NEWTON'S METHOD

Newton's Method for solving $f(x) = 0$.

- Taylors series analysis: use $f(r) = f(x_0 + r - x_0)$;

- Basic Newton's Method Algorithm: starting with approximation x_0, tolerance ϵ and maximum steps N

  1. Initialize with x_0 and $i = 0$;
  2. Set $x_{i+1} = x_i - f(x_i)/f'(x_i)$;
  3. If $|x_{i+1} - x_i| > \epsilon$ and $i < N$,
     set $i = i + 1$ and go to step 2;
  4. Stop with $x = x_{i+1}$.

  Notes:
  a) other convergences tests?
  b) needs $f(x_i)$, and $f'(x_i)$ for each step.

NEWTON'S METHOD CONTINUED

- Example: $f(x) = x^2 - \sin(x) - 0.5$;
  $f'(x) = 2x - \cos(x)$, so Newton iteration is

$$x_{i+1} = x_i - \frac{x_i^2 - \sin(x_i) - 0.5}{2x_i - \cos(x_i)}.$$

i	x	f(x)
0	2	2.59
1	1.41335678611631	0.5099
2	1.22236052594852	0.0543
3	1.19656415295535	0.000977
4	1.19608220128563	3.4e-07
5	1.19608203329716	4.1e-14
6	1.19608203329713	-2.2e-16

i	x	f(x)
0	0	-0.5
1	-0.5	0.229
2	-0.37780801587057	0.011
3	-0.370910551403399	3.9e-05
4	-0.370887340375536	4.4e-10
5	-0.370887340111992	-5.6e-17

NEWTON ERROR ANALYSIS

- Convergence Theorem: if $f \in C^2[a,b]$ and $\exists r \in (a,b)$ with $f(r) = 0$, $f'(r) \neq 0$, then $\exists \delta$ such that Newton's method converges to r for any $x_0 \in [r-\delta, r+\delta]$.
  Proof: consider $g(x) = x - f(x)/f'(x)$,
  with $g'(x) = f(x)f''(x)/(f'(x))^2$.

- Quadratic convergence:
  if a sequence $\{x_n\}$ converges to r with
  $$\lim_{n\to\infty} = \frac{|x_{n+1} - r|}{|x_n - r|^2} = M,$$
  constant, the sequence has **quadratic convergence**.
  Note: if error $e_n = x_n - r$, quadratic convergence means
  $$|e_{n+1}| \approx M|e_n|^2$$
  .

- Newton Method Quadratic Convergence
  if $f'(r) \neq 0$, $g'(r) = 0 \Rightarrow$ quadratic convergence.
  Taylor series analysis: uses $x_{n+1} = x_n - f(x_n)/f'(x_n)$.

NEWTON ERROR ANALYSIS CONT.

- Another Newton method example:
  computing 1/c without division. Solve $c - 1/x = 0$.
  Newton iteration: $x_+ = x - \frac{c-1/x}{1/x^2} = x(2 - cx)$.
  If $c = 3$, use $x_0 = .3 \rightarrow .33 \rightarrow .3333 \rightarrow .33333333\ldots$
- Multiple Roots?: let $f(x) = (x-r)^m q(x)$, $q(r) \neq 0$. With Newton's method, $g'(x) = f(x)f''(x)/(f'(x))^2$

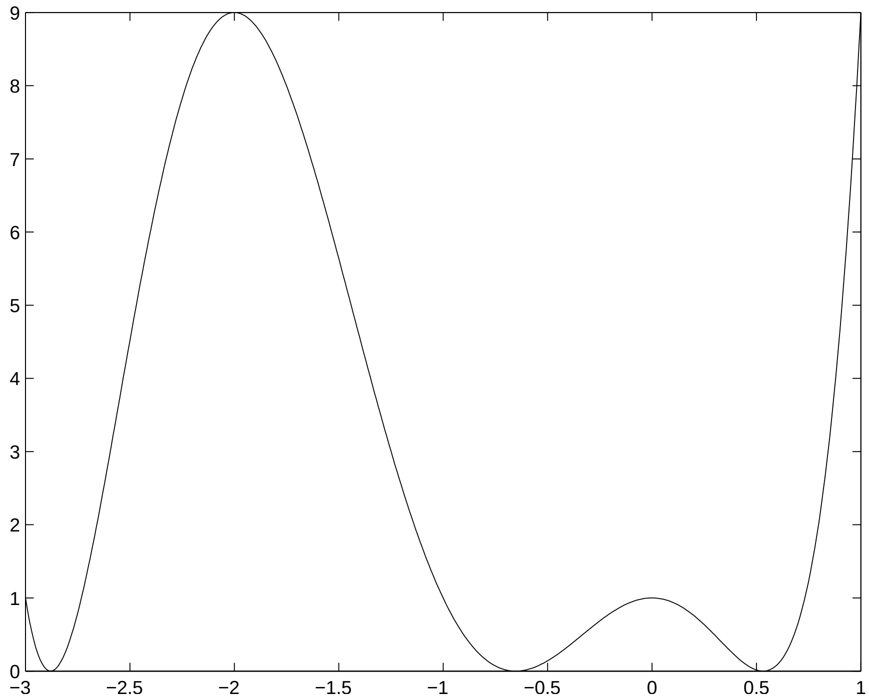
$|e_{n+1}| \approx \frac{m-1}{m}|e_n|$, **linear** convergence.
Modifed Newton Method

$$x_{i+1} = x_i - mf(x_i)/f'(x_i)$$

has quadratic convergence.

NEWTON ERROR ANALYSIS CONT.

Multiple Root Example:
$f(x) = x^6 + 6x^5 + 9x^4 - 2x^3 - 6x^2 + 1$



Matlab code with results

```
f  = @(x) x^6+6*x^5+9*x^4-2*x^3-6*x^2+1;
fp = @(x) 6*x^5+30*x^4+36*x^3-6*x^2-12*x; x = -3;
for i = 1:7, xo = x; x = x - f(x)/fp(x);
  disp([i x abs(x-xo) f(x)])
end
  i      x                  |x-x_|    f(x)
  0 -3                                1
  1 -2.944444444444445  5.6e-02  2.7e-01
  2 -2.913378139515769  3.1e-02  7.0e-02
  3 -2.896794662155538  1.7e-02  1.8e-02
  4 -2.888200262526650  8.6e-03  4.5e-03
  5 -2.883821304947564  4.4e-03  1.1e-03
  6 -2.881610539860946  2.2e-03  2.9e-04
  7 -2.880499723844134  1.1e-03  7.2e-05
```

NEWTON ERROR ANALYSIS CONT.

Modified Newton Method iteration with

$$x_+ = x - 2f(x)/f'(x)$$

i	x	|x-x_|	f(x)
0	-3		1
1	-2.888888888888889	1.1e-01	5.3e-03
2	-2.879451566951531	9.4e-03	2.5e-07
3	-2.879385244791951	6.6e-05	-7.1e-14
4	-2.879385627189817	3.8e-07	8.5e-12
5	-2.879385245597140	3.8e-07	-1.4e-13
6	-2.879385826819767	5.8e-07	2.0e-11

i	x	|x-x_|	f(x)
0	0.25		0.635
1	0.722222222222222	0.472	0.886
2	0.562590848305134	0.159	0.016
3	0.533090715782836	0.030	1.6e-05
4	0.532090025461782	0.001	2.1e-11
5	0.532088886236809	1.1e-06	3.3e-16
6	0.53210666380389	1.8e-05	5.1e-09

- Alternate Modified Newton Method: use $h(x) = f(x)/f'(x)$ instead of $f(x)$.