- **TAYLOR'S SERIES APPROXIMATION**
- Taylor's series:

Taylor Series Definition

The **Taylor series** of a function $f(x)$ at a point $x = a$ is an **infinite sum** of terms calculated from the function's derivatives at that point. It expresses $f(x)$ as:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

or in **summation notation**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where:

- $f^{(n)}(a)$ is the nth derivative of $f(x)$ evaluated at $x = a$,
- $n!$ (n factorial) is the product of all positive integers up to n.

- f(x)= sin(x)=0 at x=0
  - f'(x)= cos(x)=1 at x=0
  - f''(x) = -sin(x) =0 at x= 0
  - f'''(x)= -cos(x) = -1 at x=0
  - f^{iv}(x) = sin(x) = 0 at x=0
  - $f^{v}(x) = \cos(x) = 1$ at $x = 0$.
  - $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$. By Substitution in Taylor's series we have
    - $\sin(x) = 0 + 1(x\text{-}0) + 0 - (x-0)^3 \frac{f'''(0)}{3!} + (x-0)^4 \frac{f^{v}(0)}{4!} \;\dots\!+ \frac{f^{n}(0)}{n!}$
    - $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots$
- By using Mathematica Series we have also

In[ ]:=

```
Series[Sin[x], {x, 0, 5}]
```

Out[ ]=

$$x - \frac{x^3}{6} + \frac{x^5}{120} + O[x]^6$$

- The $0[x]^6$ is the remainder after the fifth derivative:
- If we expand the function f(x)= sin(x) at x= $\frac{\pi}{4}$ we have
  - sin(x) = sin(π/4) = $\frac{1}{\sqrt{2}}$
  - f’(x) = cos(x) = cos(π/4) = $\frac{1}{\sqrt{2}}$
  - f’’(x) = -sin(x) = -sin(π/4) = -$\frac{1}{\sqrt{2}}$
  - f’’’(x) = -cos(x) = -cos(π/4) = -$\frac{1}{\sqrt{2}}$
  - $f^{iv}(x)$ = sin(x) = sin(π/4) = $\frac{1}{\sqrt{2}}$.

    $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$ By substitution in Taylor’s series we have
  - sin(x) = $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x-\pi/4) - \frac{\frac{1}{\sqrt{2}}}{2!}(x-\pi/4)^2 - \frac{\frac{1}{\sqrt{2}}}{3!}(x-\pi/4)^3 + \frac{\frac{1}{\sqrt{2}}}{4!}(x-\pi/4)^4 + \ldots$
- Taylor’s series at x= π/4 with 5 terms:

In[ ]:=

```
Series[Sin[x], {x, π / 4, 6}]
(* At x= π/4 for 7 terms. *)
```

Out[ ]=

$$\frac{1}{\sqrt{2}} + \frac{x-\frac{\pi}{4}}{\sqrt{2}} - \frac{\left(x-\frac{\pi}{4}\right)^2}{2\sqrt{2}} - \frac{\left(x-\frac{\pi}{4}\right)^3}{6\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^4}{24\sqrt{2}} + \frac{\left(x-\frac{\pi}{4}\right)^5}{120\sqrt{2}} - \frac{\left(x-\frac{\pi}{4}\right)^6}{720\sqrt{2}} + O\left[x-\frac{\pi}{4}\right]^7$$

- ==
- $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$

```
taylorPlus =
 Normal[Series[y[x + h], {x, x, 2}]] (* Expands x+
   h in a Taylor series around the point x=x,
  up to the second order. Since (x+h)
   is a simple linear expression,
  there is no real expansion happening. ) *)
```

Series: Center point x of power series expansion involves the variable x.

Out[•]=

```
y[h + x]
```

In[•]:=

```
taylorMinus = Normal[Series[y[x - h], {x, x, 2}]]
```

Series: Center point x of power series expansion involves the variable x.

Out[•]=

```
y[-0.01 + x]
```

- Approximating the 2nd derivative of y with respect to x: $\frac{dy^2}{dx^2}$.

In[•]:=

```
Clear[h]
```

In[•]:=

```
(*Define the Taylor series expansions for y(x+h) and y(x-h)*)
taylorPlus = Normal[Series[y[x + h], {x, x, 2}]];
taylorMinus = Normal[Series[y[x - h], {x, x, 2}]];

(*Compute the second derivative approximation*)
secondDerivativeApprox = (taylorPlus - 2 y[x] + taylorMinus) / h^2 // Simplify

(*Output should be an approximation for y''(x)*)
```

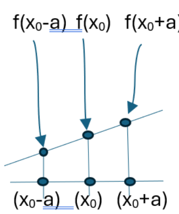
Series: Center point x of power series expansion involves the variable x.

Series: Center point x of power series expansion involves the variable x.

Out[•]=

$$\frac{-2\,y[x] + y[-h + x] + y[h + x]}{h^2}$$

(* $\frac{-2\ y[x]+y[-h+x]+y[h+x]}{h^2}$ .This is the approximation of $\frac{d^2y}{dx^2}$ which is the second derivative of y with respect to x. *)

- Taylor’s series: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$.Substitute in the Taylor’s series the following equations: (Expand about x_0 and approximate at $x_0 + a$.)
  - Let x = $x_0 + a$ or a = x - x_0 then
  - Eq. (1) $f(x_0 + a) = f(x_0) + f'(x_0)a + \frac{f''(x_0)}{2} a^2$ (up to order 2). Similarly
  - Eq. (2) $f(x_0 - a) = f(x_0) - f'(x_0)a + \frac{f''(x_0)}{2} a^2$
  - Adding equations (1) and (2) we have
  - $f(x_0 + a) + f(x_0 - a) = 2 f(x_0) + f''(x_0) a^2$ or
  - $\frac{f(x_0+a) - 2 f(x_0) + f(x_0-a)}{a^2} = f''(x_0)$
- 

- For the first derivative:

  dy/dx. $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$. Remove the remainder $+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$ from the Taylor’s series.
  - (1) $f(x_0 + a) = f(x_0) + f'(x_0)a$. Let point a be located on the right side of point of x_0. And for a point a at the left side we have
  - (2) $f(x_0 - a) = f(x_0) - f'(x_0)\, a$. Subtract equation (2) from equation (1) we have
  - $f(x_0 + a) - f(x_0 - a) = 2 f'(x_0)\, a$ or we have
  - $f'(x_0) = \frac{f(x_0+a) - f(x_0-a)}{2a}$

- If x is at the center point then to the right the point is $(x + h)$ and to the left it is $(x\text{-}h)$ so
  - (x_0+a) - $(x_0\text{-}\ a) = 2a$ (by central difference)
  - Hence $f'(x_0) = \frac{f(x_0+a)-f(x_0-a)}{2\,a}$

In[ ]:=
```
(*Forward difference approximation
 for the first derivative*)
taylorPlus = Normal[Series[y[x0 + a], {x, x, 1}]];
firstDerivativeForward =
 (taylorPlus - y[x0]) / a // Simplify
```

Series: Center point x of power series expansion involves the variable x.

Out[ ]=
$$\frac{-y[x_0]+y[a+x_0]}{a}$$

In[ ]:=
```
(*Backward difference approximation
 for the first derivative*)
taylorMinus = Normal[Series[y[x0 - a], {x, x, 1}]];
firstDerivativeBackward =
 (y[x0] - taylorMinus) / h // Simplify
```

Series: Center point x of power series expansion involves the variable x.

Out[ ]=
$$\frac{y[x_0]-y[-a+x_0]}{h}$$

In[∘]:=
```
(*Central difference approximation
 for the first derivative*)
taylorPlus = Normal[Series[y[x0 + a], {x, x, 2}]];
taylorMinus = Normal[Series[y[x0 - a], {x, x, 2}]];

firstDerivativeCentral =
 (taylorPlus - taylorMinus) / (2 a) // Simplify
```

Series: Center point x of power series expansion involves the variable x.

Series: Center point x of power series expansion involves the variable x.

Out[∘]=

$$\frac{-y[-a + x_0] + y[a + x_0]}{2\,a}$$

- Taylor’s series becomes Maclaurin’s series at x=0.

Examples of Taylor Series

1. **Exponential Function** e^x around $x = 0$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

2. **Sine Function** $\sin x$ around $x = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

3. **Cosine Function** $\cos x$ around $x = 0$:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Why Use Taylor Series?

- **Approximates complex functions** with polynomials.
- Used in **calculus, physics, and engineering** for approximations.
- Forms the basis for **Maclaurin series**, which is a Taylor series at $a = 0$.

```
Series[Exp[x], {x, 0, 10}]
(* Maclaurin's series for
exponent e^x up to the 10th derivative
but there are only 11  terms. *)
```

Out[]=

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \frac{x^7}{5040} + \frac{x^8}{40\,320} + \frac{x^9}{362\,880} + \frac{x^{10}}{3\,628\,800} + O[x]^{11}$$

```
Series[Cos[x], {x, 0, 10}]
(* Maclaurin's series for cos(x) up to the 10
th derivative but there are only 6 terms. *)
```

Out[]=

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40\,320} - \frac{x^{10}}{3\,628\,800} + O[x]^{11}$$

In[ ]:=

```
Series[Sin[x], {x, 0, 10}] (* Maclaurin's series for sin(x) up to 9th derivative. *)
```

Out[ ]=

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362\,880} + O[x]^{11}$$

Out[ ]=

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362\,880} + O[x]^{11}$$

-
- REVIEW

- ### Taylor's Approximation for the First Derivative

  To approximate the **first derivative** $f'(x)$, we use **Taylor series expansion**.

  ### 1. Forward Difference Approximation

  Expanding $f(x+h)$ using Taylor series at x:

  $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

  Ignoring higher-order terms and solving for $f'(x)$:

  $$f'(x) \approx \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h)$$

  This is the **forward difference approximation**.

- Express $f'(x)$ in terms of the other variables and divide the result by h.

2. Backward Difference Approximation

Expanding $f(x-h)$:

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \dots$$

Solving for $f'(x)$:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} + \mathcal{O}(h)$$

This is the **backward difference approximation**.

---

3. Central Difference Approximation (More Accurate)

Using both **forward** and **backward** approximations:

- $$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3)$$

  $$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3)$$

  Subtracting these:

  $$f(x+h) - f(x-h) = 2hf'(x) + \mathcal{O}(h^3)$$

  Solving for $f'(x)$:

  $$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2)$$

  This is the **central difference approximation**, which is more accurate because the error is $\mathcal{O}(h^2)$ instead of $\mathcal{O}(h)$.

- Divide by *h* also. O[*h*] is the order of accuracy.
- Taylor’s approximation for the first derivative:

In[•]:=
```
h = 0.01;
f[x_] := Sin[x];

forwardDiff[x_] := (Sin[x + h] - Sin[x]) / h;
backwardDiff[x_] := (Sin[x] - Sin[x - h]) / h;
centralDiff[x_] := (Sin[x + h] - Sin[x - h]) / (2 h);

results = {forwardDiff[Pi / 4], backwardDiff[Pi / 4], centralDiff[Pi / 4]};
results
```

Out[•]=
```
{0.703559, 0.710631, 0.707095}
```

```
D[Sin[x], x] /. x → π / 4 //
  N(* Actual value for f'(π/4). Central
   difference approximation is more accurate. *)
```

Out[•]=
```
0.707107
```

The **central difference method** is a more accurate approximation compared to forward or backward difference methods because it has a **lower truncation error** in its Taylor series expansion. Here's why:

1. Error Order Analysis (Truncation Error)

- **Forward difference:**

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{(First-order accurate: } O(h)\text{)}$$

- **Backward difference:**

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \quad \text{(First-order accurate: } O(h)\text{)}$$

- **Central difference:**

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad \text{(Second-order accurate: } O(h^2)\text{)}$$

Since the central difference has an $O(h^2)$ **error**, it means that as h decreases, the error reduces much faster than in the forward or backward difference methods, which have only $O(h)$ **error**.

2. Taylor Series Justification

Expanding $f(x+h)$ and $f(x-h)$ using Taylor series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + O(h^3)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - O(h^3)$$

Subtracting these equations:

$$f(x+h) - f(x-h) = 2hf'(x) + O(h^3)$$

Dividing by $2h$:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Since the error term is $O(h^2)$, it is **smaller** than the $O(h)$ error in forward and backward differences, making central difference more accurate.

3. Symmetry Advantage

- The forward and backward difference methods introduce an **asymmetry**, leading to larger error terms.
- The central difference method **balances errors** from both sides, reducing the impact of higher-order terms.

Conclusion

The **central difference method** is more accurate because:

1. It has a **smaller truncation error** ($O(h^2)$ instead of $O(h)$).
2. It leverages **symmetry**, which cancels out more error terms.
3. It provides a **better approximation** of the true derivative for small h.

- The central difference method balances errors from both sides, reducing the impact of higher-order terms: the second order terms O[h] are cancelled and so it reduces the impact of higher order terms.
- ==
- Taylor’s approximation for the second derivative:

Taylor's Approximation for the Second Derivative

Using **Taylor series expansion**, the second derivative $f''(x)$ can be approximated using **finite differences**.

Derivation from Taylor Series

Expanding $f(x)$ around x using Taylor series:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \mathcal{O}(h^4)$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \mathcal{O}(h^4)$$

Adding these two equations cancels out the odd-order terms:

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \mathcal{O}(h^4)$$

Solving for $f''(x)$:

- By adding $f(x+h)$ and $f(x-h)$ and dividing both sides by h^2 we have $f''(x)$ equivalence

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + \mathcal{O}(h^2)$$

Final Approximation (Central Difference Formula)

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

- **Order of accuracy:** $\mathcal{O}(h^2)$ (second-order accurate).
- **Most commonly used method** for numerical second derivatives.

- Example of &/@ and #

In[ ]:=
```
(*Using/@to apply squaring to each element*)
squaredList = #^2 & /@ {1, 2, 3, 4, 5}
```

Out[ ]=
```
{1, 4, 9, 16, 25}
```

- ==
- Example of order of accuracy which is equal to 2: $O[h^2]$

In[ ]:=
```
(*Define the function*)f[x_] := Sin[x]

(*Compute the exact second derivative at x0=π/4*)
x0 = Pi / 4;
trueValue = D[f[x], {x, 2}] /. x → x0

(*Define the finite difference approximation for f''(x)*)
finiteDifferenceSecondDerivative[x_, h_] :=
  (f[x - h] - 2 f[x] + f[x + h]) / h^2

(*Choose step sizes*)
hValues = {0.1, 0.05, 0.01};

(*Compute approximations and errors*)
approximations =
   finiteDifferenceSecondDerivative[x0, #] & /@ hValues;
errors = Abs[trueValue - #] & /@ approximations;

(*Display results in a table*)
TableForm[
 Transpose[{hValues, approximations, errors}], TableHeadings →
   {None, {"h", "Approximate f''(x0)", "Absolute Error"}}]
  (* Absolute Error =|Actual Value-FiniteDifferenceSecondDerivative| *)
```

Out[ ]=

$$-\frac{1}{\sqrt{2}}$$

Out[ ]//TableForm=

h	Approximate f''(x0)	Absolute Error
0.1	-0.706518	0.000589059
0.05	-0.706959	0.000147302
0.01	-0.707101	5.89253×10^{-6}

In[∘]:=

```
D[Sin[x], {x, 2}] /.
 x → π / 4 (* Second derivative of
   sin(x) with respect to x. *)
```

Out[∘]=

$$-\frac{1}{\sqrt{2}}$$

- ===

- Order of accuracy is 2: Reducing $h = 0.1$ to $h/2 = 0.05$ reduces the absolute error to $0.000589059/4 \approx 0.000147302$ since the error decreases quadratically as h gets smaller (order of accuracy is 2). That is by reducing h to $h/2$, error is also reduced to $\frac{\text{Error}}{2^2}$. The error term is $O[h^2]$.

In[∘]:=

```
f[x_, h_] := (Sin[x + h] - 2 * Sin[x] + Sin[x - h]) / h^2;
f[π / 4, 0.1]
```

Out[∘]=

```
-0.706518
```

Out[33]//TableForm=

h	Approximate f''(x0)	Absolute Error
0.1	-0.706518	0.000589059
0.05	-0.706959	0.000147302
0.01	-0.707101	5.89253×10^{-6}

In[∘]:=

```
f[π / 4, 0.05] // N
```

Out[∘]=

```
-0.706959
```

In[∘]:=

```
f[π / 4, 0.01]
```

Out[∘]=

```
-0.707101
```

Out[33]//TableForm=

h	Approximate f''(x0)	Absolute Error
0.1	-0.706518	0.000589059
0.05	-0.706959	0.000147302
0.01	-0.707101	5.89253×10^{-6}

In[ ]:=

```
Abs[-Sin[π / 4]] - Abs[f[π / 4, 0.1]] (* Error 1 *)
```

Out[ ]=

0.000589059

In[ ]:=

```
Abs[-Sin[π / 4]] - Abs[f[π / 4, 0.05]]
 (* Error 2 ≈ Error 1/4 *)
```

Out[ ]=

0.000147302

Out[33]//TableForm=

h	Approximate f''(x0)	Absolute Error
0.1	-0.706518	0.000589059
0.05	-0.706959	0.000147302
0.01	-0.707101	5.89253×10^{-6}

```
Abs[-Sin[π / 4]] - Abs[f[π / 4, 0.01]]
 (* Error 3. It further reduces
  the absolute error because the
  step size is further reduced. *)
```

Out[ ]=

5.89253×10^{-6}

- ==

- Approximation of e^x with step size equal to 0.01:

In[ ]:=

```
Clear[x]
```

In[◦]:=
```
(*Define step size*)h = 0.01;

(*Define function*)
f[x_] := Exp[x]; (*Example function*)

(*Second Derivative Approximation*)
secondDerivative[x_] :=
  (Exp[x + h] - 2 * Exp[x] + Exp[x - h]) / h^2;

(*Evaluate at x=π/4*)
secondDerivative[Pi / 4]
```

Out[◦]=
```
2.1933
```

In[◦]:=
```
D[Exp[x], {x, 2}] /. x → Pi / 4 //
 N (* Actual value *)
```

Out[◦]=
```
2.19328
```