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Physics III

Stern-Gerlach experiment

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1 Introduction

Being one of the most important fundamental experiments in quantum mechanics, the Stern-Gerlach-Experiment (SGE) is the key to understanding basic quantum behavior of objects in all atomic-scale systems. The historical experiment proves the directional quantization of the angular momentum of electrons by sending a beam of silver particles through an inhomogeneous magnetic field and observing their deflection. Being the first experiment of that kind, the SGE had a great impact on the physical science, refuting classical views and justifying the arising quantum theory. In addition the SGE's results initiated the development of the electron's spin.
Stern and Gerlach originally used silver atoms in their experiment due to its properties of being a chemically inactive and electrically neutral element, so that the Lorentz force won't affect the measurement. In fact silver possesses only one single electron in its outermost shell and has no orbital angular momentum in its ground state ($5s_{1/2}$ state). This results in the magnetic moment of the atom only being caused by the magnetic spin moment of the electron.
Similarly, we can find elements, which share the same properties as silver, therefore might be considered to be used in our experiment as well. Elements that come into question are every alkali metal (e.g. Natrium, Potassium...) and elements which occur in the same group as silver (e.g. Copper, Gold...).
In our modified version of the experiment, we will use potassium instead of silver atoms for various reasons. Potassium for example has a far lower evaporation temperature and corresponds with the detector we are using. The setup however will be further described in Section 3.
Our goal is to obtain results similar to the original experiment, i.e. evaluating the magnetic moment of the electron, proving the directional quantization of the angular momentum and demonstrate the existence of the electron's spin.

2 Theory

2.1 Magnetic moment

Potassium, being an alkali metal, has a single valence electron in its outermost orbit. Consequently, the magnetic moment of atoms of this element in the ground state ($4s_{1/2}$) comes exclusively from the electron of valence and is given by

$$\vec{\mu} = -\frac{e}{2 \cdot m_e} g_s \cdot \vec{S}, \tag{1}$$

where $m_e = 511\text{keV}$ is the electron's mass and $g_s \simeq 2$ is its Landé factor.
If we establish a frame of reference and we observe only the z-component of the magnetic moment, we obtain

$$\mu_z = -\mu_B \cdot m_s \cdot g_s, \tag{2}$$

where $m_s = \pm 1/2$ is the spin quantum number of the electron and

$$\mu_B = -\frac{e\hbar}{2m_e} = 9,274 \cdot 10^{-24}\text{Am}^2 \tag{3}$$

is the Bohr magneton.

2.2 Trajectory of the potassium atom

The trajectory of the atoms is as follows: they are emitted from an oven and enter a region where an inhomogeneous magnetic field is applied. Due to the magnetic moment given by the spin, the potassium atoms perform a parabolic motion. After exiting the magnetic field, the beam continues to travel in a straight line to the detector. The deviation $u - z$ suffered by the particles is given by the formula

$$u - z = \pm \frac{lL}{Mv^2} \left(1 - \frac{L}{2l}\right) m_{s,z} \frac{\partial B_z(y, z)}{\partial z}, \tag{4}$$

where l is the distance from the magnet to the detector, L is the length of the pole piece, M is the mass of the atom, $m_{s,z}$ is the z-component of the magnetic moment and $\partial B_z / \partial z$ is the gradient of the magnetic field.

From Eq. (4), we note that the velocity of the atom is indirectly proportional to the displacement, i.e. the higher the velocity the less they are deflected.

We therefore want to calculate the cumulative distribution function of the velocity inside the oven to deduce the distribution of the atoms in the plane of the detector. Using the Maxwell-Boltzmann distribution, geometric considerations and a function that describes the shape of the particles arriving in the detector's plane $\Phi_m(z)$, we find the equation

$$\mathrm{d}^2 n = \frac{\Phi_m(z)\exp(-\frac{Mv^2}{2kT})v^3 \mathrm{d}v\mathrm{d}z}{2\int_0^\infty \exp(-\frac{Mv^2}{2kT})v^3 \mathrm{d}v}, \tag{5}$$

where k the Boltzmann constant is and T is the temperature of the oven.

2.3 Current density of incoming particles

The current of the detector that we finally measure is proportional to the current density of the incoming particles $J(u)$. Using Eq. (4) and Eq. (5) and operating the following substitutions

$$q := -\frac{lL(1-\frac{L}{2l})m_{s,z}\frac{\partial B_z(y,z)}{\partial z}}{2kT}, \tag{6}$$

$$A := \frac{[lL(1-\frac{L}{2l})m_{s,z}\frac{\partial B_z(y,z)}{\partial z}]^2}{4M^2\int_0^\infty \exp(-\frac{Mv^2}{2kT})v^3 \mathrm{d}v} \tag{7}$$

we arrive at the equation

$$J(u) = A\int_{-D}^{D} J_0(z)\exp\left(-\frac{q}{|u-z|}\right)\frac{1}{|u-z|^3}\mathrm{d}z, \tag{8}$$

where D is half the width of the beam and $J_0(z) := \Phi_{+1/2}(z) + \Phi_{-1/2}(z)$.
To find the location of maximum intensity u_e, we use $\frac{\mathrm{d}J(u)}{\mathrm{d}u} = 0$ and we find

$$u_e = \pm\frac{q}{3} = \frac{lL(1-\frac{L}{2l})m_{s,z}\frac{\partial B_z(y,z)}{\partial z}}{6kT}. \tag{9}$$

By determining the gradient $\partial B_z/\partial z$, measuring the temperature of the oven T and using the given constants L and l we can evaluate $m_{s,z}$ by finding of the maxima of the intensity.

2.4 Calculation of B field and field gradient $\partial B_z/\partial z$

To produce the magnetic field we use the pole pieces sketched in figure 1. Because of their shape, they create a magnetic field identical to that produced by two parallel wires, carrying currents in the opposite direction, away from each other by $2a$.

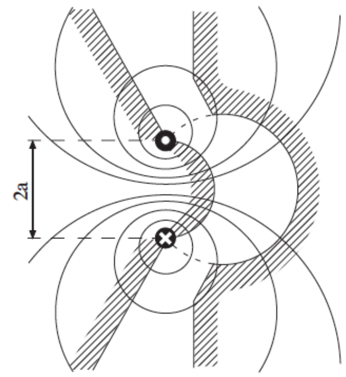

Figure 1: Pole pieces to realize inhomogeneous magnetic field B

This configuration is particularly suited to our problem since we can find a specific plane (as indicated in Fig. 2) where magnetic fields gradient $\partial B_z/\partial z$ is almost constant and B_y is constant. Simplifying the acceleration to a constant force we guarantee a symmetric current density distribution $J(u)$ in z- and y-directions.

The measuring device must imperatively be in a vacuum of at least $4\cdot 10^{-6}$mbar. Only in this environment it is possible to prevent scattering of potassium atoms in air. It is therefore difficult to take measurements without rising the probability of leaks due to openings and fixings of measuring devices.

We avoid practical problems measuring the fields gradient in z-direction by making the assumption that for small deflections of atoms in z direction in the plane mentioned above, $\partial B/\partial z$ is proportional to B. The proportionality factor is defined as

$$\epsilon = \left|\frac{\partial B}{\partial z}\right|\frac{a}{B} \tag{10}$$

and has an average value of

$$\overline{\epsilon} = 0,953 \pm 0,0026. \tag{11}$$

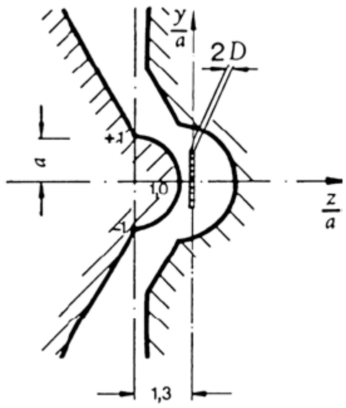

Figure 2: Field with approximately constant gradient $\partial B_z/\partial z$ and constant B_y

3 Equipment and Apparatus

3.1 Measuring Principle

We include two improvements to the original experiment that allow it to be done more efficiently:

1. Potassium is used instead of silver because,
   - Potassium evaporates at a lower temperature (63.6°C for K instead of 961.9°C for Ag).
   - a low temperature means that the Potassium atoms are moving at a lower speed and therefore are more easily deflected.
   - Potassium has a single valence electron outside its closed shells and the magnetic moment is generated only by the spin of that electron in the ground state.
   - the Langmuir-Taylor detector we are using works best with Potassium.
2. As mentioned, a Langmuir-Taylor detector is used instead of a glass plate as in the original experiment. This allows an evaluation with computers, and is advantageous due to its high sensitivity for small currents and its fast reaction time.

The apparatus of the SGE experiment used is basically composed of an oven that emits a beam of atoms, a inhomogeneous magnetic field and a detector, that are maintained at vacuum. The diagram of the main parts of the device are represented in Fig. 3.

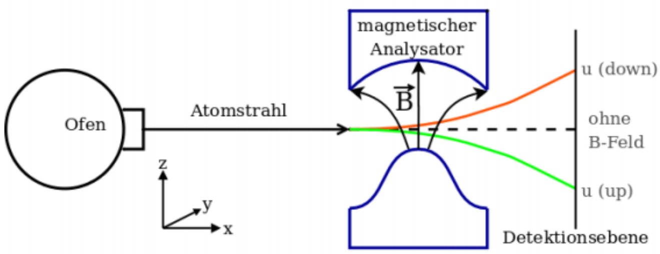

Figure 3: Scheme of the measuring apparatus

In the oven, the potassium atoms are brought to a temperature of about 170°C: this way, they will find themselves above the evaporation temperature which is located at 63.6°C. By means of shielding and slits, the atoms are collimated into a narrow beam which is directed horizontally to an inhomogeneous magnetic field. Due to the force exerted by the magnetic field on the atoms, they suffer a deflection perpendicular to the direction of motion. As demonstrated in the theory part, the acceleration depends on the magnetic momentum. The splitting of the beam shows different magnetic momenta (i.e. $m_s = \pm 1/2$).

Leaving the distance on which the atoms are accelerated, the atoms continue their motion in a straight line until they get to the detector. To measure the partial current density $J(u)$ in function of the spacial deviation u, the tube through which the beam is sent and to which the detector is fixed, is moved perpendicular to the beam (according to Fig. 3, on the z-axis). The measured partial current density varies in dependence of the position of the detector. The signal is then amplified and displayed through an external device, a computer. To regulate the temperature of the oven and to increase the magnetic field, they are used electrical circuits.

3.2 Measuring Devices

We will now illustrate in more detail the most important parts of the equipment.

In the oven the pressure is at about 10^{-3}mbar. The rest of the equipment must reach a pressure of at least $4 \cdot 10^{-6}$mbar, otherwise the potassium atoms can easily be scattered due to air molecules.

The inhomogeneous magnetic field B is realized by a solenoid with specially configured cross section. Its pole tips are shown in figure Fig. 4. The iron pole tips consist of a little convex half cylinder spaced from a bigger diameter concave half cylindrical groove. The concave circle, if extended, intersects the convex circle at the ends of its vertical diameter. These two curves are magnetic equipotentials of the magnetic field that would be present if the pole tips were replaced by two current carrying wires centered on the intersection points of the circles. The magnet is designed to approximate this field, which has a reasonably homogeneous gradient in the region the beam is allowed to pass through.

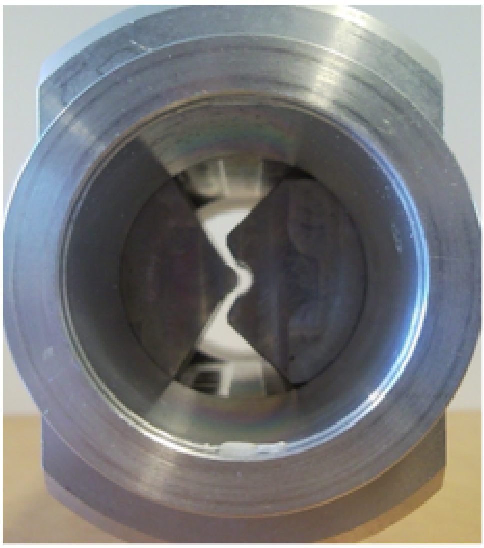

Figure 4: Cross section of pole caps

A Langmuir-Taylor detector is used to measure $J(u)$, shown schematically in Fig. 5. The detector consists of a hot tungsten filament wire surrounded by a collecting cylinder kept at a voltage of about 50 V below the wire. The collecting cylinder has a narrow slit aperture to admit the potassium atoms. The hot wire ionizes a fraction of the potassium atoms which strike it. The positive recoiling ions are then collected on a nearby negative electrode and the voltage caused by a small ionization current is measured with an electrometer.

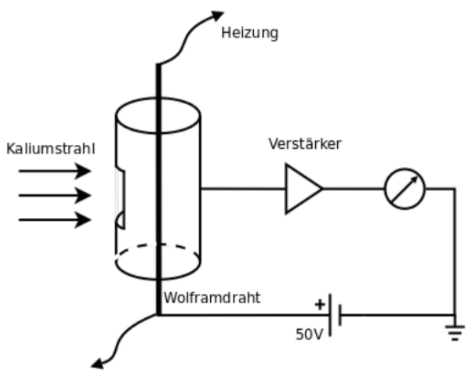

Figure 5: Langmuir detector with tungstone wire

The functioning of the detector is based on the tunnel effect. The potassium atoms, whose ionization energy is smaller than the work function of tungsten, are ionized and, because of the potential difference between the wire and the cylinder, accelerated towards the latter. A ionisation current is then produced which it is amplified by an amplifier. This current, proportional to the number of potassium atoms arrived to the wire, is then measured by the computer.

4 Measurements and Calculations

4.1 Measurements

In each measurement series samples of about 10'000 voltage measurements V_i were taken with the Langmuir-Taylor detector in function of the beam's deflection u. The Langmuir-Detector was shifted totally 1.4cm so that we have a resolution of about $1.4 \cdot 10^{-4}$cm.

For each measurement series the solenoid current I and thus the magnetic field $B(I)$ as defined in Eq. (13) was varied. By a computer interface we determined q as defined in Eq. (6) by a least square fit. Since only this value is used in the following calculations we will abstain from listing the V_i.

q [m]	I [mA]	T [°C]	m_q [m]
1.4625	300	170.0	0.0016
2.0750	405	169.8	0.0020
2.6098	500	169.5	0.0026
3.2181	600	169.1	0.0037
3.8023	700	169.0	0.0027
4.3239	800	169.0	0.0027
4.7560	900	169.0	0.0031
5.1235	1000	169.0	0.0036
1.5603	300	161.5	0.0017
2.2155	400	161.3	0.0021
2.8455	500	161.2	0.0021
3.4268	600	161.0	0.0022
4.0420	700	161.0	0.0026
4.5842	800	160.6	0.0029
5.0535	900	160.0	0.0034
5.4342	1000	159.9	0.0033

Table 1: Measurement series $\overline{T} = 169.3$ resp. 160.8 °C

The same measurement series were taken for an approximately stable average temperature $\overline{T_2} = 160.8$°C and $\overline{T_1} = 169.3$°C. The obtained values for q in a configuration with solenoid current I at temperature T are listed in Table 1. The error on q is m_q. The errors m_I and m_T on I and T are 5 mA and 0.1 °C. However, in the calculations of the two measurement series respectively the standard deviation $\sigma_{T_1} = 0.4$°C of $\overline{T_1}$ and $\sigma_{T_2} = 0.6$°C of $\overline{T_2}$ are used.

From both measurements we are able to derive μ_B by the relation in Eq. (14) making the calculations outlined in Section 4.2.

Fig. 6 shows the measurements for q as function of the magnetic field $B(I)$ for the 170 °C series in blue and the 160 °C series in red. The errors on q are too small to be shown. Furthermore, the linear fits as explained in Eq. (12) are drawn in green for the 170 °C series and in blue for the 160 °C series.

Several geometric constants define the behaviour of the solenoid magnet and thus q as is seen in Eq. (6). We list these with their respective errors in Table 2.

L	$(7.00 \pm 0.01) \cdot 10^{-2}$ m
l	0.455 ± 0.005 m
a	$(2.5 \pm 0.01) \cdot 10^{-2}$ m

Table 2: Geometric Constants of Solenoid Magnet

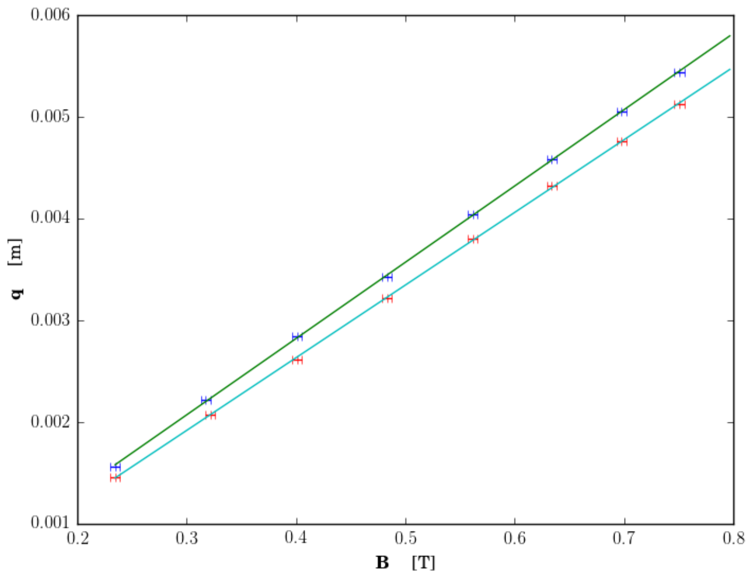

Figure 6: Measurement data and linear fits

4.2 Procedure of Calculation

Note that whenever we refer in the following derivations to the ith of N measurements of a quantity A, we write A_i.

The voltage measurements V_i are directly proportional to the current density of incoming particles $J_i = J(u_i, q)$, where q as defined in Eq. (6) is a factor that depends proportionally on geometrical quantities as well as the field gradient $\partial B_z/\partial z$ and u is the deviation perpendicular to the direction of the beam of incoming potassium atoms.

The parameter q is estimated by fitting $V(u)$ to the measurement data V_i using a least square method for N measurements V_i. $V(u)$ is gained by proportionality from Eq. (8).Thus, we want to find

$$\min_q \chi^2(q) = \min_q \sum_i^N \left(\frac{V_i - V(u_i, q)}{\sigma_{V_i}} \right)^2, \tag{12}$$

where σ_{V_i} is the uncertainty on V_i. The estimated q and the uncertainty m_q on q are calculated by a MATLAB implemented least-squares Levenberg-Marquardt algorithm.

By using the calibration curve

$$B(I) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \tag{13}$$

where $a_0 = 1.5459 \cdot 10^{-2}, a_1 = 0.6113, a_2 = 0.5146, a_3 = -0.3907$ are solenoid specific properties and I is the electric current in the solenoid, we may convert the measured current to the magnetic field. The curve is plotted in Fig. 7.

B is proportional to the solenoid current I between 0.3 A and 1 A (see for comparison Fig. 7). We have $q(B)$ with the approximation in Eq. (10). To simplify the following calculations $q(B)$ must be considered linear. However, this is only possible for configurations of B within these limits. Measurements I_i were made only within these limits.

From the definition of q and $m_{s,z}$ as well as the approximation for the field gradient $\partial B_z/\partial z$ in Eq. (6), Eq. (4) resp. Eq. (10) we conclude

$$m_{s,z} = \mu_B m_s g_s \approx \mu_B \tag{14}$$

$$= \frac{q}{B} \cdot \frac{a}{\epsilon} \frac{2k_B \overline{T}}{lL(1 - \frac{L}{2l})}, \tag{15}$$

where we defined $\overline{T}$ as the average of the temperature measurements T_i. The experiment was carried out at two different relatively constant temperatures. Thus, it is legitimate to consider the factor $T_i \approx \overline{T}$ as constant for each measurement series.

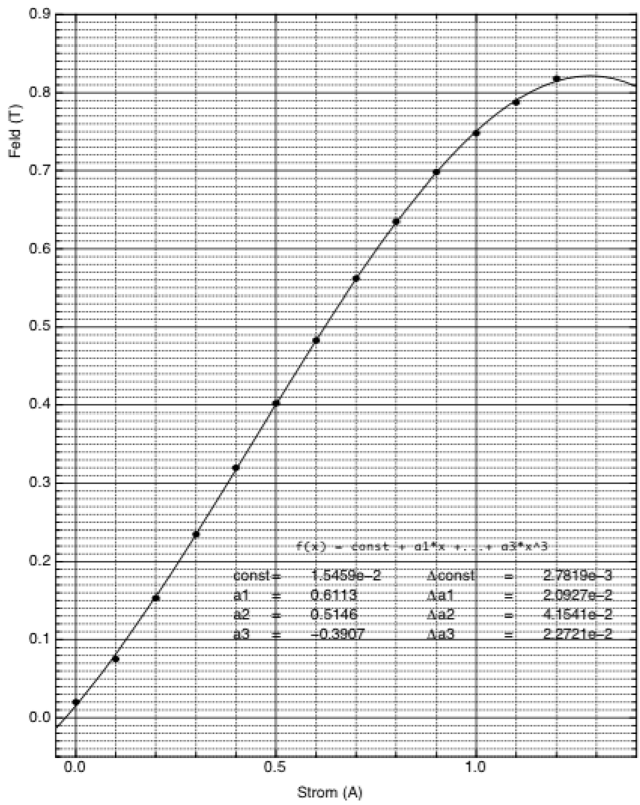

Figure 7: Calibration curve $B(A)$

Theoretically, $q(B)$ in Eq. (6) is a homogeneous linear function. Since the measurements are biased by various systematic errors resulting in a inhomogeneous behaviour (see Fig. 6), it is advantageous to find the constant fraction $\tau = q/B$, which isn't affected by these errors.

We derive τ again by a least square approach fitting $q(B_i)$ to the q_i values resulting from the first fit of Eq. (12). We estimate τ by finding

$$\min_{\tau,A} \chi^2(\tau, A) = \min_{\tau,A} \sum_i^N (q_i - f_i)^2 C_{ii}^{-2}, \quad f_i = \tau B_i + A \tag{16}$$

where A is a correcting term representing the impact of systematic errors, B_i is the magnetic field for a specific measurement and C the covariance matrix. Since the errors m_{B_i} on B_i are small compared to B_i and constant they do not affect the fitting and are neglected. C is then the diagonal matrix consisting of the uncertainties on q.

The minimum is found by calculating $\nabla\chi^2(\tau, A) = 0$ which leads to

$$(\tau, A)^t = R^I q, \tag{17}$$

where R is the matrix $(\partial f_i/\partial\tau \;\; \partial f_i/\partial A)_i^N$ and q is the vector of all q_i values. The error propagation results in

$$(m_\tau, m_A)^t = (R^t C^I R)^I. \tag{18}$$

After computing τ, $m_{s,z}$ is easily determined from Eq. (14). It's uncertainty is found by

$$m_{m_{s,z}} = \nabla m_{s,z} E \nabla m_{s,z}^t, \tag{19}$$

where E is the covariance matrix of $m_{s,z}$ and the gradient $\nabla m_{s,z}$ represents the derivatives of $m_{s,z}$ with respect to l, L, τ, ϵ and $\overline{T}$. Again, we suppose E to be diagonal, so that it only consists of the variances of these parameters.

4.3 Results

Following the calculation procedure outlined in Section 4.2 we find in the 160 °C series $\tau_2 = 0.0074958(50)$ m/T and in the 170 °C series $\tau_1 = 0.0071402(49)$ m/T.

With these results we derive $m_{s,z}$ by Eq. (14). The error is calculated according to Eq. (19). We find in the 160 °C series $m_{s,z,2} = 8.011(335) \cdot 10^{-24}$ J/T and in the 170 °C series $m_{s,z,1} = 7.781(325) \cdot 10^{-24}$ J/T.

5 Discussion

Given that this experiment is a modern version of the SGE, there are some modifications to the set-up, as we for example use a Langmuir-detector and computer to evaluate the measurements. Many elements of type ${}^1s_{1/2}$ are qualified as well to show to legitimate experimentally the theoretical prediction of electron spin. Potassium is quite a reasoned choice, as it fits the needs of the Langmuir-detector. To further motivate the set-up used in the SGE we will clarify the following points. We don't we simply use an electron beam to examine the magnetic moment, since the Lorentz force of a magnet field working on a charged particle is far bigger than its effect on the magnetic spin moment. The SGE only works with inhomogeneous magnetic fields, as there is no continued force acting on the atom in homogeneous ones. The rendering of such fields is complicated and achievable only in small environments when vacuum must be maintained in the same time.

Given that the literature value for the Bohr magneton is $\mu_B = \frac{e\hbar}{2m_e} = 9.274 \cdot 10^{-24}$J/T, we find that the values our experiment produces, vary much from the literature references. There is approximately a difference of 16% from the 170 °C and 14% from the 160 °C measurement.

There are systematical errors beside the statistical ones, as both measurements are similar and have values below the real value. Considering the temperature and the pressure inside the vacuum, both have very good values and are almost constant. We guess the error is probably not due to some mistakes we made during our measure (as both measurements give similar results), but more likely to be a calibration error on some of our devices (i.e. magnet, detector). Furthermore we made some simplifications in our calculation procedure that could have affected our result.