Product representation of perfect cubes

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Product representation of perfect powers

Definition

$F_{k,d}(n)$: maximal size of a set $A \subseteq [n]$ such that

$$a_1 a_2 \dots a_k = x^d, \ a_1 < a_2 < \dots < a_k$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x.

Product representation of perfect squares

Erdős, Sárközy, T. Sós, 1995

- $F_{2,2}(n) = \left(\frac{6}{\pi^2} + o(1)\right) n$
- $\frac{n^{3/4}}{(\log n)^{3/2}} \ll F_{4,2}(n) - \pi(n) \ll \frac{n^{3/4}}{(\log n)^{3/2}}$
- $\frac{n^{2/3}}{(\log n)^{4/3}} \ll F_{6,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right) \ll n^{7/9} \log n$
- $\frac{n^{\frac{2\ell}{4\ell-1}}}{(\log n)^{\frac{4\ell}{4\ell-1}}} \ll F_{4\ell,2}(n) - \pi(n) \ll \frac{n^{3/4}}{(\log n)^{3/2}}$
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Győri, 1997

$F_{6,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right) \ll cn^{2/3} \log n$

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$$\frac{n^{2/3}}{(\log n)^{4/3}} \ll F_{6,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right) \ll n^{7/9} \log n$$

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P., 2015

$$F_{6,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right) \ll cn^{2/3} \log n / \log\log n$$

Product representation of perfect squares

P., 2015

$$\frac{n^{3/5}}{(\log n)^{6/5}} \ll F_{8,2}(n) - \pi(n)$$

P., Vizer, 2023

- $\frac{n^{3/5}}{(\log n)^{6/5}} \ll F_{10,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right)$
- $\frac{n^{6/11}}{(\log n)^{12/11}} \ll F_{22,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right)$
- $\frac{n^{\frac{3\ell}{6\ell-2}}}{(\log n)^{\frac{3\ell}{3\ell-1}}} \ll F_{4\ell,2}(n) - \pi(n)$
- $\frac{n^{\frac{3\ell}{6\ell-1}}}{(\log n)^{\frac{6\ell}{6\ell-1}}} \ll F_{8\ell+2,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right)$
- $\frac{n^{\frac{6\ell-1}{12\ell-4}}}{(\log n)^{\frac{6\ell-1}{6\ell-2}}} \ll F_{8\ell+6,2}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right)$

Odd and even cases

Generalized multiplicative Sidon sets

For $k = 2K$:

$$a_1 a_2 \dots a_K = b_1 b_2 \dots b_K \implies a_1 a_2 \dots a_K b_1 b_2 \dots b_K = x^2$$

Erdős, Sárközy, T. Sós, 1995

- $\frac{n}{(\log n)^{1+\varepsilon}} \ll n - F_{3,2}(n) \ll n(\log n)^{\frac{e \log 2}{2} - 1 + \varepsilon}$
- $\liminf_{n \to \infty} \frac{F_{2\ell+1,2}(n)}{n} \geq \log 2 = 0.69\dots$
- $\frac{n}{(\log n)^2} \ll n - F_{2\ell+1,2}(n)$

Question

$F_{5,2}(n) = (1 - o(1))n$ or $F_{5,2}(n) < (1 - c)n$

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Tao, 2025

For $k \geq 4$ we have $F_{k,2}(n) \leq (1 - c + o(1))n$.

Granville, Soundararajan, 2001

For $\ell \geq 2$ we have $(1 - c_0 + o(1))n \leq F_{2\ell+1,2}(n)$, where $c_0 = 1 - \log(1 + \sqrt{e}) + 2\int_1^{\sqrt{e}} \frac{\log t}{t+1} dt \approx 0.1715$ is the Hall-Montgomery constant.

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Product representation of perfect powers

Conjecture (Verstraëte, 2006)

Let $f \in \mathbb{Z}[x]$ and let k be a positive integer. Then, for some constant $\rho = \rho(k, f)$ depending only on k and f, the maximal size of a set $A \subseteq [n]$ such that no product of k distinct elements of A is in the value set of f is either $(\rho + o(1))n$ or $(\rho + o(1))\pi(n)$ as $n \to \infty$.

Product representation of perfect powers

$F_{k,d}(n)$: maximal size of a set $A \subseteq [n]$ such that

$$a_1 a_2 \dots a_k = x^d,\ a_1 < a_2 < \dots < a_k$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x.

$f_{k,d}(n)$: maximal size of a set $A \subseteq [n]$ such that

$$a_1 a_2 \dots a_k = x^d$$

has no solution with $a_1, a_2, \dots, a_k \in A$ and integer x except trivial solutions where the multiset $\{a_1, a_2, \dots, a_k\}$ can be partitioned into d-tuples where each d-tuple consists of d copies of the same number. (E.g. $a^3 b^3 c^3 = x^3$ is a trivial solution, but $a^5 b^2 c^2 = x^3$ is not.) Note that $f_{k,d}(n) \leq F_{k,d}(n)$.

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Our results

- $c_1 n^{2/3} < n - F_{2,3}(n) \le n - f_{2,3}(n) < c_2 n^{2/3}$
- $f_{3,3}(n) = (c_{3,3} + o(1))n$, where $0.6224 \le c_{3,3} \le 0.6420$
- $F_{3,3}(n) = (C_{3,3} + o(1))n$, where $0.6919 \le C_{3,3} \le 0.7136$
- $\frac{n}{(\log n)^{2+\varepsilon}} < n - F_{4,3}(n) \le n - f_{4,3}(n) < \frac{n}{(\log n)^{1-\frac{e \log 3}{2\sqrt{3}}-\varepsilon}}$
- For every $d \ge 2$, $d \nmid k$, we have $\frac{n}{(\log n)^d} \ll n - F_{k,d}(n)$.
- $c_1 \frac{n^{3/4}}{(\log n)^{3/2}} < f_{6,3}(n) - \pi(n) < c_2 \frac{n^{3/4}}{(\log n)^{3/2}}$
- $F_{6,3}(n) = (1 + o(1)) \frac{n \log\log n}{\log n}$
- $\frac{n^{2/3}}{(\log n)^{4/3}} \ll f_{9,3}(n) - \pi(n) \ll n^{2/3} \log n$
- $\frac{n^{5/6}}{(\log n)^{5/3}} < F_{9,3}(n) - \left(\pi(n) + \pi\left(\frac{n}{2}\right)\right) \ll n^{5/6}$

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General k, d

k and d	growth rate of $F_{k,d}(n)$
$d \mid k,\ d < k$	$o(n)$
$k < d$	$(1 - o(1))n$
$k = d$	$(c_d + o(1))n$ with $c_d \in (0,1)$
$k = d + 1$	$(1 - o(1))n$
$k \geq d + 2$	$< (1 - c)n$ with $c > 0$

Table: Summary of results of Tao and FJKPS.

Graph theory

$F_{4,2}(n) \le G_2(n) =$
size of the largest multiplicative Sidon subset of $[n]$

Erdős (1934): $\pi(n) + \frac{c_1 n^{3/4}}{(\log n)^{3/2}} \le G_2(n) \le \pi(n) + c_2 n^{3/4}$

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Lower bound: G : C_4-free graph on $\{$primes $\le \sqrt{n}\}$

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Observe that the graph G with vertex set $[n^{2/3}] \cup \{\text{primes} \le n\}$ and edge set $E = \{u_a v_a : \ a \in A\}$ is C_4-free.

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Theorem

$F_{6,3}(n) = (1 + o(1))\frac{n \log\log n}{\log n}$

Construction:

$$A = \left\{ m : \ m = pq, \ \frac{n}{\log n} < m \leq n, \ p, q \text{ primes}, \ p < \frac{q}{\log n} \right\}.$$

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$h(m) :=$ # of triples (d_i, d_j, d_k) such that $m = d_i d_j d_k$, $1 \le i < j < k \le t$.

If $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \in M$, then

$$\Omega(m) = \Omega(d_i) + \Omega(d_j) + \Omega(d_k) \le \log\log n,$$

hence

$$h(m) \le \prod_{i=1}^{r} \binom{k_i + 2}{2} \le \prod_{i=1}^{r} 3^{k_i} \le 3^{\log\log n} = (\log n)^{\log 3}.$$

If H contains a K_4^3, then for some $d_{i_1} < d_{i_2} < d_{i_3} < d_{i_4}$ and

$$a_1 = d_{i_1} d_{i_2} d_{i_3}, \quad a_2 = d_{i_1} d_{i_2} d_{i_4}, \quad a_3 = d_{i_1} d_{i_3} d_{i_4}, \quad a_4 = d_{i_2} d_{i_3} d_{i_4}$$

we have $a_1 a_2 a_3 a_4 = (d_{i_1} d_{i_2} d_{i_3} d_{i_4})^3$.
Therefore, H does not contain any K_4^3.

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Proofs

By a result of de Caen $\exists \delta > 0$ such that there at least δt^3 triples (i, j, k) such that $\{P_i, P_j, P_k\}$ is not an edge in H.

Let $h = \max_{m \in M} h(m) \leq (\log n)^{\log 3}$.

Therefore,

$$|M| \geq \frac{\delta t^3}{h} \gg \frac{n}{(\log n)^{3+\log\frac{1}{3}-1+\varepsilon} \cdot (\log n)^{\log 3}} = \frac{n}{(\log n)^{2+\varepsilon}},$$

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Proofs

Theorem

$$\frac{n}{(\log n)^{2+\varepsilon}} < n - F_{4,3}(n) \le n - f_{4,3}(n) < \frac{n}{(\log n)^{1-\frac{e\log 3}{2\sqrt{3}}-\varepsilon}}$$

Proof (upper bound):

Construction: A is the set of the integers a such that

(i) $\frac{n}{\log n} \le a \le n$,

(ii) $d^2 \mid a$ implies $d \le \log n$, and

(iii) a cannot be written in the form $a = uvw$ with integers u, v, w such that $\frac{\sqrt[3]{n}}{(\log n)^{16}} \le u, v, w \le \sqrt[3]{n}(\log n)^{16}$.

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Questions

Let $1 < k < d$. Is it true that

$$n^{k/d} \ll n - F_{k,d}(n) \le n - f_{k,d}(n) \ll n^{k/d}?$$

Is it true that there exists a constant c such that

$$f_{2,3}(n) = n - (c + o(1))n^{2/3}?$$

Is it true that for any $d \geq 4$ and $k > d$, $d \mid k$, there exist constants $c_{k,d} > 0$ and $C_{k,d} \in \mathbb{Z}^+$ such that

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