In the name of God,
the merciful, the compassionate

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

HOMEWORK 6
(STATE FEEDBACK AND PID)

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Problem 1 Description

1. Show that:
   a. If an open-loop system is controllable, then its closed-loop system is also controllable.
   b. If the open-loop system is observable, then its closed-loop system will be observable if and only if it is minimal (i.e. there is no pole-zero cancellation).

Solution

To demonstrate that if an open-loop system is controllable, then its closed-loop system is also controllable, we can use the concept of controllability matrices.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

The controllability matrix, denoted as (C_o), is defined as:

$$C_o = [B \quad AB \quad \dots \quad A^{n-1}B]$$

Now, let's consider the closed-loop system obtained by adding a feedback controller to the open-loop system. The closed-loop system can be represented as:

$$\dot{x} = (A - BK)x + Bu$$
$$y = Cx$$

where (K) represents the feedback gain matrix. To show that the closed-loop system is also controllable, we need to examine the controllability matrix (C_c) for the closed-loop system, defined as:

$$C_o = [B \quad (A - BK)B \quad \dots \quad (A - BK)^{n-1}B]$$

$$[B \quad AB \quad \dots \quad A^{n-1}B] = [B \quad (A - BK)B \quad \dots \quad (A - BK)^{n-1}B]\mathbb{Q}$$

in which the determinant of $\mathbb{Q}$ matrix is one and shows the controllability matrix of two systems (open and closed) are the same!

$$\mathbb{Q} = \begin{bmatrix} 1 & kb & \dots & kA^{n-1}b \\ 0 & 1 & \dots & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

b) ***If part (open-loop observable => closed-loop minimal and observable):***

If the open-loop system is observable, then the observability matrix (O) has full rank. Now, let's consider the closed-loop system. If it is minimal (no pole-zero cancellation), then (A_c) is full rank. Since (C_c) is the same as (C) in this case, the observability matrix (O_c) is formed by stacking the matrices $(C), (CA_c), (CA_c^2), \dots, (CA_c^{\{n-1\}})$. Since (A_c) is full rank, the observability matrix (O_c) also has full rank, and thus the closed-loop system is observable.

Only if part (closed-loop minimal and observable => open-loop observable):

Conversely, if the closed-loop system is minimal and observable, then the observability matrix (O_c) has full rank. Now, we can express (A_c) in terms of the open-loop matrices:

$$[\, A_c = A - BK = A - BKK^{\{-1\}} = A - A_c K^{\{-1\}} \,]$$

Now, the observability matrix (O_c) is formed by stacking the matrices (C), $(CA_c), (CA_c^2), \dots, (CA_c^{\{n-1\}})$. Since (O_c) has full rank, it implies that the matrices $(C), (CA_c), (CA_c^2), \dots, (CA_c^{\{n-1\}})$ are linearly independent.

Consider (CA_c^i). If (CA_c^i) is linearly independent, then $(CA^{\{i-1\}})$ (which is a submatrix of (O)) must also be linearly independent. Therefore, the observability matrix (O) has full rank, and the open-loop system is observable.

In conclusion, if the open-loop system is observable, then the closed-loop system is observable *if and only if it is minimal*.

Problem 2 Description

2. Consider the inverted pendulum on the motor-driven cart of Prob. 2.1 Friedland, with numerical data as given in Prob. 3.6. It is desired to place the dominant poles (in a Butterworth configuration) at $s = -4, \quad s = -2 \pm 2\sqrt{3}j$
   and to leave the pole at s = -25 unchanged.
   a. Find the gain matrix that produces this set of closed-loop poles.
   b. Design a PID controller whose performance is equivalent to the state feedback controller

Solution

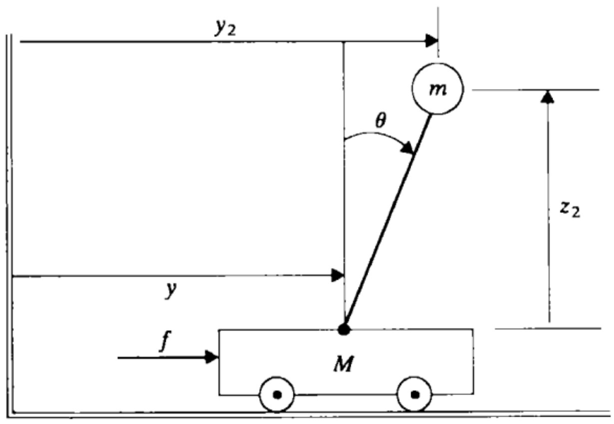


Figure 2.10 Inverted pendulum on moving cart.

a)

Considering the position of M as x instead of y according to the illustration.

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l_2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right)$$

$$U = mgl\cos\theta$$

So, the Lagrangian form of this system can be written as,

$$\frac{\partial T}{\partial \dot{x}} = M\dot{x} + m\left(\dot{x} + l\dot{\theta}\cos\theta\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right)$$

$$\frac{\partial T}{\partial \dot{\theta}} = m\left(\dot{x}l\cos\theta + l^2\dot{\theta}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\left(\ddot{x}l\cos\theta + l^2\ddot{\theta} - \dot{x}\dot{\theta}l\sin\theta\right)$$

$$\frac{\partial T}{\partial x} = 0, \frac{\partial T}{\partial \theta} = -m\dot{x}\dot{\theta}l\sin\theta$$

$$\frac{\partial U}{\partial x} = 0, \frac{\partial U}{\partial \theta} = -mgl\sin\theta$$
$$Q_x = f, Q_\theta = 0.$$

The system equation of motion can be derived as,

$$\begin{cases} M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) = f \\ m\left(\ddot{x}l\cos\theta + l^2\ddot{\theta} - \dot{x}\dot{\theta}l\sin\theta\right) + m\dot{x}\dot{\theta}l\sin\theta - mgl\sin\theta = 0 \end{cases}$$

Above equations are nonlinear, considering small rotation, the equation can be rewritten as,

$$\begin{cases} M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta} - l\dot{\theta}^2\theta\right) = f & (*) \\ \ddot{x} + l\ddot{\theta} - g\theta = 0 \rightarrow \ \ddot{\theta} = \dfrac{(g\theta - \ddot{x})}{l} & (**) \end{cases}$$

Substituting $\ddot{\theta}$ in $(*)$ yields,

$$M\ddot{x} + m\left(\ddot{x} + g\theta - \ddot{x} - l\dot{\theta}^2\theta\right) = f \rightarrow \ \ddot{x} + \frac{m}{M}\left(g\theta - l\dot{\theta}^2\theta\right) = \frac{f}{M}$$

it can be derived as,

$$\ddot{x} = \frac{f}{M} - \frac{mg}{M}\theta - \frac{ml}{M}\dot{\theta}^2\theta$$

Obtained $\ddot{x}$ can be inserted into $(**)$,

$$\ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{m}{M}\dot{\theta}^2\theta = -\frac{f}{Ml}$$

Final equation can be obtained as,

$$\begin{cases} \ddot{x} + \dfrac{mg}{M}\theta = \dfrac{f}{M} & (***) \\ \ddot{\theta} - \left(\dfrac{M+m}{Ml}\right)g\theta = -\dfrac{f}{Ml} & (****) \end{cases}$$

According to the question,

$$\tau = rf$$
$$r\theta = x \rightarrow r\omega = \dot{x}$$
$$\rightarrow f = \frac{\tau}{r} = \frac{k}{Rr}e - \frac{k^2}{Rr}\omega = \frac{k}{Rr}e - \frac{k^2}{Rr^2}\dot{x}$$

thus,

$$\begin{cases} \ddot{x} + \dfrac{k^2}{MRr^2}\dot{x} + \dfrac{mg}{M}\theta = \dfrac{k}{MRr}e & (*) \\ \ddot{\theta} - \dfrac{k^2}{MlRr^2}\dot{x} - \left(\dfrac{M+m}{Ml}\right)g\theta = -\dfrac{k}{MlRr}e & (**) \end{cases}$$

State Space representation of this system is,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\dfrac{k^2}{MRr^2} & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{k^2}{MlRr^2} & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k}{MRr} \\ 0 \\ -\dfrac{k}{MlRr} \end{bmatrix} [e]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

where,

$$m = 0.1\ \text{kg}, M = 1.0\ \text{kg}, l = 0.1\ \text{m}, g = 9.8\ \text{m.s}^{-2}, k = 1\ \text{V.s}, R = 100\ \Omega, r = 0.02\ \text{m}$$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -25 & -0.98 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 25 & 10.78 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0.5 \\ 0 \\ -0.5 \end{bmatrix}$$

According to the Ackerman Formula:

$$k^T = [0 \quad 0 \quad 0 \quad 1] Q^{-1} \Delta_c(A)$$

$$\Delta_c(s) = (s + 2 - j2\sqrt{3})(s + 2 + j2\sqrt{3})(s + 4)(s + 25)$$
$$= s^4 + 33s^3 + 232s^2 + 864s + 1600$$
$$\Delta_c(A) = A^4 + 33A^3 + 232A^2 + 864A + 1600$$
$$Q = controllability\ Matrix = [B \quad AB \quad A^2B \quad A^3B]$$
$$Q = \begin{bmatrix} 0 & 0.5 & -12.5 & 313 \\ 0.5 & -12.5 & 312.99 & -7837 \\ 0 & -0.5 & 12.5 & -317.9 \\ -0.5 & 12.5 & -317.89 & 7959.5 \end{bmatrix}$$
$$Q^{-1} = \begin{bmatrix} 50 & 2.2 & 0 & 0.2 \\ 2.2 & -5.102 & 0.2 & -5.102 \\ -5.102 & -0.204 & -5.102 & -0.204 \\ -0.204 & 0 & -0.204 & 0 \end{bmatrix}$$

$$k = [-326.5306 \quad -226.3265 \quad -812.0906 \quad -242.3265]$$

b) Designing a PID controller.

$$PID: C = P + I\frac{1}{s} + Ds = \frac{Ds^2 + Ps + I}{s}$$

For a closed-loop system,

$$\left(\frac{U}{Y}\right)_{PID} = \frac{CG}{1 + CG}$$

For a closed-loop system,

$$\left(\frac{U}{Y}\right)_{sf} = \frac{G}{1 + GH}$$

Evaluating G, or the transfer function of the original system as,

$$C(sI-A)^{-1}B+D$$

$$=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}\left(\begin{bmatrix}s&0&0&0\\0&s&0&0\\0&0&s&0\\0&0&0&s\end{bmatrix}-\begin{bmatrix}0&1&0&0\\0&-25&-0.98&0\\0&0&0&1\\0&25&10.78&0\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\0.5\\0\\-0.5\end{bmatrix}\rightarrow$$

$$G=\frac{-0.5s^2}{s^4+25s^3-10.78s^2-245s}\rightarrow G=\frac{-0.5s}{s^3+25s^2-10.78s-245}$$

PID for default system:

Using Routh stability criterion for PID controller, in order to place the poles of the system in the LHP.

Routh Criterion

s^3	1	$-10.78-0.5P$
s^2	$25-0.5D$	$245-0.5I$
s	$\frac{(25-0.5D)(-10.78-0.5P)-(-245-0.5I)}{25-0.5D}$	
1	$-245-0.5I$	

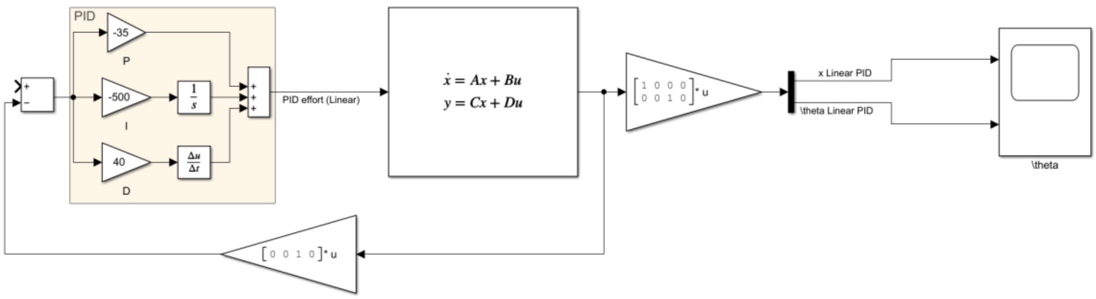
$$G_C=\frac{-0.5(Ps+Ds^2+I)}{s^3+(25-0.5D)s^2-(10.78+0.5P)s-245-0.5I}$$

Therefore,

$$25-0.5D>0\rightarrow 0.5D<25\rightarrow D<50\rightarrow D=40$$

$$-245-0.5I>0\rightarrow 0.5I<-245\rightarrow I<-490\rightarrow I=-500$$

$$\frac{(25-0.5D)(-10.78-0.5P)-(-245-0.5I)}{25-0.5D}>0\rightarrow P<-23.56\rightarrow P=-35$$



PID for the shifted system:

The system poles are,

$$\left(s+2-j2\sqrt{3}\right)\left(s+2+j2\sqrt{3}\right)(s+4)(s+25)$$
$$= s^4+33s^3+232s^2+864s+1600$$

polese located at -25, can be ignored as it is far away from the origin and others.

$$\left(s+2-j2\sqrt{3}\right)\left(s+2+j2\sqrt{3}\right)(s+4)(s+25) = s^3+8s^2+32s+64$$

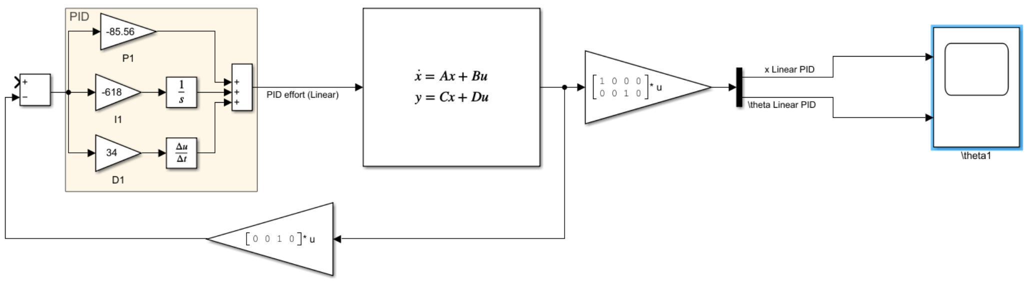
and

$$G_C = \frac{-0.5(Ps+Ds^2+I)}{s^3+(25-0.5D)s^2-(10.78+0.5P)s-245-0.5I}$$

therefore,

$$25-0.5D = 8 \rightarrow D = 34$$
$$-(10.78+0.5P) = 32 \rightarrow P = -85.56$$
$$-245-0.5I = 64 \rightarrow I = -618$$

The answers satisfy the condition of the default Routh criterion,



Problem 3 Description

3. Consider the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$
$$y = [c_1 \quad c_2]x$$

a. Show that this system is uncontrollable and find the uncontrollable mode. Is this mode stable?

b. Check the observability of the system if $c_1 = 0\,, c_2 \neq 0$.

c. Assume state feedback is $u = -[k_1 \quad k_2]x + u_{ext}$. Rewrite the state space equations for the input variable u_{ext}.

d. Show that the closed-loop system is also uncontrollable for all values of k_1, k_2. Compare uncontrollable modes of this part with part a.

e. Can you find any value for k_1, k_2 so that the closed loop system be observable? (Having Part. b assumptions.)

f. Find the $\frac{y}{u_{ext}}$ transfer function. Demonstrate that the values of $k_1, k_2, c_1, and\ c_2$ have no effect on the zero-pole cancellation. Explain your results for all parts.

Solution

a) The controllability matrix is as follows,

$$Q = [B \quad AB] = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$|Q| = 0 \rightarrow Uncontrollable$$

PBH method in finding uncontrollable mode,

$$v^T Q = 0$$

The eigenvalue of the system is equal to zero, and corresponding eigenvector is,

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[1 \quad 1] \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = 0 \rightarrow uncontrollable\ mode$$

It is marginally stable.

b) Observability matrix can be written as,

$$O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ 0 & -c_2 \end{bmatrix}$$
$$|O| = 0 \rightarrow \ it\ is\ not\ observable$$

c) Rewriting the state space equation,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x - \begin{bmatrix} -1 \\ 1 \end{bmatrix} [k_1 \quad k_2] x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{ex} = \begin{bmatrix} k_1 & 1 + k_2 \\ -k_1 & -1 - k_2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_{ex}$$

d) Forming the controllability matrix,

$$Q = [B \quad AB] = \begin{bmatrix} -1 & -k_1 + k_2 + 1 \\ 1 & k_1 - k_2 - 1 \end{bmatrix}$$
$$|Q| = 0 \rightarrow uncontrollable$$

According to the PBH test, uncontrollable mode is,

$$v^T A = \lambda v^T$$

$$v^T Q = 0$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$[1 \quad 1]\begin{bmatrix}-1\\1\end{bmatrix} = [0]$$

Similar to part a.

e) $CA = [0 \quad c_2]\begin{bmatrix}k_1 & 1+k_2\\ -k_1 & -1-k_2\end{bmatrix} = [-k_1c_2 \quad -c_2-k_2c_2]$

Observability matrix can be written as,

$$O = \begin{bmatrix}C\\CA\end{bmatrix} = \begin{bmatrix}0 & c_2\\ -k_1c_2 & -c_2-k_2c_2\end{bmatrix}$$

$$|O| = -k_1c_2^2 \neq 0 \rightarrow k_1 \neq 0$$

For $k_1 \neq 0$, it can be observable.

f) The transfer function of the system can be written as,

$$C(sI-A)^{-1}B + D = [c_1 \quad c_2]\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}k_1 & 1+k_2\\ -k_1 & -1-k_2\end{bmatrix}\right)^{-1}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \frac{[c_1 \quad c_2]}{s^2+(1-k_1+k_2)s}\begin{bmatrix}s+1+k_2 & 1+k_2\\ -k_1 & s-k_1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \frac{s(c_2-c_1)}{s^2+(1-k_1+k_2)s} = \frac{s(c_2-c_1)}{s(s+1-k_1+k_2)}$$

It can be observed that always zero-pole cancellation for $s=0$, happens in this system and systems remains uncontrollable.

Problem 4 Description

4. Consider the transfer function:

$$G(s) = \frac{-2(0.1s+1)(s+1)}{(s-2)^2(s^3+2s^2+s)}$$

a. Write the canonical controllable form of the system.

b. Find the feedback gains for two series of poles (fast and slow) with methods:

$$p_f = [-5+2i, -5-2i, -10, -15, -20]$$

$$p_s = [-0.1, -1-0.5i, -1+0.5i, -2, -5]$$

i. Bass-Gura.

ii. Ackerman.

iii. Mayne-Murdoch just for fast poles.

Solution

The transfer function can be obtained as,

$$G(s) = \frac{-2(0.1s + 1)(s + 1)}{s(s - 2)^2(s + 1)^2}$$

the canonical controllable form can be written as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [-2 \quad -2.2 \quad -0.2 \quad 0 \quad 0]x$$

therefore, controllability matrix can be written as,

$$\boldsymbol{Co} = [B \quad AB \quad A^2B \quad A^3B \quad A^4B] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 7 \\ 0 & 1 & 2 & 7 & 16 \\ 1 & 2 & 7 & 16 & 41 \end{bmatrix} \rightarrow |Co| \neq 0$$

$$\rightarrow full\ rank \rightarrow controllable$$

$$\det(sI - A) = \left| \left(\begin{bmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & -4 & 3 & 2 \end{bmatrix} \right) \right|$$

$$= \begin{vmatrix} s & -1 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & 0 & s & -1 \\ 0 & 4 & 4 & -3 & s-2 \end{vmatrix} = s^5 - 2s^4 - 3s^3 + 4s^2 + 4s$$

$$\boldsymbol{a} = [-2 \quad -3 \quad 4 \quad 4 \quad 0]$$

$$\boldsymbol{p_f} = [-5 + 2i \quad -5 - 2i \quad -10 \quad -15 \quad -20]$$

$$\boldsymbol{p_s} = [-0.1 \quad -1 - 0.5i \quad -1 + 0.5i \quad -2 \quad -5]$$

$$\Psi = \begin{bmatrix} 1 & -2 & -3 & 4 & 4 \\ 0 & 1 & -2 & -3 & 4 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Bass-Gura,

$$\boldsymbol{p_f} = [55 \quad 1129 \quad 10805 \quad 48850 \quad 87000]$$

$$\boldsymbol{p_s} = [9.1 \quad 26.15 \quad 31.275 \quad 15.3750 \quad 1.25]$$

$$k^T = (\boldsymbol{\alpha_f} - \boldsymbol{a})(\boldsymbol{Co} \times \Psi)^{-1}$$

$$= [57 \quad 1132 \quad 10801 \quad 48846 \quad 87000] \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= [87000 \quad 48846 \quad 10801 \quad 1132 \quad 57]$$

$$k^T = (\boldsymbol{\alpha_s} - \boldsymbol{a})(\boldsymbol{Co} \times \Psi)^{-1}$$

$$= [11.1 \quad 29.15 \quad 27.275 \quad 11.375 \quad 1.25] \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= [1.25 \quad 11.375 \quad 27.275 \quad 29.15 \quad 11.1]$$

Ackermann,

$$\boldsymbol{k^T} = [0 \quad 0 \quad 0 \quad 0 \quad 1]\boldsymbol{Co}^{-1} p_{df}(\boldsymbol{A})$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 1] \begin{bmatrix} 4 & 4 & -3 & -2 & 1 \\ 4 & -3 & -2 & 1 & 0 \\ -3 & -2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} (87000I + 484850(A)$$

$$+ 10805(A^2) + 1129(A^3) + 55(A^4) + A^5)$$

$$= [1 \quad 0 \quad 0 \quad 0 \quad 0] \begin{bmatrix} 87000 & 48846 & 10801 & 1132 & 57 \\ 0 & 86772 & 48618 & 10972 & 1246 \\ 0 & -4984 & 81788 & 52356 & 13464 \\ 0 & -53856 & -58840 & 122180 & 79284 \\ 0 & -317136 & -370992 & 179012 & 280748 \end{bmatrix}$$

$$= [87000 \quad 48846 \quad 10801 \quad 1132 \quad 57]$$

$$\boldsymbol{k^T} = [0 \quad 0 \quad 0 \quad 0 \quad 1]\boldsymbol{Co}^{-1}p_{df}(\boldsymbol{A})$$

$$= [0 \quad 0 \quad 0 \quad 0 \quad 1]\begin{bmatrix} 4 & 4 & -3 & -2 & 1 \\ 4 & -3 & -2 & 1 & 0 \\ -3 & -2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}(1.25I + 15.375(A) + 31.275(A^2)$$

$$+ 26.15(A^3) + 9.1(A^4) + A^5)$$

$$= [1 \quad 0 \quad 0 \quad 0 \quad 0]\begin{bmatrix} 1.25 & 11.375 & 27.275 & 29.15 & 11.1 \\ 0 & -43.15 & -33.025 & 60.575 & 51.35 \\ 0 & -205.4 & -248.55 & 121.02 & 163.27 \\ 0 & -653.1 & -858.5 & 241.27 & 447.58 \\ 0 & -1790.3 & -2443.4 & 484.22 & 1136.4 \end{bmatrix}$$

$$= [1.25 \quad 11.375 \quad 27.275 \quad 29.15 \quad 11.1]$$

Mayne-Murdoch,

:..)

Problem 5 Description

5. For the system below, design a state feedback control using the direct method in order to have a maximum overshoot less than 25%, and a settling time less than 1 second.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$
$$y = [1 \quad 0]x$$

Solution

Check controllability of the system,

$$Q = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -10 \end{bmatrix}$$
$$|Q| = -1 \neq 0 \rightarrow full\ rank$$

the system is controllable and now we can design a controller.

According to the given information,

$$t_s < 1\,s \to \frac{3}{\zeta\omega_n} < 1 \to \zeta\omega_n > 3 \;\; (5\%\; criteria)$$

Finding ζ

$$M_p < 0.25 \to e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi} < 0.25 \to \frac{\zeta\pi}{\sqrt{1-\zeta^2}} > 1.386 \to \zeta > 0.40$$

$$\to \zeta\omega_n > 3 \to (0.4)\omega_n > 3 \to \omega_n > 7.5$$

Considering $\omega_n = 8$, and $\zeta = 0.8$

For a closed-loop system we have,

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

which becomes,

$$\Delta_d = s^2 + 12.8s + 64$$

In the direct method,

$$u = -[k_1 \quad k_2]x \to A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -10 - k_2 \end{bmatrix}$$

Calculating the determinant of the $sI - (A - BK)$,

$$|sI - (A - BK)| = \begin{vmatrix} s & -1 \\ k_1 & s + 10 + k_2 \end{vmatrix} = s^2 + (10 + k_2)s + k_1$$

$$\Delta_d = s^2 + (10 + k_2)s + k_1$$

$$k_1 = 64$$

$$k_2 = 2.8$$

Verifying the results with Ackermann!

$$k = [0 \quad 1]Q^{-1}\Delta_d(A)$$

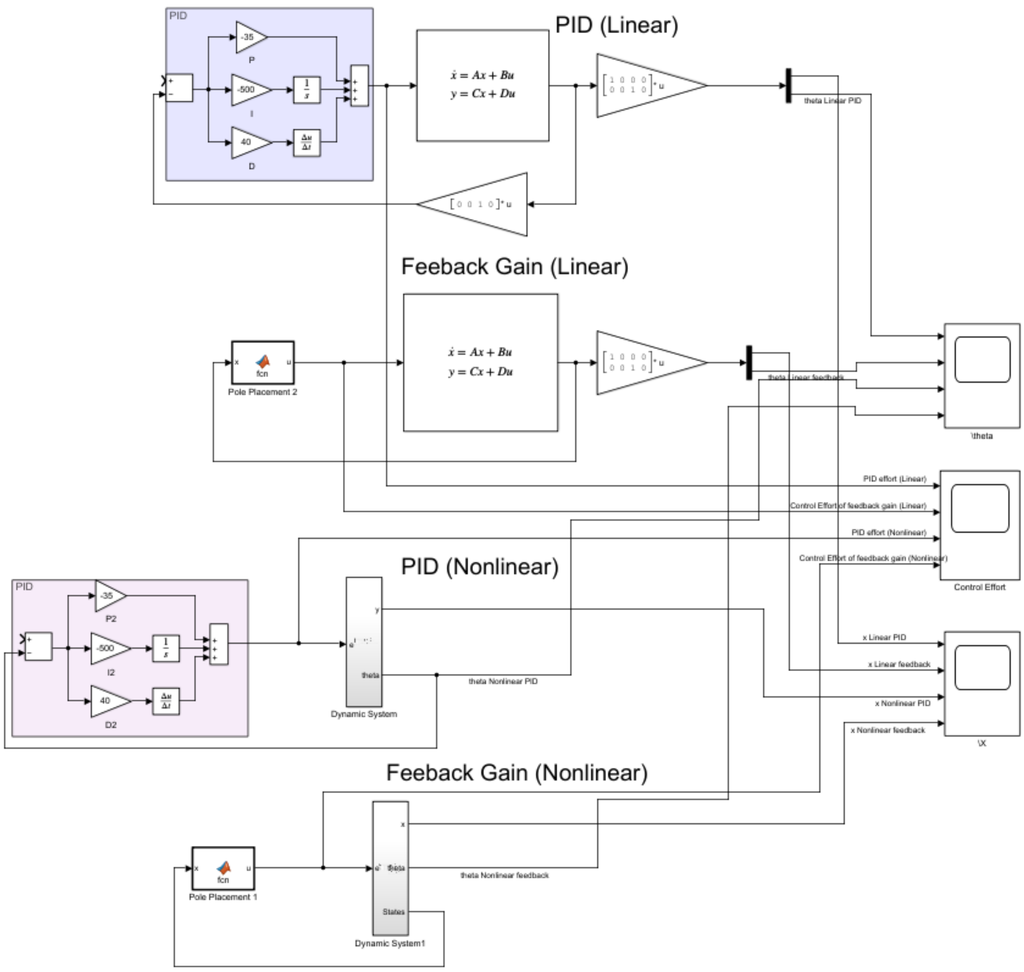
$$k = [0 \quad 1]\begin{bmatrix} 10 & 1 \\ 1 & 0 \end{bmatrix}(A^2 + 12.8A + 64I) = [0 \quad 1]\begin{bmatrix} 10 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 64 & 2.8 \\ 0 & 36 \end{bmatrix} = [64 \quad 2.8]$$

Problem 2.1 Description

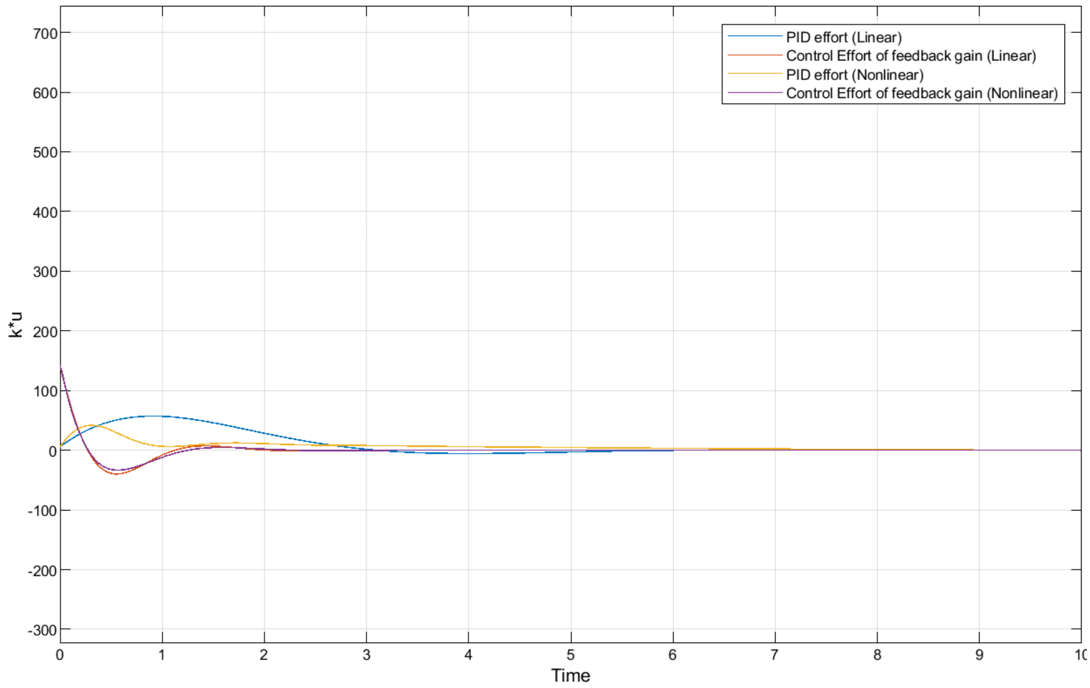
1- Implement Part1.Q2 system (linear and nonlinear equations) in Simulink and plot the system response (θ, x_c) for feedback and PID controller, Compare the linear and nonlinear dynamic responses. Consider the initial condition as $x_0 = [0\ ,0\ ,10°, 0]$ and $x_r = 0$. And for both of these controllers plot the control effort.

Solution

The system is implemented in SIMULINK and the results is as follows,

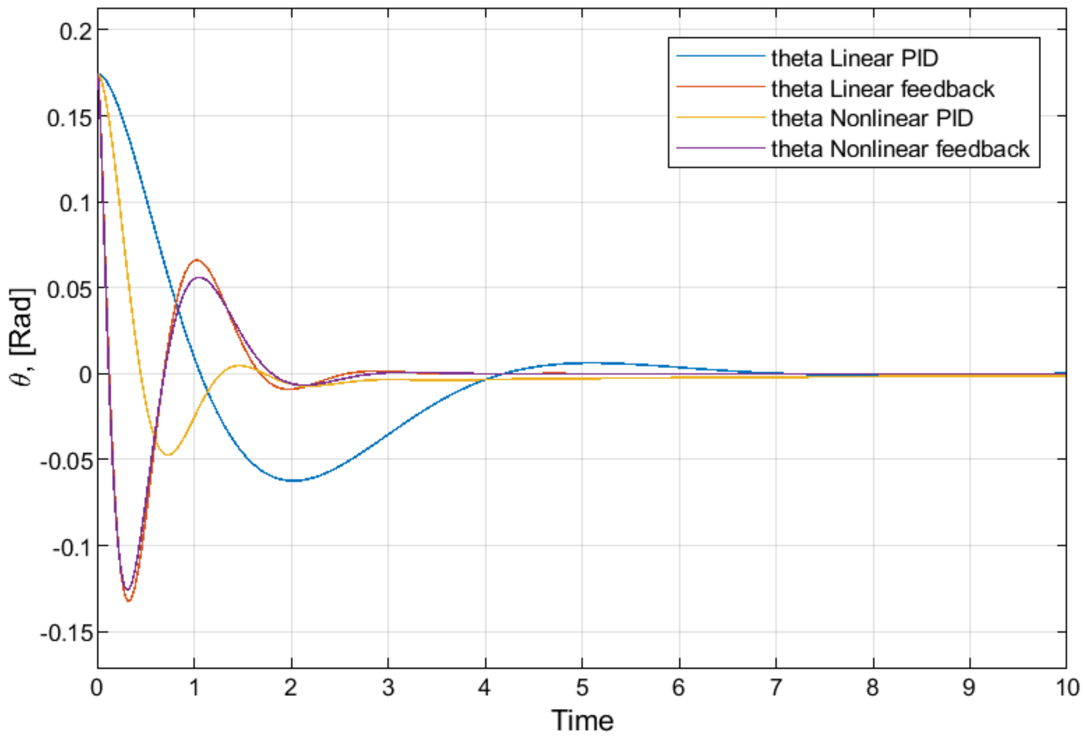


Comparing control efforts is plotted here,



If the square of the signal integral be used, PID controller has much control effort than feedback gain ones,

For comparing $\theta, and\ x$ we have,



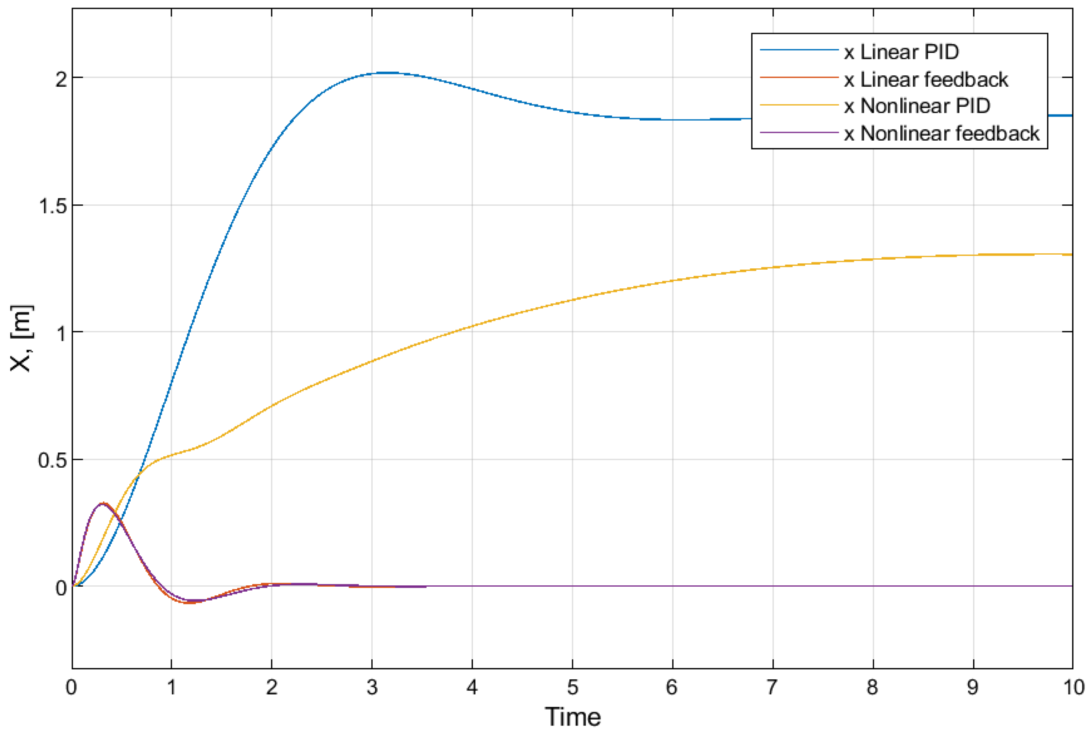


Figure 1 $\theta, and\ x$ during a 10 seconds simulation.

It is apparent that the PID controller possesses the capability to stabilize the system. However, due to the lack of feedback from the position signal of the cart, it is unable to regulate the cart's position effectively, resulting in persistent steady-state error. Furthermore, although the overshoot generated by the PID controller is less than that of feedback controllers, its response speed is significantly lower compared to feedback controllers. Furthermore, observations reveal that linear controllers have provided relatively satisfactory responses for nonlinear equations. Considering that the initial angle of the pendulum is set to 10 degrees, linearized equations approximate the same output as the nonlinear equations. Consequently, linear controllers have proven effective for nonlinear equations as well.

Problem 2.2 Description

2- For Part1.Q4 check the controllability and observability of the system with `ctrb` and `obsv`. Find the unobservable mode with command `[Abar,Bbar,Cbar,T,K]=obsvf(A,B,C)`.

a. check the results of part i., ii., iii. With MATLAB. For Ackerman method use k=acker(A,b,Poles)

b. Plot the slow and fast responses $(x_1, \dots, x_6)$ and the control effort and explain the results.

c. For Part1.Q5 check the feedback gain with command k = place(A,b,Poles).

Solution

Codes have been written in MATLAB and are as follows,

```
clc
clear
close all

syms s

p_f=[-5-2i -5+2i -10 -15 -20];
pf=poly(p_f);
pf(1)=[];
p_s=[-0.1 -1-0.5i -1+0.5i -2 -5];
ps=poly(p_s);
ps(1)=[];

A=[0 1 0 0 0;0 0 1 0 0;0 0 0 1 0;0 0 0 0 1;0 -4 -4 3
2];
B=[0;0;0;0;1];
C=[-2 -2.2 -0.2 0 0];
D=[0];
[b,aa]=ss2tf(A,B,C,D)

Co=ctrb(A,B)
fprintf('the rank of controllability matrix is %d.
\n',rank(Co))

if rank(Co)==length(Co)
    disp('The matrix is full rank and the system is
controllable!')
else
```

```
    disp('The matrix is not full rank and the system is 
uncontrollable!')
end

Ob=obsv(A,C)
fprintf('The rank of observability matrix is %d.
\n',rank(Ob))
if rank(Ob)==length(Ob)
    disp('The matrix is full rank and the system is 
Observable!')
else
    disp('The matrix is not full rank and the system is 
not observable!')
end
```

The results can be seen as follows,

```
    Co =


         0     0     0     0     1

         0     0     0     1     2

         0     0     1     2     7

         0     1     2     7    16

         1     2     7    16    41


    the rank of controllability matrix is 5.

    The matrix is full rank and the system is 
controllable!


    Ob =


       -2.0000   -2.2000   -0.2000         0         0

             0   -2.0000   -2.2000   -0.2000         0

             0         0   -2.0000   -2.2000   -0.2000
```

```
             0    0.8000    0.8000   -2.6000   -2.6000

             0   10.4000   11.2000   -7.0000   -7.8000


    The rank of observability matrix is 4.
The matrix is not full rank and the system is not
observable!
```

It can be understood that results are same as the ones obtained in the first part. The system is controllable but not observable. To find out the unobservable mode, `obsvf(A,B,C)` is used.

```
[Abar,Bbar,Cbar,T,K]=obsvf(A,B,C)
```

And the response is

```
    Abar =


      -1.0000   -0.8519   -1.5859    2.1583    1.1406

       0.0000    1.8869    2.8052   -3.7973   -3.0389

       0.0000    1.1449    0.3726   -0.5111   -0.0692

      -0.0000    0.0000    0.9335    0.1954   -0.3006

       0.0000   -0.0000   -0.0000    0.8384    0.5450



    Bbar =


       0.4472

      -0.8903

      -0.0858
```

```
          0

          0



Cbar =


    0.0000   -0.0000   -0.0000    0.0000    2.9799



T =


    0.4472   -0.4472    0.4472   -0.4472    0.4472
    0.2543   -0.2532    0.2429   -0.1399   -0.8903
   -0.3075    0.2968   -0.1898   -0.8798   -0.0858
    0.4363   -0.3206   -0.8369   -0.0801         0
   -0.6712   -0.7383   -0.0671         0         0



K =


  1     1     1     1     0
```

The results show the mode corresponding to eigen -1 is unobservable.

a) Results of parts i,ii,iii can be obtained and verified as follows,

```
a=[-2 -3 4 4 0];
Psi=[1 a(1) a(2) a(3) a(4)
```

```
      0 1 a(1) a(2) a(3)
      0 0 1 a(1) a(2)
      0 0 0 1 a(1)
      0 0 0 0 1 ];
%% Fast
k_bass_fast=(pf-a)*inv(Co*Psi)

k_acker_fast=acker(A,B,p_f)

k_acker_fast2=[0 0 0 0
1]*inv(Co)*(pf(5)*eye(5)+pf(4)*A+pf(3)*A^2+pf(2)*A^3+pf
(1)*A^4+A^5)
%% Slow
k_bass_slow=(ps-a)*inv(Co*Psi)

k_acker_slow=acker(A,B,p_s)
k_acker_slow2=[0 0 0 0
1]*inv(Co)*(ps(5)*eye(5)+ps(4)*A+ps(3)*A^2+ps(2)*A^3+ps
(1)*A^4+A^5)
```

Results are same as those calculated in part I.

```
k_bass_fast =

       87000       48846       10801        1132          57


k_acker_fast =

       87000       48846       10801        1132          57


k_acker_fast2 =

   1.0e+04 *
```

```
    8.7000    4.8846    1.0801    0.1132    0.0057


k_bass_slow =

    1.2500   11.3750   27.2750   29.1500   11.1000


k_acker_slow =

    1.2500   11.3750   27.2750   29.1500   11.1000


k_acker_slow2 =

1.2500   11.3750   27.2750   29.1500   11.1000
```

b) Slow and fast responses as well as control effort are plotted in this part,

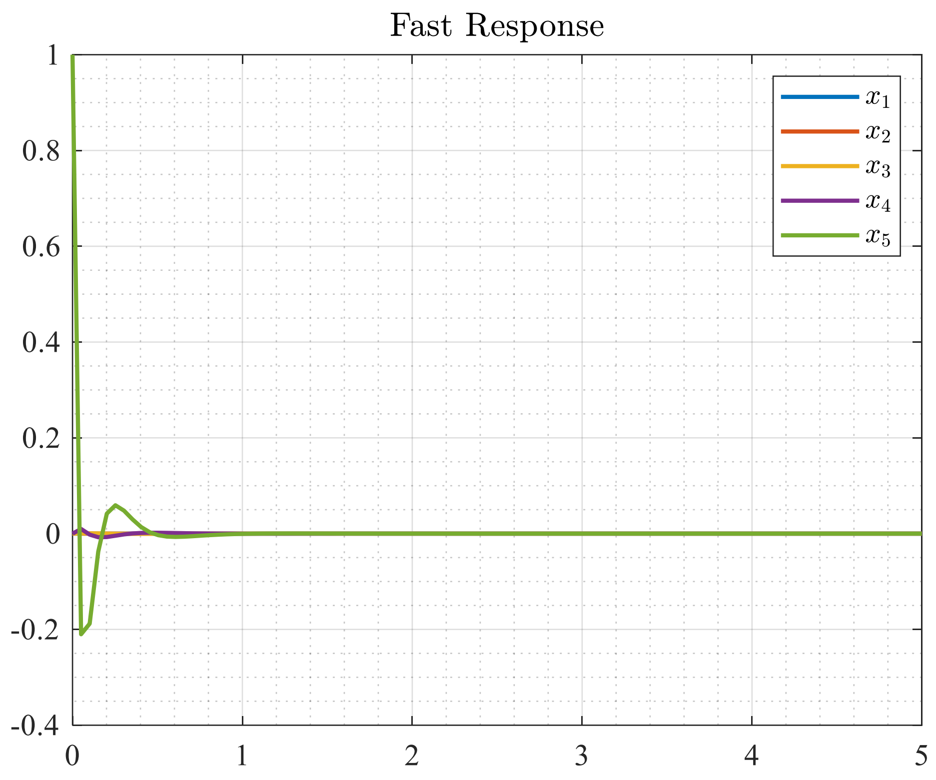


Figure 2 Fast response of the system.

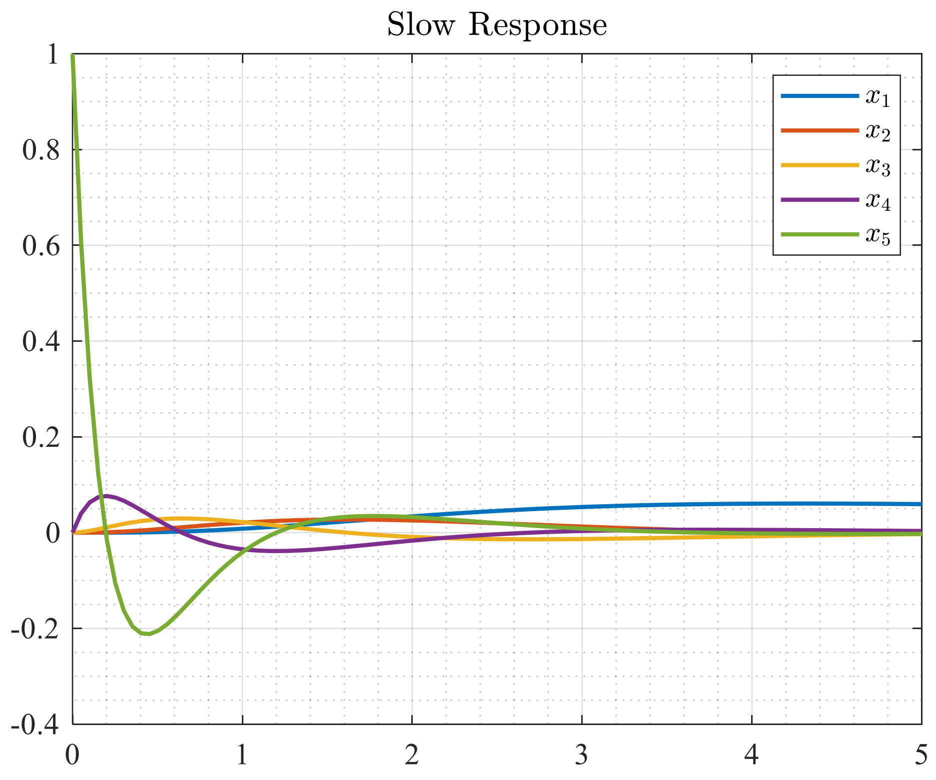


Figure 3 Slow response of the system.

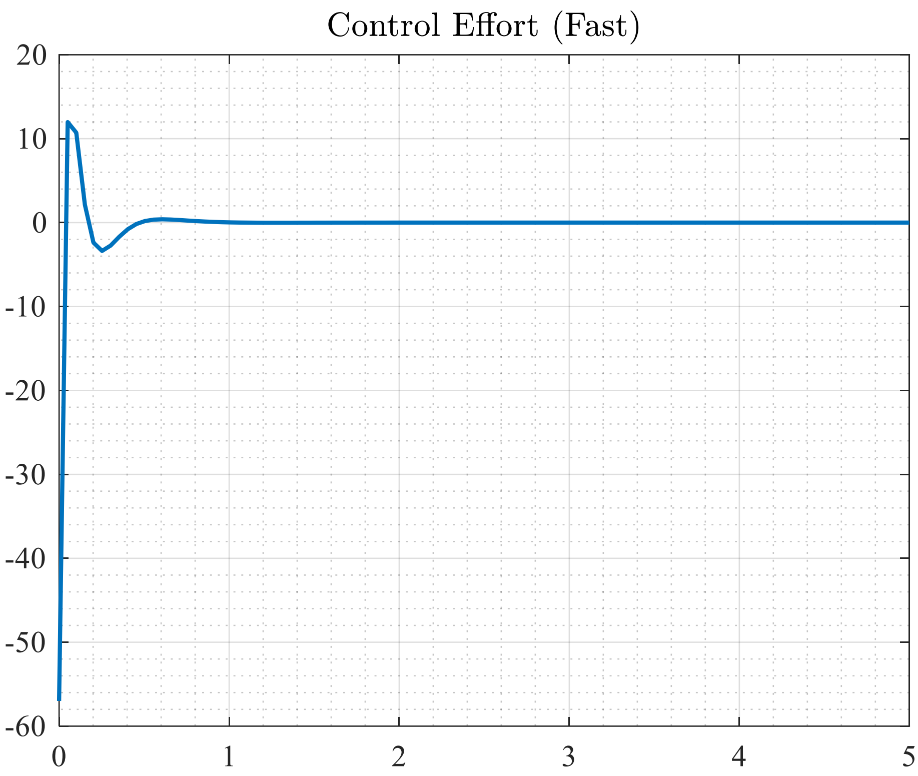


Figure 4 Control effort for fast response of the system.

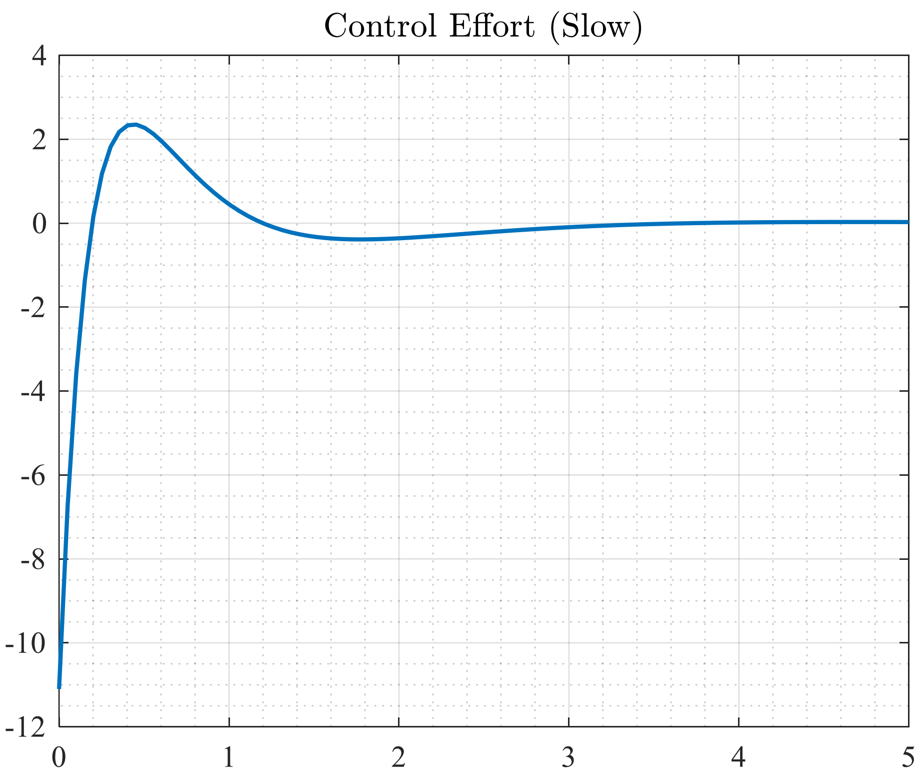


Figure 5 Control effort for slow response of the system.

The control effort required for achieving a fast response in the system is significantly higher compared to achieving a slower response. Additionally, it is evident that by utilizing fast modes, the system can reach a stable state more quickly than when relying on slower modes.

c) For part 1.Q5 we have:

```
%% Problem 5

A=[0 1; 0 -10];
B=[0;1];
C=[1 0];
Poles=roots([1 12.8 64]).';
k=place(A,B,Poles)
```

which the answer is as follows,

```
k =


64.0000    2.8000
```

which is exactly what we have obtained in part I.