Existentially Closed Hypergraphs

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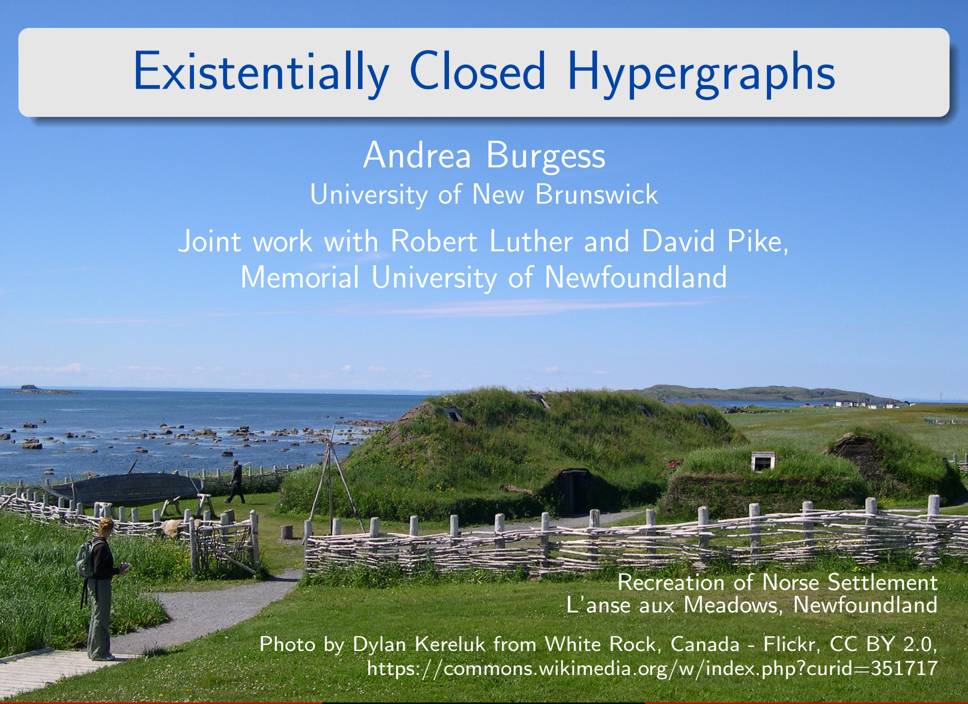
Joint work with Robert Luther and David Pike,
Memorial University of Newfoundland



Recreation of Norse Settlement
L'anse aux Meadows, Newfoundland

Existential closure in graphs

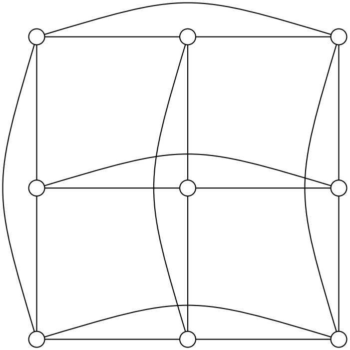
Definition

For a positive integer n, a graph G is n-existentially closed, briefly n-e.c., if for every set S of n vertices and every subset $T \subseteq S$, there is a vertex $x \notin S$ such that:

1. x is adjacent to each vertex in T
2. x is not adjacent to any vertex in $S \setminus T$.

Informally, for every set S of n vertices, there are 2^n vertices joined to S in all possible ways.

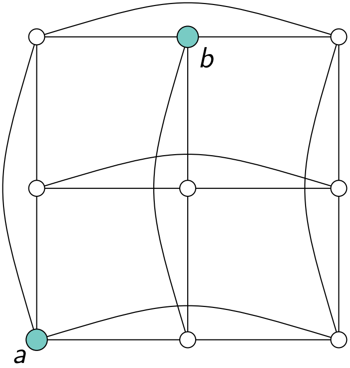
Example: $K_3 \Box K_3$ is 2-e.c.



$S = \{a, b\}$:

- Adjacent to a but not b
- Adjacent to b but not a
- Adjacent to both a and b
- Adjacent to neither a nor b

Example: $K_3 \square K_3$ is 2-e.c.



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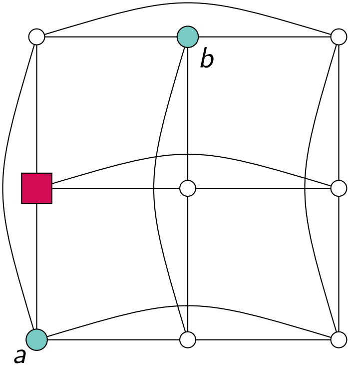

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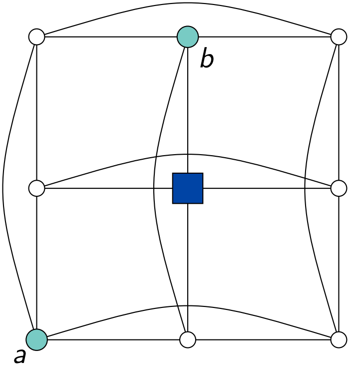

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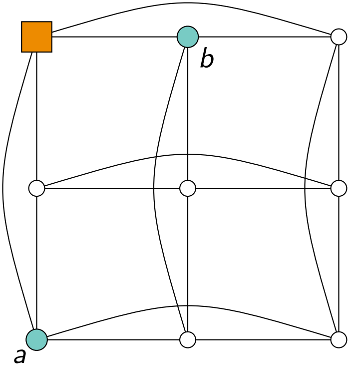

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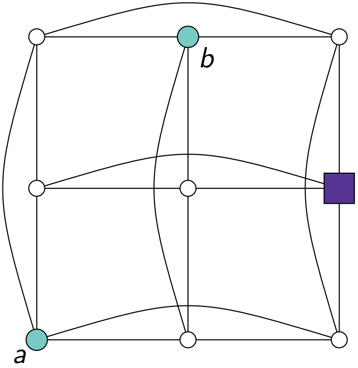

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Properties of n-e.c. graphs

Theorem (See Bonato (2009))

If G is an n-e.c. graph, then:

1. *G is m-e.c. for all $1 \leq m \leq n-1$.*
2. *G has order at least $n + 2^n$ and at least $n \cdot 2^{n-1}$ edges.*
3. *G^c is n-e.c.*
4. *Each graph of order at most $n+1$ embeds in G.*
5. *If $n > 1$, then for each vertex x of G, $G - x$, $G[N(x)]$ and $G[V(G) \setminus (N(x) \cup \{x\})]$ are $(n-1)$-e.c.*

Almost all graphs are n-e.c.

Theorem (Erdős and Rényi, 1963)

Let $n > 1$ be an integer and $p \in (0, 1)$ a real number. With probability 1 as $m \to \infty$, the Edrős-Rényi random graph $G(m, p)$ is n-e.c.

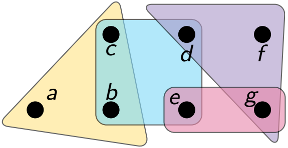
"Despite this fact, until recently only one explicit family of n-e.c. graphs was known . . . This paradoxical quality of n-e.c. graphs being both common and rare has intrigued many researchers with differing backgrounds such as graph theorists, logicians, design theorists, probabilists and geometers." (Bonato, 2009)

Explicit families of n-e.c. graphs

- Payley graphs of order greater than $n^2 \cdot 2^{n-2}$ (Blass, Exoo and Harary, 1981; Bollobás and Thomason, 1981)
- Other families of strongly regular graphs of prime power order (Cameron and Stark, 2002; Kisielewicz and Peisert, 2004)
- A family of graphs constructed from $0, 1$-matrices, with adjacencies defined by constraints. (Blass and Rossman, 2005)
- Block-intersection graphs of the following designs are 2-e.c.:
  - STS(v) with $v \geq 13$ (Forbes, Grannell and Griggs, 2005)
  - TTS(v) with $v \geq 13$ (McKay and Pike, 2007)
  - BIBD$(v, k, 1)$ with $v \geq k^2 + k + 1$ (McKay and Pike, 2007)
- Infinite families of 2-e.c. line graphs. (Burgess, Luther and Pike, 2024)

Hypergraphs

A hypergraph is a pair $(V, \mathcal{E})$, where V is a finite set of vertices and $\mathcal{E} \subseteq \mathscr{P}(V)$. Elements of $\mathcal{E}$ are hyperedges or edges.



$$V = \{a, b, c, d, e, f, g\}$$

$$\mathcal{E} = \{\{a, b, c\}, \{b, c, d, e\}, \{d, f, g\}, \{e, g\}\}$$

If all edges have size h, the hypergraph is h-uniform.

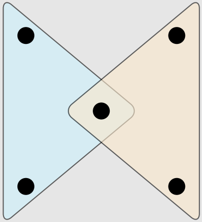
n-e.c. hypergraphs

Definition

An h-uniform hypergraph H is n-e.c. if, for any set S of n vertices and any subset $T \subseteq S$, there is an $(h-1)$-set $X \subseteq V(H) \setminus S$ such that:

1. for all $z \in T$, $X \cup \{z\}$ is an edge of H, and
2. for all $s \in S \setminus T$, $X \cup \{s\}$ is not an edge of H.

Example (A 1-e.c. 3-uniform hypergraph)



For each vertex x ($S = \{x\}$), there must be:

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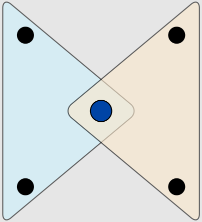

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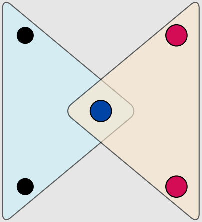

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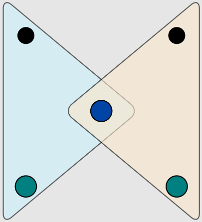

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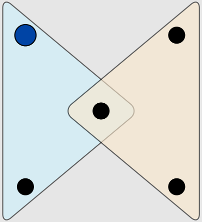

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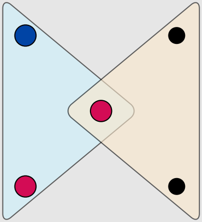

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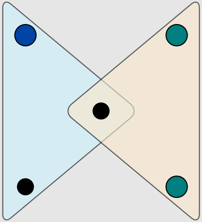

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Properties of n-e.c. hypergraphs

Theorem (Burgess, Luther and Pike, 2025)

If H is an n-e.c. h-uniform hypergraph, then:

1. *H is m-e.c. for all $1 \leq m \leq n$*
2. *H has at least $n \cdot 2^{n-1}$ edges and at least $n + \ell$ vertices, where ℓ is the smallest positive integer such that $\binom{\ell}{h-1} \geq 2^n$.*
3. *The h-uniform complement H^c of H is n-e.c.*
4. *For each vertex x, $H - x$ and $H[N(x)]$ are $(n-1)$-e.c.* [a]
5. *For each vertex x, $H[A(x)]$ is $(n-1)$-e.c., where $A(x)$ is the set of all vertices that occur together with x in at least one edge of H^c.* [b]

[a] For $Y \subseteq V(H)$, the edges of $H[Y]$ are those edges of H with all vertices in Y.
[b] If $h = 2$, then $A(x) = V(H) \setminus (N(x) \cup \{x\})$.

Random hypergraphs

Definition

The random h-uniform hypergraph $H_h(m, p)$ is the h-uniform hypergraph with m vertices in which each set of h vertices $E \subseteq V(H)$ is chosen to be an edge of H independently with probability p.

Theorem (Burgess, Luther and Pike, 2025)

Let $p \in (0, 1)$ be a real number, and let $n > 1$ and $h > 1$ be integers. With probability 1 as $m \to \infty$, $H_h(m, p)$ satisfies the n-e.c. property.

Latin squares

Definition

A Latin square of order n is an $n \times n$ array with entries from a set of size n such that each symbol occurs in each row and each column.

Two Latin squares are orthogonal if, when superimposed, the entries viewed as ordered pairs are all distinct.

A set of Latin squares in which any two are orthogonal is a set of mutually orthogonal Latin squares (MOLS).

1	2	3	4
2	1	4	3
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$(1,1)$	$(2,2)$	$(3,3)$	$(4,4)$
$(4,2)$	$(3,1)$	$(2,4)$	$(1,3)$
$(2,3)$	$(1,4)$	$(4,1)$	$(3,2)$
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Complete Sets of MOLS

It is well-known that:

- The maximum number of MOLS of order n is $n-1$.
- If n is a prime power, then there exists a set of $n-1$ MOLS of order n, called a complete sets of MOLS of order n.

Example (A complete set of MOLS of order 4)

1	2	3	4
2	1	4	3
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2-e.c. hypergraphs from complete sets of MOLS

Consider a set of h MOLS of order $h+1$.

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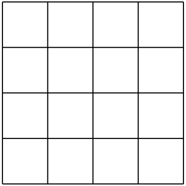

- Form an edge for each h-subset of cells in each row.
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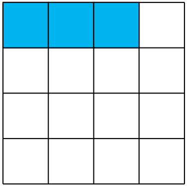

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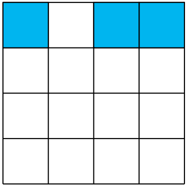

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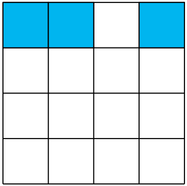

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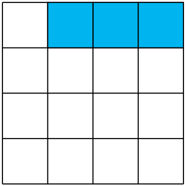

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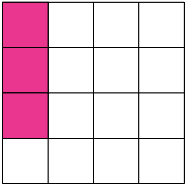

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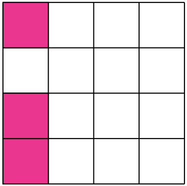

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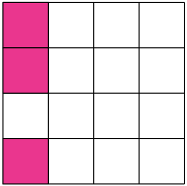

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Consider a set of h MOLS of order $h+1$.

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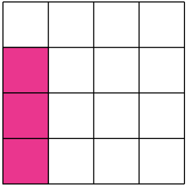

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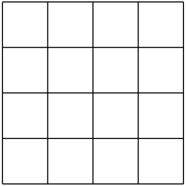

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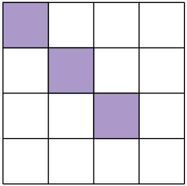

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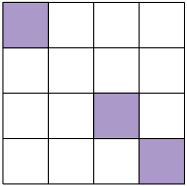

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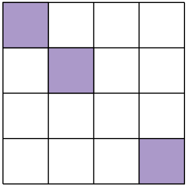

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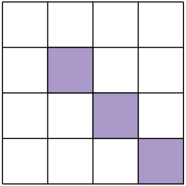

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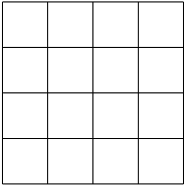

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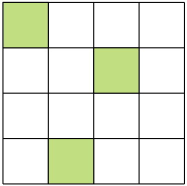

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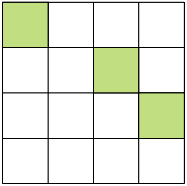

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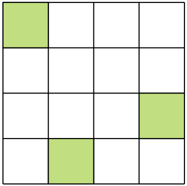

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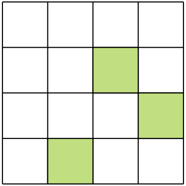

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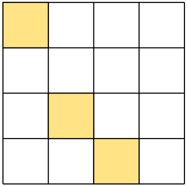

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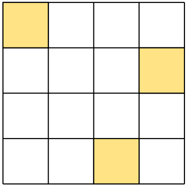

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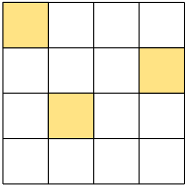

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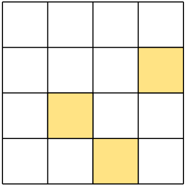

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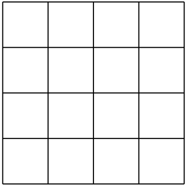

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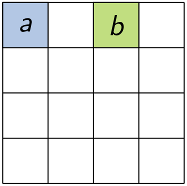

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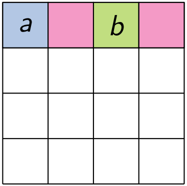

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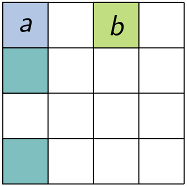

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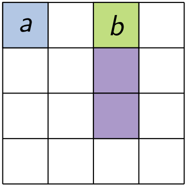

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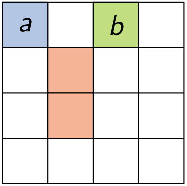

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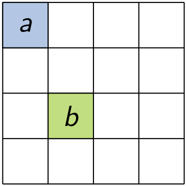

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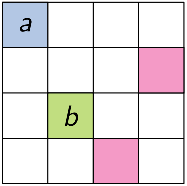

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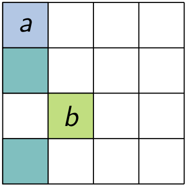

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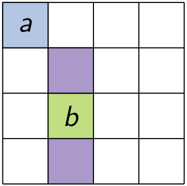
Theorem (Burgess, Luther and Pike, 2025)

Given a complete set of MOLS of order $h+1$, H_L is 2-e.c.



Given two cells a and b, we need to find sets C_1, C_2, C_3, C_4 of $h-1$ other cells such that:

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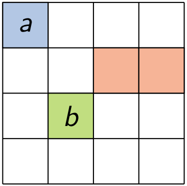

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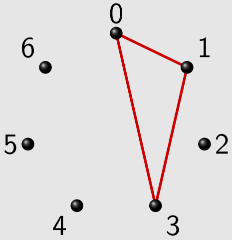
t-designs

Definition

Let $v \geq k \geq t \geq 2$ and $\lambda \geq 1$ be integers. A t-(v, k, λ) design is a pair $(V, \mathcal{B})$, where:

- V is a set of v points
- $\mathcal{B}$ is a collection of k-subsets of V with the property that each t-subset of V is contained in exactly λ blocks.

Example (A 2-$(7, 3, 1)$ design)



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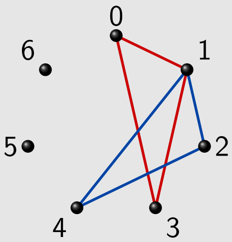

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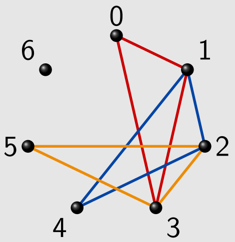

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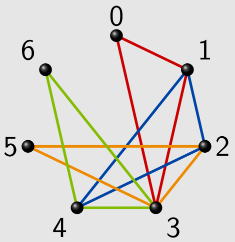

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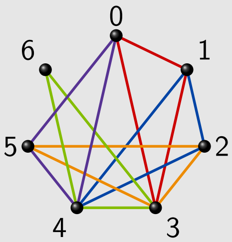

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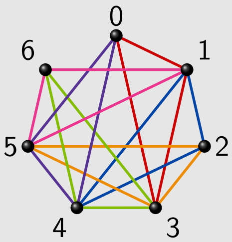

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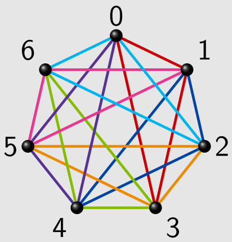

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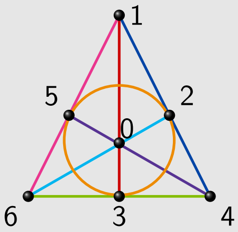

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Constructing h-uniform hypergraphs from t-designs

Let $\mathcal{D}$ be a t-$(v, k, 1)$ design, and let $2 \leq h \leq k$. Define an h-uniform hypergraph $H_{\mathcal{D},h}$:

- Vertices are the points of $\mathcal{D}$.
- Edges are h-subsets of the blocks of $\mathcal{D}$. (For each block B, we get $\binom{k}{h}$ edges.)

Remark

If $h = 2$, then $H_{\mathcal{D},h} = K_v$. If $h = k$, then $H_{\mathcal{D},h} = \mathcal{D}$.

Neither of these is 2-e.c., so we'll focus on the case $3 \leq h \leq k - 1$.

Example

$\mathcal{D}$	$\{0,1,3,9\}$	$\{1,2,4,10\}$	$\{2,3,5,11\}$	$\{3,4,6,12\}$	$\{4,5,7,0\}$
$H_{\mathcal{D},3}$	$\{0,1,3\}$	$\{1,2,4\}$	$\{2,3,5\}$	$\{3,4,6\}$	$\{4,5,7\}$
	$\{0,1,9\}$	$\{1,2,10\}$	$\{2,3,11\}$	$\{3,4,12\}$	$\{4,5,0\}$
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	$\{1,3,9\}$	$\{2,4,10\}$	$\{3,5,11\}$	$\{4,6,12\}$	$\{5,7,0\}$

$\mathcal{D}$	$\{5,6,8,1\}$	$\{6,7,9,2\}$	$\{7,8,10,3\}$	$\{8,9,11,4\}$	$\{9,10,12,5\}$
$H_{\mathcal{D},3}$	$\{5,6,8\}$	$\{6,7,9\}$	$\{7,8,10\}$	$\{8,9,11\}$	$\{9,10,12\}$
	$\{5,6,1\}$	$\{6,7,2\}$	$\{7,8,3\}$	$\{8,9,4\}$	$\{9,10,5\}$
	$\{5,8,1\}$	$\{6,9,2\}$	$\{7,10,3\}$	$\{8,11,4\}$	$\{9,12,5\}$
	$\{6,8,1\}$	$\{7,9,2\}$	$\{8,10,3\}$	$\{9,11,4\}$	$\{10,12,5\}$

$\mathcal{D}$	$\{10,11,0,6\}$	$\{11,12,1,7\}$	$\{12,0,2,8\}$
$H_{\mathcal{D},3}$	$\{10,11,0\}$	$\{11,12,1\}$	$\{12,0,2\}$
	$\{10,11,6\}$	$\{11,12,7\}$	$\{12,0,8\}$
	$\{10,0,6\}$	$\{11,1,7\}$	$\{12,2,8\}$
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Existentially closed hypergraphs from designs

Theorem (Burgess, Luther and Pike, 2025)

If $\mathcal{D}$ is a t-$(v, k, 1)$ design and $3 \leq h \leq k-1$, then $H_{\mathcal{D},h}$ is t-e.c.

Theorem (Wilson, 1975; Keevash, arxiv)

For fixed $k \geq t \geq 2$, there exists a t-$(v, k, 1)$ design for all sufficiently large admissible v.

Corollary (Burgess, Luther and Pike, 2025)

For any $t \geq 2$ and $h \geq 3$, there are infinitely many t-e.c. h-uniform hypergraphs.

Open questions

1. If a t-(v, k, λ) design with $k > t$ is n-e.c., then $n \leq \lambda$. Which designs (if any) viewed as hypergraphs are n-e.c. for $n \geq 2$?
2. Under what conditions are Paley hypergraphs (see Kocay, 1992; Potočnik and Šajna, 2009; Gosselin, 2010) existentially closed?
3. Is there an appropriate way to extend the concept of existential closure to non-uniform hypergraphs?

Thanks!