In the name of God

دانشگاه صنعتی امیرکبیر
(پلی تکنیک تهران)

HOMEWORK 4
(CONTROLLABILITY & OBSERVABILITY)

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Problem 1 Description

1- Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Is the system completely state-controllable and completely observable? (Jordan form)

Is the system completely output controllable?

Solution

a) The state space representation of this system can be written as,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, D = [0]$$

For completely state-controllable we should calculate the determinant and consequently the rank of $[B \quad AB \quad A^2B]$, which is

$$Q = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 1 & 9 & 1 \end{bmatrix}$$

$$rank(Q) = 3$$

The matrix is full rank, so it is completely state-controllable. For completely observable we should calculate the rank of $[C \quad CA \quad CA^2]^T$, which is

$$O = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

The rank of this matrix is 2, so it is not completely observable, it has 1 unobservable state.

b) For output controllability rank of $[CB \quad CAB \quad CA^2B]$ should be obtained, which is,

$$Q = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}$$

It is rank 2, which states that the system is output controllable (Maximum rank of O can be 2, as it has 2 rows).

Jordan form:

$$\dot{z} = Jz + \bar{B}u$$
$$\dot{z} = T^{-1}ATz + TBu$$

T is equal to,

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.3162 \\ 0 & 1 & 0.948 \end{bmatrix}, T^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 3.1626 & 0 \end{bmatrix}$$
$$\rightarrow J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\rightarrow \bar{B} = \begin{bmatrix} 0 & 1 \\ 0.3162 & 0 \\ 0.9487 & 1 \end{bmatrix}$$

B has 2 columns, and the last row of $\bar{B}$ is nonzero, thus it is controllable. For observability,

$$\bar{C} = CT^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.1626 & 0 \end{bmatrix}$$

It is not completely observable as there is a zero in the first column of the matrix.

Problem 2 Description

2- There is a simple scheme of the tightrope walker in Fig.1.

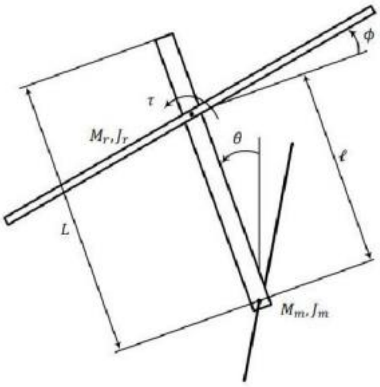

Figure 1 Tightrope walker

$M_m = 75kg,\ J_m = 3.2kgm^2,\ M_r = 2Kg\ ,\ J_r = 1.5kgm^2,\ l = 1m,\ L = 1.8m,\ \dot{\theta} = \omega_\phi.$

a. Show that its equations of motion are as below and determine the state-space equations with above parameters. (Masses are located in the middle of the rods.)

$$\ddot{\theta}\left(J_m + J_r + M_m\frac{L^2}{4} + M_r l^2\right) + \ddot{\phi}\, J_r = g sin\theta \left(M_m\frac{L}{2} + M_r l\right)$$

$$J_r(\ddot{\theta} + \ddot{\phi}) = \tau$$

b. Check the controllability of the system and find the uncontrollable modes.

c. Check the observability of the system with the below outputs and find the unobservable modes.

i. Output ϕ.

ii. Output θ.

iii. Output ϕ, θ.

Solution

a) Lagrange equation,

$$T = \frac{1}{2}J_m\dot{\theta}^2 + \frac{1}{2}J_r\left(\dot{\phi} + \dot{\theta}\right)^2 + \frac{1}{2}M_m(\frac{L}{2}\dot{\theta})^2 + \frac{1}{2}M_r(l\dot{\theta})^2$$

$$U = M_m g \left(\frac{L}{2}\right) \cos\theta + M_r g l \cos\theta$$

So, the Lagrangian form of this system can be written as,

$$\frac{\partial T}{\partial \dot{\theta}} = J_m \dot{\theta} + J_r(\dot{\phi} + \dot{\theta}) + \frac{L}{2} M_m \frac{L}{2} \dot{\theta} + l M_r l \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = J_m \ddot{\theta} + J_r \ddot{\theta} + J_r \ddot{\phi} + M_m \frac{L^2}{4} \ddot{\theta} + M_r l^2 \ddot{\theta}$$

$$= \left(J_m + J_r + M_m \frac{L^2}{4} + M_r l^2\right) \ddot{\theta} + J_r \ddot{\phi}$$

$$\frac{\partial T}{\partial \dot{\phi}} = J_r(\dot{\phi} + \dot{\theta})$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = J_r(\ddot{\theta} + \ddot{\phi})$$

$$\frac{\partial T}{\partial \theta} = 0, \frac{\partial T}{\partial \phi} = 0$$

$$\frac{\partial U}{\partial \theta} = -M_m g \left(\frac{L}{2}\right) \sin\theta - M_r g l \sin\theta = -\left(M_m \left(\frac{L}{2}\right) + M_r l\right) g \sin\theta, \frac{\partial U}{\partial \phi} = 0$$

$$Q_x = 0, Q_\theta = \tau.$$

The system equation of motion can be derived as,

$$\begin{cases} \left(J_m + J_r + M_m \frac{L^2}{4} + M_r l^2\right) \ddot{\theta} + J_r \ddot{\phi} - \left(M_m \left(\frac{L}{2}\right) + M_r l\right) g \sin\theta = 0 \\ J_r(\ddot{\theta} + \ddot{\phi}) = \tau \end{cases}$$

In linear form,

$$\begin{cases} \left(J_m + J_r + M_m \frac{L^2}{4} + M_r l^2\right) \ddot{\theta} + J_R \ddot{\phi} - \left(M_m \left(\frac{L}{2}\right) + M_r l\right) g\theta = 0 \\ J_r(\ddot{\theta} + \ddot{\phi}) = \tau \end{cases}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} = \frac{1}{J_r \left(J_m + J_r + M_m \frac{L^2}{4} + M_r l^2\right) - J_r^2} \begin{bmatrix} J_r & -J_r \\ -J_r & J_m + J_r + M_m \frac{L^2}{4} + M_r l^2 \end{bmatrix} \begin{bmatrix} \left(M_m \left(\frac{L}{2}\right) + M_r l\right) g \sin\theta \\ \tau \end{bmatrix}$$

By considering $x_1 = \theta, x_2 = \dot{\theta}, x_3 = \phi$, and $x_4 = \dot{\phi}$, the state Space representation of this system is,

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{J_r g}{k}\left(M_m \left(\frac{L}{2}\right) + M_r l\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{J_r g}{k}\left(M_m \left(\frac{L}{2}\right) + M_r l\right) & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{Bmatrix} + \begin{bmatrix} 0 \\ -\frac{J_r}{k} \\ 0 \\ \frac{J_m + J_r + M_m \frac{L^2}{4} + M_r l^2}{k} \end{bmatrix} [\tau]$$

$$\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{Bmatrix}$$

in which, $k = J_r\left(J_m + M_m\frac{L^2}{4} + M_r l^2\right)$.

Using initial values,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10.3381 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -10.3381 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ -0.0152 \\ 0 \\ 0.6818 \end{bmatrix}$$

b) Controllability and uncontrollable modes,

For controllability we should calculate the rank of $[B \quad AB \quad A^2B]$, which is

$$Q = \begin{bmatrix} 0 & -0.0152 & 0 & -0.1568 \\ -0.0152 & 0 & -0.1568 & 0 \\ 0 & 0.6818 & 0 & 0.1568 \\ 0.6818 & 0 & 0.1568 & 0 \end{bmatrix}$$

$$|Q| = 0.0109$$

$$rank(Q) = 4$$

Controllable.

c) Observability of the system,

i) Output ϕ

$$C = [0 \quad 0 \quad 1 \quad 0]$$

For observable matrix, we should calculate the rank of $[C \quad CA \quad CA^2 \quad ...]^T$, which is

$$O = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.3381 & 0 & 0 & 0 \\ 0 & -10.3381 & 0 & 0 \end{bmatrix}$$

$$|O| = 106.88 \neq 0$$

$$rank(O) = 4$$

Observable.

ii) Output θ

$$C = [1 \quad 0 \quad 0 \quad 0]$$

For observable matrix, we should calculate the rank of $[C \quad CA \quad CA^2 \quad ...]^T$, which is

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 10.3381 & 0 & 0 & 0 \\ 0 & 10.3381 & 0 & 0 \end{bmatrix}$$

$$|O| = 0$$

$$rank(O) = 2$$

Not observable.

Finding unobservable mode, $\lambda = 0,0,3.2153,-3.2153$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.21 \\ 0.6752 \\ -0.21 \\ -0.6752 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} -0.21 \\ 0.6752 \\ 0.21 \\ -0.6752 \end{bmatrix}$$

Test 4 modes with PBH:
Mode 1,

$$Cv_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

Unobservable.

Mode 2,

$$Cv_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = 0$$

Unobservable.
Mode 3,

$$Cv_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.21 \\ 0.6752 \\ -0.21 \\ -0.6752 \end{bmatrix} = 0.21$$

Observable.
Mode 3,

$$Cv_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.21 \\ 0.6752 \\ 0.21 \\ -0.6752 \end{bmatrix} = -0.21$$

Observable.

2 unobservable modes.

iii) Output ϕ, θ

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

For observable matrix, we should calculate the rank of $[C \quad CA \quad CA^2 \quad ...]^T$, which is

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 10.3381 & 0 & 0 & 0 \\ -10.3381 & 0 & 0 & 0 \\ 0 & 10.3381 & 0 & 0 \\ 0 & -10.3381 & 0 & 0 \end{bmatrix}$$

$$|O| = -1$$

$$rank(O) = 4$$

Observable.

Problem 3 Description

3- Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a. Check the controllability and observability of the system with the PBH test.

b. Suggest another C matrix to make the system observable. (make it multi-output)

Solution

a) Check the controllability and observability,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Eigenvalues can be estimated as,

$$det(\lambda I - A) = 0 \rightarrow det \begin{bmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & -3 & \lambda - 1 \end{bmatrix} = 0 \rightarrow (\lambda - 1)(\lambda - 2)^2 = 0$$

$$\lambda = 1,2,2$$

Repeated eigenvalues.

For controllability using PBH,

$$Q = [\lambda_i I - A \quad B]$$

$$i = 1, \lambda_1 \rightarrow \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \rightarrow rank(Q_1) = 2$$

or

$$W_1^T B = [0 \quad -0.9487 \quad 0.3162] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

Uncontrollable in the first mode.

$$i = 2{,}3, \lambda_{2,3} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 \end{bmatrix} \rightarrow rank(Q_1) = 2$$

or

$$W_2^T B = [0 \quad 1 \quad 0] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

Uncontrollable in the second mode.

For observability using PBH,

$$Q' = \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix}$$

$$i = 1, \lambda_1 \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow rank(Q_1) = 3$$

Another way is to determine Cv_i,

$$i = 1, \lambda_1 \rightarrow Cv_1 = [1 \quad 1 \quad 1] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \neq 0$$

Observable.

$$i = 2, \lambda_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow rank(Q_1) = 2$$

Not observable for the second/third modes.

b) Considering MIMO system, where $C = Diag[1,1,1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, now

$$Q' = \begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix}$$

$$i = 1, \lambda_1 \rightarrow \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow rank(Q_1) = 3$$

Observable.

$$i = 2, \lambda_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow rank(Q_1) = 3$$

Observable for the second/third modes.

Problem 2.1 Description

Check all Part.1 questions result with commands ctrb(A,B) and obsv(A,C).

Solution

For P1,

Table 1 Problem 1.1 MATLAB code.

```
clc
clear
close all

A=[2 0 0;0 2 0; 0 3 1];
B=[0 1;1 0;0 1];
C=[1 0 0;0 1 0];
```

```
Co=ctrb(A,B)
rank(Co)
unco = length(A) - rank(Co)

Ob = obsv(A,C)
rank(Ob)
unob = length(A) - rank(Ob)
```

The result is,

Table 2 Problem 1.1 MATLAB code result.

```
Co =

     0     1     0     2     0     4
     1     0     2     0     4     0
     0     1     3     1     9     1


ans =

     3


unco =

     0


Ob =

     1     0     0
     0     1     0
     2     0     0
     0     2     0
     4     0     0
     0     4     0


ans =

     2


unob =

     1
```

So, it is completely state-controllable, as the rank of controllability matrix is full, however, the rank of observability matrix is 2, so it is not completely observable.

For P2,

Table 3 Problem 1.2 MATLAB code.

```
clc
clear
close all

Mm=75;
Jm=3.2;
Mr=2;
Jr=1.5;
L=1.8;
l=1;
g=9.81;

k=Jr*(Jm+Mm*L^2/4+Mr*l^2);

A=[0 1 0 0; Jr*g*(Mm*L/2+Mr*l)/k 0 0 0;0 0 0 1;-
Jr*g*(Mm*L/2+Mr*l)/k 0 0 0];
B=[0;-Jr/k;0;(Jm+Jr+Mm*L^2/4+Mr*l^2)/k];
Co=ctrb(A,B)
rank(Co)
unco = length(A) - rank(Co)

C1=[0 0 1 0];
C2=[1 0 0 0];
C3=[1 0 0 0;0 0 1 0];
Ob = obsv(A,C1)
rank(Ob)
unob = length(A) - rank(Ob)

Ob2 = obsv(A,C2)
rank(Ob2)
unob = length(A) - rank(Ob2)

Ob3 = obsv(A,C3)
rank(Ob3)
unob = length(A) - rank(Ob3)
```

The result is,

Table 4 Problem 1.2 MATLAB code result.

```
Co =

         0   -0.0152         0   -0.1568
   -0.0152         0   -0.1568         0
         0    0.6818         0    0.1568
    0.6818         0    0.1568         0


ans =

     4


unco =

     0


Ob =

         0         0    1.0000         0
         0         0         0    1.0000
  -10.3381         0         0         0
         0  -10.3381         0         0


ans =

     4


unob =

     0


Ob2 =

    1.0000         0         0         0
         0    1.0000         0         0
   10.3381         0         0         0
         0   10.3381         0         0


ans =

     2
```

```
unob =

     2


Ob3 =

    1.0000         0         0         0
         0         0    1.0000         0
         0    1.0000         0         0
         0         0         0    1.0000
   10.3381         0         0         0
  -10.3381         0         0         0
         0   10.3381         0         0
         0  -10.3381         0         0


ans =

     4


unob =

     0
```

So, it is controllable, as the rank of controllability matrix is full, and there is no uncontrollable mode. Having said that, the rank of observability matrix is varied, for output ϕ, $C = [0 \quad 0 \quad 1 \quad 0]$ it is observable.

for output θ, $C = [1 \quad 0 \quad 0 \quad 0]$ it is not observable. Finally, if outputs are ϕ, θ, C is equal to $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which the rank of observability matrix is full and shows observability.

For P3,

Table 5 Problem 1.3 MATLAB code.

```
clc
clear
close all
syms l
A=[2 0 0;0 2 0;0 3 1];
```

```
[V D W]=eig(A)
B=[1;0;0];
C=[1,1,1];

Ob = obsv(A,C)
rank(Ob)
eq=det(l*eye(3)-A)==0;
l=solve(eq,l);
Q1=[l(1)*eye(3)-A B]
rank(Q1)
W(:,2).'*B

Q2=[l(2)*eye(3)-A B]
rank(Q2)
W(:,1).'*B

O1=[l(1)*eye(3)-A ;C]
rank(O1)

O2=[l(2)*eye(3)-A ;C]
rank(O2)

%% Part b
C=[1,1,1];
C=diag(C);
disp('***Part b.***')
O1=[l(1)*eye(3)-A ;C]
rank(O1)

O2=[l(2)*eye(3)-A ;C]
rank(O2)
```

The result is,

Table 6 Problem 1.3 MATLAB code result.

```
V =

    1.0000         0         0
         0         0    0.3162
         0    1.0000    0.9487


D =

     2     0     0
     0     1     0
     0     0     2
```

```
W =

    1.0000         0         0
         0   -0.9487    1.0000
         0    0.3162         0


Ob =

     1     1     1
     2     5     1
     4    13     1


ans =

     2


Q1 =

[-1,  0, 0, 1]
[ 0, -1, 0, 0]
[ 0, -3, 0, 0]


ans =

     2


ans =

     0


Q2 =

[0,  0, 0, 1]
[0,  0, 0, 0]
[0, -3, 1, 0]


ans =

     2
```

```
ans =

     1


O1 =

[-1,  0, 0]
[ 0, -1, 0]
[ 0, -3, 0]
[ 1,  1, 1]


ans =

     3


O2 =

[0,  0, 0]
[0,  0, 0]
[0, -3, 1]
[1,  1, 1]


ans =

     2

Part b.

O1 =

[-1,  0, 0]
[ 0, -1, 0]
[ 0, -3, 0]
[ 1,  0, 0]
[ 0,  1, 0]
[ 0,  0, 1]


ans =

     3


O2 =
```

```
[0,  0, 0]
[0,  0, 0]
[0, -3, 1]
[1,  0, 0]
[0,  1, 0]
[0,  0, 1]


ans =

     3
```

Controllability and observability for modes have been checked, and it is verified that it is not observable and controllable. Moreover, if $C = Diag[1,1,1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, be considered the system would be observable, as it is full rank.

Problem 2.2 Description

2. For the system below check the observability on paper and with code `obsv(A,C)` then find the unobservable state variable and check this result with the command `null(obsv(A,C))`.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution

Check observability,

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Observability matrix,

$$O = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -2 \\ -1 & 0 & 4 \end{bmatrix}, |O| = 0, rank(O) = 2$$

It is not observable.

The second column of this matrix is zero. Therefore, null space of a matrix contains vectors x that satisfy $Ax = 0$.

```
clc
clear
close all

A=[-1 0 -2;0 -1 1;1 0 -1];
B=[0 1 0].';
C=[1 0 0];
Ob = obsv(A,C)
rank=rank(Ob)
null=null(Ob,'r')
[V D]=eig(A);
C*V(:,1)
C*V(:,2)
C*V(:,3)
```

which indicates that $null(Obsv(A, C)) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

```

Ob =

     1     0     0
    -1     0    -2
    -1     0     4


rank =

     2


null =

     0
     1
     0



ans =
```

```
     0


ans =

    0.7559


ans =

    0.7559
```

Eigenvalues can be determined as follows,

$$|\lambda I - A| = \left|\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -2 \\ -1 & 0 & 4 \end{bmatrix}\right| = \begin{vmatrix} \lambda+1 & 0 & 2 \\ 0 & \lambda+1 & -1 \\ -1 & 0 & \lambda+1 \end{vmatrix} = 0$$

$$\lambda_1 = 1 \rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -1.0000 \ + \ 1.4142i \rightarrow v_2 = \begin{bmatrix} 0.7559 \\ -0.3780 \\ -0.5345\, i \end{bmatrix}$$

$$\lambda_3 = -1.0000 - \ 1.4142i \rightarrow \ v_3 = \begin{bmatrix} 0.7559 \\ -0.3780 \\ 0.5345\, i \end{bmatrix}$$

finding the observability modes.

$$Cv_i = 0$$

$$Cv_1 = [1 \quad 0 \quad 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

It is a non-observable mode (first mode, $\lambda_1 = 1$).

$$Cv_2 = [1 \quad 0 \quad 0] \begin{bmatrix} 0.7559 \\ -0.3780 \\ -0.5345\, i \end{bmatrix} = 0.7559$$

It is an observable mode.

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It is an observable mode.

$$T = \begin{bmatrix} 0.7559 & 0.7559 & 0 \\ -0.3780 & -0.3780 & 1 \\ -0.5345\,i & 0.5345\,i & 0 \end{bmatrix}$$

$$\bar{A} = T^{-1}AT = \begin{bmatrix} -1+\ 1.4142i & 0 & 0 \\ 0 & -1-\ 1.4142i & 0 \\ 0 & 0 & -1 \end{bmatrix}, \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{C} = [0.7559 \quad 0.7559 \quad 0]$$

Problem 2.3 Description

3. For Part.1-Q3 find the decomposition of controllable/uncontrollable subsystems with command ctrbf(A,B,C) and explain your result.

Solution

The MATLAB code is as follows,

Table 7 MATLAB code of Problem 2.3.

```
clc
clear
close all

A=[2 0 0;0 2 0;0 3 1];
[V D W]=eig(A)
B=[1;0;0];
C=[1,1,1];

[Abar,Bbar,Cbar,T,k]=ctrbf(A,B,C)
```

The result is as follows,

Table 8 MATLAB result for Problem 2.3.

```
    Abar =



         1     3     0
```

```
     0     2     0
     0     0     2


Bbar =

     0
     0
     1


Cbar =

     1     1     1


T =

     0     0     1
     0     1     0
     1     0     0


k =
```

```
        1       0       0
```

$$\bar{A} = \begin{bmatrix} A_{un} & 0 \\ A_{21} & A_c \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \bar{C} = \begin{bmatrix} C_{nc} & C_c \end{bmatrix}$$

Table 8 indicates that $(2,1)$ is controllable. Furthermore, all eigenvalues of $A_{un} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ are uncontrollable, which are $\lambda = 1,2$. sum(k) is the number of states in $\boldsymbol{A_c}$, the controllable portion of Abar, which is equal to 1. Finally, T is the similarity transformation matrix.