A Construction of $\mathbb{R}$ from $\mathbb{Q}$

Real Analysis Final Project

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Abstract

Throughout our study of Real Analysis we have been reliant upon the Axiom of Completeness for our construction and manipulations of $\mathbb{R}$. But what if we could prove it? In this paper we will show that $\mathbb{R}$ can be constructed from $\mathbb{Q}$ by defining the real numbers to be the set of all Dedekind cuts. Indeed, this construction of $\mathbb{R}$ is a set which has the properties of an ordered field and contains $\mathbb{Q}$ as a subfield. We can replace our assumption of the Axiom of Completeness with a rigorous proof of this set's completeness, placing our study of Real Analysis on a firm and certain foundation.

A Dedekind Cut Construction of $\mathbb{R}$

Our goal is to fill the gaps in the rational numbers by proving the existence of the real numbers.

Theorem 8.6.1 (Existence of the Real Numbers). *There exists an ordered field in which every nonempty set that is bounded above has a least upper bound. In addition, this field contains $\mathbb{Q}$ as a subfield.*

To prove this theorem we begin by working with Dedekind Cuts. These cuts will be our nonempty sets that are bounded above. Later in this paper we will prove that these cuts have least upper bounds. For now we will work a few examples to familiarize ourselves with the properties of cuts.

Problem 1. Exercise 8.6.1a. Fix $r \in \mathbb{Q}$. Show that $C_r = \{t \in \mathbb{Q} : t < r\}$ is a cut.

C_r must exhibit three properties to be a cut:

(c1): **C_r is nonempty and C_r is not the set of all rationals $\mathbb{Q}$.**
Trivially, C_r is nonempty and is not the entire set $\mathbb{Q}$ since for any $r \in \mathbb{Q}$, there exists a $p > r$ and $p \notin C_r$ by definition.

(c2): **If $s \in C_r$, then C_r contains all rationals $q < s$.**
Let $s \in C_r$. By definition, $s < r$ for the r which defines C_r. This means for any rational q with $q < s$, it is also true that $q < r$. Therefore, $q < r$, so $q \in C_r$.

(c3): **C_r has no maximum. That is, for any $n \in C_r$, there exists an $m \in C_r$ with $n < m$.**
Assume for the sake of contradiction that q is the maximum of C_r. This means for any rational $d \in C_r$, it is true that $q > d$. However, since C_r contains all rationals less than r, there must exist a rational b with $q < b < r$ by the Archimedean Property. This contradicts our assumption that q is the maximum, therefore, C_r has no maximum.

We now use a few concrete examples to illustrate the application of these properties, and investigate the properties of rational numbers in relation to the cuts.

Problem 2. Exercise 8.6.1 bcd. Determine which of the following sets are cuts.

(b) The set $S = \{t \in \mathbb{Q} : t \leq 2\}$ is not a cut.

For any $b \in S$, we have $b \leq 2$. However, since we also know $2 \in S$, then S has a maximum, and is not a cut.

(c) The set $T = \{t \in \mathbb{Q} : t^2 < 2 \text{ or } t < 0\}$ is a cut.

T contains 1, so it is nonempty and doesn't contain 4, so it is not all of $\mathbb{Q}$. Since T includes all rationals less than 0, we know it is not bounded below. Finally, since T contains all rationals strictly less than some value (either $\sqrt{2}$ or 0), we know it does not have a maximum. Therefore, T is a cut.

(d) The set $U = \{t \in \mathbb{Q} : t^2 \leq 2 \text{ or } t < 0\}$ is a cut.

Similar to T, the cut U contains 1, so it is nonempty and doesn't contain 4, so it is not all of $\mathbb{Q}$. Since U includes all rationals less than 0, we know it is not bounded below. Finally, since $\sqrt{2}$ is irrational, we know that no rational q can satisfy $q^2 = 2$. This means that U contains all rationals strictly less than $\sqrt{2}$, so U does not have a maximum. Therefore, U is a cut.

Problem 3. Exercise 8.6.2.

Proposition. Let A be a cut. Show that if $r \in A$ and $s \notin A$, then $r < s$.

Proof. We assume A is a cut and $r \in A$ while $s \notin A$. Now, we will also assume for the sake of contradiction that $s \leq r$. We will now proceed by cases.

Case 1: $s = r$. Then, $r = s$, and $r \in A$, which implies then that $s \in A$. But, by the assumptions, $s \notin A$, which is a contradiction.

Case 2: $s < r$. By Cut Property (c2), if $r \in A$ and $q < r$, then $q \in A$, so s must be an element of A. However we assumed $s \notin A$ and find another contradiction.

Thus we have proven that if A is a cut and $r \in A$ while $s \notin A$, then $r < s$. □

By this point we can continue in our quest to prove the existence of the real numbers. We define $\mathbb{R}$ as the set of all cuts in $\mathbb{Q}$. This may feel strange to define a set of numbers as a set of sets. However in the next few problems we will show that these sets obey the properties of a field and that we can work with these sets using our familiar notions of addition and multiplication. First we remind ourselves of field properties.

Problem 4. Exercise 8.6.3. Verify field properties for $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$.

Properties:

(f1) (commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

(f2) (associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

(f3) (identities exist) There exist two special elements 0 and 1, with $0 \neq 1$ such that $x + 0 = x$ and $x \cdot 1 = x$ for all $x \in F$.

(f4) (inverses exist) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = 0$. If $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.

(f5) (distributive property) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Properties Possessed by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$:

$\mathbb{N}$:
- (a) **Commutativity:** Yes for addition and multiplication.
- (b) **Associativity:** Yes for addition and multiplication.
- (c) **Identities Exist:** Yes for multiplication. We know that $1 \in \mathbb{N}$, thus $1n = n$ for all natural numbers. No for addition. Whether or not $0 \in \mathbb{N}$ depends on the definition we use, though generally we consider $0 \notin \mathbb{N}$. Thus, there does not exist an element 0 in $\mathbb{N}$ such that for any $n \in \mathbb{N}$, $n + 0 = n$.
- (d) **Inverses Exist:** No for addition and multiplication. If you take $-n$ for any $n \in \mathbb{N}$, $-n \notin \mathbb{N}$. If you take n^{-1} for any $n \in \mathbb{N}$, you find a fraction of the form $\frac{1}{n}$. This is not a natural number.
- (e) **Distributive Property:** Yes.

$\mathbb{Z}$:
- (a) **Commutativity:** Yes for addition and multiplication.

(b) **Associativity:** Yes for addition and multiplication.

(c) **Identities Exist:** Yes for addition and multiplication. $1, 0 \in \mathbb{Z}$, and for any $z \in \mathbb{Z}$, $1z = 1$ and $z + 0 = z$.

(d) **Inverses Exist:** Yes for addition. No for multiplication. For any $z \in \mathbb{Z}$, $-z \in \mathbb{Z}$ and $z + -z = 0$. However, for multiplication if you take z^{-1} for any $z \in \mathbb{Z}$, you find a fraction of the form $\frac{1}{z}$. This is not an integer.

(e) **Distributive Property:** Yes.

$\mathbb{Q}$: The rationals are a field. Thus the following are true:

(a) **Commutativity:** Yes for addition and multiplication.

(b) **Associativity:** Yes for addition and multiplication.

(c) **Identities Exist:** Yes. $1, 0 \in \mathbb{Q}$, and for any $q \in \mathbb{Q}$, $1 \cdot q = 1$ and $q + 0 = q$.

(d) **Inverses Exist:** Yes for addition and multiplication. If you take q^{-1} for any $q \in \mathbb{Q}$ (with the exception of 0), you find a fraction of the form $\frac{1}{q}$. By definition, this is a rational number. What's more, for any $q \in \mathbb{Q}$, $-q \in \mathbb{Q}$ and $q + (-q) = 0$.

(e) **Distributive Property:** Yes.

Problem 5. Exercise 8.6.5a. Demonstrate that the set of cuts is closed under addition.

Proposition. Let A and B be cuts. Show that $A + B = \{a + b : a \in A, b \in B\}$ is a cut.

Proof. We will show that $A + B$ satisfies properties (c1) and (c3).

(c1): Let $a \in A$ and $b \in B$. By definition, $a + b \in A + B$. This means $A + B$ is nonempty.
Since A is a cut, we know there exists $c \notin A$. Similarly, we know there exists $d \notin B$. By definition, $c + d \notin A + B$, so $A + B$ is not all of $\mathbb{Q}$.

(c3): Let $a \in A$ and $b \in B$. By definition, $a + b \in A + B$. By contradiction, assume $a + b$ is the maximum of $A + B$. Since A and B are cuts, there exist $a' \in A$ with $a < a'$ and $b \in B$ with $b < b'$. By definition, $a' + b' \in A + B$. But $a + b < a' + b'$ contradicting our assumption that $a + b$ is the maximum of $A + B$.

The proof that $A + B$ exhibits property (c2) is given in Abbott. Therefore, $A + B$ is a cut. □

Problem 6. Exercise 8.6.5b. Demonstrate that addition on cuts is commutative and associative.

Proposition. Let A, B, and C be cuts. Show $A + B = B + A$ and $(A + B) + C = A + (B + C)$.

Proof. We will first show addition is commutative, that is, $A + B = B + A$. Let $a \in A$ and $b \in B$.

$\subseteq$: By definition, $a + b \in A + B$. Since $\mathbb{Q}$ is an ordered field, $a + b = b + a$. By definition, $b + a \in B + A$. This means $A + B \subseteq B + A$.

$\supseteq$: By definition, $b + a \in B + A$. We have shown $b + a = a + b$, and $a + b \in A + B$. Therefore, $B + A \subseteq A + B$. Therefore $A + B$ has no maximum.

Thus we have shown using double containment that $A + B = B + A$. Therefore addition on $\mathbb{R}$ is commutative.

We next show addition is associative, that is, $(A + B) + C = A + (B + C)$. Let $a \in A$, $b \in B$, $c \in C$.

$\subseteq$: We know that pairwise cut addition yields a cut, so $(A+B)$ and $(B+C)$ are cuts, and thereby $A+(B+C)$ and $(A + B) + C$ are cuts. By definition, $a + b \in A + B$ and $c \in C$, so let $(a + b) + c \in (A + B) + C$. We know that $\mathbb{Q}$ is an ordered field, so $(a + b) + c = a + (b + c)$. Since $a \in A$ and $b + c \in B + C$ by definition, we know $(a + b) + c \in A + (B + C)$. Therefore, $(A + B) + C \subseteq A + (B + C)$.

$\supseteq$: Let $a + (b + c) \in A + (B + C)$. Since $\mathbb{Q}$ is an ordered field, we know $a + (b + c) = (a + b) + c$. We have also already shown that $(a + b) + c \in (A + B) + C$, so $a + (b + c) \in (A + B) + C$. Therefore, $A + (B + C) \subseteq (A + B) + C$.

Thus by double containment, $A+(B+C) = (A+B)+C$, so we conclude that addition on $\mathbb{R}$ is associative. $\square$

We will now move our exploration of the ordered field properties of $\mathbb{R}$ to the discussion of identities and inverses. That is, we propose cuts which may act as the additive and multiplicative identities, and the additive inverse.

Problem 7. Exercise 8.6.5d. Show the cut $O = \{p \in \mathbb{Q} : p < 0\}$ is the additive identity for the set of cuts.

Proposition. If A is a cut, then $A + O = A$.

Proof. Let A be a cut, and O be the additive identity cut as defined.

$\subseteq$: Let $a + p \in A + O$. Since $p \in O$, we know $p < 0$, so $a + p < a$. This implies $a + p \in A$, therefore, $A + O \subseteq A$.

$\supseteq$: Let $a \in A$. Since A is a cut, and therefore has no maximum, we can find $b \in A$ with $a < b$. This means $a - b < 0$, and hence $a - b \in O$. Since $b \in A$ and $a - b \in O$, we know $b + (a - b) \in A + O$. Since $b + (a - b) = b$, then $b \in A + O$. Therefore, $A \subseteq A + O$.

By double containment, we have shown $A = A + O$. $\square$

Problem 8. Exercise 8.6.7b. Define multiplication on positive cuts $A \geq O$ and $B \geq O$ to be $AB = \{ab : a \in A, b \in B \text{ with } a, b \geq 0\} \cup \{q \in \mathbb{Q} : q < 0\}$. Propose a good candidate for the multiplicative identity on $\mathbb{R}$ and show that this works for all cuts $A \geq O$.

Proposition. Define the multiplicative identity cut on $\mathbb{R}$ as $I = \{r \in \mathbb{Q} : r < 1\}$. Then, for some positive cut A where $A \geq O$, and $A \subseteq \mathbb{R}$, verify that $A \cdot I = A$.

Proof. We aim to prove that $A \cdot I = A$ by double containment.

$\subseteq$: First, we will show that $AI \subseteq A$. Let $ab \in AI$ where $a \in A$ and $b \in I$. Also, note that $A \geq O$. Next, we find that $a, b > 0$. Then because $b \in \mathbb{I}$, by the construction of $\mathbb{I}$, $0 < b < 1$, so $ab < a$, and so, $ab \in A$ by property 8.6.2(c2). As such, $AI \subseteq A$.

$\supseteq$: Next, we will show that $A \subseteq AI$. Let $a, b \in A$, and without loss of generality let $a < b$, and that $a, b > 0$. Note then that $0 < \frac{a}{b} < 1$, so $\frac{a}{b} \in I$. Then, $b \in A$ and $b \cdot \frac{a}{b} \in AI$ by the construction of AI. Also, $b \cdot \frac{a}{b} = a$, and thus $a \in AI$. As such, $A \subseteq AI$.

Therefore, by double containment, $A \cdot I = A$, and so $I = \{r \in \mathbb{Q} : r < 1\}$ is the multiplicative identity cut on $\mathbb{R}$. $\square$

Problem 9. Exercise 8.6.6c. Show that additive inverses exist on the cuts.

Proposition. Define the cut $-A = \{r \in \mathbb{Q} : \text{there exists } t \notin A \text{ with } t < -r\}$. Show that $A + -(A) = O$.

Proof. We show set equality by proving double containment.

$\subseteq$: In order to show that $A + (-A) \subseteq O$, let $a \in A, r \in -A$ such that $a + r \in A + (-A)$. Then $a \in A$ and $r \in -A$, and further $a < -r$ by the definition of $-A$. It follows that $a + r < 0$ and therefore $a + r \in O$ so $A + (-A) \subseteq O$.

$\supseteq$: Next we show that $O \subseteq A + (-A)$. Let $x \in O$, which implies $x < 0$ by the definition of O. Let $x = a + r$ for some $a \in A, r \in -A$. Since $a < -r$ by the definition of $-A$, we know $a + r < 0$ is a valid representation of x. Thus $x = a + r \in A + (-A)$ by the properties of cut addition and $O \subseteq A + (-A)$.

Therefore we have shown that $A + (-A) = O$, so for all $A \subseteq \mathbb{Q}$, there exists a $-A \subseteq \mathbb{Q}$ such that $A + (-A) = O$ and thus additive inverses exist on the cuts. $\square$

To continue to make our notion of the real numbers familiar, we introduce the concept of order to the set of cuts.

Problem 10. Exercise 8.6.4. Define $A \leq B$ to mean $A \subseteq B$. Verify one property from Definition 8.6.5 to show that this defines an ordering on $\mathbb{R}$.

Chosen Property: If $x \leq y$ and $y \leq x$ then $x = y$. (Property 8.6.5(o2) in Abbott)

Proposition. If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof. We assume that $A \subseteq B$ and $B \subseteq A$. Then, for any element $a \in A$, we know $a \in B$ also because $A \subseteq B$. Next, we also know that for any element $b \in B$, $b \in A$ because $B \subseteq A$. Thus, all elements of A are in B and all elements of B are also in A.

There does not exist an element in either set that is not in the other set so the two sets must contain the same elements and as such, are equal.

Thus we have proven that if $A \subseteq B$ and $B \subseteq A$, then $A = B$. Thus we may extend the familiar order property that $x \leq y$ and $y \leq x \rightarrow x = y$ to our definition of $\mathbb{R}$ as the set of Dedekind cuts. ☐

At this point we have shown that $\mathbb{R}$ obeys the properties of an ordered field and can be worked with in a familiar manner. Now we take it upon ourselves to show that this definition of the real numbers has suprema, and is thus complete. This is the final step to establish the existence of the reals!

Problem 11. Exercise 8.6.8a.

Proposition. Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the union of all $A \in \mathcal{A}$. Show $S \in \mathbb{R}$ by showing it is a cut.

Proof. We will show that the three properties of cuts apply to S.

(c1): Since $\mathcal{A} \subseteq \mathbb{R}$ and $\emptyset \not\subseteq \mathbb{R}$ then $\mathcal{A} \neq \emptyset$. Since S is the union of $A \in \mathcal{A}$, then $S \neq \emptyset$. Also $\mathcal{A}$ has an upper bound by (c2) and S has the same upper bound. Since $\mathbb{Q}$ has no upper bound, $S \neq \mathbb{Q}$.

(c2): Since S is the union of $A \in \mathcal{A}$ and each $A \in \mathcal{A}$ obeys (c2), then S also obeys (c2).

(c3): We know $A \in \mathcal{A}$. Let $A' \in \mathcal{A}$. Then $A, A' \in S$. Without loss of generality, let $A < A'$ since $\mathcal{A}$ follows (c3). Then all elements of S follow (c3).

Since (c1), (c2), and (c3) apply to S, S is a cut. ☐

Problem 12. Exercise 8.6.8b. Show that a supremum exists for the set $\mathcal{A}$.

Proposition. Let $\mathcal{A} \subseteq \mathbb{R}$ be nonempty and bounded above, and let S be the union of all $A \in \mathcal{A}$. S is the least upper bound for $\mathcal{A}$.

Proof. First, $\mathcal{A} \subseteq \mathbb{R}$, so $\mathcal{A}$ is a cut. We aim to show that S is a least upper bound, by showing that (i) S is an upper bound, and that (ii) for any upper bound U, we have $S \leq U$.

(i): We aim to show that for all $A \in \mathcal{A}$, $A \leq S$. Assume for the sake of contradiction that there exists an $A \in \mathcal{A}$ such that $S < A$. Then, by Definition 8.6.2(c2), $A \notin S$. But S is the union of all $A \in \mathcal{A}$, so any $A \in \mathcal{A}$ must also be in S. So we find a contradiction that $A \in \mathcal{A}$ and $A \notin S$. As such, for all $A \in \mathcal{A}$, $A \leq S$, and so S is an upper bound for $\mathcal{A}$.

(ii): Let B be another upper bound for $\mathcal{A}$. Note by Definition 8.6.2(c3), $\mathcal{A}$ can not have a maximum, and so $B \notin \mathcal{A}$. Next, note by the same logic that $\mathcal{A} < S$, because S is an upper bound for $\mathcal{A}$.
Now, assume for the sake of contradiction that B is an upper bound less than S, that is $B < S$.

Then, by the construction of S, there exists some $A > B$, and so by Definition 8.6.2(c2), $B \in \mathcal{A}$. This contradicts the fact that $B \notin \mathcal{A}$. Thus, we find a contradiction and so S is the least upper bound for $\mathcal{A}$.

☐

We have proven the existence and completeness of the reals. What was once only an axiom is now a proven fact! However Theorem 8.6.1 contains an extra condition that $\mathbb{R}$ contained $\mathbb{Q}$ as a subfield. We claim the subset $\mathbb{Q}$ of $\mathbb{R}$ is the set of rational cuts where $C_r = \{t \in \mathbb{Q} : t < r\}$ and $r \in \mathbb{Q}$. Indeed $\mathbb{R}$ contains other non-rational cuts which is how we have defined the irrational numbers. We need to prove that these rational cuts have the same properties as $\mathbb{Q}$, that is, these rational cuts are an ordered field. Here we prove a few of these properties.

Problem 13. Exercise 8.6.9a. Verify that addition and multiplication are commutative on the rational cuts.

Proposition. $C_r + C_s = C_{r+s}$ for all $r, s \in \mathbb{Q}$ and $C_r C_s = C_{rs}$ for all $r, s \geq 0$.

Proof. First, to prove $C_r + C_s = C_{r+s}$, we fix $r, s \in \mathbb{Q}$ and show containment in both directions for the cuts $C_r + C_s$ and C_{r+s}.

$\subseteq$: To prove that $C_r + C_s \subseteq C_{r+s}$, we aim to show that an element in $C_r + C_s$ is less than $r + s$. This would imply that this element is in C_{r+s} using property $(c2)$ and the construction of rational cuts.

Thus, we first construct arbitrary elements in C_r and C_s. So, let $q \in C_r$ and $p \in C_s$. By construction we know any element in C_r is strictly less than r, and equivalently for C_s. Then we know that $q < r$ and $p < s$. Thus, $q + p < r + s$, and so for any arbitrary element, $C_r + C_s \subseteq C_{r+s}$.

$\supseteq$: To prove that $C_{r+s} \subseteq C_r + C_s$ we proceed similarly to the forward direction and aim to show that an arbitrary element in C_{r+s} is less than the sum $x + y$ where $x \in C_r$ and $y \in C_s$.

Let $b \in C_{r+s}$. Then by the construction of C_{r+s}, $b < r + s$ which means $b - r < s$. Since $b - r < s$, there exists some $\alpha_1 \in \mathbb{Q}$ such that $b - r < b - r + \alpha_1 < s$ by property $(c3)$ that cuts can not have a maximum. Also, there exists an $\alpha_2 \in \mathbb{Q}$ such that $r - \alpha_2 < r$. Let $\alpha = \min\{\alpha_1, \alpha_2\}$.

Next let $x = r - \alpha$. We know that $x \in C_r$ by property $(c2)$ because $r - \alpha < r$. Also, let $y = b - r + \alpha$. We know that $y \in C_s$ also by property $(c2)$ because $b - r + \alpha < s$.

Note that $x + y = r - \alpha + b - r + \alpha = b$. So, $x + y = b$. We know that $x + y \in C_r + C_s$, which means that $b \in C_r + C_s$, and so $C_{r+s} \subseteq C_r + C_s$.

Thus $C_r + C_s \subseteq C_{r+s}$ and $C_{r+s} \subseteq C_r + C_s$ as desired, which by double containment means that $C_r + C_s = C_{r+s}$. Next, to prove $C_r C_s = C_{rs}$, we fix $r, s \geq 0$ and show double containment of the cuts $C_r C_s$ and C_{rs}.

$\subseteq$: For containment in the forwards direction, let $a \in C_r C_s$ where $a = xy$ such that $x < r$ and $y < s$. Then $a = xy < rs$ so we conclude that $a \in C_{rs}$ and therefore $C_r C_s \subseteq C_{rs}$.

$\supseteq$: In the opposite direction, let $b \in C_{rs}$. Then $b < rs$, and by $(c2)$ we know that there exists an $\alpha_1 \in \mathbb{Q}$ such that $\frac{b}{s} < \frac{b}{\alpha_1 s} < r$. Further, there exists an $\alpha_2 \in \mathbb{Q}$ such that $\alpha_2 s < s$ since $s \in \mathbb{Q}^+$. If we take $\alpha = \min\{\alpha_1, \alpha_2\}$ and let $x = \alpha s$ and $y = \frac{b}{\alpha s}$, it follows that $x \in C_s$ and $y \in C_r$. Further since $xy = (\alpha s)\left(\frac{b}{\alpha s}\right) = b$, we conclude that $xy \in C_r C_s$ and therefore $b \in C_r C_s$. Thus $C_{rs} \subseteq C_r C_s$.

Thus we have proven that $C_r C_s = C_{rs}$ when $r, s \geq 0$. □

Problem 14. Exercise 8.6.9b. Verify that the rational cuts obey the ordering property of fields.

Proposition. Let C_r, C_s be cuts and $, rs \in \mathbb{Q}$. Then $C_r \subseteq C_s \Leftrightarrow r \leq s$ in $\mathbb{Q}$.

Proof. We will proof this statement by showing it is true in both directions.

$(\rightarrow)$ In the case where $C_r \subset C_s$, there exists $t \in C_s$ where $t \notin C_r$ such that $r < t < s$. Thus $r < s$. In the case where $C_r = C_s$, $r = s$ by definition of cuts.

$(\leftarrow)$ Let $a \in C_r$. Then $a < r \leq s$ so $a < s$ and $a \in C_s$. Thus $C_r \subseteq C_s$

□

Thus ends our proof of Theorem 8.6.1. We have constructed a set which we proved to satisfy the properties of an order field and which contain least upper bounds. We have demonstrated that this field contains $\mathbb{Q}$ as a subfield by verifying some of the field properties of the subset of rational cuts (the remaining properties we leave as exercises to the reader). The Axiom of Completeness, on which our entire study of Real Analysis was based, is now proven and we may rest assured that our work based on its assumption was not in vain.