On Proofs of Generalized Knuth's Old Sum

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Introduction

For non-negative integers n, the Reed-Dawson identity is given by

$$\sum_{k=0}^{n} \left(-\frac{1}{2}\right)^{k} \binom{n}{k} \binom{2k}{k} = \begin{cases} \frac{1}{2^n}\binom{n}{n/2} & \text{for } n \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

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- Notation: For $n \in \mathbb{N}$, $x \in \mathbb{R}$, $(x)_n := x(x+1)\cdots(x+n-1)$ and $(x)_0 := 1$.

Introduction

Rathie-Kim-Paris

$$\sum_{k=0}^{2v}(-1)^k\binom{2v+i}{k+i}\binom{2k}{k}2^{-k}$$
$$=\pi\,(2v+1)_i\frac{2^{2i}i!}{(2i)!}\sum_{r=0}^{i}\frac{2^{-r}\binom{i}{r}\left(\frac{1}{2}+\frac{1}{2}(i-r)\right)_v}{(i-r)!\Gamma^2\left(\frac{1}{2}+\frac{1}{2}(r-i)\right)\left(1+\frac{1}{2}(i-r)\right)_v} \quad (1)$$

$$\sum_{k=0}^{2v+1}(-1)^k\binom{2v+1+i}{k+i}\binom{2k}{k}2^{-k}$$
$$=2\pi\,(2v+2)_i\frac{2^{2i}i!}{(2i)!}\sum_{r=0}^{i}\frac{2^{-r}\binom{i}{r}\left(1+\frac{1}{2}(i-r)\right)_v}{(i-r+1)!\Gamma^2\left(\frac{1}{2}(r-i)\right)\left(\frac{3}{2}+\frac{1}{2}(i-r)\right)_v}. \quad (2)$$

Introduction

Applying the basic properties of rising factorials, the Gamma function and the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

Rathie-Kim-Paris identities can be formulated as follows.

Main Results

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Identity 1

For all $m, n \in \mathbb{N}_0$,

$$\sum_{k=0}^{2n} (-1)^k \binom{4m+2}{2m+1} \binom{2n+2m+1}{k+2m+1} \binom{2k}{k} 2^{2n-k}$$
$$= \sum_{i=0}^{m} \binom{2n+2m+1}{2n} \binom{2m+1}{2i+1} \binom{2n+2m-2i}{n+m-i} 2^{2i+1}.$$

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Identity 2

For all $m, n \in \mathbb{N}_0$,

$$\sum_{k=0}^{2n}(-1)^k \binom{4m}{2m}\binom{2n+2m}{k+2m}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{m}\binom{2n+2m}{2n}\binom{2m}{2i}\binom{2n+2m-2i}{n+m-i}2^{2i}.$$

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Identity 3

For all $m, n \in \mathbb{N}_0$,

$$\sum_{k=0}^{2n+1} (-1)^k \binom{4m}{2m} \binom{2n+2m+1}{k+2m} \binom{2k}{k} 2^{2n-k}$$
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$$= \sum_{i=0}^{m} \binom{2n+2m+2}{2n+1} \binom{2m+1}{2i} \binom{2n+2m-2i+2}{n+m-i+1} 2^{2i}.$$

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Lemma 1

For a fixed $n \in \mathbb{N}$, let S_n be the set of all words in the alphabet $\{a, b, c, d\}$ of length n such that $\#a$'s $= \#$ b's. Then $|S_n| = \binom{2n}{n}$.

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Proof

Let T_n be a set of bit strings of length $2n$ such that $\#\ 0$'s $= \#\ 1$'s. Then, $|T_n| = \binom{2n}{n}$. Define a bijection f between S_n and T_n as follows. For w in S_n, read w from left to right and replace a by 00, b by 11, c by 01 and d by 10. Thus, $|S_n| = \binom{2n}{n}$.

$\sum_{k=0}^{2n}(-1)^k\binom{4m+2}{2m+1}\binom{2n+2m+1}{k+2m+1}\binom{2k}{k}2^{2n-k} = \sum_{i=0}^{m}\binom{2n+2m+1}{2n}\binom{2m+1}{2i+1}\binom{2n+2m-2i}{n+m-i}2^{2i+1}.$

For fixed $n, m \in \mathbb{N}_0$, consider a set S of words in the alphabet

$$\{a,\ b,\ c,\ d,\ C,\ D,\ ?,\ .,\ !,\ *\}$$

such that w is in S if and only if the following conditions all hold:

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Example: Let n $= 3$, m $= 1$.

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Example: Let n = 3, m = 1. Then the word $w = ab?!adbC.$ is in S.

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$$\sum_{w\in S} Wt(w) = \sum_{k=0}^{2n} L_k(w)\, Wt(w) = \sum_{k=0}^{2n} L_k(w)\,(-1)^k = \sum_{k=0}^{2n} U_{2n-k}(w)\,(-1)^k,$$

where $U_{2n-k}(w) = \#$ w's with $2n-k$ uppercase letters.

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w	$\sigma(w)$	$Wt(w)$	$Wt(\sigma(w))$
$ab?!adbC.$	$ab?!aDbC.$	-1	1

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To find $|T|$, we condition on the number of $!$'s and $*$'s in a word $w \in T$. Consequently, we get

$$|T| = \sum_{i=0}^{m}\binom{2n+2m+1}{2m+1}\binom{2m+1}{2i+1}\binom{2(n+m-i)}{n+m-i}2^{2i+1}.$$

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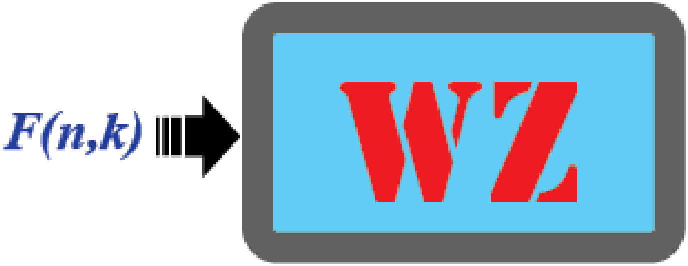
- Find recurrences for $L(n)$ and $R(n)$, and see whether they coincide, and check the initial conditions.
- The production of the recurrences is done by applying the Wilf-Zeilberger (WZ) Algorithm on $F(n,k)$ and $H(n,k)$.

Computerized Proof: The WZ Style



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$$Ope(F(n,k)) = G(n,k+1) - G(n,k)$$

Proof of Identity 3

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Identity 3

$$\sum_{k=0}^{2n+1}(-1)^k\binom{4m}{2m}\binom{2n+2m+1}{k+2m}\binom{2k}{k}2^{2n-k}$$
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Denote the summands of the left and right sides of the equation in Identity 3 by $F_1(n,m,k)$ and $F_2(n,m,i)$, respectively.

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Since $F_1(n,m,k)=0$ for $k>2n+1$ and $F_2(n,m,i)=0$ for $i\geq m$, the equation in Identity 3 is equivalent to

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Denote the left and right sides of Equation (3) by $S(n,m)$ and $T(n,m)$, respectively.

To show that $S(n,m) = T(n,m,)$ for all $m,n \in \mathbb{N}_0$, it suffices to show that both $S(n,m)$ and $T(n,m)$ satisfy the same recurrence relation with the same initial conditions.

Proof of Identity 3

Applying the WZ algorithm on $F_1(n,m,k)$ we get the WZ-equation

$$p_2(n,m)F_1(n+2,m,k)+p_1(n,m)F_1(n+1,m,k)+p_0(n,m)F_1(n,m,k) \\ = G_1(n,m,k+1)-G_1(n,m,k), \quad (4)$$

where

$$p_2(m,n) = 2n^2+9n+10,$$
$$p_1(n,m) = -(16n^2+(16m+56)n+8m^2+28m+50),$$
$$p_0(n,m) = 32n^2+(64m+80)n+32m^2+80m+48,$$
$$G_1(n,m,k) = (-1)^k 2^{2n+2-k}\binom{4m}{2m}\binom{2n+2m+4}{k+2m-1}\binom{2k}{k}R_1(n,m,k),$$
$$R_1(n,m,k) = \frac{k(k^2+(2m-4n-9)k+(-8m+2)n-18m+4)}{n+m+2}.$$

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By applying the above procedure to $F_2(n,m,k)$ and $T(m,n)$, we get the same result.

Problem:

Generalized Knuth's old sum

$$S(m,n,i) := \sum_{k=0}^{n} (-1)^k \binom{n+i}{k+i} \binom{mk}{k} m^{-k} = ?Nice(m,n,i).$$

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