A Littlewood-Type Identity for Robbins Polynomials

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Prelude

Generalising the determinant

- The classical notion of a determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc.$$

- In the early 1980s, Robbins and Rumsey wanted to generalise the classical determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}_{\lambda} := ad + \lambda bc.$$

- Through Dodgson condensation, we obtain the notion of λ-determinant for any square matrix.

Leibniz formula for λ-determinants

Theorem (Leibniz formula)
The determinant of an $n \times n$ matrix M is

$$|M| = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\sigma)} \prod_{i=1}^{n} m_{i,\sigma(i)}.$$

Theorem (Robbins, Rumsey 1986)
The λ-determinant of an $n \times n$ matrix M is

$$|M|_\lambda = \sum_{A \in \mathcal{ASM}_n} \lambda^{\mathrm{inv}(A)} (1 + \lambda^{-1})^{\mathcal{N}(A)} \prod_{i,j=1}^{n} m_{i,j}^{a_{i,j}},$$

where we sum over all alternating sign matrices $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$.

Alternating sign matrices

Definition (Robbins, Rumsey 1986)

An alternating sign matrix (ASM) of order n is an $n \times n$-matrix with entries -1, 0 or $+1$ such that

- the entries in each row and each column sum to 1, and
- the nonzero entries alternate in sign along each row and each column.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Every permutation matrix is an ASM!

Alternating sign matrices

- Mills, Robbins and Rumsey conjectured the number of ASMs of order n to be

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = 1, 2, 7, 42, 429, 7436, \ldots$$

  which was first established by Zeilberger in 1996.

- Alternative proofs by Kuperberg 1996, Fischer 2007.

- There are three other classes of combinatorial objects, that are enumerated by the same formula:
  - descending plane partitions
  - totally symmetric self-complementary plane partitions
  - alternating sign triangles

Descending plane partitions

Definition (Andrews 1979)

A descending plane partition (DPP) of order n is the filling of a shifted Young diagram with positive integers less than or equal to n such that

- the entries weakly decrease along rows
- and strictly decrease down columns, and
- the first part in each row is strictly larger than the length of the row
- but less than or equal to the length of the previous row.

11	10	10	10	7	5	4	4	3
	7	7	6	5	3	1		
		5	5	4	2			
			4	3	1			
				2				

ASMs and DPPs are equinumerous

Theorem (Andrews 1979, Zeilberger 1996)
ASMs and DPPs of the same order are equinumerous.

Example for $n = 3$:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

$$\varnothing\quad \begin{array}{|c|}\hline 2\\\hline\end{array}\quad \begin{array}{|c|}\hline 3\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&1\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\quad \begin{array}{cc}\boxed{3}&\boxed{3}\\ &\boxed{2}\end{array}$$

Problem
Construct a bijection between ASMs and DPPs!

Introduction

Semistandard Young tableaux

Schur functions s_λ are generating functions of semistandard Young tableaux (SSYT) of shape $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_n > 0)$:

$$\leqslant$$

$$\wedge \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 4 & 6 \\ \hline 2 & 3 & 3 & 5 \\ \cline{1-4} 4 & 5 & 7 \\ \cline{1-3} 6 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

$$\lambda = (5, 4, 3, 3, 1)$$

`weight:` $\displaystyle\prod_{i \geqslant 1} x_i^{\#i} = x_1^3 x_2 x_3^2 x_4^2 x_5^2 x_6^3 x_7^2 x_8$

The classical Littlewood identities

The sums of Schur polynomials over certain classes of integer partitions λ admit nice factorisations:

Theorem (Schur, Littlewood)

- $$\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}$$
- $$\sum_{\lambda \text{ even}} s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i^2} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}$$
- $$\sum_{\lambda' \text{ even}} s_\lambda(x_1, \ldots, x_n) = \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}$$

$\longrightarrow$ Proofs by Robinson-Schensted-Knuth correspondence (RSK)

Combinatorial interpretation of the Littlewood identity

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1-x_i x_j}$$

Schur functions $s_\lambda(x_1, \ldots, x_n)$: generating function of semistandard Young tableaux of shape λ

generating function of symmetric matrices $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ with non-negative integer entries via $\frac{1}{1-x_i} = \sum_{a_{i,i} \geqslant 0} x_i^{a_{i,i}}$ and $\frac{1}{1-x_i x_j} = \sum_{a_{i,j} \geqslant 0} (x_i x_j)^{a_{i,j}}$

Robinson–Schensted–Knuth correspondence

pairs (P, Q) of SSYT of the same shape $\xleftrightarrow{\text{RSK}}$ matrices with non-negative integer entries

Symmetry of RSK:

$$(P, Q) \xleftrightarrow{\text{RSK}} A \iff (Q, P) \xleftrightarrow{\text{RSK}} A^\top.$$

RSK on symmetric matrices A implies the Littlewood identity:

$$\sum_{\lambda} s_\lambda(x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{1}{1 - x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1 - x_i x_j}.$$

Goal: Littlewood identity related to ASMs

Classical identities for symmetric functions	$\longleftrightarrow$	Alternating sign matrices

- Warnaar (2008):

Cauchy identity for Hall–Littlewood polynomials	$\longleftrightarrow$	partition function of ASMs

- Betea, Wheeler, Zinn-Justin (2015/16):

Various Cauchy/Littlewood identities	$\longleftrightarrow$	partition function of symmetry classes of ASMs

- Fischer, H. (2025):

Littlewood identity for Robbins polynomials (generalised Hall–Littlewood polynomials)	$\longleftrightarrow$	Diagonally symmetric alternating sign matrices

Improvement: Alternating sign matrices on both sides!

Main result I:
Littlewood identity for Robbins polynomials

Gelfand–Tsetlin patterns

Semistandard Young tableau | Gelfand–Tsetlin pattern

1	1	1	4	6
2	3	3	5	
4	5	7		
6	6	8		
7				

$$\longleftrightarrow$$

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 3 & & & & & & & \\
 & & & & & & 1 & & 3 & & & & & & \\
 & & & & & 0 & & 3 & & 3 & & & & & \\
 & & & & 0 & & 1 & & 3 & & 4 & & & & \\
 & & & 0 & & 0 & & 2 & & 4 & & 4 & & & \\
 & & 0 & & 0 & & 2 & & 2 & & 4 & & 5 & & \\
 & 0 & & 0 & & 1 & & 2 & & 3 & & 4 & & 5 & \\
0 & & 0 & & 0 & & 1 & & 3 & & 3 & & 4 & & 5
\end{array}$$

$$\leqslant$$

$$x_1^3 x_2 x_3^2 x_4^2 x_5^2 x_6^3 x_7^2 x_8$$

`weight:` $\prod_{i=1}^{n} x_i^{\#i}$ $\qquad$ $\prod_{i=1}^{n} x_i^{\sum \text{entries in row } i - \sum \text{entries in row } (i-1)}$

Definition

A down-arrowed monotone triangle (DAMT) is a Gelfand–Tsetlin pattern with strict increase along rows where each entry, except for those in the bottom row, is decorated with either $\swarrow$, $\downarrow$ or $\searrow$ subject to the following rule:

If an entry is equal to one of the entries in the row below, then those entries have to be connected by a slanted arrow ($\swarrow$ or $\searrow$).

$$
\begin{array}{ccccccccccccc}
 & & & & & & 3 & & & & & & \\
 & & & & & & & \searrow & & & & & \\
 & & & & 2 & & & & 5 & & & & \\
 & & & & \downarrow & & & & & \searrow & & & \\
 & & 1 & & & & 3 & & & & 5 & & \\
 & \swarrow & & & & & & \searrow & & & \downarrow & & \\
1 & & & & 2 & & & & 4 & & & & 7
\end{array}
$$

Modified Robbins Polynomials

Definition
The (modified) Robbins polynomial $R^*_k(x_1, ..., x_n; u, v, w)$ is the generating function of DAMTs with bottom row k with weight

$$u^{\#\searrow} v^{\#\swarrow} w^{\#\downarrow}$$
$$\times \prod_{i=1}^{n} x_i^{\sum \text{entries in row } i - \sum \text{entries in row } (i-1) + \#\searrow \text{ in row } (i-1) - \#\swarrow \text{ in row } (i-1)}.$$

$$\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & & & \searrow & & \\
 & & 2 & & 5 & & \\
 & & \downarrow & & & \searrow & \\
 & 1 & & 3 & & 5 & \\
\swarrow & & & & \searrow & \downarrow & \\
1 & & 2 & & 4 & & 7
\end{array}
\qquad u^3 v w^2 x_1^3 x_2^5 x_3^3 x_4^5$$

Main result I

We establish a Littlewood identity for Robbins polynomials:

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w)$$
$$= \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i<j \leqslant n} \frac{x_i + x_j + w x_i x_j}{x_j - x_i}$$
$$\times \operatorname*{Pf}_{\chi_{\text{even}}(n) \leqslant i<j \leqslant n} \left(\begin{cases} 1, & i=0, \\ \frac{(x_j-x_i)(1+(1+w)x_i x_j)}{(x_i+x_j+w x_i x_j)(1-x_i x_j)}, & i \geqslant 1, \end{cases} \right)$$

where Pf denotes the Pfaffian of an upper triangular array and $\chi_{\text{even}}(n)$ equals 1 if n is even and 0 otherwise.

Pfaffians

- Consider all $(2n-1)!!$ partitions of $\{1, 2, \ldots, 2n\}$ into pairs.
- They can be written as $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ with $i_1 < \cdots < i_n$ and $i_k < j_k$ for all $1 \leqslant k \leqslant n$.
- For a triangular array $A = (a_{i,j})_{1 \leqslant i < j \leqslant 2n}$, the Pfaffian $\mathrm{Pf}(A)$ is defined as

$$\sum_{\{(i_1,j_1),\ldots,(i_n,j_n)\}} \operatorname{sgn}(i_1 j_1 \ldots i_n j_n) a_{i_1,j_1} \cdots a_{i_n,j_n},$$

  where we sum over all pairings in consideration.
- If we complete A to the uniquely determined skew-symmetric matrix M with A being its upper triangular part, then it is well known that

$$\mathrm{Pf}(A)^2 = \det(M).$$

Symmetric functions

Robbins polynomials are connected to other symmetric functions:

- Schur polynomials
- symmetric Grothendieck polynomials
- Hall–Littlewood polynomials
- fully inhomogeneous spin Hall–Littlewood symmetric rational functions (Borodin, Petrov 2018)

Related Littlewood identities

- Littlewood identities for Hall–Littlewood polynomials by Macdonald
- Refinement by Betea, Wheeler, Zinn-Justin (2015)
- Littlewood identity for spin Hall-Littlewood symmetric functions (Gavrilova 2023)

$\rightarrow$ These identities are of different type than ours!
We conjecture a generalisation of our identity at the level of spin Hall-Littlewood symmetric functions, which Fischer and Gangl have (almost) proved.

Monotone triangles and alternating sign matrices

$$
\begin{array}{ccccccc}
 & & & 3 & & & \\
 & & 2 & & 3 & & \\
 & 1 & & 2 & & 4 & \\
1 & & 2 & & 3 & & 4
\end{array}
\quad\longleftrightarrow\quad
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}
\longleftrightarrow
\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

Observation (Mills, Robbins, Rumsey 1983)

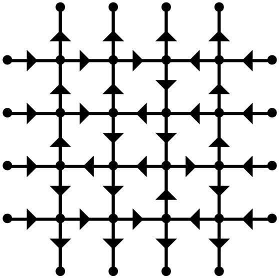
ASM of order n are in bijective correspondence with monotone triangles with bottom row $(1, 2, \dots, n)$.

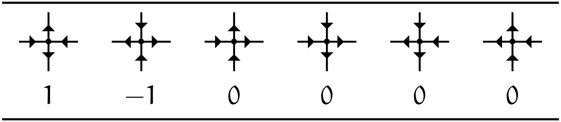
Main result II:
Combinatorial interpretation of the right-hand side of the Littlewood identity

Six-vertex model

Alternating sign matrices are in correspondence with six-vertex model configurations with domain wall boundary conditions:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$

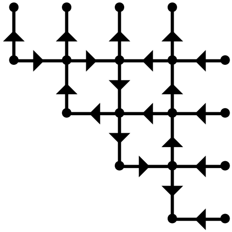




Diagonally symmetric alternating sign matrices

Diagonally symmetric alternating sign matrices (DSASMs) correspond to six-vertex model configurations on a triangular grid:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longleftrightarrow$$



The generating function of all such six-vertex model configurations of size n, denoted by $6V_{\triangledown}(n)$, is called the partition function $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$.

Diagonally symmetric alternating sign matrices

- There are eight different symmetry classes of ASMs that are induced by the symmetry group of the square. The enumeration of these symmetry classes was initiated by Stanley.
- In $5\frac{1}{2}$ cases, a product formula has been established (Behrend, Fischer, Konvalinka, Kuperberg, Razumov, Stroganov, Okada, Zeilberger).
- DSASMs are the first and only of the remaining symmetry classes for which an enumeration formula is known (Behrend, Fischer, Koutschan 2023):

$$\mathrm{DSASM}(n) = \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \leqslant i < j \leqslant n-1} \left(\langle u^i v^j \rangle \frac{(v-u)(2+uv)}{(1-uv)(1-u-v)} \right),$$

where $\langle u^i v^j \rangle f(u,v)$ denotes the coefficient of $u^i v^j$ in the expansion of $f(u,v)$.

Main result II

We relate the Littlewood identity for Robbins polynomial to the partition function of diagonally symmetric alternating sign matrices:

Theorem (Fischer, H. 2025)
Let n be a positive integer. Then

$$\sum_{0 \leqslant k_1 < \cdots < k_n} R^*_{(k_1,\ldots,k_n)}(x_1,\ldots,x_n;1,1,w) \\ = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1 \leqslant i < j \leqslant n} \frac{1}{1-x_i x_j} Z_{\mathrm{DSASM}}(x_1,\ldots,x_n).$$

Main result III: Coefficient of the highest term in the polynomial expansion of $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$

Main result III

The partition function $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ is a symmetric polynomial in $x_1, \ldots, x_n$. What can we say about its Schur expansion?
$\longrightarrow$ Work in progress

Theorem (Fischer, H. 2025)
The coefficient of $x_1^{n-1} \cdots x_n^{n-1}$ in $Z_{\mathrm{DSASM}}(x_1, \ldots, x_n)$ is given by the generating function

$$\sum_{6V_{\bigtriangledown}(n)} w^{\# + \#},$$

which equals

$$w^{\binom{n}{2}} \operatorname*{Pf}_{\chi_{\mathrm{odd}}(n) \leqslant i < j \leqslant n-1} \left(\langle u^i v^j \rangle \frac{(v-u)(1+uv+w)}{(1-uv)(w-u-v)} \right).$$

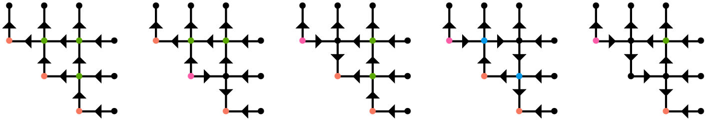
Open problem

Problem

Find a bijective proof of the following identity:

$$\sum_{6V_{\triangledown}(n)} (-1)^{\# + \#} w^{\# + \#} (w+2)^{\#} = \sum_{6V_{\triangledown}(n)} w^{\# + \#}.$$

Here is an illustration of the case $n = 3$, for which both sides sum to $1 + w + 2w^2 + w^3$.

DSASM	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$
$6V_{\triangledown}$					
LHS	-1	$w+2$	$w+2$	$w^2(w+2)$	$-(w+2)$
RHS	w^3	w^2	w^2	1	w

Do you want to learn more about ASMs?

Workshop **42 years of alternating sign matrices**

- University of Ljubljana (Slovenia), 22-26 September 2025
- Invited speakers:
  - Roger Behrend
  - Philippe Di Francesco
  - Ilse Fischer
  - Christian Krattenthaler
  - Anna Weigandt
- Organised by Matjaž Konvalinka, Florian Schreier-Aigner and Jessica Striker
- More information: `https://tinyurl.com/42yearsASMs`

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Takk fyrir!