and the error has been evaluated. The step has been given to the system as an input.


Estimated
[xs]
[Stp]
[Td]
Mux_xs_u
$\dot{x} = Ax + Bu$
$y = Cx + Du$
XE
[xe]
Input
-5
[Td]
[Stp]
[xe]
[Stp]
[Td]
Mux_xe_s
XS
[xs]
Scope
[xhs]
Demux
x1
x2
Scope1
Mux the Inputs
XH
[xhs]


$\times 10^{-14}$
x1, x2
x1
x2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
9
10
Time

Problem 2.3 Description

The following MATLAB functions are useful for observer design:

a. $\mathrm{L} = \mathrm{place}(A', C', ObsEig)'$

Solve for the observer gain matrix L to place the desired eigenvalues ObsEig of the observer error dynamics matrix $\mathrm{A} - \mathrm{LC}$.

b. $\mathrm{L} = \mathrm{acker}(A', C', PbsEig)'$

Solve for the observer gain matrix L to place the desired eigenvalues ObsEig of the observer error dynamics matrix $\mathrm{A} - \mathrm{LC}$, using Ackermann's formula, for single-input, single-output systems only.

Check the results of Part1.Q3 with a. and b. commands.

Solution

Codes have been written in MATLAB, and the results are as follows,

```
clc
clear
close all


A=[0 1 0;0 0 1;-6 -11 -6];
C=[1 0 0];
ObsEig=[-5 -2-3.464i -2+3.464i];




L1=place(A.',C.',ObsEig).'
L2=acker(A.',C.',ObsEig).'
```

results,

```
L1 =

    3.0000
    6.9993
   -0.9993


L2 =
```

```
   3.0000
   6.9993
  -0.9993
```

It can be seen that the results are the same.

Problem 2.4 Description

Implement Part1.Q4 in MATLAB and check your controller and observer results.

Solution

In order to check our obtained results, they are implemented in MATLAB Simulink,

For instance, in the case of controllability, setteling time should be estimated near

$$t_s = \frac{4}{\sigma} = \frac{4}{1.50} = 2.667 \text{ s}$$

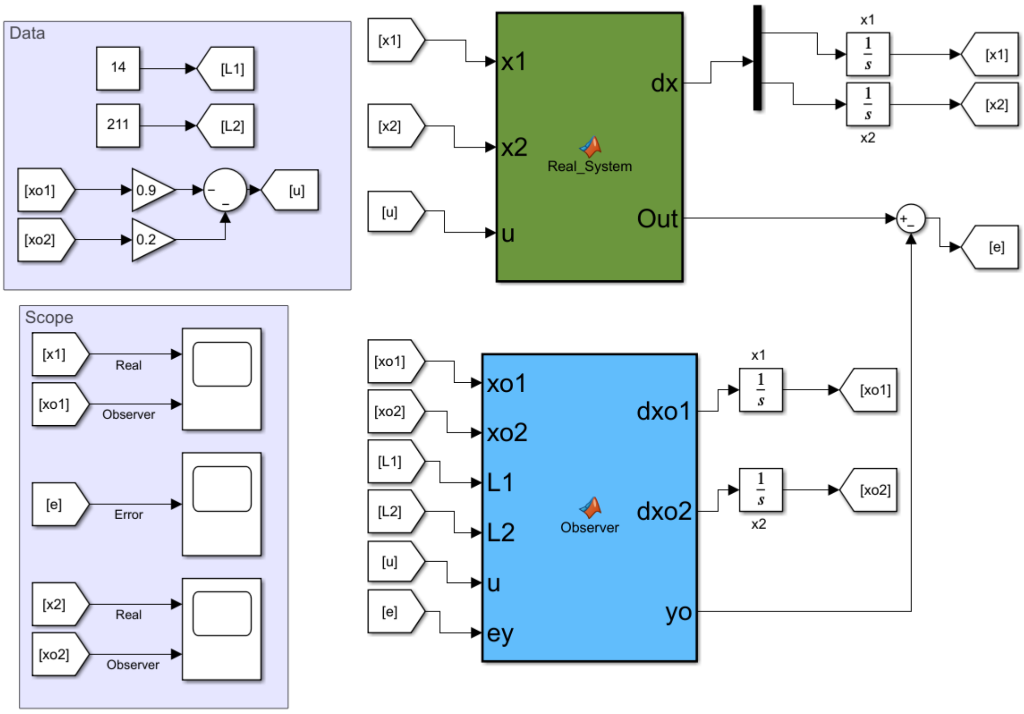
Step plot is considered, and the settling time is reported here,

$$t_s = 2.7 \text{ s}$$

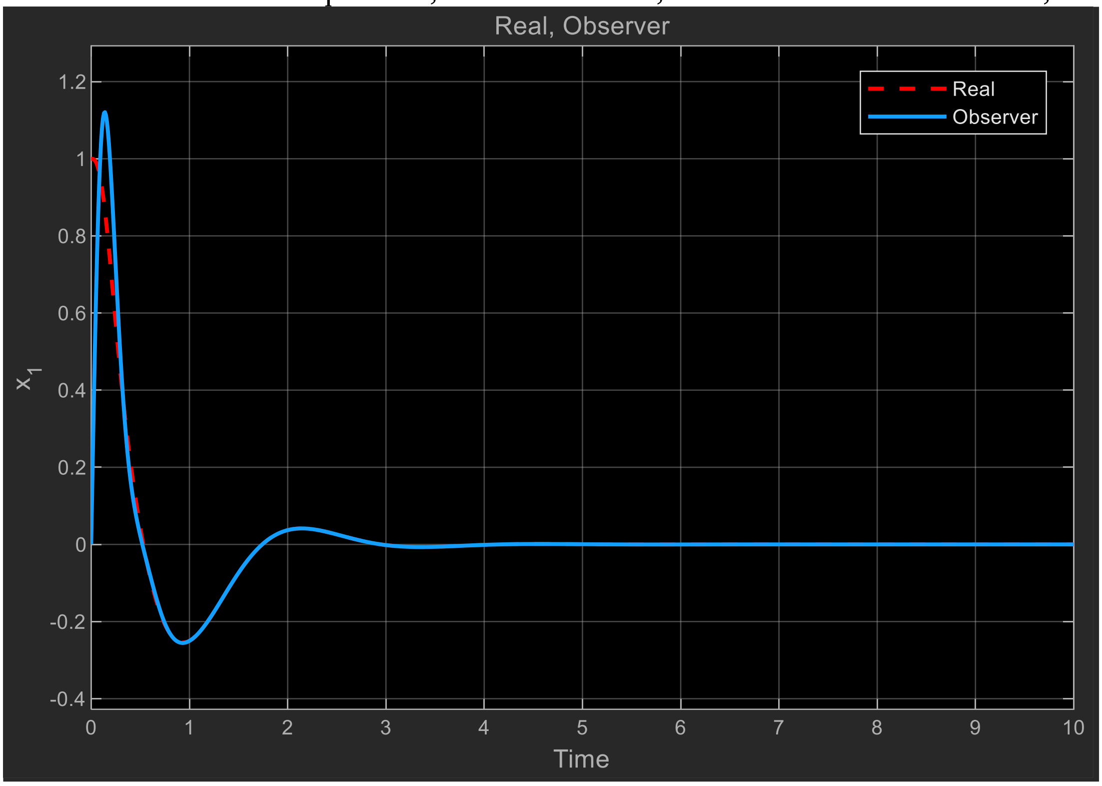
Therefore, it is correctly proposed!

Then, in the case of observer,

$$t_s = \frac{4}{\sigma} = \frac{4}{7.5} = 0.53 \text{ s}$$



The initial condition is equal to 1, for the first state, and the results are as follows,



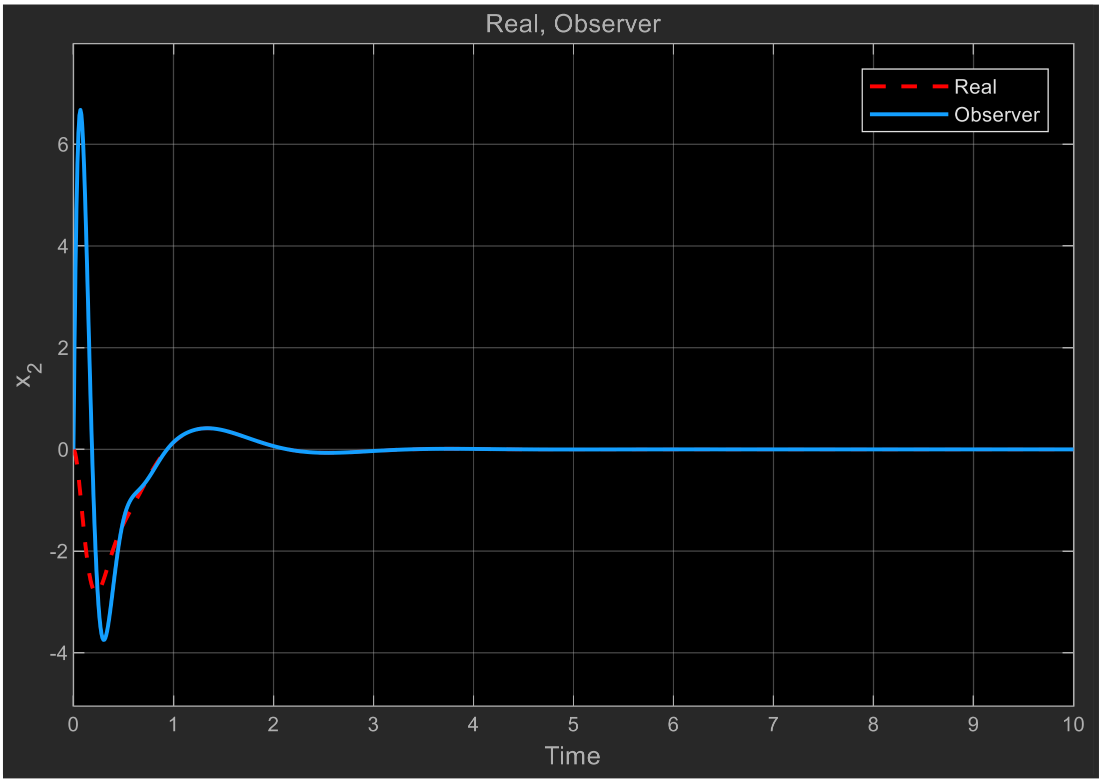


Figure 1 The states response due to an initial condition.