Cops and Robber Pebbling on Graphs

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Rules of Graph Pebbling

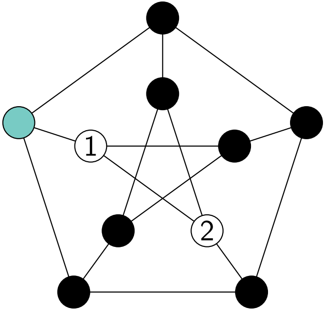
- A configuration C of pebbles on a graph G is a function from the vertices of G to the non-negative integers.
- Its size equals $|C| = \sum_{v \in G} C(v)$.
- For adjacent vertices u and v with $C(u) \geq 2$, a *pebbling step* from u to v removes two pebbles from u and adds one pebble to v, while, when $C(u) \geq 1$, a free step from u to v removes one pebble from u and adds one pebble to v.
- In the context of moving pebbles, we use the word move to mean *move via pebbling steps*.

Rules of Graph Pebbling

The pebbling number of a graph G, denoted $\pi(G)$, is the minimum number m so that, from any configuration of size m, one can move a pebble to any specified target vertex.

The optimal pebbling number of a graph G, denoted $\pi^*(G)$, is the minimum number m so that, from some configuration of size m, one can move a pebble to any specified target vertex.

We note that in the definitions of the pebbling number and the optimal pebbling number, free moves are not allowed.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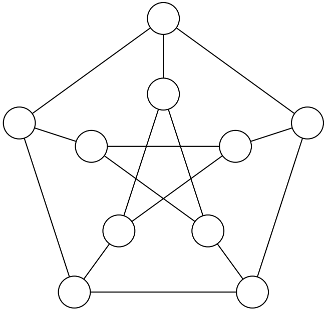
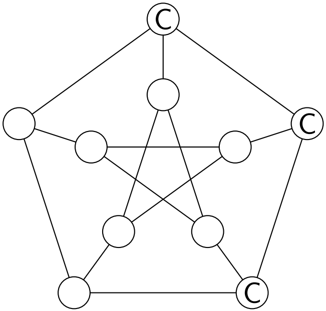
Rules of the Cops and Robber Game

- two opposing sides, $k > 0$ cops and a single robber
- both sides play with perfect information
- cops begin the game by each choosing a vertex to occupy then robber chooses a vertex
- opposing sides move alternately, where a move for the cop side consists of any positive number of them making a free step and a move for the robber consists of making a free step or not
- cops win if at least one of them occupies the same vertex as the robber after a finite number of moves, i.e. they capture the robber

The copnumber of a graph G, denoted $c(G)$, is the minimum number of cops that suffice to guarantee a win on G.

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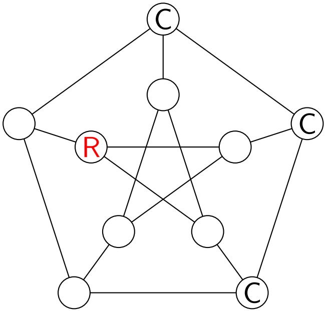
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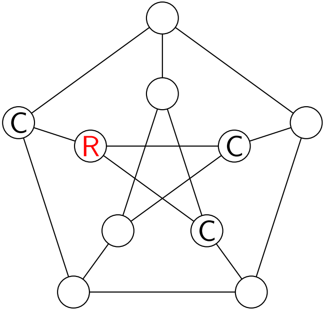
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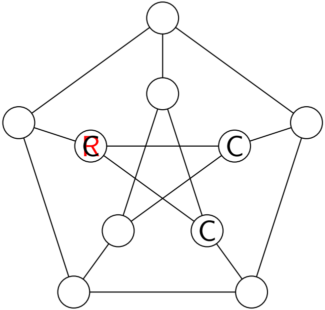
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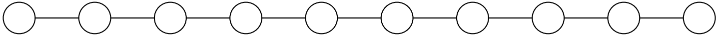
Rules of Cops and Robber Pebbling

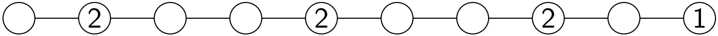
The cop pebbling number $\pi^{c}(G)$ is defined as the minimum number m so that, from some configuration of m cops, it is possible to capture any robber via pebbling steps.

Note: $\pi^{c}(G) \leq n(G)$.

We call an instance of a graph G, configuration (of cops) C, and robber vertex R a game, and say that the cops win the game if they can capture the robber; else the robber wins.

Note that, since we lose a cop with each pebbling step, the cops must catch the robber within at most $|C| - 1$ turns to win the game.

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Theorem If G is a graph with $\delta = \delta(G) \geq 27$, and with $\text{gir}(G) \geq 4t + 1$ and $n(G) \leq \delta^{2t+1}$ for some $t \geq 3$, then $c(G) > \pi^{*}(G)$.

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Theorem For all d there exists a graph G such that $\text{gir}(G) = 5$, $\pi^*(G) \leq 4$ and $c(G) \geq d$.

Upper Bounds

Theorem Let G be a graph, S a subset of its vertices, and define $S' = V - N[S]$. Then $\pi^c(G) \leq 2|S| + |S'|$. In particular, $\pi^c(G) \leq 2\gamma(G)$.

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Corollary Every graph G satisfies $\pi^{\mathsf{c}}(G) \leq n - \Delta(G) + 1$. In particular, if $n(G) \leq 2$ then $\pi^{\mathsf{c}}(G) = n$, if $n(G) \geq 3$ then $\pi^{\mathsf{c}}(G) \leq n - 1$, and if $n(G) \geq 6$ then $\pi^{\mathsf{c}}(G) \leq n - 2$.

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Theorem (Capture) If G is a copwin graph with $\mathrm{capt}(G) = t$, then $\pi^{\mathrm{c}}(G) \leq 2^t$. More generally, if $c(G) = k$ and $\mathrm{capt}_k(G) = t$ then $\pi^{\mathrm{c}}(G) \leq k2^t$.

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Corollary If G is a chordal graph with radius r, then $\pi^{\mathsf{c}}(G) \leq 2^r$.

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Example For integers k and d, the spider $S = S(k, d)$ has $c(S) = 1$ and $\mathrm{capt}(S) = d$, with $n = kd + 1$. Thus our capture time result (Theorem:Capture) yields $\pi^{\mathrm{c}}(S) \leq 2^d$, while our tree result (Theorem:Treebound) yields $\pi^{\mathrm{c}}(S) \leq \lceil (2kd + 2)/3 \rceil$. Hence one bound is stronger than the other depending on how k compares, roughly, to $3 \cdot 2^{d-1}/d$.

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Example For integers $k, t \geq 1$, let T be the complete k-ary tree of depth t. Then $n(T) = \sum_{i=0}^{t} k^i = (k^{t+1} - 1)/(k - 1)$. Thus Theorem:Capture is stronger than Theorem:Treebound for $k \geq 3$ and for $k = 2$ with $t \geq 2$, while Theorem:Treebound is stronger than Theorem:Capture when $k = 1$ and $t \geq 5$ (because $\mathrm{capt}(P_t) = \lceil t/2 \rceil$).

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Example For $1 \leq i \leq 3$, define the tree T_i to be the complete binary tree of depth $d - 1$, rooted at vertex v_i, and define the tree T to be the union of the three T_i with an additional root vertex adjacent to each v_i. Then $\gamma_d(T) = d$, and $n = 3(2^d - 1) + 1$.

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Theorem For every graph G we have $\pi^{\mathsf{c}}(G \Box K_t) \leq t\pi^{\mathsf{c}}(G)$.

Cartesian Products

A famous conjecture of Graham postulates that every pair of graphs G and H satisfy $\pi(G \Box H) \leq \pi(G)\pi(H)$. This relationship was shown by Shiue to hold for optimal pebbling.

Theorem (Shiue) Every pair of graphs G and H satisfy $\pi^*(G \Box H) \leq \pi^*(G)\pi^*(H)$.

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Theorem There exist graphs G and H such that $\pi^{\mathsf{c}}(G\square H) > \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$.

Cartesian Products of Wheels

Theorem Let $n \geq 4$ and let $G = W_n \Box W_n$. Then $\pi^{\mathsf{c}}(G) \leq 14$ and if $n \geq 67$ then $\pi^{\mathsf{c}}(G) = 14$.

Corollary For $n \geq 67$ we have $\pi^{\mathsf{c}}(W_n \Box W_n) = \frac{7}{2}\pi^{\mathsf{c}}(W_n)\pi^{\mathsf{c}}(W_n)$.

Open Questions

Can the bounds $.178 \approx \frac{5092}{28593} \leq \lim_{k,m\to\infty} \pi^{\mathsf{c}}(P_k \square P_m)/km \leq .4$ from Theorem:CopGrids be improved?

Is there an infinite family of graphs $\mathcal{G}$ for which $\pi^{\mathsf{c}}(G \square H) \leq \pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all $G, H \in \mathcal{G}$?

Is there some constant $a \geq 7/2$ such that $\pi^{\mathsf{c}}(G \square H) \leq a\pi^{\mathsf{c}}(G)\pi^{\mathsf{c}}(H)$ for all G and H?

Are there constant upper bounds on $\pi^{\mathsf{c}}(G)$ when G is planar or outerplanar? If $k = \pi^{\mathsf{c}}(G)$ then is $\mathrm{capt}_k(G)$ linear?

Is there a similar, narrow range of values of $\pi^{\mathsf{c}}(G)$ over all diameter two graphs G?

Meyniel's Conjecture

Finally, Meyniel conjectured in 1985 that every graph G on n vertices satisfies $c(G) = O(\sqrt{n})$.

Some evidence in support of this (Bollobás, Kun, Leader): for $G \in \mathcal{G}_{n,p}$, when $0 < \epsilon < 1$ and $p > 2(1+\epsilon)\log(n)/n$, we have $c(G) < \frac{10^3}{\epsilon^3} n^{\frac{1}{2}} \log(n)$ almost surely. (In fact, they also show that when $p \gg 1/n$ we have $c(G) > \frac{1}{(pn)^2} n^{\frac{1}{2}\left(\frac{\log\log(pn) - 9}{\log\log(pn)}\right)}$ almost surely.)

Conjecture Every graph G on n vertices satisfies $\pi^{\mathsf{c}}(G) \leq 2n/3 + o(n)$; i.e. Ï$^{\mathsf{c}}(G) \geq n/3 - o(n)$.

Thanks

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