# On the sharpest upper bound on the MP-ratio: approaching the problem by geometric insights

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joint work with Bojan Bašić

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- $|u|_v = |\{i : 1 \leq i \leq |u| - |v| + 1, u[i, i + |v| - 1] = v\}|$.

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## Theorem (A., Bašić; 2021)

*The optimal upper bound on the MP-ratio is not less than $2n$.*

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Now we know that the optimal upper bound on the MP-ratio for $n \geqslant 4$ is somewhere between

$$2n \quad \text{and} \quad \sim 2^{\frac{n}{2}} n,$$

and it remains an open problem to narrow (or even better, close) this gap.

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$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \; w \; 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

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*Let* $u \in \{1,2\}^*$, $t, v \in 2^*$ *and let* $p$ *and* $q$ *be subpalindromes of* $tu$ *and* $uv$*, respectively. If* $|p| + |q| > 2|u|$,

# The main theorem

## Theorem (A., Bašić; 2021)

*The MP-ratio is well-defined in the ternary case, it is bounded from above by 6 and this upper bound is the best possible.*

$$f(w) = 0^{2|w|-|w|_0} 2^{2|w|-|w|_2-g'(w,0,2)} \; w \; 2^{g'(w,0,2)} 1^{2|w|-|w|_1}$$

$$f'(w) = 1^{2|w|-|w|_1} 2^{g'(\widetilde{w},0,2)} \; w \; 2^{2|w|-|w|_2-g'(\widetilde{w},0,2)} 0^{2|w|-|w|_0}$$

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$$|u|_1 \leqslant \frac{|tv| - 1}{|tv|} |tuv|_2.$$

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Let us call them $f_1(w)$ and $f_2(w)$, respectively. We have $|f_1(w)| = |f_2(w)| = 8|w|$, and we prove that at least one of the words $f_1(w)$ and $f_2(w)$ does not have a subpalindrome whose length exceeds $2|w|$. ■

# Propositions 1–2

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## Proposition

*The length of an arbitrary subpalindrome of the form $0p0$ in each of the words $f_1(w)$ and $f_2(w)$ is less than or equal to $2|w|$.*

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- In proving this result, we used a geometric approach in one part of the proof. We defined certain mappings on sequences of letters in a word, inspired by Euclidean isometries—especially reflections and translations—and used these mappings in our analysis.

# Propositions 4–5

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*One of the following is true: the word $f_1(w)$ does not contain a subpalindrome of the form $2p2$ longer than $2|w|$, or the word $f_2(w)$ does not contain a subpalindrome of the form $3p3$ longer than $2|w|$. Analogously, the word $f_1(w)$ does not contain a subpalindrome of the form $3p3$ longer than $2|w|$, or the word $f_2(w)$ does not contain a subpalindrome of the form $2p2$ longer than $2|w|$.*

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Recall: we claim that at least one of the words $f_1(w)$ and $f_2(w)$ does not have a subpalindrome whose length exceeds $2|w|$.

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