Totally nonnegative matrices, chain enumeration and zeros of polynomials

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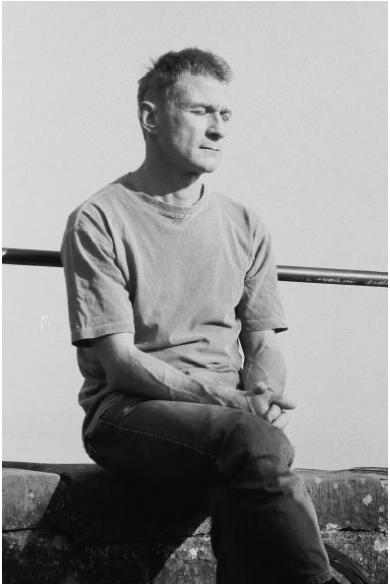
joint work with
Leonardo Saud Maia Leite

Einar fest, Reykjavik, June 18, 2025

Til hamingju Einar!

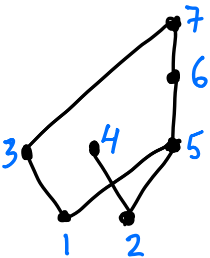


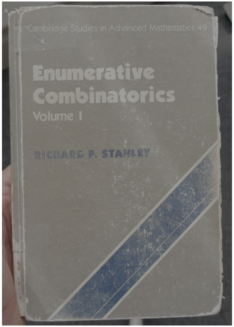
"The one contribution of mine that I hope will be remembered has consisted in pointing out that all sorts of problems of combinatorics can be viewed as problems of the location of the zeros of certain polynomials..."

Gian-Carlo Rota

Chain enumeration in posets

- A central theme in enumerative and algebraic combinatorics is counting chains in posets.





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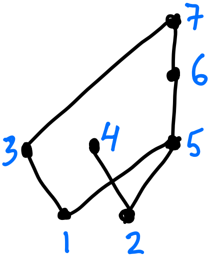

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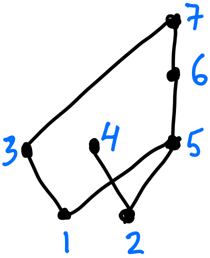

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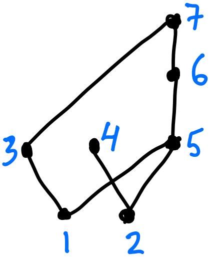

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- Example. If $P = \{1 < 2 < \cdots < n\}$, then $c_P(t) = (1+t)^n$.

Chain enumeration in posets

- Example. If P is the Boolean algebra $\mathbb{B}_n$, then

$$c_P(t) = (1+t)^2 \sum_{k=0}^{n-1} (k+1)! \cdot S(n, k+1) t^k.$$

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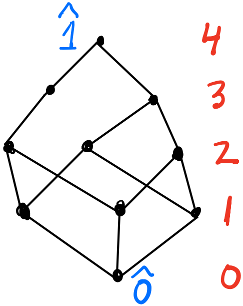
- Chain polynomials tend to be real-rooted:
- The Poset Conjecture (Neggers, 1978): P is a distributive lattice. Counterexamples by B. and Stembridge (2004, -06).
- $(3+1)$-free posets (Stanley, 1998).
- Cohen-Macaulay simplicial complexes (Brenti-Welker, B., 2006).
- Shellable cubical complexes (Athanasiadis, 2021).
- Geometric lattices (Conjecture of Athanasiadis and Kalampogia-Evangelinou, 2023).

Rank uniform posets

- We consider finite, bounded and graded posets P, i.e.,
- P has a least element $\hat{0}$, and a largest element $\hat{1}$, and
- P admits a rank function $\rho : P \to \mathbb{N}$ such that $\rho(\hat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever y covers x.



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- These posets generalize binomial- and Sheffer-posets.
- If P is rank uniform define a matrix $R(P) = (r_{n,k}(P))_{n,k}$, where

  $$r_{n,k}(P) = |\{x \leq y : \rho(x) = k\}|,$$

  where y is an element of P of rank n.

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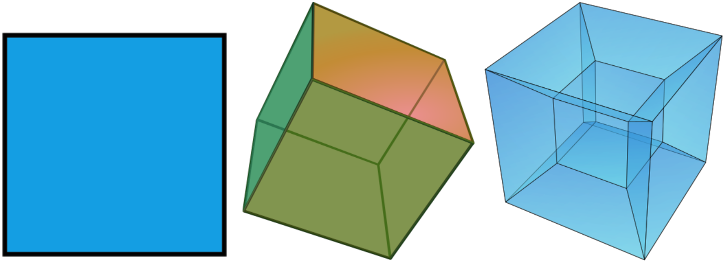
- Example. $r_{n,k}(Q) = 2^{n-k}\binom{n-1}{k-1}$, where Q is the face lattice of a hypercube.



What do these matrices have in common?

Totally nonnegative matrices

- A matrix with real entries is totally nonnegative (TN) if all of its minors are nonnegative.
- Important in linear algebra, statistics, analysis, geometry, algebraic combinatorics and physics.

Felix Gantmacher

Mark Krein

Samuel Karlin

TN-matrices and TN-posets

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- Suppose P is a bounded and graded poset of rank r, and that $S \subseteq [r-1]$. Then

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- If P is TN, then so is P_S.

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- Corollary. Rank selected subposets of the above posets have real-rooted chain polynomials.

The shape of characteristic polynomials

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- Can we describe all linear inequalities satisfied by all characteristic polynomials of hyperplane arrangements in $\mathbb{F}_q^n$?
- Theorem (B., Saud). The convex hull of all characteristic polynomials of hyperplane arrangements in $\mathbb{F}_q^n$ is equal to the simplex with vertices

$$t^{n-k}(t-1)(t-q)\cdots(t-q^{k-1}), \qquad 0 \le k \le n.$$

The shape of chromatic polynomials

- Problem. Describe the convex hull of all chromatic polynomials of graphs on $V = \{1, 2, \ldots, n\}$.

Congratulations Einar!