In the name of God

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

HOMEWORK 3 (CONTROLLABILITY)

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Problem 1 Description

1- Consider the electrical circuit in Fig.1. $v_s(t)$ is the input and $v_x(t)$ is the output.

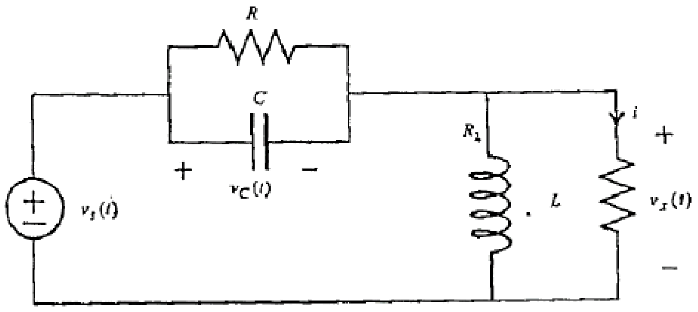


Fig. 1

a. Find the state-space equations.

b. Find the relationship between the parameters where the system is uncontrollable.

c. Write the transfer function. Is there any Pole-Zero cancelation possible? If so, determine the relationship between the parameters and explain parts b and c.

Solution

a) The equation can be written as,

$$v_s(t) = v_x(t) + v_c(t)$$

where,

$$v_x(t) = L\frac{di(t)}{dt}$$

thus,

$$v_s(t) - L\frac{di(t)}{dt} = v_c(t)$$

For current it can be derived as,

$$i_c(t) = C\frac{dv_c(t)}{dt}$$

$$\left[i_L(t) + i_R(t) - \frac{v_c(t)}{R}\right] = C\frac{dv_c(t)}{dt}$$

$$\left[i_L(t) + \frac{v_s(t) - v_c(t)}{R} - \frac{v_c(t)}{R}\right] = C\frac{dv_c(t)}{dt}$$

$$\left[i_L(t) + \frac{-2v_c(t)}{R} + \frac{v_s(t)}{R}\right] = C\frac{dv_c(t)}{dt}$$

Therefore,

$$\frac{dv_c(t)}{dt} = \left[\frac{i_L(t)}{C} + \frac{-2v_c(t)}{RC} + \frac{v_s(t)}{RC}\right]$$

Considering, $\mathbf{x}(t) = [v_c(t) \quad i_L(t)]^T$, and v_s as an input, the state space representation can be stated as,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\frac{2}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix}u(t)$$
$$y(t) = [-1 \quad 0]\mathbf{x}(t) + u(t)$$

If $R_1 \neq R$, it can be reintroduced as,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\frac{1}{C}\left(\frac{1}{R} + \frac{1}{R_1}\right) & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix}u(t)$$
$$y(t) = [-1 \quad 0]\mathbf{x}(t) + u(t)$$

b) Controllability matrix can be written as,

$$Q_c = [B \quad AB] = \begin{bmatrix} \frac{1}{RC} & \frac{-2}{C^2R^2} + \frac{1}{CL} \\ \frac{1}{L} & -\frac{1}{RLC} \end{bmatrix}$$

where,

$$|Q_c| = -\left(\frac{1}{RLC}\right)\left(\frac{1}{RC}\right) - \left(\frac{-2}{C^2R^2L} + \frac{1}{CL^2}\right) = \frac{2}{C^2R^2L} - \frac{1}{CL^2}$$

If $R_1 \neq R$, it can be represented as,

$$Q_c = [B \quad AB] = \begin{bmatrix} \frac{1}{R_1 C} & \frac{-1}{C^2 R_1}\left(\frac{1}{R} + \frac{1}{R_1}\right) + \frac{1}{CL} \\ \frac{1}{L} & -\frac{1}{R_1 LC} \end{bmatrix}$$

in which,

$$|Q_c| = -\left(\frac{1}{R_1 LC}\right)\left(\frac{1}{R_1 C}\right) + \frac{1}{R_1 LC^2}\left(\frac{1}{R} + \frac{1}{R_1}\right) = \frac{2}{C^2 R^2 L} - \frac{1}{CL^2}$$

The system is uncontrollable, if $R = \sqrt{\frac{L}{C}}$, which yields $|Q_c|$ to be zero. If $R_1 \neq R$, $R_1 R = \frac{L}{C}$.

c) The transfer function of the system can be determined as,

$$H = C(sI - A)^{-1}B + D = [-1 \quad 0]\begin{bmatrix} s + \frac{2}{RC} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} + 1 = \frac{s\left(s + \frac{1}{RC}\right)}{s^2 + \frac{2}{RC}s + \frac{1}{LC}}$$

$$H = \frac{sl(R_1^3 + RR_1^2 + CRR_1 s - R)}{R_1(LR_1^2 s + LRR_1 s + CLRs^2 + R)}$$

System poles can be calculated as,

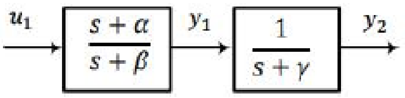
$$s_{1,2} = -\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{1}{LC}}$$

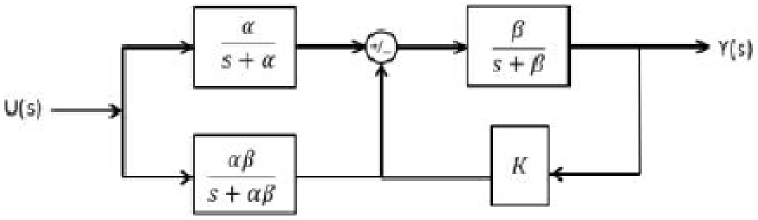
Pole-zero-cancellation can be happened if $R = \sqrt{\frac{L}{C}}$ holds, where R represents the resistance, L denotes the inductance, and C signifies the capacitance. Under these circumstances, both poles of the system are positioned at -1/RC, thereby resulting in notable consequences, (becomes a first order control system).

$$H = \frac{s(s + \frac{1}{RC})}{\left(s + \frac{1}{RC}\right)^2} = \frac{s}{s + 1/RC}$$

Problem 2 Description

2- Specify the controllability conditions for the bellow systems. $(\alpha, \beta > 0)$

a. 

b. 

Solution

a) Using state space representation,

$$\dot{y}_1 + \beta y_1 = \dot{u}_1 + \alpha u_1$$
$$x_1 = y_1 - u_1$$
$$\rightarrow \dot{x}_1 = -\beta x_1 + (\alpha - \beta) u_1$$

then,

$$\dot{y}_2 + \gamma y_2 = u_2$$
$$x_2 = y_2$$
$$\rightarrow \dot{x}_2 = -\gamma x_2 + u_2$$

therefore,

$$\dot{x} = \begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix} x + \begin{bmatrix} \alpha - \beta \\ 1 \end{bmatrix} u_1$$
$$y = [0 \quad 1] x$$
$$A = \begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix}, B = \begin{bmatrix} \alpha - \beta \\ 1 \end{bmatrix}, C = [0 \quad 1], D = 0$$

Controllability,

$$Q = [B \quad AB] = \left[\begin{matrix} \alpha-\beta \\ 1 \end{matrix} \quad \begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} \alpha-\beta \\ 1 \end{bmatrix}\right] = \begin{bmatrix} \alpha-\beta & -\beta(\alpha-\beta) \\ 1 & \alpha-\beta-\gamma \end{bmatrix} \rightarrow |Q|$$
$$= (\alpha-\beta)(\alpha-\gamma)$$

The rank of Q is 2, when $\alpha \neq \beta, \alpha \neq \gamma$

Observability,

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} [0 \quad 1] \\ [0 \quad 1]\begin{bmatrix} -\beta & 0 \\ 0 & -\gamma \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix}$$

It is observable.

b) Similarly, the state space can be obtained as,

$$\dot{x}_1 = \alpha U - \alpha x_1$$
$$\dot{x}_2 = -\beta x_3 - \beta x_1 - (K\beta + \beta)x_2$$
$$\dot{x}_3 = \alpha\beta U - \alpha\beta x_3$$

therefore,

$$A = \begin{bmatrix} -\alpha & 0 & 0 \\ -\beta & -(K\beta+\beta) & -\beta \\ 0 & 0 & -\alpha\beta \end{bmatrix}, B = \begin{bmatrix} \alpha \\ 0 \\ \alpha\beta \end{bmatrix}$$

Controllability,

$$Q = [B \quad AB \quad A^B]$$
$$= \left[\begin{matrix} \alpha \\ 0 \\ \alpha\beta \end{matrix} \quad \begin{bmatrix} -\alpha & 0 & 0 \\ -\beta & -(K\beta+\beta) & -\beta \\ 0 & 0 & -\alpha\beta \end{bmatrix}\begin{bmatrix} \alpha \\ 0 \\ \alpha\beta \end{bmatrix} \quad \begin{bmatrix} -\alpha & 0 & 0 \\ -\beta & -(K\beta+\beta) & -\beta \\ 0 & 0 & -\alpha\beta \end{bmatrix}^2 \begin{bmatrix} \alpha \\ 0 \\ \alpha\beta \end{bmatrix}\right]$$
$$Q = \begin{bmatrix} \alpha & -\alpha^2 & \alpha^3 \\ 0 & -\alpha\beta^2 - \alpha\beta & \alpha\big(\alpha\beta + \beta(\beta+\beta K)\big) + \alpha\beta\big(\alpha\beta^2 + \beta(\beta+\beta K)\big) \\ \alpha * \beta & -\alpha^2\beta^2 & \alpha^3\beta^3 \end{bmatrix}$$
$$|Q| = 2\alpha^5\beta^3 - \alpha^4\beta^3 + \alpha^4\beta^5 - 2\alpha^5\beta^4 - \alpha^4\beta^3 K + \alpha^4\beta^5 K$$
$$|Q| = \alpha^4\beta^3(2\alpha - 1 + \beta^2 - 2\alpha\beta - K + \alpha\beta^2 K)$$

According to the problem, $\alpha, \beta > 0$,

$$2\alpha - 1 + \beta^2 - 2\alpha\beta - K + \alpha\beta^2 K = (\beta-1)(\beta - 2\alpha + K + \beta K + 1) \neq 0$$

One is $\beta \neq 1$

$$\beta - 2\alpha + K + \beta K + 1 \neq 0$$

No feedback gain, $K = 0$,

$$\beta - 2\alpha \neq -1$$

Problem 3 Description

3- For the system in Fig.2: ($l_1 = l_2 = l$)

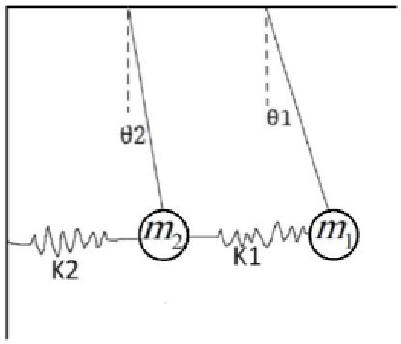


Fig. 2

a. Find the equations of motion with Lagrange method.

b. Write the state-space form of the equations. $q_1 = \theta_1, q_2 = \dot{\theta}_1, q_3 = \theta_2, q_4 = \dot{\theta}_2$.

c. Check the controllability if a torque u_1 just be applied to the 1st pendulum. And again, consider a torque u_2 just on the second pendulum check the controllability.

$k_1 = 20\frac{N}{m}$, $K_2 = 15\frac{N}{m}$, $m_1 = 2kg$, $m_2 = 1kg$, $l = 0.5m$, $g = 9.8\frac{m}{s^2}$

Solution

a) The potential energy can be obtained as,

$$U = \frac{1}{2}k_1(l\sin\theta_1 - l\sin\theta_2)^2 + \frac{1}{2}k_2(l\sin\theta_2)^2 + m_1gl(1-\cos\theta_1) + m_2gl(1-\cos\theta_2)$$

The kinetic energy can be derived as,

$$T = \frac{1}{2}I\dot{\theta}^2$$

$$T = \frac{1}{2}m_1l^2\dot{\theta}_1^2 + \frac{1}{2}m_2l^2\dot{\theta}_2^2$$

The nonlinear governing equation can be achieved as,

$$\frac{\partial T}{\partial\dot{\theta}_1} = m_1l^2\dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = m_1 l^2 \ddot{\theta}_1$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2 l^2 \dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = m_2 l^2 \ddot{\theta}_2$$

$$\frac{\partial T}{\partial \theta_1} = 0$$

$$\frac{\partial T}{\partial \theta_2} = 0$$

$$\frac{\partial U}{\partial \theta_1} = k_1(l\sin\theta_1 - l\sin\theta_2)(l\cos\theta_1) + m_1 g l \sin\theta_1$$

$$\frac{\partial U}{\partial \theta_2} = -k_1(l\sin\theta_1 - l\sin\theta_2)(l\cos\theta_2) + k_2(l\sin\theta_2)(l\cos\theta_2) + m_2 g l \sin\theta_2$$

Consequently,

$$m_1 l^2 \ddot{\theta}_1 + k_1(l\sin\theta_1 - l\sin\theta_2)(l\cos\theta_1) + m_1 g l \sin\theta_1 = 0$$

$$m_2 l^2 \ddot{\theta}_2 - k_1(l\sin\theta_1 - l\sin\theta_2)(l\cos\theta_2) + k_2(l\sin\theta_2)(l\cos\theta_2) + m_2 g l \sin\theta_2 = 0$$

In linear form,

$$m_1 l^2 \ddot{\theta}_1 + k_1 l^2(\theta_1 - \theta_2) + m_1 g l \theta_1 = 0$$
$$m_2 l^2 \ddot{\theta}_2 - k_1 l^2(\theta_1 - \theta_2) + k_2 l^2 \theta_2 + m_2 g l \theta_2 = 0$$

b) Considering $q_1 = \theta_1, q_2 = \dot{\theta}_1, q_3 = \theta_2, q_4 = \dot{\theta}_2$.

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = \left(-\frac{k_1}{m_1} - \frac{g}{l}\right) q_1 + \frac{k_1}{m_1} q_3$$
$$\dot{q}_3 = q_4$$
$$\dot{q}_4 = -\left(\frac{k_1 + k_2}{m_2} + \frac{g}{l}\right) q_3 + \frac{k_1}{m_2}\, q_1$$

State-space can be derived as,

$$\begin{bmatrix}\dot q_1\\ \dot q_2\\ \dot q_3\\ \dot q_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ \left(-\frac{k_1}{m_1}-\frac{g}{l}\right) & 0 & \frac{k_1}{m_1} & 0\\ 0 & 0 & 0 & 1\\ \frac{k_1}{m_2} & 0 & -\left(\frac{k_1+k_2}{m_2}+\frac{g}{l}\right) & 0\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 1 & 0 & 0\\ -29.6 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\\ 20 & 0 & 54.6 & 0\end{bmatrix}$$

c) Controllability

For $B=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$.

$$Q=[B \quad AB \quad A^2B]=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0 & -29.6 & 0\\ 1 & 0 & 0 & 0 & -29.6 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0\end{bmatrix}$$

$$\rightarrow \begin{vmatrix}0 & 1 & 0 & -29.6\\ 1 & 0 & -29.6 & 0\\ 0 & 0 & 0 & 10\\ 0 & 0 & 20 & 0\end{vmatrix}=100\neq 0$$

It is controllable, as the rank of the matrix is full.

For $B=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$.

$$Q=[B \quad AB \quad A^2B]=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 20\\ 0 & 0 & 0 & 0 & 0 & 20 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 54.6\\ 0 & 1 & 0 & 0 & 0 & 54.6 & 0 & 0\end{bmatrix}$$

$$\rightarrow \begin{vmatrix}0 & 0 & 0 & 20\\ 0 & 0 & 20 & 0\\ 0 & 1 & 0 & 54.6\\ 1 & 0 & 54.6 & 0\end{vmatrix}=100\neq 0$$

It is controllable, as the rank of the matrix is full.

Problem 4 Description

4- For the system below check the controllability for different values of a, b, c and specify the uncontrollable modes in the uncontrollable condition.

$$\dot{x} = \begin{bmatrix} s_1 & 1 & 0 \\ 0 & s_2 & 1 \\ 0 & 0 & s_3 \end{bmatrix} x(t) + \begin{bmatrix} a \\ b \\ c \end{bmatrix} u(t)$$

$$y(t) = [d \quad e \quad f]\, x(t) + D\, u(t)$$

Solution

As $s_1 = s_2 = s_3, J = \begin{bmatrix} s_1 & 1 & 0 \\ 0 & s_2 & 1 \\ 0 & 0 & s_3 \end{bmatrix}$ is Jordan form of the system,

It is controllable if,

1- and only if the row vector "B" associated with distinct eigenvalues possesses a non-zero value.

2- In the case of same eigenvalues, where only a single corresponding Jordan block exists, it is essential for the last row of matrix "B" to be non-zero. Here, $c \neq 0$.

3- When confronted with Jordan blocks that share the same eigenvalues, the last row of matrix "B" associated with these blocks should be linearly independent. Additionally, we need to have independent columns in matrix "B" as many as the number of blocks with the same eigenvalues.

According to the second point, and the given system, c should be non-zero in B matrix in order to be controllable. Corresponding third mode of the system can be uncontrollable as results of $c = 0$.

Another way,

$$Q = [B \quad JB \quad J^2B] = \begin{bmatrix} a & as + b & as^2 + 2bc + c \\ b & bs + c & bs^2 + 2cs \\ c & cs & cs^2 \end{bmatrix}$$

$$|Q| = -c^3$$

$|Q|$ should be non-zero in order to be a full rank matrix, therefore, $c \neq 0$.

Problem 5 Description

5- Consider the inverted pendulum on a motor driven cart described in probs. 2.1 and 3.6 Friedland. Determine whether or not it is output controllable with the following sets of observations:

a. Cart displacement: $y = x_1$

b. Cart velocity and pendulum angle: $y_1 = x_2,\ y_2 = x_3$

c. Cart displacement and pendulum angle: $y_1 = x_1,\ y_2 = x_3$

Solution

State Space representation of this system is,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{k^2}{MRr^2} & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{k^2}{MlRr^2} & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k}{MRr} \\ 0 \\ -\dfrac{k}{MlRr} \end{bmatrix} [e]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

where,

$$m = 0.1\ \mathrm{kg}, M = 1.0\ \mathrm{kg}, l = 0.1\ \mathrm{m}, g = 9.8\ \mathrm{m.s^{-2}}, k = 1\ \mathrm{V.s}, R = 100\ \Omega, r = 0.02\ \mathrm{m}$$

a) The system can be represented as,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -12.5 & -0.49 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 12.5 & 10.29 & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ -0.25 \end{bmatrix} [e]$$

$$\boldsymbol{y} = [1 \quad 0 \quad 0 \quad 0] \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

For complete output controllability the following matrix should be rank m.

$$[CB \quad CAB \quad CA^2B \dots]$$
$$[0 \quad 0.25 \quad -3.125 \quad 39.185]$$
$$rank = 1,$$

Hence, the system is not completely output controllable.

b) The system can be represented as,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -12.5 & -0.49 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 12.5 & 10.29 & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ -0.25 \end{bmatrix} [e]$$
$$\boldsymbol{y} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

For complete output controllability the following matrix should be rank m.

$$[CB \quad CAB \quad CA^2B \dots]$$
$$\begin{bmatrix} 0.25 & -3.125 & 39.185 & -491.185 \\ 0 & -0.25 & 3.125 & -41.125 \end{bmatrix}$$
$$rank = 2,$$

Hence, the system is completely output controllable.

c) The system can be represented as,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -12.5 & -0.49 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 12.5 & 10.29 & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0.25 \\ 0 \\ -0.25 \end{bmatrix} [e]$$
$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

For complete output controllability the following matrix should be rank m.

$$[CB \quad CAB \quad CA^2B \dots]$$
$$\begin{bmatrix} 0 & 0.25 & -3.125 & 39.185 & \dots \\ 0 & -0.25 & 3.125 & -41.6351 & \end{bmatrix}$$
$$rank = 2,$$

Hence, the system is completely output controllable.

Problem 2.1 Description

1. Implement the linear and nonlinear equations of part.1-Q3 in Simulink and plot the results for the two initial conditions $\theta_1(0) = 5, and\ \theta_1(0) = 25.$ $(\theta_2(0) = 0)$. Plot the linear and nonlinear responses of θ_1, θ_2 on each other. Explain the results.

Solution

Simulink of this problem can be seen in **Figure 1**.

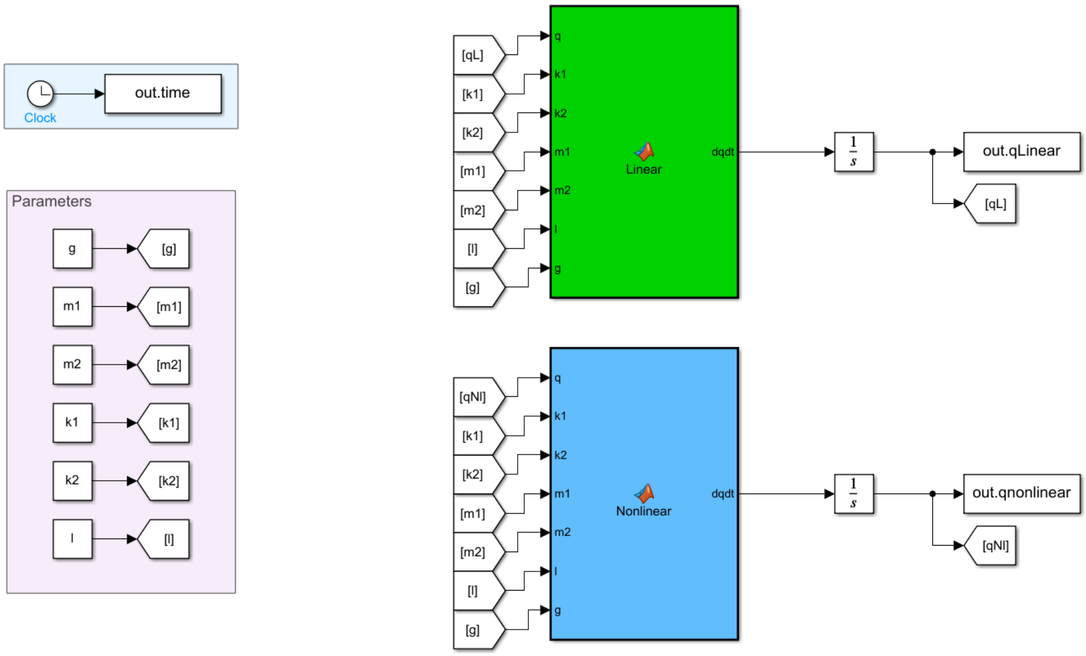


Figure 1 Problem 2.1 MATLAB Simulink

It can be observed that it contains of two main MATLAB function, i.e., Linear and Nonlinear ones. These functions were introduced as follows,

Table 1 Linear function

```
function dqdt = linear2(q,k1,k2,m1,m2,l,g)


dqdt=[q(2);
    -k1/m1*(q(1)-q(3))-g*q(1)/l;
    q(4);
    k1*(q(1)-q(3))/m2-k2/m2*q(3)-g*q(3)/l;];
end
```

Table 2 Nonlinear function

```
function dqdt = nonlinear2(q,k1,k2,m1,m2,l,g)


dqdt=[q(2);
    -k1*(l*sin(q(1))-
l*sin(q(3))*(l*cos(q(1))))/(m1*l^2)-
m1*g*l*sin(q(1))/(m1*l^2);
    q(4);
```

```
    +k1*(l*sin(q(1))-
l*sin(q(3))*(l*cos(q(3))))/(m2*l^2)-
k2*(l*sin(q(3)))*(l*cos(q(3)))/(m2*l^2)-
m2*g*l*sin(q(3))/(m2*l^2);];
end
```

Furthermore, the parameters are presented in ‘parameters’ area,

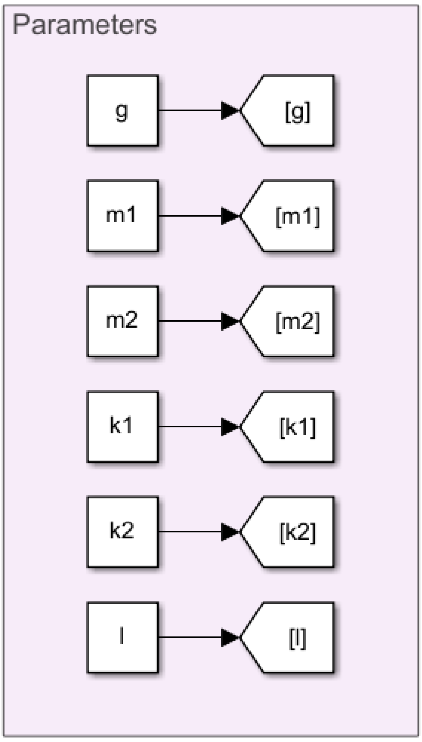


Figure 2 Parameters in Simulink

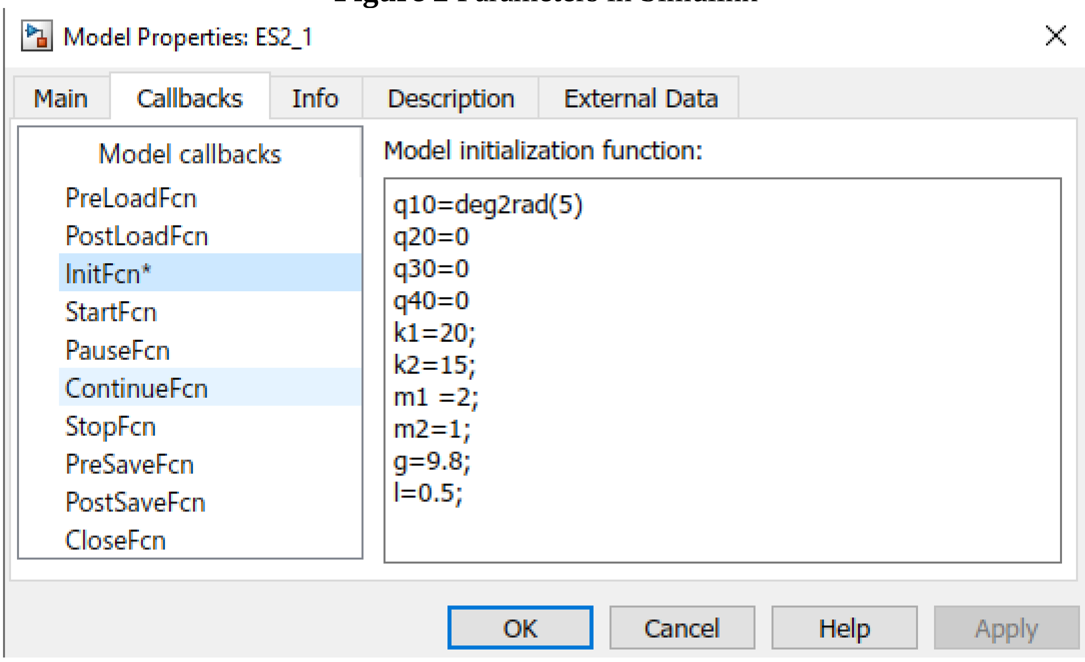


Figure 3 Model Properties InitFcn

The outputs of this Simulink are nonlinear and linear qs, which are plotted in MATLAB environment, for $\theta = 5^o$, it becomes

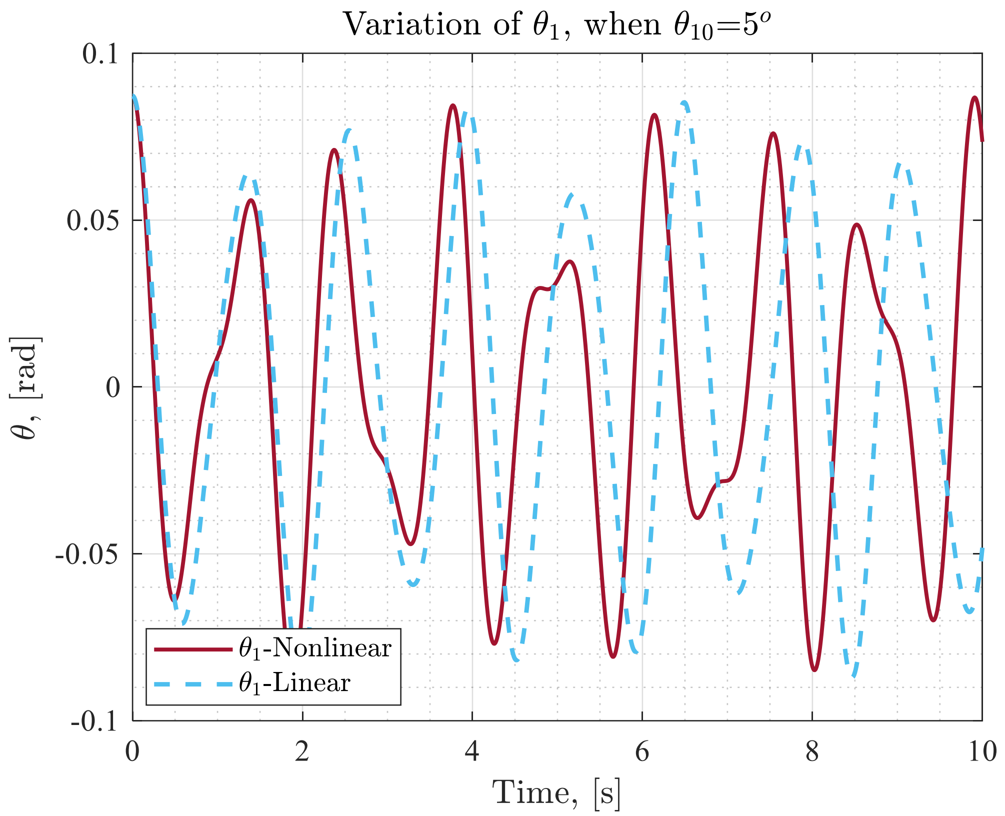


Figure 4 Variation of θ_1 for 10-second maneuver, when $\theta_{10} = 5$.

Same plot for θ_2 can be seen in **Figure 5**.

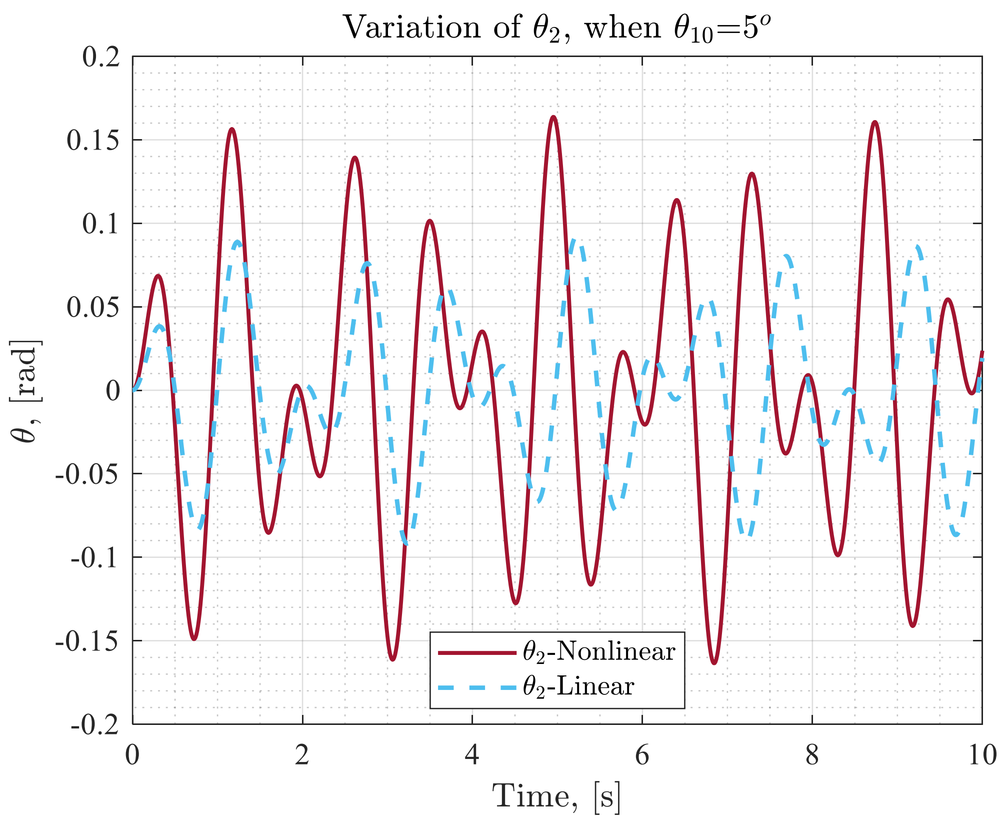


Figure 5 Variation of θ_2 for 10-second maneuver.

For $\theta = 25^o$ we have,

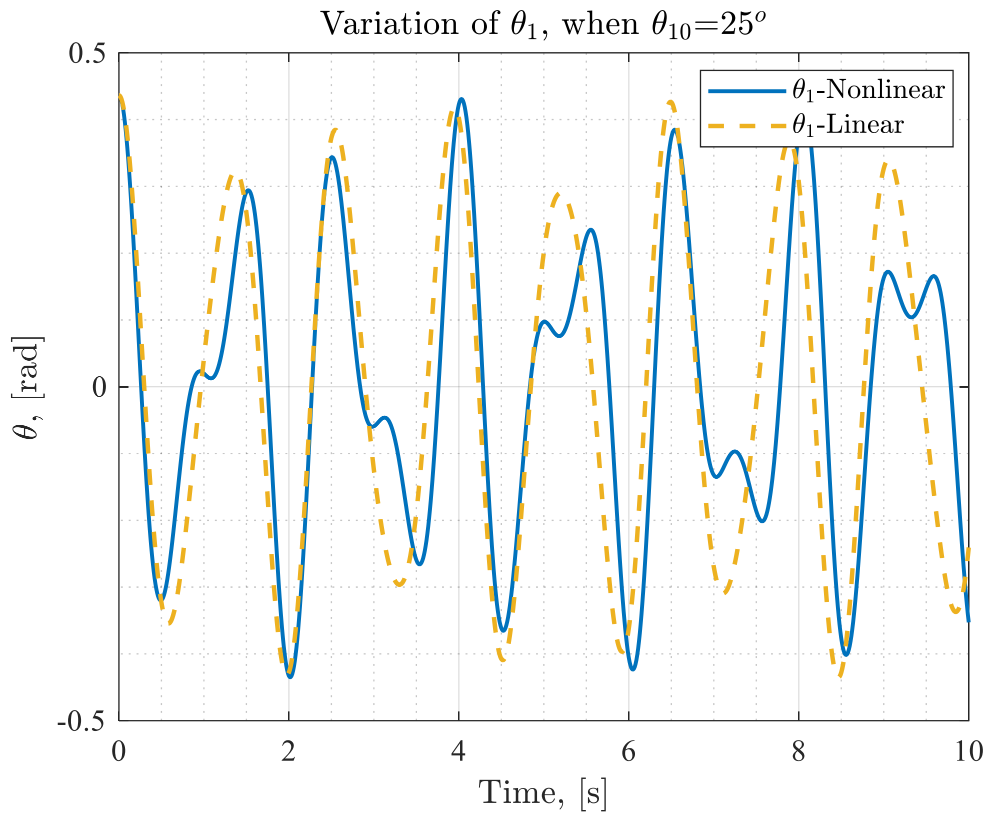


Figure 6 Variation of θ_1 for 10-second maneuver, when $\theta_{10} = 25$.

Next, $\theta = 40^o$

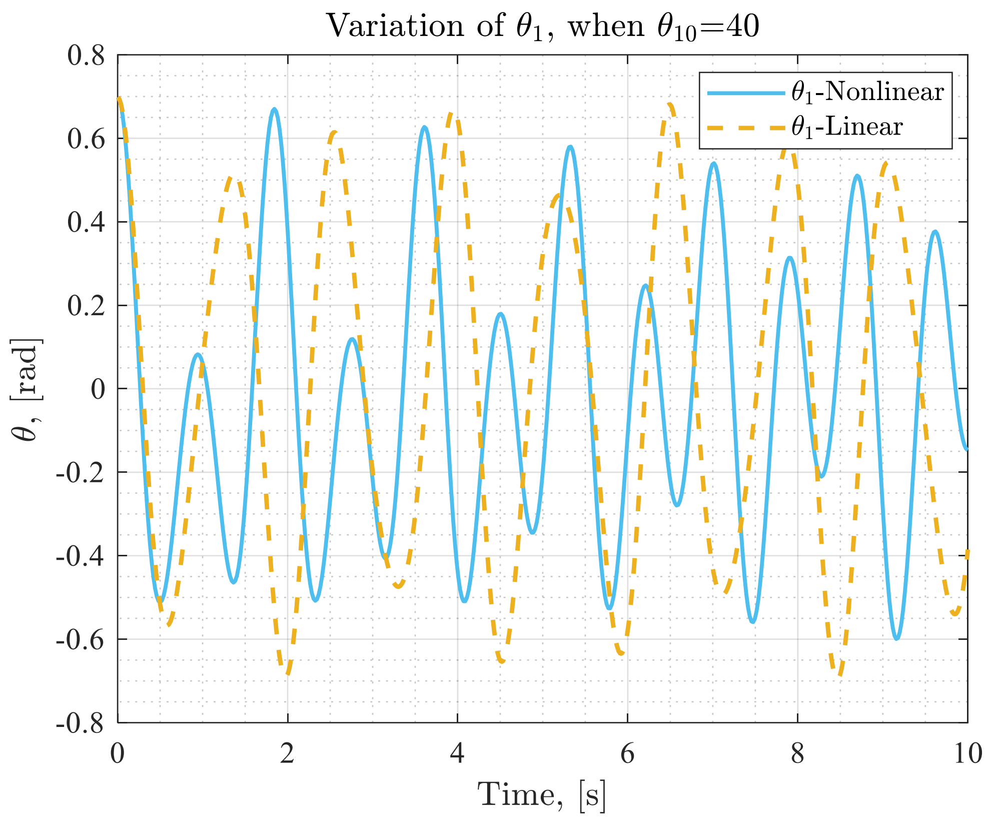


Figure 7 Variation of θ_1 for 10-second maneuver, when $\theta_{10} = 40$.

This discrepancy arises due to the inherent limitations of linearization, which assumes small rotation angles for pendulums, typically $\theta \leq 6$ degrees. As the angular displacement surpasses this critical threshold, the linear approximation fails to capture the nonlinear effects that become increasingly significant.

Problem 2.2 Description

2. Check the result of part.1-Q3 controllability with MATLAB `ctrb(A,B)` command.

Solution

The code is as follows,

```
clc
clear
close all;
```

```
%% Initial Parameters.
k1=20;
k2=15;
m1=2;
m2=1;
l=0.5;
g=9.8;

%% State Space

A=[0 1 0 0;
   (-k1/m1-g/l) 0 k1/m1 0;
   0 0 0 1;
   k1/m1 0 ((k1+k2)/m2+g/l) 0];
B1=[0 0;1 0;0 0;0 0];
B2=[0 0;0 0;0 0;0 1];
B3=[0 0;1 0;0 0;0 1];

q11=ctrb(A,B1);
q1=rank(q11)
unco1 = length(A) - rank(q11)

q22=ctrb(A,B2);
q2=rank(q22)
unco2 = length(A) - rank(q22)

q33=ctrb(A,B3);
q3=rank(q33)
unco3 = length(A) - rank(q33)
```

MATLAB `ctrb(A,B)` command output is controllability matrix, which the rank of it should be calculated in order to find out the controllability of the system. Therefore, `rank(q11)` can be seen, which determines the rank of `q11`(controllability matrix). `unco1 = length(A) - rank(q11)` estimates the number of uncontrollable modes.

```
q1 =

     4


unco1 =

     0


q2 =

     4
```

```
unco2 =

     0
q3 =

     4


unco3 =

     0
```

In this code, q_{11}, q_{22}, and q_{33}, denotes that only u_1, only u_2, and u_1 and u_2 are applied. Which all of them make the controllable.