Permutations,
Patterns, Posets, Processes,
Positivity

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Joint works with **Slim Kammoun** (Poitiers) and
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Positivity (Moments)

$$n! = \int_0^\infty x^n \underbrace{exp(-x)}_{\geq 0} dx$$

$$B_n = \sum_{k \geq 0} k^n \underbrace{\frac{e^{-1}}{k!}}_{\geq 0}$$

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$$m_n = \int_{\mathbb{R}} x^n \, d\mu(x).$$

Hamburger

Answer: (Hamburger, ~100 years ago)

μ exists if and only if the Hankel matrices

$$H_n = \begin{bmatrix} m_0 & m_1 & \dots & m_n \\ m_1 & m_2 & \dots & m_{n+1} \\ \vdots & & & \vdots \\ m_n & m_{n+1} & \dots & m_{2n} \end{bmatrix}$$

are positive semi-definite for all $n \geq 1$.

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If $\det H_1, \dots, \det H_k > 0$ and $\det H_n = 0$ for all $n > k$, then μ is supported on k elements.

If $\det H_n > 0$ for all n, μ exists (need not be unique).

Test your intuition

$$n! = \int_{[0,\infty)} x^n \, e^{-x} \, dx$$

$$a_n := \#\{\sigma \in S_n \mid \nexists i \in \{1, \dots, n-2\} \text{ s.t. } \sigma(i) < \sigma(i+1) < \sigma(i+2)\}$$

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YES 

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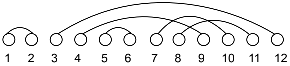
NO ☹

Probability 101: some fundamental laws

Notation: $[n] := \{1, \ldots, n\}$

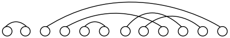
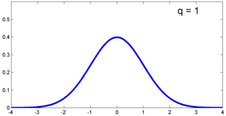
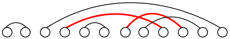
$$\int_{-\infty}^{\infty} x^{2n} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\ \text{perfect matchings on } [2n]$$



$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\ \text{non-crossing perfect matchings on } [2n]$$

$$\sum_{k \geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\ \text{set partitions on } [n]$$

$$\int_{0}^{\infty} x^n exp(-x)\, dx \quad = \quad \#\ \text{permutations on } [n]$$

Comm. rel.	Law of $a_i + a_i^*$ on $\bigoplus_n H^{\otimes n}$	CLT & Moments
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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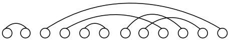
Fock 1932, Segal 1956

$\mathcal{N}(0,1)$



Classical CLT

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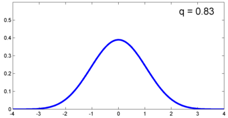
Physics ‘50s/‘70s

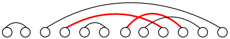
Maths ‘90s

Bożejko & Speicher 1991

q-Gaussian

$q \in [-1,1]$





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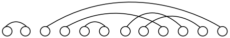
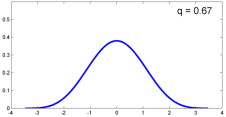
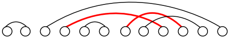
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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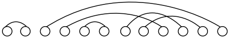
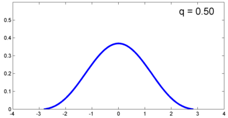
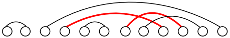
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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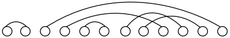
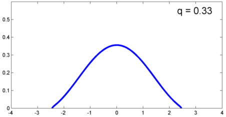
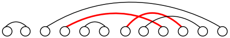
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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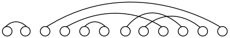
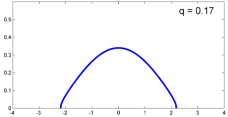
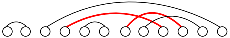
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow q\text{-Gauss}$ Speicher 1992

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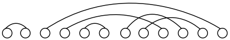
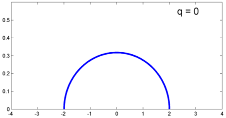
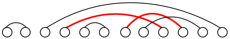
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1+\ldots+X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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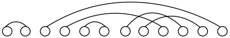
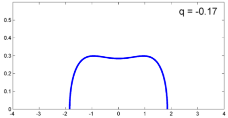
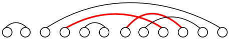
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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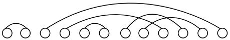
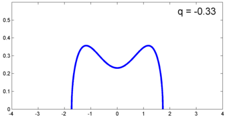
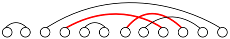
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics ‘50s/‘70s Maths ‘90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ “iid” $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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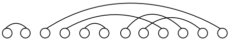
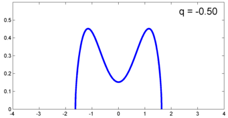
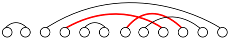
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics ‘50s/‘70s Maths ‘90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ “iid” $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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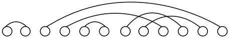
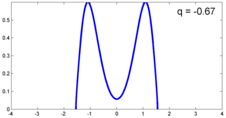
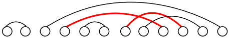
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \dots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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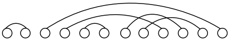
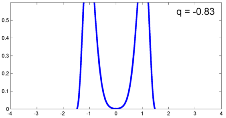
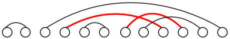
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \dots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1,1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \implies q\text{-Gauss}$ Speicher 1992

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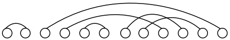
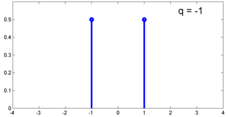
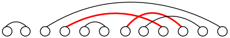
$a_i a_j^* - a_j^* a_i = \delta_{i,j}$ Fock 1932, Segal 1956	$\mathcal{N}(0,1)$ 	Classical CLT
$a_i a_j^* - q a_j^* a_i = \delta_{i,j}$ Physics '50s/'70s Maths '90s Bożejko & Speicher 1991	q-Gaussian $q \in [-1,1]$   $q^{\#\text{crossings}}$	$X_1, X_2, \ldots$ "iid" $\varphi(X_i) = 0, \varphi(X^2) = 1$ $X_i X_j = s(j,i) X_j X_i$ $s(j,i) \in \{-1, 1\}$ $\frac{X_1 + \ldots + X_N}{\sqrt{N}} \Longrightarrow q\text{-Gauss}$ Speicher 1992

$$a_i a_j^* - q a_j^* a_i = t^N \delta_{i,j}$$

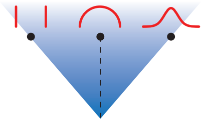
Physics ‘90s

Bożejko & Yoshida ’06

B. 2012 JFA

‘(q, t)-Gaussian’

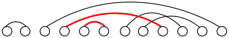
$$|q| \leq t$$



$$X_i^{\epsilon} X_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i) X_j^{\epsilon'} X_i^{\epsilon}$$

$$\mu_{\epsilon',\epsilon}(j,i) \in \mathbb{R}$$

B. 2014 AIHP



$$q^{\#\text{crossings}} \, t^{\#\text{nestings}}$$

$$a_i a_j^* - q a_j^* a_i$$
$$= \delta_{i,j} + \alpha \langle e_i, \Pi_0 e_j \rangle q^{2N}$$

Bożejko, Ejsmont,

& Hasebe 2015

‘type B’ Gaussian

$$q, \alpha \in (-1, 1)$$

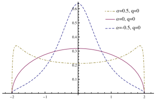


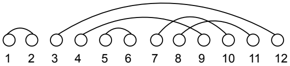
Fig. 1. $q = 0$.

$$X_i X_j = s(j,i) X_j X_i$$
$$Y_i Y_j = s(j,i) Y_j Y_i$$
$$X_i Y_j = r(j,i) Y_j X_i$$
$$s(j,i), r(j,i) \in \{-1, 1\}$$

B. - Ejsmont 2019 JMAA

Unifying perspective?

$$\int_{-\infty}^{\infty} x^{2n} \frac{exp(-x^2/2)}{\sqrt{2\pi}}\, dx \quad = \quad \#\text{ perfect matchings on } [2n]$$



$$\int_{-2}^{2} x^{2n} \frac{\sqrt{4-x^2}}{2\pi}\, dx \quad = \quad \#\text{ non-crossing perfect matchings on } [2n]$$

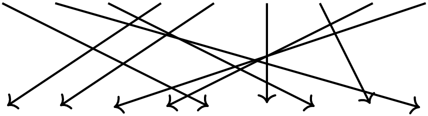
$$\sum_{k\geq 0} k^n \frac{e^{-1}}{k!} \quad = \quad \#\text{ set partitions on } [n]$$

$$\int_{0}^{\infty} x^n exp(-x)\, dx \quad = \quad \#\text{ permutations on } [n]$$

A combinatorial perspective on positivity (B. & Steingrímsson '21, Trans. Amer. Math. Soc.)

Question: Which combinatorial statistics on permutations give rise to moment sequences?

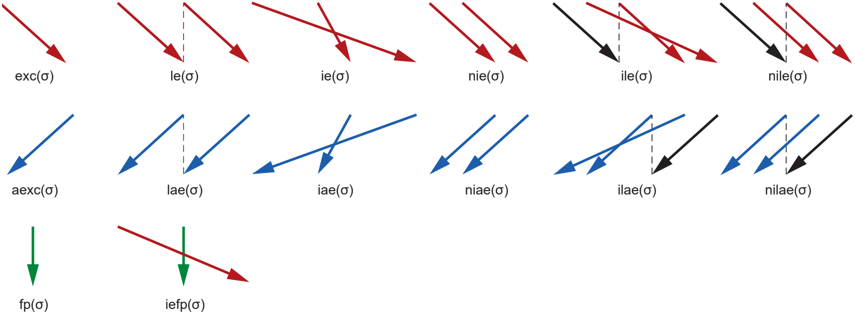
Represent $\sigma \in S_n$ in 2-line notation: i [top] $\mapsto \sigma(i)$ [bottom]



$\sigma = 597126843$

Definition (B. & Steingrímsson '21)

For $\sigma \in S_n$,



Theorem (B.-Steingrímsson, 2021) As formal power series,

$$\mathcal{C}(z) =$$

$$\sum_{n\geq 0} \sum_{\sigma\in\mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)}$$
$$\times\, \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)} z^n$$

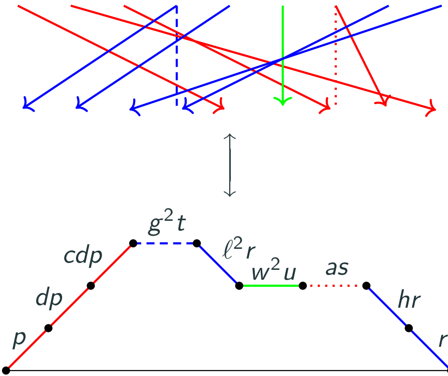
$$= \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with $\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}$, $\qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell}$

where $[n]_{x,y} = \sum_{k=0}^{n-1} x^k y^{n-1-k}$. (For $x \neq y$, $[n]_{x,y} = \frac{x^n-y^n}{x-y}$.)

Proof (à la Flajolet)

Exhibit a bijection:



Related to/extends: Françon-Viennot '79, Foata-Zeilberger '90, Biane '93, de Médicis-Viennot '94, Simion-Stanton '94 Clarke-Steingrímsson-Zeng '96, Randrianarivony '98, Elizalde '18.

Contemporaneous: Sokal & Zeng '22 ($\sim$ 120 p.).

Positivity

Fix $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$ and let

$$m_n = \sum_{\sigma \in \mathcal{S}_n} a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)}$$
$$\times \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}.$$

Is (m_n) a moment sequence of some positive Borel measure on $\mathbb{R}$?

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}} = m_0 + m_1 z + m_2 z^2 + \dots$$

with

$$\alpha_n = u\,w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

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Answer: Hamburger, rephrased.

$\alpha_n \in \mathbb{R}$ for all n.

If $\beta_1, \dots, \beta_{k-1} > 0$ and $\beta_n = 0$ for all $n \geq k$, measure supp. on k elements.

If $\beta_n > 0$ for all n, measure exists (need not be unique).

Quick example: recovering other combinatorial objects

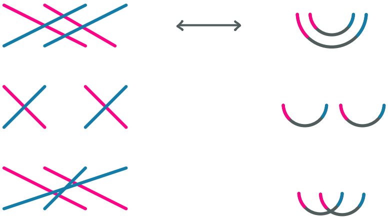
$c = s = t = u = 0$, $a = b = d = f = g = \ell = w = p = r = 1$.

Free parameter: h.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}$

$n = 4$:



Quick example: recovering other combinatorial objects

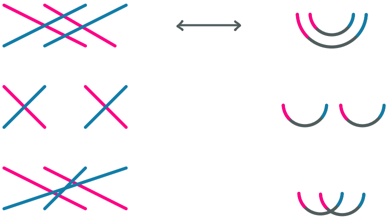
$c = s = t = u = 0$, $a = b = d = f = g = w = p = r = 1$.

Free parameters: h, ℓ.

No: fixed points, linked excedances, linked antiexcedances, inversions among excedances.

Yes: $h^{\#\text{inversions among anti-excedances}}\ \ell^{\#\text{restricted non-inversions among anti-excedances}}$

$n = 4$:



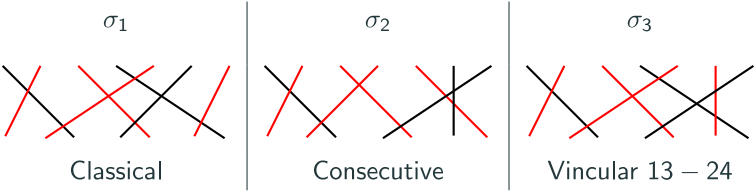
Loads more examples (classical and new)

Set partitions by # blocks (Stirling, second kind)	$\longleftrightarrow$	Poisson
Non-crossing set partitions by # blocks (Narayana)		Free Poisson (Marchenko-Pastur)
Eulerian polynomials $\sum_\sigma q^{\mathrm{des}(\sigma)}$		Geometric
Derangements		e.g. Martin & Kearney '15
Alternating permutations		Sokal '18
Little Schröder numbers		Młotkowski & Penson '13
Various combinatorial sequences		Moments of known OP
***k*-arrangements** (new def.)		Shifted exponentials
Inv, Exc, FP on *k*-colored perm.		B. & Steingrímsson '21

An intriguing observation

Definition. Permutation pattern: $\pi \in S_k$ and a containment rule

Example: $\pi = 1324$ classical, consecutive, or vincular



Let $\mathrm{Av}_\pi(n) := \#$ permutations on $[n]$ avoiding π.

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Theorem (MacMahon 1915, Knuth 1968).
For any classical pattern π of length 3,

$$\mathrm{Av}_\pi(n) = \frac{1}{n+1}\binom{2n}{n}.$$

Conjecture (Stanley-Wilf) / Theorem (Marcus & Tardos '04)
For any classical permutation pattern π,

$$(\mathrm{Av}_\pi(n))^{1/n} \to c_\pi > 0 \quad \text{as } n \to \infty.$$

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Only 3 symmetry classes for patterns of length 4:

1234 Gessel '90 (exact) | 1342 Bona '97 (exact) | 1324 wide open, $c = ?$

Theorem (Rains '98). Let $\pi = 1\,2\,3\ldots k$ classical ($k \in \mathbb{N}$). Then,

$$\mathrm{Av}_\pi(n) = \mathbb{E}_{U\in\mathbb{U}(k)}(|Tr(U)^2|^n).$$

Conjecture (B. & Steingrímsson, Elvey Price & Guttmann).
For any classical pattern π, $(\mathrm{Av}_\pi(n))_n$ is a moment sequence.

Numerical evidence for patterns of length 4: Bostan, Elvey Price, Guttmann, Maillard '20

Numerical evidence for patterns of length 5: Clisby, Conway, Guttmann, Inoue '21+

$$\mathcal{C}(z) = \mathcal{C}_{a,b,c,d,f,g,h,\ell,p,r,s,t,u,w}(z) := \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

with

$$\alpha_n = u\, w^n + s\,[n]_{a,b} + t\,[n]_{f,g}, \qquad \beta_n = p\, r\,[n]_{c,d}\,[n]_{h,\ell}.$$

Theorem (B. & Steingrímsson, '21) For any pattern of length 3 (classical, consecutive, vincular), $(\mathrm{Av}_\pi(n))_n$ is a moment sequence $\iff \sum_{n\geq 0} \mathrm{Av}_\pi(n) z^n$ is a special case of $\mathcal{C}(z)$.

Refines further.

Classical vs Consecutive

$$a_n := \#\{\sigma \in S_n \mid \nexists i \in \{1, \ldots, n-2\} \text{ s.t. } \sigma(i) < \sigma(i+1) < \sigma(i+2)\}$$

YES

$$b_n := \#\{\sigma \in S_n \mid \nexists i \in \{1, \ldots, n-2\} \text{ s.t. } \sigma(i) < \sigma(i+2) < \sigma(i+1)\}$$

NO ☹

~~Avoiding~~ Packing consecutive permutation patterns

B., Kammoun, Steingrímsson, '20-'25+ (~~WIP~~ in prep.)

Definition

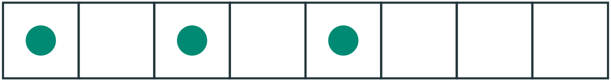
A <u>K-pattern</u> is a consecutive permutation pattern $\pi \in S_K$.

Definition (Pattern packing)

Fix $K, j \in \mathbb{N}$, π a K-pattern, and $m \in [K]$ ("overlap"). We say that a permutation σ is <u>m-packing j occurrences of π</u> if σ contains occurrences of π starting in positions

$\{1, 1 + K - m, 1 + 2(K - m), \ldots, 1 + (j - 1)(K - m)\}$.

Example: 2-packing 3 occurrences of $\pi = 1324$. ($K = 4$, $m = 2$, $j = 3$.)



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1	5	2	6	3	7	4	8

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1	5	2	6	3	**7**	**4**	**8**

1	5	2	6				
		2	6	3	7		
				3	7	4	8

The definition is natural.

Permutations 'covered' by π (i.e. not imposing regularity):
clusters in the cluster method and avoidance of consecutive patterns.

Elizalde & Noy '03, Elizalde & Noy '12, Beaton, Conway, Guttmann '18

Non-overlapping patterns (i.e. $m = 1$ as the only possibility):
enumerative results, algebraic results.

Elizalde '04, Duane & Remmel '11, Elizalde & Noy '12, Dotsenko & Khoroshkin '13

The descent set (i.e. $K = 2$, not requiring regularity): for $\sigma \in S_n$,

$$DES(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$$

The number of permutations $\#\sigma \in S_n$ with

$$DES(\sigma) = \{c_1 < c_2 < \ldots < c_k\}$$

is

$$\det\left(\binom{a_i}{b_j}\right)_{1\le i,j\le k+1}$$

where

$(a_1, \ldots, a_{k+1}) = (c_1, \ldots, c_k, n)$ and $(b_1, \ldots, b_{k+1}) = (0, c_1, \ldots, c_k)$.

MacMahon 1908

Interpretations:

Non-intersecting paths (**Gessel & Viennot '85**)

Determinantal point processes (**Borodin, Diaconis & Fullman '10**)

Recall (pattern packing): A permutation m-packing j occurrences of K-pattern π must contain occurrences of π starting in positions

$$\{1, 1+K-m, 1+2(K-m), \ldots, 1+(j-1)(K-m)\}.$$

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Proposition (**Elizalde & Noy '12**) The number of permutations m-packing j occurrences of

$$134\ldots(m+2)2(m+3)(m+4)\ldots(2m+1)$$

equals $\frac{1}{mj+1}\binom{(m+1)j}{j}$.

pattern	overlaps 1	overlaps 2	overlaps 3
312	$(n-2)!_2$		
1243	$(n-3)!_3$ Coro. 1	–	
1324	1	C_{j+1} Prop. 2	
1342	$\frac{(3j)!}{j!(3!)^j}$	–	
1423	$(n-3)!_3$ Coro. 1	1	
2143	$1, 9, 234, 12204, 1067040, \ldots$	1	
2413	$1, 9, 234, 12204, 1067040, \ldots$	C_{j+1}, Prop. 2	
12354	$(n-4)!_4$ Coro. 1	–	–
12435	1	1	–
12453	$\prod_{i=0}^{j} \binom{4i+2}{2}$ Prop. 1	–	–
12534	$(n-4)!_4$ Coro. 1	$(n-4)!_3$	–
13254	$(n-4)!_4$ Coro. 1	–	1
13425	1	$\frac{1}{2j+3}\binom{3j+3}{j+1}$ Prop. 3	–
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Table 1: Enumerating sequences: equivalence classes of patterns of length ≤ 5.

Some early results

B. Kammoun, Steingrímsson, FPSAC '23

Lemma

The number of permutations m-packing a K-pattern π depends only on the m-prefix and m-suffix of π.

Lemma

Let $\pi = \mathbf{pws}$ be a K-pattern, where $\mathbf{p}$ (resp. $\mathbf{s}$) is the m-prefix (resp. m-suffix) of π. Given $\rho \in S_m$, let $\pi' = \rho(\mathbf{p})\,\mathbf{w}\,\rho(\mathbf{s})$. Then, number of permutations m-packing π equals that m-packing π'.

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Let π be a K-pattern. Count permutations m-packing j occurrences of π.

Proposition. When $\pi_1 = 1$ and $m = 1$, the count is $\prod_{i=0}^{j} \binom{i(k-1)+d}{d}$, where $d = K - \pi_K$.

(See also **Dotsenko & Khoroshkin '13** and **Elizalde & Noy '12** with $a = 1$.)

Corollary. When $p_K = K - 1$, the count is $((j-1)(K-1)+1)!_{K-1}$.

Proposition. Suppose $K = 2k$, for $k \geq 2$, and let $\pi = 1(k+1)2(k+2)\ldots k(2k)$, $m = 2k-2$. The count is C_j.

This same count also holds for the pattern $\psi = (k+1)1(k+2)2\ldots(2k)k$.

The same is true when $(2k+1)$ is appended to ψ, with overlaps $2k-1$, and also when 1 is prepended to π and each letter of π incremented by 1, with overlaps $2k-1$.

Proposition. If $\pi = 14523$ and $m = 2$, the count is $(2j-1)!_2$.

A slew of general results

All results refer to

B., Kammoun, Steingrímsson, '20-'25+ (~~WIP~~ in prep.)

Theorem 1. (FPSAC '23) Fix a K-pattern π. Fix the positions of occurrences. Obtain a general recursive formula for enumerating permutations m-packing π (where $m \leq K/2$).

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Theorem 1. (FPSAC '23) Fix a K-pattern π. Fix the positions of occurrences. Obtain a general recursive formula for enumerating permutations m-packing π (where $m \leq K/2$).

Theorem 2. Fix a K-pattern π. The number of permutations 1-packing π is obtained by taking derivatives of the same master object.

E.g. $\#$ permutations with 2 occurrences of π is the coefficient of

$x_1^{\pi_1 - 1} x_2^{\pi_1 - 1} y_1^{K - \pi_K} y_2^{K - \pi_K} z_1^{\pi_K - \pi_1 - 1} z_2^{\pi_K - \pi_1 - 1}$

in $(1 - x_1 - x_2)^{-1}(1 - y_1 - x_2)^{-1}(1 - z_1 - y_2)^{-1}(1 - z_1 - z_2)^{-1}$.

Going forward: σ is m-packing j occurrences of π.

Observe: packing π forces relations between the elements of σ.

Example: Any σ 1-packing two occurrences of $\pi = 2413$ must satisfy:

$\sigma_1\,\sigma_2\,\sigma_3\,\sigma_4 \sim \pi$, which is equivalent to $\sigma_3 < \sigma_1 < \sigma_4 < \sigma_2$ (first occurrence),

and also

$\sigma_4\,\sigma_5\,\sigma_6\,\sigma_7 \sim \pi$, which is equivalent to $\sigma_6 < \sigma_4 < \sigma_7 < \sigma_5$ (second occurrence).

Lemma. A permutation σ is m-packing J occurrences of a K-pattern π if and only if it satisfies following J inequalities:

$$\sigma(\pi_1^{-1}) < \sigma(\pi_2^{-1}) < \ldots < \sigma(\pi_K^{-1}) \tag{1}$$

$$\sigma(\pi_1^{-1} + K - m) < \sigma(\pi_2^{-1} + K - m) < \ldots < \sigma(\pi_K^{-1} + K - m), \tag{2}$$

$$\ldots,$$

$$\sigma(\pi_1^{-1} + (J-1)(K-m)) < \ldots < \sigma(\pi_K^{-1} + (J-1)(K-m)) \tag{J}$$

Lemma. A permutation σ is m-packing J occurrences of a K-pattern π if and only if it satisfies following J inequalities:

$$
\begin{aligned}
&\sigma(\pi_1^{-1}) < \sigma(\pi_2^{-1}) < \ldots < \sigma(\pi_K^{-1}) && (1)\\
&\sigma(\pi_1^{-1} + K - m) < \sigma(\pi_2^{-1} + K - m) < \ldots < \sigma(\pi_K^{-1} + K - m), && (2)\\
&\ldots,\\
&\sigma(\pi_1^{-1} + (J-1)(K-m)) < \ldots < \sigma(\pi_K^{-1} + (J-1)(K-m)) && (\mathrm{J})
\end{aligned}
$$

Immediately suggests the following definition:

Definition

Given a consecutive permutation pattern π, overlap m and number of occurrences J, the poset $\mathcal{P}_\pi^m(J)$ defined by the inequalities (1)-(J) is said to be a **pattern-packing poset**.

The relationship between pattern-packing permutations and pattern-packing posets is immediate:

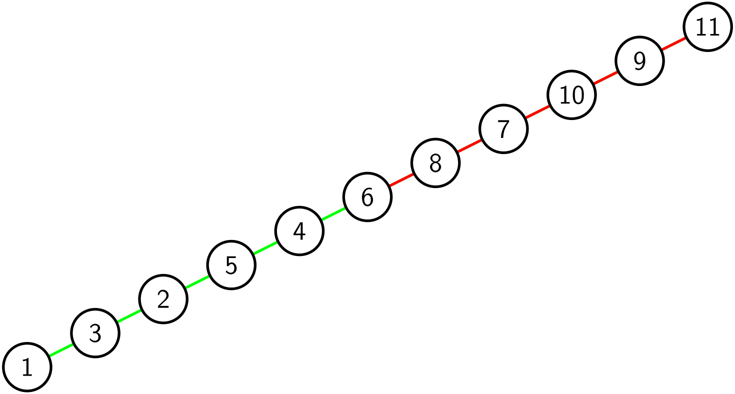
Theorem
For every K-pattern π, $m \in [K]$ and $J \in \mathbb{N}$, the set of linear extensions of the poset $\mathcal{P}_\pi^m(J)$ is in natural bijection with the set of permutations m-packing J occurrences of π.

The permutation structure translates to basic properties of the poset. For example:

Theorem
An element in $\mathcal{P}_\pi^m(J)$ can cover at most two elements and can be covered by at most two elements.

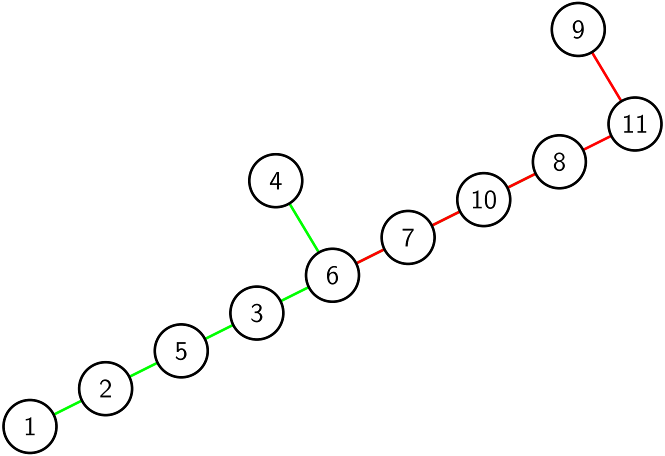
A pattern can give a chain.

Example: $\pi = 132546$ (whose inverse is 132546), $m = 1$, $J = 2$.



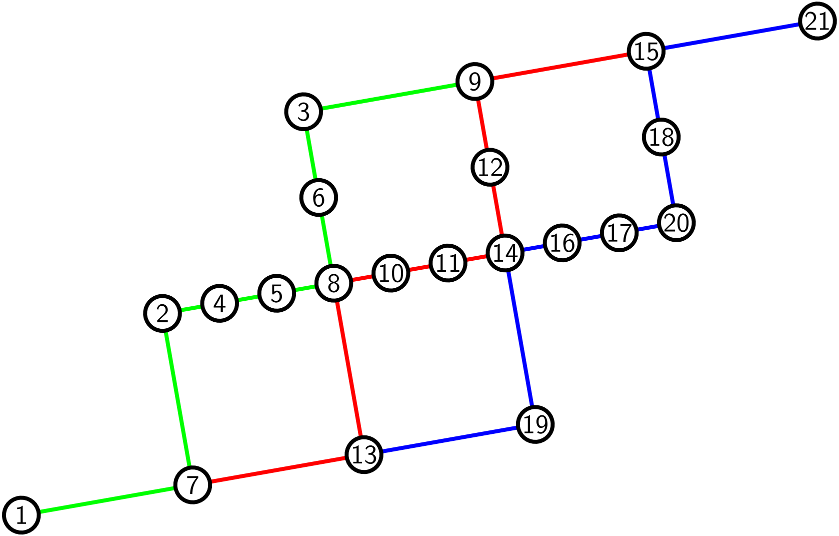
A pattern can give a tree.

Example: $\pi = 124635$ (whose inverse is 125364), $m = 1$, $J = 2$.



A pattern can give a multi-level grid.

Example: $\pi = 138457269$ (whose inverse is 172458639), $m = 3$, $J = 3$.



And many other fun shapes!

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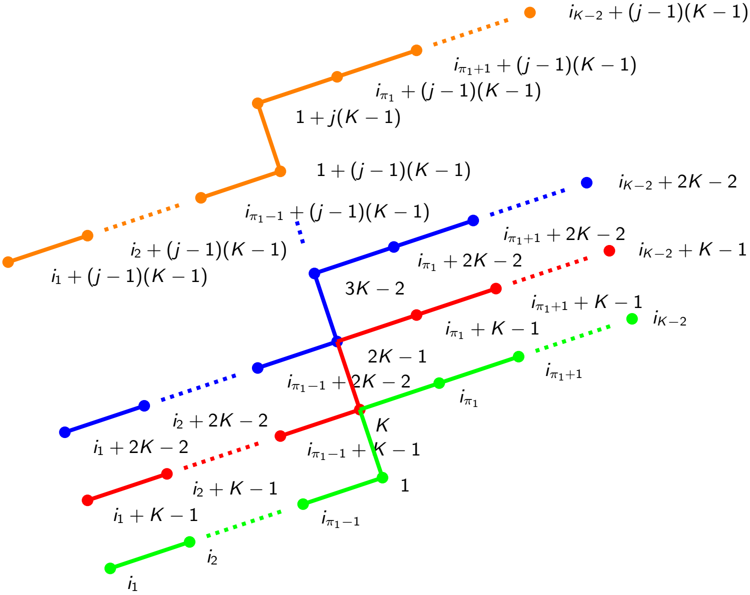
General observations:

- For any pattern, overlap $m=1$ always produces a tree. (**Proposition**)
- For any m, can characterize patterns that produce trees. (**Theorem**)
- For any m, can characterize patterns that produce grids. (**Theorem**)
- Diagrams can get complicated for long patterns with larger overlaps.
- Diagrams need not be regular, but we conjecture that for J large enough, **regularity emerges**.

Counting via order statistics

Idea:

Draw a sequence of iid random variables. Ask about the probability of them respecting a given ordering.



$i_{K-2} + (j-1)(K-1)$
$i_{\pi_1+1} + (j-1)(K-1)$
$i_{\pi_1} + (j-1)(K-1)$
$1 + j(K-1)$
$1 + (j-1)(K-1)$
$i_{\pi_1-1} + (j-1)(K-1)$
$i_2 + (j-1)(K-1)$
$i_1 + (j-1)(K-1)$
$i_{K-2} + 2K - 2$
$i_{\pi_1+1} + 2K - 2$
$i_{\pi_1} + 2K - 2$
$3K - 2$
$i_{K-2} + K - 1$
$i_{\pi_1+1} + K - 1$
$i_{\pi_1} + K - 1$
$2K - 1$
$i_{\pi_1-1} + 2K - 2$
$i_2 + 2K - 2$
$i_1 + 2K - 2$
i_{K-2}
i_{π_1+1}
i_{π_1}
K
$i_{\pi_1-1} + K - 1$
$i_2 + K - 1$
$i_1 + K - 1$
1
i_{π_1-1}
i_2
i_1

Observation #1: regardless of the individual distributions, the symmetry makes the probability problem equivalent to the counting problem.

Observation #2: the counting problem is that of counting linear extensions of the underlying poset.

Remark:

- Essentially, the same as the 'density method' of **Banderier and Wallner** (e.g. FPSAC '21).
- We arrive at it from a different probabilistic intuition: order statistics (any distribution) vs. volumes of polytopes (uniform distribution).

Proposition
The number of permutations 1*-packing* J *occurrences of* $\pi = 2413$ *is:*

$$\frac{(1+3J)!}{(J-1)!}\int_0^1\int_y^1 y(1-z)\left(\frac{1}{6}(2y^3-3y^2+3z^2-6z^3)\right)^{J-1}dz\,dy.$$

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Theorem

Let π be a K-pattern with $\pi(K)-\pi(1)=1$. The number of permutations 1*-packing J occurrences of π is*

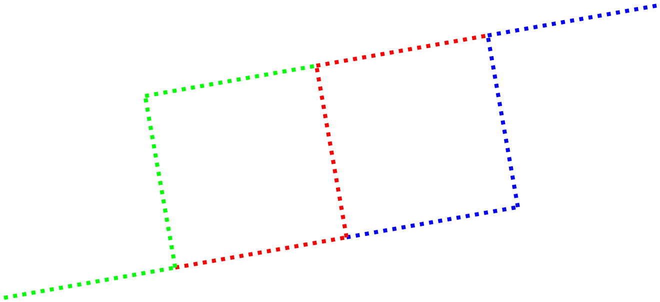
$$\frac{(1+J(K-1))!}{(J-1)!\,((\pi_1-1)!(K-\pi_1-1)!)^J}$$

times

$$\int_{-\infty}^{\infty}\int_y^{\infty}F(y)(1-F(z))\left(\int_y^z F(x)^{\pi_1-1}(1-F(x))^{K-\pi_1-1})f(x)\,dx\right)^{J-1}f(y)f(z)\,dz\,dy,$$

where f is any probability density on $\mathbb{R}$ and F its cumulative distribution function.

Two-level grid:



Let ξ the number of vertices in the lower horizontal segment, υ vertical, and ζ upper horizontal.

Theorem

Let π be a K-pattern whose m-packings give rise to a two-level grid poset parameterized by (ξ, υ, ζ). Assume $\zeta = 2$ and define:

$$n = K + J(K-2),$$
$$A = (a_{i,\ell})_{1 \le i,\ell \le n} \quad \textit{where} \quad a_{i,\ell} = \binom{\upsilon - 1 + \ell}{\upsilon} \cdot 1_{\{\ell > i - K + 2\}}.$$

Then the number permutations of $[n]$ that are m-packing J occurrences of π is $\left(A^J\right)_{1,1}$, i.e. the $(1,1)$-entry of A^J.

Generalizes (mildly more messily) to other values of ζ.

No longer mystery sequences

pattern	overlaps 1	overlaps 2	overlaps 3
312	$(n-2)!_2$		
1243	$(n-3)!_3$ Coro. 1	–	
1324	1	C_{j+1} Prop. 2	
1342	$\frac{(3j)!}{j!(3!)^j}$	–	
1423	$(n-3)!_3$ Coro. 1	1	
2143	$1, 9, 234, 12204, 1067040, \ldots$	1	
2413	$1, 9, 234, 12204, 1067040, \ldots$	C_{j+1}, Prop. 2	
12354	$(n-4)!_4$ Coro. 1	–	–
12435	1	1	–
12453	$\prod_{i=0}^{j}\binom{4i+2}{2}$ Prop. 1	–	–
12534	$(n-4)!_4$ Coro. 1	$(n-4)!_3$	–
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Table 1: Enumerating sequences: equivalence classes of patterns of length ≤ 5.

$$\int_{\mathbb{R}^k} (f(x))^J dx$$

$$(A^J)_{i,j}$$

A bold conjecture

Conjecture (FPSAC '23) For any K-pattern π and overlap m, the enumerating sequence for permutations m packing J occurrences of π is a moment sequence, i.e. can be expressed as

$$\int_{\mathbb{R}} (\, f(x)\,)^{J} g(x)\, dx$$

for $g \geq 0$, or, equivalently,

$$(A^{J})_{1,1}$$

where $A = A^{\mathsf{T}}$.

A few facts:

- For $K \leq 5$, we can show, for all but 3 sequences, that these are moment sequences. (Some are famous, some require work.)

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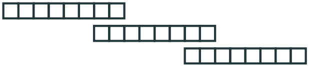
- For $K \leq 5$, we can show, for all but 3 sequences, that these are moment sequences. (Some are famous, some require work.)
- The remaining (elusive) sequences numerically give positive determinants. (Hamburger.) Two sequences have explicit formulas.
- There is a $K = 10$ sequence that is not a moment sequence.

But not so fast!

- Original intuition: regularity + luck $\implies$ positivity

$$\int_{\mathbb{R}} (\,f(x)\,)^{J} g(x)dx = \int_{\mathbb{R}} \underbrace{f(x)\cdots f(x)}_{J} g(x)dx$$



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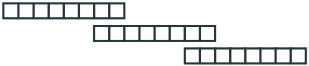

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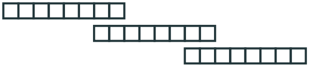

- But posets aren't always immediately regular. Regularity starts past $J \geq J_0$, where J_0 depends on the pattern.

Conjecture (2025) Fix π, m as before. Start the sequence at J_0. The sequence will be a moment sequence.

Theorem (2025) From new enumerative formulas (obtained via posets), can **prove positivity** for a number of general families.

A “right” way to look at positivity for consecutive permutation patterns, which is not pattern-dependent.

For comparison:

$$a_n := \#\{\sigma \in S_n \mid \nexists i \in \{1, \dots, n-2\} \text{ s.t. } \sigma(i) < \sigma(i+1) < \sigma(i+2)\}$$

YES 

$$b_n := \#\{\sigma \in S_n \mid \nexists i \in \{1, \dots, n-2\} \text{ s.t. } \sigma(i) < \sigma(i+2) < \sigma(i+1)\}$$

NO ☹

One more perspective shift

(last one, promise)

A different probabilistic question

Now draw a permutation at random and mark the starting points of the occurrences of π.

(Note: no requirement of regularity.)

Fact: This is a point process. (A general class of random process.)

Recall: The descent point process is extremely nice!

- What are some probabilistic properties of this more general process?
- How do they depend on π?
- How do they depend on the choice of what we mean by 'random permutation'?

Definition:

σ_n is conjugacy-invariant if for any ρ,

$$\rho \sigma_n \rho^{-1} \stackrel{d}{=} \sigma_n.$$

- Example 0: Uniform permutation.
- Example 1: Ewens

$$\mathbb{P}(\sigma_n = \sigma) \sim \theta^{\#\text{cycles}(\sigma)}.$$

- Example 2: Generalized Ewens

$$\mathbb{P}(\sigma_n = \sigma) \sim \prod_k \theta_k^{\#\text{k-cycles}(\sigma)}.$$

- Example 3: Uniform permutation within a conjugacy class.

Universality for consecutive patterns

Theorem (B., Kammoun, Steingrímsson)

Suppose

- *for all n, σ_n is conjugacy invariant.*
- $\frac{\mathrm{fix}(\sigma_n)}{n}$ *converges in probability to $c \geq 0$.*

Then, $\pi(\sigma_n)$ converges to a process $X_{\pi,c}$ (i.e. the correlation functions converge).

The correlation functions of $X_{c,\pi}$ are "explicit", e.g.

$$\mathbb{P}(E \subset X_{21,c}) = \det[k_c(i-j)]_{i,j\in E}$$

$$\mathbb{P}(E \subset X_{12,c}) = \det[k'_c(i-j)]_{i,j\in E}$$

with

$$\sum_{\ell} k_c(\ell) z^{\ell} = \frac{1}{1-(1+c)ze^{(1-c)z}}.$$

In the proofs, the k-arrangements occur naturally!

Conclusion (for now)

Permutations

Patterns

Posets

Processes

Positivity

LEVERHULME TRUST

Engineering and Physical Sciences Research Council

Heilbronn Institute for Mathematical Research

A Pattern Plays in Probability's Palm,
Permutations Parade in Predictable Calm.
Points Pivot on a Plane so Precise,
Each Path a Puzzle, a Price, or a Paradise.

– Chat GPT

Happy Birthday, Einar!