Determinantal Varieties, Linear Codes, and Rook Placements

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Based on joint work with Mahir Bilen Can and previously with Peter Beelen and Sartaj Ul Hasan as well as with Peter Beelen

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Determinantal Varieties

Fix a prime power q, positive integers t, ℓ, m, and define:

- $X = (X_{ij})$: an $\ell \times m$ matrix with variable entries
- $\mathbb{F}_q[X]$: polynomial ring over $\mathbb{F}_q$ in the ℓm variables X_{ij}
- $\mathbb{M}_{\ell \times m}(\mathbb{F}_q)$: set of all $\ell \times m$ matrices with entries in $\mathbb{F}_q$
- $\mathcal{I}_{t+1}$: ideal of $\mathbb{F}_q[X]$ generated by all $(t+1) \times (t+1)$ minors
- $\mathcal{D}_t = \mathcal{D}_t(\ell, m) = \{M \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q) : \mathrm{rk}(M) \le t\}$.
- $\widehat{\mathcal{D}}_t = \widehat{\mathcal{D}}_t(\ell, m)$: corres. projective variety $\mathbb{P}(\mathcal{D}_t) \subseteq \mathbb{P}^{\ell m - 1}(\mathbb{F}_q)$.

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We will also consider open (quasi-affine/quasi-projective) varieties:

- $\mathcal{E}_t = \mathcal{E}_t(\ell, m) = \{M \in \mathbb{M}_{\ell\times m}(\mathbb{F}_q) : \mathrm{rk}(M) > t\}$
- $\widehat{\mathcal{E}}_t = \widehat{\mathcal{E}}_t(\ell, m) = \mathbb{P}\left(\widehat{\mathcal{E}}_t\right) \subseteq \mathbb{P}^{\ell m-1}(\mathbb{F}_q)$.

(Linear) Codes

- $\mathbb{F}_q$: finite field with q elements.
- $[n, k]_q$-code: a k-dimensional subspace C of $\mathbb{F}_q^n$.
- C is nondegenerate if $C \not\subseteq$ coordinate hyperplane of $\mathbb{F}_q^n$.
- Hamming weight of $c = (c_1, \ldots, c_n) \in \mathbb{F}_q^n$:

$$\mathrm{wt}(c) := |\{i : c_i \neq 0\}|.$$

- Minimum distance of a (linear) code C:

$$d(C) := \min\{\mathrm{wt}(c) : c \in C, \ c \neq 0\}.$$

- Spectrum or the Weight distribution of a $[n, k]_q$-code C:

  the sequence $(A_i)_{i \geq 0}$ where $A_i := \#\{c \in C : \mathrm{wt}(c) = i\}$.

  or equivalently, the polynomial $\sum_{i=0}^{n} A_i Z^i$.

Determinantal Codes

Fix an ordering $M_1, \ldots, M_n$ of $\mathcal{D}_t$ and consider the evaluation map

$$\mathrm{Ev} : \mathbb{F}_q[X]_1 \to \mathbb{F}_q^n \quad \text{defined by} \quad \mathrm{Ev}(f) = c_f := (f(M_1), \ldots, f(M_n)),$$

Define $C_{\det}(t; \ell, m) := \mathrm{im}(\mathrm{Ev})$.

Also, let $P_1, \ldots, P_{\hat{n}}$ be an ordering of $\widehat{\mathcal{D}}_t$ and $\widehat{M}_1, \ldots, \widehat{M}_{\hat{n}}$ be their fixed representatives in $\mathbb{M}_{\ell \times m}(\mathbb{F}_q)$. Consider the evaluation map

$$\widehat{\mathrm{Ev}} : \mathbb{F}_q[X]_1 \to \mathbb{F}_q^{\hat{n}} \quad \text{defined by} \quad \widehat{\mathrm{Ev}}(f) = \hat{c}_f := \left(f(\widehat{M}_1), \ldots, f(\widehat{M}_{\hat{n}})\right).$$

Determinantal code: $\widehat{C}_{\det}(t; \ell, m) := \mathrm{im}(\widehat{\mathrm{Ev}})$.

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Determinantal code: $\widehat{C}_{\det}(t; \ell, m) := \mathrm{im}(\widehat{\mathrm{Ev}})$.

One can also define similarly the linear code $C(\widehat{\mathcal{E}}_t(\ell, m))$ corresponding to the open determinantal variety $\widehat{\mathcal{E}}_t(\ell, m)$ and refer to it as an open determinantal code.

We will usually assume (WLOG) that $1 \le t \le \ell \le m$.

Relation between $C_{\det}(t;\ell,m)$ and $\widehat{C}_{\det}(t;\ell,m)$

Proposition

Write $C = C_{\det}(t;\ell,m)$ and $\widehat{C} = \widehat{C}_{\det}(t;\ell,m)$. Let n, k, d, and A_i (resp. $\hat{n}$, $\hat{k}$, $\hat{d}$, and $\hat{A}_i$) denote, respectively, the length, dimension, minimum distance and the number of codewords of weight i of C (resp. $\widehat{C}$). Then

$$n = 1 + \hat{n}(q-1), \qquad k = \hat{k} \qquad \text{and} \qquad d = \hat{d}(q-1).$$

Further,

$$A_{i(q-1)} = \hat{A}_i \quad \text{for } 0 \le i \le \hat{n}.$$

Moreover $A_n = 0$ and $A_j = 0$ whenever $0 \le j \le n$ and $(q-1) \nmid j$.

Question: Determine explicitly the length, dimension, and the minimum distance and more generally, the weight distribution of the determinantal code $\widehat{C}_{\det}(t;\ell,m)$.

Length and Dimension

Proposition (Landsberg (1893))

$\widehat{C}_{\det}(t;\ell,m)$ is nondegenerate of dimension $\hat{k}=\ell m$ and length

$$\hat{n}=\sum_{j=1}^{t}\hat{\mu}_j(\ell,m) \quad \textit{where} \quad \hat{\mu}_j(\ell,m)=\frac{\mu_j(\ell,m)}{q-1}$$

where $\mu_j(\ell,m)$ is the number of matrices in $\mathbb{M}_{\ell\times m}(\mathbb{F}_q)$ of rank j:

$$\mu_j(\ell,m)=q^{\binom{j}{2}}\prod_{i=0}^{j-1}\frac{\left(q^{\ell-i}-1\right)\left(q^{m-i}-1\right)}{q^{i+1}-1}.$$

Notation: For integers a,b with $0<b\le a$, define

$$[a]_q:=\frac{q^a-1}{q-1}, \quad [a]!:=[a]_q[a-1]_q\cdots[1]_q \quad \text{and} \quad \begin{bmatrix} a\\ b\end{bmatrix}:=\frac{[a]!}{[b]![a-b]!}.$$

By convention, $[0]_q:=1=[0]!$ and $\left[\begin{smallmatrix} a\\ b\end{smallmatrix}\right]:=0$ if $b<0$ or $b>a\ge 0$.

(i) $t = \ell = \min\{\ell, m\}$: Here $\widehat{C}_{\det}(t; \ell, m)$ is a simplex code. So

$$\hat{n} = \frac{q^{\ell m} - 1}{q - 1}, \quad \hat{k} = \ell m \quad \text{and} \quad \hat{d} = q^{\ell m - 1}.$$

(ii) $\ell = m = t + 1$: Here $\mathcal{D}_t = \mathbb{M}_{\ell \times m} \setminus \mathrm{GL}_\ell(\mathbb{F}_q)$ while $\widehat{\mathcal{D}}_t$ is the hypersurface in $\mathbb{P}^{\ell^2 - 1}$ given by $\det(X) = 0$. Thus

$$\hat{d} = \hat{n} - \max_H |\widehat{\mathcal{D}}_t \cap H|, \quad \text{where} \quad \hat{n} = |\widehat{\mathcal{D}}_t| = \frac{q^{\ell^2} - 1}{q - 1} - q^{\binom{\ell}{2}} \prod_{i=2}^{\ell} (q^i - 1)$$

The irreducible polynomial $\det(X)$, when restricted to H gives rise to a (possibly reducible) hypersurface in $\mathbb{P}(H) \simeq \mathbb{P}^{\ell^2 - 2}$ of degree $\leq \ell$. Hence by Serre's inequality (1991)

$$|\widehat{\mathcal{D}}_t \cap H| \leq \ell q^{\ell^2 - 3} + \frac{q^{\ell^2 - 3} - 1}{q - 1}.$$

Example (ii) continued

Hence we get a bound on the minimum distance of $\widehat{C}_{\text{det}}(t;\ell,\ell)$:

$$\hat{d} \geq q^{\ell^2-1} + q^{\ell^2-2} - (\ell-1)q^{\ell^2-3} - q^{\binom{\ell}{2}} \prod_{i=2}^{\ell}(q^i - 1).$$

In the special case when $\ell = m = 2$ and $t = 1$, we find

$$|\widehat{\mathcal{D}}_t \cap H| \leq 2q + 1 \quad \text{and} \quad \hat{d} \geq q^2.$$

The Serre bound $2q+1$ is attained if we take H to be any of the coordinate hyperplanes. Hence $d\left(\widehat{C}_{\mathrm{det}}(1;2,2)\right) = q^2$.

Remark: In general, the Serre bound gives a rather crude bound on the minimum distance of the determinantal code $\widehat{C}_{\text{det}}(\ell-1;\ell,\ell)$.

Weight Distribution of Determinantal Codes

Lemma (Beelen–G–Hasan, 2015)

Let $f(X) = \sum_{i=1}^{\ell} \sum_{j=1}^{m} f_{ij} X_{ij} \in \mathbb{F}_q[X]_1$ *and let* $F = (f_{ij})$ *be the coefficient matrix of* f*. Then the Hamming weights of the corresponding codewords* c_f *of* $C_{\mathrm{det}}(t;\ell,m)$ *and* $\hat{c}_f$ *of* $\widehat{C}_{\mathrm{det}}(t;\ell,m)$ *depend only on* $\mathrm{rk}(F)$*. In fact,* $\mathrm{wt}(c_f) = \mathrm{wt}(c_{\tau_r})$ *and* $\mathrm{wt}(\hat{c}_f) = \mathrm{wt}(\hat{c}_{\tau_r})$*, where* $r = \mathrm{rk}(F)$ *and* $\tau_r := X_{11} + \cdots + X_{rr}$ *.*

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Corollary

Each of the codes $C_{\det}(t; \ell, m)$ *and* $\widehat{C}_{\det}(t; \ell, m)$ *have at most* $\ell + 1$ *distinct weights,* $w_0, w_1, \ldots, w_\ell$ *and* $\hat{w}_0, \hat{w}_1, \ldots, \hat{w}_\ell$ *respectively, given by* $w_r = \mathrm{wt}(c_{\tau_r})$ *and* $\hat{w}_r = \mathrm{wt}(\hat{c}_{\tau_r}) = w_r/(q-1)$ *for* $r = 0, 1, \ldots, \ell$. *Moreover, the weight enumerator polynomials* $A(Z)$ *of* $C_{\det}(t; \ell, m)$ *and* $\hat{A}(Z)$ *of* $\widehat{C}_{\det}(t; \ell, m)$ *are given by*

$$A(Z) = \sum_{r=0}^{\ell} \mu_r(\ell, m) Z^{w_r} \quad \text{and} \quad \hat{A}(Z) = \sum_{r=0}^{\ell} \mu_r(\ell, m) Z^{\hat{w}_r},$$

Case of 2×2 minors $[t = 1]$

Using an elementary approach, we obtain rather easily the complete weight distribution of determinantal codes in the case $t = 1$:

Theorem (Beelen–G–Hasan, 2015)

The nonzero weights of $\widehat{C}_{\det}(1; \ell, m)$ are $\hat{w}_1, \ldots, \hat{w}_\ell$, given by

$$\hat{w}_r = \mathrm{wt}(\hat{c}_{\tau_r}) = q^{\ell+m-2} + q^{\ell+m-3} + \cdots + q^{\ell+m-r-1}$$

for $r = 1, \ldots, \ell$. In particular, $\hat{w}_1 < \hat{w}_2 < \cdots < \hat{w}_\ell$ and the minimum distance of $\widehat{C}_{\det}(1; \ell, m)$ is $q^{\ell+m-2}$.

Remark: The exponent $\ell + m - 2$ of q in the minimum distance $\widehat{C}_{\det}(1; \ell, m)$ is precisely the dimension of the determinantal variety $\widehat{\mathcal{D}}_t$ when $t = 1$. Also, the relative distance $\delta = d/n$ of $\widehat{C}_{\det}(1; \ell, m)$ is asymptotically equal to 1 as $q \to \infty$. On the other hand, the rate $R = k/n$ is quite small as $q \to \infty$, but it tends to 1 as $q \to 1$.

Formulas for possible weights in the general case

- Thanks to the above Lemma, the possible nonzero weights of $\widehat{C}_{\det}(t;\ell,m)$ and $C_{\det}(t;\ell,m)$ are precisely

$$\widehat{w}_r(t;\ell,m) = \frac{w_r(t;\ell,m)}{q-1} \quad \text{and} \quad w_r(t;\ell,m) = \sum_{s=1}^{t} \mathfrak{w}_r(s;\ell,m)$$

for $r = 1, \ldots, \ell$, where $\mathfrak{w}_r(s;\ell,m)$ is the number of $\ell \times m$ matrices $M \in \mathbb{M}_{\ell \times m}(\mathbb{F}_q)$ of rank s for which $\tau_r(M) \neq 0$

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- Delsarte (1978), using an explicit determination of characters of the Schur ring of an association scheme corresponding to bilinear forms, showed that $\mathfrak{w}_r(s;\ell,m)$ equals

$$\frac{q-1}{q}\left(\mu_s(\ell,m) - \sum_{i=0}^{\ell}(-1)^{s-i} q^{im+\binom{s-i}{2}} \begin{bmatrix}\ell-i\\ \ell-s\end{bmatrix}\begin{bmatrix}\ell-r\\ i\end{bmatrix}\right).$$

- Ravagnani (2016) gave an alternative approach to Delsarte's formula using MacWilliams identities for suitable Delsarte rank metric codes.

More Formulas for the possible weights of det'l codes

- Ravagnani's formula for $\mathfrak{w}_r(s;\ell,m)$ is as follows.

$$\frac{1}{q}\sum_{i=0}^{\ell}(-1)^{t-i}q^{mi+\binom{s-i}{2}}\begin{bmatrix}\ell-i\\ \ell-s\end{bmatrix}\left(\begin{bmatrix}\ell\\ i\end{bmatrix}+(q-1)\begin{bmatrix}\ell-r\\ i\end{bmatrix}\right).$$

- Equivalence of Delsarte and Ravagnani's formula follows using

$$\mu_s(\ell,m)=\sum_{i=0}^{\ell}(-1)^{s-i}q^{mi+\binom{t-i}{2}}\begin{bmatrix}\ell-i\\ \ell-s\end{bmatrix}\begin{bmatrix}\ell\\ i\end{bmatrix}.$$

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- Beelen–G (2020) obtained two more formulas for $\mathfrak{w}_r(s;\ell,m)$:

$$\frac{q-1}{q}\sum_{j=1}^{r}q^{\binom{j}{2}}\left(\frac{[m]!}{[m-s]!}-(-1)^j\frac{[m-j]!}{[m-s]!}\right)q^{j(\ell-r)}q^{\binom{s-j}{2}}\begin{bmatrix}r\\ s\end{bmatrix}\begin{bmatrix}\ell-r\\ s-j\end{bmatrix}$$

$$=\frac{q-1}{q}\left(\mu_s(\ell,m)-\sum_{j=0}^{r}q^{\binom{j}{2}}(-1)^j\frac{[m-j]!}{[m-s]!}q^{j(\ell-r)}q^{\binom{s-j}{2}}\begin{bmatrix}r\\ j\end{bmatrix}\begin{bmatrix}\ell-r\\ s-j\end{bmatrix}\right).$$

Issues about using these formulas for possible weights

- It is far from obvious whether or not the possible weights $\widehat{w}_r(t;\ell,m)$ of $\widehat{C}_{\rm det}(t;\ell,m)$ are distinct.
- It is also not clear which among the ℓ possible nonzero weights $\widehat{w}_1(t;\ell,m),\dots,\widehat{w}_\ell(t;\ell,m)$ has the least value (so that it would give the minimum distance).
- In general, it would be interesting to know how the weights $\widehat{w}_1(t;\ell,m),\dots,\widehat{w}_\ell(t;\ell,m)$ are ordered.
- Recall that in the simple case when $t=1$, all these questions have nice answers since $\widehat{w}_1(1;\ell,m)<\cdots<\widehat{w}_\ell(1;\ell,m)$.
- Even in the very simple case $t=\ell$, where $\widehat{C}_{\rm det}(t;\ell,m)$ is the simplex code of dimension ℓm, and

$$\widehat{w}_1(\ell;\ell,m)=\cdots=\widehat{w}_\ell(\ell;\ell,m)=q^{\ell m-1},$$

  the above formulas give a much more complicated expression.

A partial solution and some Conjectures

Theorem (Beelen–G (2020))

Suppose $1 < r \le \ell$ *and* $1 \le t < \ell$. *Then*

$$\widehat{w}_r(t;\ell,m) - \widehat{w}_1(t;\ell,m) = q^t \widehat{\mathfrak{w}}_{r-1}(t;\ell-1,m-1),$$

Consequently, $\widehat{w}_1(t;\ell,m) < \widehat{w}_r(t;\ell,m)$. *Moreover,*

$$\widehat{w}_1(t;\ell,m) = q^{\ell+m-2}\sum_{s=1}^{t}\mu_{s-1}(\ell-1,m-1).$$

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Conjecture (Beelen–G (2020))

Assume that t,ℓ,m *are integers with* $1 < t < \ell \le m$*. Then:*

1. *All weights* $\widehat{w}_1(t;\ell,m),\ldots,\widehat{w}_\ell(t;\ell,m)$ *are mutually distinct.*
2. $\widehat{w}_1(t;\ell,m) < \widehat{w}_2(t;\ell,m) < \cdots < \widehat{w}_{\ell-t+1}(t;\ell,m)$.
3. *For all* $\ell-t+2 \le r \le \ell$*, the weight* $\widehat{w}_r(t;\ell,m)$ *lies between* $\widehat{w}_{r-2}(t;\ell,m)$ *and* $\widehat{w}_{r-1}(t;\ell,m)$*.* [Interlacing Conjecture]

In a distinct, but in hindsight, related development, Mahir Bilen Can (2023) considered the linear code, say $C(m)$, associated to $\mathrm{GL}_m(\mathbb{F}_q)$ given by the evaluations of homogeneous linear polynomials in m^2 (matrix of) variables on elements of $\mathrm{GL}_m(\mathbb{F}_q)$. He showed that;

$$\begin{aligned} \text{length}(C(m)) &= q^{\binom{m}{2}}(q-1)^m[m]!, \quad \dim(C(m)) = m^2, \text{ and} \\ \text{min. dist.}(C(m)) &= q^{\binom{m}{2}-1}(q-1)^{m-1}\left((q-1)^2[m]! - [m-2]!\right) \end{aligned}$$

For example, $C(4)$ is a $[q^4 - q^3 - q^2 + q,\ 4,\ q^4 - 2q^3 + q]$-code. In fact, Can not only found the minimum (nonzero) weight of $C(m)$, but also the maximum weight of $C(m)$. This was done by analyzing for $r = 1, \ldots, m$, the function

$$f_r(m) := |\mathrm{GL}_m(\mathbb{F}_q) \cap \{M \in M_m(\mathbb{F}_q) : \tau_r(M) = m_{11} + \cdots + m_{rr} = 0\}|$$

and showing that

$$\max_{1 \le r \le m} f_r(m) = f_2(m) \quad \text{and} \quad \min_{1 \le r \le m} f_r(m) = f_1(m).$$

Main Result

Theorem (Can–G)

All three parts of the conjecture hold in the affirmative. In other words, if $1 < r < \ell \leq m$, then the weights $\widehat{w}_r = \widehat{w}_r(t; \ell, m)$, $r = 1, \ldots, \ell$, of $\widehat{C}_{\mathrm{det}}(t; \ell, m)$ satisfy:

1. [Distinctness] $\widehat{w}_1, \ldots, \widehat{w}_\ell$ *are mutually distinct.*
2. [Partial Monotonicity] $\widehat{w}_1 < \widehat{w}_2 < \cdots < \widehat{w}_{\ell-t+1}$.
3. [Interlacing] $\widehat{w}_r$ *lies between* $\widehat{w}_{r-2}$ *and* $\widehat{w}_{r-1}$ *for all* $\ell - t + 2 \leq r \leq \ell$.

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Remark: The assertions on the weights $\widehat{w}_r = \widehat{w}_r(t;\ell,m)$ are equivalent to similar assertions for the "Delsarte weights" $\mathfrak{w}_r = \mathfrak{w}_r(t;\ell,m)$ since for $1 \leq s \leq r \leq \ell$ and $1 \leq t < \ell$,

$$\widehat{w}_r - \widehat{w}_s = q^t\left(\mathfrak{w}_{r-1}(t;\ell-1,m-1) - \mathfrak{w}_{s-1}(t;\ell-1,m-1)\right)$$

Moreover, the first assertion is a consequence of the second and third assertions.

Some Consequences

Corollary

We know the complete weight distribution of $\widehat{C}_{\mathrm{det}}(t;\ell,m)$. In particular,

$$\textit{min.wt}\left(\widehat{C}_{\mathrm{det}}(t;\ell,m)\right) = \widehat{w}_1(t;\ell,m)$$

and

$$\textit{max.wt}\left(\widehat{C}_{\mathrm{det}}(t;\ell,m)\right) = \widehat{w}_{\ell-t+1}(t;\ell,m).$$

Corollary

The result of Can (2023) on the minimum and maximum weights of the code associated to $\mathrm{GL}_m(\mathbb{F}_q)$ follows as a special case. More generally, we obtain the complete weight distribution of open determinantal codes $C(\widehat{\mathcal{E}}_t(\ell,m))$ and results on weights similar to those for determinantal codes, and in particular, the explicit determination of its minimum and maximum weights.

Final Remark: There are also nice connections of (the minimum distance of) determinantal codes with the rook monoids, Bruhat-Chevalley-Renner double coset decompositions, H-polynomials of certain configurations of rook placements on an $m \times m$ board, and Garsia-Remmel q-rook polynomials. But more about that some other time!

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Thank you!