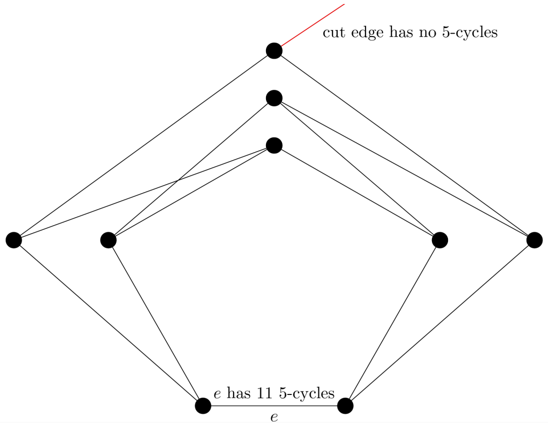
Maximizing Subgraph Counts on Regular Graphs

Arturo Ortiz San Miguel

Advisor: Gabor Lippner

NORCOM 2025

Example: Maximizing 5-cycles, $d = 3$



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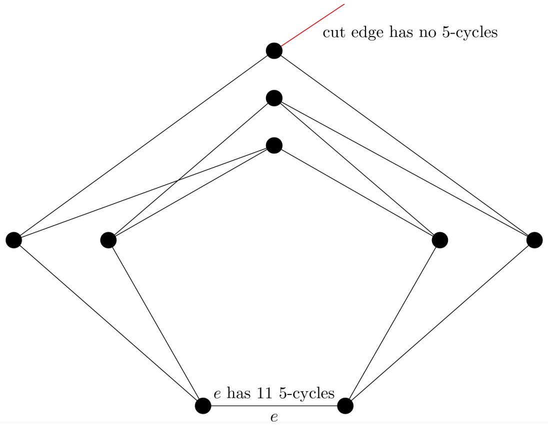

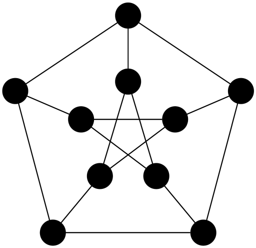

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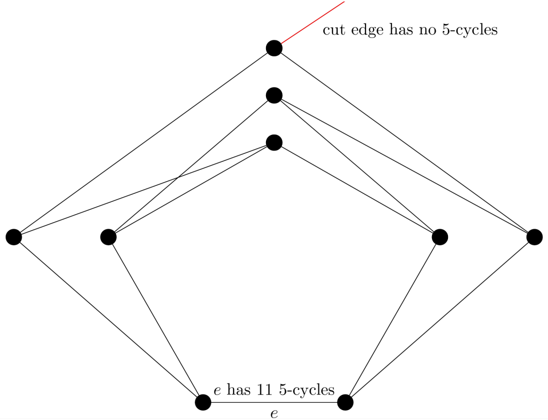

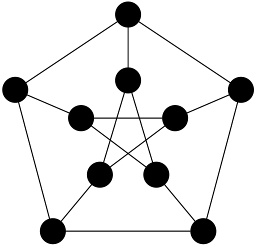

The same method on K_3, C_4 gives copies of $K_{d+1}, K_{d,d}$.

A Spectral Approach

$$\# \text{ of closed walks of length } k \text{ of } G = \operatorname{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k$$

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For odd k, maximized at spectrum of copies of K_{d+1}.

For even k, maximized at copies of $K_{d,d}$.

Graph Homomorphism Numbers

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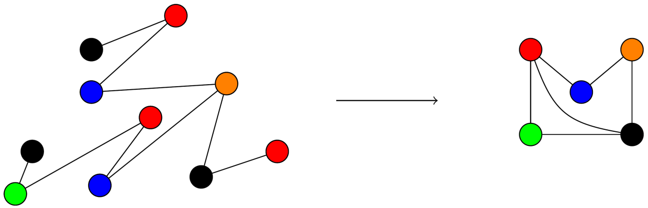
Graph Quotient: H a graph, P a partition of $V(H)$. Then, H/P is



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Theorem (Lovasz)

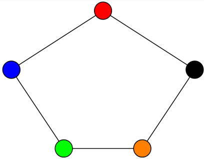
$$\hom(H, G) = \sum_P \mathrm{inj}(H/P, G)$$

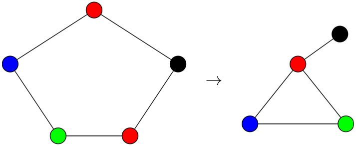
$$\mathrm{inj}(H, G) = \sum_P \mu_P \cdot \hom(H/P, G)$$

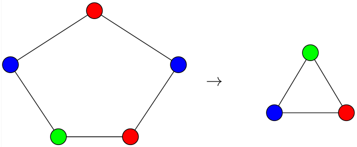
$$\mu_P = (-1)^{v(G)-|P|} \prod_{S \in P} (|S| - 1)$$

Example Computation

$$\mathrm{inj}(C_5, G) = \mathrm{hom}(C_5, G) - 5 \cdot \mathrm{hom}(K_3 + e) + 5 \cdot \mathrm{hom}(K_3, G)$$







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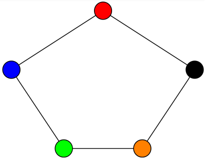

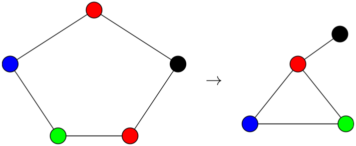

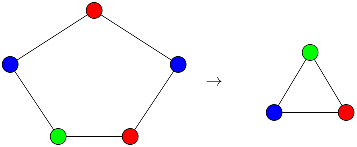

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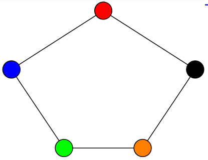
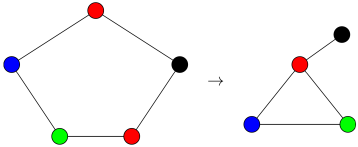
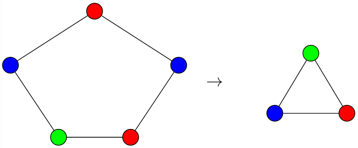
$$= \sum_{i=1}^{n} \lambda_i^5 + (5 - 5d)\lambda_i^3$$







$$\mathrm{hom}(K_3 + e, G) = d \cdot \mathrm{hom}(K_3, G)$$

Optimization Lemma for Odd k

Lemma

$p(x) = c_0(d) + c_1(d)x + ... + c_{2k-1}(d)x^{2k-1} + x^{2k+1}$, where the $c_i(d)$ are degree at most $d - i - 1$ polynomials in d. For sufficiently large d and $n = c(d+1)$,

$$\max \sum_{i=1}^{n} p(x), \quad \textit{subject to} \quad \sum_{i=1}^{n} \lambda_i = 0, \sum_{i=1}^{n} \lambda_i^2 = nd, \lambda_{\max} = d, |\lambda_i| \leq d$$

is solved uniquely by the spectrum of c copies of K_{d+1},

$$x_1 = ... = x_c = d, x_{c+1} = ... = x_n = -1,$$

.

5-cycles

$$\sum_{i=1}^{n} \lambda_i^5 + (5 - 5d)\lambda_i^3$$

Theorem (Lippner, O.)

For $d \geq 7$, the d-regular graph on $n = c(d+1)$ vertices with the maximal number of 5-cycles is c copies of K_{d+1}.
For $d = 3$ and $n = 10c$, the optimal graph is a collection of Petersen graphs.

Counting 5-cycles

Corollary (Lippner, O.)

Given a graph G with adjacency matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, the number of 5-cycles in G is

$$\frac{1}{10}\left(\left[\sum_{i=1}^{n} \lambda_i^5 + 5\lambda_i^3\right] - 5 \cdot \mathrm{tr}\left(\mathrm{diag}\left(A^3\right) D\right)\right).$$

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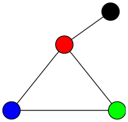
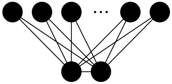


Using Homomorpshim Number Inequalities

$$\bigwedge^{m} K_3 =$$ 

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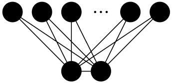
$$\bigwedge^{m} K_3 =$$ 

Lemma

$$\hom\left(\bigwedge^{m} K_3\right) \leq (d-1)\hom\left(\bigwedge^{m-1} K_3\right).$$

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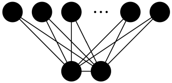

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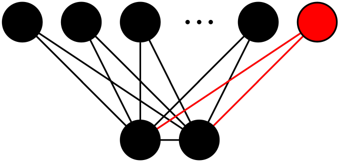


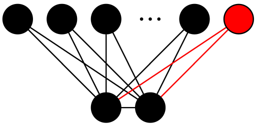
Maximizing $\bigwedge^m K_3$

Lemma

$$\hom\left(\bigwedge^m K_3\right) \leq (d-1)\hom\left(\bigwedge^{m-1} K_3\right).$$

$$\operatorname{inj}\left(\bigwedge^m K_3\right) = \hom\left(\bigwedge^m K_3\right) - \binom{m}{2}\hom\left(\bigwedge^{m-1} K_3\right) + \left[\binom{m}{2}\binom{m-2}{2} + \binom{m}{3}\right]\hom\left(\bigwedge^{m-2} K_3\right) - \dots$$

$$\leq p_m(d)\hom(K_3),$$



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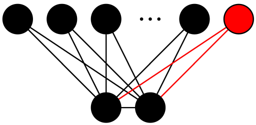


Theorem (Lippner, O.)

For any $m \geq 1$ and sufficiently large d, the d-regular graph on $n = c(d+1)$ vertices with the most $\bigwedge^m K_3$ subgraphs is c copies of K_{d+1}

Maximizing Bipartite Subgraph Count

Theorem (Lippner, O.)

Let H be a finite connected bipartite graph. Then, for sufficiently large d, the d-regular graph on $n = 2cd$ vertices with the highest H subgraph count is c copies of $K_{d,d}$.

Optimization Lemma for Even k

Lemma

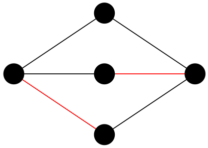
Let $p(x) = x^{2k} + c_1 x^{2k-1} + \ldots$ be a degree $2k$ monic polynomial where the c_i are degree $i-1$ polynomials in d. Then, for $n = 2cd$ and d sufficiently large,

$$\max \sum_{i=1}^{n} p(x_i) \quad \textit{subject to the same constraints}$$

is solved uniquely by the spectrum of c copies of $K_{d,d}$,

$$x_1 = \ldots = x_c = d, x_{c+1} = \ldots = x_{2c} = -d, x_{2c+1} = \ldots = x_n = 0.$$

Proof that $K_{d,d}$ is Optimal



$$\begin{aligned}\mathrm{inj}(H) &= \hom(H) - \sum_P \mathrm{inj}(H/P) \leq \hom(H) - \sum_B \mathrm{inj}(H/B) \\ &= \hom(H) - \sum_B \hom(H/B) + \sum_{B,Q} \mathrm{inj}((H/B)/Q)\end{aligned}$$

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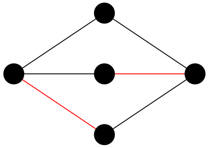


$$\mathrm{inj}(H) = \mathrm{hom}(H) - \sum_P \mathrm{inj}(H/P) \leq \mathrm{hom}(H) - \sum_B \mathrm{inj}(H/B)$$

$$= \mathrm{hom}(H) - \sum_B \mathrm{hom}(H/B) + \sum_{B,Q} \mathrm{inj}((H/B)/Q)$$

$$\leq nd^{|H|-1} + \sum_B \left(\sum_{i=1}^{m_B} \mathrm{hom}(T \cup e_i) - (m_B - 1)nd^{|H/B|-1} \right)$$

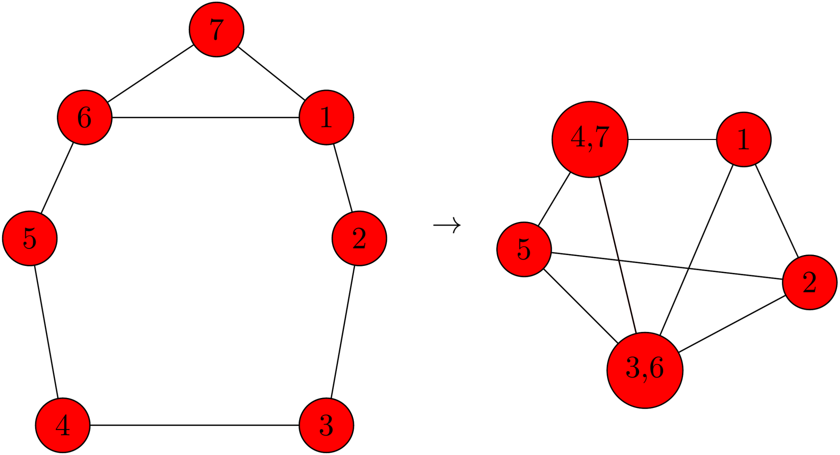
$$+ \sum_{B,Q} \mathrm{inj}((H/B)/Q)$$

Odd Cycles

Theorem (Lippner, O.)

Given odd k, for sufficiently large d, the d-regular graph on $n = c(d+1)$ vertices with the most k cycles is c copies of K_{d+1}.

Non-backtracking Homomorphisms



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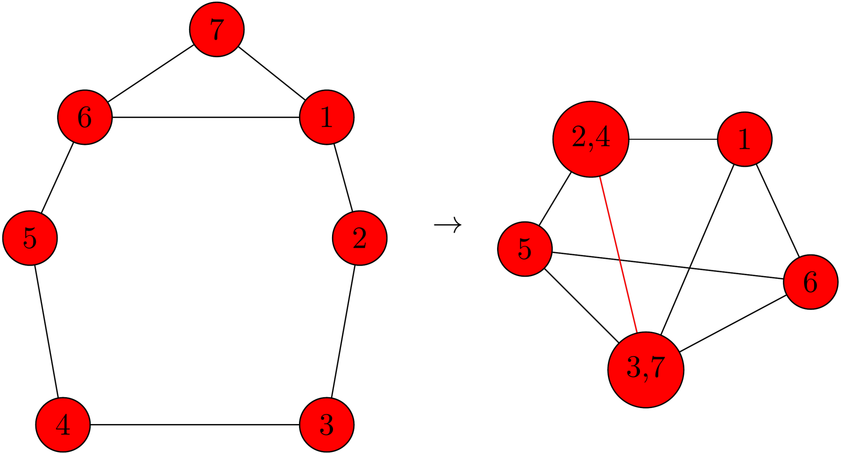


Non-backtracking Homomorphism Numbers

Theorem (Lippner, O.)

$$\mathrm{nob}(H, G) = \sum_{Q} \mathrm{inj}(H/Q, G), \quad \textit{no common neighbors}$$

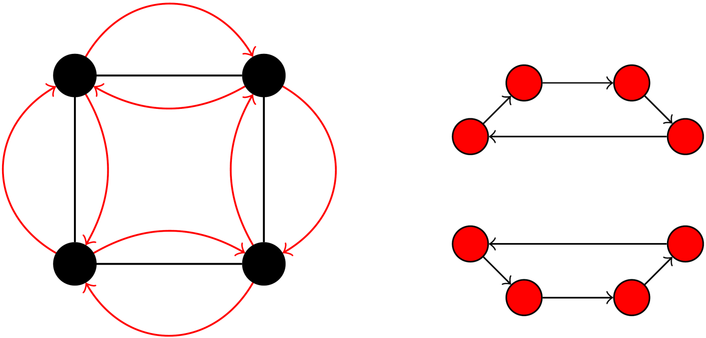
$$\mathrm{hom}(H, G) = \mathrm{nob}(H, G) + \mathrm{bac}(H, G)$$

$$\mathrm{bac}(H, G) = \sum_{S} \mathrm{inj}(H/S, G), \quad \textit{some part with common neighbors.}$$

Lemma

$$\mathrm{nob}(C_k, G) = (\#\ \textit{closed non-backtracking walks of length } k \textit{ in } G).$$

Non-backtracking Spectrum



$$B_{(u,v),(x,y)} = \begin{cases} 1, & \text{if } v = x, u \neq y \\ 0, & \text{otherwise} \end{cases}$$

Non-backtracking Spectrum

Theorem (Glover, Kempton 2021)

Let G be a d-regular graph. Then, the eigenvalues of its non-backtracking matrix B are

$$\pm 1, \quad \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4(d-1)}}{2},$$

where λ_i are the eigenvalues of A and ± 1 each have multiplicity $|E| - n$.

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$$\max_{\lambda} \sum_{i=1}^{n} \left(\frac{\lambda_i + \sqrt{\lambda_i^2 - 4(d-1)}}{2} \right)^k + \left(\frac{\lambda_i - \sqrt{\lambda_i^2 - 4(d-1)}}{2} \right)^k$$

$$= \max_{\lambda} \sum_{j=1}^{n} \sum_{i=1}^{\lfloor k/2 \rfloor} \binom{k}{2i} 4 \left(\frac{\lambda_j}{2} \right)^{k-2i} \frac{(\lambda_j^2 - 4(d-1))^i}{2^{2i}}$$

Maximizing Non-backtracking

Theorem (Lippner, O.)

For odd k, sufficiently large d, and $n = c(d+1)$, the d-regular graph on n vertices with the most non-backtracking closed walks of length k is c copies of K_{d+1}.

For even k, sufficiently large d and $n = 2cd$, the d-regular graph on n vertices with the most non-backtracking closed walks of length k is c copies of $K_{d,d}$.

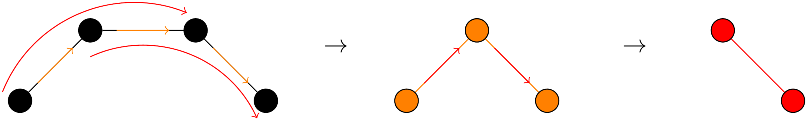
Future Work

Prove for non-bipartite case (done for a few families)

How big does d have to be?

Expected structure of a near-optimal graph

Non-triangulating and non-ℓ-tracking homomorphisms and spectra



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