Chow functions for partially ordered sets

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Build a new class of polynomials associated to various objects in combinatorics.

- Do these polynomials have interesting features?
- When you specialize these polynomials to specific cases do you recover any known result?

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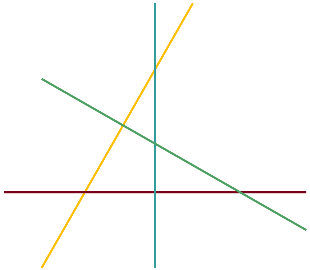
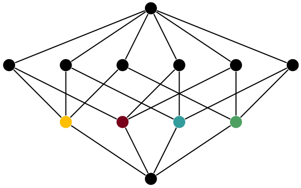
Joint work with Luis Ferroni (IAS – University of Pisa) and Jacob Matherne (NCSU).

Posets

Hyperplane arrangements

Consider a hyperplane arrangement over some field $\mathbb{K}$.

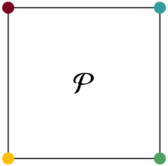


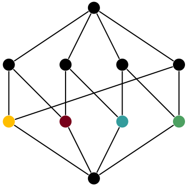


This is a geometric lattice (atomistic and semimodular).

Polytopes

Let $\mathcal{P}$ be a polytope. Consider its face lattice





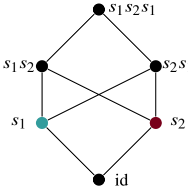
This is an Eulerian poset.

Bruhat order

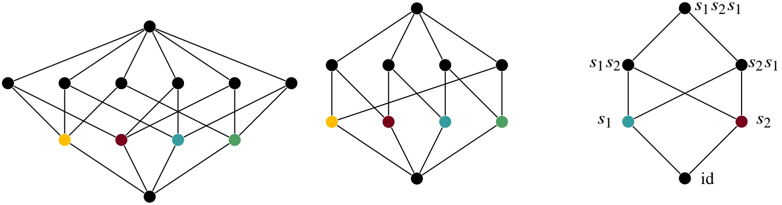
Let (W,S) be a Coxeter group and $T = \{wsw^{-1} \mid w \in W, s \in S\}$.
The *Bruhat order* is defined as

$$u \leqslant v \Leftrightarrow \exists w_0, \ldots, w_n \text{ such that } w_0 = u, w_n = v, w_i^{-1} w_{i+1} \in T.$$

$$\mathfrak{S}_3 = \left\langle s_1, s_2 \mid {s_1}^2 = {s_2}^2 = (s_1 s_2 s_1)^2 = 1 \right\rangle$$

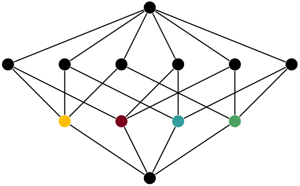


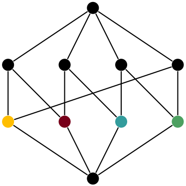
Polynomials for posets

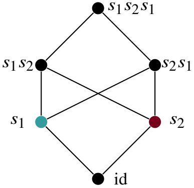


These posets are bounded and graded.

Polynomials for posets







These posets are bounded and graded.
Usually, one associates polynomials to posets using the language of the incidence algebra.

Incidence algebra

Incidence algebra

$$I(P) = \{f : P \times P \to \mathbb{Z}[t],\ f([x, y]) = f_{xy}(t)\}$$

- $(f + g)_{xy}(t) = f_{xy}(t) + g_{xy}(t)$
- $(fg)_{xy}(t) = \sum_{x \leqslant z \leqslant y} f_{xz}(t) g_{zy}(t)$

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- Identity

$$\delta_{xy}(t) = \begin{cases} 1 & x = y \\ 0 & \text{otherwise} \end{cases}$$

Incidence algebra (ctd.)

Restrict to the subalgebra
$\mathcal{I}(P) = \{f \in I(P) \mid \deg f_{xy}(t) \leqslant \rho(y) - \rho(x)\}$.

$$\mathrm{rev} : \mathcal{I}(P) \to \mathcal{I}(P), \qquad f^{\mathrm{rev}}_{xy}(t) = t^{\rho(y)-\rho(x)} f_{xy}(t^{-1}).$$

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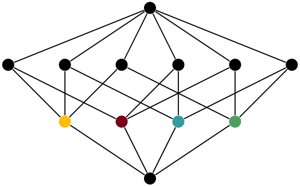
Definition (Kazhdan–Lusztig, Stanley, Brenti)

An element $\kappa \in \mathcal{I}(P)$ is a P-kernel if and only if

$$\kappa^{-1} = \kappa^{\mathrm{rev}}.$$

Characteristic polynomial

$$\chi_{xy}(t) = \sum_{x \leqslant z \leqslant y} \mu_{xz} t^{\rho(y)-\rho(z)}.$$



In this case

$$\chi_P(t) = t^3 - 4t^2 + 6t - 3.$$

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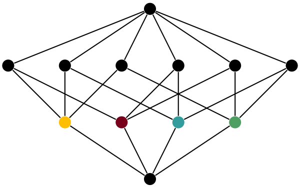


In this case

$$\chi_P(t) = t^3 - 4t^2 + 6t - 3.$$

χ is a P-kernel for every poset!

Eulerian P-kernel

Theorem (Stanley)

A poset P is Eulerian if and only if

$$\varepsilon : [x, y] \mapsto (t-1)^{\rho(y)-\rho(x)}$$

is a P-kernel.

Coxeter P-kernel

For every $x \leqslant y \in W$, if $\rho(ys) < \rho(y)$ we define

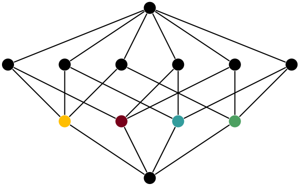
$$R_{xy}(t) := \begin{cases} 1 & x = y \\ R_{xs,ys}(t) & \rho(xs) < \rho(x) \\ tR_{xs,ys}(t) + (t-1)R_{x,ys}(t) & \rho(xs) > \rho(x) \end{cases}$$

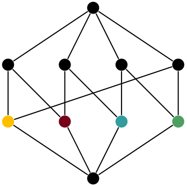
Theorem (Björner–Brenti)

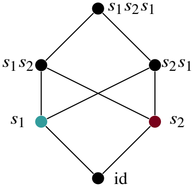
The function $R : [x, y] \mapsto R_{xy}(t)$ is a P-kernel for every Coxeter group.

The new stuff

What we have so far







$$\chi_P(t) = (t-1)(t^2-3t+3) \qquad \varepsilon_P(t) = (t-1)^3 \qquad R_P(t) = (t-1)(t^2-t+1)$$

Reduced P-kernels

Lemma

If κ is a P-kernel, then $(t-1) \mid \kappa_{xy}(t)$ for every $x < y$.

Definition (Ferroni–Matherne–V.)

$$\overline{\kappa}_{xy}(t) := \begin{cases} -1 & x = y \\ \frac{\kappa_{xy}(t)}{t-1} & x < y \end{cases}$$

Chow functions

Definition (Ferroni–Matherne–V.)

$$\mathrm{H} := (-\overline{\kappa})^{-1}.$$

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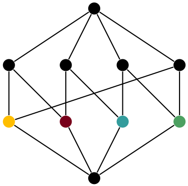
$$\mathrm{H} := (-\overline{\kappa})^{-1}.$$

$$\mathrm{H}_{xy}(t) = \sum_{x<z\leqslant y} \overline{\kappa}_{xz}(t)\mathrm{H}_{zy}(t)$$

$$\mathrm{H}_P(t) := \mathrm{H}_{\widehat{01}}(t).$$

Example

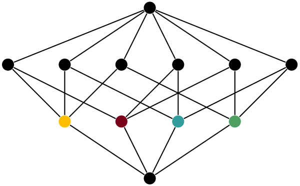


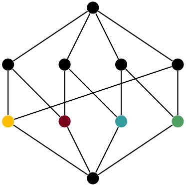
$$\mathrm{H}_{xy}(t) = \sum_{x<z\leqslant y} \overline{\kappa}_{xz}(t)\mathrm{H}_{zy}(t)$$

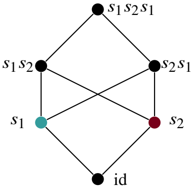
$$\overline{\kappa}_{xy}(t) = (t-1)^{\rho(y)-\rho(x)-1}$$

$$\begin{aligned}
&\mathrm{H}_P(t)\\
&= 4(t-1)^0\,\mathrm{H}_{x\widehat{1}}(t) + 4(t-1)^1\,\mathrm{H}_{y\widehat{1}}(t) + (t-1)^2\,\mathrm{H}_{\widehat{1}\widehat{1}}(t)\\
&= 4\cdot 1\cdot (t+1) + 4\cdot (t-1)\cdot 1 + (t-1)^2\cdot 1\\
&= t^2 + 6t + 1.
\end{aligned}$$

Examples







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$$\mathrm{H}_P(t) = t^2 + 7t + 1 \qquad \mathrm{H}_P(t) = t^2 + 6t + 1 \qquad \mathrm{H}_P(t) = t^2 + 3t + 1$$

Palindromicity...

Theorem (Ferroni–Matherne–V.)

For every P and κ, $\mathrm{H}_P(t)$ is a palindromic polynomial of degree at most $\rho(P) - 1$.
If κ is monic then so is H.

… and more

Theorem (Ferroni–Matherne–V.)

For matroids, polytopes and Bruhat intervals, H *is non-negative and unimodal.*

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This is related to the Kazhdan–Lusztig–Stanley functions of the poset.

Upshot

We now have a new way of computing polynomials that are

- non-negative,
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at least for every matroid, polytope and Coxeter group.

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Theorem (Ferroni–Matherne–Stevens–V.)

The characteristic Chow polynomial of a matroid coincides with the Hilbert–Poincaré series of its Chow ring.

Chow ring

Feichtner–Yuzvinsky define

$$\underline{\mathrm{CH}}(M) = \frac{\mathbb{Q}[x_F \mid F \in P \setminus \widehat{0}]}{I + J}$$

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The Chow ring of a matroid satisfies the Kähler package, i.e. (PD), (HL), (HR).

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(PD) $\implies$ palindromicity (HL) $\implies$ unimodality

Real-rootedness

Conjecture (Huh–Stevens, Ferroni–Schröter)

The Chow polynomial of M only has real roots.

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Proved for

- Uniform matroids [Brändén–V.]
- Matroids of rank at most 5 [Ferroni–Matherne–Stevens–V.]

γ-positivity

If $\mathrm{H}(t)$ is palindromic, then we say that it is γ-positive if

$$\mathrm{H}(t) = \sum_{i \geqslant 0} \gamma_i t^i (1+t)^{r-2i}, \qquad \gamma_i \geqslant 0.$$

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$$\gamma_{\mathrm{H}}(t) = \sum_i \gamma_i t^i.$$

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Theorem (Ferroni–Matherne–V.)

If P is a Cohen–Macaulay poset, then the characteristic Chow polynomial is γ-positive.

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Proof: Combinatorial interpretation for the coefficients.

Conjecture (Ferroni–Matherne–V.)

The characteristic Chow polynomial of a Cohen–Macaulay poset only has real roots.

Eulerian Chow functions

The f-polynomial of a simplicial complex is given by

$$f_\Delta(t) = \sum_{F \in \Delta} t^{\dim F + 1}$$

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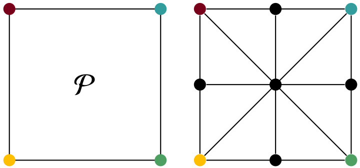
Theorem (Ferroni–Matherne–V.)

If P is an Eulerian poset, the Eulerian Chow polynomial coincides with the h-polynomial of the order complex of P.

Example

In the simpler case of a face lattice of a polytope $\mathcal{P}$, this corresponds to the baricentric subdivision $\mathrm{sd}(\mathcal{P})$.



$$\begin{aligned} f_\Delta(t) &= 8t^3 + 16t^2 + 9t + 1 \\ &= 1(t^3 + 3t^2 + 3t + 1) \\ &+ 6t(t^2 + 2t + 1) \\ &+ t^2(t + 1) \end{aligned}$$

$$h_\Delta(t) = 1 + 6t + t^2.$$

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Conjecture (Brenti–Welker)

The Eulerian Chow polynomial of a polytope only has real roots.

Proved when $\mathcal{P}$ is a simplicial polytope.

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Proved when $\mathcal{P}$ is a simplicial polytope.

Conjecture (Athanasiadis–Kalampogia-Evangelinou)

The Eulerian Chow polynomial of an Eulerian Cohen–Macaulay poset only has real roots.

γ-positivity

Theorem

If P is an Eulerian Cohen–Macaulay poset, then the Eulerian Chow polynomial is γ-positive.

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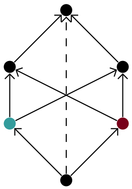
Proof

- Gal shows that $\gamma_P(t) = \Phi(1, 2t)$,
  where $\Phi(c, d)$ is the *cd-index*.
- Karu proves that $\Phi(c, d)$ is non-negative when P is Eulerian and Cohen–Macaulay.

Coxeter Chow functions

Let $B(x, y)$ be the Bruhat graph of $[x, y]$, where $z_1 \longrightarrow z_2$ if $z_1^{-1} z_2 \in T$.



Theorem (Ferroni–Matherne–V.)

H *enumerates paths in the Bruhat graph,*

$$\mathrm{H}_{xy}(t) = \sum_{\Delta \in B(x,y)} t^{\frac{\rho(y)-\rho(x)-\ell(\Delta)}{2} + \mathrm{des}(\Delta)}.$$

Real-rootedness

Conjecture (Ferroni–Matherne–V.)

The Coxeter Chow polynomials only have real roots.

Checked on all intervals of $\mathfrak{S}_n$ for $n \leqslant 7$.

γ-positivity

- Billera–Brenti define a more general version of the cd-index called the *complete* cd*-index* $\widetilde{\Psi}(c, d)$.

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Theorem (Ferroni–Matherne–V.)

$$\gamma(t^2) = t^{\rho(y)-\rho(x)}\widetilde{\Psi}(t^{-1},2).$$

Conjecture (Billera–Brenti, Ferroni–Matherne–V.)

The complete cd-index (resp. γ) is non-negative.

Conclusion/Open questions

- What are other nice P-kernels that provide well-behaved families of Chow polynomials?
- What does real-rootedness mean for a Hilbert series?
- This new language lets us collect under the same object a number of real–rootedness conjectures.
- Borrowing tools from one area and applying them to another seems to be effective (cd-index and complete cd-index).

Thank you! ☺