Magic squares in finite Abelian groups

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Magic squares

Definition
A magic square of order n is an $n \times n$ array with entries $1, 2, \ldots, n^2$, each appearing once, such that the sum of each row, column, and both main diagonals is equal to $n(n^2 + 1)/2$.

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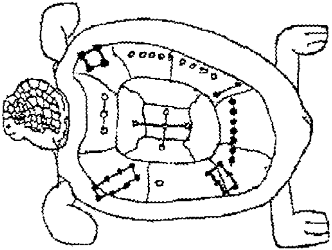

Lo Shu magic square, 2800 B.C

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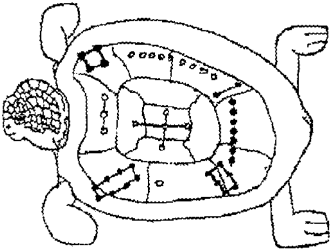

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Magic squares

Magic squares

Magic rectangles

Definition

A magic rectangle $\mathrm{MR}(a, b)$ is an $a \times b$ array with entries from the set $\{1, 2, \ldots, ab\}$, each appearing once, with all its row sums equal to a constant δ and with all its column sums are equal to a constant η.

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8	2	3	5

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$$\delta = 18, \eta = 9.$$

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Example: $\mathsf{MR}(2, 4)$

1	7	6	4
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$$\delta = 18, \eta = 9.$$

Theorem (Harmuth, 1881)

A magic rectangle $\mathsf{MR}(a, b)$ exists if and only if $a, b > 1$, $ab > 4$, and $a \equiv b \pmod 2$.

Notation

- Γ - Abelian group
- an involution – an element of Γ of order 2
- $\mathcal{G}$ – the set consisting of all Abelian groups which are of odd order or contain more than one involution

Γ-magic square

Definition

A Γ-magic square $\text{MS}_\Gamma(n)$ is an $n \times n$ array with entries from an Abelian group Γ of order n^2, each appearing once, with all its row, column and diagonal sums equal to a constant $\delta \in \Gamma$.

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Example: $\mathsf{MS}_{\mathbb{Z}_2^4}(4)$

(0, 0, 0, 0)	(0, 1, 0, 0)	(1, 0, 0, 0)	(1, 1, 0, 0)
(0, 0, 1, 0)	(0, 1, 1, 0)	(1, 0, 1, 0)	(1, 1, 1, 0)
(0, 0, 0, 1)	(0, 1, 0, 1)	(1, 0, 0, 1)	(1, 1, 0, 1)
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(0, 0, 1, 0)	(0, 1, 1, 0)	(1, 0, 1, 0)	(1, 1, 1, 0)
(0, 0, 0, 1)	(0, 1, 0, 1)	(1, 0, 0, 1)	(1, 1, 0, 1)
(0, 0, 1, 1)	(0, 1, 1, 1)	(1, 0, 1, 1)	(1, 1, 1, 1)

$$\delta = (0, 0, 0, 0)$$

Γ-magic rectangle

Definition

A Γ-magic rectangle $\mathrm{MR}_\Gamma(m, n)$ is an $m \times n$ array with entries from an Abelian group Γ of order mn, each appearing once, with all its row sums equal to a constant $\delta \in \Gamma$ and with all its column sums are equal to a constant $\eta \in \Gamma$.

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Example: $\mathrm{MR}_{\mathbb{Z}_4}(2, 2)$

0	1
3	2

Γ-magic rectangle

Definition

A Γ-magic rectangle MR$_\Gamma(m, n)$ is an $m \times n$ array with entries from an Abelian group Γ of order *mn*, each appearing once, with all its row sums equal to a constant $\delta \in \Gamma$ and with all its column sums are equal to a constant $\eta \in \Gamma$.

Example: MR$_{\mathbb{Z}_4}(2, 2)$

0	1
3	2

$$\delta = 1, \eta = 3$$

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Claim (Sun, Yihui, 1997)
Γ-magic squares $\mathsf{MS}_\Gamma(n)$ exist for all Abelian groups Γ of order n^2 for any $n > 2$.

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$\mathbb{Z}_n \times \mathbb{Z}_n$-magic squares $\mathrm{MS}_{\mathbb{Z}_n \times \mathbb{Z}_n}(n)$ exist for all odd $n > 1$.

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Theorem (SC, Hinc, 2021)
Let Γ be an Abelian group of order $\Gamma = mn$. There exists a non-trivial Γ-magic rectangle $\mathrm{MR}_\Gamma(m, n)$ if and only if $m > 1, n > 1$ and m and n are both even or $\Gamma \in \mathcal{G}$.

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A Kotzig array is a $j \times k$ grid, each row being a permutation of $\{0, 1, \ldots, k-1\}$ and each column having the same sum.

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4	3	2	1	0

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Lemma (Wallis, 2001)

A Kotzig array of size $j \times k$ exists whenever $j > 1$ and $j(k-1)$ is even.

Definition

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For an Abelian group Γ of order k we define a Γ-Kotzig array $\mathsf{KA}_\Gamma(j,k)$ of size $j \times k$ as a $j \times k$ grid, each row being a permutation of Γ and each column having the same sum.

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Lemma (SC, 2018)

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Lemma (SC, 2018)

A Γ-Kotzig array of size $j \times k$ exists whenever $j > 1$ and j is even or $\Gamma \in \mathcal{G}$.

Example: $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-Kotzig array 3×4

(0,0)	(1,0)	(1,1)	(0,1)
(1,0)	(0,1)	(1,1)	(0,0)
(0,1)	(0,0)	(1,1)	(1,0)

Γ-magic square

Theorem (SC, Froncek, 2025+)
Γ-magic squares $\mathsf{MS}_\Gamma(n)$ exist for all groups Γ of order n^2 for any $n > 2$.

Γ-magic square

Lemma (SC, Froncek, 2025+)
Let Γ be an Abelian group of order n^2. Let $\Gamma \cong \Gamma_0 \oplus H$ for some group $|\Gamma_0| = m^2$, $m > 1$ and $|H| = k^2$. If there exists a Γ_0-magic square $\mathrm{MS}_{\Gamma_0}(m)$ with the magic sum δ and an H-Kotzig array of size $m \times k^2$, then there exists a Γ-magic square $\mathrm{MS}_{\Gamma}(n)$ with the magic sum $(k\delta, 0)$.

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$\mathrm{MS}_{\mathbb{Z}_9}(3)$

8	1	6
3	5	7
4	9	2

$\mathrm{KA}_{(\mathbb{Z}_2)^2}(3,4)$

(0,0)	(1,0)	(1,1)	(0,1)
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Γ-magic square

STEP 1:

Using Kotzig array we build k^2 different $m \times m$ *H-residual squares* $R_H^s(m)$

KA$_{(\mathbb{Z}_2)^2}(3,4)$

(0,0)	(1,0)	(1,1)	(0,1)
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$R^1_{(\mathbb{Z}_2)^2}(3)$

(0,0)		
(0,0)		
(0,0)		

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$R^1_{(\mathbb{Z}_2)^2}(3)$

(0,0)	(0,0)	(0,0)
(0,0)	(0,0)	(0,0)
(0,0)	(0,0)	(0,0)

$R^2_{(\mathbb{Z}_2)^2}(3)$

(1,0)	(1,1)	(0,1)
(1,1)	(0,1)	(1,0)
(0,1)	(1,0)	(1,1)

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$R^1_{(\mathbb{Z}_2)^2}(3)$

(0,0)	(0,0)	(0,0)
(0,0)	(0,0)	(0,0)
(0,0)	(0,0)	(0,0)

$R^2_{(\mathbb{Z}_2)^2}(3)$

(1,0)	(1,1)	(0,1)
(1,1)	(0,1)	(1,0)
(0,1)	(1,0)	(1,1)

$R^3_{(\mathbb{Z}_2)^2}(3)$

(1,1)	(0,1)	(1,0)
(0,1)	(1,0)	(1,1)
(1,0)	(1,1)	(0,1)

$R^4_{(\mathbb{Z}_2)^2}(3)$

(0,1)	(1,0)	(1,1)
(1,0)	(1,1)	(0,1)
(1,1)	(0,1)	(1,0)

STEP 2:

„Glue" each *H*-residual square $R_H^s(m)$ with the magic square $\mathrm{MS}_{\Gamma_0}(m)$

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(1,1)	(0,1)	(1,0)
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(1,0)	(1,1)	(0,1)
(1,1)	(0,1)	(1,0)

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STEP 2:

„Glue” each H-residual square $R_H^s(m)$ with the magic square $\mathrm{MS}_{\Gamma_0}(m)$

$\mathrm{MS}_{\mathbb{Z}_9}(3)$

8	1	6
3	5	7
4	9	2

$S^1_{\mathbb{Z}_3\oplus(\mathbb{Z}_2)^2}(3)$

(8,0,0)	(1,0,0)	(6,0,0)
(3,0,0)	(5,0,0)	(7,0,0)
(4,0,0)	(9,0,0)	(2,0,0)

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(8,1,0)	(1,1,1)	(6,0,1)
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(3,1,0)	(5,1,1)	(7,0,1)
(4,1,1)	(9,0,1)	(2,1,0)

STEP 3:

Permute rows in *k* squares

$S^1_{\mathbb{Z}_3\oplus(\mathbb{Z}_2)^2}(3)$

(8,0,0)	(1,0,0)	(6,0,0)
(3,0,0)	(5,0,0)	(7,0,0)
(4,0,0)	(9,0,0)	(2,0,0)

$S^2_{\mathbb{Z}_3\oplus(\mathbb{Z}_2)^2}(3)$

(8,1,0)	(1,1,1)	(6,0,1)
(3,1,1)	(5,0,1)	(7,1,0)
(4,0,1)	(9,1,0)	(2,1,1)

$S^3_{\mathbb{Z}_3\oplus(\mathbb{Z}_2)^2}(3)$

(4,1,0)	(9,1,1)	(2,0,1)
(3,0,1)	(5,1,0)	(7,1,1)
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(4,1,1)	(9,0,1)	(2,1,0)
(3,1,0)	(5,1,1)	(7,0,1)
(8,0,1)	(1,1,0)	(6,1,1)

STEP 4:

Glue squares **properly**

$\mathsf{MS}^{1}_{\mathbb{Z}_9\oplus(\mathbb{Z}_2)^2}(6)$

(8,0,0)	(1,0,0)	(6,0,0)	(4,1,0)	(9,1,1)	(2,0,1)
(3,0,0)	(5,0,0)	(7,0,0)	(3,0,1)	(5,1,0)	(7,1,1)
(4,0,0)	(9,0,0)	(2,0,0)	(8,1,1)	(1,0,1)	(6,1,0)
(4,1,1)	(9,0,1)	(2,1,0)	(8,1,0)	(1,1,1)	(6,0,1)
(3,1,0)	(5,1,1)	(7,0,1)	(3,1,1)	(5,0,1)	(7,1,0)
(8,0,1)	(1,1,0)	(6,1,1)	(4,0,1)	(9,1,0)	(2,1,1)

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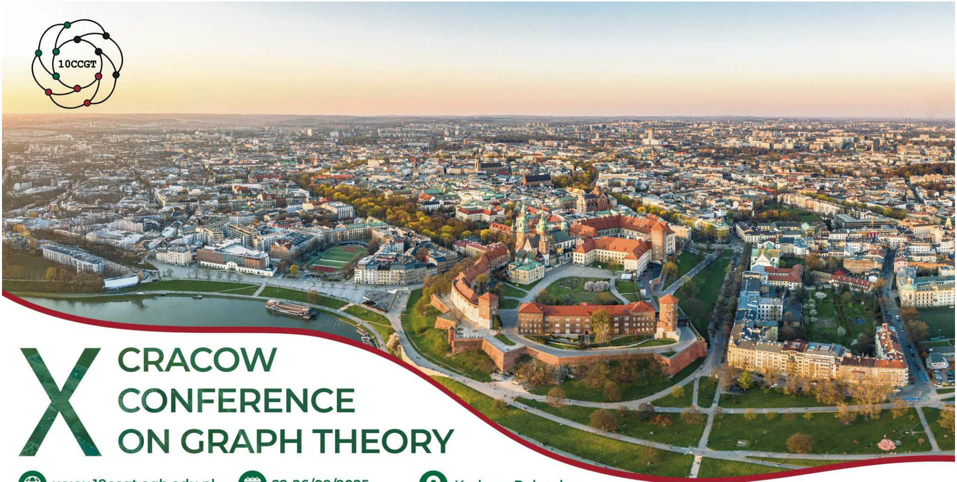
Thank you

Þakka þér fyrir

Dziękuję



X CRACOW CONFERENCE ON GRAPH THEORY

www.10ccgt.agh.edu.pl | 22-26/09/2025 | Krakow, Poland

Keynote speakers:

Noga Alon	USA
Maria Axenovich	Germany
Zdeněk Dvořák	Czech Republic
Bojan Mohar	Canada
Xuding Zhu	China

Thematic sessions | Invited speakers:

1. Algebraic Graph Theory	Robert Jajcay (Slovakia)
2. Algorithmic Graph Theory	Paweł Rzążewski (Poland)
3. Design Theory	Anita Pasotti (Italy)
4. Domination Graph Theory	Michael Henning (RSA)
5. Extremal Graph Theory	Andrzej Grzesik (Poland)
6. Graphs Colouring	Jarosław Grytczuk (Poland)
7. Graph Product	Iztok Peterin (Slovenia)
8. Labelings of Graphs	Rinovia Simanjuntak (Indonesia)
9. Probabilistic Methods	Paweł Prałat (Canada)
10. General	Ingo Schiermeyer (Poland)

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