Bridging Computability and Measure Theory: A Proof of RCA $<$ WWKL $<$ WKL

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A THESIS

in

Mathematics

Presented to the Faculties of the University of Pennsylvania
in Partial Fulfillment of the Requirements for the Degree of
Bachelor of Arts

2024

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Contents

Introduction

Overview of the Thesis Topic

Reverse mathematics is a field within mathematical logic that seeks to understand which axioms are necessary to prove particular theorems of mathematics. This area of study involves analyzing subsystems of second-order arithmetic and determining the equivalence of theorems with these subsystems. Among these, the Recursive Comprehension Axiom (RCA), Weak Weak König's Lemma (WWKL), and Weak König's Lemma (WKL) play fundamental roles. Each system has its own strengths and limitations, contributing to a hierarchy that informs our understanding of mathematical foundations. The relationships among these systems can reveal deep insights into the nature of mathematical proofs and the underlying logic required for various types of mathematical reasoning.

Thesis Purpose

The goal of this thesis is to provide an expository exploration that demonstrates the strict hierarchy RCA $<$ WWKL $<$ WKL within the context of reverse mathematics. In particular, we aim to explain why WWKL is strictly stronger than RCA yet strictly weaker than WKL. This exploration will involve concepts from computability theory and measure theory, referencing and building upon a measure-theoretic proof by Conidis that relates to Turing incomparability in the context of WWKL [1]. The proof, combined with other foundational concepts, serves as a means to illustrate the unique position of WWKL between RCA and WKL.

Outline of the Thesis

- **Background and Preliminaries** - We introduce the basic concepts and definitions from computability theory, including Turing machines, Turing reducibility, and degrees, as well as measure theory and algorithmic randomness. This section also includes a brief overview of reverse mathematics and its main subsystems relevant to our study.

- **Proof that RCA $<$ WWKL** - This section provides an exposition of why WWKL proves the existence of mathematical objects that RCA cannot, using the framework of Martin-Löf randomness and effectively closed classes.

- **Proof that WWKL $<$ WKL** - We construct an ω-model that satisfies WWKL but fails to satisfy WKL, showing that WWKL is indeed weaker than WKL. This part highlights the construction of computable trees without infinite paths and utilizes measure-theoretic principles.

The thesis aims to present an in-depth expository study on the proof that RCA $<$ WWKL $<$ WKL, situating these relationships within the broader field of reverse mathematics and computability theory. By exploring the different strengths of these subsystems of second-order arithmetic, the thesis will clarify how concepts such as computability, randomness, and measure-theoretic arguments are interwoven in mathematical logic. Through this structured exposition, it is hoped that this work will contribute to a clearer understanding of how different axioms relate in terms of proving the existence of certain mathematical objects and properties.

1 Background and Preliminaries

1.1 Computability Theory Basics

Computability theory is the branch of mathematical logic that studies the capabilities and limits of algorithmic computation. It seeks to understand what problems can be solved by computers (or more abstractly, by Turing machines) and what problems cannot be solved due to inherent limitations. In this section, we introduce some of the fundamental concepts that form the basis for the discussions in this thesis, including Turing machines, Turing degrees, Turing incomparability, and Scott sets. These ideas are foundational for understanding the measure-theoretic proof of Turing incomparability discussed in the later chapters.

1.1.1 Turing Machines

Definition 1.1 (Turing Machine)**.**

A *Turing machine* is an abstract model of computation that formalizes the concept of an algorithm. Introduced by Alan Turing in 1936 as part of his effort to answer the Entscheidungsproblem (decision problem) posed by David Hilbert, a Turing machine consists of the following components:

- An **infinite tape** divided into cells, each of which can hold a symbol from a finite alphabet. each capable of holding a symbol from a finite alphabet S. The tape serves as the machine's memory, where both input and output are recorded.
- A **tape head** that can read and write symbols on the tape and move left or right. The head interacts with the tape to manipulate data according to the machine's rules.
- A **finite set of states** $Q = \{q_0, q_1, \ldots, q_n\}$, where q_0 is the halting state. These states dictate the machine's current "configuration." The machine's behavior depends on its current state and the symbol it is reading from the tape.
- A **transition function** $\delta : Q \times S \to Q \times S \times \{L, R\}$, which determines the machine's actions based on its current state and the symbol being read. Specifically, the transition function determines:
  - The new symbol to write on the tape.
  - The direction to move the tape head (left or right).
  - The next state the machine should enter.

If δ is undefined for the current state and symbol, the machine halts.

If the machine halts and the tape contains a meaningful output, we say the machine has *computed* a result.

1.1.2 Turing Machine Example

Here's an example of a Turing machine and how it follows transition rules. Consider a Turing machine M designed to compute the function $f(x) = x + 3$. Its operation can be described by the following transition rules:

$$q_1\ 1 \ \rightarrow \ q_1,\ 1,\ R \tag{1}$$
$$q_1\ B \ \rightarrow \ q_2,\ 1,\ R \tag{2}$$
$$q_2\ B \ \rightarrow \ q_0,\ 1,\ R \tag{3}$$

In this machine:

- In state q_1, if the machine reads symbol 1, it writes 1, moves right, and remains in q_1 (Equation 1).
- In state q_1, if the machine reads blank (B), it writes 1, moves right, and transitions to state q_2 (Equation 2).
- In state q_2, if the machine reads blank (B), it writes 1, moves right, and transitions to the halting state q_0 (Equation 3).

Starting with input x (represented by $x + 1$ ones), the machine moves right, copying each 1 until it reaches a blank. It then writes three additional 1's before halting, effectively computing $x + 3$.[9].

1.1.3 Configurations and Computations

The **instantaneous configuration** of M during each step is determined by:

- The current state q_i of the machine.
- The symbol s_1 being scanned.
- The symbols on the tape to the right of s_1 up to the last 1, i.e., $s_2, s_3, \ldots, s_k$.
- The symbols to the left of s_1 up to the first 1, i.e., $t_1, t_2, \ldots, t_m$.

This configuration is written as:

$$c = t_m t_{m-1} \ldots t_1\ q_i\ s_1\ s_2\ s_3\ \ldots\ s_k. \tag{4}$$

Configurations formalize the notion of computation steps and are essential for analyzing the machine's behavior over time.[9].

1.2 Computability and Partial Functions

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be **computable** if there exists a Turing machine that computes f. In other words, given any input $n \in \mathbb{N}$, the Turing machine will halt and output $f(n)$ after a finite number of steps. However, not all functions are total; some Turing machines may not halt on certain inputs. This leads to the concept of **partial computable functions**, where the function is defined only on the inputs for which the machine halts.

Definition 1.2 (Computable Function)**.** A function $f : \mathbb{N} \to \mathbb{N}$ is *computable* if there exists a Turing machine M such that for all $x \in \mathbb{N}$, M halts on input x and outputs $f(x)$.

Definition 1.3 (Partial Computable Function)**.** A partial function $\varphi : \mathbb{N} \to \mathbb{N}$ is *partial computable* if there exists a Turing machine M such that for all $x \in \mathbb{N}$:

- If $\varphi(x)$ is defined, then M halts on input x and outputs $\varphi(x)$.
- If $\varphi(x)$ is undefined, then M does not halt on input x.

Turing machines serve as the foundation for defining computability and form the basis for understanding more complex notions like Turing reducibility and Turing degrees [9].

1.2.1 Turing Degrees

Turing degrees provide a way to classify problems (or sets of natural numbers) based on their relative computational complexity. Given two sets of natural numbers A and B, we say that A is **Turing reducible** to B (denoted $A \leq_T B$) if there is a Turing machine with access to an oracle for B that can compute the membership function for A. This means that, given access to the set B, we can decide whether any given number belongs to A.

More formally, $A \leq_T B$ if there exists an oracle Turing machine M^B such that for every input $n \in \mathbb{N}$, the machine M^B halts and correctly determines whether $n \in A$. Intuitively, this means that the information in B is sufficient to "compute" the set A. If both $A \leq_T B$ and $B \leq_T A$, we say that A and B are **Turing equivalent** (denoted $A \equiv_T B$).

The **Turing degree** of a set A is the equivalence class of all sets that are Turing equivalent to A. This classification captures the idea that certain sets (or problems) have the same level of computational complexity. For example, the halting problem—the set of all Turing machines that eventually halt on a given input—has a Turing degree that is strictly greater than that of any computable set, as it is not computable itself. The study of Turing degrees is fundamental in computability theory and helps us understand the structure of the partially ordered set of degrees, often referred to as the *Turing degrees lattice* [9, 5].

1.2.2 Turing Incomparability

Two sets A and B are said to be **Turing incomparable** if neither is Turing reducible to the other. That is, $A \not\leq_T B$ and $B \not\leq_T A$, meaning that the information contained in A is insufficient to compute B, and vice versa. Turing incomparability is a crucial concept in computability theory because it shows that there are sets of natural numbers that, while both incomputable, represent fundamentally different types of incomputability.

The concept of Turing incomparability is central to the study of algorithmic randomness and measure theory. For example, the Kučera-Slaman theorem demonstrates that for any incomputable set A, there exists another set B such that A and B are Turing incomparable. This result, along with its extensions, reveals the intricate structure of the Turing degrees and the diversity of incomputable sets [2]. In the context of this thesis, Turing incomparability will be proven using techniques from measure theory rather than the more traditional forcing methods.

1.3 Measure Theory

Measure theory is a fundamental branch of mathematical analysis that extends the intuitive notions of length, area, and volume to more complex and abstract sets. It provides the rigorous mathematical foundation for concepts such as integration, probability, and real analysis. The primary objective of measure theory is to assign a non-negative real number, called a *measure*, to subsets of a given space in a way that generalizes our intuitive understanding of size.

Lebesgue Measure One of the most significant measures in mathematics is the *Lebesgue measure*, denoted by μ. Developed by Henri Lebesgue in the early 20th century, it extends the concept of length from intervals to a wide class of subsets of the real line $\mathbb{R}$ and higher-dimensional spaces $\mathbb{R}^n$. The Lebesgue measure possesses several key properties:

- **Non-negativity**: For any measurable set $E \subseteq \mathbb{R}^n$, the measure $\mu(E) \geq 0$.
- **Null empty set**: The measure of the empty set is zero, i.e., $\mu(\emptyset) = 0$.
- **Countable additivity (δ-additivity)**: If $\{E_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint measurable sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

The Lebesgue measure aligns with our intuitive understanding for simple sets; for example, the measure of an interval $[a, b]$ in $\mathbb{R}$ is simply its length $b - a$. However, Lebesgue's framework allows us to measure much more complicated sets, including those that are uncountable or have fractal structures.

Sets of Measure Zero A set $E \subseteq \mathbb{R}^n$ is said to have *measure zero* (or be a *null set*) if, intuitively, it is "negligibly small" in terms of Lebesgue measure. Formally, E has measure zero if for every $\epsilon > 0$, there exists a countable collection of rectangles $\{R_i\}_{i=1}^{\infty}$ such that:

- $E \subseteq \bigcup_{i=1}^{\infty} R_i$.
- The sum of the volumes of these rectangles is less than ϵ, i.e., $\sum_{i=1}^{\infty} \mu(R_i) < \epsilon$.

Examples of measure-zero sets include:

- **Single points**: Any finite or countable set of points in $\mathbb{R}^n$ has measure zero.
- **The Cantor set**: Despite being uncountable and having no intervals, the classic Cantor set in $\mathbb{R}$ has measure zero.
- **Nowhere-dense sets**: Sets that are "thinly spread" throughout $\mathbb{R}^n$ can have measure zero even if they are uncountable.

Sets of measure zero are significant in analysis and probability because they are the sets over which properties can fail without affecting "most" of the space. For instance, a property that holds *almost everywhere* holds on the complement of a measure-zero set.

1.3.1 Algorithmic Randomness

Algorithmic randomness is a field that merges computability theory with probability theory to rigorously define what it means for an individual infinite sequence to be random. Unlike classical probability, which deals with events and their probabilities, algorithmic randomness focuses on the randomness of individual objects (typically sequences or real numbers) from a computational perspective.

Intuitive Notion of Randomness Intuitively, a sequence is random if it lacks any discernible pattern or regularity that can be exploited to compress it or predict future elements based on past observations. In other words, a random sequence should be *incompressible* and exhibit statistical properties expected of a typical sequence generated by a fair random process.

Martin-Löf Randomness Per Martin-Löf introduced a rigorous and widely accepted definition of randomness for infinite binary sequences, known as *Martin-Löf randomness*. A sequence is Martin-Löf random if it cannot be "singled out" by any effective null set—a set that is effectively describable and has measure zero.

Martin-Löf Tests

A *Martin-Löf test* is a uniformly computable sequence of effectively open sets $\{U_n\}_{n=1}^{\infty}$ in the Cantor space $\{0,1\}^{\mathbb{N}}$ such that:

- The measure of each U_n is bounded above by 2^{-n}, i.e., $\mu(U_n) \leq 2^{-n}$.
- Each U_n is effectively open, meaning it can be expressed as a computable union of basic open sets (cylinder sets) determined by finite binary strings.

A sequence $x \in \{0,1\}^{\mathbb{N}}$ *fails* the test if it belongs to the intersection $\bigcap_{n=1}^{\infty} U_n$. The intersection of all U_n is an effectively null set because its measure is at most $\sum_{n=1}^{\infty} 2^{-n} = 1$, ensuring that the total measure can be made arbitrarily small.[9]

An infinite binary sequence x is *Martin-Löf random* if it passes all Martin-Löf tests, i.e., it is not contained in any effectively null Σ_1^0 class of measure zero.

1.4 Reverse Mathematics Overview

1.4.1 Introduction to Reverse Mathematics

Reverse mathematics is a program in mathematical logic and foundations that seeks to determine the minimal axiomatic frameworks necessary to prove theorems of mathematics. Initiated by Harvey Friedman in the 1970s and further developed by Stephen Simpson and others, reverse mathematics works by analyzing the equivalence between mathematical theorems and subsystems of second-order arithmetic

Traditional mathematical practice typically involves proving theorems from a fixed, robust set of axioms, such as those found in Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) or Peano Arithmetic (PA). In contrast, reverse mathematics reverses this approach by starting with known theorems and asking: *Which axioms are precisely necessary to prove this theorem?* This methodology not only identifies the weakest possible axioms required but also often reveals surprising equivalences between different mathematical statements and axiomatic systems.

At the heart of reverse mathematics is the study of subsystems of second-order arithmetic. Second-order arithmetic extends first-order arithmetic by including quantification over sets of natural numbers, not just over individual numbers. This framework is sufficiently expressive to formalize most of classical mathematics, including analysis, combinatorics, and topology.

1.4.2 Subsystems of Second-Order Arithmetic

The subsystems of second-order arithmetic are categorized based on the strength of their axioms, particularly the comprehension (existence) axioms for sets of natural numbers. Each subsystem captures a different level of mathematical reasoning, allowing for a fine-grained analysis of the logical requirements of theorems. The most studied subsystems in reverse mathematics are known as the "Big Five," but for the purposes of this thesis, we focus on three key systems: RCA, WKL, and WWKL.

RCA (Recursive Comprehension Axiom) RCA, the *Recursive Comprehension Axiom*, serves as the base system in reverse mathematics. It embodies the principle that any set of natural numbers definable by a computable (recursive) property exists. We proceed to use the definition for RCA in Simpson's book in I.7[8]:

- **Basic Axioms of Peano Arithmetic (I.2.4(i))**: These axioms define the foundational properties of the natural numbers, including the basic operations of addition and multiplication, along with the axioms for order and identity. These are the fundamental properties required to discuss natural numbers formally within the system.

- **Δ^0_1-Comprehension Scheme (Definition I.7.3)**: This scheme allows us to construct sets whose membership criteria are defined by a formula that can be expressed both as a Σ^0_1 (existential) and as a Π^0_1 (universal) property. Formally, for any number variable n and formulas $\varphi(n)$ (a Σ^0_1 formula) and $\psi(n)$ (a Π^0_1 formula), if

$$\forall n(\varphi(n) \leftrightarrow \psi(n)),$$

then there exists a set X such that

$$\forall n\,(n \in X \leftrightarrow \varphi(n)).$$

This means that X contains exactly those elements n for which $\varphi(n)$ (or equivalently, $\psi(n)$) holds. Here, $\varphi(n)$ and $\psi(n)$ may include parameters, meaning they can have free set or number variables beyond n. In this way, the Δ^0_1-Comprehension Scheme ensures that we can form sets whose membership is defined by computable properties in a way that remains consistent across both existential and universal quantifications.

- **Σ^0_1-Induction Scheme (Definition I.7.2)**: This is a restricted form of induction that applies only to Σ^0_1 (existential) properties. Specifically, for any Σ^0_1 formula $\varphi(n)$, the scheme allows the use of induction to conclude that φ holds for all n, given that it holds at 0 and is preserved under the successor function (i.e., $\varphi(n) \to \varphi(n+1)$). Formally, it includes the universal closure of:

  $$(\varphi(0) \wedge \forall n\,(\varphi(n) \to \varphi(n+1))) \to \forall n\,\varphi(n).$$

  This restriction to Σ^0_1 formulas is significant because Σ^0_1 properties correspond to existentially definable sets, which are important in computability theory for describing recursively enumerable sets.

RCA encapsulates mathematics that can be developed using computable methods. It suffices for much of elementary number theory and combinatorics but is inadequate for proving theorems that require non-constructive existence arguments, such as those involving infinite objects guaranteed by compactness principles.

WKL (Weak König's Lemma) WKL augments RCA by incorporating *Weak König's Lemma*, a fundamental combinatorial principle with significant implications in analysis and topology. Weak König's Lemma states that every infinite subtree of the binary tree $\{0,1\}^{<\omega}$ has an infinite path. Specifically, if T is an infinite, finitely branching tree of finite binary strings (closed under initial segments), then there exists an infinite sequence $x \in \{0,1\}^\omega$ such that every finite initial segment of x is in T.

WKL consists of:

- All the axioms of RCA.
- **Weak König's Lemma (WKL)**: The assertion that every infinite, finitely branching tree has an infinite path.[8]

The addition of WKL elevates the system's strength, enabling proofs of theorems that are equivalent to compactness arguments, such as the Heine-Borel Theorem for closed intervals in $\mathbb{R}$. WKL is pivotal for results that require the existence of infinite objects derived from finite conditions, bridging the gap between computable mathematics and classical analysis.

WWKL (Weak Weak König's Lemma) WWKL further refines the logical landscape by introducing *Weak Weak König's Lemma*, which connects combinatorial principles with measure theory. WWKL asserts that any infinite binary tree of positive measure possesses an infinite path. In this context, the measure is defined using the standard Lebesgue measure on the Cantor space $\{0, 1\}^\omega$, where the measure of a basic open set (cylinder set) determined by a finite binary string s is $2^{-|s|}$.

WWKL comprises:

- All the axioms of RCA.
- **Weak Weak König's Lemma (WWKL)**: If $T \subseteq \mathrm{Seq}_2$ is a tree without a path, then

$$\lim_{n\to\infty} \sum_{\substack{s\in T \\ \mathrm{lh}(s)=n}} \frac{1}{2^{\mathrm{lh}(s)}} = 0.$$

  Here, Seq_2 denotes the full binary tree, i.e., the tree of all finite sequences of 0's and 1's ordered by inclusion.[10]

  1. **Tree $T \subseteq \mathbf{Seq}_2$**: T represents a binary tree that is a subset of Seq_2, where Seq_2 is the set of all finite binary sequences (sequences consisting of 0's and 1's). The elements of this set are ordered by inclusion, meaning each sequence can be extended by adding more 0's or 1's.
  2. **Tree without a path**: The statement specifies that T is a tree that does not have an infinite path. In other words, there is no infinite sequence of 0's and 1's such that all its finite initial segments are in T.
  3. **Sum condition**:

$$\lim_{n\to\infty} \sum_{\substack{s\in T \\ \mathrm{lh}(s)=n}} \frac{1}{2^{\mathrm{lh}(s)}} = 0.$$

     This part of the definition deals with the "weight" of the tree at different levels:
     - $\mathrm{lh}(s)$ denotes the length of a finite sequence s.
     - The sum $\sum_{\substack{s\in T \\ \mathrm{lh}(s)=n}} \frac{1}{2^{\mathrm{lh}(s)}}$ calculates the total weight of all sequences of length n that are in T. Each sequence s contributes $\frac{1}{2^{\mathrm{lh}(s)}}$ to the sum, which represents its "probability" under a uniform distribution over all binary sequences.
     - The limit condition $\lim_{n\to\infty}$ states that, as n (the length of the sequences) approaches infinity, the total weight of the sequences in T must approach 0. This implies that T becomes "sparse" as the level n increases.

  In other words, every infinite binary tree with positive Lebesgue measure has an infinite path.

WWKL is instrumental in the study of algorithmic randomness and measure theory within a logical framework. It is weaker than WKL but strong enough to prove the existence of Martin-Löf random sequences and to develop a substantial portion of measure theory. WWKL allows for probabilistic methods in reverse mathematics, providing a bridge between computability and randomness.

2 Proof that RCA < WWKL

We aim to show that WWKL is strictly stronger than RCA, meaning there are statements provable in WWKL that are not provable in RCA.

2.1 Definitions and Notations

1. **Subsets of ω:**

   - Let $X, Y, Z, \ldots$ denote subsets of the natural numbers ω.
   - Each set $X \subseteq \omega$ is identified with its **characteristic function** $\chi_X : \omega \to \{0, 1\}$ defined by:
   $$\chi_X(n) = \begin{cases} 1 & \text{if } n \in X, \\ 0 & \text{if } n \notin X. \end{cases}$$

2. **Join Operation ($X \oplus Y$):**

   - For sets $X, Y \subseteq \omega$, define the **join** $X \oplus Y$ as the set Z such that:
   $$Z(2n) = X(n), \quad Z(2n+1) = Y(n), \quad \text{for all } n \in \omega.$$
   - This operation interleaves the bits of X and Y to form Z.

3. **Turing Reducibility ($\leq_T$):**

   - A set A is **Turing reducible** to a set B (denoted $A \leq_T B$) if there exists a Turing machine which, given oracle access to B, computes A.

4. A **Turing Ideal** is a non-empty collection of subsets of ω (the natural numbers) satisfying the following properties:

   - Closure under Turing Reducibility:
   If $A \in \mathcal{I}$ and $B \leq_T A$ (i.e., B is Turing reducible to A), then $B \in \mathcal{I}$.
   - Closure under Join:
   If $A, B \in \mathcal{I}$, then the join $A \oplus B \in \mathcal{I}$
   This essencially means that combining any two sets in $\mathcal{I}$ through the join operation results in another set in $\mathcal{I}$.

5. Connection Between **Turing Ideals** and **ω-Models of RCA**:

   An ω-model $\mathcal{M}$ of RCA consists of the standard natural numbers ω and a collection of subsets of ω. The system RCA includes:

   - The axioms of Recursive Comprehension: If a set A can be defined by a computable (recursive) process using existing sets in the model, then A is included in the model.

- Restricted forms of Induction which we discussed in the earlier section of definition on RCA

6. **ω-Models of RCA:** An ω-model $\mathcal{M}$ of RCA consists of the standard natural numbers ω and a collection of subsets of ω. The system RCA includes:
   - **Recursive Comprehension:** If a set A can be defined by a computable process using existing sets in the model, then A is included in the model.
   - **Restricted Induction:** RCA includes induction restricted to formulas involving computable functions.

2.2 Remark: The Minimal Turing Ideal

The minimal Turing ideal consists of exactly the **decidable sets**, i.e., those subsets of ω that have computable characteristic functions. This property is fundamental to the structure of RCA, as the recursive comprehension axiom in RCA ensures that only computable sets and those derived from them through recursive operations are included in the model.

Implication for WWKL: Since WWKL requires the existence of non-computable elements in certain effectively closed sets of positive measure (such as Martin-Löf random reals), it cannot hold in the minimal Turing ideal. This discrepancy highlights why RCA does not imply WWKL: the former is restricted to computable sets, while the latter demands the existence of non-computable objects.

2.3 Constructing a Π_1 Class of Positive Measure

2.3.1 The Universal Martin-Löf Test

A *Martin-Löf test* is a sequence of effectively open sets $\{U_n\}_{n\in\omega}$ where each set U_n captures the sequences that fail to be random at a certain level n. For a sequence to pass the Martin-Löf randomness test, it must avoid all these sets U_n for all n. The measure of each U_n is constrained such that $\mu(U_n) \leq 2^{-n}$, where μ represents the *Lebesgue measure* on 2^ω (Cantor space).

The *universal Martin-Löf test* $\{U_n\}$ is designed to catch all sequences that fail some form of Martin-Löf randomness test. If a sequence appears in any U_n, it fails to be random according to the Martin-Löf criterion.

2.3.2 Purpose of U_n

U_n is designed to capture sequences that fail the randomness criteria up to level n. This means that each sequence in U_n displays some kind of predictable pattern that makes it non-random by the n-th stage of the test.

2.3.3 Why U_n is Effectively Open (Σ_1)

Each set U_n is constructed to represent sequences that are "non-random" by meeting specific failure conditions. Because these failure conditions are computably enumerable, they can be represented by a union of finite segments (initial segments of paths in Cantor space).

- Let F_n be a c.e. set of finite binary strings that identify the failure conditions for randomness at level n.

- Then, U_n can be represented as:

$$U_n = \bigcup_{\sigma \in F_n} [\![\sigma]\!]$$

where $[\![\sigma]\!]$ denotes the *cylinder set* of all paths in 2^ω that start with the finite string σ.

- Since U_n is the union of basic cylinder sets $[\![\sigma]\!]$, it is *open* in Cantor space. Each cylinder set $[\![\sigma]\!]$ depends only on a finite initial segment σ, satisfying the definition of an open class.

2.3.4 Constructing the Π_1 Class

To define a set of paths that pass the test up to a certain level, we consider the *complement* of U_n:

- For each $n \in \omega$, define:

$$V_n = 2^\omega \setminus U_n$$

Here, V_n represents all paths in Cantor space that do *not* fail the randomness test up to level n. Since U_n is effectively open, its complement V_n is *effectively closed*, meaning it's a Π_1 set. Here, we can take any $n > 0$.

2.3.5 Why the Complement $V_n = 2^\omega \setminus U_n$ is Closed

Since U_n is effectively open, its complement $V_n = 2^\omega \setminus U_n$ is *effectively closed* in Cantor space.

- *Closed Sets in Cantor Space*: In Cantor space, a closed set (Π_1 class) is one that includes all paths avoiding certain patterns specified by a c.e. set.

- *Defining V_n*: Concretely, we can write V_n as:

$$V_n = \{f \in 2^\omega \mid \forall x \, \neg R(f \upharpoonright x)\}$$

where R is the relation capturing the failure conditions in U_n. Thus, V_n consists of all paths that do not match any of the failure conditions in F_n, making it effectively closed.

2.3.6 Properties of V_n

- Π_1 *Class*: Because U_n is effectively open (Σ_1), V_n, as its complement, is effectively closed (Π_1). In Cantor space, a Π_1 class is one that can be defined as the set of all sequences that avoid certain conditions determined by a computably enumerable (c.e.) set.

- *Positive Measure*: Since $\mu(U_n) \leq 2^{-n}$, it follows that:

$$\mu(V_n) = 1 - \mu(U_n) \geq 1 - 2^{-n}$$

which is greater than zero when n > 0. Therefore, V_n has *positive measure.*

2.4 Why V_n Has Positive Measure

1. *Measure of U_n*: By construction, each U_n has measure at most 2^{-n}. This keeps U_n relatively small in measure as n increases.

2. *Measure of $V_n = 2^\omega \setminus U_n$*: Since $\mu(U_n) \leq 2^{-n}$, the measure of V_n is:

$$\mu(V_n) = 1 - \mu(U_n) \geq 1 - 2^{-n}$$

which is greater than zero when $n > 0$. This confirms that V_n is a Π_1 class of *positive measure*.

V_n Contains Only Martin-Löf Random Reals: By definition, V_n includes only those sequences that pass the first n levels of the Martin-Löf test. We ensure that V_n excludes all computable reals, since every computable real fails some level of the test. Consequently, V_n consists entirely of sequences that exhibit a high degree of randomness and can be viewed as containing only Martin-Löf random reals.

Also see Corollary 11.2.2 p. 191 of Soare[9].
Statement: There is a nonempty Π^0_1 class all of whose elements are ML-random.

Let $U_0, U_1, \ldots$ be a universal Martin-Löf test. For every $n > 0$, U_n is a proper Σ^0_1 subclass of 2^ω, implying that $\overline{U_n}$ is a nonempty Π^0_1 class. By the definition of a universal Martin-Löf test,

$$\overline{U_n} \subseteq \bigcup_{n\in\omega} \overline{U_n} = \overline{\bigcap_{n\in\omega} U_n} = \{X \in 2^\omega : X \text{ is ML-random}\} = V_n,$$

2.5 Why RCA Cannot Prove the Existence of an Element in V_n

In RCA, we are limited to proving the existence of sets that are computable or can be derived from computable operations. Since RCA does not guarantee the existence of non-computable objects, it cannot ensure the existence of sequences in V_n, which contains only non-computable, Martin-Löf random reals.

- *Implication*: Since V_n contains no computable elements, RCA cannot prove $V_n \neq \emptyset$.

- In an *ω-model* of RCA that includes only computable reals, V_n would appear empty, as there would be no elements satisfying the conditions to belong to V_n within the constraints of RCA.

2.6 Why WWKL Proves the Existence of an Element in V_n

WWKL, on the other hand, includes an axiom that extends beyond RCA: *For any $\Pi^{0,A}_1$-class $X \subseteq 2^\omega$ with positive measure, $X \neq \emptyset$.*[1] This guarantees that any effectively closed set of positive measure will contain at least one element.

- *Application to V_n*: Since V_n is a Π_1 class with $\mu(V_n) > 0$, WWKL asserts that $V_n \neq \emptyset$. This means that WWKL can prove the existence of at least one Martin-Löf random real in V_n.
- *Implication*: WWKL, therefore, proves that V_n contains a non-computable element, something RCA cannot prove.

Since WWKL can guarantee the existence of non-computable reals (specifically, Martin-Löf random reals) that RCA cannot, we conclude that RCA is strictly weaker than WWKL.

3 Proof that (WWKL) < (WKL)

3.1 Introduction

We aim to demonstrate that Weak Weak König's Lemma (WWKL) is strictly weaker than Weak König's Lemma (WKL) by constructing an ω-model $\mathcal{M}$ that satisfies WWKL but does not satisfy WKL. This construction is discussed in this paper [10]. Here, we will provide an exposition of this argument involving use of computability theory and measure theory where we present a model of RCA in which every binary tree of positive measure has an infinite path, but there is a certain infinite binary tree without an infinite path.

3.2 Definitions and Notations

1. **Subsets of ω:**
   - Let $X, Y, Z, \ldots$ denote subsets of the natural numbers ω.
   - Each set $X \subseteq \omega$ is identified with its **characteristic function** $\chi_X : \omega \to \{0, 1\}$ defined by:
   $$\chi_X(n) = \begin{cases} 1 & \text{if } n \in X, \\ 0 & \text{if } n \notin X. \end{cases}$$
2. **Join Operation ($X \oplus Y$):**
   - For sets $X, Y \subseteq \omega$, define the **join** $X \oplus Y$ as the set Z such that:
   $$Z(2n) = X(n), \quad Z(2n+1) = Y(n), \quad \text{for all } n \in \omega.$$
   - This operation interleaves the bits of X and Y to form Z.
3. **Randomness Over a Set:**
   - A set X is said to be random over another set Y if, for every Δ_1^{1Y} class $\mathcal{B} \subseteq 2^\omega$ (i.e., a Borel subset of 2^ω with a code recursive in Y), if $\mu(\mathcal{B}) = 1$, then $X \in \mathcal{B}$. This randomness notion implies that X satisfies all Y-computable random tests that are Borel in Y.

4. **Turing Reducibility ($\leq_T$):**

   - A set A is **Turing reducible** to a set B (denoted $A \leq_T B$) if there exists a Turing machine which, given oracle access to B, computes A.

5. A **Turing Ideal** is a non-empty collection of subsets of ω (the natural numbers) satisfying the following properties:

   - Closure under Turing Reducibility:
     If $A \in \mathcal{I}$ and $B \leq_T A$ (i.e., B is Turing reducible to A), then $B \in \mathcal{I}$.
   - Closure under Join:
     If $A, B \in \mathcal{I}$, then the join $A \oplus B \in \mathcal{I}$
     This essencially means that combining any two sets in $\mathcal{I}$ through the join operation results in another set in $\mathcal{I}$.

6. Connection Between **Turing Ideals** and **ω-Models of RCA**:

   An ω-model $\mathcal{M}$ of RCA consists of the standard natural numbers ω and a collection of subsets of ω. The system RCA includes:

   - The axioms of Recursive Comprehension: If a set A can be defined by a computable (recursive) process using existing sets in the model, then A is included in the model.
   - Restricted forms of Induction which we discussed in the earlier section of definition on RCA

   Remark: Full Induction in ω-Models. Since the natural number part of an ω-model is precisely the standard natural numbers ω, ω-models satisfy *full induction* for all arithmetic formulas. This is because the principle of induction, when applied to the standard natural numbers, holds without restriction. However, in the context of ω-models of RCA, the induction axiom schema included in the formal system is limited to Σ^0_1 formulas.

7. Density of a Tree

   The **density** of a tree $T \subseteq 2^{<\omega}$ is defined as:

   $$\text{density}(T) = \limsup_{n\to\infty} \frac{|T \cap 2^n|}{2^n},$$

   where $T \cap 2^n$ denotes the set of all strings in T of length n, and $|T \cap 2^n|$ is the cardinality of this set.

   Interpretation: Intuitively, the density measures how "large" T is at each level of the binary tree $2^{<\omega}$, relative to the total number of strings of a given length. A tree with higher density at deeper levels corresponds to a "larger" set of infinite paths.

   Measure of a Tree's Paths: $[T]$

The set of infinite paths through T is denoted by:

$$[T] = \{Z \in 2^\omega : \forall n\ (Z \upharpoonright n \in T)\}.$$

The measure of $[T]$ under the Lebesgue measure on 2^ω is given by:

$$\mu([T]) = \sum_{s \in T} 2^{-|s|},$$

where $|s|$ is the length of the string s. This measure corresponds to the total weight of all strings in T, with each string weighted inversely proportional to its length.

Lemma: Density and Measure of a Tree

Lemma: The density of a tree T is equal to the measure of its paths:

$$\text{density}(T) = \mu([T]).$$

Proof: Consider the n-th level of T, denoted $T \cap 2^n$. Each string $s \in T \cap 2^n$ contributes a weight of 2^{-n} to the measure $\mu([T])$. Therefore:

$$\mu([T]) = \sum_{n=0}^{\infty} \sum_{s \in T \cap 2^n} 2^{-|s|}.$$

Since $|s| = n$ for all $s \in T \cap 2^n$, we can rewrite the measure as:

$$\mu([T]) = \sum_{n=0}^{\infty} \left(|T \cap 2^n| \cdot 2^{-n}\right).$$

For sufficiently large n, the density of T is given by:

$$\text{density}(T) = \limsup_{n \to \infty} \frac{|T \cap 2^n|}{2^n}.$$

Observe that:

$$\frac{|T \cap 2^n|}{2^n} = \sum_{s \in T \cap 2^n} 2^{-n},$$

which matches the contribution of $T \cap 2^n$ to $\mu([T])$. Taking the limit superior as $n \to \infty$, we conclude that:

$$\text{density}(T) = \mu([T]).$$

$\square$

Observation

A collection of subsets of ω is an ω-model of RCA if and only if it is a Turing ideal.

Necessity. If $\mathcal{M}$ is an ω-model of RCA, then it must include all sets computable from its members (closure under Turing reducibility) and be closed under join (since the join of two computable sets is computable). This closure under Turing reducibility and join is precisely the structure of a Turing ideal.

Sufficiency. If $\mathcal{M}$ is a Turing ideal, it satisfies the recursive comprehension axiom because it contains all sets computable from its members. The closure under join ensures that the model can construct sets defined by computable operations on existing sets, which is necessary for satisfying RCA.

8. **The Model $\mathcal{M}$:**
   - Define $\mathcal{M} = \{Y \subseteq \omega \mid \exists n \in \omega \text{ such that } Y \leq_T X^n\}$, where X^n are sets constructed recursively as described below.

3.3 Construction of the Sequence $\{X^n\}_{n\in\omega}$

We define a sequence of sets X^n for $n \in \omega$ as follows:

- **Base Case:**
  - Let $X^0 = \emptyset$ (the empty set).
- **Inductive Step:**
  - For each $n \geq 0$:
    1. **Random Set X_n:** Choose X_n to be a set that is random over X^n.
    2. **Define X^{n+1}:** Let $X^{n+1} = X^n \oplus X_n$.

3.4 Proof that $\mathcal{M}$ is an ω-Model of RCA

Recall that $\mathcal{M} = \{Y \subseteq \omega \mid \exists n \in \omega \text{ such that } Y \leq_T X^n\}$

Non-Emptiness: $X^0 = \emptyset$ is recursive (computable without any oracle), so $\mathcal{M}$ contains at least all recursive sets (since any recursive set Y satisfies $Y \leq_T X^0$).

Closure under Turing Reducibility: Let $A \in \mathcal{M}$ and $B \leq_T A$. There exists n such that $A \leq_T X^n$. By transitivity of Turing reducibility, $B \leq_T A \leq_T X^n$, so $B \leq_T X^n$. Therefore, $B \in \mathcal{M}$.

Closure under Join: Let $A, B \in \mathcal{M}$. Then $A \leq_T X^{n_1}$ and $B \leq_T X^{n_2}$ for some $n_1, n_2 \in \omega$. Let $n = \max(n_1, n_2)$.

We prove by induction on n that for all $m < n$, $X^m \leq_T X^n$.

Base Case ($n = 1$):

- For $n = 1$, the only $m < n$ is $m = 0$.
- $X^0 = \emptyset$, which is computable without any oracle (i.e., recursive).

- Since X^1 is an oracle, $X^0 \leq_T X^1$ trivially, as any recursive set is reducible to any oracle.

Thus, the claim holds for $n = 1$.
Inductive Step: Assume that for some n, the claim holds: for all $m < n$, $X^m \leq_T X^n$. We need to prove that for all $m < n+1$, $X^m \leq_T X^{n+1}$.

- **Case 1:** $m < n$**:**
  - By the inductive hypothesis, $X^m \leq_T X^n$.
  - Since $X^n \leq_T X^{n+1}$ (by definition, X^n is computable from X^{n+1}, as X^{n+1} includes all the computational power of X^n plus potentially more),

$$X^m \leq_T X^n \quad \text{and} \quad X^n \leq_T X^{n+1} \implies X^m \leq_T X^{n+1}.$$

- **Case 2:** $m = n$**:**
  - By definition, $X^n \leq_T X^{n+1}$.

Conclusion: By induction, for all n and for all $m < n$, $X^m \leq_T X^n$.

Since $X^{n_1}, X^{n_2} \leq_T X^n$, we have $A, B \leq_T X^n$. The join $A \oplus B$ is computable from X^n (since X^n can compute A and B). Therefore, $A \oplus B \leq_T X^n$, so $A \oplus B \in \mathcal{M}$.

Since $\mathcal{M}$ is non-empty and closed under Turing reducibility and join, $\mathcal{M}$ is a Turing ideal. Therefore, by the observation above, $\mathcal{M}$ is an ω-model of RCA.

3.5 Theorem 1: $\mathcal{M}$ Satisfies WWKL$_0$

Statement: The model $\mathcal{M}$ satisfies WWKL$_0$; that is, every infinite tree $T \subseteq 2^{<\omega}$ of positive measure has an infinite path $X \in \mathcal{M}$.

Proof:

As mentioned in Yu and Simpson's paper[10], the proof is the following:

Suppose T is a binary tree in this model, i.e., there is n such that $T \leq_T X^n$. Let $\mathcal{D} = \{Y : \exists Z \equiv_T Y(Z \text{ is a path of } T)\}$. $\mathcal{D}$ is $\Delta_1^{1X^n}$. If the set of paths through T is of positive measure, then by the 0-1 law (cf. Halmos [3] p. 201 and Sect. 10 of Sacks[6]), $\mu(\mathcal{D}) = 1$, so that $X_n \in \mathcal{D}$. From the definition of $\mathcal{D}$, there is Z, a path of T, such that $Z \equiv_T X_n$. Thus T has a path in the model. This shows that $\mathcal{M}$ is a model of WWKL

Let's break it down:

Let T be a binary tree of positive density in $\mathcal{M}$, i.e., there exists n such that $T \leq_T X^n$. We need to show that T has an infinite path $Z \in \mathcal{M}$.

Define:

$$\mathcal{D} = \{Y : \exists Z \equiv_T Y(Z \text{ is a path of } T)\}$$

The Borel hierarchy is built from countable unions, intersections, and complements of basic open sets in 2^ω. Here's why $\mathcal{D}$ fits into this hierarchy:

1. **Paths of T are Borel:**
   From the earlier explanation, $Paths(T)$ is defined as:

   $$Paths(T) = \{Z \in 2^\omega : \forall n\ (Z \upharpoonright n \in T)\},$$

   where $Z \upharpoonright n$ is the prefix of Z of length n.

   This definition implies that Paths(T) is Π^0_1 in T. Therefore, since T is recursive in X^n(there is n such that $T \leq_T X^n$), Paths(T) is Π^0_1 in X^n. Thus Paths(T) is **Borel in** X^n.

2. **Turing Equivalence Preserves Borelness:**
   Turing equivalence does not introduce complexity beyond the Borel hierarchy because it involves computable relationships. Hence, the union over Turing equivalence relations preserves Borelness. Thus closure of Paths(T) under Turing equivalence is **Borel in** X^n

The **0-1 law** states that for a measurable class $\mathcal{C}$ that is **closed under finite variations**, the measure $\mu(\mathcal{C})$ must be either 0 or 1.(p71, Proposition 1.9.12 [5]) Let us apply this to $\mathcal{D}$ in the context of the proof.

Step 1: Why $\mathcal{D}$ Is Measurable and Closed Under Finite Variations

$$\mathcal{D} = \{Y : \exists Z \equiv_T Y\ (Z \text{ is a path of } T)\}$$

is defined based on paths of T, which are subsets of 2^ω, the space of infinite binary sequences.

Closed under finite variations means that if a set $Y \in \mathcal{D}$ differs from another set Y' in only finitely many places, then Y' must also be in $\mathcal{D}$. This property holds for $\mathcal{D}$ because if $Z \equiv_T Y$, then any finite variant of Z is turing equivalent to Y.

Step 2: Why $\mu(\mathcal{D}) > 0$

The proof assumes that T, the binary tree, has positive measure. This means the set of all infinite paths through T, Paths(T) has measure > 0. Since every set is $\equiv_T$ to itself, $\mathcal{D}$ contains Paths(T), the measure of $\mathcal{D} \geq \mu(Paths(T)) \geq 0$.

The **0-1 law** implies that $\mu(\mathcal{D})$ must be 1. This follows from the proposition provided: if a measurable set $\mathcal{C}$ is closed under finite variations and has $\mu(\mathcal{C}) > 0$, then $\mu(\mathcal{C}) = 1$. Since by definition, X_n is random over X^n and $\mathcal{D}$ is $\Delta^{1X^n}_1$, $X_n \in \mathcal{D}$ The existence of Z as a path of T means Z is an infinite sequence that satisfies $Z \upharpoonright n \in T$ for all n. Since $Z \equiv_T X_n$ and X_n is part of the model $\mathcal{M}$, Z must also belong to $\mathcal{M}$. Therefore, T has a path Z that is defined within the model $\mathcal{M}$, satisfying the requirement that every infinite binary tree with positive measure has a path.

3.6 Theorem 2: $\mathcal{M}$ Does Not Satisfy WKL

Statement: The model $\mathcal{M}$ does not satisfy WKL_0; that is, there exists an infinite subtree $T \subseteq 2^{<\omega}$ such that T has no infinite path in $\mathcal{M}$.

Proof:

As mentioned in Yu and Simpson's paper[10], the proof is the following:

Let A and B be two disjoint recursively enumerable subsets of ω which are not recursively separable. We claim that there is no $Y \in \mathcal{M}$ which separates A and B. To prove this, it suffices to show that for all $n \in \omega$, there is no $Y \leq_T X^n$ which separates A and B. We prove this by induction on n. For $n = 0$, it follows from our assumption. Suppose that our claim has been proved for n. Let:

$$\mathcal{D} = \{Z : \exists Y \leq_T^{X^n} Z(A \subseteq Y \wedge B \cap Y = \emptyset)\}.$$

Since A and B are inseparable by any $Y \leq_T X^n$, from Theorem 5.3 of Jockusch and Soare [4] relativized to X^n, we see that $\mu(\mathcal{D}) = 0$. Hence $X_n \notin \mathcal{D}$ because $\mathcal{D}$ is $\Delta_1^{1X^n}$. If $Y \leq_T X^{n+1}$, then $Y \leq_T^{X^n} X^n$, and hence Y cannot separate A and B. This proves our claim. Since weak König's lemma is equivalent to Σ_1^0 separation (Lemma 2.6 of Simpson [7]), it follows that $\mathcal{M}$ is not a model of WKL. $\square$

Let's break it down:

1. Goal of the Proof

The goal is to show that $\mathcal{M}$ **does not satisfy** WKL (Weak König's Lemma).

- Weak König's Lemma states that **every infinite binary tree has an infinite path**.
- Here, the equivalent formulation being used is:

  WKL is equivalent to Σ_1^0-separation (Lemma 2.6 of Simpson[7]).

- The proof shows that in $\mathcal{M}$, there is no Y that separates two specific disjoint r.e. sets A and B, thereby contradicting WKL.

2. The Claim to Prove

There is no $Y \in \mathcal{M}$ that separates A and B, where:

- $A \subseteq Y$, meaning A is fully contained in Y,
- $B \cap Y = \emptyset$, meaning B has no overlap with Y.

To prove this, it suffices to show that **for all** $n \in \omega$, there is no $Y \leq_T X^n$ that separates A and B.

3. Proof by Induction

Base Case ($n = 0$): For $n = 0$, $X^0 = \emptyset$.

- By assumption, A and B are **not recursively separable**, meaning there is no recursive set (e.g., Y) that separates A and B.
- Hence, the claim holds.

Inductive Step: Assume the claim holds for n, i.e., there is no $Y \leq_T X^n$ that separates A and B.

- **Define $\mathcal{D}$:**
$$\mathcal{D} = \{Z : \exists Y \leq_T^{X^n} Z \, (A \subseteq Y \wedge B \cap Y = \emptyset)\}.$$
- $\mathcal{D}$ represents the set of all potential "candidates" Z that could compute a separating set Y (relative to X^n) of A and B. If $\mathcal{D} \cap \mathcal{M} \neq \emptyset$, it would imply the existence of a separating set $Y \in \mathcal{M}$.

- **Measure-Theoretic Argument:** By Theorem 5.3 of Jockusch and Soare[4] (relativized to X^n), $\mu(\mathcal{D}) = 0$. The theorem states that If A and B are disjoint recursively inseparable sets and $\mathcal{S}$ is the collection of all sets which separate A and B, then $\mu(\mathcal{U}(\mathcal{S})) = 0$.

  This definition aligns $\mathcal{D}$ with the relativized upward cone $\mathcal{U}(\mathcal{S})$ under X^n. Specifically, $\mathcal{D}$ captures all sets Z that can compute (relative to X^n) a separating set $Y \in \mathcal{S}$. By Theorem 5.3, $\mu(\mathcal{U}(\mathcal{S})) = 0$, and since $\mathcal{D}$ corresponds to $\mathcal{U}(\mathcal{S})$ in the relativized setting, it follows that:
$$\mu(\mathcal{D}) = 0.$$
  We can show that $\mathcal{D}$ is $\Delta_1^{1X^n}$ by similar reasoning as we showed before

- **Implication for X^{n+1}:** If $Y \leq_T X^{n+1}$, then $Y \leq_T^{X^n} X_n$ because $X^{n+1} = X^n \oplus X_n$, and X_n is random over X^n. Therefore, Y cannot separate A and B.

4. Conclusion

- By induction, for all $n \in \omega$, there is no $Y \leq_T X^n$ that separates A and B.
- Hence, there is no $Y \in \mathcal{M}$ that separates A and B.

  Lemma 2.6 of Simpson [7] states that the following two statements are equivalent:

  1. WKL
  2. **Σ_1^0-Separation:**

– Let $\varphi_i(n)$, $i = 0, 1$, be Σ^0_1 formulas in which X does not occur freely. If

$$\neg\exists n\,(\varphi_0(n) \wedge \varphi_1(n)),$$

then

$$\exists X\,\forall n\,\big[(\varphi_0(n) \to n \in X) \wedge (\varphi_1(n) \to n \notin X)\big].$$

Let $\varphi_i(n)$ (for $i = 0, 1$) be Σ^0_1-formulas (a class of formulas where the existence of a natural number witness can verify membership). The second statement establishes that if $\neg\exists n(\varphi_0(n) \wedge \varphi_1(n))$ (i.e., φ_0 and φ_1 are disjoint sets of natural numbers), then there exists a set X such that X "separates" φ_0 and φ_1: it contains all elements of φ_0 but none of φ_1.

- Since WKL is equivalent to Σ^0_1-separation, this proves that $\mathcal{M}$ is **not a model of** WKL.

References

[1] Chris J. Conidis. A measure-theoretic proof of turing incomparability. *Elsevier B.V*, pages 83–88, 2010.

[2] Rod Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer, 2010.

[3] P.R. Halmos. Measure theory. *New York: Van Nostrand*, 1950.

[4] Soare R.I. Jockusch, C.G. π_1^0 classes and degrees of theories. *Transactions of the American Mathematical Society*, pages 33–56, 1972.

[5] Andre Nies. *Computability and Randomness*. Oxford University Press, 2009.

[6] G.B. Sacks. *Degrees of Únsolvability*. Princeton University Press, 1963.

[7] Stephen G. Simpson. Which set existence axioms are needed to prove the cauchy/peano theorem for ordinary differential equations? *The Journal of Symbolic Logic, vol. 49, no. 3*, page 783–802, 1984.

[8] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Cambridge University Press, 2009.

[9] Robert Soare. *Turing Computability: Theory and Applications*. Springer, 2016.

[10] Xiaokang Yu and Stephen G. Simpson. Measure theory and weak könig's lemma. *Archive for Mathematical Logic*, 30:171–180, 1990.