The Path Poset and Multipath Homology

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Nottingham Trent University

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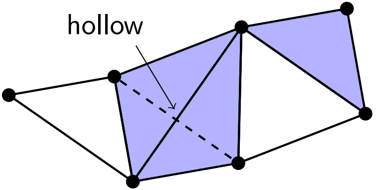
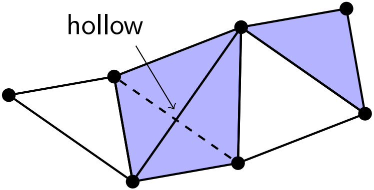
Some Topology

- Simplicial Complex Δ = n-dimensional triangles glued along faces



Some Topology

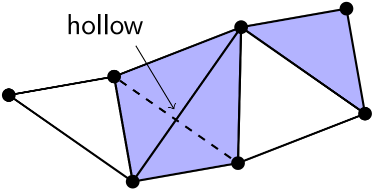
- Simplicial Complex Δ = n-dimensional triangles glued along faces
- Homology = "Holes", β_k = number of holes in dimension k
  β_0 = connected components, β_1 = holes, β_2 = spheres, ...



$\beta_0 = 1 \quad \beta_1 = 2 \quad \beta_2 = 1$

Some Topology

- Simplicial Complex Δ = n-dimensional triangles glued along faces
- Homology = "Holes", β_k = number of holes in dimension k
  β_0 = connected components, β_1 = holes, β_2 = spheres, ...
- Euler Characteristic $\chi(\Delta) = \beta_0 - \beta_1 + \beta_2 - \beta_3 + \cdots$



$\beta_0 = 1 \quad \beta_1 = 2 \quad \beta_2 = 1$

$\chi(\Delta) = 1 - 2 + 1 = 0$

Graph Homology Theories

What is the homology of a graph?

Reachability Homology

Magnitude Homology

Chromatic Homology

Graph Homology

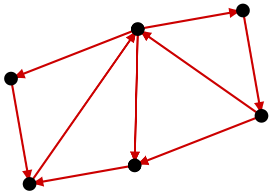
Path Homology

Hochschild Homology

Flag complex Homology

Multipaths

Directed graph: bidirectional edges allowed, no self loops, no multiedges.



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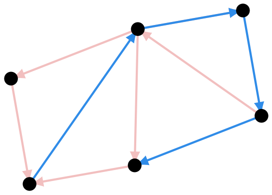
No repeat vertices, not a cycle.



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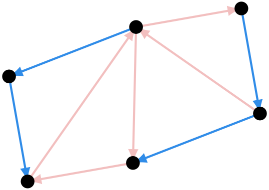

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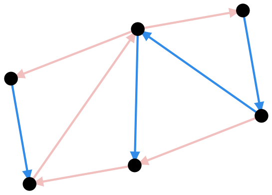

Multipath Poset and Complex

Graph G

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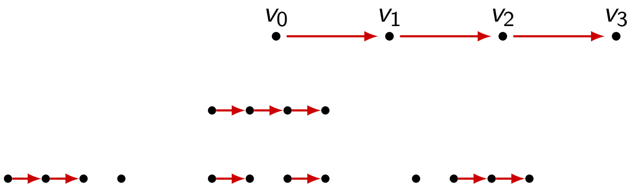

Multipath Poset and Complex

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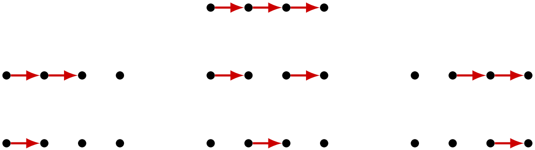

Multipath Poset and Complex

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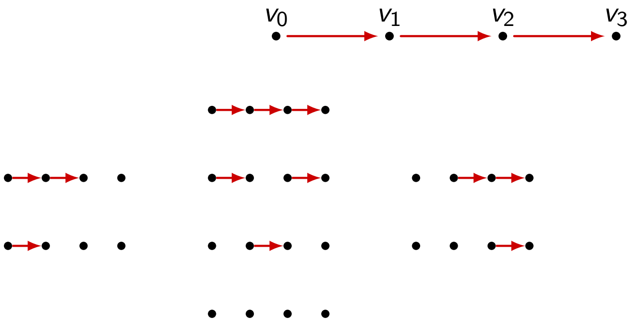

Multipath Poset and Complex

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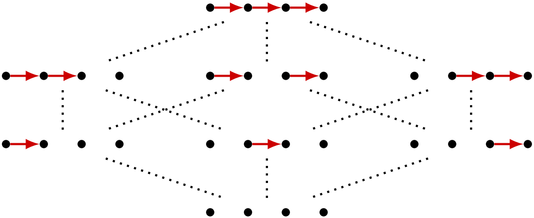
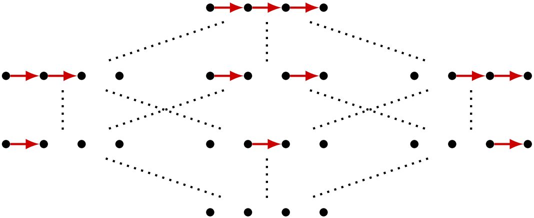


Multipath poset $P(G)$

Multipath Poset and Complex

Graph G



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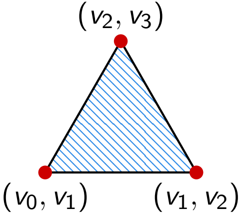
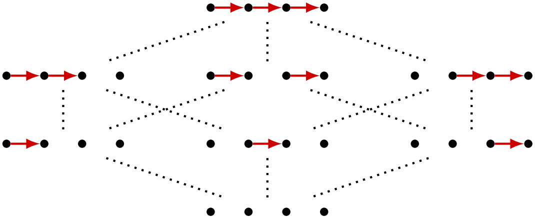


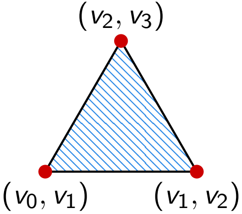
Multipath complex $X(G)$

Multipath Poset and Complex

Graph G



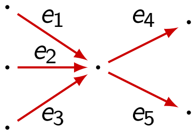
Multipath poset $P(G)$



Multipath complex $X(G)$

$$\beta_0 = 1, \qquad \beta_k = 0 \ {\scriptstyle \forall k>0}, \qquad \chi(G) := \chi(X(G)) = 1$$

Another Example



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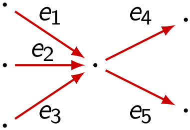

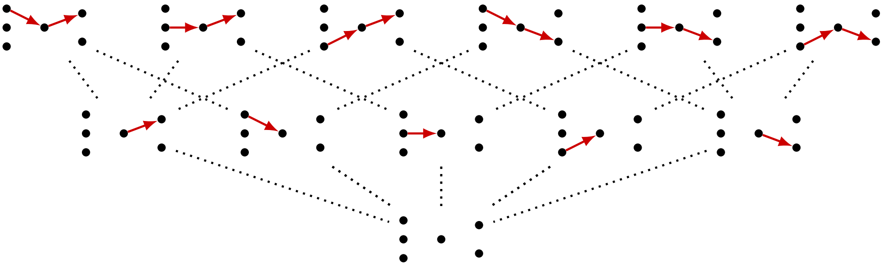

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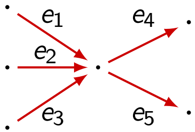

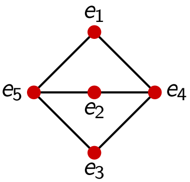

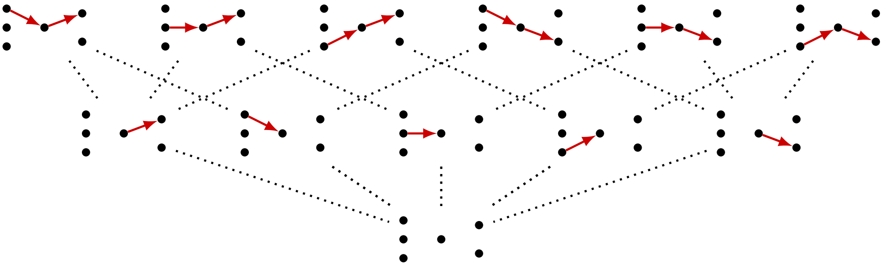

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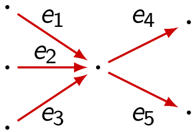
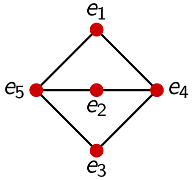
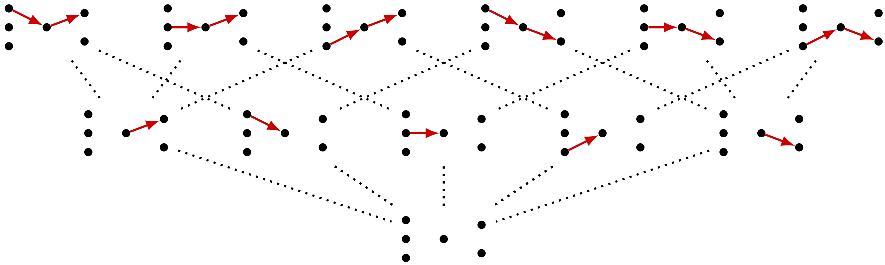
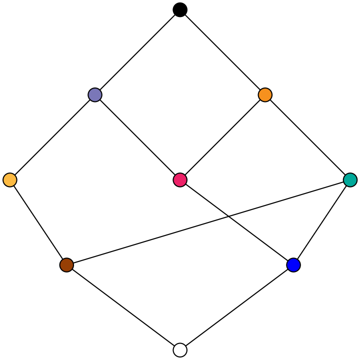






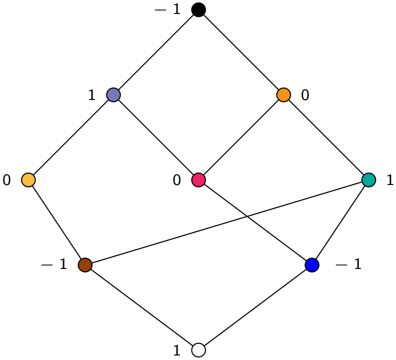
$$\beta_0 = 1, \qquad \beta_1 = 2, \qquad \chi(G) = -1$$

Möbius Function and Order Complex



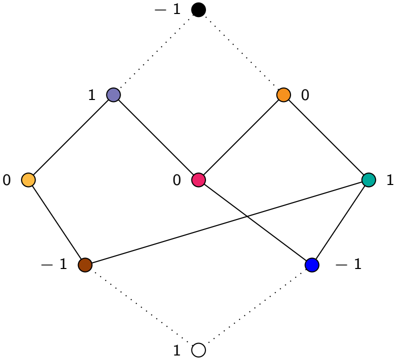
P

Möbius Function and Order Complex



$$\mu(P) = \mu(\circ, \bullet) = -1$$

Möbius Function and Order Complex



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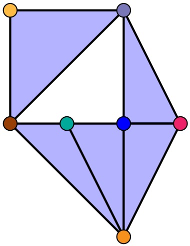
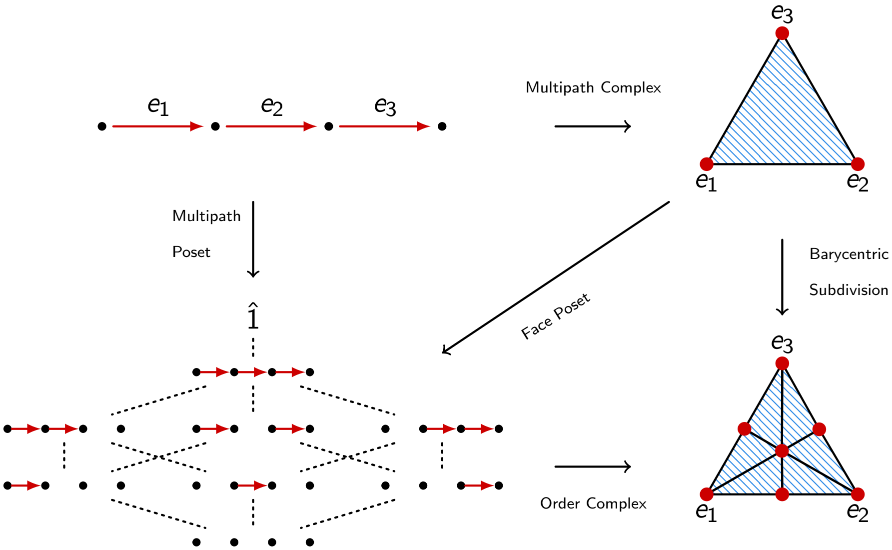


$\Delta(P)$

Order Complex

Multipath Poset and Complex



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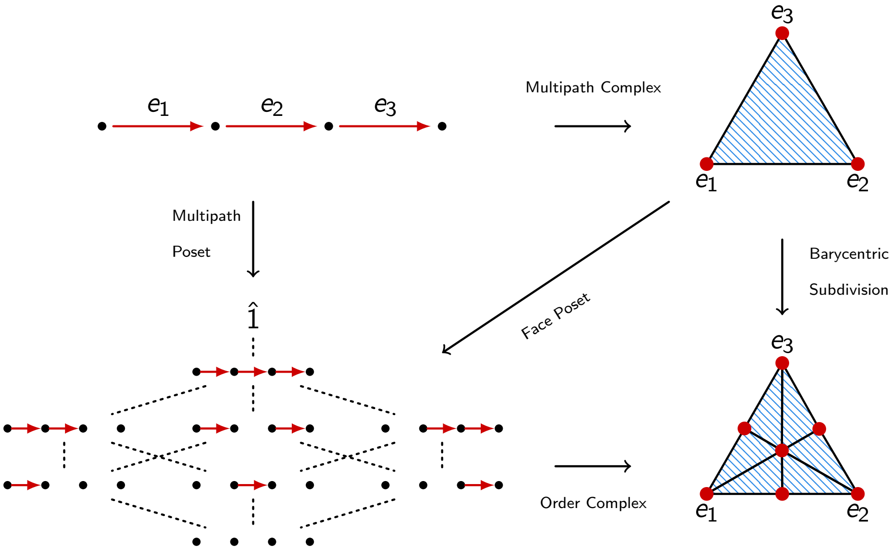
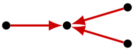
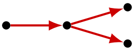
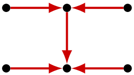
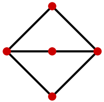


$$\mu(P(G)) = \chi(G)$$

Simple Examples

Digraph	Complex	χ	β_0	β_1	$\beta_k \ \forall k \geq 2$
	n-simplex	1	1	0	0
		2	2	0	0
		3	3	0	0
		1	1	0	0
		-1	1	2	0

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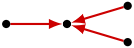
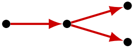
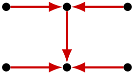
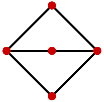
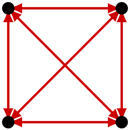
		3	3	0	0
		1	1	0	0
		-1	1	2	0

Existing homology theories cannot distinguish trees.

Complete Bidirectional Graph K_n

Bidirectional edges between every pair



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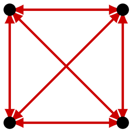

Theorem

$$\chi(\mathtt{K}_n) = \sum_{k=1}^{n} (-1)^{n-k-1} \binom{n-1}{k-1} \frac{n!}{k!}$$

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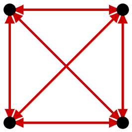

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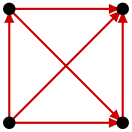
Proof Sketch:

1. multipaths $\leftrightarrow$ ordered set partitions
2. $\mu(\emptyset, m) = (-1)^{n-k-1}$, where $k = \#$ edges
3. $\#$ ordered partitions $=$ Lah numbers
4. $\chi(G) =$ sum of Lah numbers $\square$

Transitive Tournament T_n



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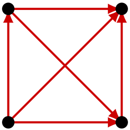
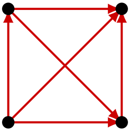


Theorem

$$\chi(\mathtt{T}_n) = \sum_{k=1}^{n} (-1)^{n-k-1} S(n, k) \ ,$$

where $S(n, k)$ are the Stirling numbers of the second kind.

Transitive Tournament T_n



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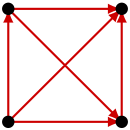
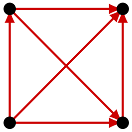


Theorem

The multipath complex of the transitive tournament is either contractible, or homotopy equivalent to a wedge of spheres.

Transitive Tournament T_n



Theorem

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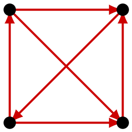
Proof Sketch:
I have a truly marvelous demonstration of this theorem which this slide is too small to contain

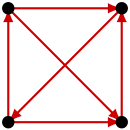
Almost Transitive Tournament R_n

Reverse the edge between source and sink



Almost Transitive Tournament R_n

Reverse the edge between source and sink



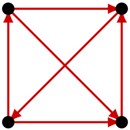
Theorem

$$\chi(\mathtt{R}_n) = \sum_{k=1}^{n-2} (-1)^{n-k-1} k S(n-2, k) \ ,$$

where $S(n, k)$ are the Stirling numbers of the second kind.

Almost Transitive Tournament R_n

Reverse the edge between source and sink



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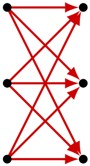
where $S(n,k)$ are the Stirling numbers of the second kind.

Proof Sketch:

1. Delete vertices 1 and n from $\mathtt{R}_n$ gives $\mathtt{T}_{n-2}$
2. Connect 1 and n to any path of multipath, yielding $kS(n-2,k)$ $\square$

Complete Bipartite Graph $K_{n,m}$



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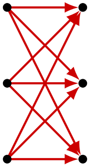
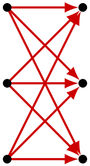


Theorem

$$\chi(K_{n,m}) = \sum_{k=0} (-1)^{k+1} \binom{m}{k} \binom{n}{k} k!$$

Complete Bipartite Graph $K_{n,m}$



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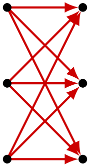
Proof Sketch:

1. Multipaths = matchings, there are $\binom{m}{k} \frac{n!}{(n-k)!}$
2. $\mu(m) = (-1)^{k+1}$

$\square$

Complete Bipartite Graph $K_{n,m}$



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Proof Sketch:

1. Multipaths = matchings, there are $\binom{m}{k} \frac{n!}{(n-k)!}$
2. $\mu(m) = (-1)^{k+1}$ $\square$

This is equivalent to the matching complex and the chessboard complex, for which it is proven in: A. Björner, et al. Chessboard complexes and matching complexes. J. London Math. Soc. (2), 49(1):25–39, 1994.

Grid Graph $\mathtt{I}_n \times \mathtt{I}_m$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet & \cdots \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet & \cdots & \\
\bullet \arrow[r] \arrow[d] & \bullet & \cdots & & \\
\bullet & \cdots & & &
\end{tikzcd}$$

Proposition

$$X(\mathtt{I}_n \times \mathtt{I}_m) \simeq \begin{cases} * & \textit{if} \;\; n, m \neq 1 \\ S^n & \textit{if} \;\; m = 1 \\ S^m & \textit{if} \;\; n = 1 \end{cases}$$

Future work and open questions

- Conjecture: For any digraph $\mathtt{G}$ on n vertices $\chi(\mathtt{K}_n) \geq \chi(\mathtt{G})$.
- Betti numbers of Transitive Tournament?
- Characterisation of contractible multipath complexes
- Applications in Topological Data Analysis

The collaborators:

Luigi Caputi	Carlo Collari	Sabino Di Trani
University of Bologna	University of Pisa	Sapienza University of Rome

The paper:

On the homotopy type of multipath complexes.

Mathematika (2023), 70: e12235. https://doi.org/10.1112/mtk.12235

The code:

`https://github.com/JasonPSmith/path_poset`

The advert:

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The collaborators:

Luigi Caputi	Carlo Collari	Sabino Di Trani
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The paper:

On the homotopy type of multipath complexes.

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Thanks for Listening!