Continuous Elementary Signal

$$rect(\frac{t}{\tau}) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$

$$u(t) = \begin{cases} 1 & |t| > 0 \\ 0 & |t| < 0 \end{cases} \quad r(t) = \begin{cases} t & |t| > 0 \\ 0 & |t| < 0 \end{cases}$$

Unit Impulse Function

1. $\delta(0) \to \infty$
2. $\delta(t) = 0, t \neq \infty$
3. $\int\limits_{-\infty}^{\infty} \delta(t)\, dt = 1$
4. $\delta(t)$ is an even function, i.e., $\delta(t) = \delta(-t)$

- **Sifting** $\int\limits_{t_1} t_2 x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0) & t_1 < t_0 < t_2 \\ 0 & \text{otherwise} \end{cases}$
- **Sampling** $x(t)\delta(t - t_0) = x(t_0)\delta(t - t0)$
- **Scaling** $\delta(at + b) = \frac{1}{|a|}\delta(t + \frac{b}{a})$

$$\frac{du(t)}{dt} = \delta(t) \quad \int\limits_{-\infty}^{\infty} \delta(\tau)\, d\tau = u(t)$$

Periodic Signal

Continous

$x(t) = x(t + nT)$ T is period, $T = \frac{2\pi}{|\omega_0|}$

$z(t) = ax(t) + by(t) \quad x(t) = x(t + kT_1) \quad y(t) = y(t + lT_2)$

Will be Periodic if $T = kT_1 = lT_2$ where k, l are integers

Discrete

$x[n] = x[n + N]$ for all n, some integer $N > 0$

Continuous Time System

4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega - \theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd

Discrete Elementary Signal

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \quad u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} = \sum_{k=0}^{\infty} \delta[k]$$

$$u[n] - u[n-1] = \delta[n] \quad x[n]\delta[n-k] = x[k]\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Convolution

Continuous

$$y(t) = \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = x(t)*h(t), \quad x(t)*\delta(t-a) = x(t-a)$$

$$x(t) * u(t) = \int\limits_{-\infty}^{\infty} x(\tau)u(t-\tau)\, d\tau = \int\limits_{-\infty}^{t} x(\tau)\, d\tau$$

Discrete

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$x[n] * u[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=-\infty}^{n} x[k]$$

Continuous Fourier Transform

$$rect(\frac{t}{\tau}) \leftrightarrow \tau sinc(\frac{\omega\tau}{2}) \quad e^{-\alpha t} \leftrightarrow \frac{1}{\alpha + j\omega}$$

$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\text{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier Transform for DT (DTFT)

Property	Aperiodic Signal	Fourier Transform
	$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{Re}\{X(e^{j\omega})\} = \mathcal{Re}\{X(e^{-j\omega})\} \\ \mathcal{Im}\{X(e^{j\omega})\} = -\mathcal{Im}\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{Re}\{X(e^{j\omega})\}$ $j\mathcal{Im}\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos\omega_0 n$	$\pi\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin\omega_0 n$	$\frac{\pi}{j}\sum_{l=-\infty}^{+\infty}\{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \dots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi\sum_{l=-\infty}^{+\infty}\delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & N_1 < \lvert n\rvert \leq N/2 \end{cases}$ and $x[n + N] = x[n]$	$2\pi\sum_{k=-\infty}^{+\infty} a_k\delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N\sin[2\pi k/2N]}, k \neq 0, \pm N, \pm 2N, \dots$ $a_k = \frac{2N_1 + 1}{N}, k = 0, \pm N, \pm 2N, \dots$
$\sum_{k=-\infty}^{+\infty}\delta[n - kN]$	$\frac{2\pi}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n]\begin{cases} 1, & \lvert n\rvert \leq N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \leq \lvert\omega\rvert \leq W \\ 0, & W < \lvert\omega\rvert \leq \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty}\pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n + 1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n + r - 1)!}{n!(r - 1)!}a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

Discrete Fourier Transform (DFT)

Matrix Representation

$$X[k] = \sum_{n=0}^{N-1} x[n]W_n^{nk}\,; k = 0, 1, ..., N - 1$$

Convolution Example

$$x[n] = \{1, 2, 0, -1\} \; h[n] = \{1, 3, -1, 2\} \; N = 4 \; e^{\frac{-2\pi}{4}j} = -j$$

$$X = Wx = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-3j \\ 0 \\ 1+3j \end{bmatrix}$$

$$H = Wh = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2-5j \\ -1 \\ 2+5j \end{bmatrix}$$

$$Y = XH = \begin{bmatrix} 2 \\ -13-11j \\ 0 \\ -13+11j \end{bmatrix} \; W^{-1} = \frac{1}{N}W^*$$

$$y = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & 1 & j \end{bmatrix} = \begin{bmatrix} 2 \\ -13-17j \\ 0 \\ -13+11j \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 7 \\ -5 \end{bmatrix}$$

Sampling

Nyquist rate $2\omega_0$

Impulse Train Sampling

$x(t)$ low pass signal and no freq. $> \omega_B$
$p(t)$ periodic impulse train
T_s sampling period
Output of the sampler $x_s(t)$
Fundamental freq. (sampling freq.) of $p(t)$ is $\omega_s = \frac{2\pi}{T_s}$

$$x_s(t) = x(t)p(t) \text{ where } p(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$

Aliasing

$\omega_s < 2\omega_B \rightarrow$ Aliasing
Reconstructed frequency f_p of signal frequency f which is sampled at f_s

$f_p = |f - kf_s|\,; k$ is the rounded value of $\frac{f}{f_s}$