Naturally labelled posets and a hierarchy related to interval orders

Sergey Kitaev

University of Strathclyde

Joint work with David Bevan and Gi-Sang Cheon

Einar Fest

The Party After My PhD Defense (January 2003)

The Party After My PhD Defense (January 2003)

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The Party After My first student's PhD Defense (2019)

Meet the research team!

David Bevan

Gi-Sang Cheon

SK

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On naturally labelled posets and permutations avoiding 12–34

David Bevan[a], Gi-Sang Cheon[b], Sergey Kitaev[a]

[a] Department of Mathematics and Statistics, University of Strathclyde, Glasgow, Scotland, United Kingdom

[b] Department of Mathematics, Sungkyunkwan University, Suwon, South Korea

Naturally labelled posets

A partially ordered set (poset) of cardinality n is labelled if its elements are in $[n] := \{1, 2, \ldots, n\}$.

Naturally labelled poset

A poset $(P, \prec)$ is naturally labelled (or natural for brievity) if $i \prec j$ implies $i < j$ (in the usual order).

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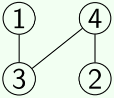
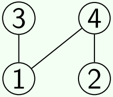
Non-example of a naturally labelled poset



$(3 \prec 1$ but $3 \not< 1)$

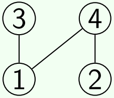
Example of a naturally labelled poset



Naturally labelled posets

Natural posets on $[n]$ are **in bijection** with lower triangular binary matrices with **ones on the main diagonal** that contain no $\left(\begin{smallmatrix} 1 & \mathbf{1} \\ 0 & 1 \end{smallmatrix}\right)$ submatrix whose **upper right entry** (shown in bold) is on the **main diagonal**.

Example



$$\leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

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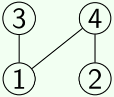
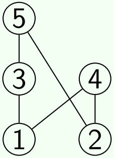


$$\leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 1 & 0\\ 1 & 1 & 0 & 1 \end{pmatrix}$$

These are counted by A006455 in OEIS. **Connected** natural posets are counted by A323502 in OEIS.

Brightwell, Prömel and Steger ("The average number of linear extensions of a partial order. *J. Comb. Theory Ser. A*, 73(2):193–206, 1996.") prove that the number of natural posets on $[n]$ is asymptotically $C_{\text{even}}\, n\, 2^{n^2/4}$ for even n and $C_{\text{odd}}\, n\, 2^{n^2/4}$ for odd n, where the constants differ by less than 10^{-4} and both are approximately equal to 12.7636.

Non-example of a 3-free natural poset



(elements $1, 3, 5$ form a chain of length 3)

Natural posets on $[n]$ of **height at most 2** are in bijection with **lower triangular binary matrices** with **ones on the main diagonal** that contain no $\begin{pmatrix} 1 & \mathbf{1} \\ & 1 \end{pmatrix}$ whose **upper right entry** is on the **main diagonal**.

3-free naturally labelled posets

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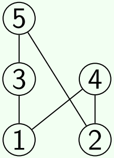

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Stanley graphs

A labelled graph on $[n]$ is a Stanley graph **iff** no vertex v has **both** left neighbours ($u < v$) and right neighbours ($u > v$). Clearly, every Stanley graph is **bipartite**. Assigning the **minimal elements** of a 3-free natural poset to one part of a bipartite graph and the remaining elements to the other part gives a **bijection** with Stanley graphs.

3-free naturally labelled posets

Stanley graphs are counted by A135922 in OEIS. These numbers grow like $7.3719688... \cdot 2^{n^2/4}$.

The **generating function** for the number of **3-free natural posets** with k **minimal elements** is

$$P_k(t) = \frac{t^k}{(1-t)(1-3t)\cdots(1-(2^k-1)t)}, \quad (k = 0, 1, \ldots).$$

Hence, the **generating function** for all **3-free natural posets** is given by

$$\sum_{k\geq 0} P_k(t) = \sum_{k\geq 0} \frac{t^k}{\prod_{i=0}^{k}(1-(2^i-1)t)}.$$

Alternatively, the **generating function** is $F(t,1)$, where

$$F(t,y) = 1 + t((y-1)F(t,y) + F(t,2y)).$$

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(For connected such posets the **generating function** is $G(1,0)$, where $G(u,v) = 1 + t((v-1)G(u,v) + G(2u, u+v))$; A323843 in OEIS.)

An influential paper

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MR2652101 (2011h:05022) Reviewed

Bousquet-Mélou, Mireille (F-BORD-LB); Claesson, Anders (ICE-RU-IM); Dukes, Mark (ICE-UICE-SI); Kitaev, Sergey (ICE-RU-IM)

$(2+2)$**-free posets, ascent sequences and pattern avoiding permutations.** (English summary)

J. Combin. Theory Ser. A 117 (2010), no. 7, 884–909.

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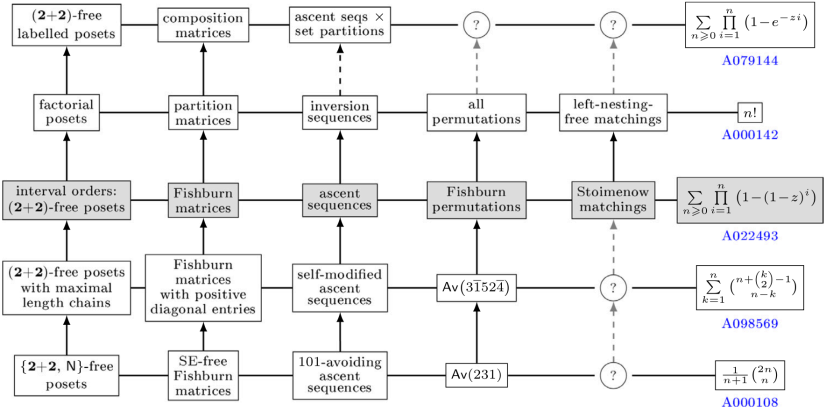

2+2-free naturally labelled posets

2+2-free poset

A poset is 2+2-free if it does not contain an induced subposet that is isomorphic to 2+2, the **union of two disjoint 2-element chains**.

2+2-free naturally labelled posets

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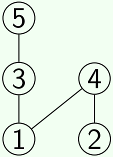
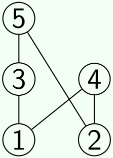
Non-example of a 2+2-free natural poset



(the elements 3, 5 and 2, 4 form 2+2)

Example of a 2+2-free natural poset



2+2-free naturally labelled posets

Brute-force generation yields the following enumeration of 2+2-free natural posets that is **not** in OEIS: 1, 2, 7, 37, 272, 2637, 32469, ...

Open problem

Find a **recurrence** / **functional equation** / **generating function** / **formula** for 2+2-free naturally labelled posets.

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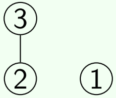
This problem is **hard**, but imposing **extra restrictions** may give interesting results. Claesson and Linusson ("n! matchings, n! posets. *Proc. Amer. Math. Soc.*, **139(2)**:435–449, 2011") prove that there are exactly n! factorial posets, i.e. naturally labelled posets s.t. $i < j \prec k \Rightarrow i \prec k$ (these posets are proved to be **2+2-free**).

Non-example of a factorial poset



Naturally labelled, but $1 < 2 \prec 3$ while $1 \not\prec 3$.

$\{3,2+2\}$-free naturally labelled posets

Brute-force generation yields the following for the enumeration of **$\{3,2+2\}$-free natural posets** on $[n]$:

$$1, 2, 6, 23, 107, 585, 3669, 25932, \ldots$$

This matches A113226 in OEIS enumerating permutations avoiding the vincular pattern 12-34 ($= \underline{12}\,\underline{34}$); a permutation $\pi_1\pi_2\cdots\pi_n$ avoids the pattern if there is no $i < j-1$ such that $\pi_i < \pi_{i+1} < \pi_j < \pi_{j+1}$.

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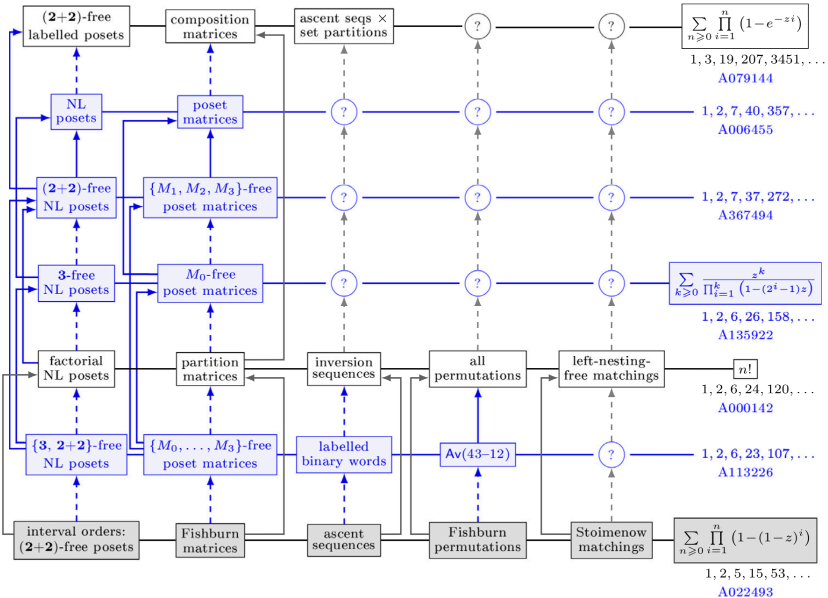
We proved the connection **bijectively** via the **decomposition** of **unlabelled** 2+2-free posets, and using a bijection between the set of **{3,2+2}-free** n**-element natural posets** and certain decorated binary words of length n.

Theorem

$\{3, 2+2\}$-free naturally labelled posets are in bijection with 12-34-avoiding permutations.

The place of naturally labelled posets in the hierarchy



Thank you for your attention!

Any questions?