Monochromatic structures in two-edge-colored ordered complete graphs

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joint work with András Gyárfás and Géza Tóth

Co-authors

Preliminary results

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Every 2-colored complete graph has a monochromatic spanning tree.

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Every 2-colored complete graph K_{3n-1} contains a monochromatic matching M_n and this is not true for K_{3n-2}.

Preliminary geometric results

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Theorem (KPT)

Every 2-colored complete geometric graph has a monochromatic plane spanning tree.

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Every 2-colored complete geometric graph K_{3n-1} contains a monochromatic plane matching M_n.

Here a plane subgraph is one, whose edges in the embedding do not have common internal points.

Our starting point

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Conjecture: Every 2-colored complete simple drawing has a monochromatic plane spanning tree.

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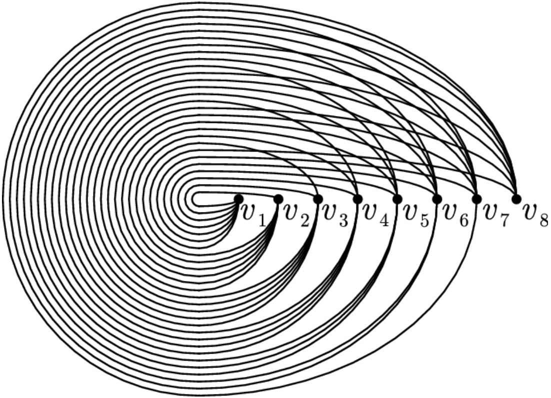

Ordered graphs

Ordered graphs

An ordered graph G is a simple graph with $V(G) = [m] = \{1, 2, \ldots, m\}$.
We also use $[i, j] = \{i, i + 1, \ldots, j\}$
The vertex set is considered with the natural ordering and the edges are denoted by (i, j), where we always assume $i < j$.
The length of (i, j) is $j - i$.

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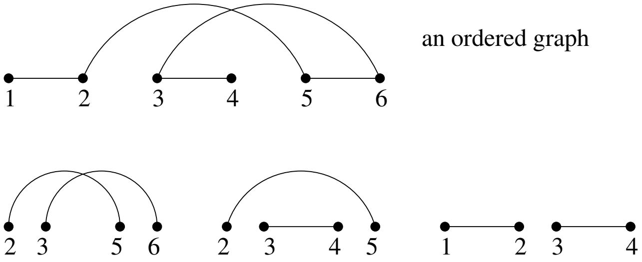
- Edges (a, b) and (c, d) are crossing if either $a < c < b < d$ or $c < a < d < b$.
- Edges (a, b) and (c, d) are nested if either $a < c < d < b$ or $c < a < b < d$.
- Edges (a, b) and (c, d) are separated if either $a < b < c < d$ or $c < d < a < b$.



6 types of questions

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	length of the	longest path	
	lower bound	upper bound	
nested	$n/4$	$n/2+1$	
crossing	$n/4$	$n/2$	
separated	$\sqrt{n}$	$\sqrt{n}+2$	
non−nested	$n/4$	$2n/3$	$\rightarrow$ twisted drawing
non−crossing	$\lfloor\frac{n+1}{2}\rfloor$	$\lfloor\frac{n+1}{2}\rfloor$	$\rightarrow$ convex drawing
non−separated	$n/4$	$n/2$	

Monochromatic spanning trees

Monochromatic spanning trees

Theorem (JB, AGy, GT)

In every 2-coloring of the ordered complete graph, there exists
(i) a monochromatic non-crossing spanning tree.
(ii) a monochromatic non-nested spanning tree.
(iii) a monochromatic non-separated spanning tree.

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(ii) connection to twisted drawings

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Otherwise there is an i, $1 < i < n$ such that $(i-1, i)$ and $(i, i+1)$ have different colors.

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Delete vertex i. Use induction.

Non-nested spanning tree

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Let c be any 2-coloring of the edges. $(1, 2)$ is blue. If $(1, i)$ is blue for every i, $2 \leq i \leq n$, then we are done.

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Otherwise, let s be the smallest number such that $(1, s)$ is red. We now change the coloring c to $\tilde{c}$ as follows: we recolor each edge induced by $[s-1]$ blue, and keep c otherwise. Consider the coloring $\tilde{c}$ on $[2, n]$ and apply the induction hypothesis.

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Suppose first that we find a blue spanning tree B without nested edges. Delete the edges in B induced by $[2, s-1]$. The resulting graph can also be found in the original coloring c. Now add the blue edges $(1,2),(1,3),\ldots,(1,s-1)$.

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Suppose first that we find a blue spanning tree B without nested edges. Delete the edges in B induced by $[2, s-1]$. The resulting graph can also be found in the original coloring c. Now add the blue edges $(1,2),(1,3),\ldots,(1,s-1)$.
Suppose now that we found a red spanning tree R on $[2,n]$. It cannot contain any edges induced by $[2,s-1]$ since they are blue. So, R can also be found in the original coloring c. Simply add edge $(1,s)$.

Non-separated spanning tree

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Let c be any 2-coloring of the edges. $(1, n)$ is blue. If $(1, i)$ is blue for every i, $2 \leq i \leq n$, then we are done,
Otherwise, let s be the largest number such that $(1, s)$ is red.

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Otherwise, let s be the largest number such that $(1, s)$ is red. We now change the coloring c to $\tilde{c}$ as follows: we recolor each edge induced by $[s+1, n]$ blue, and keep c otherwise. Consider the coloring $\tilde{c}$ on $[2, n]$ and apply the induction hypothesis.

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Suppose first that we find a blue spanning tree B without separated edges. Delete the edges of B induced by $[s+1, n]$. The resulting graph can also be found in the original coloring c. Now add the blue edges $(1, s+1), \ldots, (1, n)$.

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In every 2-coloring of the ordered complete graph on $[n]$, there exists (iii) a monochromatic non-separated spanning tree.

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Suppose now that we found a red spanning tree R on $[2, n]$. It cannot contain any edges induced by $[s+1, n]$ since they are blue. So, R can also be found in the original coloring c. Simply add edge $(1, s)$.

Monochromatic spanning subgraphs

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Theorem (JB, AGy, GT)

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Proposition (JB, AGy, GT)

(i) There is a 2-coloring of the ordered complete graph on $[n]$, which does not contain a non-crossing monochromatic subgraph with n edges.
(ii) There is a 2-coloring of the ordered complete graph on $[n]$, which does not contain a non-nested monochromatic subgraph with n edges.

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There is a 2-coloring of the ordered complete graph on $[n]$, which does not contain a non-crossing monochromatic subgraph with n edges.

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Let H be a non-crossing subgraph with red edges.

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Consider the convex drawing of K_n, where the blue edges are on the outer cycle. Now H becomes a planar subgraph of the convex drawing.

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Let us add all edges of the outer cycle ($n-1$ blue and possibly one red) to H to get H'.

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It is still an outerplanar graph. Therefore, H' has at most $2n-3$ edges, $n-1$ of them are blue.

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It is still an outerplanar graph. Therefore, H' has at most $2n-3$ edges, $n-1$ of them are blue.
Thus H can have at most $n-2$ edges.

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The argument is the same for the blue edges.

Non-separated subgraph

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Proposition (JB, AGy, GT)

In any 2-coloring of the ordered complete graph on $[n]$, there is a non-separated monochromatic subgraph of $\lfloor n^2/8 \rfloor$ edges.

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Consider all edges (i, j) with $i \leq \lfloor n/2 \rfloor < j$.
\# edges $= \frac{n^2}{4}$

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no separated pairs
majority color

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Conjecture

Every 2-colored ordered complete graph on $[3n-1]$ contains a monochromatic non-nested matching M_n

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Theorem (JB, AGy, GT)

If an ordered complete graph on $[3n-1]$ contains either
(i) a red K_{2n-1} or
(ii) a blue $K_{n-1,2n}$ as a subgraph, then there is a monochromatic non-nested M_n.

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use induction.

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Consider the complete bipartite graph on $A = [1, n]$ and $B = [2n, 3n-1]$.

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check the new edge and the non-separated property

Open questions

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Problem 1

What is the minimum number m such that every 2-colored ordered complete graph on $[m]$ contains a monochromatic non-nested M_n?

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Problem 2

Show that every 2-colored ordered complete graph on $[11]$ contains a monochromatic non-nested M_4.

Results on cycles

Results on cycles

	length of the	longest cycle	
	lower bound	upper bound	
nested	3	3	
crossing	$\log n$	$n/2$	
separated	3	3	
non−nested	$\log n$	$2n/3$	$\rightarrow$ twisted drawing
non−crossing	$\sqrt{n/2}$	$\sqrt{n}$	$\rightarrow$ convex drawing
non−separated	$n/8$	$n/2$	