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HOMEWORK 1
(SOLVE STATE EQUATIONS)

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Problem 1 Description

1- A mechanical system is represented by the three degree-of-freedom linear time-invariant system shown in Figure 1. There are three input forces $u_i(t)$ and three output displacements $y_i(t), i = 1, 2, 3$. The constant parameters are the masses $m_i, i = 1, 2, 3$, the spring coefficients k_j , and the damping coefficients c_j , $j = 1, 2, 3, 4$.

a) Derive the mathematical model for this system, i.e., draw the free-body diagrams and write the correct number of independent ordinary differential equations. All motion is constrained to be horizontal. Outputs $y_i(t)$ are each measured from the neutral spring equilibrium location of each mass m_i. Also express the results in matrix-vector form $M\ddot{y}(t) + C\dot{y}(t) + Ky(t) = u(t)$.

b) Derive a valid state-space realization for the system. That is, specify the state variables and derive the coefficient matrices A, B, C, and D. Write out your results in matrix-vector form. Give the system order and matrix/vector dimensions of your result. Consider three distinct cases:

i. Multiple-input, multiple-output: three inputs, three displacement outputs.

ii. Multiple-input, multiple-output: two inputs [$u_1(t)$ *and* $u_3(t)$only], all three displacement outputs.

iii. Single-input, single-output: input $u_2(t)$ and output $y_3(t)$.

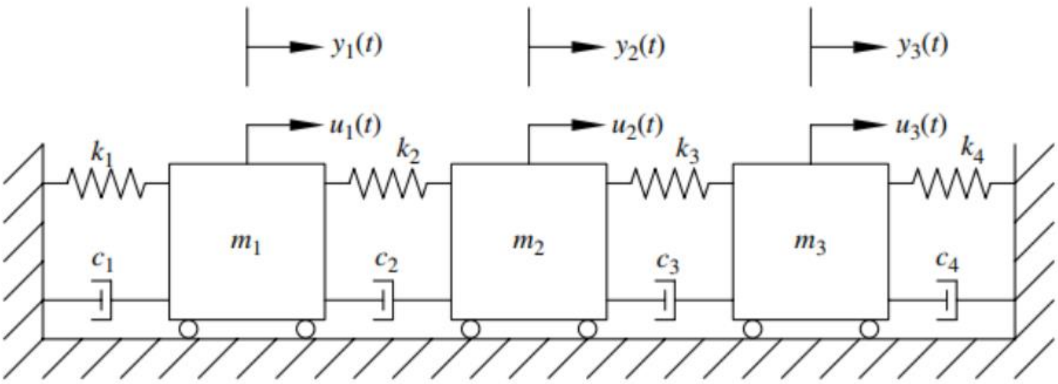
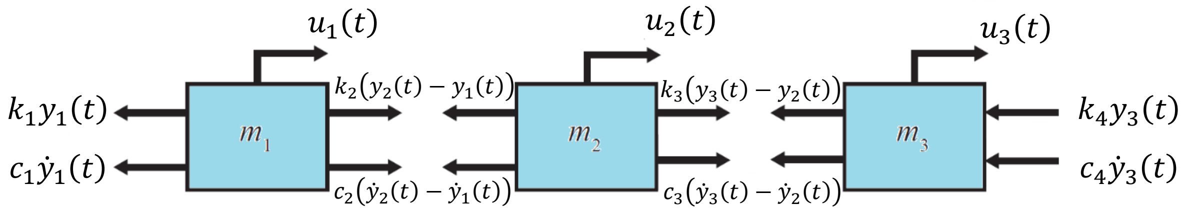


Figure 1 Mechanical system Q.1

Solution

First by considering the following illustration, the system mathematical model, i.e., the equation of motion, can be derived as,



$$\text{Mass 1} \overset{+}{=\to} \sum F = m_1\ddot{y}_1 = -k_1y_1 + k_2(y_2-y_1) - c_1\dot{y}_1 + c_2(\dot{y}_2-\dot{y}_1) + u_1$$

$$\text{Mass 2} \overset{+}{=\to} \sum F = m_2\ddot{y}_2$$
$$= -k_2(y_2-y_1) + k_3(y_3-y_2) - c_2(\dot{y}_2-\dot{y}_1) + c_3(\dot{y}_3-\dot{y}_2) + u_2$$

$$\text{Mass 3} \overset{+}{=\to} \sum F = m_3\ddot{y}_3 = -k_3(y_3-y_2) - c_3(\dot{y}_3-\dot{y}_2) + u_2 - k_4y_3 - c_4\dot{y}_3$$

Rearranging and put it in matrix form, the equations of motion are obtained as,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \\ \ddot{y}_3(t) \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix} \begin{Bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{Bmatrix}$$
$$+ \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix} \begin{Bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{Bmatrix} = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix}$$

These equations are three linear, coupled, second-order ordinary differential equations. Moreover, it is a multiple-input, multiple output system with three inputs $u_i(t)$ and three outputs $y_i(t)$. In a short form the equations can be written as,

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) = \boldsymbol{u}(t)$$

in which,

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix},$$
$$\mathbf{K} = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}$$

i. Multiple-input, Multiple-output

Energy storage can be written as,

$$\begin{aligned} x_1(t) &= y_1(t) \\ x_2(t) &= \dot{x}_1(t) = \dot{y}_1(t) \\ x_3(t) &= y_2(t) \\ x_4(t) &= \dot{x}_3(t) = \dot{y}_2(t) \\ x_5(t) &= y_3(t) \\ x_6(t) &= \dot{x}_5(t) = \dot{y}_3(t) \end{aligned}$$

Their derivatives can be obtained as,

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \frac{k_2x_3(t) + c_2x_4(t) - (k_1 + k_2)x_1(t) - (c_1 + c_2)x_2(t)}{m_1} + \frac{u_1(t)}{m_1}$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{k_2x_1(t) + c_2x_2(t) + k_3x_5(t) + c_3x_6(t) - (k_2 + k_3)x_3(t) - (c_2 + c_3)x_4(t)}{m_2} + \frac{u_2(t)}{m_2}$$

$$\dot{x}_5(t) = x_6(t)$$

$$\dot{x}_6(t) = \frac{k_3x_3(t) + c_3x_4(t) - (k_3 + k_4)x_5(t) - (c_3 + c_4)x_6(t)}{m_3} + \frac{u_3(t)}{m_3}$$

Now, the state equation can be achieved as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \\ \dot{x}_6(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & -\frac{c_1 + c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2 + k_3}{m_2} & -\frac{c_2 + c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3 + k_4}{m_3} & -\frac{c_3 + c_4}{m_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}$$

Substituting in the form of $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du},$

$$
\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & -\frac{c_1 + c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2 + k_3}{m_2} & -\frac{c_2 + c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3 + k_4}{m_3} & -\frac{c_3 + c_4}{m_3} \end{bmatrix}
$$

$$
\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}
$$

$$
\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
\boldsymbol{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

This representation indicates 6-dimensional multiple-input, multiple output system with 6 states.

ii. Multiple-input, Multiple-output, two outputs, three inputs

The state equation can be rewrite as, (Next Page)

$$\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\\\dot{x}_4(t)\\\dot{x}_5(t)\\\dot{x}_6(t)\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2}\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3}\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{1}{m_1} & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & \frac{1}{m_3}\end{bmatrix}\begin{bmatrix}u_1(t)\\u_3(t)\end{bmatrix}$$

$$\begin{bmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}u_1(t)\\u_3(t)\end{bmatrix}$$

Substituting in the form of $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}$,

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2}\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3}\end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix}0 & 0\\ \frac{1}{m_1} & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & \frac{1}{m_3}\end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

iii. Single-input, Single-output, one outputs, one inputs

The state equation can be rewrite as,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \\ \dot{x}_5(t) \\ \dot{x}_6(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \\ 0 \\ 0 \end{bmatrix} u_2(t)$$

$$y_3(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{bmatrix} + [0]u_2(t)$$

Substituting in the form of $\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}; \boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du},$

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2+k_3}{m_2} & -\frac{c_2+c_3}{m_2} & \frac{k_3}{m_2} & \frac{c_3}{m_2} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_3}{m_3} & \frac{c_3}{m_3} & -\frac{k_3+k_4}{m_3} & -\frac{c_3+c_4}{m_3} \end{bmatrix}$$

$$
\boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \\ 0 \\ 0 \end{bmatrix}
$$
$$
\boldsymbol{C} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0]
$$
$$
\boldsymbol{D} = [0]
$$

Problem 2 Description

Problem 2.1 Motor-driven cart with inverted pendulum

The cart carrying the inverted pendulum of Example 2E is driven by an electric motor having the characteristics described in Example 2B. Assume that the motor drives one pair of wheels of the cart, so that the whole cart, pendulum and all, becomes the "load" on the motor. Show that the differential equations that describe the entire system can be written

$$
\ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta = \frac{k}{MRr}e
$$
$$
\ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{k^2}{Mr^2Rl}\dot{x} = -\frac{k}{MRrl}e
$$

where k is the motor torque constant, R is the motor resistance (both as described in Example 2B), r is the ratio of motor torque to linear force applied to the cart ($\tau = rf$), and e is the voltage applied to the motor.

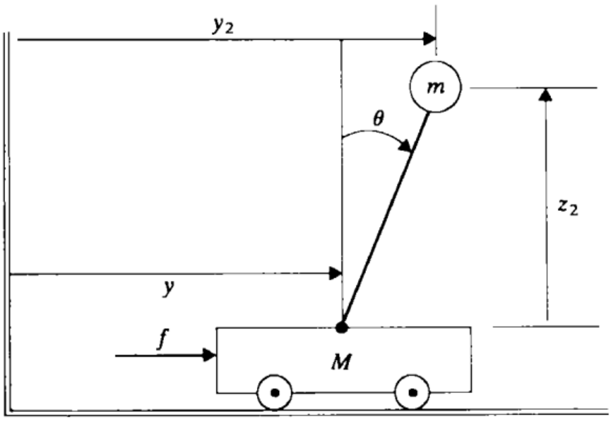


Figure 2.10 Inverted pendulum on moving cart.

Considering the position of M as x instead of y according to the illustration.

$$
T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l_2\dot{\theta}^2 + 2\dot{x}l\dot{\theta}\cos\theta\right)
$$
$$
U = mgl\cos\theta
$$

So, the Lagrangian form of this system can be written as,

$$\frac{\partial T}{\partial \dot{x}} = M\dot{x} + m(\dot{x} + l\dot{\theta}\cos\theta)$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = M\ddot{x} + m(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)$$
$$\frac{\partial T}{\partial \dot{\theta}} = m(\dot{x}l\cos\theta + l^2\dot{\theta})$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m(\ddot{x}l\cos\theta + l^2\ddot{\theta} - \dot{x}\dot{\theta}l\sin\theta)$$
$$\frac{\partial T}{\partial x} = 0, \frac{\partial T}{\partial \theta} = -m\dot{x}\dot{\theta}l\sin\theta$$
$$\frac{\partial U}{\partial x} = 0, \frac{\partial U}{\partial \theta} = -mgl\sin\theta$$
$$Q_x = f, Q_\theta = 0.$$

The system equation of motion can be derived as,

$$\begin{cases} M\ddot{x} + m(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta) = f \\ m(\ddot{x}l\cos\theta + l^2\ddot{\theta} - \dot{x}\dot{\theta}l\sin\theta) + m\dot{x}\dot{\theta}l\sin\theta - mgl\sin\theta = 0 \end{cases}$$

Above equations are nonlinear, considering small rotation, the equation can be rewritten as,

$$\begin{cases} M\ddot{x} + m(\ddot{x} + l\ddot{\theta} - l\dot{\theta}^2\theta) = f & (*) \\ \ddot{x} + l\ddot{\theta} - g\theta = 0 \rightarrow \ddot{\theta} = \dfrac{(g\theta - \ddot{x})}{l} & (**) \end{cases}$$

Substituting $\ddot{\theta}$ in $(*)$ yields,

$$M\ddot{x} + m(\ddot{x} + g\theta - \ddot{x} - l\dot{\theta}^2\theta) = f \rightarrow \ddot{x} + \frac{m}{M}(g\theta - l\dot{\theta}^2\theta) = \frac{f}{M}$$

it can be derived as,

$$\ddot{x} = \frac{f}{M} - \frac{mg}{M}\theta - \frac{ml}{M}\dot{\theta}^2\theta$$

Obtained $\ddot{x}$ can be inserted into $(**)$,

$$\ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{m}{M}\dot{\theta}^2\theta = -\frac{f}{Ml}$$

Final equation can be obtained as,

$$\begin{cases} \ddot{x} + \dfrac{mg}{M}\theta = \dfrac{f}{M} & (***) \\ \ddot{\theta} - \left(\dfrac{M+m}{Ml}\right)g\theta = -\dfrac{f}{Ml} & (****) \end{cases}$$

According to the question,

$$\tau = rf$$
$$r\theta = x \rightarrow r\omega = \dot{x}$$
$$\rightarrow f = \frac{\tau}{r} = \frac{k}{Rr}e - \frac{k^2}{Rr}\omega = \frac{k}{Rr}e - \frac{k^2}{Rr^2}\dot{x}$$

thus,

$$\begin{cases} \ddot{x} + \dfrac{k^2}{MRr^2}\dot{x} + \dfrac{mg}{M}\theta = \dfrac{k}{MRr}e & (*) \\ \ddot{\theta} - \dfrac{k^2}{MlRr^2}\dot{x} - \left(\dfrac{M+m}{Ml}\right)g\theta = -\dfrac{k}{MlRr}e & (**) \end{cases}$$

State Space representation of this system is,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\dfrac{k^2}{MRr^2} & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{k^2}{MlRr^2} & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k}{MRr} \\ 0 \\ -\dfrac{k}{MlRr} \end{bmatrix} [e]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

Problem 3 Description

What are minimal and nonminimal systems? provide an example for each. Where do nonminimal systems come from?

A minimal system is a state-space model that is controllable and observable while maintaining equivalent input-output behavior as the corresponding transfer function. The term "minimal" refers to the fact that it represents the system with the fewest possible states. The minimum number of state variables required to describe a system is equal to the order of the differential equation, although additional state variables beyond the minimum can be defined. For instance, a second-order system can be described using two or more state variables, but a minimal realization would only employ two.

In contrast, a non-minimal system is a state-space representation that arises as a natural description of a discrete-time transfer function, where its dimension is determined by the complete structure of the model. Unlike minimal state-space descriptions, which consider only the order of the denominator, non-minimal systems incorporate state variables that often represent combinations of input and output signals.

Non-minimal systems commonly originate from practical engineering applications and are often derived through precise algorithmic computations, frequently involving numerical optimization techniques.

To illustrate this with an example, consider a simple mass-spring-damper system. The minimal representation of this system would include only the position and velocity of the mass as state variables, as these two variables fully describe the system's behavior. Any additional state variables beyond position and velocity would make the representation nonminimal.

For a nonminimal system example, suppose we have a control system where the dynamics of the system can be represented by a first-order differential equation. However, if we choose to represent this system using a higher-order differential equation, including multiple integrators, the resulting representation would be nonminimal. The additional integrators would not contribute to the essential behavior of the system but would increase the complexity of the representation.

Problem 4 Description

Find $X(t) = e^{At}$ for the systems bellow. a) Series method. b) Cayley-Hamilton and Silvester (find in references.) methods for repeated eigenvalues. c) Diagonalization method for complex eigenvalues.

a) $\begin{bmatrix} -5 & 0 \\ 1 & -5 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & -10 \\ 1 & -1 \end{bmatrix}$

a) $A = \begin{bmatrix} -5 & 0 \\ 1 & -5 \end{bmatrix}, \Phi(t) = e^{At} = \sum \frac{1}{n!} A^n t^n$

$$A^2 = \begin{bmatrix} 25 & 0 \\ -10 & 25 \end{bmatrix}, A^3 = \begin{bmatrix} -125 & 0 \\ -15 & -125 \end{bmatrix}, A^4 = \begin{bmatrix} 625 & 0 \\ -20 & 625 \end{bmatrix}, \ldots, A^n$$

$$= \begin{bmatrix} (-5)^n & 0 \\ n(-5)^{n-1} & (-5)^n \end{bmatrix}.$$

$$e^{At} = I + \begin{bmatrix} -5 & 0 \\ 1 & -5 \end{bmatrix} t + \frac{1}{2!} \begin{bmatrix} 25 & 0 \\ -10 & 25 \end{bmatrix} t^2 + \cdots$$

$$e^{At} = I + \sum \frac{t^n}{n!} \begin{bmatrix} (-5)^n & 0 \\ n(-5)^{n-1} & (-5)^n \end{bmatrix} = \begin{bmatrix} e^{-5t} & 0 \\ te^{-5t} & e^{-5t} \end{bmatrix}$$

b) $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \Delta(s) = |sI - A| = \begin{vmatrix} s-2 & -1 & -4 \\ 0 & s-2 & 0 \\ 0 & -3 & s-1 \end{vmatrix}$

$$\rightarrow (s-2)^2(s-1) \rightarrow s = 2,2,1 \; \textit{Repeated Eigenvalues}.$$

$$f(s_j) = g(s_j) = \alpha_0 + \alpha_1 s_j + \alpha_2 s_j^2 = e^{s_j t}$$

$$\frac{d}{ds}(e^{st})_{s=s_j} = \frac{d}{ds}\big(g(s)\big)_{s=s_j} \quad For\ Repeated\ Eigens.$$

$$1)\ a_0 + a_1 + a_2 = e^t$$

$$2)\ a_0 + 2a_1 + 4a_2 = e^{2t}$$

$$3)(Repeated)a_1 + 2a_2 s_j = te^{s_j t} \to a_1 + 4a_2 = te^{2t}$$

$$\to a_0 = 4e^t - 3e^{2t} + 2te^{2t}$$

$$\to a_1 = 4e^{2t} - 4e^t - 3te^{2t}$$

$$\to a_2 = 4e^t - e^{2t} + te^{2t}$$

$$f(A) = e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$f(A) = \begin{bmatrix} e^{2t} & 12e^t - 12e^{2t} + 13te^{2t} & 4e^{2t} - 4e^t \\ 0 & e^{2t} & 0 \\ 0 & 3e^{2t} - 3e^t & e^t \end{bmatrix}$$

c) $A = \begin{bmatrix} 0 & -10 \\ 1 & -1 \end{bmatrix}, |sI - A| = \begin{vmatrix} s & 10 \\ -1 & s+1 \end{vmatrix} = s^2 + s + 10 = 0,$

$$s = -0.5 \pm 3.12i$$

$$s_1 = -0.5 + 3.12i \to (A - s_1 I)v_1 = 0 \to \left(\begin{bmatrix} 0 & -10 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} -0.5 + 3.12i & 0 \\ 0 & -0.5 + 3.12i \end{bmatrix}\right) v_1 = 0 \to \left(\begin{bmatrix} 0.5 - 3.12i & -10 \\ 1 & 0.5 - 3.12i \end{bmatrix}\right) v_1 = 0 \to$$

$v_1 = \begin{bmatrix} -0.9535 + 0.0000i \\ -0.0477 + 0.2977i \end{bmatrix}$, similarly, $v_1 = \begin{bmatrix} -0.9535 + 0.0000i \\ -0.0477 - 0.2977i \end{bmatrix}$

$$e^{At} = Te^{\Lambda t}T^{-1}$$

$$= \begin{bmatrix} -0.9535 & -0.9535 \\ -0.0477 + 0.2977i & -0.0477 - 0.2977i \end{bmatrix} e^{\begin{bmatrix} -0.5+3.12i & 0 \\ 0 & -0.5-3.12i \end{bmatrix} t} \begin{bmatrix} -0.9535 & -0.9535 \\ -0.0477 + 0.2977i & -0.0477 - 0.2977i \end{bmatrix}^{-1}$$

$$\begin{bmatrix} -0.9535 & -0.9535 \\ -0.0477 + 0.2977i & -0.0477 - 0.2977i \end{bmatrix} \begin{bmatrix} e^{(-0.5+3.12i)t} & 0 \\ 0 & e^{(-0.5-3.12i)t} \end{bmatrix} \begin{bmatrix} -0.9535 & -0.9535 \\ -0.0477 + 0.2977i & -0.0477 - 0.2977i \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (0.5 - 0.08006i)e^{(-0.5+3.12i)t} + (0.5 + 0.08006i)e^{(-0.5-3.12i)t} & -(1.6i)e^{(-0.5-3.12i)t} + (1.6i)e^{(-0.5+3.12i)t} \\ (0.160i)e^{(-0.5-3.12i)t} - (0.6i)e^{(-0.5+3.12i)t} & (0.5 - 0.08006i)e^{(-0.5-3.12i)t} + (0.5 + 0.08006i)e^{(-0.5+3.12i)t} \end{bmatrix}$$

Problem 2.1 Description

Implement the equations of part.1-Q2 in Simulink. Use Friedland problem 3.6 data.

State Space representation of this system is,

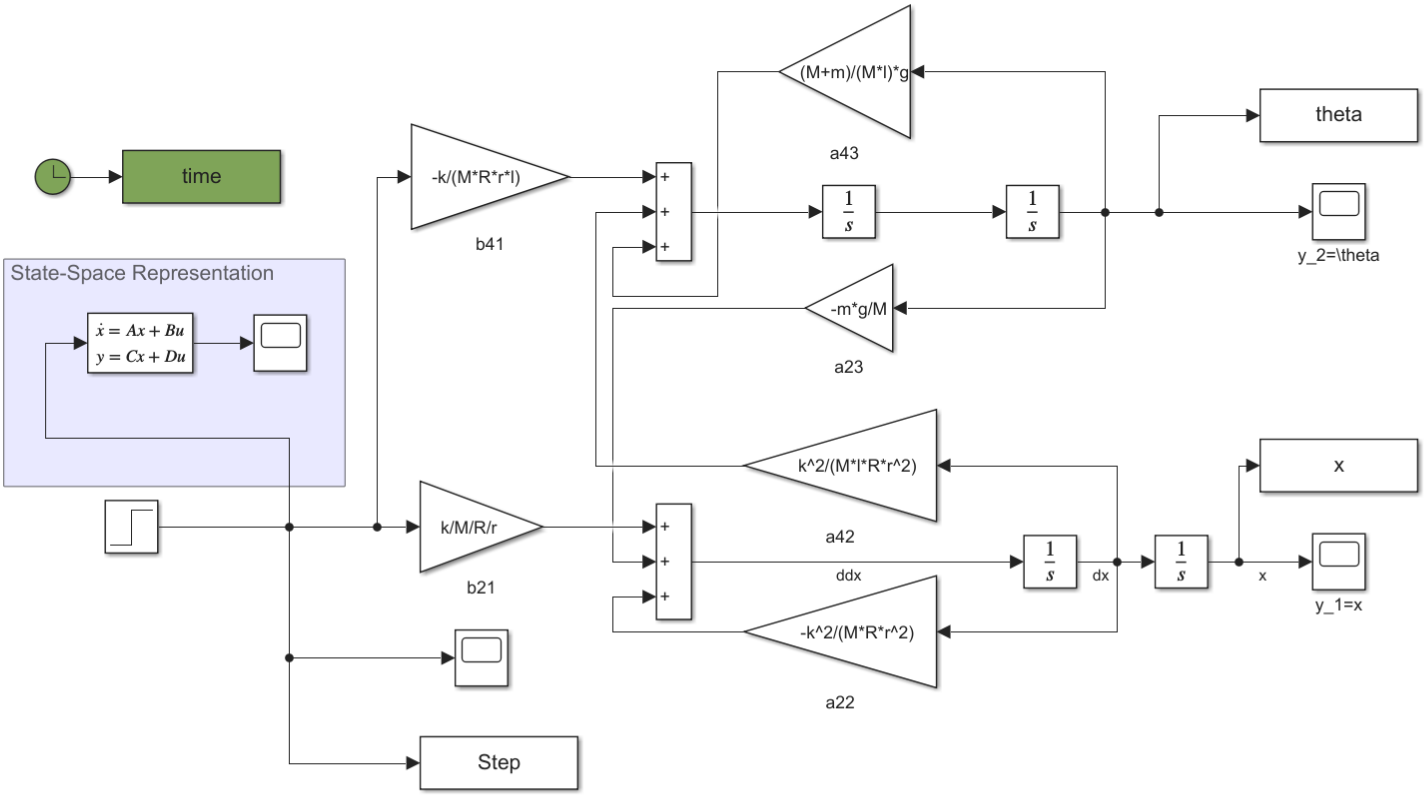
$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\dfrac{k^2}{MRr^2} & -\dfrac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{k^2}{MlRr^2} & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{k}{MRr} \\ 0 \\ -\dfrac{k}{MlRr} \end{bmatrix} [e]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} [e]$$

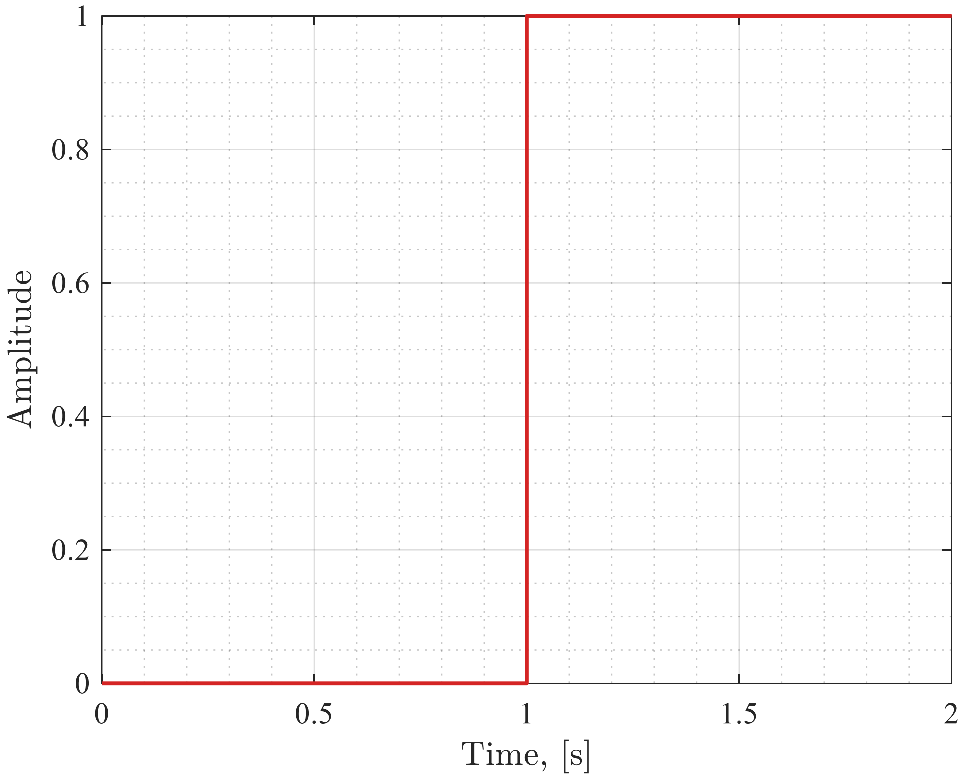
where,

$$m = 0.1 \text{ kg}, M = 1.0 \text{ kg}, l = 0.1 \text{ m}, g = 9.8 \text{ m.s}^{-2}, k = 1 \text{ V.s}, R = 100\ \Omega, r = 0.02 \text{ m}$$

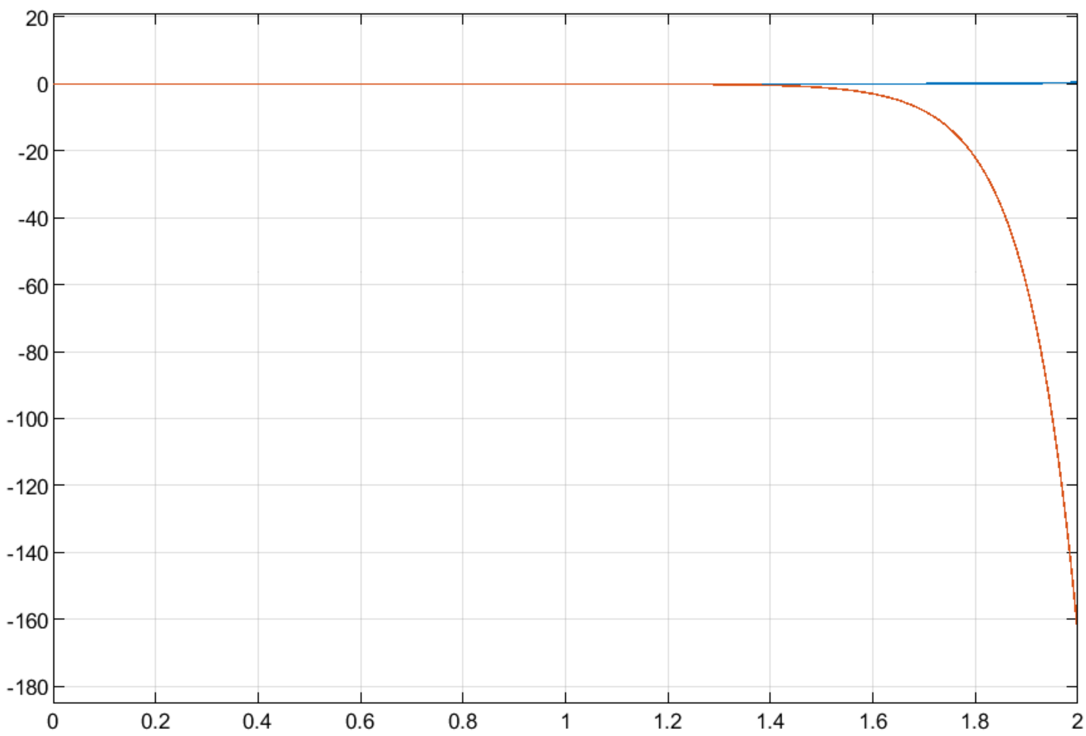
The equations can be presented in state-space form or integrating the equations of motion as,



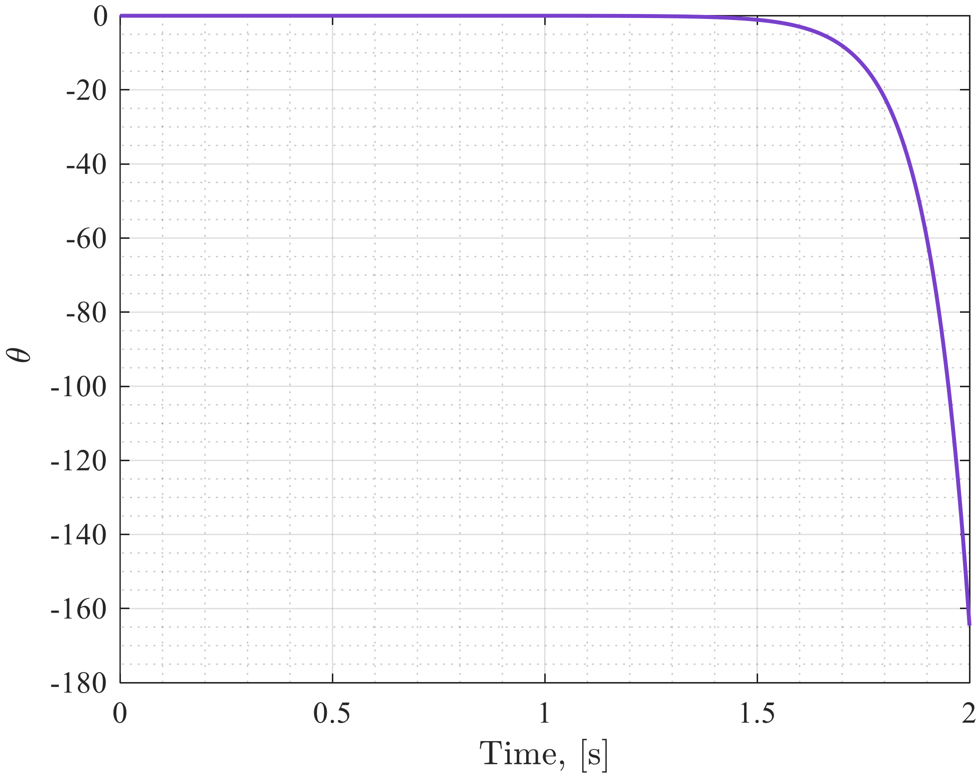
As an input, step function is considered and is illustrated here,

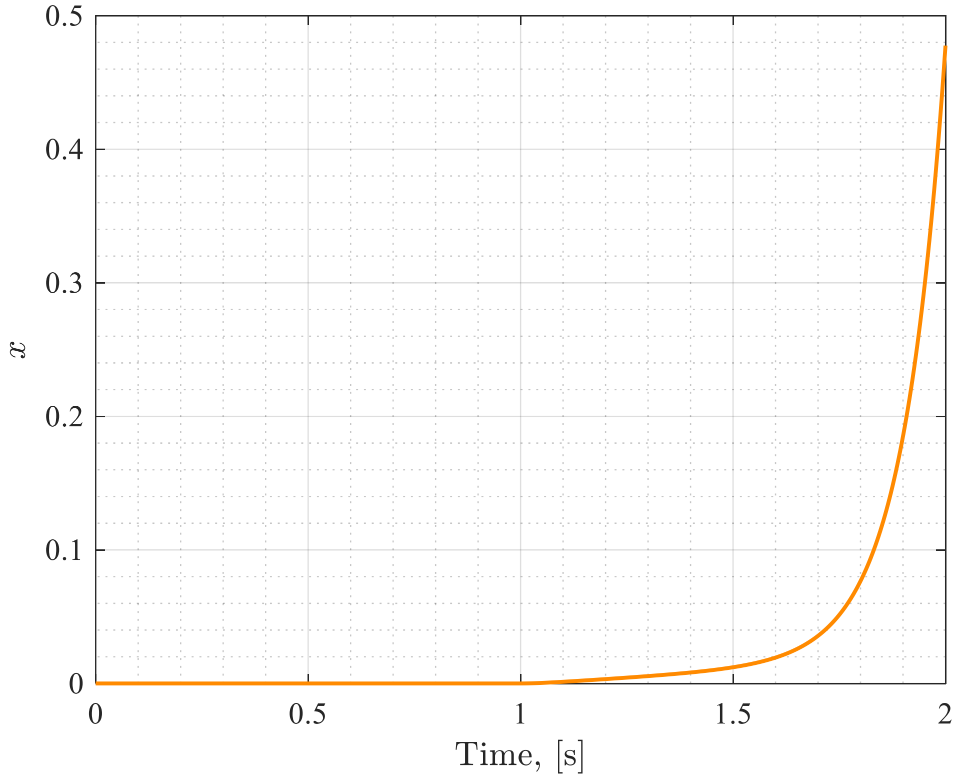


Here, the state space scope is provided (this is scope output),



The results of full presentation from integration can be observed as, which the first figure demonstrates the variation of θ, and the second one depicts for x.





which can be seen there is no significant disparency between the responses.

Problem 2.2 Description

Validate your results of part.1-Q4.c with the **Syms, eig, and expm** commands in MATLAB.

MATLAB code is provided in **Table 1**.

Table 1 MATLAB CODE FOR PROBLEM 1.Q4.C

```
clc
clear
close all
%%
syms t v1 v2
A=[0,-10;1,-1];
[q,d]=eig(A); %eig function
disp(q) %eigenvectors
disp(d) %eigenvalues
%%
eAt=vpa((expm(A*t)),4);
```

Table 2 MATLAB CODE RESULTS

$$\begin{pmatrix} \sigma_1\ (0.5+0.08006\,\mathrm{i})+\sigma_2\ (0.5-0.08006\,\mathrm{i}) & -1.601\ \sigma_1\,\mathrm{i}+1.601\ \sigma_2\,\mathrm{i} \\ 0.1601\ \sigma_1\,\mathrm{i}-0.1601\ \sigma_2\,\mathrm{i} & \sigma_1\ (0.5-0.08006\,\mathrm{i})+\sigma_2\ (0.5+0.08006\,\mathrm{i}) \end{pmatrix}$$

where

$$\sigma_1 = \mathrm{e}^{t\ (-0.5-3.122\,\mathrm{i})}$$

$$\sigma_2 = \mathrm{e}^{t\ (-0.5+3.122\,\mathrm{i})}$$

Problem 2.3 Description

Consider the following nonlinear dynamic equation. (Lorenz system.)

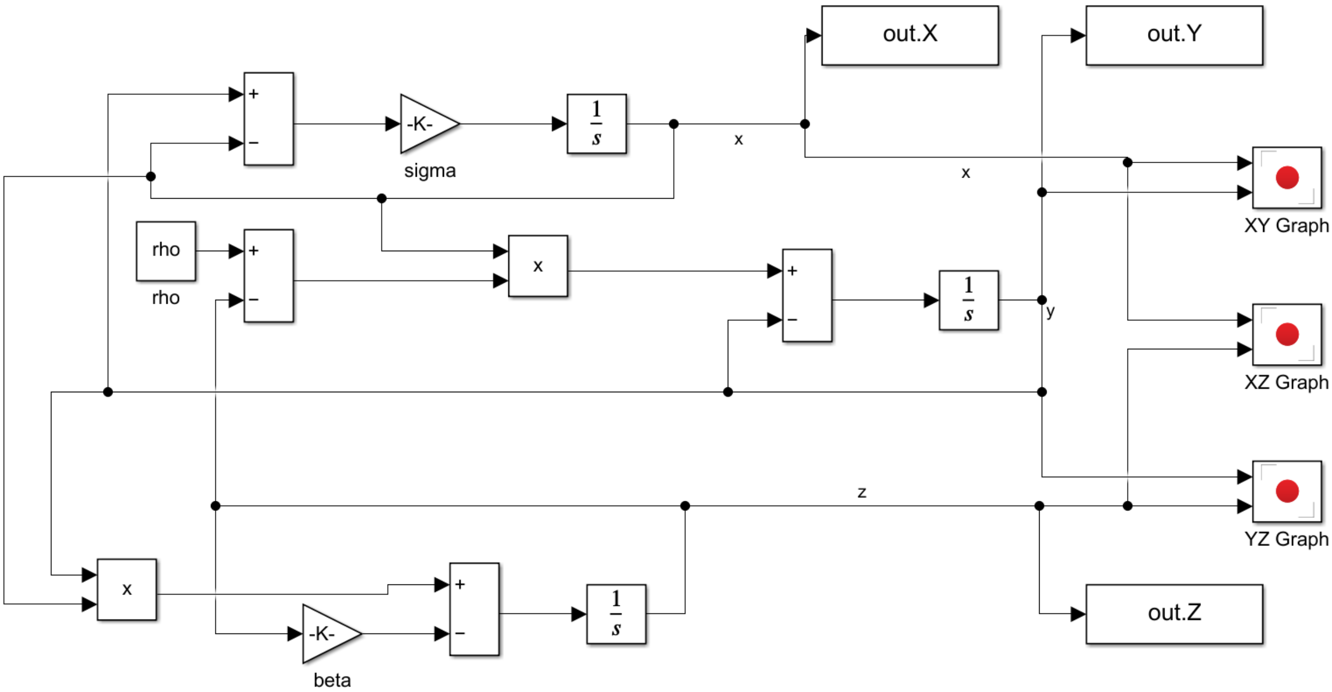
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x),$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\rho - z) - y,$$

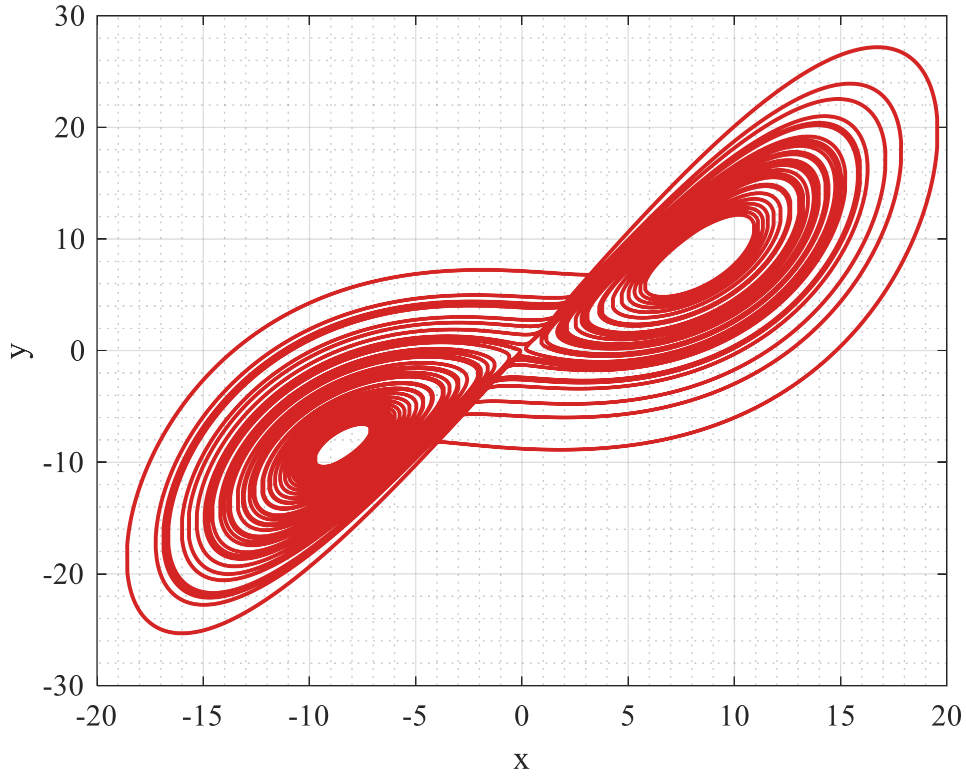
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - \beta z.$$

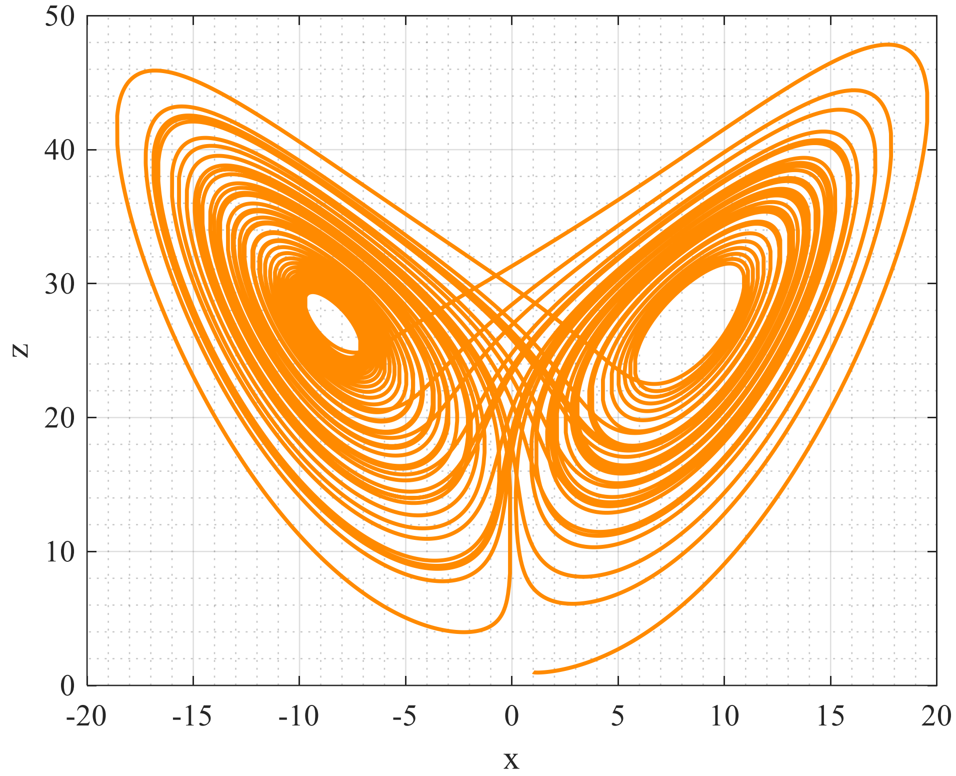
Implement this system in Simulink and plot its phase plane ((x,y) ,(x,z), (y,z)). Constants coefficients are $\rho = 28, \sigma = 10,\ and\ \beta = 8/3$. For the initial condition use: [1 1 1].

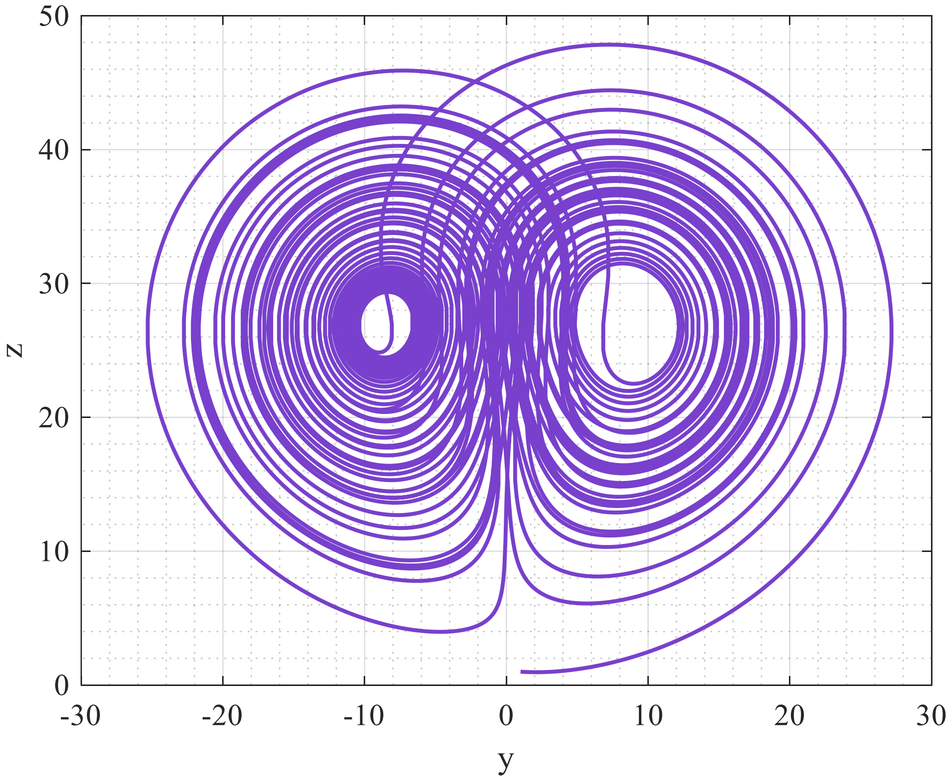
This can be implemented as,



The results for ((x,y) ,(x,z), (y,z)) are provided.


30
20
10
0
-10
-20
-30
y
-20
-15
-10
-5
0
5
10
15
20
x


50
40
30
20
10
0
z
-20
-15
-10
-5
0
5
10
15
20
x


50
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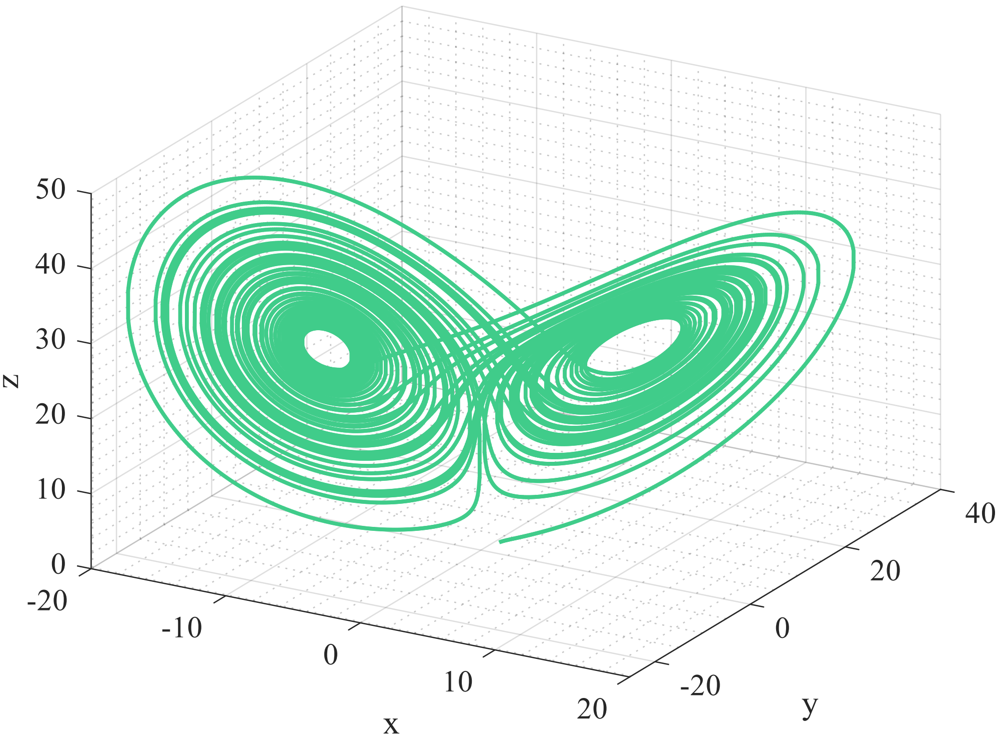
10
0
z
-30
-20
-10
0
10
20
30
y


50
40
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-10
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x
40
20
0
-20
y