A Formal Power Series Approach to Multiplicative Dynamic Feedback Interconnection

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Some Preliminary Facts

- $X = \{x_0, x_1, x_2, \ldots, x_m\}$: non-commutative alphabet.
- X^* : free monoid of X (empty word : $\mathbf{1}$).
- $\mathbb{R}^n\langle X\rangle$: n-tuple of non-commutative polynomials over X.
- $(\mathbb{R}\langle X\rangle, \Delta_{\sqcup\sqcup}, \emptyset)$ is a cofiltered connected coalgebra.
- $\Delta_{\sqcup\sqcup}$ is primitive on X and extended multiplicatively (along catenation product)

$$\Delta_{\sqcup\sqcup}(x_i) = x_i \otimes \mathbf{1} + \mathbf{1} \otimes x_i$$

- The convolution algebra of linear maps from $(\mathbb{R}\langle X\rangle, \Delta_{ш})$ to $\mathbb{R}$, is given by the space of formal power series denoted by $\mathbb{R}\langle\langle X\rangle\rangle$.
- $\mathbb{R}_p\langle\langle X\rangle\rangle := \{c \in \mathbb{R}\langle\langle X\rangle\rangle : c(\mathbf{1}) = 0\}$
- The dual basis is given by $\{\emptyset\} \cup X^+$, such that $\eta(\xi) = 1$ if $\eta = \xi$ in X^+, and zero else.
- An element $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is represented by

$$c = c(\mathbf{1})\emptyset + \sum_{\eta \in X^+} c(\eta)\eta.$$

- The convolution product on $\mathbb{R}\langle\langle X\rangle\rangle$ is the shuffle product, which is defined for all $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$ and $p \in \mathbb{R}\langle X\rangle$ by

$$(c ш d)(p) = m_{\mathbb{R}} \circ (c \otimes d) \circ \Delta_{ш}(p).$$

Chen–Fliess series

- Given a word $\eta = x_{i_1} x_{i_2} \cdots x_{i_k}$ and an m-vector of integrable inputs $u = (u_1, u_2, \cdots, u_m)$ on $[0, T]$, then for $t \leq T$:

$$F_\eta[u](t) := \int_0^t d\tau_1 u_{i_1}(\tau_1) \int_0^{\tau_1} d\tau_2 u_{i_2}(\tau_2) \cdots \int_0^{\tau_{k-1}} d\tau_k u_{i_k}(\tau_k),$$

  where $u_0 := 1$ and $F_\emptyset[u](t) := 1$.

  Then for all $c \in \mathbb{R}\langle\langle X \rangle\rangle$ and an integrable function u the corresponding Chen–Fliess series F_c is : (Fliess 1981)

$$y(t) = F_c[u](t) = c(\mathbf{1}) + \sum_{\eta \in X^+} c(\eta)\, F_\eta[u](t).$$

- Chen–Fliess series are input-output maps for nonlinear dynamical systems and provide some key intuitions about interconnections of nonlinear systems (Fliess, Reutenauer, Gray, Duffaut Espinosa, Ebrahimi–Fard, Thitsa, V etc..)

Shuffle Product

- The shuffle product of two words $x_i\eta \sqcup\!\sqcup\, x_j\gamma$ is defined as

$$x_i\eta \sqcup\!\sqcup\, x_j\gamma = x_i\left(\eta \sqcup\!\sqcup\, x_j\gamma\right) + x_j\left(x_i\eta \sqcup\!\sqcup\, \gamma\right),$$
$$\eta \sqcup\!\sqcup\, \emptyset = \eta \sqcup\!\sqcup\, \emptyset = \eta.$$

- Examples:

$$x_1 \sqcup\!\sqcup\, x_0 = x_1x_0 + x_0x_1.$$
$$x_1^2 \sqcup\!\sqcup\, x_0 = x_0x_1^2 + x_1^2x_0 + x_1x_0x_1.$$
$$x_1x_0 \sqcup\!\sqcup\, x_0x_1 = 2x_1x_0x_1x_0 + 4x_1^2x_0^2.$$

- $F_\eta.F_\gamma[u] = F_{\eta \sqcup\!\sqcup\, \gamma}[u]$. This relation encodes integration by parts rule.

“Adorned” Shuffle Product

- For $\ell \geq 2$, $\mathbb{R}^\ell\langle\langle X\rangle\rangle$ can also inherit associative (but not commutative) algebra structure via “adorned” shuffle products, $\sqcup\!\sqcup_k$ where the subscript $k = 1, 2, \ldots, \ell$ (Foissy 2016).
- For $c, d \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$

$$c \sqcup\!\sqcup_k d = \begin{pmatrix} c_1 \sqcup\!\sqcup d_k \\ c_2 \sqcup\!\sqcup d_k \\ \vdots \\ c_\ell \sqcup\!\sqcup d_k \end{pmatrix}.$$

- In general, for a given $\mathsf{a} = (a_1, a_2, \ldots, a_\ell) \in \mathbb{R}^\ell$; define $c \sqcup\!\sqcup_{\mathsf{a}} d = \sum_{i=1}^{\ell} a_i\,(c \sqcup\!\sqcup_i d)$, then $\left(\mathbb{R}^\ell\langle\langle X\rangle\rangle, \sqcup\!\sqcup_{\mathsf{a}}\right)$ is an associative algebra. For all $c, d, e \in \mathbb{R}^\ell\langle\langle X\rangle\rangle$:

$$c \sqcup\!\sqcup_i (d \sqcup\!\sqcup_j e) = (c \sqcup\!\sqcup_i d) \sqcup\!\sqcup_j e = (c \sqcup\!\sqcup_j e) \sqcup\!\sqcup_i d.$$

Multiplicative Dynamic Feedback

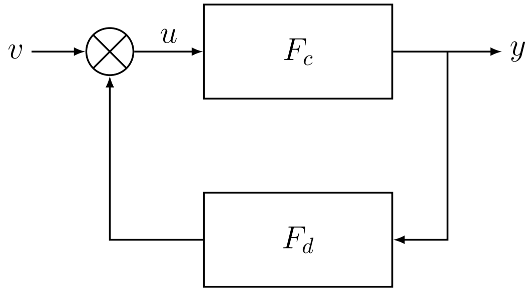

Figure 1: F_c in multiplicative output feedback with F_d

- The notion that feedback can be described in mathematical terms as a transformation group acting on the plant is well established in control theory due to Brockett (1978).

- Strictly speaking, the right statement as the recent works reveal, is that associated with every feedback there is a pre/post group and the transformation group for the feedback is its Grossman-Larson group.

Multiplicative feedback group

- Let $M^m := \{\, \mathbb{1} + c \ : c \in \mathbb{R}^m_p \langle\langle X \rangle\rangle \}$, where $\mathbb{1} = (1\emptyset, 1\emptyset, \ldots, 1\emptyset)$. Note that $(M^m, \sqcup\!\sqcup, \mathbb{1})$ is an Abelian group.

- $(M^m, \star, \mathbb{1})$ is the transformation group associated with multiplicative dynamic feedback where

$$c \star d := d \sqcup\!\sqcup (c \curvearrowleft d)$$

- For all $c, d, e \in M^m$

$$(c \curvearrowleft d) \curvearrowleft e = c \curvearrowleft (d \star e)$$
$$(c \sqcup\!\sqcup d) \curvearrowleft e = (c \curvearrowleft e) \sqcup\!\sqcup (d \curvearrowleft e)$$

- $(M^m, \star)$ is the Grossman-Larson group of the pre-group $(M^m, \sqcup\!\sqcup, \curvearrowleft)$

2. Hopf Algebra of Coordinate functions

- The vector space V of coordinate maps on $\mathbb{R}^m\langle\langle X\rangle\rangle$ is spanned by a_η^j, where $\eta \in X^*$ and $j = 1, 2, \ldots, m$.
- For $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$

$$a_\eta^j\,(c) = c_j\,(\eta)\,, \quad \forall\, j = 1, 2, \ldots, m.$$

- The vector space $V = \bigoplus_{n\geq 0} V_n$ is graded, where V_n is spanned by a_η^j, $j = 1, 2, \ldots, m$, $|\eta| = n$.
- For all $k = 0, 1, 2, \ldots, m$ define a linear endomorphism $\theta_k : V \longrightarrow V$ such that $\theta_k(a_\eta^j) = a_{x_k\eta}^j$, for all $j = 1, 2, \ldots, m$.

- $\mathcal{B} :=$ graded symmetric algebra with grading induced by V and product is denoted by $\boldsymbol{m}$ and the unit is $\mathbf{1}$.

- $\mathcal{B}$ is a bialgebra with the cocommutative coproduct $\Delta_{\sqcup\sqcup}$ defined as: for all $c, d \in \mathbb{R}^m\langle\langle X\rangle\rangle$

$$\Delta_{\sqcup\sqcup}\left(a_\eta^j\right)(c \otimes d) = \left(c \sqcup\sqcup d\right)_j(\eta) = \left(c_j \sqcup\sqcup d_j\right)(\eta)\,.$$

- By extending the usual unshuffle coproduct on words, $\Delta_{\sqcup\sqcup}(\eta) = \sum_{(\eta)} \eta' \otimes \eta''$ (employing Sweedler's notation), it is understood that for all $a_\eta^j \in V$,

$$\Delta_{\sqcup\sqcup}\left(a_\eta^j\right) = \sum_{(\eta)} a_{\eta'}^j \otimes a_{\eta''}^j.$$

- The counit ν is defined as

$$\nu\left(h\right)=\begin{cases}1; & \text{if } h=\mathbf{1},a_{\mathbf{1}}^{1},a_{\mathbf{1}}^{2},\ldots,a_{\mathbf{1}}^{m}\\ 0; & \text{otherwise.}\end{cases}$$

Theorem 1: (Foissy 2015) On V

$$\Delta_{\sqcup\sqcup}\circ\theta_{k}=\left(\theta_{k}\otimes\mathbf{id}+\mathbf{id}\otimes\theta_{k}\right)\circ\Delta_{\sqcup\sqcup},$$

for all $k=0,1,2,\ldots,m$.

- Observe that $(\mathcal{B},\boldsymbol{m},\mathbf{1},\Delta_{\sqcup\sqcup},\nu)$ is not a connected graded bialgebra as the elements $a_{\mathbf{1}}^{j}$, $j=1,2,\ldots,m$, are group-like but not invertible.

Denote $\mathfrak{s}_\mathfrak{i} := a_{\mathbf{1}}^{i} - \mathbf{1}$ for $i = 1, 2, \ldots, m$. The ideal $(\mathfrak{s}_\mathfrak{1}, \mathfrak{s}_\mathfrak{2}, \ldots, \mathfrak{s}_\mathfrak{m})$, is a bi-ideal. Define $\mathcal{H} = \mathcal{B}/\left(\mathfrak{s}_\mathfrak{1}, \mathfrak{s}_\mathfrak{2}, \ldots, \mathfrak{s}_\mathfrak{m}\right)$.

Theorem 2: $(\mathcal{H}, \boldsymbol{m}, 1, \Delta_{\sqcup\sqcup}, \nu)$ is a graded connected bialgebra. The character group of $(\mathcal{H}, \Delta_{\sqcup\sqcup}, \nu)$ is isomorphic to the shuffle group

$$(M^m, \sqcup\sqcup) \cong \underbrace{(M, \sqcup\sqcup) \times (M, \sqcup\sqcup) \times \cdots \times (M, \sqcup\sqcup)}_{m \text{ times}}.$$

- There is another coalgebra compatible with the graded augmented algebra of $\mathcal{H}$ (dualizing multiplicative feedback group product)

2.1 Multiplicative Feedback Bialgebra

- Define an unital algebra map $\rho : \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}$ such that

$$\rho\left(a_\eta^j\right)(c \otimes d) = (c \curvearrowleft d)_j(\eta) = (c_j \curvearrowleft d)(\eta),$$

  for all $c, d \in \mathbb{R}^m\langle\langle X\rangle\rangle$.

- The map ρ is not coassociative.

Theorem 3: For all $i = 0, 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$;

(i) $\rho\left(a_{\mathbf{1}}^i\right) = a_{\mathbf{1}}^i \otimes \mathbf{1}$.

(ii) $\rho \circ \theta_0 = (\theta_0 \otimes \mathbf{id}_{\mathcal{B}}) \circ \rho$.

(iii) $\rho \circ \theta_k(a_\eta^j) = (\theta_k \otimes \boldsymbol{m}) \circ (\rho \otimes \mathbf{id}_{\mathcal{B}}) \circ \sum_{(\eta)} a_{\eta'}^j \otimes a_{\eta''}^k$,

for all $j, k = 1, 2, \ldots, m$ and $\eta \in X^*$.

- The coproduct $\Delta : \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}$ is defined such that

$$\Delta\left(a_\eta^j\right)(c \otimes d) = (c \star d)_j\,(\eta)\,,$$

for all $c, d \in \mathbb{R}^m\langle\langle X \rangle\rangle$. Since the product, $\star$, is associative, the map Δ is coassociative.

Theorem 4: The coproduct on V is defined as

$$\Delta = (\mathbf{id}_{\mathcal{B}} \otimes \boldsymbol{m}) \circ (\rho \otimes \mathbf{id}_{\mathcal{B}}) \circ \Delta_{\sqcup\sqcup}\,.$$

Theorem 5: For all $n \geq 0$;

$$\Delta(\mathcal{V}_n) \subseteq \bigoplus_{i+j=n} \mathcal{V}_i \otimes \mathcal{B}_j$$

Remark: Theorem 5 asserts that the graded bialgebra $(\mathcal{B}, \Delta)$ is right–handed.

Theorem 6: $(\mathcal{B}, \boldsymbol{m}, 1, \Delta_{\sqcup\sqcup}, \nu)$ is a right graded comodule bialgebra of $(\mathcal{B}, \boldsymbol{m}, 1, \Delta, \nu)$ with the coaction map ρ.

- $\mathfrak{s}_{\mathfrak{i}} = a^{i}_{\mathbf{1}} - 1$ and the ideal $(\mathfrak{s}_{1}, \mathfrak{s}_{2}, \ldots, \mathfrak{s}_{\mathfrak{m}})$ is a bi–ideal of the bialgebra $(\mathcal{B}, \Delta)$.
- Thus, $\mathcal{H} = \mathcal{B}/\left(\mathfrak{s}_{1}, \mathfrak{s}_{2}, \ldots, \mathfrak{s}_{\mathfrak{m}}\right)$ gives us a connected structure with $\mathcal{H}_0 \cong \mathbb{R}\mathbf{1}$, thus making $(\mathcal{H}, \Delta)$ a Hopf algebra.
- $$\rho(\mathfrak{s}_{\mathfrak{i}}) \subseteq \mathfrak{s}_{\mathfrak{i}} \otimes \mathcal{B}.$$

Therefore, $\rho : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is a right coaction map on Hopf algebra $(\mathcal{H}, \Delta_{\sqcup\sqcup})$ by the Hopf algebra $(\mathcal{H}, \Delta)$.

Summary so far

- $(\mathcal{H}, \boldsymbol{m}, 1, \Delta_{\sqcup\!\sqcup}, \nu)$ is a graded connected bialgebra.
- The character group of $(\mathcal{H}, \Delta_{\sqcup\!\sqcup}, \nu)$ is isomorphic to the group $(M^m, \sqcup\!\sqcup)$.
- $(\mathcal{H}, \boldsymbol{m}, 1, \Delta, \nu)$ is a graded connected right–handed bialgebra.
- The character group of $(\mathcal{H}, \Delta, \nu)$ is isomorphic to the group $(M^m, \star)$.
- $(\mathcal{H}, \boldsymbol{m}, 1, \Delta_{\sqcup\!\sqcup}, \nu)$ is a right graded comodule Hopf algebra of $(\mathcal{H}, \boldsymbol{m}, 1, \Delta, \nu)$ with the (graded) coaction map ρ.

3. pre-Lie Structure on $\mathbb{R}_p^m\langle X\rangle$

Let k be a field of characteristic zero and V be a k–vector space.

Definition 1: $(V,\bullet)$ is a (right) pre-Lie algebra if the magmatic map $\bullet : V^{\otimes 2} \longrightarrow V$ satisfies for all $a,b,c \in V$ the (right) pre-Lie identity

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = (a \bullet c) \bullet b - a \bullet (c \bullet b)\,.$$

Define $[a,b]_\bullet := a \bullet b - b \bullet a$, then $\big(V, [\cdot,\cdot]_\bullet\big)$ is a Lie algebra.

Definition 2: $(V,\mathring{\delta})$ is a (right) pre-Lie coalgebra if $\mathring{\delta} : V \longrightarrow V^{\otimes 2}$ satisfies

$$(\mathbf{id}_V \otimes \mathbf{id}_V \otimes \mathbf{id}_V - \tau_{(23)}) \circ \Big(\big(\mathring{\delta} \otimes \mathbf{id}_V\big) - \big(\mathbf{id}_V \otimes \mathring{\delta}\big) \Big) \circ \mathring{\delta} = 0,$$

where $\tau_{(23)} : V^{\otimes 3} \to V^{\otimes 3}$, $\tau_{(23)}(a \otimes b \otimes c) = a \otimes c \otimes b$.

Let $\mathcal{S}(V)$ be the free symmetric algebra generated by the vector space V, with $\boldsymbol{m}$ denoting the symmetric product.

Theorem 1: Let $(\mathcal{S}(V), \boldsymbol{m}, \delta, \Delta, \epsilon, \rho)$ be graded connected cointeracting bialgebra where

(i) $(\mathcal{S}(V), \boldsymbol{m}, \delta, \epsilon)$ is a graded connected Hopf algebra in the category of $(\mathcal{S}(V), \boldsymbol{m}, \Delta, \epsilon)$ right comodule with coaction map ρ.

(ii) $\Delta = (\mathbf{id} \otimes \boldsymbol{m}) \circ (\rho \otimes \mathbf{id}) \circ \delta$.

(iii) $\delta'(V) \subseteq \mathcal{V} \otimes \mathcal{S}^{+}(V)$.

(iv) For all $x \in V$: $\rho'(x) := \rho(x) - x \otimes 1 \subseteq V \otimes \mathcal{S}^{+}(V)$.

Then,

1. $(\mathcal{S}(V), \boldsymbol{m}, \Delta, \epsilon)$ is a right–handed bialgebra.

2. On V: $\mathring{\Delta} = \mathring{\rho} + \mathring{\delta}$, where $\mathring{\upsilon} := (\pi_V \otimes \pi_V) \circ \upsilon$ for all $\upsilon \in Hom\,(\mathcal{S}(V), \mathcal{S}(V) \otimes \mathcal{S}(V))$.

- The Hopf algebra $\mathcal{H} \cong \mathcal{S}(V^+)$ as $\mathbb{R}$-algebras where $V^+ = \bigoplus_{n\geq 1} V_n$.

- V^+ is a graded right pre–Lie coalgebra with the pre-lie coproducts $\mathring{\Delta}$ and $\mathring{\Delta}_{\sqcup\sqcup}$ with
$$\mathring{\Delta} = \mathring{\rho} + \mathring{\Delta}_{\sqcup\sqcup}\,. \tag{1}$$

- The graded dual of V^+ is identified with proper polynomials $\mathbb{R}^m_p\langle X\rangle \subsetneq \mathbb{R}^m\langle\langle X\rangle\rangle$; with dual basis ηe_j where $\eta \in X^+$ and e_j for $j = 1, 2, \ldots, m$ are standard unit vectors in $\mathbb{R}^m$ such that $a^j_\eta\,(\zeta e_k) = \delta_{\eta,\zeta}\delta_{j,k}$ for all $\zeta \in X^+$.

- The vector space $\mathbb{R}^m_p\langle X\rangle$ is equipped with a magmatic product $\triangleleft : \mathbb{R}^m_p\langle X\rangle^{\otimes 2} \longrightarrow \mathbb{R}^m_p\langle X\rangle$ such that for $c, d \in \mathbb{R}^m_p\langle X\rangle$
$$(c \triangleleft d)_i\,(\eta) = a^i_\eta\,(c \triangleleft d) = \mathring{\rho}\left(a^i_\eta\right)(c \otimes d)\,.$$

Theorem 2: For all $c, d \in \mathbb{R}_p^m\langle X\rangle$ and $j = 1, 2, \ldots, m$.

(i) $x_0 e_j \triangleleft d = 0$

(ii) $x_k e_j \triangleleft d = x_k d_k e_j \qquad \forall\, k = 1, 2, \ldots, m.$

(iii) $x_0 c \triangleleft d = x_0\,(c \triangleleft d)$.

(iv) $x_k c \triangleleft d = x_k\,(c \triangleleft d) + x_k\,(c \sqcup\!\sqcup_k d) \qquad \forall\, k = 1, 2, \ldots, m.$

- Define $\bullet : \mathbb{R}_p^m\langle X\rangle^{\otimes 2} \longrightarrow \mathbb{R}_p^m\langle X\rangle$ as

$$\mathring{\Delta} a_\eta^i\,(c \otimes d) = (c \bullet d)_i\,(\eta)$$

Theorem 3: $\left(\mathbb{R}_p^m\langle X\rangle, \bullet\right)$ is a graded right pre–lie algebra such that

$$c \bullet d = (c \triangleleft d) + (c \sqcup\!\sqcup d)\,, \tag{2}$$

for all $c, d \in \mathbb{R}_p^m\langle X\rangle$.

4. com-pre-Lie Algebra on $\mathbb{R}_p^m\langle X\rangle$ associated with a linear Endomorphism

Definition 1: (Foissy 2015) $(\mathcal{A}, \oslash, \bullet)$ is a (right) com-pre-Lie algebra if

(i) $(\mathcal{A}, \oslash)$ is an associative and commutative algebra.

(ii) $(\mathcal{A}, \bullet)$ is a right pre-Lie algebra.

and for all $a, b, c \in \mathcal{A}$

$$(a \oslash b) \bullet c = (a \bullet c) \oslash b + a \oslash (b \bullet c) \,.$$

Theorem 1:

1. If $(\mathcal{A}, \oslash, \bullet)$ is right com-pre-Lie, then $\mathcal{A}$ inherits another right pre-Lie product, denoted by $\diamond$ and defined for all $a, b \in \mathcal{A}$ as $a \diamond b = a \bullet b + a \oslash b$.
2. $\left(\mathcal{A}, [\cdot,\cdot]_\diamond\right)$ is the derived Lie algebra of right pre-Lie algebra $(\mathcal{A}, \diamond)$ with $[a, b]_\diamond = [a, b]_\bullet$.

Remark: There are two pre-Lie products, $\diamond$ and $\bullet$, whose derived Lie algebras are identical and with the difference of a commutative product.

- Let $g \in \mathrm{End}\,(\mathbb{R}X)$, where $\mathbb{R}X$ is the $\mathbb{R}$-span of the alphabet X.
- Let e_j for $j = 1, 2, \ldots, m$ denote the set of standard unit vectors in $\mathbb{R}^m$.

Definition 2: Define a magmatic product $\triangleleft$ on the vector space $\mathbb{R}_p^m\langle X\rangle$ by induction on the degree of polynomials:

$$\begin{aligned} x_i e_j \triangleleft d &= g(x_i) d_i e_j \\ x_i c \triangleleft d &= x_i\,(c \triangleleft d) + g(x_i)\,(c \sqcup\!\sqcup_i d) \end{aligned} \tag{3}$$

where $d_0 := 0$, $x_i \in X$ and $c, d \in \mathbb{R}_p^m\langle X\rangle$.

Theorem 2: For all $c, d, h \in \mathbb{R}_p^m\langle X\rangle$:

(i) $(c \sqcup\!\sqcup\, d) \triangleleft h = (c \triangleleft h) \sqcup\!\sqcup\, d + c \sqcup\!\sqcup\, (d \triangleleft h)$.

(ii) $(c \sqcup\!\sqcup_k d) \triangleleft h = (c \triangleleft h) \sqcup\!\sqcup_k d + c \sqcup\!\sqcup_k (d \triangleleft h)$.

Theorem 3: $\left(\mathbb{R}_p^m\langle X\rangle, g, \triangleleft\right)$ is a pre-Lie algebra if and only if g is of the form $g(x_i) = \alpha_i x_i + \beta_i x_0$, for all $i = 1, 2, \ldots, m$.

With $X = \{x_0, x_1, x_2, \ldots, x_m\}$ in natural order, the matrix representation of the endomorphism g, denoted by $[g]_X$ for which $\left(\mathbb{R}_p^m\langle X\rangle, g, \triangleleft\right)$ becomes a right pre-Lie algebra is

$$[g]_X = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & \ldots & a_{0m} \\ a_{10} & a_{11} & 0 & 0 & \ldots & 0 \\ a_{20} & 0 & a_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_{m0} & 0 & 0 & 0 & \ldots & a_{mm} \end{pmatrix} \tag{4}$$

where the non–zero (not necessarily zero) elements $a_{ij} \in \mathbb{R}$. The submatrix of $[g]_X$ when restricted to $X \setminus \{x_0\}$ is a diagonal matrix.

Theorem 4: For $g \in \operatorname{End}(\mathbb{R}X)$ whose matrix representation is of the form in (4),

(i) $\left(\mathbb{R}_p^m\langle X\rangle, g, \sqcup\!\sqcup, \triangleleft\right)$ is a right com-pre-Lie algebra. Thus, $\left(\mathbb{R}_p^m\langle X\rangle, \bullet\right)$ is a right pre-Lie algebra where $c \bullet d = (c \triangleleft d) + (c \sqcup\!\sqcup d)$ for all $c, d \in \mathbb{R}_p^m\langle X\rangle$.

(ii) The derived Lie algebras of both right pre-Lie algebras $\left(\mathbb{R}_p^m\langle X\rangle, g, \triangleleft\right)$ and $\left(\mathbb{R}_p^m\langle X\rangle, \bullet\right)$ are identical.

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