Enumerating 1324-avoiders with few inversions

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Based on joint work with

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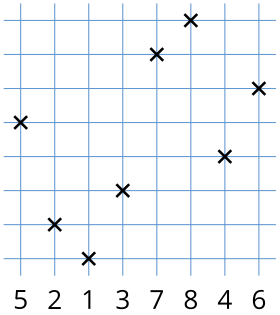
arXiv: `2408.15075`

Pattern avoiding permutations

A permutation $\pi \in S_n$ *contains* a pattern $p \in S_m$ if π has a subsequence that is order-isomorphic to p. Otherwise π *avoids* p.



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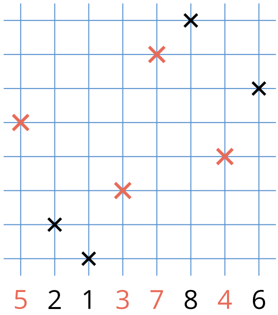

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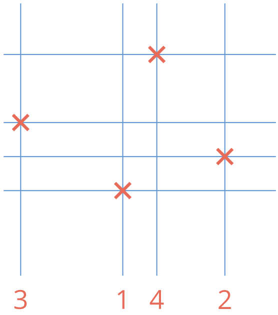

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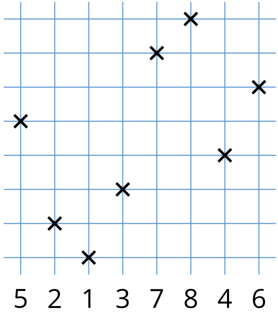

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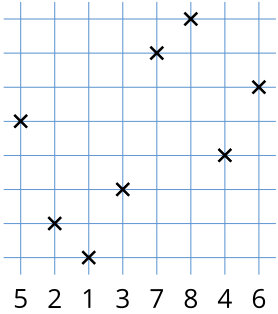

π contains 3142

π avoids 1324

We write $\mathrm{Av}(p)$ for the set of all permutations that avoid p,

$$\mathrm{Av}_n(p) = \mathrm{Av}(p) \cap S_n \qquad \text{and} \qquad \mathrm{av}_n(p) = |\mathrm{Av}_n(p)|.$$

History

Early history

MacMahon (1915) showed that

$$\mathrm{av}_n(123) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

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Erdős–Szekeres theorem (1935): any permutation of length at least $(a-1)(b-1)+1$ contains the pattern

$$[1, 2, \ldots, a] \quad \text{or} \quad [b, b-1, \ldots, 1].$$

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Knuth (1968) showed that the *stack-sortable* permutations are exactly the 231-avoiders, and $\mathrm{av}_n(231) = C_n$.

The Renaissance

Major actors:

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Developments:

- The Marcus–Tardos theorem (2004).
- Enumeration of $\mathrm{Av}(1234)$ and $\mathrm{Av}(1342)$.
- Generalizations: vincular patterns, mesh patterns, …
- Books: Bóna's *Combinatorics of Permutations* (2004), Kitaev's *Patterns in Permutations and Words* (2011).
- Annual conference *Permutation Patterns* (2003–present).

A Big Problem

The questions we ask

Holy Grail. Enumerate $\mathrm{av}_n(p)$ for some pattern p.

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Gessel (1990):

$$\mathrm{av}_n(1234) = 2\sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}^2 \frac{3k^2+2k+1-n-2nk}{(k+1)^2(k+2)(n-k+1)}.$$

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Bóna (1997):

$$\mathrm{av}_n(1342) = (-1)^{n-1}\cdot\frac{7n^2-3n-2}{2} + 3\sum_{k=2}^{n}(-1)^{n-k}\cdot 2^{k+1}\cdot\frac{(2k-4)!}{k!(k-2)!}\cdot\binom{n-i+2}{2}.$$

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Theorem (Marcus–Tardos, 2004)**.** For any pattern p, the *Stanley–Wilf limit*

$$L(p) := \lim_{n\to\infty} \mathrm{av}_n(p)^{1/n}$$

exists and is finite.

Examples: $L(132) = 4$, $L(1234) = 9$, $L(1342) = 8$.

What is known

Known bounds for $L(1324)$.

	Lower	Upper
Bóna (2004)		288
Bóna (2005)	9	
Albert–Elder–Rechnitzer–Westcott–Zabrocki (2006)	9.35	
Claesson–Jelínek–Steingrímsson (2012)		16
Bóna (2014)		13.93
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An innocent conjecture

Inversion = 21-pattern.

$$\mathrm{Av}_n^{k}(p) = \{\pi \in \mathrm{Av}_n(p) : \mathrm{inv}(\pi) = k\}\,, \qquad \mathrm{av}_n^k(p) = |\mathrm{Av}_n^k(p)|.$$

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Conjecture (Claesson–Jelínek–Steingrímsson, 2012)**.**

$$\mathrm{av}_n^k(1324) \leqslant \mathrm{av}_{n+1}^k(1324)$$

for all k, n, i.e. 1324 is *inversion monotone*.

Corollary. $L(1324) < 13.002$.

Another view

Numbers $\mathrm{av}_n^k(1324)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
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4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	1582	2305	3284	4617	6374	8665	11521
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
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13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
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CJS conjecture: increasing

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Claesson–Jelínek–Steingrímsson (2012) finds the blue sequence.

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CJS conjecture: increasing

Claesson–Jelínek–Steingrímsson (2012) finds the blue sequence.

Linusson–V. (2025) proves the conjecture in the red region.

Another view

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
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9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

Row differences

CJS conjecture: nonnegative

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14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
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14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
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Row differences

CJS conjecture: nonnegative

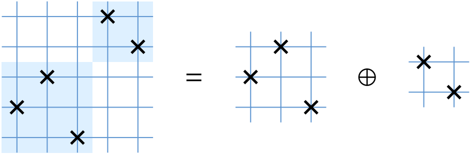
A constant sequence appears in the red region.

Why?

Decomposability

Decomposable permutations

Permutations can be decomposed with respect to the *direct sum*:



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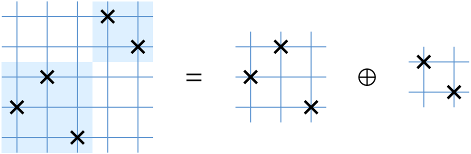
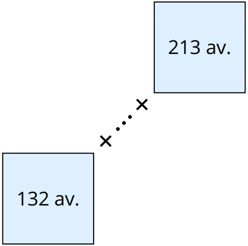


Fact. A decomposable $\pi \in S_n$ avoids 1324 if and only if

$$\pi = \sigma \oplus \mathrm{id} \oplus \tau,$$

where $\sigma \in \mathrm{Av}(132)$, $\tau \in \mathrm{Av}(213)$.

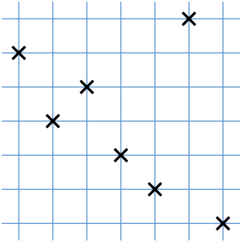


The limit sequence

Fact. If $n \geqslant k + 1$, then

$$\mathrm{av}_n^k(132) = p(k),$$

the number of integer partitions of k.



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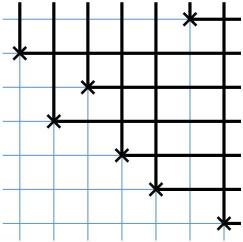

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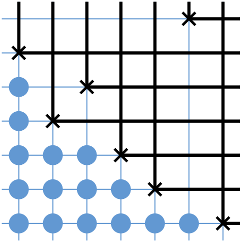

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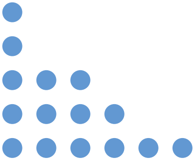

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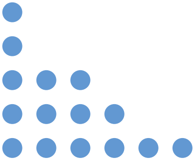

Combining the facts: if $n \geqslant k+2$, then

$$\mathrm{av}_n^k(1324) = \sum_{i=0}^{k} p(i)p(k-i) = [x^k]\big(P(x)^2\big).$$

This is the blue sequence

1, 2, 5, 10, 20, 36, 65, 110, 185, 300, 481, 752, 1165, ...

from the table!

Almost decomposable permutations

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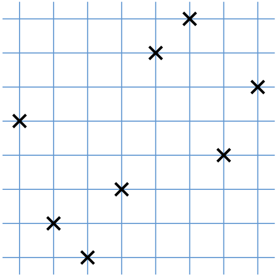

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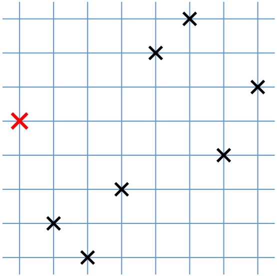

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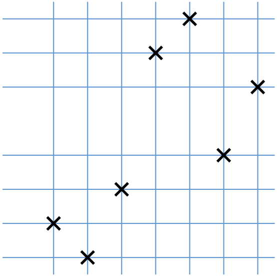

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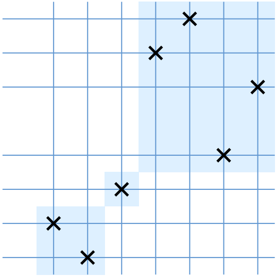

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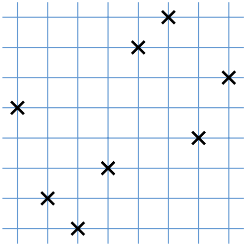

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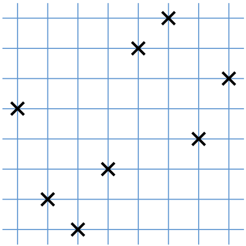

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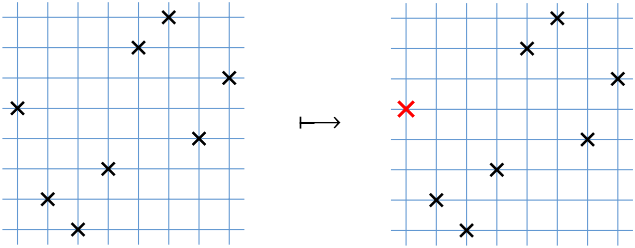

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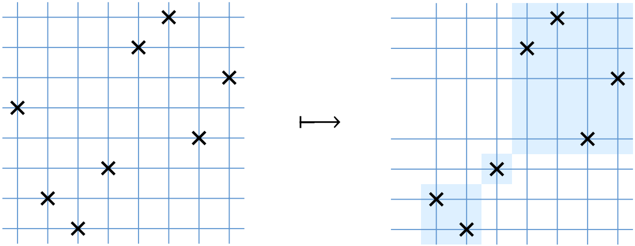

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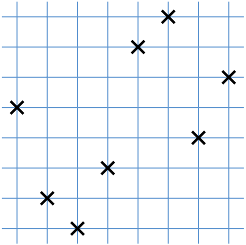

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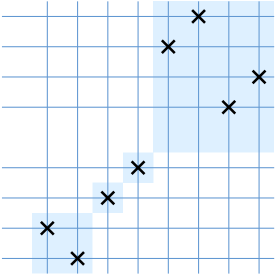

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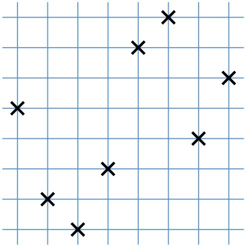

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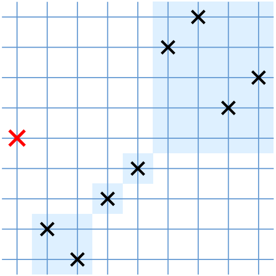

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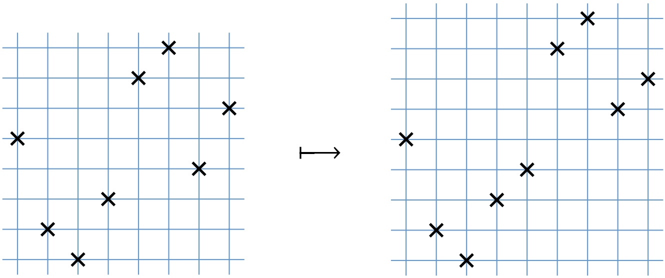

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Corollary. The conjecture holds when $n \geqslant \frac{k+7}{2}$.

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Thank you, and

Happy Birthday to Einar!

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Based on joint work with

Svante Linusson – KTH Royal Institute of Technology

Enumerating 1324-avoiders with few inversions

arXiv: 2408.15075

Numbers $\mathrm{av}_n^k(1324)$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	1																			
2	1	1																		
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4	1	2	5	6	5	3	1													
5	1	2	5	10	16	20	20	15	9	4	1									
6	1	2	5	10	20	32	51	67	79	80	68	49	29	14	5	1				
7	1	2	5	10	20	36	61	96	148	208	268	321	351	347	308	241	165	98	49	20
8	1	2	5	10	20	36	65	106	171	262	397	568	784	1019	1264	1478	1628	1681	1619	1441
9	1	2	5	10	20	36	65	110	181	286	443	664	985	1416	1988	2715	3589	4579	5631	6654
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11	1	2	5	10	20	36	65	110	185	300	477	738	1127	1682	2477	3584	5134	7240	10100	13915
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	1732	2577	3768	5450	7766	10976	15312
13	1	2	5	10	20	36	65	110	185	300	481	752	1161	1756	2627	3868	5634	8098	11526	16216
14	1	2	5	10	20	36	65	110	185	300	481	752	1165	1766	2651	3918	5734	8282	11858	16786
15	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2661	3942	5784	8382	12042	17118
16	1	2	5	10	20	36	65	110	185	300	481	752	1165	1770	2665	3952	5808	8432	12142	17302
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6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	303	240	165	98	49	20
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	956	1237	1463	1583	1570	1421
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	724	1237	1961	2898	4012	5213
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	317	569	1028	1795	3034	4867
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	172	300	517	866	1435	2394
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	316	526	876	1397
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	550	904
13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

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CJS conjecture: nonnegative

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13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

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13	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332	570
14	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184	332
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100	184
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50	100
17	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	10	24	50

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