On the second and third minimum weights of projective Reed-Muller codes

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Notations:

The following notations will be used throughout the presentation.

- $\mathbb{A}^k(\mathbb{F}_q)$ = affine space of dimension k over $\mathbb{F}_q$.
- $\mathbb{P}^k(\mathbb{F}_q)$ = projective space of dimension k over $\mathbb{F}_q$.
- For $k \geq 0$, we define $p_k := |\mathbb{P}^k(\mathbb{F}_q)| = 1 + q + \cdots + q^k$.
- A k-dimensional linear subspace C of $\mathbb{F}_q^n$ is called an $[n, k]_q$ linear code. The parameter n is called the length of the C. An element of a code C is called a codeword.
- For an element $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_q^n$, (expressed in terms of the usual basis of $\mathbb{F}_q^n$), we define the Hamming weight of $\mathbf{a}$, denoted by $\mathrm{wt}(\mathbf{a})$, as follows:

$$\mathrm{wt}(\mathbf{a}) := \#\{i : a_i \neq 0\}.$$

Reed-Muller Codes

Throughout, m and d are positive integers with $d \leq q-1$.

Definition: Reed-Muller codes Let $n = q^m$. We define an evaluation map

$$\mathrm{ev} : R_{\leq d} \to \mathbb{F}_q^n \quad \text{given by} \quad f \mapsto (f(P_1), \ldots, f(P_n)),$$

where $P_1, \ldots, P_n$ are a fixed enumeration of all the points in $\mathbb{A}^m(\mathbb{F}_q)$. Note that $R_{\leq d}$ is a vector space over $\mathbb{F}_q$ and the map ev is an $\mathbb{F}_q$-linear transformation. We define the Reed-Muller code, denoted by $\mathrm{RM}(d,m)$, as

$$\mathrm{RM}(d,m) = \mathrm{ev}(R_{\leq d}).$$

We also note that the evaluation map is injective, thanks to the underlying hypothesis $d \leq q-1$. It follows that $\mathrm{RM}(d,m)$ is a linear code of

- length $n = q^m$,
- dimension $\binom{m+d}{d}$.

Projective Reed-Muller Codes

Definition: Projective Reed-Muller Codes As before, we fix positive integers m and d. Let $N = p_m$. Each point of $\mathbb{P}^m(\mathbb{F}_q)$ admits a unique representative in $\mathbb{F}_q^{m+1}$ in which the first nonzero coordinate is 1. Let $P_1, \ldots, P_N$ be an ordered listing of such representatives in $\mathbb{F}_q^{m+1}$of points of $\mathbb{P}^m(\mathbb{F}_q)$. The Projective Reed-Muller code of order d and length N, denoted by $\mathrm{PRM}(d, m)$, is defined by the image of the Evaluation map,

$$\mathrm{Ev} : S_d \to \mathbb{F}_q^N \quad \text{given by} \quad F \mapsto (F(P_1), \ldots, F(P_N)).$$

That is,

$$\mathrm{PRM}(d, m) := \{(F(P_1), \ldots, F(P_N)) : F \in S_d\}.$$

As in the case of Reed-Muller codes, the Evaluation map is an $\mathbb{F}_q$-linear transformation. Moreover, it is injective, thanks to the assumption that $d \leq q - 1$. Thus $\mathrm{PRM}(d, m)$ is a linear code of

- length $N = p_m$,
- dimension $\binom{m+d}{d}$.

Hamming weights of codewords and $\mathbb{F}_q$-rational points on Hypersurfaces

Note that an codeword of $\mathrm{RM}(d,m)$ is uniquely given by $\mathrm{ev}(f)$ for some $f \in R_{\leq d}$, where

$$\mathrm{ev}(f) = (f(P_1), \ldots, f(P_n)).$$

Thus, the Hamming weight of $(\mathrm{ev}(f))$ is given by

$$wt(\mathrm{ev}(f)) = |\{P \in \mathbb{A}^m(\mathbb{F}_q) : f(P) \neq 0\}| = q^m - |Z(f)|,$$

where $Z(f)$ is the set of zeroes of f in $\mathbb{A}^m(\mathbb{F}_q)$. In particular, the determination of the minimum weight of $\mathrm{RM}(d,m)$ is equivalent to the determination of the maximum number of $\mathbb{F}_q$-rational zeroes of a polynomial in $R_{\leq d}$ of degree at most d.

- In algebro-geometric terminology, we see that the determination of the minimum weight of $\mathrm{RM}(d,m)$ is equivalent to the determination of the maximum number of $\mathbb{F}_q$-rational points on an affine hypersurface in $\mathbb{A}^m$ of degree d defined over $\mathbb{F}_q$.
- Similarly, the determination of the minimum weight of $\mathrm{PRM}(d,m)$ is equivalent to the determination of the maximum number of $\mathbb{F}_q$-rational points on an projective hypersurface in $\mathbb{P}^m$ of degree d defined over $\mathbb{F}_q$.

Minimum weight of (projective) Reed-Muller codes and the main question

As mentioned, we fix positive integers m, d with $d \leq q-1$.

- (Ore's Inequality) If $f \in R_{\leq d}$, and $f \neq 0$, then $|Z(f)| \leq dq^{m-1}$.
- (Serre's Inequality) If $f \in S_d$, and $F \neq 0$, then $|V(F)| \leq dq^{m-1} + p_{m-2}$.

Thus, the minimum weight of $\mathrm{RM}(d,m)$ and $\mathrm{PRM}(d,m)$, denoted by $d(\mathrm{RM}(d,m))$ and $d(\mathrm{PRM}(d,m))$ respectively, are given by

$$d(\mathrm{RM}(d,m)) = q^m - dq^{m-1} \text{ and } d(\mathrm{PRM}(d,m)) = p_m - dq^{m-1} - p_{m-2}.$$

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Question: Determine the weight distributions of $\mathrm{RM}(d,m)$ and $\mathrm{PRM}(d,m)$.

Broader goal

To complete the following "frequency tables":

Reed-Muller codes

w	$\mathcal{X} \subset \mathbb{A}^m$ affine hypersurface of degree d, $\lvert\mathcal{X}(\mathbb{F}_q)\rvert = w$
dq^{m-1}	Union of d parallel hyperplanes in $\mathbb{A}^m$ each defined over $\mathbb{F}_q$
?	?

Projective Reed-Muller codes

w	$\mathcal{X} \subset \mathbb{P}^m$ projective hypersurface of degree d, $\lvert\mathcal{X}(\mathbb{F}_q)\rvert = w$
$dq^{m-1} + p_{m-2}$	Union of d hyperplanes in $\mathbb{P}^m$ each defined over $\mathbb{F}_q$, containing a codimension 2 linear subspace in common
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Remarks: These problems are easy when either $m = 1$ or $d = 1$. On the other hand, when $d = 2$, the tables can be completed using the well-known classification of quadrics over finite fields. We will thus assume from now on that $m \geq 2$ and $d \geq 3$.

Second minimum weight of Reed-Muller codes

1. (Cherdieu and Rolland, 1996) If $f \in R_{\leq d}$ is given by a product of d linear polynomials defined over $\mathbb{F}_q$ and $|Z(f)| < dq^{m-1}$, then $|Z(f)| \leq dq^{m-1} - (d-1)q^{m-2}$. Moreover, if $|Z(f)| = dq^{m-1} - (d-1)q^{m-2}$ and f is a product of d linear polynomials defined over $\mathbb{F}_q$, then $Z(f)$ is given by one of the following configurations:
   (a) **Type I:** $Z(f)$ is a union of d affine hyperplanes passing through a common codimension 2 affine subspace in $\mathbb{A}^m$, or
   (b) **Type II:** $Z(f)$ is a union of $d-1$ parallel hyperplanes and another hyperplane intersecting each of the $d-1$ hyperplanes at distinct codimension 2 affine subspaces.

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2. (Geil, 2008) If $f \in R_{\leq d}$ and $|Z(f)| < dq^{m-1}$, then $|Z(f)| \leq dq^{m-1} - (d-1)q^{m-2}$.

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3. (Leducq, 2012) If $q \geq 3$ and $f \in R_{\leq d}$ with $|Z(f)| = dq^{m-1} - (d-1)q^{m-2}$, then f is a product of d linear polynomials defined over $\mathbb{F}_q$.

Summary

Reed-Muller codes

w	$\mathcal{X} \subset \mathbb{A}^m$ affine hypersurface of degree d, $\lvert\mathcal{X}(\mathbb{F}_q)\rvert = w$
dq^{m-1}	Union of d parallel hyperplanes in $\mathbb{A}^m$ each defined over $\mathbb{F}_q$
$dq^{m-1} - (d-1)q^{m-2}$	Type I or Type II
?	?

Concerning the projective Reed-Muller codes: A basic (in)equality

Question: Determine $N_2 = \max\{|V(F)| : F \in S_d, |V(F)| < dq^{m-1} + p_{m-2}\}$. As mentioned, we will assume that $m \geq 2$ and $d \geq 3$.

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Known results:

- Rodier and Sboui, (2008) If $d \leq \frac{q}{2} + 1$, then $N_2 = dq^{m-1} + p_{m-2} - (d-2)q^{m-2}$.
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Basic inequality: Suppose $F \in S_d$ containing at least one linear polynomial defined over $\mathbb{F}_q$ as a linear factor defined over $\mathbb{F}_q$, and $|V(F)| < dq^{m-1} + p_{m-2}$, then

$$|V(F)| \leq dq^{m-1} + p_{m-2} - (d-1)q^{m-2}.$$

The equality is attained by $V(F)$ if and only if $V(F)$ contains a hyperplane Π and $V(F) \setminus \Pi$ is an affine hypersurface of degree $d-1$ with

$$|V(F) \setminus \Pi| = (d-1)q^{m-1} - (d-2)q^{m-2}.$$

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Question: What happens when F does not contain any linear factor defined over $\mathbb{F}_q$?

Homma-Kim elementary bound

(Homma-Kim elementary bound, 2012) If $X \subset \mathbb{P}^m$ is a projective hypersurface of degree $d \leq q+1$ not containing any $\mathbb{F}_q$-linear component, then

$$|\mathcal{X}(\mathbb{F}_q)| \leq (d-1)q^{m-1} + dq^{m-2} + p_{m-3}.$$

Remark: This bound was also obtained by Carvalho and Neumann, although in a different avatar, in 2016.

Now, a direct comparison shows that

$$(d-1)q^{m-1} + dq^{m-2} + p_{m-3} < dq^{m-1} + p_{m-2} - (d-2)q^{m-2} \iff d < \frac{q+3}{2}.$$

(Carvalho and Neumann, 2018) If $d \leq \frac{q+2}{2}$, then $N_2 = dq^{m-1} + p_{m-2} - (d-2)q^{m-2}$.

Remark: Combining everything above and the result by Leducq, we can get the geometric structure of $V(F)$ where $|V(F)| = dq^{m-1} + p_{m-2} - (d-2)q^{m-2}$ in the case when $d \leq \frac{q+1}{2}$.

An elementary improvement of Homma-Kim elementary bound

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$$|\mathcal{X}(\mathbb{F}_q)| \leq (d-1)q^{m-1} + dq^{m-2} + p_{m-3}.$$

(Tironi, 2017) Let $\mathcal{X} \subset \mathbb{P}^m$ be a projective hypersurface of degree $d \leq q+1$, not containing any $\mathbb{F}_q$-linear component, and $|\mathcal{X}(\mathbb{F}_q)| = (d-1)q^{m-1} + dq^{m-2} + p_{m-3}$, then $d = 2, \sqrt{q}+1$ (in this case q must be a perfect square) or $q+1$.

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(Datta, 2024) If $d \neq 2, \sqrt{q}+1$, and $\mathcal{X} \subset \mathbb{P}^m$ is a projective hypersurface of degree $d \leq q$ not containing any $\mathbb{F}_q$-linear component, then

$$|\mathcal{X}(\mathbb{F}_q)| \leq (d-1)q^{m-1} + dq^{m-2} + p_{m-3} - (d-2)q^{m-3}.$$

Remark: With the new bound mentioned above, and observing that $\sqrt{q}+1 < \frac{q+3}{2}$, now we can show that for $d \leq \frac{q+3}{2}$, we have $N_2 = dq^{m-1} + p_{m-2} - (d-2)q^{m-2}$ and obtain the geometric classification of all hypersurfaces of degree $\leq \frac{q+3}{2}$ achieving N_2.

Unfortunately, nothing more about N_2 is known when $m \geq 3$ yet!!

The case $m = 2$

From now on, let $\mathcal{C} \subset \mathbb{P}^2$ denote a plane curve of degree $d \leq q$ defined over $\mathbb{F}_q$.

1. Serre's inequality shows that $|\mathcal{C}(\mathbb{F}_q)| \leq dq + 1$.
2. Previously mentioned results on N_2 show that if $d \leq \frac{q+3}{2}$ and $|\mathcal{C}(\mathbb{F}_q)| < dq + 1$, then

   $$|\mathcal{C}(\mathbb{F}_q)| \leq dq - d + 3.$$

   We can also classify the curves of degree d attaining the bound.
3. The basic inequality shows that if $\mathcal{C}$ contains a line defined over $\mathbb{F}_q$, then $|\mathcal{C}(\mathbb{F}_q)| \leq dq - d + 3$.

What can we say about plane curves that do not contain a line defined over $\mathbb{F}_q$?

Homma-Kim-Sziklai bound

(Conjecture, P. Sziklai, 2008) Let $\mathcal{C} \subset \mathbb{P}^2$ be a plane curve of degree $d \leq q+1$ not containing any lines defined over $\mathbb{F}_q$. Then $|\mathcal{C}(\mathbb{F}_q)| \leq (d-1)q+1$.

(Homma-Kim, 2009-10) The conjecture is true, except when $d = q = 4$. Moreover, when $d = q = 4$, then up to projective equivalence, there is exactly one curve $\mathcal{C}$ for which the conjecture is false.

The bound is known to be attained in the following cases:

1. $d = 2$: hyperbolic quadric
2. q is a square and $d = \sqrt{q}+1$: nonsingular Hermitian curve (in this case, Hermitian curves are the only curves attaining this bound, which is known thanks to (Hirschfeld, Storme, Thas and Voloch, 1991)
3. $d = q-1$ any curve of degree $q-1$ attaining the bound is equivalent to the plane curve given by the equation:
$$\alpha X^{q-1} + \beta Y^{q-1} + \gamma Z^{q-1} = 0,$$
where $\alpha, \beta, \gamma \in \mathbb{F}_q^{\times}$, and $\alpha+\beta+\gamma = 0$. This classification was obtained by Ferreira and Speziali in 2023.
4. $d = q, q+1$. Some examples are known.

Back to N_2 when $m = 2$

Note that $(d-1)q + 1 < dq - d + 3$, whenever $d \leq q + 1$.

(Datta, 2024) For $m = 2$, and $d \leq q$, we have $N_2 = dq - d + 3$. Moreover, if $\mathcal{C}$ is a plane curve of degree d defined over $\mathbb{F}_q$ such that $|\mathcal{C}(\mathbb{F}_q)| = dq - d + 3$, then $\mathcal{C}$ is a union of d distinct lines, say $\ell_1, \ldots, \ell_d$ satisfying the following condition:

$\ell_1, \ldots, \ell_{d-1}$ intersect at a common point P and ℓ_d intersect each of $\ell_1, \ldots, \ell_{d-1}$ at a point different from P.

N_3: One more step

Question: Determine $N_3 = \max\{|\mathcal{C}(\mathbb{F}_q)| : \mathcal{C}$ a plane curve, $\deg \mathcal{C} = d, |\mathcal{C}(\mathbb{F}_q)| < dq - d + 3\}$.

Known results:

- Sboui, 2009 If $\mathcal{C}$ is a union of d distinct lines and $|\mathcal{C}(\mathbb{F}_q)| < dq - d + 3$, then $|\mathcal{C}(\mathbb{F}_q)| \leq dq + 1 - 2(d-3)$.
- Sboui, 2009 For $d \leq \frac{q}{3} + 2$, we have $N_3 = dq + 1 - 2(d-3)$.
- Rodier and Sboui, 2008 If q is a prime and $2 < d < q - 2$, then $N_3 = dq + 1 - 2(d-3)$.

Remark: The last result is wrong. Note that the curve $\mathcal{C}$ defined over $\mathbb{F}_q$, given by a union of $d-1$ lines passing through a common point, with one of the lines repeated twice, has degree d and satisfies $|\mathcal{C}(\mathbb{F}_q)| = (d-1)q + 1$.

But one can easily check that

$$dq + 1 - 2(d-3) < (d-1)q + 1 < dq - d + 3,$$

the first inequality holds when $d > \frac{q+6}{2}$ and, as observed before, the second one is always true.

Recent result

Theorem (Datta, 2024)

For $5 \leq d \leq q-1$ and $m = 2$, we have

$$N_3 = \begin{cases} dq + 1 - 2(d-3) & \text{if } d \leq \frac{q+5}{2} \\ \leq (d-1)q + 2 & \text{if } d \geq \frac{q+6}{2}. \end{cases}$$

Remarks:

- In fact, we proved that for $d \geq \frac{q+6}{2}$, we have $N_3 = (d-1)q + 1$ or $(d-1)q + 2$.
- A classification of degree curves d attaining the bound is still open. For example, let $\mathcal{C}$ be a curve of degree d. Write $\mathcal{C} = \mathcal{L} \cup \mathcal{N}$, where $\mathcal{L}$ consists of the union of all lines defined over $\mathbb{F}_q$ contained in $\mathcal{C}$, and $\mathcal{N}$ is the "line-free" part of $\mathcal{C}$. Suppose $\deg \mathcal{N} = s$. Then $|\mathcal{C}(\mathbb{F}_q)| = (d-1)q + 2$ if and only if
  - $\mathcal{L}$ is a union of $d - s$ lines passing through a common point,
  - $|\mathcal{N}(\mathbb{F}_q)| = (s-1)q + 1$, and
  - $(\mathcal{L} \cap \mathcal{N})(\mathbb{F}_q) = \emptyset$
- Datta, 2024 When $q = p$ or p^2, where p is a prime, and $\frac{q+6}{2} \leq d \leq q-1$, then $N_3 = (d-1)q + 1$.

Open Questions:

1. Determination for N_2 for $m \geq 3$.
2. Determination of N_3 for $m \geq 2$.

Our approach essentially suggests investigations on the following questions:

1. Classify all the plane curves not containing a line defined over $\mathbb{F}_q$ that attain the Homma-Kim-Sziklai bound.
2. Find the best possible upper bounds on the number of possible $\mathbb{F}_q$-rational points on absolutely irreducible hypersurfaces of degree d defined over $\mathbb{F}_q$.

These problems are known to be very difficult problems.

Thank you for your attention!