Reverse Mathematics: RCA, WWKL, and WKL

Exploring the Hierarchy of Logical Systems

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Introduction

- Studies subsystems of second-order arithmetic (RCA, WWKL, WKL).
- **Main Goal:** Show the hierarchy RCA $<$ WWKL $<$ WKL.

Key Questions

- What are the computability-theoretic and measure-theoretic differences among RCA, WWKL, and WKL?
- How do we use randomness and Π^0_1-classes to separate these systems?

Disclaimer

Normally, these subsystems of second-order arithmetic are denoted with a subscript 0, which indicates a restricted induction scheme (restricted to Σ_1^0 formulas only). However, since we are working exclusively within ω-models, we are implicitly assuming unrestricted induction for all formulas. Consequently, we omit the subscripts, as they hold no meaning for our purpose.

Key Subsystems

- RCA: Recursive Comprehension Axiom.
  - Base system: only computable sets and functions are allowed.
- WWKL: Weak Weak König's Lemma.
  - Every infinite binary tree of positive measure has an infinite path.
- WKL: Weak König's Lemma.
  - Every infinite binary tree has an infinite path.

Why RCA $<$ WWKL $<$ WKL?

We construct a model satisfying RCA but not WWKL and another model satisfying WWKL but not WKL.

Proof: RCA $<$ WWKL

- Goal: Prove that WWKL proves the existence of objects RCA cannot.
- Key Idea: Model - minimal turing ideal, or recursive set, that satisfies RCA not WWKL,
- Construct a Π_1-class V_n of positive measure and show it contains Martin-Löf random sequences.

Universal Martin-Löf Test

- A **Martin-Löf test** is a descending chain of effectively open sets $\{U_n\}_{n\in\omega}$ capturing sequences that fail randomness at level n.
- For a sequence to be Martin-Löf random, it must avoid all U_n.
- Each U_n satisfies $\mu(U_n) \leq 2^{-n}$, where μ is the Lebesgue measure on 2^ω (Cantor space).

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Purpose of U_n

- U_n captures sequences non-random up to level n.
- Sequences in U_n exhibit predictable patterns, failing randomness criteria at stage n.

Why U_n is Effectively Open (Σ_1)

- Each U_n represents sequences meeting computably enumerable failure conditions.
- Let F_n be a c.e. set of finite binary strings representing failure conditions.
- Then,

$$U_n = \bigcup_{\sigma \in F_n} [\sigma],$$

where $[\sigma]$ denotes the cylinder set of sequences starting with σ.

- As a union of basic open sets, U_n is effectively open.

Constructing the Π_1 Class

- Define $V_n = 2^\omega \setminus U_n$.
- V_n consists of sequences passing the test up to level n.
- Since U_n is effectively open (Σ_1), V_n is effectively closed (Π_1).

Properties of V_n

- V_n is a Π_1 class in Cantor space.
- It has positive measure:

$$\mu(V_n) = 1 - \mu(U_n) \geq 1 - 2^{-n} > 0 \quad \text{for } n > 0.$$

- Contains sequences passing the first n levels of the randomness test.

Why RCA Cannot Prove $V_n \neq \emptyset$

- RCA cannot guarantee the existence of non-computable objects.
- Since V_n contains only non-computable, random sequences, RCA cannot prove $V_n \neq \emptyset$.
- In an ω-model of RCA with only computable reals, V_n appears empty.

Why WWKL Proves $V_n \neq \emptyset$

- WWKL includes the axiom: Every Π^0_1 class of positive measure is non-empty.
- Since V_n is a Π^0_1 class with positive measure, WWKL proves $V_n \neq \emptyset$.
- Therefore, WWKL proves the existence of non-computable, random sequences in V_n.

WWKL v.s. WKL

- Show that WWKL is strictly weaker than WKL.
- Construct an ω-model $\mathcal{M}$ satisfying WWKL but not WKL.

Subsets of ω

- Let $X, Y, Z, \ldots$ denote subsets of ω.
- Each set $X \subseteq \omega$ is identified with its characteristic function χ_X.

Join Operation ($X \oplus Y$)

- For sets $X, Y \subseteq \omega$, define the join $X \oplus Y$ by interleaving:

$$Z(2n) = X(n), \quad Z(2n+1) = Y(n).$$

Randomness Over a Set

- A set X is random over Y if, for every $\Delta_1^{1,Y}$ class $\mathcal{B} \subseteq 2^\omega$ with $\mu(\mathcal{B}) = 1$, we have $X \in \mathcal{B}$.

Construction of $\{X^n\}$

- Base case: $X^0 = \emptyset$.
- Inductive step: For $n \geq 0$,

1. Choose X_n random over X^n.
2. Define $X^{n+1} = X^n \oplus X_n$.

Construction of Model $\mathcal{M}$

- $\mathcal{M} = \{Y : \exists n, Y \leq_T X^n\}$.
- Observe that $\mathcal{M}$ is an ω-Model of RCA(we show this by showing that $\mathcal{M}$ is a turing ideal)

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Non-Emptiness

$X^0 = \emptyset$ is recursive (computable without any oracle), so $\mathcal{M}$ contains at least all recursive sets (since any recursive set Y satisfies $Y \leq_T X^0$).

Closure Under Turing Reducibility

- If $A \in \mathcal{M}$ and $B \leq_T A$, then $B \leq_T X^n$ for some n, so $B \in \mathcal{M}$.

Closure Under Join

- If $A, B \in \mathcal{M}$, then $A \leq_T X^{n_1}$ and $B \leq_T X^{n_2}$.
- Let $n = \max(n_1, n_2)$, then $A \oplus B \leq_T X^n$.
- Thus, $A \oplus B \in \mathcal{M}$.

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Properties of $\mathcal{M}$

- $\mathcal{M}$ is a Turing ideal.
- Satisfies RCA.
- Satisfies WWKL.
- Fails WKL: Cannot separate certain sets A and B.

Why $\mathcal{M}$ Satisfies WWKL

- Let $T \subseteq 2^{<\omega}$ be a binary tree of positive density in $\mathcal{M}$, i.e., there exists n such that $T \leq_T X^n$.
- Define a set $\mathcal{D}$ capturing paths of T.
- Show $\mathcal{D}$ is measurable and Borel in X^n.
- Apply the **0-1 law** to prove $\mu(\mathcal{D}) = 1$.
- Conclude $\mathcal{M}$ contains an infinite path Z of T.
- Thus, $\mathcal{M}$ satisfies WWKL.

Define $\mathcal{D}$

$$\mathcal{D} = \left\{ Y : \exists Z \equiv_T Y \left(Z \text{ is a path of } T \right) \right\}$$

- $\mathcal{D}$ captures all sets Y Turing equivalent to a path Z of T.
- If T has paths of positive measure, $\mathcal{D}$ also has positive measure.

Paths of T are Borel

$$\mathit{Paths}(T) = \{Z \in 2^\omega : \forall n\ (Z \upharpoonright n \in T)\}$$

- $\mathit{Paths}(T)$ is the set of all infinite paths in T. where $Z \upharpoonright n$ is the prefix of Z of length n.
- Note that Paths(T) is Π^0_1 in T, and hence Borel in X^n.

D is Borel

- $\mathcal{D}$ is the closure of Paths(T) under Turing equivalence.
- Exercise: Show that D is Borel in X^n. [Hint: think about the definition of Turing Equivalence]

The 0-1 Law

Statement: For any measurable class $\mathcal{C}$ closed under finite variations:

$$\mu(\mathcal{C}) \in \{0, 1\}.$$

(p71, Proposition 1.9.12 Nies, Computability and Randomness 2009)

- $\mathcal{D}$ is measurable and closed under finite variations:
  - If $Y \in \mathcal{D}$, any finite variant of Z would be turing equivalent to Y and thus also belongs to $\mathcal{D}$.

$\mu(\mathcal{D}) = 1$

- The set $\mathit{Paths}(T) \subseteq \mathcal{D}$, so $\mu(\mathcal{D}) \geq \mu(\mathit{Paths}(T)) > 0$.
- By the 0-1 law, $\mu(\mathcal{D}) = 1$.

Putting it All Together

- Since by definition, X_n is random over X^n and $\mathcal{D}$ is $\Delta_1^{1X^n}$, $X_n \in \mathcal{D}$.
- The existence of Z as a path of T means Z is an infinite sequence that satisfies $Z \upharpoonright n \in T$ for all n. Since $Z \equiv_T X_n$ and X_n is part of the model $\mathcal{M}$, Z must also belong to $\mathcal{M}$.
- Therefore, T has a path Z that is defined within the model $\mathcal{M}$, satisfying the requirement that every infinite binary tree with positive measure has a path.

Why $\mathcal{M}$ Fails WKL

- **Weak König's Lemma (WKL):** Every infinite binary tree has an infinite path.
- Let A and B be disjoint r.e. sets not recursively separable.
- Claim: No $Y \in \mathcal{M}$ separates A and B.
- Inductive proof shows that for all n, no $Y \leq_T X^n$ separates A and B.
- Therefore, $\mathcal{M}$ does not satisfy WKL, since WKL implies Σ_1^0 separation over RCA, as discussed in class.

Base Case ($n = 0$)

- $X^0 = \emptyset$.
- Assumption: A and B are not recursively separable.
- Therefore, no recursive set Y separates A and B.
- **Conclusion:** Claim holds for $n = 0$.

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Inductive Hypothesis

Inductive Hypothesis:

- Assume the claim holds for n: No $Y \leq_T X^n$ separates A and B.

Goal:

- Prove the claim for $n + 1$: No $Y \leq_T X^{n+1}$ separates A and B.

Defining $\mathcal{D}$

$$\mathcal{D} = \{Z : \exists Y \leq_T^{X^n} Z \,(A \subseteq Y \wedge B \cap Y = \emptyset)\}.$$

- $\mathcal{D}$: Set of all Z that can compute (relative to X^n) a separating set Y.

Measure-Theoretic Argument

Theorem 5.3 (Jockusch and Soare):

- If *A* and *B* are disjoint recursively inseparable sets, and $\mathcal{S}$ is the collection of all sets that separate *A* and *B*,
$$\mu\left(\mathcal{U}(\mathcal{S})\right) = 0.$$
- $\mathcal{U}(\mathcal{S})$: Upward cone of $\mathcal{S}$ (sets Z such that Z computes some $Y \in \mathcal{S}$).

Application:

- Relativize to X^n: $\mu(\mathcal{D}) = 0$.
- Since $\mathcal{D}$ is $\Delta_1^{1X^n}$ [Exercise], $X_n \notin \mathcal{D}$.

Inductive Step

Implication for X^{n+1}:

- $X^{n+1} = X^n \oplus X_n$.
- X_n is random over X^n.
- If $Y \leq_T X^{n+1}$, then $Y \leq_T^{X^n} X_n$.
- Y cannot separate A and B.

Conclusion: $\mathcal{M}$ does not satisfy *WKL*.

Conclusion

- RCA $<$ WWKL $<$ WKL hierarchy demonstrated.
- Highlighted the role of randomness and measure theory in reverse mathematics.
- Explored the interplay between computability and logical systems.

Implications

- Foundational insights into mathematical theorems.
- Applications to algorithmic randomness and measure theory.

Works Cited I

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