In the name of God

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

HOMEWORK 5 (STABILITY)

By

Reza Nopour
402126924

Advanced Automatic Control
Mechanical Engineering Department
Amirkabir University of Technology
05 Dec. 23

Problem 1 Description

1. A mechanical system is represented by the three degree-of-freedom linear time-invariant system shown in Figure 1. There are three input forces $u_i(t)$ and three output displacements $y_i(t), i = 1,2,3$. The constant parameters are the masses $m_i, i = 1,2,3$, the spring coefficients k_j , and the damping coefficients $c_j , j = 1,2,3,4$. (You have derived the equations of motion in HW1-Part.1-Q.1)

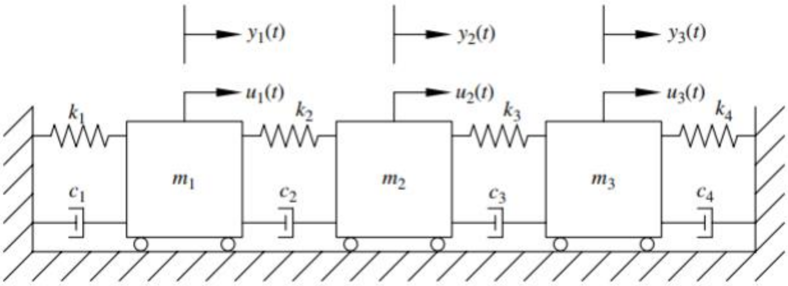


Figure 1 Mechanical system Q.1

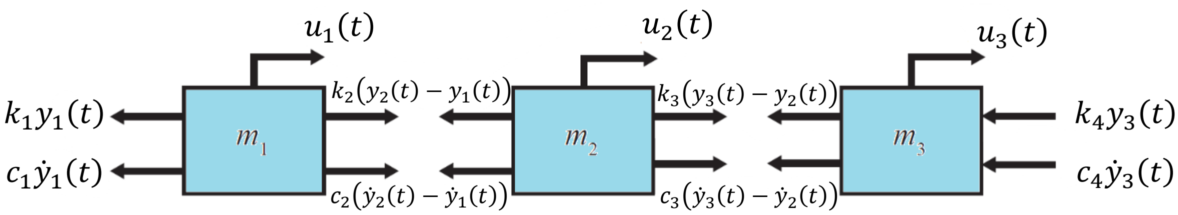
a. Assess system stability using second method Lyapunov analysis. Compare this result with eigenvalue analysis. Explain the results. ($m_i = 1, K_i = 10, C_i = 0$.)

b. Check the stability of the system like part.a for $C_i = 1$.

c. Check the stability of the system like part.a for $C_i = -1$.

Solution

The system mathematical model, i.e., the equation of motion, can be derived as,



$$\text{Mass 1} \overset{+}{\Rightarrow} \sum F = m_1 \ddot{y}_1 = -k_1 y_1 + k_2(y_2 - y_1) - c_1 \dot{y}_1 + c_2(\dot{y}_2 - \dot{y}_1) + u_1$$

$$\text{Mass 2} \overset{+}{\Rightarrow} \sum F = m_2 \ddot{y}_2$$

$$= -k_2(y_2 - y_1) + k_3(y_3 - y_2) - c_2(\dot{y}_2 - \dot{y}_1) + c_3(\dot{y}_3 - \dot{y}_2) + u_2$$

$$\text{Mass 3} \overset{+}{\Rightarrow} \sum F = m_3 \ddot{y}_3 = -k_3(y_3 - y_2) - c_3(\dot{y}_3 - \dot{y}_2) + u_2 - k_4 y_3 - c_4 \dot{y}_3$$

Rearranging and put it in matrix form, the equations of motion are obtained as,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \\ \ddot{y}_3(t) \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \begin{Bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{Bmatrix}$$
$$+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{Bmatrix} = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$

These equations are three linear, coupled, second-order ordinary differential equations. Moreover, it is a multiple-input, multiple output system with three inputs $u_i(t)$ and three outputs $y_i(t)$. In a short form the equations can be written as,

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \boldsymbol{u}(t)$$

in which,

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix}$$

Energy storage can be written as,

$$\begin{aligned} x_1(t) &= y_1(t) \\ x_2(t) &= \dot{x}_1(t) = \dot{y}_1(t) \\ x_3(t) &= y_2(t) \\ x_4(t) &= \dot{x}_3(t) = \dot{y}_2(t) \\ x_5(t) &= y_3(t) \\ x_6(t) &= \dot{x}_5(t) = \dot{y}_3(t) \end{aligned}$$

Their derivatives can be obtained as,

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{k_2 x_3(t) + c_2 x_4(t) - (k_1 + k_2) x_1(t) - (c_1 + c_2) x_2(t)}{m_1} + \frac{u_1(t)}{m_1} \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= \frac{k_2 x_1(t) + c_2 x_2(t) + k_3 x_5(t) + c_3 x_6(t) - (k_2 + k_3) x_3(t) - (c_2 + c_3) x_4(t)}{m_2} + \frac{u_2(t)}{m_2} \\ \dot{x}_5(t) &= x_6(t) \\ \dot{x}_6(t) &= \frac{k_3 x_3(t) + c_3 x_4(t) - (k_3 + k_4) x_5(t) - (c_3 + c_4) x_6(t)}{m_3} + \frac{u_3(t)}{m_3} \end{aligned}$$

Now, the state equation can be achieved as,

$$\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\\\dot{x}_4(t)\\\dot{x}_5(t)\\\dot{x}_6(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3} \end{bmatrix} \begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{bmatrix}$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{bmatrix}$$

Substituting in the form of $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du},$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3} \end{bmatrix}$$

Considering Lyapunov function to be an expression of the energy of the system.

$$V(x) = \frac{k}{2}x_1^2 + \frac{k}{2}(x_1 - x_3)^2 + \frac{k}{2}(x_3 - x_5)^2 + \frac{k}{2}x_5^2 + \frac{1}{2}mx_2^2 + \frac{1}{2}mx_4^2 + \frac{1}{2}mx_6^2$$

a) $\boldsymbol{A}$ can be obtained as, where $m_i = 1$, $Ki = 10$, $C_i = 0$

$$
\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{10+10}{1} & -\dfrac{0+0}{1} & \dfrac{10}{1} & \dfrac{0}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \dfrac{10}{1} & \dfrac{0}{1} & -\dfrac{10+10}{1} & -\dfrac{0+0}{1} & \dfrac{10}{1} & \dfrac{0}{1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{10}{1} & \dfrac{0}{1} & -\dfrac{10+10}{1} & -\dfrac{0+0}{1} \end{bmatrix}
$$

which is equal to,

$$
\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -20 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 0 & -20 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 0 & -20 & 0 \end{bmatrix}
$$

therefore,

$$
\dot{x} = Ax = \begin{bmatrix} x_2 \\ -20x_1 + 10x_3 \\ x_4 \\ 10x_1 - 20x_3 + 10x_5 \\ x_6 \\ 10x_3 - 20x_5 \end{bmatrix}
$$

$grad(V(x))$ can be estimated as,

$$
grad\big(V(x)\big) = \begin{bmatrix} 2kx_1 - kx_3 \\ mx_2 \\ 2kx_3 - kx_1 - kx_5 \\ mx_4 \\ 2kx_5 - kx_3 \\ mx_6 \end{bmatrix} \xrightarrow{k=10,m=1} \begin{bmatrix} 20x_1 - 10x_3 \\ x_2 \\ 20x_3 - 10x_1 - 10x_5 \\ x_4 \\ 20x_5 - 10x_3 \\ x_6 \end{bmatrix}
$$

Consequently, $\dot{V}(x)$ can be obtained as,

$$
\dot{V}(x) = \dot{x}^T grad\big(V(x)\big) = \begin{bmatrix} x_2 \\ -20x_1 + 10x_3 \\ x_4 \\ 10x_1 - 20x_3 + 10x_5 \\ x_6 \\ 10x_3 - 20x_5 \end{bmatrix}^T \begin{bmatrix} 20x_1 - 10x_3 \\ x_2 \\ 20x_3 - 10x_1 - 10x_5 \\ x_4 \\ 20x_5 - 10x_3 \\ x_6 \end{bmatrix}
$$

$$
\begin{aligned}
&= \cancel{20x_1x_2} - \cancel{10x_2x_3} - \cancel{20x_1x_2} + \cancel{10x_2x_3} + \cancel{20x_3x_4} - \cancel{10x_1x_4} \\
&- \cancel{10x_4x_5} + \cancel{10x_1x_4} - \cancel{20x_3x_4} + \cancel{10x_4x_5} + \cancel{20x_5x_6} - \cancel{10x_3x_6} \\
&+ \cancel{10x_3x_6} - \cancel{20x_5x_6} = 0
\end{aligned}
$$

No result. Now, eigenvalues of $\boldsymbol{A}$ can be found here, which are located imaginary axis, the system is marginally stable.

```
-0.0000 + 5.8431i

-0.0000 - 5.8431i

-0.0000 + 4.4721i

-0.0000 - 4.4721i

 0.0000 + 2.4203i

 0.0000 - 2.4203i
```

b) For $C_i = 1$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{10+10}{1} & -\dfrac{1+1}{1} & \dfrac{10}{1} & \dfrac{1}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \dfrac{10}{1} & \dfrac{1}{1} & -\dfrac{10+10}{1} & -\dfrac{1+1}{1} & \dfrac{10}{1} & \dfrac{1}{1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{10}{1} & \dfrac{1}{1} & -\dfrac{10+10}{1} & -\dfrac{1+1}{1} \end{bmatrix}$$

which is equal to,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -20 & -2 & 10 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & 1 & -20 & -2 & 10 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & 1 & -20 & -2 \end{bmatrix}$$

$$\dot{x} = Ax = \begin{bmatrix} x_2 \\ -20x_1 - 2x_2 + 10x_3 + x_4 \\ x_4 \\ 10x_1 + x_2 - 20x_3 - 2x_4 + 10x_5 + x_6 \\ x_6 \\ 10x_3 + x_4 - 20x_5 - 2x_6 \end{bmatrix}$$

$\dot{V}(x)$ is similar to part (a),

$\dot{V}(x) = \dot{x}^T grad\big(V(x)\big)$

$$= \begin{bmatrix} x_2 \\ -20x_1 - 2x_2 + 10x_3 + x_4 \\ x_4 \\ 10x_1 + x_2 - 20x_3 - 2x_4 + 10x_5 + x_6 \\ x_6 \\ 10x_3 + x_4 - 20x_5 - 2x_6 \end{bmatrix}^T \begin{bmatrix} 20x_1 - 10x_3 \\ x_2 \\ 20x_3 - 10x_1 - 10x_5 \\ x_4 \\ 20x_5 - 10x_3 \\ x_6 \end{bmatrix}$$

$$= -2x_2^2 - 2x_4^2 - 2x_6^2 + 2x_2x_4 + 2x_4x_6$$

$$= -x_2^2 - x_6^2 - (x_2 - x_4)^2 - (x_4 - x_6)^2$$

$which\ is < 0, so,$

it is proved that $\dot{V}(x)$ is negative definite and the system is stable according to the Lyapunov theorem.

Eigenvalues are here, which are at LHP, all negative, and demonstrate the stability of the system.

```
-1.7071 + 5.5882i

-1.7071 - 5.5882i

-1.0000 + 4.3589i

-1.0000 - 4.3589i

-0.2929 + 2.4025i

-0.2929 - 2.4025i
```

which are all negative and located and LHP. Therefore, the system is stable.

c) For $C_i = -1$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{10+10}{1} & -\frac{-1-1}{1} & \frac{10}{1} & -\frac{1}{1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{10}{1} & -\frac{1}{1} & -\frac{10+10}{1} & -\frac{1+1}{1} & \frac{10}{1} & \frac{-1}{1} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{10}{1} & -\frac{1}{1} & -\frac{10+10}{1} & -\frac{-1-1}{1} \end{bmatrix}$$

which is equal to,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -20 & 2 & 10 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 10 & -1 & -20 & 2 & 10 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 10 & -1 & -20 & 2 \end{bmatrix}$$

$$\dot{x} = Ax = \begin{bmatrix} x_2 \\ -20x_1 + 2x_2 + 10x_3 - x_4 \\ x_4 \\ 10x_1 - x_2 - 20x_3 + 2x_4 + 10x_5 - x_6 \\ x_6 \\ 10x_3 - x_4 - 20x_5 + 2x_6 \end{bmatrix}$$

$\dot{V}(x)$ is similar to part (a),

$$\dot{V}(x) = \dot{x}^T grad\big(V(x)\big)$$

$$= \begin{bmatrix} x_2 \\ -20x_1 + 2x_2 + 10x_3 - x_4 \\ x_4 \\ 10x_1 - x_2 - 20x_3 + 2x_4 + 10x_5 - x_6 \\ x_6 \\ 10x_3 - x_4 - 20x_5 + 2x_6 \end{bmatrix}^T \begin{bmatrix} 20x_1 - 10x_3 \\ x_2 \\ 20x_3 - 10x_1 - 10x_5 \\ x_4 \\ 20x_5 - 10x_3 \\ x_6 \end{bmatrix}$$

$$= 2x_2^2 + 2x_4^2 + 2x_6^2 - 2x_2x_4 - 2x_4x_6$$

$$= x_2^2 + x_6^2 + (x_2 - x_4)^2 - (x_4 - x_6)^2$$

$which\ is > 0, so,$

it is proved that $\dot{V}(x)$ is positive definite and the system is not stable according to the Lyapunov theorem.

Eigenvalues are,

```
1.7071 + 5.5882i

1.7071 - 5.5882i

1.0000 + 4.3589i

1.0000 - 4.3589i

0.2929 + 2.4025i

0.2929 - 2.4025i
```

which are all positive and located and RHP. Therefore, the system is not stable.

Problem 2 Description

2. Using Lyapunov equation ($A^T P + PA = -Q$)check the stability of below systems for $-\infty < k < \infty$.

a. $\dot{x} = \begin{bmatrix} 0 & k \\ -1 & -2 \end{bmatrix}$

b. $\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & k \end{bmatrix}$

Solution

a) $\boldsymbol{A}$ is equal to,

$$A = \begin{bmatrix} 0 & k \\ -1 & -2 \end{bmatrix}$$

Using Lyapunov equation,

$$A^T P + PA = -Q$$
$$\begin{bmatrix} 0 & -1 \\ k & -2 \end{bmatrix} \begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} a & b \\ b & d \end{bmatrix} \begin{bmatrix} 0 & k \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} -b & -d \\ ka - 2b & kb - 2d \end{bmatrix} + \begin{bmatrix} -b & ak - 2b \\ -d & bk - 2d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$-2b = -1 \rightarrow b = \frac{1}{2}$$
$$-d + ak - 2b = 0$$
$$ka - 2b - d = 0$$
$$2bk - 4d = -1$$
$$d = \frac{k+1}{4}, a = \frac{k+5}{4k}$$
$$P = \begin{bmatrix} \frac{k+5}{4k} & 1/2 \\ 1/2 & \frac{k+1}{4} \end{bmatrix}$$

Using Sylvester's method,

$$|P_{11}| > 0$$
$$\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} > 0$$

$$\frac{k+5}{4k} > 0 \rightarrow k > 0 \; or \; k < -5$$
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \left(\frac{k+5}{4k}\right)\left(\frac{k+1}{k}\right) - \frac{1}{4} = \frac{k^2 + 6k + 5 - 4k}{16k} = \frac{k^2 + 2k + 5}{16k}$$

It should be greater than zero, therefore, $k > 0$.
It can be concluded if $k > 0$, the system is stable.

b) $\boldsymbol{A}$ is equal to,

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 0 & k\end{bmatrix}$$

Using Lyapunov equation,

$$A^TP+PA=-Q$$

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 0 & k\end{bmatrix}\begin{bmatrix}a & b & d\\ b & c & e\\ d & e & f\end{bmatrix}+\begin{bmatrix}a & b & d\\ b & c & e\\ d & e & f\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2 & 0 & k\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

$$a=-\frac{k^2}{4}+\frac{k}{2}-1$$

$$b=-\frac{1}{2}$$

$$c=\frac{\mathrm{k}^3}{8}-\frac{\mathrm{k}^2}{4}+\frac{\mathrm{k}}{2}-\frac{1}{4}$$

$$d=\frac{1}{4}$$

$$e=-\frac{k^2}{8}+\frac{k}{4}-\frac{1}{2}$$

$$f=\frac{k}{8}-\frac{1}{4}$$

Using Sylvester's method,

$$|P_{11}|>0$$

$$\begin{vmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{vmatrix}>0$$

$$\begin{vmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{vmatrix}>0$$

so,

$$|P_{11}|>0$$

$$-\frac{k^2}{4}+\frac{k}{2}-1>0$$

$$k=1\pm\sqrt{3}i$$

then,

$$\begin{vmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{vmatrix}>0\rightarrow\ -\frac{k^5}{32}+\frac{k^4}{8}-\frac{(3\,k^3)}{8}+\frac{9\,k^2}{16}-\frac{(5\,k)}{8}>0$$

$$\begin{gathered}1.49\ +\ 1.49i\\ 1.49\ -\ 1.49i\\ 0.514\ -\ 2.07i\\ 0.514\ +\ 2.07i\\ 0\end{gathered}$$

and

$$\begin{vmatrix}P_{11} & P_{12} & P_{13}\\ P_{21} & P_{22} & P_{23}\\ P_{31} & P_{32} & P_{33}\end{vmatrix}=\frac{k^4}{64}-\frac{k^3}{16}+\frac{13\,k^2}{64}-\frac{5\,k}{16}+\frac{25}{64}>0$$

$$1.5\ +\ 1.66i$$

$$1.5 - 1.66i$$
$$0.5 - 2.18i$$
$$0.5 + 2.18i$$

finally,

$$k > 0 \rightarrow eig\ positive, the\ system\ is\ not\ stable$$
$$k < 0 \rightarrow undifined \rightarrow No\ result$$

Problem 3 Description

3. Consider the system below:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 - x_2 sat(x_2^2 - x_3^2)$$
$$\dot{x}_3 = x_3 sat(x_2^2 - x_3^2)$$

sat is the term for saturation and the saturation limit is $-\alpha \leq (x_2^2 - x_3^2) \leq \alpha$.

a) Check the local stability around the equilibrium point using linearization. (In case sat is not active)

b) Check the global stability for all 3 cases. ($-\alpha$ saturation is active, saturation is not active, and α saturation is active.)

Solution

a) Finding EP,

$$\dot{x}_1 = x_2 = 0$$
$$\dot{x}_2 = -x_1 - x_2 sat = 0 \rightarrow x_1 = 0$$
$$\dot{x}_3 = x_3 = 0$$

Jacobian matrix for linearization,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & x_3^2 - 3x_2^2 & 2x_2x_3 \\ 0 & 2x_2x_3 & x_2^2 - 3x_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow eigens\ are\ located\ at\ imag. axis \rightarrow Marginal\ Stable$$

$$\lambda_1 = 0, \lambda_{2,3} = \pm i$$

b) Global stability by considering Lyapunov function,

$$V(x) = x_1^2 + x_2^2 + x_3^2 \rightarrow grad\big(V(x)\big) = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\dot{V}(x) = \dot{x}^T grad\big(V(x)\big) = \begin{bmatrix} x_2 \\ -x_1 + \alpha x_2 \\ -\alpha x_2 \end{bmatrix}^T \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = 2x_1x_2 - 2x_1x_2 + 2\alpha x_2^2 - 2\alpha x_3^2$$

$$= 2\alpha(x_2^2 - x_3^2) = 2(x_2^2 - x_3^2)(x_2^2 - x_3^2) < 0 \rightarrow negative$$

The system is stable if $-\alpha$ saturation is active.

In the case of no active saturation,

$$\dot{V}(x) = \dot{x}^T grad\big(V(x)\big) = \begin{bmatrix} x_2 \\ -x_1 - x_2^3 + x_2x_3^2 \\ x_3x_2^2 - x_3^2 \end{bmatrix}^T \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = 2(-x_2^4 - x_3^4 + 2x_3^2x_2^2)$$

$$= -2(x_2^2 - x_3^2)^2 < 0 \rightarrow negative$$

The system is stable if *no* saturation is active.

In the case of α active saturation,

$$\dot{V}(x) = \dot{x}^T grad\big(V(x)\big) = \begin{bmatrix} x_2 \\ -x_1 - \alpha x_2 \\ \alpha x_2 \end{bmatrix}^T \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = 2x_1x_2 - 2x_1x_2 - 2\alpha x_2^2 + 2\alpha x_3^2$$

$$=> -2\alpha(x_2^2 - x_3^2) > -2\alpha^2$$

No result.

Problem 4 Description

4. Consider the system below:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u$$

$$y = [1 \quad -0.5 \quad 1]x$$

a. Is it internal stable?

b. Find the transfer function $\frac{y}{u}$ and show that the system is BIBO stable.

c. Consider a disturbance added to the input:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

$$y = [1 \quad -0.5 \quad 1]x$$

Show that in this case the transfer function $\frac{y}{w}$ is not BIBO stable.

Solution

a) For the given system, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, the eigenvalues can be calculated as,

$$|sI - A| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right| = \left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 0 & s \end{bmatrix} \right| = s^3 - 1 = 0 \rightarrow$$

$$s_1 = 1, s_{2,3} = -0.5 \pm 0.86i$$

Because of one pole at RHP, it is not asymptotically stable.

b) The transfer function can be calculated as,

$$H(s) = C(sI - A)^{-1}B = [1 \quad -0.5 \quad 1] \left(\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 0 & s \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1.5(s-1)}{s^3 - 1}$$

$$= \frac{1.5\cancel{(s-1)}}{\cancel{(s-1)}(s^2 + s + 1)} = \frac{1.5}{s^2 + s + 1} \rightarrow BIBO: s^2 + s + 1 = 0$$

$s = -0.5 \pm 0.86i \rightarrow LHP$

$$\rightarrow it\ is\ Bounded\ input - Bounded\ Output\ (BIBO)\ stable.$$

c) The transfer function of the second system can be derived as,

d) $G(s) = C(sI - A)^{-1}B_2 = \begin{bmatrix} 1 & -0.5 & 1 \end{bmatrix} \left(\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 0 & s \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\frac{s^2 - 0.5s + 1}{(s^3 - 1)} \rightarrow Not\ stable\ as\ s = 1\ is\ located\ at\ RHP.$$

Problem 2.1 Description

Plot phase portraits for Part1-Q.1 (3 Dof system) to reinforce your stability results. And plot the total energy stored in the system for each part. Explain the energy and phase portrait results. (All plots in MATLAB)

Solution

Check phase portraits, here, first we can define phase portraits, a phase portrait represents the directional behavior of a system of ordinary differential equations (ODEs). The phase portrait can indicate the stability of the system.

Unstable: Most of the system's solutions tend towards ∞ over time.

Asymptotically stable: All of the system's solutions tend to 0 over time.

Neutrally stable: None of the system's solutions tend towards ∞ over time, but most solutions do not tend towards 0 either.

If $c_i = 0$, the system is marginally (neutrally) stable. It can be seen that none of the system's solutions tend towards ∞ over time, but most solutions do not tend towards 0 either.

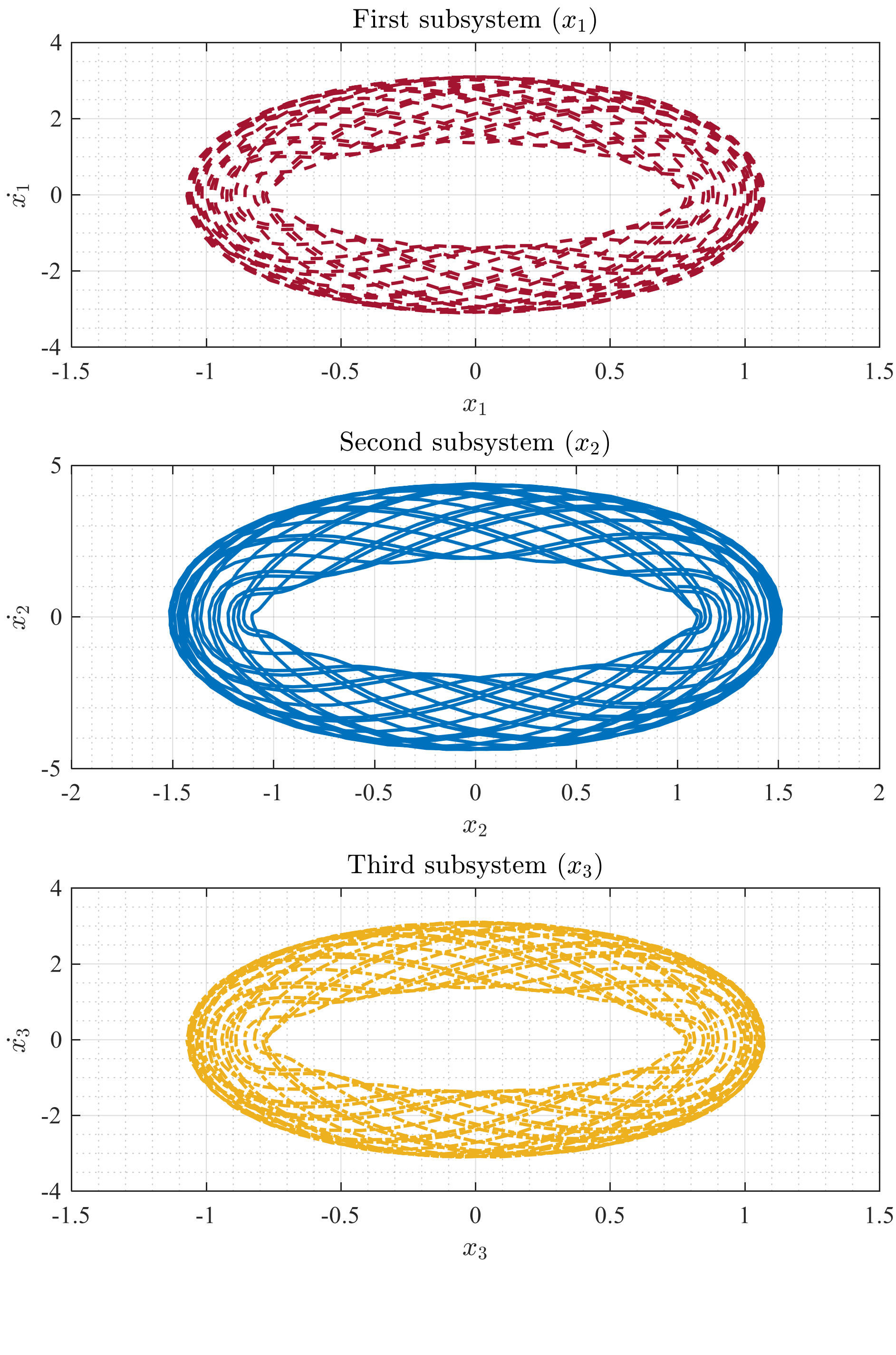


Figure 1 Phase portrait when there is no damper acting on the system.

Energy stored of the system can be observed that fluctuated around 11.5 [J].

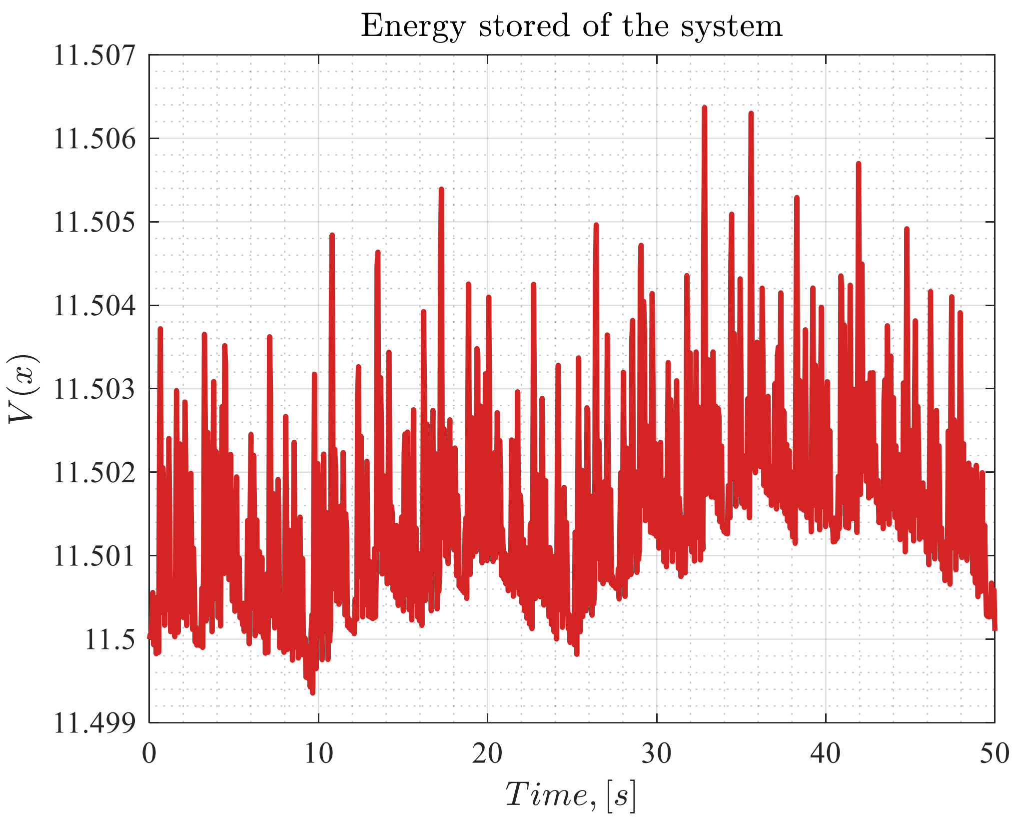


Figure 2 Energy stored of the system when there is no damper acting on the system.

If $c_i = 1$, the system is stable. It can be seen that all of the system's solutions tend to 0 over time.

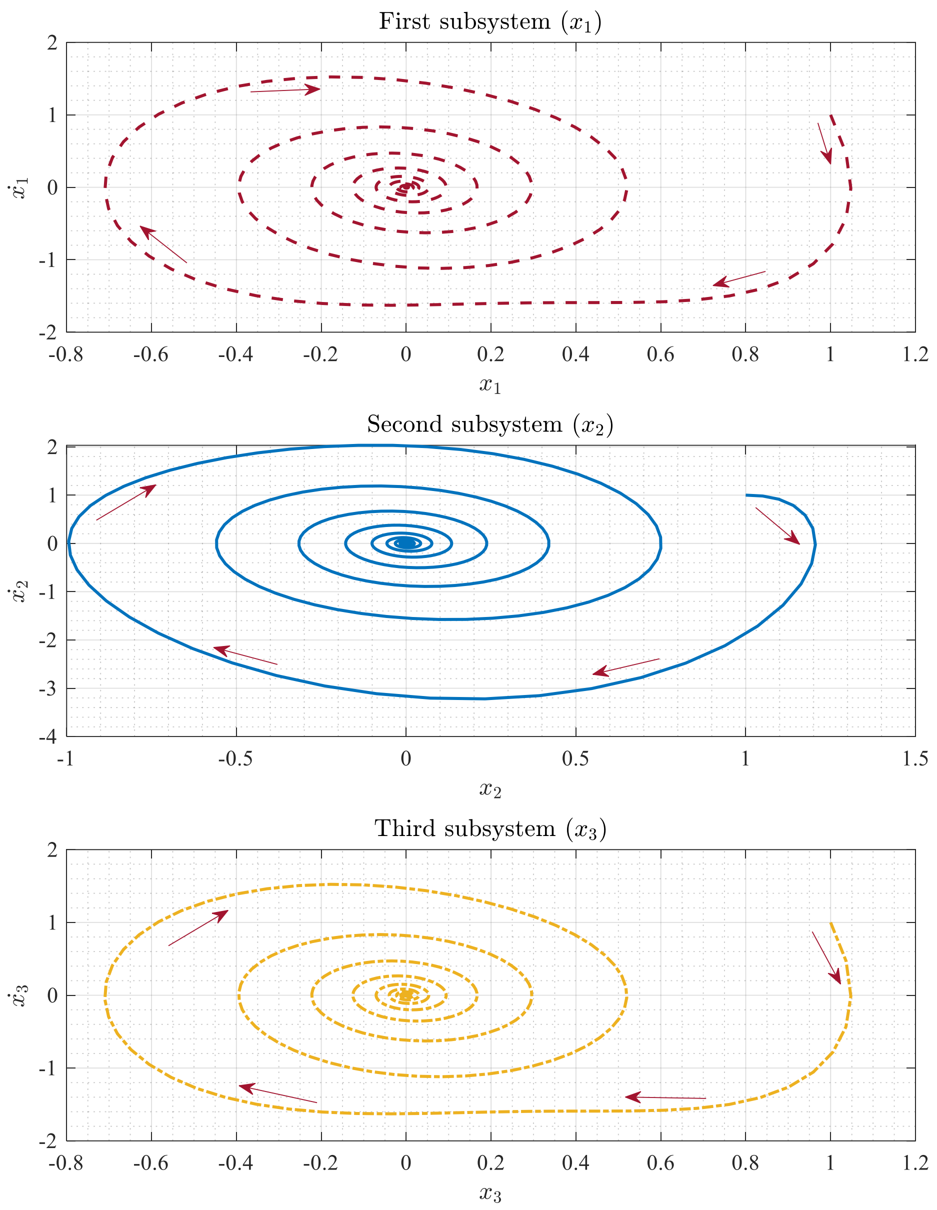


Figure 3 Phase portrait when there are positive dampers acting on the system.

Similarly, the stored energy is plotted. It can be seen that the energy becomes almost zero after 21.42 seconds.

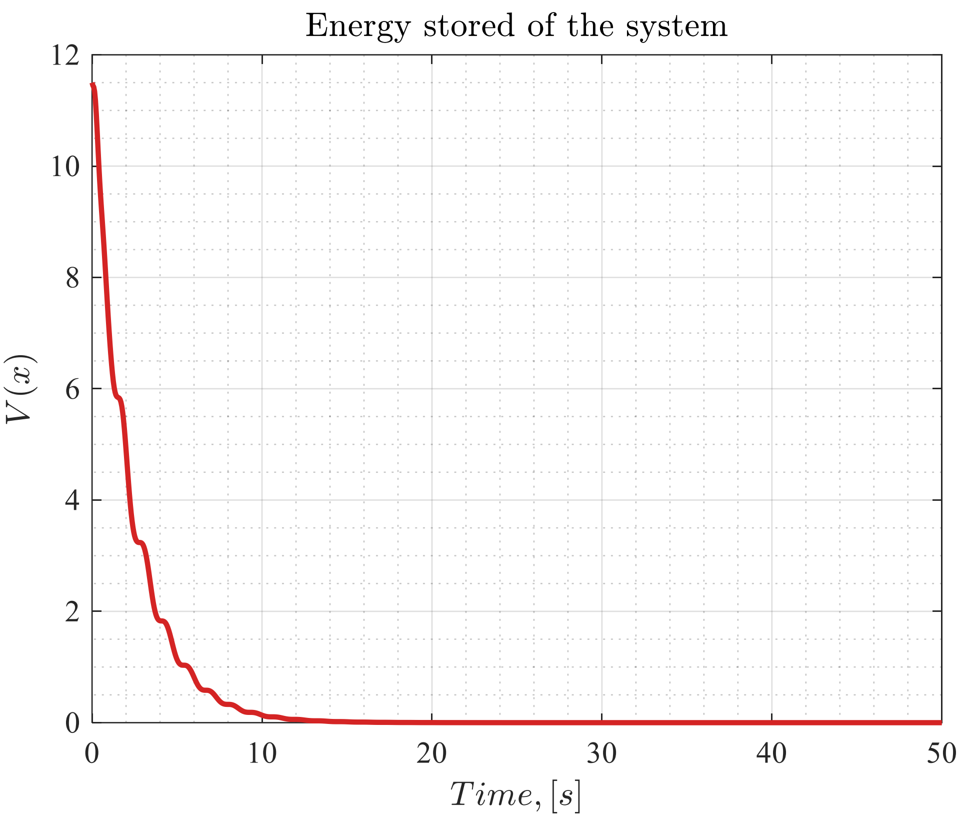


Figure 4 Energy stored of the system when there are positive dampers acting on the system.

If $c_i = -1$, the system is unstable. It can be seen that system's solutions tend towards ∞ over time.

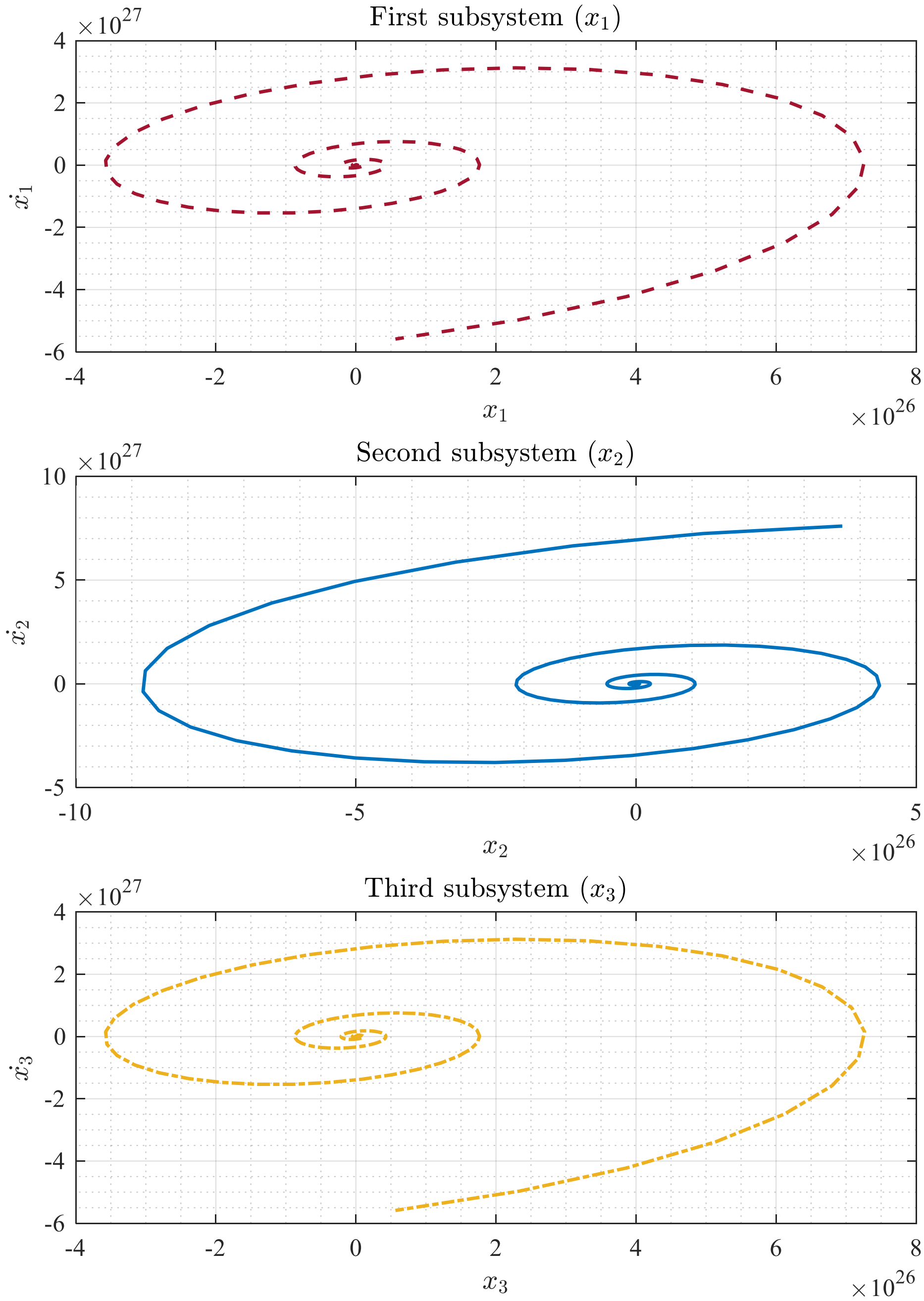


Figure 5 Phase portrait when there are negative dampers acting on the system.

From energy stored illustration, it can be found the system energy is significantly increasing, which as results make the system unstable.

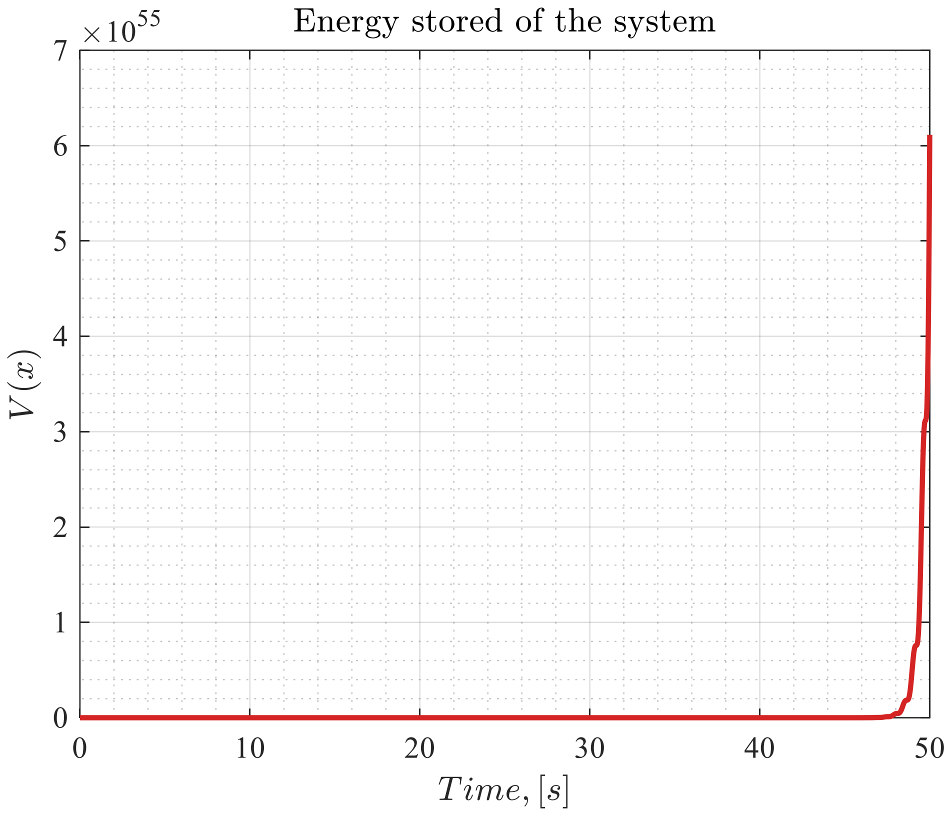


Figure 6 Energy stored of the system when there are negative dampers acting on the system.

One code is as follows; others can be found in the attached folder.

Table 1 Code for problem 2.1.

```
clc
clear
close all
%% CODE FOR COLORS
ccc=["#A2142F";"#0072BD";"#EDB120";"#77AC30"]; %code for
colour
ccc1=["#c1272d";"#0000a7";"#eecc16";"#008176";"b3b3b3"];
%code for colour
c1=[0.83 0.14 0.14];
c2=[1.00 0.54 0.00];
c3=[0.47 0.25 0.80];
c4=[0.25 0.80 0.54];
%% Parameters
m=1;
k=10;
c=1;
%%
```

```
time=linspace(0,50,1000);
x0=ones(6,1);
[t, x] = ode45(@MSD3,time,x0);
figure
title('Phase Portraits','Interpreter','latex')
subplot(3,1,1)
plot(x(:,1),x(:,2),'--','color',ccc(1),'LineWidth' , 1.6);
title('First subsystem ($x_1$)','Interpreter','latex')
xlabel('$x_1$','Interpreter','latex')
ylabel('$\dot{x_1}$','Interpreter','latex')
grid minor
grid on
set(gca,'FontSize',12)
set(gca,'fontname','Times New Roman')
subplot(3,1,2)
plot(x(:,3),x(:,4),'-','color',ccc(2),'LineWidth' , 1.6);
title('Second subsystem ($x_2$)','Interpreter','latex')
xlabel('$x_2$','Interpreter','latex')
ylabel('$\dot{x_2}$','Interpreter','latex')
grid minor
grid on
set(gca,'FontSize',12)
set(gca,'fontname','Times New Roman')
subplot(3,1,3)
plot(x(:,5),x(:,6),'-.','color',ccc(3),'LineWidth' , 1.6);
title('Third subsystem ($x_3$)','Interpreter','latex')
xlabel('$x_3$','Interpreter','latex')
ylabel('$\dot{x_3}$','Interpreter','latex')
grid minor
grid on
set(gca,'FontSize',12)
set(gca,'fontname','Times New Roman')
%%
figure
V_x = (k/2*(x(:,1).^2)) + (k/2*((x(:,1)-x(:,3)).^2)) +
(k/2*((x(:,3)-x(:,5)).^2))+ (k/2*(x(:,5).^2)) +
(m/2*(x(:,2).^2)) +(m/2*(x(:,4).^2)) + (m/2*(x(:,6).^2));
plot(t,V_x,'-','color',c1,'LineWidth' , 2);
title('Energy stored of the system','Interpreter','latex')
xlabel('$Time, [s]$','Interpreter','latex')
ylabel('$V(x)$','Interpreter','latex')
grid minor
grid on
set(gca,'FontSize',12)
set(gca,'fontname','Times New Roman')
for i=1:1000
if V_x(i)<=0.001
fprintf('%d.\n',i)
dd=i;
```

```
break
end
end
fprintf('Energy becomes almost zero at %.2f [sec].\n',t(dd))
%%
function x_dot = MSD3(t,x)
m=1;
k=10;
c=1;
x_dot=[ x(2);
(k*(x(3)-x(1))+c*(x(4)-x(2))-k*x(1)-c*x(2))/m;
x(4);
(k*(x(5)-x(3))+c*(x(6)-x(4))-k*(x(3)-x(1)))/m;
x(6);
(-k*x(5)-c*(x(6))-k*(x(5)-x(3))-c*(x(6)-x(4)))/m; ];
end
```

Problem 2.2 Description

Command `lyap(A',Q)` Solve $A'P + PA = -Q$ for matrix P, given a positive-definite matrix Q. With this in mind check the results of Part.1-Q.1 with `lyap` command. Make sure to put A transpose as an input for `lyap`. (Command `det(matrix)` can be used if needed.)

Solution

For $c_i = 0$, we have the following code,

```
clc
clear
close all
%% Parameters
m=1;
k=10;
c=0;
%%
A =[ 0 1 0 0 0 0;
-2*k/m -2*c/m k/m c/m 0 0;
0 0 0 1 0 0;
k/m c/m -2*k/m -2*c/m k/m c/m;
0 0 0 0 0 1;
0 0 k/m c/m -2*k/m -2*c/m];
```

```
Q=eye(6);
P=lyap(A',Q)
```

which results in

```
Error using lyap

The solution of this Lyapunov equation does not exist or is not
unique.
```

It is similar as calculated in this assignment. There is no result according to the Lyapunov method.

For $c_i = 1$, we have the following code,

```
clc
clear
close all
%% Parameters
m=1;
k=10;
c=1;
%%
A =[ 0 1 0 0 0 0;
-2*k/m -2*c/m k/m c/m 0 0;
0 0 0 1 0 0;
k/m c/m -2*k/m -2*c/m k/m c/m;
0 0 0 0 0 1;
0 0 k/m c/m -2*k/m -2*c/m];


Q=eye(6);
P=lyap(A',Q)
eig(P)
```

which results in

```
P =

   5.4250    0.0375    0.2500    0.0250    0.1250    0.0125

   0.0375    0.4188    0.0250    0.3000    0.0125    0.1563

   0.2500    0.0250    5.5500    0.0500    0.2500    0.0250
```

0.0250	0.3000	0.0500	0.5750	0.0250	0.3000
0.1250	0.0125	0.2500	0.0250	5.4250	0.0375
0.0125	0.1563	0.0250	0.3000	0.0375	0.4188

Obtaining eigenvalues as,

```
0.1507

0.2624

0.9978

5.1965

5.3001

5.9050
```

It is similar as calculated in this assignment. The matrix P is having positive eigenvalues, so, it is P.D and finally the system is asymptotically stable.

For $c_i = -1$, we have the following code,

```
clc
clear
close all
%% Parameters
m=1;
k=10;
c=-1;
%%
A =[ 0 1 0 0 0 0;
-2*k/m -2*c/m k/m c/m 0 0;
0 0 0 1 0 0;
k/m c/m -2*k/m -2*c/m k/m c/m;
0 0 0 0 0 1;
0 0 k/m c/m -2*k/m -2*c/m];


Q=eye(6);
P=lyap(A',Q)
eig(P)
```

which results in

P =

-5.4250	0.0375	-0.2500	0.0250	-0.1250	0.0125
0.0375	-0.4188	0.0250	-0.3000	0.0125	-0.1563
-0.2500	0.0250	-5.5500	0.0500	-0.2500	0.0250
0.0250	-0.3000	0.0500	-0.5750	0.0250	-0.3000
-0.1250	0.0125	-0.2500	0.0250	-5.4250	0.0375
0.0125	-0.1563	0.0250	-0.3000	0.0375	-0.4188

Obtaining eigenvalues as,

-5.9050

-5.3001

-5.1965

-0.9978

-0.2624

-0.1507

It is similar as calculated in this assignment. The matrix P is having negative eigenvalues, so it is N.D and finally the system is unstable.