Irrational enumeration

Analytic combinatorics for objects of irrational size

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Einar Fest, NORCOM 2025
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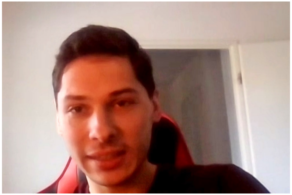
18th June 2025

Co-conspirators

This is joint work with



Julien Condé *and* Andrew Elvey Price

of

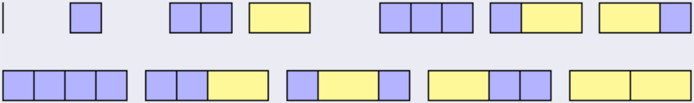
l'Université de Tours.

Enumerative combinatorics

Combinatorial classes

How many distinct tilings are there of a strip of length n using squares and dominoes?

The answer is the $(n+1)^{\text{th}}$ Fibonacci number: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$



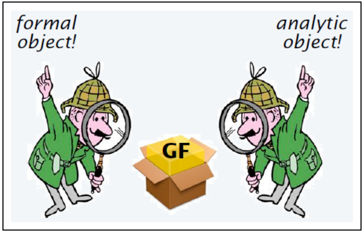
Generating functions

For Fibonacci tilings, $\mathcal{T}^{(2)}$, we have

$$f_{\mathcal{T}^{(2)}}(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \ldots = \frac{1}{1 - z - z^2}.$$

Analytic combinatorics

Derive asymptotics of $|\mathcal{C}_n|$ by treating $f_{\mathcal{C}}(z)$ as a complex function.



- Singularities give full information on growth of coefficients.
- Analytic combinatorics can also tell us what a typical large object looks like.

Analytic combinatorics

Asymptotics

If $f_{\mathcal{C}}(z)$ has *unique* dominant singularity ρ, and

$$f_{\mathcal{C}}(z) \;=\; g(z) + \frac{h(z)}{(1 - z/\rho)^{\alpha}},$$

where $\alpha \notin -\mathbb{N}$, and both g and h are analytic on $\overline{D}(0,\rho)$. Then,

$$|\mathcal{C}_n| \;=\; [z^n] f_{\mathcal{C}}(z) \;\sim\; \frac{h(\rho)}{\Gamma(\alpha)} \rho^{-n} n^{\alpha-1}.$$

Fibonacci tilings

$f_{\mathcal{T}^{(2)}}(z) \;=\; \dfrac{1}{1 - z - z^2}$ has dominant singularity (a simple pole) at φ^{-1}.

$$|\mathcal{T}_n^{(2)}| \;\sim\; \frac{\varphi}{\sqrt{5}} \varphi^n.$$

Irrational enumeration

What happens if we relax the requirement that objects have integer sizes?

How many distinct tilings are there of a strip of length x using square tiles and tiles of length $\beta \notin \mathbb{Q}$, if the final tile may be a partial tile?

A strip of length π^2 tiled with tiles of length 1 and $\sqrt{2}$

Irrational generating functions

Irrational generating function (IGF) for $\mathcal{C}$:

$$f_{\mathcal{C}}(z) \;=\; \sum_{\mathfrak{c}\in\mathcal{C}} z^{|\mathfrak{c}|} \;=\; \sum_{\lambda\in\Lambda_{\mathcal{C}}} |\mathcal{C}_\lambda| z^\lambda \;=\; c_1 z^{\lambda_1} + c_2 z^{\lambda_2} + c_3 z^{\lambda_3} + \ldots,$$

where $\Lambda_{\mathcal{C}} = \{|\mathfrak{c}| : \mathfrak{c} \in \mathcal{C}\} = \{\lambda_1 < \lambda_2 < \lambda_3 < \ldots \to \infty\}$.

- A formal power series that *admits irrational exponents*.
- We call these Ribenboim series after Paulo Ribenboim (1928–).
  - He investigated these series in the 1990s.

Irrational generating functions

Strip tilings with tiles of length either 1 or $\beta \notin \mathbb{Q}$ (without final partial tiles).

$$f_{\mathcal{T}}(z) \;=\; \frac{1}{1 - z - z^{\beta}}.$$

$\beta = \sqrt{2}$:

$$\begin{aligned} f_{\mathcal{T}}(z) \;=\; 1 &+ z + z^{\sqrt{2}} + z^2 + 2z^{1+\sqrt{2}} + z^{2\sqrt{2}} + z^3 + 3z^{2+\sqrt{2}} \\ &+ 3z^{1+2\sqrt{2}} + z^4 + z^{3\sqrt{2}} + 4z^{3+\sqrt{2}} + 6z^{2+2\sqrt{2}} + \ldots \end{aligned}$$

Asymptotics of irrational classes

Coefficients in an IGF can fluctuate wildly:

Seven consecutive terms in $f_{\mathcal{T}(\sqrt{2})}(z)$:

$$8z^{1+7\sqrt{2}},\ \ z^{11},\ \ 126z^{4+5\sqrt{2}},\ \ 120z^{7+3\sqrt{2}},\ \ z^{8\sqrt{2}},\ \ 11z^{10+\sqrt{2}},\ \ 84z^{3+6\sqrt{2}}.$$

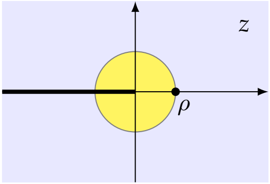
We consider the number of objects of size *at most* a given value:

- $\mathcal{C}_{\leqslant x} = \{\mathfrak{c} \in \mathcal{C} : |\mathfrak{c}| \leqslant x\}$.
- If $f(z) = \sum_{\lambda\in\Lambda} c_\lambda z^\lambda$, then $[z^{\leqslant x}]f(z) = \sum_{\lambda\leqslant x} c_\lambda$.
- $[z^{\leqslant x}]f_{\mathcal{C}}(z) = |\mathcal{C}_{\leqslant x}|$.

Analytic properties of Ribenboim series

$$f(z) = \sum_{\lambda \in \Lambda} c_\lambda z^\lambda$$

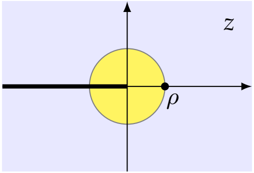
- Radius of convergence in $[0, \infty]$.
- Analytic within disk of convergence, except for branch cut on negative real axis.
- Nonnegative coefficients and finite radius of convergence ρ $\implies$ singularity at $z = \rho$.



Exponential transform & Dirichlet generating functions

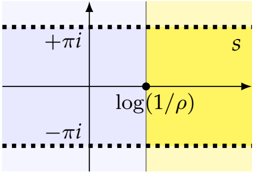
If $f(z) = \sum_{\lambda \in \Lambda} c_\lambda z^\lambda$, then its exponential transform is $F(s) = \sum_{\lambda \in \Lambda} c_\lambda e^{-\lambda s}$.

- This is a (generalised) Dirichlet series.
- Dirichlet generating function (DGF): Exponential transform of an IGF.
- Cut-disk of convergence maps into the right half plane $\{s : \mathfrak{Re}\, s > \log(1/\rho)\}$.
- $F(s)$ analytic to the right of its line of convergence $\mathfrak{Re}\, s = \log(1/\rho)$.



$$z \mapsto e^{-s}$$

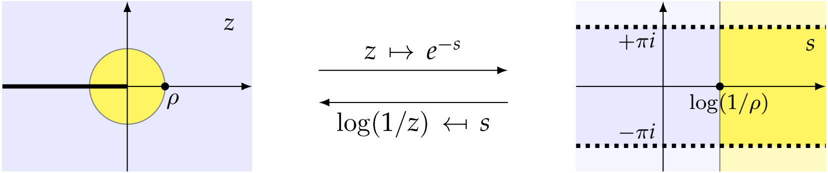
$$\log(1/z) \leftarrow s$$



Intrinsic irrationality

We need to restrict the IGFs we consider.

A class is intrinsically irrational if it has an IGF with radius of convergence $\rho \in (0, 1)$, and $\log(1/\rho)$ is the *unique* singularity of its DGF on its line of convergence.



- We have a nice sufficient condition for intrinsic irrationality.
- If $\mathcal{C}$ has an OGF, then its DGF is periodic in the imaginary direction and has singularities at $\{\log(1/\rho) + 2\pi k i : k \in \mathbb{Z}\}$.

Asymptotics of intrinsically irrational classes

Theorem

If $\mathcal{C}$ is intrinsically irrational with radius of convergence ρ and IGF

$$f_{\mathcal{C}}(z) \;=\; g(z) + \frac{h(z)}{(1 - z/\rho)^{\alpha}},$$

where $\alpha \notin -\mathbb{N}$, and both g and h are analytic on $\overline{D}(0,\rho) \setminus \mathbb{R}^{\leqslant 0}$. Then,

$$|\mathcal{C}_{\leqslant x}| \;=\; \left[z^{\leqslant x}\right] f_{\mathcal{C}}(z) \;\sim\; \frac{h(\rho)}{\log(1/\rho)\,\Gamma(\alpha)}\,\rho^{-x} x^{\alpha-1}.$$

- Irrational classes that aren't intrinsically irrational may exhibit periodic oscillations in their asymptotics.

Strip tiling: two tiles

Square tiles and tiles of length $\beta \notin \mathbb{Q}$:

$$f_{\mathcal{T}}(z) \;=\; \frac{1}{1 - z - z^{\beta}}.$$

$\mathcal{T}$ is intrinsically irrational when β is irrational.

Asymptotics: $$|\mathcal{T}_{\leqslant x}| \;\sim\; \frac{\rho^{-x}}{H(\rho)},$$

where ρ is the positive root of $1 - z - z^{\beta}$, and

$$H(x) = -x \log x - (1 - x) \log(1 - x).$$

- A mysterious connection to entropy.

Strip tiling: any irrational set Γ of tile lengths

A set $S \subset \mathbb{R}$ is irrational if there is no ω such that $S \subseteq \omega\mathbb{Z}$.

$\mathcal{T}^{\Gamma}$: Tilings with tiles whose lengths are drawn from an irrational set Γ.

$$f_{\mathcal{T}^{\Gamma}}(z) \;=\; \frac{1}{1-\sum_{\gamma\in\Gamma} z^{\gamma}}.$$

$$\left|\mathcal{T}^{\Gamma}_{\leqslant x}\right| \;\sim\; \frac{1}{\log(1/\rho)\sum_{\gamma\in\Gamma}\gamma\,\rho^{\gamma}}\,\rho^{-x},$$

where ρ is the unique positive root of $\sum_{\gamma\in\Gamma} z^{\gamma} = 1$.

Strip tiling: partial final tiles

Sets of tilings with a final partial tile, $\mathcal{V}^{\Gamma}$, contain tilings of *every* nonnegative real size, so don't have an IGF.

$$\left|\mathcal{V}^{\Gamma}_{x}\right| \;=\; \big(|\Gamma| - 1\big)\left|\mathcal{T}^{\Gamma}_{<x}\right| \;+\; 1.$$

- Each tiling in $\mathcal{V}^{\Gamma}_{x}$ can be created uniquely from a partial tiling $\mathfrak{t} \in \mathcal{T}^{\Gamma}_{<x}$ by adding a run of tiles of the same colour.
- This colour must differ from the colour of the last tile in $\mathfrak{t}$.
- There are $|\Gamma| - 1$ choices, unless $\mathfrak{t}$ is empty when any colour is OK.

$$\left|\mathcal{V}^{\Gamma}_{x}\right| \;\sim\; \big(|\Gamma| - 1\big)\left|\mathcal{T}^{\Gamma}_{\leqslant x}\right|.$$

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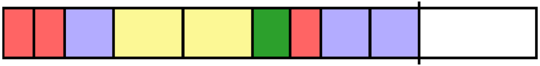

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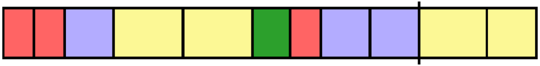

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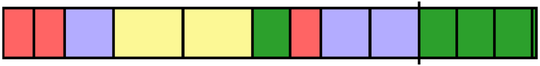

$$\left|\mathcal{V}_x^\Gamma\right| \;=\; \big(|\Gamma|-1\big)\left|\mathcal{T}_{<x}^\Gamma\right| \;+\; 1.$$

- Each tiling in $\mathcal{V}_x^\Gamma$ can be created uniquely from a partial tiling $\mathfrak{t} \in \mathcal{T}_{<x}^\Gamma$ by adding a run of tiles of the same colour.
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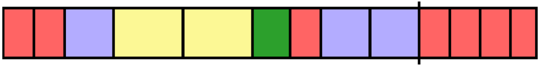

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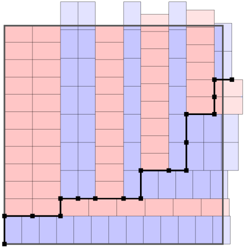
$$\left|\mathcal{V}^{\Gamma}_{x}\right| \;\sim\; \big(|\Gamma|-1\big)\left|\mathcal{T}^{\Gamma}_{\leqslant x}\right|.$$

Open question: Tiling a floor

Tiling a square floor with $\beta \times \gamma$ tiles, where $\beta/\gamma \notin \mathbb{Q}$.

- Tiles may be laid in either orientation, starting in the southwest corner, partial tiles being permitted along the east and north walls.



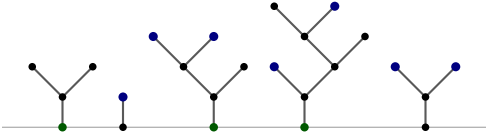
A tiling of a square with sides of length 4π by rectangular tiles of dimension $1 \times \varphi$

Question

Asymptotically, how many tilings are there of an $x \times x$ square using $\beta \times \gamma$ tiles, when $\beta/\gamma \notin \mathbb{Q}$?

Forests of rooted binary trees

- Root of degree one, of size either 1 or γ.
- Leaves of size either 1 or β.
- Internal vertices of degree three and size 1.



A forest of rooted binary trees in $\mathcal{F} = \mathcal{F}^{(\beta,\gamma)}$

IGF:
$$f_{\mathcal{F}}(z) = \frac{2z}{z - z^{\gamma} + (z + z^{\gamma})\sqrt{1 - 4z(z + z^{\beta})}}.$$

Consider the *family* of combinatorial classes, $\{\mathcal{F}^{(\beta,\gamma)} : \beta, \gamma > 0\}$.

Phase transitions: Forests of rooted binary trees

IGF: $$f_{\mathcal{F}}(z) = \frac{2z}{z - z^{\gamma} + (z + z^{\gamma})\sqrt{1 - 4z(z + z^{\beta})}}.$$

- $\mathcal{F}$ intrinsically irrational if $\gamma < 1$ and $\gamma \notin \mathbb{Q}$ or if $\gamma \geqslant 1$ and $\beta \notin \mathbb{Q}$.
- ρ_{β}: positive root of $1 - 4z^2 - 4z^{1+\beta}$.
- ρ_{γ}: positive root of $z^{\gamma} - z - (z + z^{\gamma})\sqrt{1 - 4z(z + z^{\beta})}$, if $\gamma < 1$.

Asymptotic enumeration

Three phases, with distinct critical exponents:

$$\left|\mathcal{F}_{\leqslant x}\right| \sim \begin{cases} c_1\,\rho_{\gamma}^{-x}, & \text{if } \gamma < 1 \text{ and } \gamma \notin \mathbb{Q}, \\ c_2\,\rho_{\beta}^{-x} x^{-1/2}, & \text{if } \gamma = 1 \text{ and } \beta \notin \mathbb{Q}, \\ c_3\,\rho_{\beta}^{-x} x^{-3/2}, & \text{if } \gamma > 1 \text{ and } \beta \notin \mathbb{Q}, \end{cases}$$

for constants c_1, c_2 and c_3, which depend only on β and γ.

Phase transitions: Forests of rooted binary trees

IGF: $$f_{\mathcal{F}}(z) = \frac{2z}{z - z^\gamma + (z + z^\gamma)\sqrt{1 - 4z(z + z^\beta)}}.$$

- $\mathcal{F}$ intrinsically irrational if $\gamma < 1$ and $\gamma \notin \mathbb{Q}$ or if $\gamma \geqslant 1$ and $\beta \notin \mathbb{Q}$.
- ρ_β: positive root of $1 - 4z^2 - 4z^{1+\beta}$.
- ρ_γ: positive root of $z^\gamma - z - (z + z^\gamma)\sqrt{1 - 4z(z + z^\beta)}$, if $\gamma < 1$.

Asymptotic structure

$T(\mathfrak{f})$ is the number of trees in a forest $\mathfrak{f}$:

$$\mathbb{E}_{\leqslant x}[T] \sim \begin{cases} t_1 x, & \text{if } \gamma < 1, \\ t_2\sqrt{x}, & \text{if } \gamma = 1, \\ t_3, & \text{if } \gamma > 1, \end{cases}$$

where t_1, t_2 and t_3 depend only on β and γ.

Thanks!

Thanks for listening!

Takk fyrir að hlusta!

Reference

David Bevan and Julien Condé. Introducing irrational enumeration: analytic combinatorics for objects of irrational size. arXiv:2412.14682.