Structure of braid graphs for reduced words in Coxeter groups

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Coxeter systems

Definition

A Coxeter system consists of a Coxeter group W generated by a set of involutions S together with a function $m : S \times S \to \mathbb{N} \cup \{\infty\}$ such that for $s \neq t$:

$$\left.\begin{array}{lcc} m(s,t) = 2 & \iff & st = ts \end{array}\right\} \quad \text{commutation relation}$$

$$\left.\begin{array}{lcc} m(s,t) = 3 & \iff & sts = tst \\ m(s,t) = 4 & \iff & stst = tsts \\ & \vdots & \end{array}\right\} \quad \text{braid relations}$$

Reduced expressions & Matsumoto's Theorem

Definition

A word $\boldsymbol{\alpha} = s_{x_1} s_{x_2} \cdots s_{x_\ell} \in S^*$ is called an expression for $w \in W$ if it is equal to w when considered as a group element. If ℓ is minimal among all expressions for w, $\boldsymbol{\alpha}$ is a called a reduced expression.

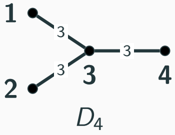
Matsumoto's Theorem

Any two reduced expressions for $w \in W$ differ by a sequence of commutation & braid moves.

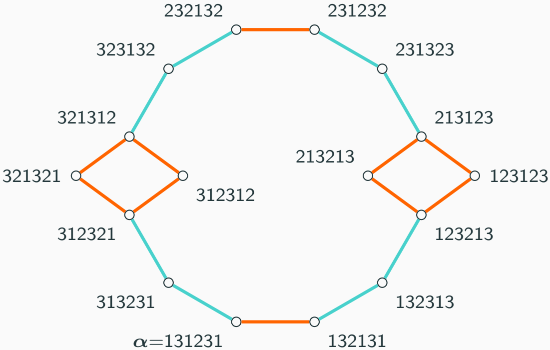
Matsumoto graphs

Example

Consider the reduced expression $\alpha = 131231$ in the Coxeter system of type D_4.



Coxeter graph



Matsumoto graph

Braid equivalence & braid graphs

Definition

Reduced expressions α and β are braid equivalent iff they are related by a sequence of braid moves. The corresponding equivalence classes are called braid classes, denoted $[\alpha]$.

Definition

We can encode a braid class $[\alpha]$ in a braid graph, denoted $\mathcal{B}(\alpha)$:

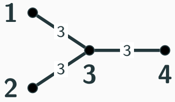
- Vertex set $= [\alpha]$
- $\{\gamma, \beta\}$ is an edge iff γ and β are related via a single braid move

Braid graphs are the maximal blue connected components in the Matsumoto graph.

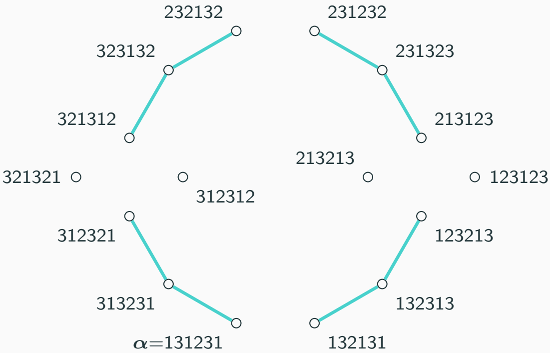
Braid graphs

Example

Consider the reduced expression $\alpha = 131231$ in type D_4 from earlier.



D_4



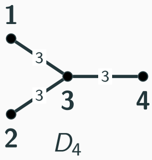
Eight braid graphs

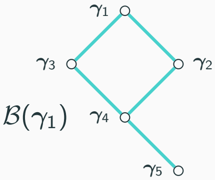
Braid graphs (continued)

Example

In the Coxeter system of type D_4, the expression $\gamma_1 = 2321434$ is reduced and its braid class consists of the following reduced expressions:

$$\gamma_1 = \underline{232}1\underline{434},\ \gamma_2 = \underline{323}1\underline{434},\ \gamma_3 = \underline{232}1\underline{343},\ \gamma_4 = \underline{32\overline{3}}\overline{1}\overline{\underline{3}}\underline{43},\ \gamma_5 = 32\underline{131}43.$$





Example of Fibonacci cube

Braid shadows

Notation

For $i \leq j$, we define the interval

$$[\![i, j]\!] := \{i, i+1, \ldots, j-1, j\}.$$

Definition

Let $\boldsymbol{\alpha}$ be a reduced expression.

- $[\![i, j]\!]$ is a braid shadow for $\boldsymbol{\alpha}$ if $\boldsymbol{\alpha}_{[\![i,j]\!]} = \underbrace{st\cdots}_{m(s,t)\geq 3}$
- $\mathcal{S}(\boldsymbol{\alpha}) :=$ set of braid shadows for $\boldsymbol{\alpha}$
- Collection of braid shadows for braid class $[\boldsymbol{\alpha}]$:

$$\mathcal{S}([\boldsymbol{\alpha}]) := \bigcup_{\boldsymbol{\beta}\in[\boldsymbol{\alpha}]} \mathcal{S}(\boldsymbol{\beta})$$

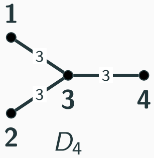
- $\text{rank}(\boldsymbol{\alpha}) := |\,\mathcal{S}([\boldsymbol{\alpha}])|$

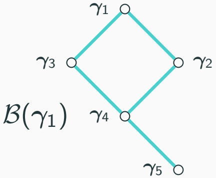
Links

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{232}1\underline{434},\ \gamma_2 = \underline{323}1\underline{434},\ \gamma_3 = \underline{232}1\underline{343},\ \gamma_4 = \underline{32\overline{3}}\overline{1}\underline{\overline{3}43},\ \gamma_5 = 32\underline{131}43.$$





We see that

$$\mathcal{S}(\gamma_1) = \{[\![1,3]\!], [\![5,7]\!]\} \text{ and } \mathcal{S}([\gamma_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}.$$

Links (continued)

Theorem
Braid shadows are either disjoint or overlap by a single position.

Definition
If α is a reduced expression, then α is a link provided it either consists of a single generator or

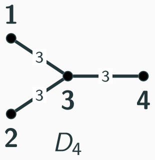
$$\mathcal{S}([\alpha]) = \{[\![1, \ell_1]\!], [\![\ell_1, \ell_2]\!], \ldots, [\![\ell_{d-1}, \ell_d]\!]\}$$

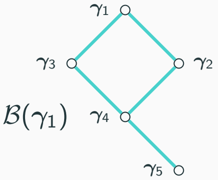
with $1 < \ell_1 < \ell_2 < \cdots < \ell_d$.

Example

Recall the reduced expression $\gamma_1 = 2321434$ in the Coxeter system of type D_4 with braid class:

$$\gamma_1 = \underline{232}1\underline{434},\ \gamma_2 = \underline{323}1\underline{434},\ \gamma_3 = \underline{232}1\underline{343},\ \gamma_4 = \underline{32\overline{3}}\overline{1}\underline{\overline{3}43},\ \gamma_5 = 32\underline{131}43.$$





Recall

$$\mathcal{S}([\gamma_1]) = \{[\![1,3]\!], [\![3,5]\!], [\![5,7]\!]\}.$$

So, γ_1 is a link of rank 3.

Link factorizations

Definition

If $\boldsymbol{\alpha}$ is a reduced expression, then $\boldsymbol{\beta}$ is a link factor of $\boldsymbol{\alpha}$ if:

- $\boldsymbol{\beta}$ is factor of $\boldsymbol{\alpha}$,
- $\boldsymbol{\beta}$ is a link, and
- $\boldsymbol{\beta}$ is not a proper factor of a link that is a factor of $\boldsymbol{\alpha}$.

Theorem (ABEPV)

Every reduced expression for a nonidentity group element can be written uniquely as a product of link factors.

For emphasis, we write the link factorization as:

$$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_m.$$

Braid graphs for link factorizations

Theorem (ABEPV)

If α is reduced expression with link factorization

$$\boldsymbol{\alpha} = \boldsymbol{\beta}_1 \mid \boldsymbol{\beta}_2 \mid \cdots \mid \boldsymbol{\beta}_m,$$

then $\mathcal{B}(\boldsymbol{\alpha})$ is the box product of the braid graphs for each $\boldsymbol{\beta}_i$.

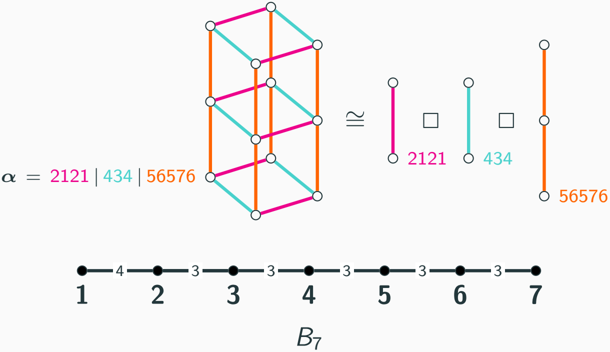
Upshot

If you want to understand the structure of braid graphs, you can first characterize braid graphs for links.

Example

Consider the reduced expression $\boldsymbol{\alpha} = 212143456576$ in type B_7 with link factorization:

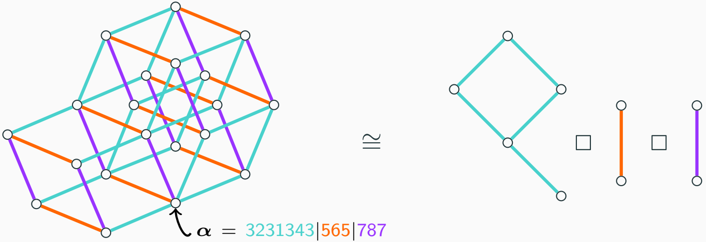
$$2121 \mid 434 \mid 56576.$$

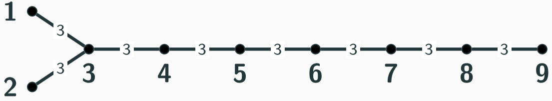


Example

Consider the reduced expression $\boldsymbol{\alpha} = 3231343565787$ in type D_9 with link factorization:

$$\color{turquoise}{3231343} \mid \color{orange}{565} \mid \color{purple}{787}.$$





D_9

Core of a braid shadow

Definition

If $[\![i, j]\!]$ is the kth braid shadow of $[\boldsymbol{\alpha}]$, then the core of the shadow at $\boldsymbol{\alpha}$ is the factor of $\boldsymbol{\alpha}$ at $[\![i+1, j-1]\!]$, denoted $\Theta_k(\boldsymbol{\alpha})$.

Example

Consider the reduced expression $\boldsymbol{\beta}_1 = 21213243$ in the Coxeter system of type B_4 with braid class:

$$\boldsymbol{\beta}_1 = \underline{2121}3243,\ \boldsymbol{\beta}_2 = \underline{121\overline{2}}\overline{32}43,\ \boldsymbol{\beta}_3 = 121\underline{32\overline{3}}\overline{43},\ \boldsymbol{\beta}_4 = 12132\overline{434}.$$

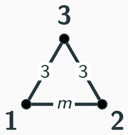
B_4

Then for example:

$$\Theta_1(\boldsymbol{\beta}_1) = 12, \Theta_2(\boldsymbol{\beta}_1) = 3, \Theta_3(\boldsymbol{\beta}_1) = 4.$$

Definition

For $m \geq 3$, a Coxeter system (W, S) is $(3, 3, m)$-avoiding, written $\overset{\Delta}{m}$-avoiding, if its Coxeter graph avoids the following subgraph:



Theorem (ABEPV)

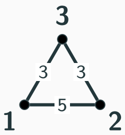
If (W, S) is $\overset{\Delta}{m}$-avoiding and α is a link of rank at least one, then $\Theta_k(\beta)$ is an *st*-string or *ts*-string for a unique pair s and t for every $\beta \in [\alpha]$.

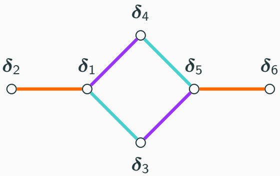
Example

Consider the link $\delta_1 = 12121312121$ in the Coxeter system given below with braid class:

$$\delta_1 = \underline{1212\overline{1}}\overline{3}\underline{\overline{1}2121},\ \delta_2 = 1212\overline{313}2121,\ \delta_3 = \underline{21212}3\underline{12121}$$

$$\delta_4 = \underline{12121}3\underline{21212},\ \delta_5 = \underline{2121\overline{2}}\overline{3}\underline{\overline{2}1212},\ \delta_6 = 2121\overline{323}1212$$





Theorem (ABEPV)

Suppose (W, S) is $\underset{m}{\triangle}$-avoiding and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be braid equivalent links. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ iff $\Theta_k(\boldsymbol{\alpha}) = \Theta_k(\boldsymbol{\beta})$ for all k.

Example

Recall the reduced expression $\boldsymbol{\beta}_1 = 21213243$ in the Coxeter system of type B_4 with braid class:

$$\boldsymbol{\beta}_1 = \underbrace{\underline{2121}3243}_{(12,3,4)}, \ \boldsymbol{\beta}_2 = \underbrace{\underline{121\overline{2}}\overline{32}43}_{(21,3,4)}, \ \boldsymbol{\beta}_3 = \underbrace{121\underline{32\overline{3}}\overline{43}}_{(21,2,4)}, \ \boldsymbol{\beta}_4 = \underbrace{12132\underline{434}}_{(21,2,3)}.$$

Observation

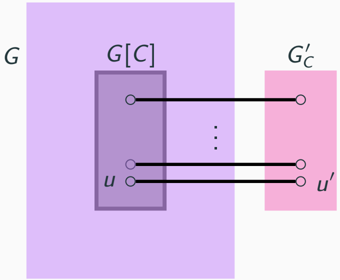
In $\underset{m}{\triangle}$-avoiding Coxeter systems, $|\mathcal{B}(\boldsymbol{\alpha})| \leq 2^{\operatorname{rank} \boldsymbol{\alpha}}$.

Convex expansions

Definition

Given a graph G and a convex set $C \subseteq V(G)$, we define the expanded graph relative to C:

- Start with a graph G;
- Make an isomorphic copy of $G[C]$, denoted G'_C, where each $u \in C$ corresponds to $u' \in C' := V(G'_C)$;
- For each $u \in C$, join u and u' with an edge.



Median graphs

Definition

A graph is median if every three vertices x, y, z have a unique median: a vertex $\mathrm{med}(x, y, z)$ that belongs to geodesics between each pair.

Proposition (Mulder)

A graph is median iff it can be obtained from a single vertex by a sequence of convex expansions.

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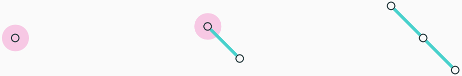

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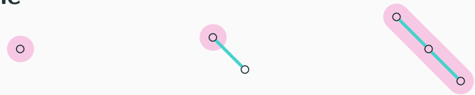

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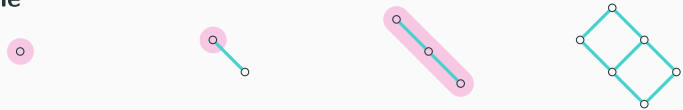

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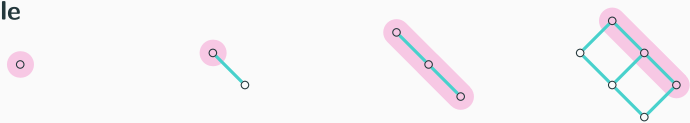

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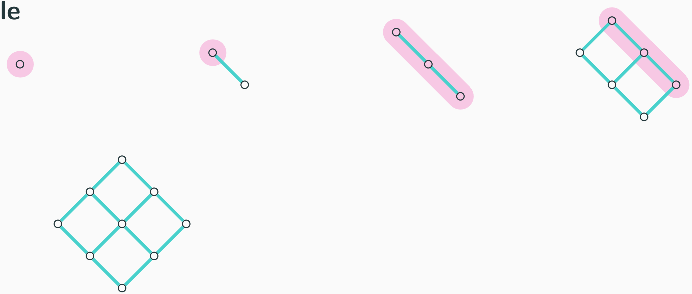

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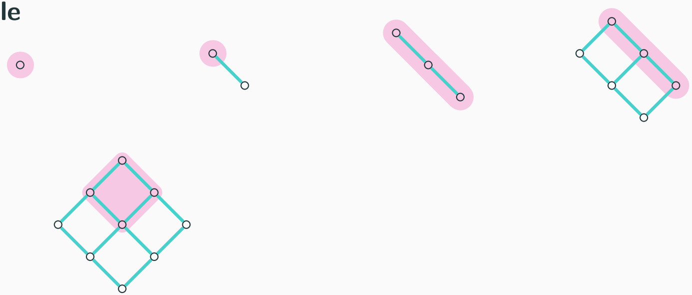

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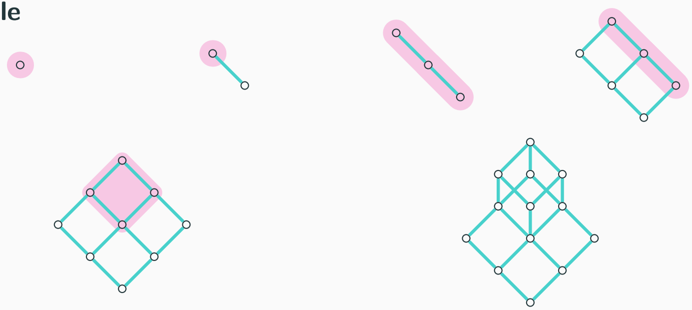


Convex expansions

Earth, Moon, & Shadow

Definition

Suppose (W, S) is $\underset{m}{\triangle}$-avoiding and $\boldsymbol{\alpha}$ is a link of rank $r \geq 2$, and let $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ such that the two rightmost braid shadows exist in $\boldsymbol{\sigma}$. Define

$$\hat{\boldsymbol{\sigma}} := \text{"chop off at last core in } \boldsymbol{\sigma}\text{"}.$$

For example: $\boldsymbol{\sigma} = \underline{21\overline{2}}\overline{3232} \Rightarrow \hat{\boldsymbol{\sigma}} = 212$.

$$\begin{aligned}
\text{Earth} &:= \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \Theta_r(\boldsymbol{\beta}) = \Theta_r(\boldsymbol{\sigma})\} \\
\hat{\text{Earth}} &:= [\hat{\boldsymbol{\sigma}}] \\
\text{Moon} &:= \{\boldsymbol{\beta} \in [\boldsymbol{\alpha}] \mid \Theta_r(\boldsymbol{\beta}) \neq \Theta_r(\boldsymbol{\sigma})\} \\
\text{Shadow} &:= \{\boldsymbol{\beta} \in \text{Earth} \mid \text{rightmost braid shadow exists in } \boldsymbol{\beta}\}
\end{aligned}$$

For simplicity, we refer to the corresponding induced subgraphs using the same names.

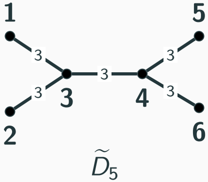
Earth, Moon, & Shadow are convex

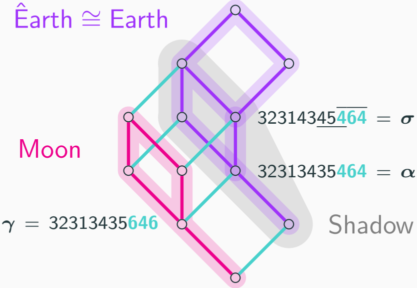
Theorem (ABEPV)
Suppose (W, S) is $\underset{m}{\Delta}$-avoiding and $\boldsymbol{\alpha}$ is a link of rank at least two. Choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ according to previous definition. Then

- Earth, Moon, and Shadow are convex.
- $\hat{\boldsymbol{\sigma}}$ is a link with rank one less than $\boldsymbol{\sigma}$.
- $\boldsymbol{\beta} \in$ Earth iff $\hat{\boldsymbol{\beta}} \in \hat{\text{E}}\text{arth} = [\hat{\boldsymbol{\sigma}}]$.
- $\hat{\text{E}}\text{arth} \xhookrightarrow{\text{isometric}} \mathcal{B}(\boldsymbol{\alpha})$ with $\hat{\text{E}}\text{arth} \cong$ Earth
- Shadow $\cong$ Moon

Example

Consider the link $\alpha = 32313435464$ in the Coxeter system of type $\widetilde{D}_5$.





Braid graphs for links are median

Theorem (ABEPV)
If (W, S) is $\underset{m}{\triangle}$-avoiding and $\boldsymbol{\alpha}$ is a link, then $\mathcal{B}(\boldsymbol{\alpha})$ is median.

Outline of Proof

- We induct on rank. Base cases check out.
- Choose $\boldsymbol{\sigma} \in [\boldsymbol{\alpha}]$ with the last two braid shadows locally available.
- By induction, Earth $\cong$ Êarth is median.
- $\mathcal{B}(\boldsymbol{\alpha})$ is obtained from Earth via a convex expansion relative to Shadow.

Braid graphs for reduced expressions are median

Proposition

If graphs G_1 and G_2 are median, then $G_1 \square G_2$ is also median.

Theorem (ABEPV)

If (W, S) is $\underset{m}{\triangle}$-avoiding and α a reduced expression, then $\mathcal{B}(\alpha)$ is median. The median of any three reduced expressions is computed by taking majority across sequence of cores.

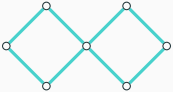
Corollary (ABEPV)

If (W, S) is $\underset{m}{\triangle}$-avoiding and α a reduced expression, then $\mathcal{B}(\alpha)$ is a partial cube with isometric dimension equal to rank (also equal to diameter).

Closing remarks

Example

Not every median graph can be realized as the braid graph for a reduced expression!



Braid graphs are "special" median graphs. What is "special"???

Hot off the press!

If (W, S) is $\underset{m}{\triangle}$-avoiding and α is a reduced expression, then $\mathcal{B}(\alpha)$ is the underlying graph for the Hasse diagram of a distributive lattice. But not every distributive lattice arises in this way!

To do!

Deal with the pesky $\underset{m}{\triangle}$-avoiding obstruction! We conjecture that braid graphs are still median (and distributive lattice?) in this context.