Heffter arrays from a discrete tomography perspective

15th NORCOM

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Reykjavík, June 16, 2025

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Last trip to Iceland

Heffter arrays

Definition

A Heffter array $H(m, n; h, k)$ is an $m \times n$ matrix with entries in $\mathbb{Z}_{2nk+1}$, such that

- each row contains h filled cells and each column contains k filled cells,
- for any $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, either x or $-x$ appears in the matrix,
- the sum of the entries on each row and column is zero (in $\mathbb{Z}_{2nk+1}$).

Necessary conditions for the existence of $H(m, n; h, k)$:

- $mh = nk$,
- $3 \leq k \leq m$,
- $3 \leq h \leq n$.

Special cases

If $m = n$: $H(n; k)$.

If all entries are filled: $H(m, n)$.

Definition

Shiftable Heffter array: the number of positive entries equals the number of negative entries in each row and column.

Integer Heffter array: rows and columns sum to zero in $\mathbb{Z}$.

Some results and generalizations

- $H(n;k)$ exists iff $3 \leq k \leq n$ [Archdeacon at al. 2015, Dinitz et al. 2017, Cavenagh et al. 2019];
- integer $H(n;k)$ exists iff $3 \leq k \leq n$ and $nk \equiv 0 \mod 4$ [Archdeacon at al. 2015, Dinitz et al. 2017];
- $H(m,n)$ exists iff $m, n \geq 3$ [Archdeacon et al. 2017];
- integer $H(m,n)$ exists iff $m, n \geq 3$ and $mn \equiv 0 \mod 4$ [Archdeacon et al. 2017].

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Variants, generalizations and related concepts:

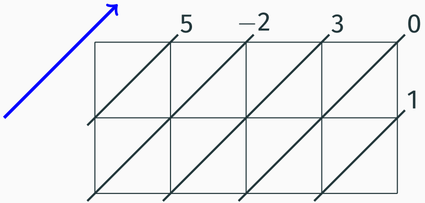
- weak Heffter arrays [Archdeacon 2015];
- λ-fold relative Heffter arrays [Costa et al. 2020, Costa et al. 2021];
- signed magic arrays [Khodkar et al. 2017];
- Archdeacon arrays [Costa et al. 2020];
- near alternating sign matrices [Mella et al. 2024].

Discrete tomography

(One of) the origins: H.J. Ryser. *Combinatorial properties of matrices of zeros and ones*. Canad. J. Math. (1957).

Aim: reconstruct an $m \times n$ matrix A from the knowledge of its line sums along a set of directions.

In a discrete tomographic language: find $f : A \longrightarrow B$ from the projections along given directions.



Ambiguities

In general, the reconstruction problem is ill-posed.

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Definition

A matrix is said to be a ghost w.r.t. a set S of directions (briefly, an S-ghost) if it has null sums along the lines with directions in S.

The sum of a matrix, which is a solution to a tomographic problem, and a (non-trivial) ghost gives another solution to the problem.

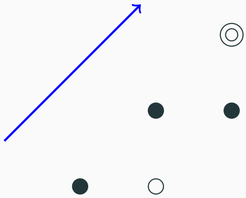
$$\begin{array}{|c|c|c|c|}\hline 5 & 1 & 0 & 2 \\ \hline -3 & 3 & -2 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 0 & -1 & 0 & 3 \\ \hline 1 & 0 & -3 & 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 5 & 0 & 0 & 5 \\ \hline -2 & 3 & -5 & 1 \\ \hline \end{array}$$

Switching components

Definition

A switching component w.r.t. S if a pair of (multi-)sets of points (Z, W) such that each line with direction in S meets the same number of points of Z and W.

If we attach opposite values to the elements of the two (multi-)sets, we get an S-ghost.



$$\begin{array}{ccc} 0 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{array}$$

Binary ghosts

Remark: A switching component can always be modified so that Z and W are sets, in a larger region.

Equivalently, a ghost can always be made binary.

1	0	-2	1

$$\downarrow$$

1	1	-1	0	-1	-1	1

Known construction revised 1/3

Idea from [Archdeacon et al. 2015]: Use sub-rectangles.

Let G be a binary S-ghost with the minimal bounding rectangle (say $m_G \times n_G$) and s non-zero entries. Let A_0 be an $m_G \times n_G$ matrix such that

$$a_0(i,j) = \begin{cases} 0 & \text{if } g(i,j) = 0, \\ \in \{1, 3, \ldots, s-1\} & \text{if } g(i,j) = 1, \\ \in \{-s, -s+2, \ldots, -2\} & \text{if } g(i,j) = -1 \end{cases}$$

and such that the non-zero entries of A_0 are all distinct.

Example. Let $S = \{(1,0),(0,1)\}$ and $G = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ ($s = 4$). Then a possible A_0 is $\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$.

Known construction revised 2/3

For $\ell = 1, 2, \ldots, s-1$, let $A_\ell = A_0 + \ell\, s\, G$.

Example.

$$A_1 = \begin{bmatrix} 5 & -8 \\ -6 & 7 \end{bmatrix},\ A_2 = \begin{bmatrix} 9 & -12 \\ -10 & 11 \end{bmatrix},\ A_3 = \begin{bmatrix} 13 & -16 \\ -14 & 15 \end{bmatrix}.$$

Replace the positive entries of G with a matrix in $\{A_0, \ldots, A_{\frac{s}{2}-1}\}$ (all different, arbitrary assignment). Similarly, replace the negative entries of G with a matrix in $\{-A_{\frac{s}{2}}, \ldots, -A_{s-1}\}$ (all different, arbitrary assignment) and the null entries of G with the $m_G \times n_G$ zero matrix.

Known construction revised 3/3

What we get is a (shiftable integer) Archdeacon array of size $m_G^2 \times n_G^2$.

Example. $H = \begin{bmatrix} A_1 & -A_3 \\ -A_2 & A_0 \end{bmatrix} = \begin{bmatrix} 5 & -8 & -13 & 16 \\ -6 & 7 & 14 & -15 \\ -9 & 12 & 1 & -4 \\ 10 & -11 & -2 & 3 \end{bmatrix}.$

In this case, it is a Heffter array $H(16, 16)$, with entries in $\mathbb{Z}_{33}$. It has a stronger structure since S contains only the horizontal and the vertical directions.

Different ghosts w.r.t. the same directions

Example. Again $S = \{(1,0),(0,1)\}$. Let $G = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$,

$A_0 = \begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}$, $A_\ell = A_0 + 4\,\ell\,G'$ for $\ell = 1, \ldots, 5$, where

$G' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Replace

- the positive entries of G with distinct elements of $\{A_0, A_1, A_2\}$,
- the negative entries with distinct elements of $\{-A_3, -A_4, -A_5\}$,
- the null entries with the 2×2 zero matrix.

Different ghosts w.r.t. the same directions

Therefore

$$H = \begin{bmatrix} A_0 & \mathbf{0} & -A_3 \\ \mathbf{0} & -A_4 & A_1 \\ -A_5 & A_2 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & -4 & 0 & 0 & -13 & 16 \\ -2 & 3 & 0 & 0 & 14 & -15 \\ 0 & 0 & -17 & 20 & 5 & -8 \\ 0 & 0 & 18 & -19 & -6 & 7 \\ -21 & 24 & 9 & -12 & 0 & 0 \\ 22 & -23 & -10 & 11 & 0 & 0 \end{bmatrix}$$

which is an $H(6;4)$ with entries in $\mathbb{Z}_{49}$.

Example of a "strict" Archdeacon array

$$S = \{(1,0), (0,1), (1,2)\}, \quad G = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

The previous construction leads to a Heffter array when

- $S = \{(1,0),(0,1),(1,1)\}$, with corresponding minimal ghost $G = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$,
- $S = \{(1,0),(0,1),(1,-1)\}$, with corresponding minimal ghost $G = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$,
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How to get Heffter arrays

Our aim is to construct Heffter arrays having zero sum even along directions other than the coordinate ones.

Other directions do not preserve the number of filled (i.e., non-zero) cells in every row/column.

Idea: Construct ghosts from tessellations.

Gluing

Type 1 move: vertex gluing (VG).

$$
\begin{array}{ccc}
 & 1 & -1 \\
-1 & & 1 \\
1 & -1 &
\end{array}
\quad \longrightarrow \quad
\begin{array}{ccccc}
 & & & 1 & -1 \\
 & & -1 & & 1 \\
 & 1 & \times & -1 & \\
-1 & & 1 & & \\
1 & -1 & & &
\end{array}
$$

Gluing

Type 2 move: edge gluing along the diagonal side (EG).

$$
\begin{array}{ccc}
 & 1 & -1 \\
-1 & & 1 \\
1 & -1 &
\end{array}
\quad \longrightarrow \quad
\begin{array}{cccc}
 & 1 & -1 & \\
-1 & & \times & 1 \\
1 & \times & & -1 \\
 & -1 & 1 &
\end{array}
$$

Combining the moves

Theorem

An integer linear combination of VG and EG produces a ghost which gives rise to a Heffter array with zero sums along the horizontal, the vertical and the (anti-)diagonal directions.

Note that the number of filled cells in the diagonal directions is not guaranteed to be constant.

Further work

How to proceed:

- find construction which give Archdeacon arrays with *few* different values for the number of filled cells in rows/columns;
- in particular, consider the case of empty rows/columns;
- exploit other known discrete tomographic constructions, such as boundary ghosts;
- consider toroidal projections (Mojette transform).

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Thank you for your attention!