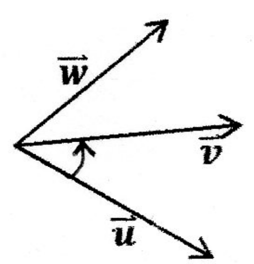
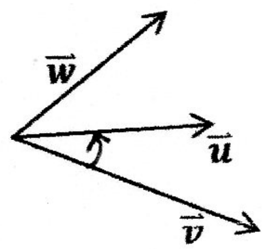
Section 4.4 THE VECTOR (OR CROSS) PRODUCT

Definition: An ordered triple $\{\vec{u},\vec{v},\vec{w}\}$ of three independent vectors is said to be right-handed or left-handed if the vectors are situated as in the figure below.

Right-handed triple $\{\vec{u},\vec{v},\vec{w}\}$



Left-handed triple $\{\vec{u},\vec{v},\vec{w}\}$



Remark: *The notion of left-handed and right-handed triple is not defined if all three vectors lie on the same plane.*

Definition: Two sets of ordered triples of vectors are said to be similarly oriented if and only if both sets are right-handed or both are left-handed. Otherwise they are oppositely oriented.

Theorem: If $\{\vec{u},\vec{v},\vec{w}\}$ is a right-handed triple, then $\{\vec{v},\vec{u},-\vec{w}\}$ is a right handed triple and $\{c_1\vec{u},c_2\vec{v}, c_3\vec{w}\}$ is a right-handed triple provided that $c_1c_2c_3 > 0$.

Definition: Let $\vec{u}$ and $\vec{v}$ be vectors. The **vector or cross** product $\vec{u}\times\vec{v}$ is defined as follows:

(i) $\vec{u}\times\vec{v} = \mathbf{0}$, if either $\vec{u}$ or $\vec{v}$ is 0 or $\vec{u}$ is proportional to $\vec{v}$ (i.e. $\vec{u} \parallel \vec{v}$);

(i) Otherwise, $\vec{u}\times\vec{v} = \vec{w}$, where $\vec{w}$ has three properties:

a. $\vec{w}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

b. $\vec{w}$ has magnitude $|\vec{w}| = |\vec{u}||\vec{v}|\sin\theta$, where θ is the angle between $\vec{u}$ and $\vec{v}$.

c. The direction of $\vec{w}$ is selected so that $\{\vec{u},\vec{v},\vec{w}\}$ is a right-handed triple.

Remark: For any vector $\vec{v}$, $\vec{v}\times\vec{v} = \mathbf{0}$.

Definition: Suppose that $\vec{u}$ and $\vec{v}$ are any vectors, $\{i,j,k\}$ is a right-handed triple, and that t is any number. Then

(i) $\vec{v}\times\vec{u} = -(\vec{u}\times\vec{v})$

(ii) $(\overrightarrow{tu})\times\vec{v} = t(\vec{u}\times\vec{v}) = \vec{u}\times(t\vec{v})$

(iii) $i\times j = -j\times i = k$
$j\times k = -k\times j = i$
$k\times i = -i\times k = j$

(iv) $i\times i = j\times j = k\times k = 0$

Theorem: If $\vec{u}$, $\vec{v}$, $\vec{w}$ are any vectors, then: (i) $\vec{u}\times(\vec{v}+\vec{w}) = (\vec{u}\times\vec{v}) + (\vec{u}\times\vec{w})$ - Right
(ii) $(\vec{v}+\vec{w})\times\vec{u} = (\vec{v}\times\vec{u}) + (\vec{v}\times\vec{w})$ - Left } Dist. Laws

Theorem: If $\vec{u} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, $\vec{v} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$$

Theorem: The area of a parallelogram with adjacent sides $\overline{AB}$ and $\overline{CD}$ is given by $|\vec{v}(\overline{AB})\times\vec{v}(\overline{AC})|$.
The area of the triangle ABC is then $\frac{1}{2}|\vec{v}(\overline{AB})\times\vec{v}(\overline{AC})|$.

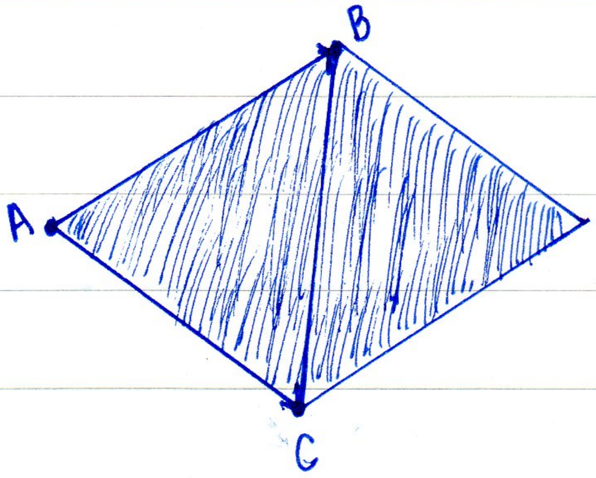
Examples:

1. Given $\vec{u} = i + 3j - k$ and $\vec{v} = 2i - j + k$. Find $\vec{u} \times \vec{v}$.

2. Find the area of ΔABC and the equation of the plane thru $A(1,-2,3)$, $B(3,1,2)$, $C(2,3,-1)$ using vector methods.
3. Find the perpendicular distance between the lines $L_1: \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ & $L_2: \frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$.

4. Use vector methods to find the equations of the line (in symmetric form) passing through the point $P(-1,3,2)$ & parallel to the two planes $m_1: 2x - 2y + 4z + 2 = 0$ & $m_2: 2x + y - z = 0$.

5. Use vector methods to find an equation of the plane through the point $P(2,-1,-3)$ and parallel to the lines $L_1: \frac{x-1}{3} = \frac{y+2}{2} = \frac{z}{-4}$ and $L_2: \frac{x}{2} = \frac{y-1}{-3} = \frac{z-2}{2}$.



Solutions:

1. $\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 3 & -1 \\ 2 & -1 & 1 \end{vmatrix} = (3i - 2j - k) - (6k + i + j) = \boxed{2i - 3j - 7k}$

2. $\vec{v}(\overrightarrow{AB}) = (3-1)i + (1+2)j + (2-3)k = 2i + 3j - k$

$\vec{v}(\overrightarrow{AC}) = (2-1)i + (3+2)j + (-1-3)k = i + 5j - 4k$

$\vec{v}(\overrightarrow{AB}) \times \vec{v}(\overrightarrow{AC}) = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & 5 & -4 \end{vmatrix} = (-12i - j + 10k) - (3k - 8j - 5i)$

$= -7i + 7j + 7k$

Thus, the area of the parallelogram is

$\left| \vec{v}(\overrightarrow{AB}) \times \vec{v}(\overrightarrow{AC}) \right| = \sqrt{(-7)^2 + 7^2 + 7^2} = 7\sqrt{3}$ sq. units.

∴ The area of the triangle ABC is $\frac{7\sqrt{3}}{2}$ sq. units and the equation of the plane (thru A, B & C) with attitude numbers $-7, 7, 7$ is

$-7(x-1) + 7(y+2) + 7(z-3) = 0$

$\boxed{x - y - z = 0}$.

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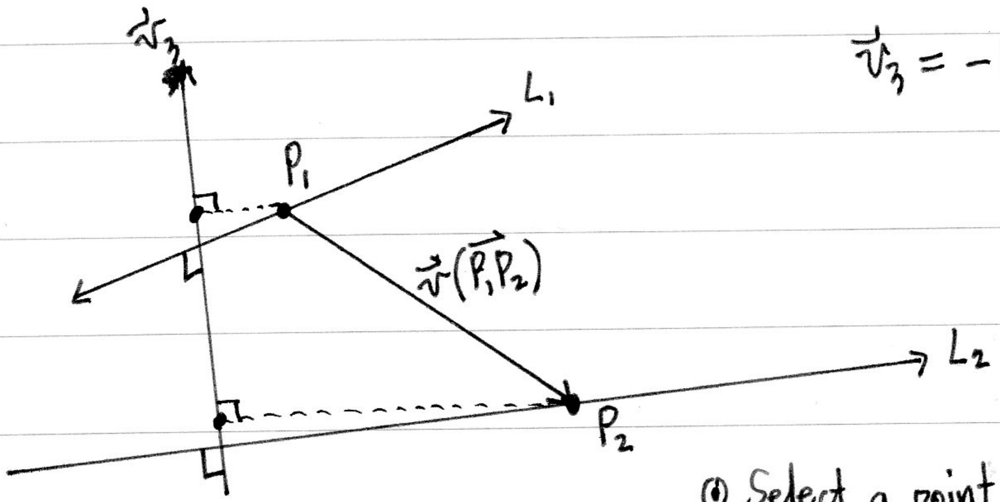
3. vector $\parallel$ to L_1: $\vec{v}_1 = 3i + 2j + 5k$

vector $\parallel$ to L_2: $\vec{v}_2 = 4i + 3j - 2k$

$\Rightarrow$ vector $\perp$ to both $\vec{v}_1$ & $\vec{v}_2$ is $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 3 & 2 & 5 \\ 4 & 3 & -2 \end{vmatrix}$

$\vec{v}_3 = -19i + 26j + k$



① Select a point on L_1: $P_1(1,-1,1)$

② Select a point on L_2: $P_2(-2,1,-1)$

$\Rightarrow \vec{v}(\overrightarrow{P_1P_2}) = (-2-1)i + (1+1)j + (-1-1)k$

$= -3i + 2j - 2k$

$\therefore$ The desired distance is the projection of the vector $\vec{v}(\overrightarrow{P_1P_2})$ on $\vec{v}_3$, i.e. the perpendicular distance between the given lines L_1 & L_2 is

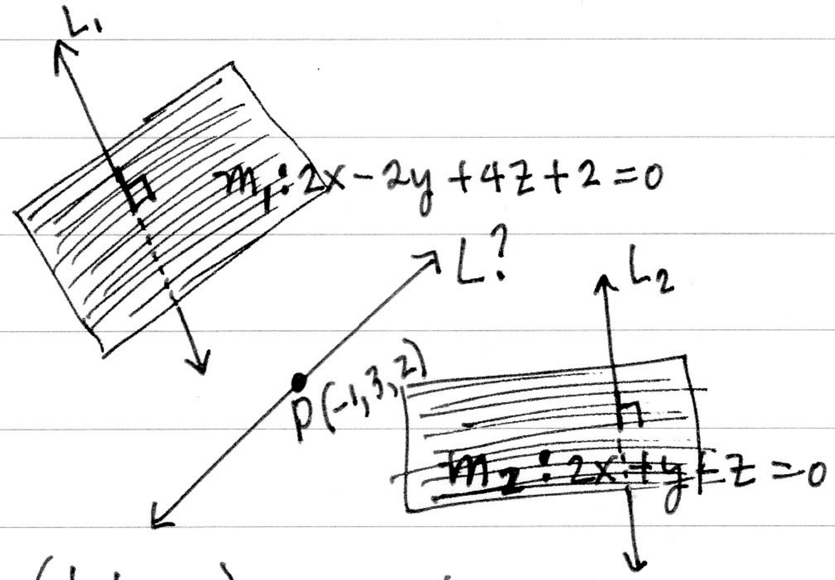
$$\text{Proj}_{\vec{v}_3} \vec{v}(\overrightarrow{P_1P_2}) = \frac{\vec{v}(\overrightarrow{P_1P_2}) \cdot \vec{v}_3}{|\vec{v}_3|} = \frac{(-19)(-3) + 26 \cdot 2 + 1 \cdot (-2)}{\sqrt{(-19)^2 + 26^2 + 1^2}}$$

$$= \boxed{\frac{107}{\sqrt{1038}} \text{ units}}$$

4. Find: $L: \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$

Given: $(x_0, y_0, z_0) = (-1, 3, 2)$

Find: dir. nos.: a, b, c



dir. nos. of L_1 ($\perp$ to m_1): $2, -2, 4$

dir. nos. of L_2 ($\perp$ to m_2): $2, 1, -1$

Since $L \parallel m_1$ and $L \parallel m_2$, $L \perp L_1$ and $L \perp L_2$.

vector $\parallel$ to L_1: $\vec{v}_1 = 2i - 2j + 4k$

vector $\parallel$ to L_2: $\vec{v}_2 = 2i + j - k$

$\Rightarrow$ vector parallel to L: $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 2 & -2 & 4 \\ 2 & 1 & -1 \end{vmatrix}$

$= (2i + 8j + 2k) - (-4k - 2j + 4i)$

$= -2i + 10j + 6k.$

$\Rightarrow$ dir. nos. of L are $-2, 10, 6$.

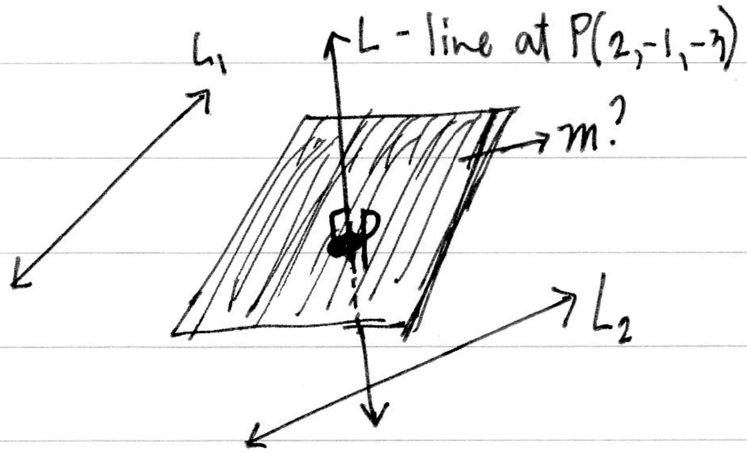
$\therefore$ The equations of the line L (in symmetric form) are

$$\frac{x+1}{-2} = \frac{y-3}{10} = \frac{z-2}{6}.$$

5. Find: $m: A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

Given: $(x_0, y_0, z_0) = P(2, -1, -3)$

Find: Attitude nos.: A, B, C.



dir. nos. of L_1 ($\parallel m$) : 3, 2, -4

dir. nos. of L_2 ($\parallel m$) : 2, -3, 2

$\Rightarrow$ vector $\parallel$ to L_1: $\vec{v}_1 = 3i + 2j - 4k$

vector $\parallel$ to L_2: $\vec{v}_2 = 2i - 3j + 2k$

vector $\parallel$ to L and $\perp$ to m: $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 3 & 2 & -4 \\ 2 & -3 & 2 \end{vmatrix}$

$\vec{v}_3 = -8i - 14j - 13k$

$\Rightarrow$ dir. nos. of the line L are $-8, -14, -13$.

$\therefore$ The equation of the plane is

$$-8(x-2) + (-14)(y+1) + (-13)(z+3) = 0$$

$$-8x - 14y - 13z - 37 = 0$$

$$\boxed{8x + 14y + 13z + 37 = 0}$$

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