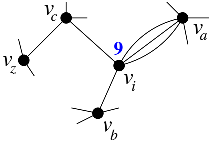
Symmetries in the sandpile model and the shuffle conjecture

Mark Dukes

School of Mathematics and Statistics, University College Dublin, Ireland

EinarFest @ NORCOM 2025



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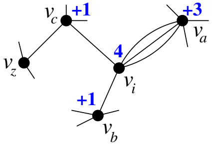
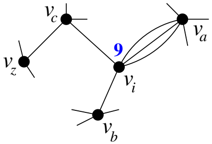
vertex v_i unstable $\Rightarrow$

$$\begin{cases} x_i \to x_i - d_i \\ x_j \to x_j + e_{ij} \quad \forall j \neq i \end{cases}$$



1. Chip-firing on $K_{m,n}$ and sandpile dynamics



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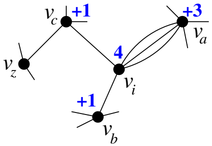

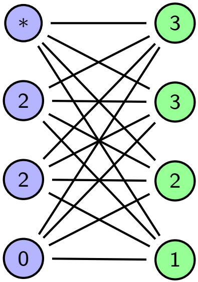

recurrent
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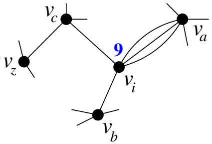

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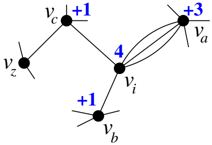

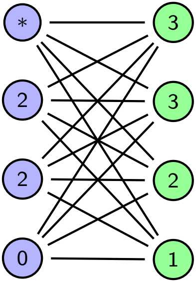

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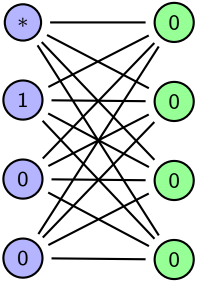

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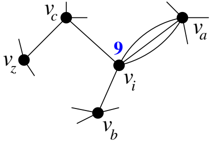

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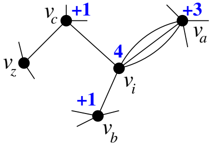

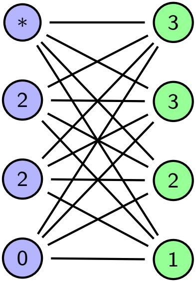

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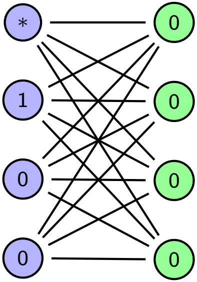
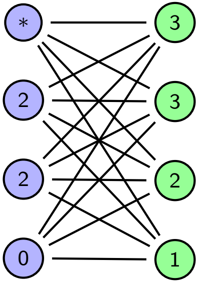
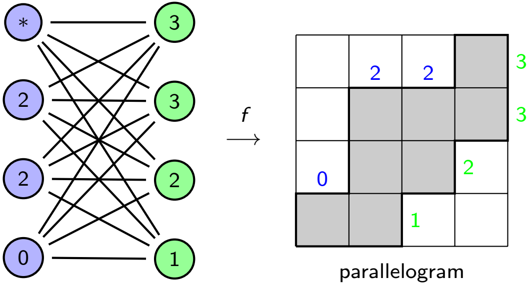


non-recurrent configuration

$\mathrm{rec}(K_{m,n}) :=$
set of all (weakly-decreasing) recurrent configurations

2. Characterizing recurrent configurations





parallelogram
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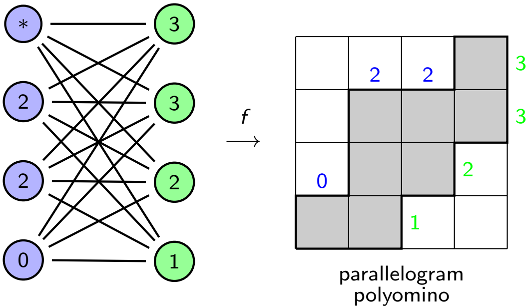
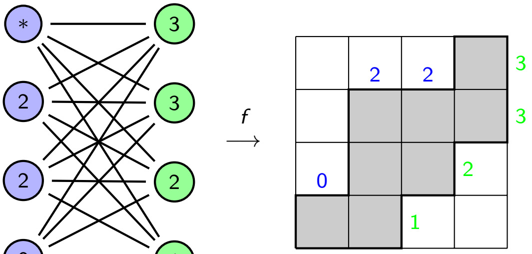


Theorem (D and Le Borgne 2013)

$c \in \mathrm{rec}(K_{m,n})$ iff $f(c)$ is a $m \times n$ parallelogram polyomino.



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Corollary

If $c \in \mathrm{rec}(K_{m,n})$ then $\sum c_i = \mathrm{area}(f(c)) + (m + n - 3)$.

3. The parabounce of a polyomino

Fix a canonical toppling on recurrent configurations. Initially add 1 to all vertices ($\equiv$ toppling the sink)

1. Topple all unstable right vertices.
2. Topple all unstable left vertices.
3. If any vertices are unstable then go to 1.

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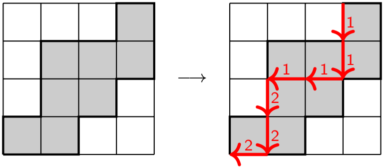

$$\begin{aligned}\text{parabounce}(P) =& 1+1+1+1 \\ & +2+2+2 \\ =& 10 \\ \text{area}(P) =& 8.\end{aligned}$$

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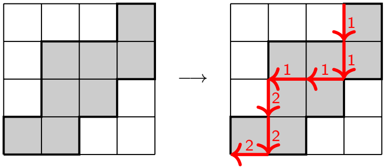

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Define

$$\mathrm{Nara}_{m,n}(q,t) := \sum_{P \in \mathrm{Para}_{m,n}} q^{\mathrm{area}(P)} t^{\mathrm{parabounce}(P)}.$$

$\mathrm{Nara}_{m,n}(1,1)$ gives the Narayana numbers hence our naming 'q,t-Narayana polynomials'.

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Conjecture (D and Le Borgne 2013)

$\mathrm{Nara}_{m,n}(q,t)$ is symmetric in both m,n and q,t:

$$\mathrm{Nara}_{m,n}(q,t) = \mathrm{Nara}_{m,n}(t,q)$$
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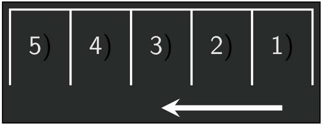
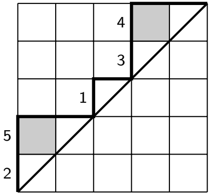
5. Parking functions

A sequence $(t_1, \ldots, t_n)$ of non-negative integers is an n-parking function if there exists a permutation π such that $t_{\pi(i)} \le i$ for all $1 \le i \le n$.

The interpretation is that t_i is the preferred parking spot of car i. If their spot is taken then the car parks in the next available ($> t_i$) spot. If all cars starting with car 1 and ending in car n can park without 'going around' then t is a parking function.



$$\begin{array}{rl} & \;\;\, 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \\ t = & (3, 1, 4, 4, 1) \end{array}$$



$$\begin{aligned} \text{area}(t) &= 2 \\ \mathrm{dinv}(t) &= 4 \\ \mathrm{word}(t) &= (4, 5, 3, 1, 2) \\ F_{5, ides(t)} &= \sum_{i_1 \le i_2 < i_3 < i_4 \le i_5} z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} \\ &= z_1^2 z_2 z_3^2 + z_1^2 z_2 z_3 z_4 + z_1^2 z_2 z_3 z_5 + \ldots \end{aligned}$$

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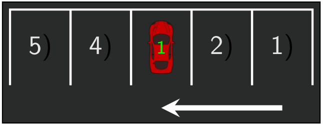

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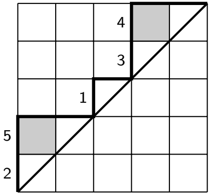

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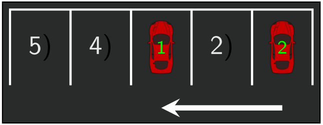

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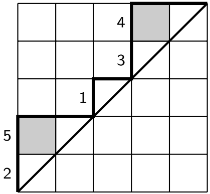

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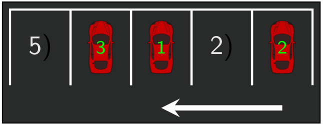

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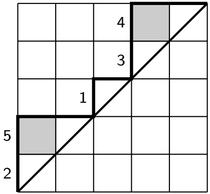

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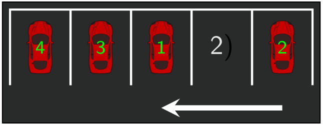

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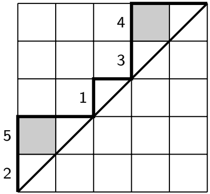

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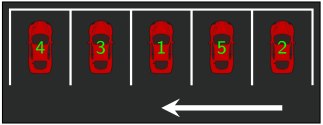

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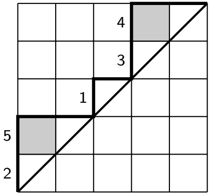

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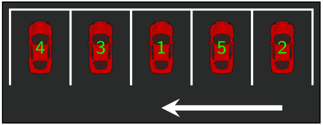
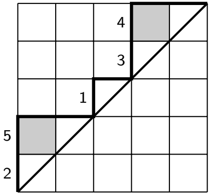


$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$

$$t = (3, 1, 4, 4, 1)$$

is a 5-parking function



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6. The shuffle conjecture: setting

Diagonal harmonic polynomials are the solutions to a system of PDEs:

$$DH_n := \left\{ f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \ : \ \sum_{i=1}^{n} \partial_{x_i}^a \partial_{y_i}^b f = 0 \text{ for all } a + b > 0 \right\}.$$

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E.g.

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Set $DH_n^{c,d}$ to be those $f \in DH_n$ whose total x degree is c and whose total y degree is d. The *bigraded Frobenius characteristic* of DH_n is the power series in q, t and $Z = \{z_1, z_2, \ldots\}$:

$$DH_n(Z; q, t) = \sum_{c,d \geq 0} t^c q^d \sum_{\lambda \vdash n} s_\lambda \mathrm{mult}(\chi^\lambda, \mathrm{char} DH_n^{c,d}) \ = \ \nabla e_n,$$

where s_λ is a Schur function in variables Z, χ^λ is the character of $DH_n^{c,d}$.

6. The shuffle conjecture: setting

Diagonal harmonic polynomials are the solutions to a system of PDEs:

$$DH_n := \left\{ f \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \;:\; \sum_{i=1}^{n} \partial_{x_i}^a \partial_{y_i}^b f = 0 \text{ for all } a + b > 0 \right\}.$$

E.g.

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$$\begin{aligned} DH_2(Z, q, t) &= \nabla e_2 = s_2 + (q + t)s_{11} \\ DH_3(Z, q, t) &= \nabla e_3 = s_3 + (q^2 + qt + t^2 + q + t)s_{21} \\ &\qquad + (q^3 + q^2 t + qt^2 + t^3 + qt)s_{111}. \end{aligned}$$

7. The shuffle conjecture: statement + special cases

The shuffle conjecture (and now theorem!) is a combinatorial interpretation for this power series:

Theorem (Haglund et al. 2003 (publ. 2005))

$$\nabla e_n = \sum_{\pi \in PF_n} t^{\mathsf{area}(\pi)} q^{\mathrm{dinv}(\pi)} F_{n, ides(\pi)},$$

where PF_n is the set of parking functions of order n, and $\mathsf{area}(\pi)$ *and* $\mathrm{dinv}(\pi)$ *are two statistics on parking functions.* $F_{n,ides(\pi)}$ *is a fundamental quasi-symmetric function, and each such expression can be written in terms of Schur functions.*

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Theorem (Haglund 2005)

$$\langle \nabla e_{m+n-2}, h_{m-1} h_{n-1} \rangle = \sum_{\pi \in \mathrm{Park}_{m-1,n-1}} t^{\mathsf{area}(\pi)} q^{\mathrm{dinv}(\pi)} =: \mathrm{Park}_{m-1,n-1}(q,t),$$

where $\mathrm{Park}_{m-1,n-1}$ *is the set of all parking functions* π *of order* $m+n-2$ *whose reading word* $\sigma(\pi)$ *is a shuffle of the sequences* $(1, \ldots, m-1)$ *and* $(m, \ldots, m+n-2)$.

8. Symmetry established

By conditioning on the first 'bounce' in a parallelogram polyomino, we find the following recursion holds for $\mathrm{Nara}_{m,n}(q,t)$:

$$\mathrm{Nara}_{m,n}^{(r,s)}(q,t) = t^{m+n-1} q^r \sum_{h=1}^{n-r} \sum_{k=0}^{m-s-1} q^s \binom{s+r-1}{s}_q \binom{s+h-1}{h}_q \mathrm{Nara}_{n-r,m-s}^{(h,k)}(q,t).$$

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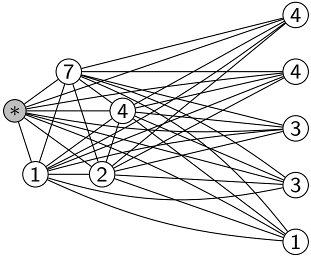
Corollary

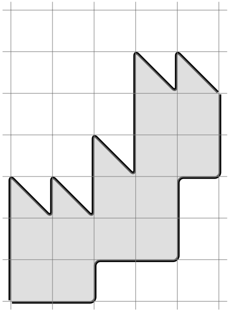
$\mathrm{Nara}_{m,n}(q,t) = \mathrm{Nara}_{m,n}(t,q)$ *and* $\mathrm{Nara}_{m,n}(q,t) = \mathrm{Nara}_{n,m}(q,t)$.

9. Chip-firing on the complete split graph

Complete split graph $S_{m,n}$:



$\xrightarrow{g}$



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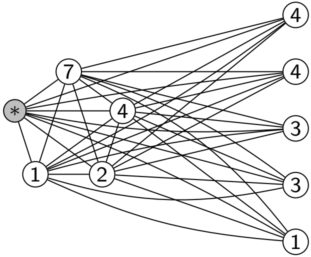

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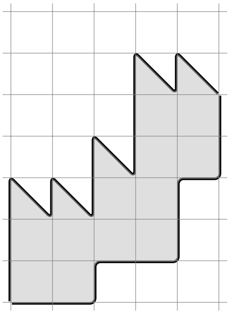
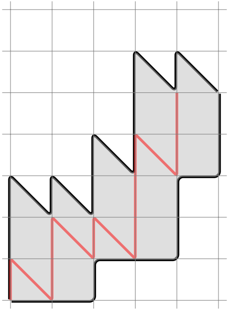


A representation of the recurrent states $\mathrm{rec}(S_{m,n})$ comes in the form of a new type of polyomino that we call *sawtooth polyominos*.

Theorem (Derycke, D, and Le Borgne 2024)

$c \in \mathrm{rec}(S_{m,n})$ *iff* $g(c) \in \mathrm{Sawtooth}_{m,n}$.

10. Another surprise

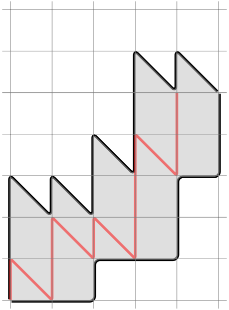


$$\mathrm{area}(P) = 12$$

$$\begin{aligned}\mathrm{parabounce}(P) &= 1\cdot(2+1) + 2\cdot(2+1) \\ &\quad + 3\cdot(0+1) + 4\cdot(1+1) \\ &= 20.\end{aligned}$$

Let

$$F^{ITC}_{n,d}(q,t) := \sum_{P\in \mathrm{Sawtooth}_{m,n}} q^{\mathrm{area}(P)} t^{\mathrm{itcbounce}(P)}.$$



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Theorem (Derycke, D, and Le Borgne 2024)

$$\begin{aligned}F^{ITC}_{n,d}(q,t) &= \sum_{k=1}^{n} \sum_{\substack{(b_1,\ldots,b_{k+1})\vDash^*_k d \\ (a_1,\ldots,a_k)\vDash_k n}} \prod_{i=1}^{k+1} q^{\binom{a_i}{2}} \binom{a_i+b_i}{b_i}_q \binom{a_i+b_i+a_{i-1}-1}{a_{i-1}-1}_q t^{(i-1)(a_i+b_i)} \\ &= \langle \nabla e_{n+d}, e_n h_d \rangle.\end{aligned}$$

Symmetry in q and t follows.

11. A more general result

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be a pair of integer compositions with $n = |\lambda| + |\mu|$.

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Let $G_{\lambda,\mu}$ be the graph that consists of cliques $K_{\lambda_1}, K_{\lambda_2}, \ldots$, and independent sets $I_{\mu_1}, I_{\mu_2}, \ldots$: every pair of vertices that are members of different sets (be they cliques/independent sets) are connected by an edge.

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Theorem (D'Adderio, D, Iraci, Lazar, Le Borgne, Vanden Wyngaerd 2024)

For every pair of compositions λ, μ such that $n = |\lambda| + |\mu|$,

$$\langle \nabla e_n, e_\lambda h_\mu \rangle = \sum_{c \in \mathrm{rec}(G_{\lambda,\mu})} q^{\mathrm{level}(c)} t^{\mathrm{delaybounce}(c)}.$$

Symmetry in q and t follows from this form. (Each of the coefficients of ∇e_n when written as a linear combination of Schur functions is a symmetric function.)

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Symmetry in q and t follows from this form. (Each of the coefficients of ∇e_n when written as a linear combination of Schur functions is a symmetric function.)

Special cases of the above theorem include the results for the complete bipartite graph and the complete split graph.

12. Comments

- These instances of symmetry, that are related to the shuffle conjecture, suggest something more general is afoot. An interesting question to consider if whether there is a parameterized graph whose bivariate q, t-polynomial (in the sense of what we have discussed) corresponds to other instances of the inner product of ∇e_n with some other symmetric functions.

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Thanks for your attention!