Random processes and 321-avoiding permutations

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Framework

- **Large ($n \to \infty$) random structures**
  Random permutations, $\mathcal{S}_n$
  Erdös-Renyi graphs, $G(n, p)$
  Preferential attachment graphs, $G(n, m, \alpha)$ etc.

- **Properties**
  Avoidance of a certain pattern, existence of a fixed point etc.
  Existence of a cycle, Hamiltonicity, connectivity etc.

- **Probability that a given property holds**
  Does there exist a limit for the probability?
  If so, is the probability different from 0 or 1? etc.

Pattern-avoiding permutations

- Two sequences $s = s(1) \cdots s(m)$ and $t = t(1) \cdots t(m)$ are **order-isomorphic** if and only if
$$s(i) < s(j) \Leftrightarrow t(i) < t(j)$$
for all $1 \leq i < j \leq n$. For example,
$$5648 \text{ is order-isom. to } 2419.$$
- A permutation $\pi \in S_n$ contains $\rho \in S_m$ as a **pattern** if π has a subpermutation order-isomorphic to ρ. Otherwise π **avoids** ρ.

Example

$\pi = 1234$ avoids 21.

$\pi = 152436798$ avoids 231.

$\pi = 426351$ contains 231.

Random permutations

If σ_n is a uniformly random permutation of length n, then

$$\mathrm{P}(\sigma_n \text{ avoids } 231) = \mathrm{P}(\sigma_n \text{ avoids } 321) = \frac{C_n}{n!} \to 0,$$

$$\text{where } C_n = \frac{1}{n+1}\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi}n^{3/2}} \text{ is the nth Catalan number.}$$

Theorem (Foy, Woods, '90)

There exists a first-order property φ, e.g. avoiding a given pattern, such that

$$\lim_{n\to\infty} \mathrm{P}(\sigma_{2n} \text{ satisfies } \varphi) = 1 \text{ and } \lim_{n\to\infty} \mathrm{P}(\sigma_{2n+1} \text{ satisfies } \varphi) = 0.$$

That is to say, $\lim_{n\to\infty} \mathrm{P}(\sigma_n \text{ satifies } \varphi)$ *does not exist.*

Random pattern avoiding permutations

Example

Let φ_{max} mean the last entry of the permutation is the largest. We can show that

$$\lim_{n\to\infty} \mathbf{P}(\sigma_n^{231} \vDash \varphi_{max}) = \lim_{n\to\infty} \mathbf{P}(\sigma_n^{321} \vDash \varphi_{max}) = \lim_{n\to\infty} \frac{C_{n-1}}{C_n} = \frac{1}{4}.$$

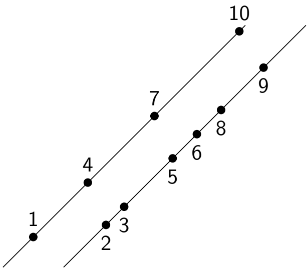
where σ_n^{231} (σ_n^{321}) is a uniformly random 231(321)-avoiding permutation of length n.

Theorem (Albert, Bouvel, Féray, Noy '22)

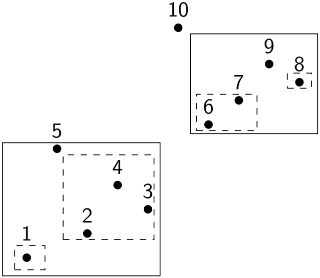
Let σ_n^{231} is a randomly chosen 231-avoiding permutation and φ is a first-order property on permutations. Then

$$\lim_{n\to\infty} \mathrm{P}(\sigma_n^{231} \vDash \varphi) \text{ exists.}$$

The proof of the result above uses the recursive pattern on the right.



$\pi = 1\,4\,2\,3\,7\,5\,6\,8\,10\,9 \in \mathrm{AV}(321)$



$\pi = 1\,5\,2\,4\,3\,10\,6\,7\,9\,8 \in \mathrm{AV}(231)$

Figure: Two increasing subsequences vs. the recursive pattern

Theorem (Ö.,'23)

For any first-order property φ, $\lim_{n\to\infty} \mathrm{P}(\sigma_n^{321} \models \varphi)$ exists.

Quantifying properties

The quantifier depth of a first-order property is defined recursively as

- If φ is atomic, then $\mathrm{qd}(\varphi) = 0$.
- If $\psi = \neg\varphi$, then $\mathrm{qd}(\psi) = \mathrm{qd}(\varphi)$.
- If $\psi = \forall x \varphi$ or $\psi = \exists x \varphi$, then $\mathrm{qd}(\psi) = \mathrm{qd}(\varphi) + 1$.
- If $\psi = \varphi_1 \vee \varphi_2, \psi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \Rightarrow \varphi_2$ then $\mathrm{qd}(\psi) = \max\{\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2)\}$.

Example

$\varphi_{max} = \exists x \forall y[\neg(x = y) \Rightarrow (y <_{position} x) \wedge (y <_{value} x)]$ has quantifier depth 2.

Elementarily equivalence

Definition

Given two comparable structures Σ and Σ', we say $\Sigma \equiv_k \Sigma'$ if for any first-order property φ with $\mathrm{qd}(\varphi) \leq k$

$$\Sigma \models \varphi \text{ if and only if } \Sigma' \models \varphi.$$

Σ and Σ' are **elementarily equivalent** if $\Sigma \equiv_k \Sigma'$ for all k.

Example

- $\sigma = 1\,2\,3\,8\,5\,6\,7 \equiv_2 1\,2\,3\,7\,5\,6 = \sigma'$
- $(\mathbb{Z}, <)$ and $(\mathbb{Z}^2, <_{\mathrm{lex}})$ are elementarily equivalent but not isomorphic.

Theorem (Gurevich, '83)

There are finitely many equivalence classses of $\equiv_k$ for any structure with only relational symbols (permutations, graphs, matroids etc.)

Binary words

We want to define a process over the equivalence classes of $\equiv_k$ to study the limiting probability as $n \to \infty$.

Example (Lynch, '93)

Consider the set of binary words where at any position 1 occurs with probability $p \in [0, 1]$ and 0 with probability $1 - p$. For example,

$$\mathbf{P}(00011) = p^3(1 - p)^2.$$

For any words v and w and $s \in \{0, 1\}$,

$$v \equiv_k w \implies vs \equiv_k ws.$$

So if w_n is a random binary word of length n, for all $w \equiv_k w'$,

$$\mathbf{P}(w_{n+1} \in L \,|\, w_n = w) = \mathbf{P}(w_{n+1} \in L \,|\, w_n = w')$$

for any equivalance class L. That gives a *Markov chain* on equivalance classes with transition probabilities p or $1 - p$.

Markov chains

Definition

A sequence of random variables $\{X_0, X_1, X_2, \ldots\}$ is a **Markov chain** with state space S and the transition matrix P if

$$\mathbf{P}(X_{n+1} = j \,|\, X_n = i) = P(i,j) \text{ for all } i, j \in S \text{ and } n = 0, 1, 2, \ldots$$

A chain is **irreducible** if for all $i, j \in S$, $\exists\, m$ such that $P^m(i,j) > 0$.

A chain is **aperiodic** if for all $i \in S$, $\gcd\{n \in \mathbb{N} : P^n(i,i) > 0\} = 1$.

Theorem (Perron-Frobenius)

If a Markov chain defined on a finite state space S is irreducible and aperiodic, it has a unique stationary distribution π on S,

$$\pi P = \pi.$$

Example

Let

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ 1/2 & 1/2 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1/4 & 3/4 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

P_2 is reducible and P_3 is periodic. While P_1 is neither and has

$$\pi = [1/9, 2/3, 2/9]$$

as its stationary distribution.

Some applications of finite-state space MCs in this context:

- (Lynch '93) Limit laws for random binary words,
- (Braunfeld and Kukla, '21) Convergence for layered permutations (direct sum of decreasing permutations).

321-avoiding permutations

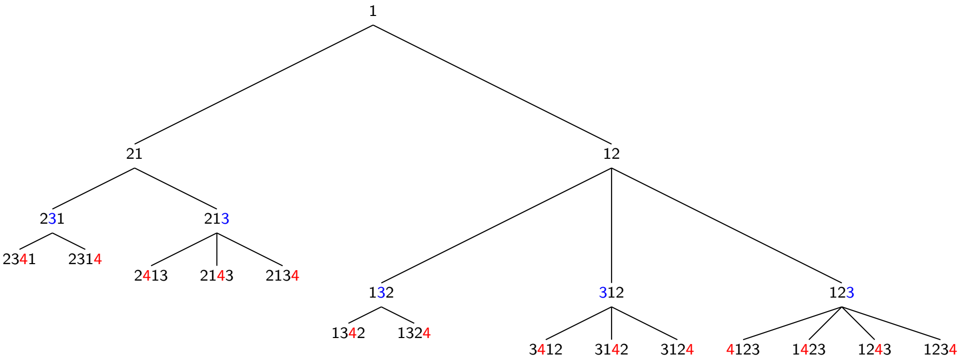


Figure: The children of each vertex of rank n are obtained by inserting $n+1$ in a position such that the new permutation does not contain 321 pattern.

A finer state space

Recall σ_n^{321} is a uniformly random 321-avoiding permutation of length n.
Even if $\sigma \equiv_k \sigma'$, it can be the case that

$$\mathbf{P}(\sigma_{n+1}^{321} \in L \,|\, \sigma_n^{321} = \sigma) \neq \mathbf{P}(\sigma_{n+1}^{321} \in L \,|\, \sigma_n^{321} = \sigma').$$

So we first space than the set of elementary equivalance classes.

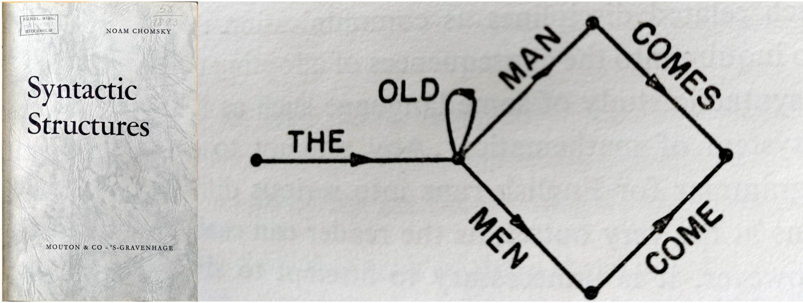


Figure: Language as a Markov chain

Tail configuration

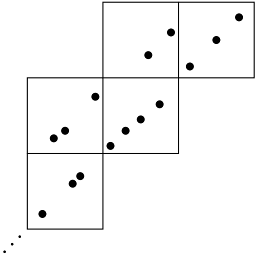


Figure: The tail configuration of size $k = 5$ of a permutation in $AV(321)$.

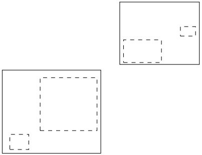


Figure: Recursive depth of size 2 of a permutation in $AV(231)$.

A finer state space

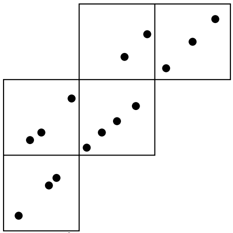
Recall σ_n^{321} is a uniformly random 321-avoiding permutation of length n. Even if $\sigma \equiv_k \tau$, it can be the case that

$$\mathbf{P}(\sigma_{n+1}^{321} \in L \mid \sigma_n^{321} = \sigma) \neq \mathbf{P}(\sigma_{n+1}^{321} \in L \mid \sigma_n^{321} = \tau)$$

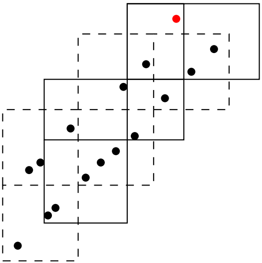
However, if, in addition, the tail configurations of size k of σ and τ agree, then

$$\mathbf{P}(\sigma_{n+1}^{321} \in L \mid \sigma_n^{321} = \sigma) = \mathbf{P}(\sigma_{n+1}^{321} \in L \mid \sigma_n^{321} = \tau).$$

Tail configuration



$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)$



$\psi \to \psi'$

Figure: The tail configuration ψ evolves into ψ' following the insertion. The red dot represent the peak and $k = 5$ in this example.

Lemma (Well-definedness)

Let $\pi, \sigma \in \mathrm{AV}(321)$ with a common tail configuration ψ and $\pi \equiv_k \sigma$ for a fixed k. Then the logical classes of the permutations obtained by insertion depend only on the insertion location.

The distance to the rightmost descent, Q_n

The number of leaves with i branches at the nth level of the Catalan tree is counted by the *ballot numbers*:

$$q_{n,i} = \frac{i-1}{n}\binom{2n-i}{n-1} \quad \text{for } i = 2, \ldots, n+1,$$

which can be obtained from

$$q_{n,i} = [z^n]\left(\frac{1-\sqrt{1-4z}}{2}\right)^r$$

Therefore, $\mathbf{P}(Q_n = i) = q_{n,i}/C_n$ and

(Stationary distribution) $\pi_i = \lim_{n\to\infty} \mathbf{P}(Q_n = i) = \frac{i}{2^{i+1}}$ for $i = 2, 3, \ldots$

Note that

$$\mathbf{E}[Q_n] = \frac{C_{n+1}}{C_n} \to 4 \text{ as } n \to \infty.$$

Countable state-space Markov chains

Definition

Let $\tau_{ii} := \min_n\{X_n = i \,|\, X_0 = i\}$, the first return time.

- A chain is **positive recurrent** if $\mathbf{E}[\tau_{ii}] < \infty$.
- It is **null recurrent** if $\mathbf{P}(\tau_{ii} < \infty) = 1$ $\mathbf{E}[\tau_{ii}] = \infty$.
- It is **transitive** if $\mathbf{P}(\tau_{ii} < \infty) < 1$.

Example

1. (Symmetric random walk) $S = \mathbb{Z}$, $P(i,j) = 1/2$ for $j = i-1, i+1$. The chain is null recurrent.
2. (Geometric walk) $S = \mathbb{Z}_{\geq 0}$, $P(i,0) = q$ and $P(i,i+1) = p$ where $p + q = 1$. The chain is positive recurrent with $\pi = [q, qp, qp^2, \ldots]$.
3. (Symmetric random walk on $\mathbb{Z}^d$ for $d \geq 3$) $S = \mathbb{Z}^d$, $P(i,j) = 1/2d$ if $|i - j| = 1$. Positive probability of no return.

Countable state-space Markov chains

Theorem

If a countable state-space chain is **irreducible, aperiodic** and **positive recurrent,** it has a unique stationary distribution**.**

Some applications of countable state-space MCs in this context is

- (Muller, Skerman, Verstraaten, '23) Logical limit law with respect to the Mallows distribution on permutations.
- (Ö., '24) Limit law for preferential attachment graphs

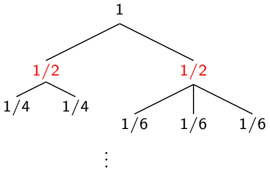


Figure: Uniform distribution over $AV_2(321)$ but not over $AV_3(321)$

Countable state-space Markov chains

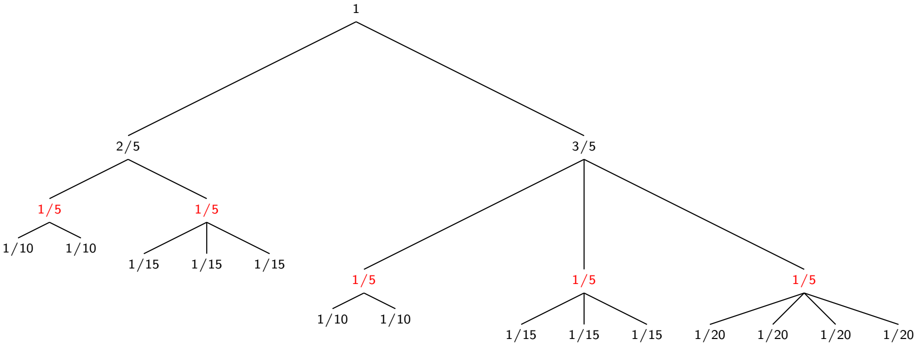


Figure: Uniform distribution over $AV_3(321)$ but at no other stage

Countable state-space Markov chains

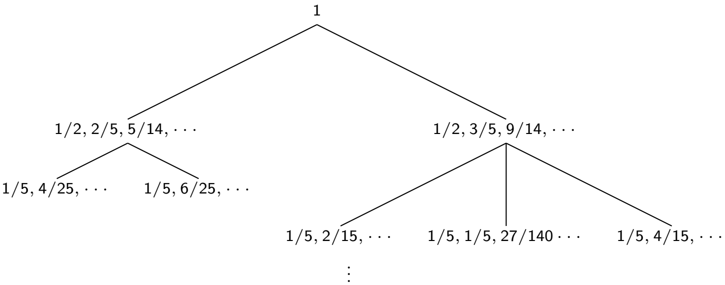


Figure: Ratios of descendants at the same level as the tree branches out

For any $\sigma \in \mathrm{AV}_n(321)$,

$$\lim_{N \to \infty} \frac{\left|\text{descendants of } \sigma \text{ at the } N^{\text{th}} \text{ level}\right|}{\left|\text{descendants of all } \mathrm{AV}_n(321) \text{ at the } N^{\text{th}} \text{ level}\right|} > 0$$

A limiting distribution

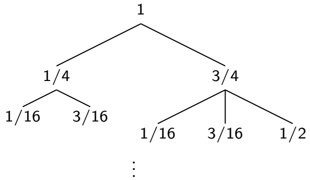


Figure: The limiting ratios are non-uniform at any stage

We can define a Markov chain for the statistic Q_n in this case. The transition probabilities of the Markov chain are

$$P(i,j) = \frac{j}{i \cdot 2^{i-j+2}} \quad \text{for } j = 2, \ldots, i, i+1.$$

However, $\mathbf{E}[Q_n] \to \infty$. In fact, the chain is null-recurrent.

Symbolic chains (Infinite transfer matrices)

Define a directed graph on some V with an irreducible and aperiodic adjacency matrix A. Let

$$\text{(Perron value)} \quad \lambda = \sqrt[n]{A^n(i,j)}$$

and

$$\text{(Left and right eigenvectors of } \lambda) \ \vec{l} \cdot A = \lambda \vec{l} \quad A \cdot \vec{r} = \lambda \vec{r}.$$

- Observe that for stochastic matrices (MC matrices), $\lambda = 1$, $\vec{r} = \mathbf{1}$ and $\vec{l}$ is the stationary distribution if exists.

Let $\pi_i(n)$ denote the frequency of paths leading to i at the nth stage.

Theorem (Kitchens '98)

Suppose A is irreducible and aperiodic on V. If $\vec{l} \cdot \vec{r} < \infty$, then $\lim_{n\to\infty} \pi_i(n) \to \pi_i > 0$ for some i.

Symbolic chains

We let $V = \{2, 3, \dots\}$ and

$$A(i,j) = \begin{cases} 1 & \text{if } j = 2, 3, \dots, i+1 \\ 0 & \text{otherwise} \end{cases}$$

The Perron value of A is 4 and it has left and right eigenvectors:

- $l = l = \left(1, 1, \frac{3}{4}, \cdots, \frac{n}{2^{n-1}}, \cdots\right)$
- $r = (1, 3, 8, \dots, (1+n)2^{n-2}, \dots)$.

$l \cdot r = \infty \Rightarrow$ not *positive recurrent* according to Kitchens '98. In fact, this chain is classified as *transitive*.

However, we know that $\pi_i(n) = \mathbf{P}(Q_n = i) > 0$.

Operator viewpoint

Lemma

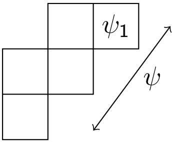
Let $\Gamma = (V, E)$ *be a locally finite, strongly connected and non-partite directed graph,* Δ^V *be the probability simplex on the set of vertices and* A *be the adjacency matrix of* Γ*. Define*

$$T : \Delta^V \to \Delta^V \text{ as } T(w) = \frac{w^T A}{\|w^T A\|_1}.$$

If $T(K) \subseteq K$ *for some non-empty, compact and convex* $K \subseteq \Delta^V$*, then there exists a unique* $w^* \in K$ *such that* $\lim_{n\to\infty} T^n(w_0) = w^*$ *for all* $w_0 \in \Delta^V$.

irreducible	$\Leftrightarrow$	strongly connected
aperiodic	$\Leftrightarrow$	non-partite
positive recurrence	$\Leftrightarrow$	compactness

Compact set



We take $V = L \times \Psi_k$ instead of $\{2, 3, \dots\}$ where

- L is the set of all elementary equivalence classes (finite)
- Ψ_k is the set of all tail configurations for a fixed k (countable)

The subset for the stationary distribution for $|\psi_1|$ (or π as $\lim_n Q_n$) :

$$\Pi := \left\{ w \in \Delta^V \,:\, \mathrm{P}_w(|\psi_1| = i) = \frac{i}{2^{i+1}} \text{ for } i = 2, 3, \dots \right\},$$

and the convex, compact set in the theorem:

$$K_A := \{ w \in \Pi \,:\, \mathrm{E}_w(|\psi_1| \cdot |\psi|) \le A(k) \} \subset \Delta^V.$$