The EHZ-capacity of polytopes is NP-hard

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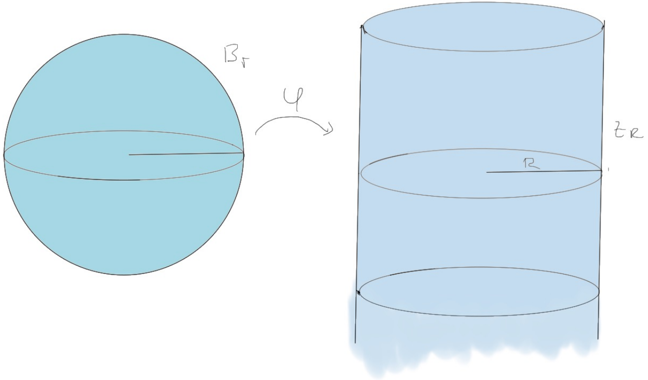
Welcome to Symplectic Geometry!

A sympletic space is $\mathbb{R}^{2n}$ equipped with an antisymmetric bilinear form:

$$\omega(u, v) = u^T \underbrace{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}_{:=J} v \quad \text{for all } u, v \in \mathbb{R}^{2n}.$$

Gromov's Non-Squeezing Theorem



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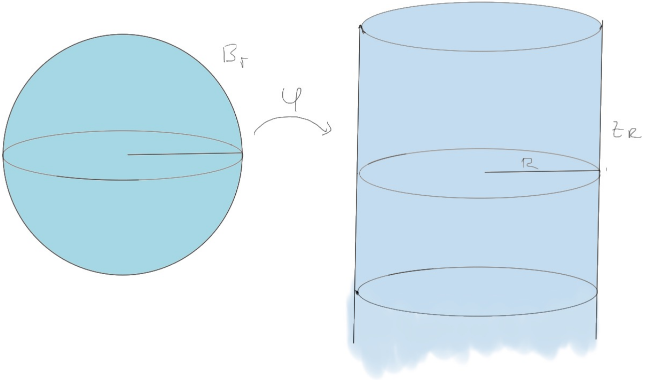

Symplectic capacities are important invariants!

EHZ Capacity of Simplices

Let $P \subset \mathbb{R}^{2n}$ be a simplex denoted by

$$P = P(B, \mathbf{e}) = \{x \in \mathbb{R}^{2n} : Bx \leq \mathbf{e}\} \quad \text{for} \quad B \in \mathbb{R}^{(2n+1)\times(2n)},\ \mathbf{e} \in \mathbb{R}^{2n+1}.$$

Theorem (Haim-Kislev (2019))

The EHZ capacity of a simplex can be computed by

$$c_{\mathrm{EHZ}}(P(B, \mathbf{e})) = \frac{(2n+1)^2}{2}\left(\max\left\{\sum_{1 \leq j < i \leq 2n+1} (P_\sigma^\top W P_\sigma)_{ij} \ :\ \sigma \in \mathfrak{S}_{2n+1}\right\}\right)^{-1}$$

for $W = BJB^\top$ and where P_σ is the permutation matrix corresponding to the permutation $\sigma \in \mathfrak{S}_{2n+1}$.

Theorem (Vallentin, L. (2024))

Computing the EHZ capacity of polytopes is NP*-hard.*

Proof Strategy

We want to prove that problem $\mathcal{A}$ is NP-hard.

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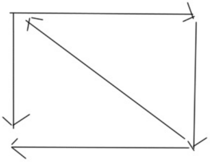
4. If $\mathcal{A}$ would not be NP-hard, $\mathcal{B}$ could not be NP-hard.
5. **Contradiction!**

Maximum Acyclic Subgraph and Feedback Arc Set

Let $D = (V, A)$ be a directed graph.

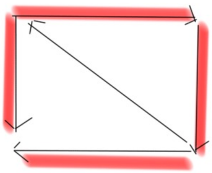


Maximum Acyclic Subgraph and Feedback Arc Set

Let $D = (V, A)$ be a directed graph.

The Maximum Acyclic Subgraph problem (MAS) is:

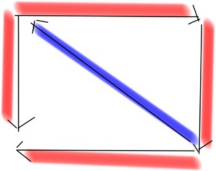
$$|\text{MAS}| = \max_{A' \subset A} \{|A'| : D(V, A') \text{ is acyclic}\}$$



Maximum Acyclic Subgraph and Feedback Arc Set

$$|\text{MAS}| = \max_{A' \subset A} \{|A'| : D(V, A') \text{ is acyclic}\}$$

$$|\text{FAS}| = \min_{\bar{A} \subset A} \{|\bar{A}| : D(V, A \setminus \bar{A}) \text{ is acyclic}\}$$



Bipartite Tournaments

Definition

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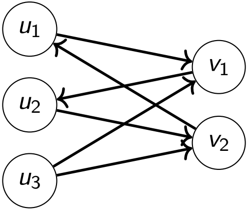

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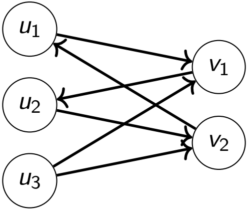

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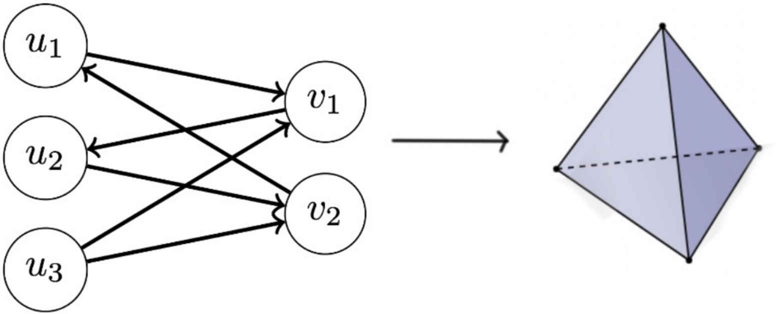
From Bipartite Tournament to Simplices



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Let $D = (U \cup V, A)$ be a complete bipartite tournament with $n = |U|$ and $m = |V|$, where we assume $n \geq m$.

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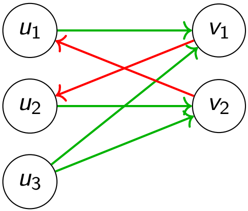

$$S_{ij} = \begin{cases} 1 & \text{if } (u_i, v_j) \in A, \\ -1 & \text{if } (v_j, u_i) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

$B \in \mathbb{Z}^{(2n+1)\times(2n)}$ by

$$B = \begin{bmatrix} I_n & 0 \\ 0 & S \\ -\mathbf{e}^\top & -\mathbf{e}^\top S \end{bmatrix},$$

$$s_1 = (1, -1, 0)$$
$$s_2 = (-1, 1, 0)$$
$$s_3 = (1, 1, 0)$$

$$P = P(B, \mathbf{e}) = \{x \in \mathbb{R}^{2n} : Bx \leq \mathbf{e}\}$$

The EHZ Capacity of this Simplex

$W = BJB^T$ is of the form

$$W = \begin{bmatrix} 0 & S^\mathsf{T} & -S^\mathsf{T}\mathbf{e} \\ -S & 0 & S\mathbf{e} \\ \mathbf{e}^\mathsf{T}S & -\mathbf{e}^\mathsf{T}S^\mathsf{T} & 0 \end{bmatrix}.$$

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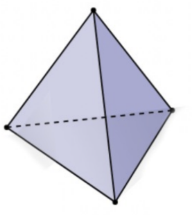
The main computational task in the EHZ capacity is:

$$\max\left\{ \sum_{1\leq j<i\leq 2n+1} (P_\sigma^{\mathsf{T}} W P_\sigma)_{ij} \ : \ \sigma \in \mathfrak{S}_{2n+1} \right\}.$$

Helpful Properties of this Construction

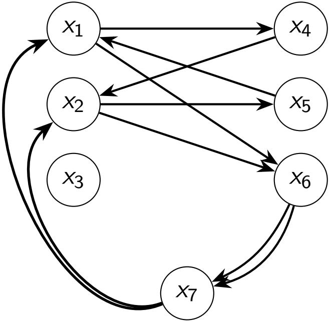


$$\xrightarrow[\sum_{i>j}(P_\sigma^T W P_\sigma)_{ij}]{W_{ij} = s_i J s_j}
\left(\begin{array}{ccc|ccc|c}
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Matrix Representation and Graph

Define matrix M: $M_{ij} = \max\{0, W_{ij}\}$. Then $W = M - M^{\mathsf{T}}$. From M as an adjacency matrix we obtain a new graph $\tilde{D} = (\tilde{V}, \tilde{A})$:

$$M = \left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$



Eulerian Graphs

A directed graph D is Eulerian if and only if:

- For every pair of vertices u, v, there is a path from u to v (i.e., D is **strongly connected**).
- The **indegree** of every vertex equals its **outdegree**.

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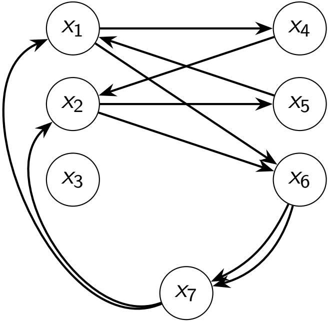
In our case, the graph $\tilde{D}$ is Eulerian.

$$W = \left(\begin{array}{ccc|ccc|c} 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \\ \hline 1 & 1 & 0 & 0 & 0 & -2 & 0 \end{array}\right)$$



Maximizing over an Antisymmetric Matrix

- For any permutation matrix P_σ, with $\sigma \in \mathfrak{S}_{2n+1}$, we have

$$P_\sigma^{\mathsf{T}} W P_\sigma = P_\sigma^{\mathsf{T}}(M - M^{\mathsf{T}})P_\sigma = 2P_\sigma^{\mathsf{T}} M P_\sigma - P_\sigma^{\mathsf{T}}(M + M^{\mathsf{T}})P_\sigma.$$

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$$\max_{\sigma\in\mathfrak{S}} \left\{ \sum_{1\le j<i\le 2n+1} (P_\sigma^{\mathsf{T}} W P_\sigma)_{ij} \right\} = 2\max_{\sigma\in\mathfrak{S}} \left\{ \sum_{1\le j<i\le 2n+1} (P_\sigma^{\mathsf{T}} M P_\sigma)_{ij} \right\} - \Delta.$$

MAS with the Adjacency Matrix

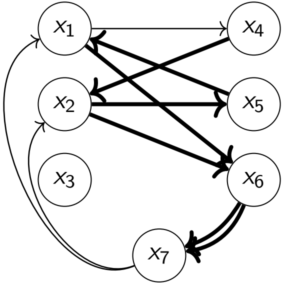
The maximization problem

$$\max \left\{ \sum_{1 \le j < i \le 2n+1} (P_\sigma^\mathsf{T} M P_\sigma)_{ij} \;:\; \sigma \in \mathfrak{S}_{2n+1} \right\}$$

determines a maximum acyclic subgraph in $\tilde{D}$.

Upper Triangles and Aylic Graphs

	x_4	x_2	x_5	x_1	x_6	x_7	x_3
x_4	0	1	0	0	0	0	0
x_2	0	0	1	0	1	0	0
x_5	0	0	0	1	0	0	0
x_1	1	0	0	0	1	0	0
x_6	0	0	0	0	0	2	0
x_7	0	1	0	1	0	0	0
x_3	0	0	0	0	0	0	0



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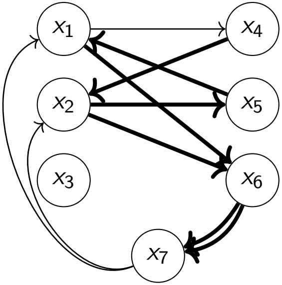
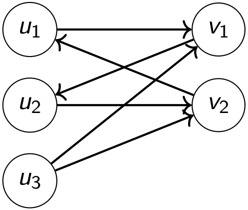
- It is Eulerian by construction.
- Max over W is equivalent to Max over M.
- Max over M gives us $\mathrm{MAS}(\tilde{D})$.

How do we obtain a MAS on D?





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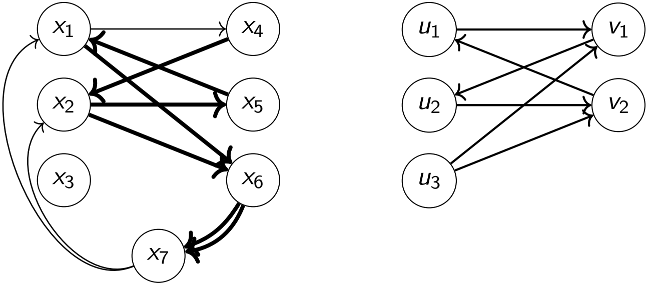
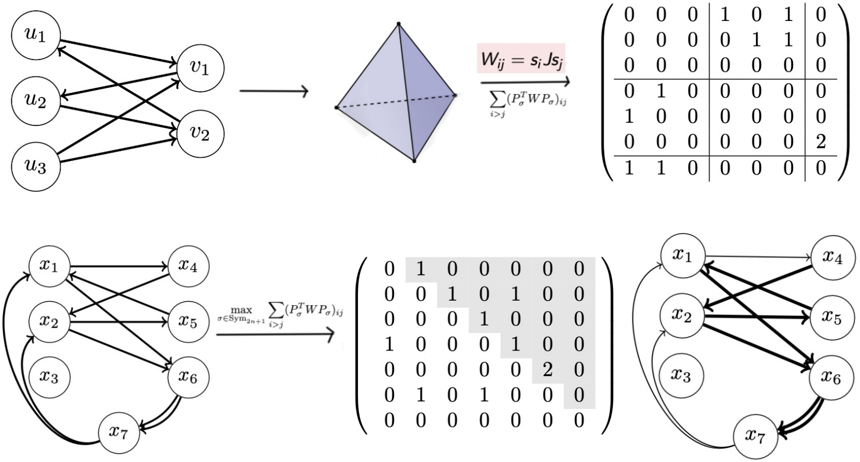


Lemma (Perrot, Van-Pham (2015))

Let $\tilde{D}$ be a directed Eulerian graph. Then:
$MAS(D) = MAS(\tilde{D}) - |\delta^{\mathrm{out}}(x_{2n+1})|$.

Conclusion Formula

$$\mathrm{MAS}(D) = \frac{1}{2}\left(\left\lfloor \frac{(2n+1)^2}{2c_{\mathrm{EHZ}}(P(\tilde{B}, \mathbf{e}))} + \frac{1}{2} \right\rfloor + \Delta\right) - \left|\delta^{\mathrm{out}}(x_{2n+1})\right|.$$



u_1
u_2
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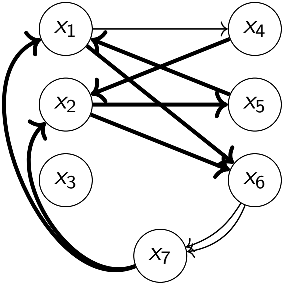
x_1
x_2
x_3
x_4
x_5
x_6
x_7
$\max_{\sigma \in \mathrm{Sym}_{2n+1}} \sum_{i>j} (P_\sigma^T W P_\sigma)_{ij}$

Thank you for your attention!

Proof

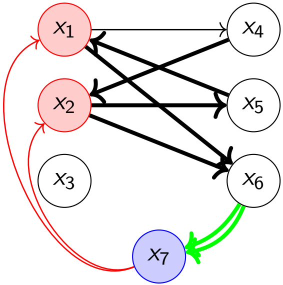
$$\mathsf{MAS}(D) \cup \delta^{\mathrm{out}}(x_{2n+1}) = \mathsf{MAS}(\tilde{D}).$$



Constructing A'' Without Cycles

For the other direction:

- Let A' be a maximum acyclic subgraph of $\tilde{D}$ and R be the set of vertices reachable from x_{2n+1} via walks in A'.



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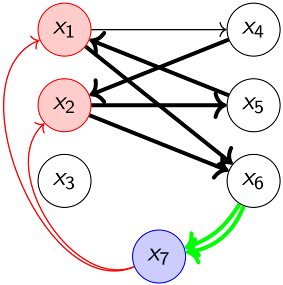
$$A'' = (A' \setminus \delta^{\text{in}}(R)) \cup \delta^{\text{out}}(R)$$



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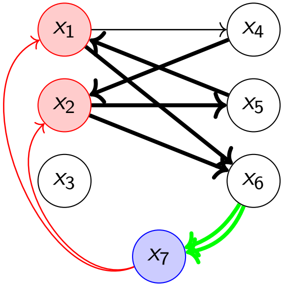

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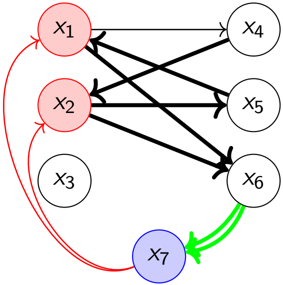
- $\delta^{\text{out}}(R) \cap A' = \emptyset$, so:

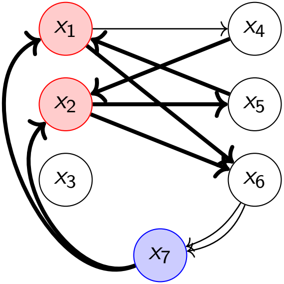
$$|A''| \geq |A'| - |\delta^{\text{in}}(R)| + |\delta^{\text{out}}(R)|.$$



Conclusion Using Eulerian Property

- Since $\tilde{D}$ is Eulerian (ignoring isolated vertices):

$$|\delta^{\text{in}}(R)| = |\delta^{\text{out}}(R)|.$$



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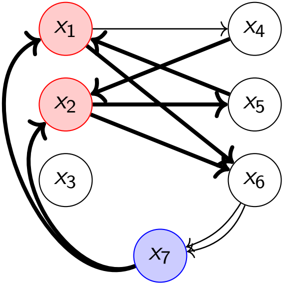

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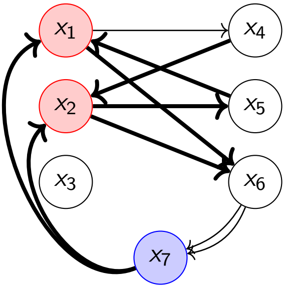

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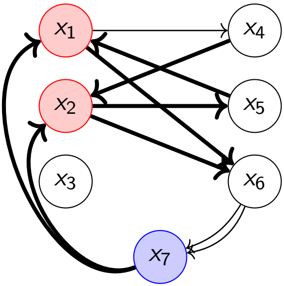

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- A'' is a maximum acyclic subgraph of $\tilde{D}$ and excludes incoming arcs to x_{2n+1}.
- Since A'' is also maximum, it must contain all arcs leaving x_{2n+1}.
- Removing these arcs yields an acyclic subgraph of D of size:

$$|A'| - |\delta^{\text{out}}(x_{2n+1})|.$$