Enumerating finite models of Hilbert's incidence axioms

Nikolina Miholjčić

Faculty of Sciences, University of Novi Sad, Serbia

Joint work with K. Ago and B. Bašić

15th Nordic Combinatorial Conference
June 17 2025, Reykjavik, Iceland

Hilbert's incidence axioms

Hilbert's incidence axioms

- **Primitive terms:** point, line, plane.

Hilbert's incidence axioms

- **Primitive terms:** point, line, plane.
- **Primitive relation:** incidence.

Hilbert's incidence axioms

- **Primitive terms:** point, line, plane.
- **Primitive relation:** incidence.
- Axioms of incidence (denoted by $\mathcal{A}$):

I_1: For every two points A, B, there exists a line a that contains both A and B.

I_2: For every two points A, B, there exists at most one line that contains both A and B.

I_3: There exist at least two points on a line. There exist at least three points that do not lie on a single line.

I_4: For any three non-collinear points A, B, C, there exists a plane α that contains all three. Every plane contains at least one point.

I_5: For any three non-collinear points A, B, C, there exists at most one plane that contains them.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

I_7: If two planes α, β share a point A, then they share at least one more point B.

I_8: There exist at least four points that do not lie in the same plane.

Finite models of $\mathcal{A}$

Finite models of $\mathcal{A}$

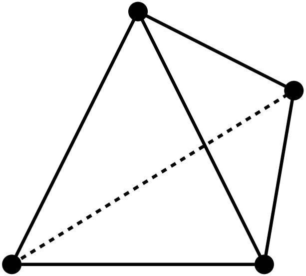
- We are interested in **finite models** (P, L, PI) of $\mathcal{A}$.

Finite models of $\mathcal{A}$

- We are interested in **finite models** (P, L, Pl) of $\mathcal{A}$.
- **Open problem**: Given a natural number n, how many non-isomorphic finite models of $\mathcal{A}$ exists with exactly n points?

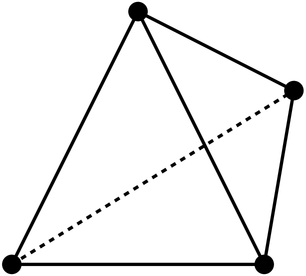
Finite models of $\mathcal{A}$

- We are interested in **finite models** (P, L, PI) of $\mathcal{A}$.
- **Open problem**: Given a natural number n, how many non-isomorphic finite models of $\mathcal{A}$ exists with exactly n points?
- The smallest finite model of $\mathcal{A}$:



Finite models of $\mathcal{A}$

- We are interested in **finite models** (P, L, Pl) of $\mathcal{A}$.
- **Open problem**: Given a natural number n, how many non-isomorphic finite models of $\mathcal{A}$ exists with exactly n points?
- The smallest finite model of $\mathcal{A}$:



K. Ago & B. Bašić & M. Maksimović & M. Šobot, On finite models of Hilbert's incidence geometry, *Discrete Math.* **347** (2024), Article No. 114159

Tetrahedron-models

Tetrahedron-models

- The *Tetrahedron-model* is obtained by starting from the 4-point model and then adding some more points on a pair of two disjoint lines.

Tetrahedron-models

- The *Tetrahedron-model* is obtained by starting from the 4-point model and then adding some more points on a pair of two disjoint lines.

Theorem

Let n be an integer, $n \geqslant 4$. Let i be an integer, $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Let:

$$P = \{1, 2, \ldots, n\},$$

$$L = \{\{1, 2, \ldots, i\}, \{i+1, i+2, \ldots, n\}\} \cup \{\{x, y\} : 1 \leqslant x \leqslant i,\ i+1 \leqslant y \leqslant n\},$$

$$\mathrm{Pl} = \{\{1, 2, \ldots, i, x\} : i+1 \leqslant x \leqslant n\} \cup \{\{i+1, i+2, \ldots, n, y\} : 1 \leqslant y \leqslant i\}.$$

Then (P, L, Pl) is a model of $\mathcal{A}$.

Tetrahedron-models

- The *Tetrahedron-model* is obtained by starting from the 4-point model and then adding some more points on a pair of two disjoint lines.

Theorem

Let n be an integer, $n \geqslant 4$. Let i be an integer, $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Let:

$$P = \{1, 2, \ldots, n\},$$

$$L = \{\{1, 2, \ldots, i\}, \{i+1, i+2, \ldots, n\}\} \cup \{\{x, y\} : 1 \leqslant x \leqslant i,\ i+1 \leqslant y \leqslant n\},$$

$$\mathrm{Pl} = \{\{1, 2, \ldots, i, x\} : i+1 \leqslant x \leqslant n\} \cup \{\{i+1, i+2, \ldots, n, y\} : 1 \leqslant y \leqslant i\}.$$

Then (P, L, Pl) is a model of $\mathcal{A}$.

Proposition

There are $\lfloor \frac{n-2}{2} \rfloor$ nonisomorphic tetrahedron-models of $\mathcal{A}$.

Tetrahedron-models

- The *Tetrahedron-model* is obtained by starting from the 4-point model and then adding some more points on a pair of two disjoint lines.

Theorem

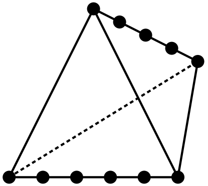
Let n be an integer, $n \geqslant 4$. Let i be an integer, $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$. Let:

$$P = \{1, 2, \ldots, n\},$$
$$L = \{\{1, 2, \ldots, i\}, \{i+1, i+2, \ldots, n\}\} \cup \{\{x, y\} : 1 \leqslant x \leqslant i,\ i+1 \leqslant y \leqslant n\},$$
$$\mathrm{Pl} = \{\{1, 2, \ldots, i, x\} : i+1 \leqslant x \leqslant n\} \cup \{\{i+1, i+2, \ldots, n, y\} : 1 \leqslant y \leqslant i\}.$$

Then (P, L, Pl) is a model of $\mathcal{A}$.

Proposition

There are $\lfloor \frac{n-2}{2} \rfloor$ nonisomorphic tetrahedron-models of $\mathcal{A}$.



Projective-space-models

Projective-space-models

- $PG(n, q)$: the $(n+1)$-dimensional vector space over the field of order q, where points are interpreted as 1-dimensional subspaces, and lines as 2-dimensional subspaces.

Projective-space-models

- $PG(n, q)$: the $(n+1)$-dimensional vector space over the field of order q, where points are interpreted as 1-dimensional subspaces, and lines as 2-dimensional subspaces.
- For $n \geq 3$, these are exactly all projective spaces.

Projective-space-models

- $PG(n, q)$: the $(n+1)$-dimensional vector space over the field of order q, where points are interpreted as 1-dimensional subspaces, and lines as 2-dimensional subspaces.
- For $n \geq 3$, these are exactly all projective spaces.
- For $n = 2$, $PG(2, q)$ are examples of projective planes (though there exist projective planes that are not of this type).

Projective-space-models

- $PG(n, q)$: the $(n+1)$-dimensional vector space over the field of order q, where points are interpreted as 1-dimensional subspaces, and lines as 2-dimensional subspaces.
- For $n \geq 3$, these are exactly all projective spaces.
- For $n = 2$, $PG(2, q)$ are examples of projective planes (though there exist projective planes that are not of this type).

Theorem

Let F^4 be a 4-dimensional vector space over some finite field F of order q. Let P be the set of 1-dimensional subspaces of F^4, let L be the set of 2-dimensional subspaces, and let Pl *be the set of 3-dimensional subspaces. Then (P, L, Pl) is a model of $\mathcal{A}$.*

Projective-space-models

- $PG(n, q)$: the $(n+1)$-dimensional vector space over the field of order q, where points are interpreted as 1-dimensional subspaces, and lines as 2-dimensional subspaces.
- For $n \geq 3$, these are exactly all projective spaces.
- For $n = 2$, $PG(2, q)$ are examples of projective planes (though there exist projective planes that are not of this type).

Theorem

Let F^4 be a 4-dimensional vector space over some finite field F of order q. Let P be the set of 1-dimensional subspaces of F^4, let L be the set of 2-dimensional subspaces, and let Pl be the set of 3-dimensional subspaces. Then (P, L, Pl) is a model of $\mathcal{A}$.

Proposition

Up to isomorphism, there is one n-element projective-space-model of $\mathcal{A}$ for each number n of the form $q^3 + q^2 + q + 1$, where q is a prime power.

Projective-plane-models

Projective-plane-models

Theorem

Let P' *and* L' *be the set of points and the set of lines of some projective plane. Let:*

$$P = P' \cup \{X\}, \text{ where } X \notin P';$$
$$L = L' \cup \{\{Y, X\} : Y \in P'\};$$
$$\mathrm{PI} = \{P'\} \cup \{l \cup \{X\} : l \in L'\}.$$

Then (P, L, PI) *is a model of* $\mathcal{A}$*.*

Projective-plane-models

Theorem

Let P' and L' be the set of points and the set of lines of some projective plane. Let:

$$P = P' \cup \{X\}, \text{ where } X \notin P';$$
$$L = L' \cup \{\{Y, X\} : Y \in P'\};$$
$$\mathrm{PI} = \{P'\} \cup \{l \cup \{X\} : l \in L'\}.$$

Then (P, L, PI) is a model of $\mathcal{A}$.

Proposition

For each n of the form $q^2 + q + 2$, where q is a number such that there exists a projective plane of order q, there are as many n-element projective-plane-models of $\mathcal{A}$ as there are nonisomorphic projective planes with $n - 1$ points.

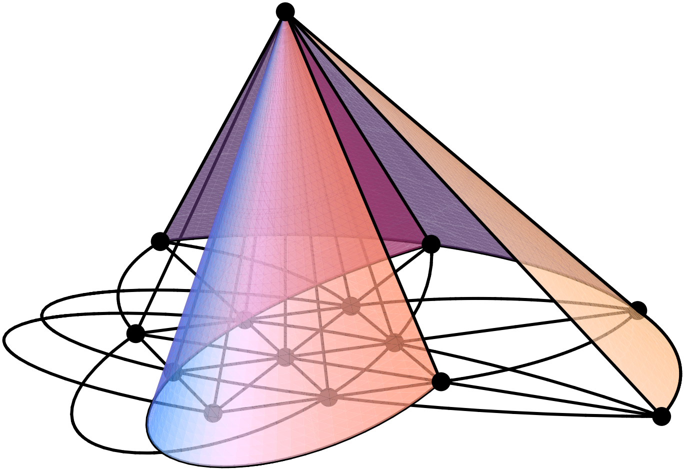
Projective-plane-models

Projective-plane-models

The projective-plane-model with 14 points:

Projective-plane-models

The projective-plane-model with 14 points:



Matroids to the rescue

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.
- The method heavily relies on existing databases of matroids with specified properties, which are currently available only for $n \leq 12$.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.
- The method heavily relies on existing databases of matroids with specified properties, which are currently available only for $n \leq 12$.
- Beyond $n = 12$, the number of matroids becomes too large for exhaustive methods to remain tractable.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.
- The method heavily relies on existing databases of matroids with specified properties, which are currently available only for $n \leq 12$.
- Beyond $n = 12$, the number of matroids becomes too large for exhaustive methods to remain tractable.

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, $n = 1, 2, \ldots, 12$, is given in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
HilbInc(n)	0	0	0	1	1	2	2	5	3	4	4	6

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.
- The method heavily relies on existing databases of matroids with specified properties, which are currently available only for $n \leq 12$.
- Beyond $n = 12$, the number of matroids becomes too large for exhaustive methods to remain tractable.

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, $n = 1, 2, \ldots, 12$, is given in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
HilbInc(n)	0	0	0	1	1	2	2	5	3	4	4	6

- This summarizes the current state of knowledge on the topic.

Matroids to the rescue

- A **matroid** is an abstract combinatorial structure designed to capture the essence of independence across different mathematical settings.
- In short, a neat duality has been established between matroids satisfying certain properties and finite models of subsets of the axiom set $\mathcal{A}$.
- Consequently, counting finite models of $\mathcal{A}$ reduces to counting matroids that satisfy additional structural conditions.
- The method heavily relies on existing databases of matroids with specified properties, which are currently available only for $n \leq 12$.
- Beyond $n = 12$, the number of matroids becomes too large for exhaustive methods to remain tractable.

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, $n = 1, 2, \ldots, 12$, is given in the following table:

n	1	2	3	4	5	6	7	8	9	10	11	12
HilbInc(n)	0	0	0	1	1	2	2	5	3	4	4	6

- This summarizes the current state of knowledge on the topic.
- **Can this boundary be pushed further to count such models for larger** n?

A new approach for larger values

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.
- This reformulates the model-counting problem as a propositional satisfiability task.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.
- This reformulates the model-counting problem as a propositional satisfiability task.
- The **SAT** (*Boolean satisfiability*) **problem** asks whether variables in a propositional formula can be assigned values so that the formula is true.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.
- This reformulates the model-counting problem as a propositional satisfiability task.
- The **SAT** (*Boolean satisfiability*) **problem** asks whether variables in a propositional formula can be assigned values so that the formula is true.
- SAT was the first problem proven to be *NP-complete*.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.
- This reformulates the model-counting problem as a propositional satisfiability task.
- The **SAT** (*Boolean satisfiability*) **problem** asks whether variables in a propositional formula can be assigned values so that the formula is true.
- SAT was the first problem proven to be *NP-complete*.
- Despite theoretical hardness, modern SAT solvers efficiently handle large instances in practice.

A new approach for larger values

- Motivated by the need for a computational approach, we designed a new axiomatic system which is logically equivalent to Hilbert's incidence axioms but formulated to support algorithmic model counting.
- The new system consists of a set of quantified formulas.
- We prove a one-to-one correspondence between:
  - finite models of the incidence axioms, and
  - satisfying truth assignments of these formulas.
- This reformulates the model-counting problem as a propositional satisfiability task.
- The **SAT** (*Boolean satisfiability*) **problem** asks whether variables in a propositional formula can be assigned values so that the formula is true.
- SAT was the first problem proven to be *NP-complete*.
- Despite theoretical hardness, modern SAT solvers efficiently handle large instances in practice.
- Leveraging such tools, we adapted a state-of-the-art SAT solver to our specific setting and used it to compute the exact number of non-isomorphic finite models of $\mathcal{A}$ for all $n \leq 18$.

A new axiomatic system

A new axiomatic system

We introduce a new system, $\mathcal{B}$, defined by the six axioms which are as follows:

A new axiomatic system

We introduce a new system, $\mathcal{B}$, defined by the six axioms which are as follows:

B_1: $(\exists a, b, c)\ \neg\,\mathrm{col}(\{a, b, c\})$

B_2: $(\forall a, b, c, d)\ \big(\mathrm{col}(\{a, b, c\}) \wedge \mathrm{col}(\{a, b, d\}) \Rightarrow \mathrm{col}(\{a, c, d\})\big)$

B_3: $(\forall a, b, c, d, e)\ \big(\mathrm{cop}(\{a, b, c, d\}) \wedge \mathrm{cop}(\{a, b, c, e\}) \wedge \neg\,\mathrm{col}(\{a, b, c\}) \Rightarrow \mathrm{cop}(\{a, b, d, e\})\big)$

B_4: $(\forall a, b, c, d)\ \big(\mathrm{col}(\{a, b, c\}) \wedge \neg\,\mathrm{col}(\{a, b, d\}) \Rightarrow \mathrm{cop}(\{a, b, c, d\})\big)$

B_5: $(\forall a, b, c, d, e)\ \big(\neg\,\mathrm{col}(\{a, b, c\}) \wedge \neg\,\mathrm{col}(\{a, d, e\}) \Rightarrow \mathrm{cop}(\{a, b, c, d\}) \vee \mathrm{cop}(\{a, b, c, e\}) \vee \mathrm{cop}(\{a, d, e, b\}) \vee \mathrm{cop}(\{a, d, e, f\}) \vee (\exists f)\ (\mathrm{cop}(\{a, b, c, f\}) \wedge \mathrm{cop}(\{a, d, e, f\}))\big)$

B_6: $(\exists a, b, c, d)\ \neg\,\mathrm{cop}(\{a, b, c, d\})$

A neat duality

A neat duality

Theorem

Let Mod, Mod$=(P, L, \mathrm{Pl})$*, be a finite model of* $\mathcal{A}$*. We define two relations, col and cop, on 3-element and 4-element subsets of a given point set* P*, respectively, as follows:*

- *For every subset* $S \subseteq P$ *such that* $|S| = 3$*,* $\mathrm{col}(S)$ *if and only if the points in* S *are collinear;*
- *For every subset* $K \subseteq P$ *such that* $|K| = 4$*,* $\mathrm{cop}(K)$ *if and only if the points in* K *are coplanar.*

Then the formulas from $\mathcal{B}$ *will be satisfied.*

A neat duality

Theorem

Let Mod, Mod$=(P, L, \mathrm{Pl})$, be a finite model of $\mathcal{A}$. We define two relations, col and cop, on 3-element and 4-element subsets of a given point set P, respectively, as follows:

- *For every subset $S \subseteq P$ such that $|S| = 3$, $\mathrm{col}(S)$ if and only if the points in S are collinear;*
- *For every subset $K \subseteq P$ such that $|K| = 4$, $\mathrm{cop}(K)$ if and only if the points in K are coplanar.*

Then the formulas from $\mathcal{B}$ will be satisfied.

B_1: $(\exists a, b, c)\, \neg \mathrm{col}(\{a, b, c\})$

A neat duality

Theorem

Let Mod, Mod$=(P, L, \mathrm{Pl})$*, be a finite model of* $\mathcal{A}$*. We define two relations, col and cop, on 3-element and 4-element subsets of a given point set* P*, respectively, as follows:*

- *For every subset* $S \subseteq P$ *such that* $|S| = 3$, $\mathrm{col}(S)$ *if and only if the points in* S *are collinear;*
- *For every subset* $K \subseteq P$ *such that* $|K| = 4$, $\mathrm{cop}(K)$ *if and only if the points in* K *are coplanar.*

Then the formulas from $\mathcal{B}$ *will be satisfied.*

B_1: $(\exists a, b, c)\, \neg\, \mathrm{col}(\{a, b, c\})$

follows directly from

I_3: There exist at least three points that do not lie on a single line.

A neat duality

Theorem

Let Mod, Mod$=(P, L, \mathrm{Pl})$, be a finite model of $\mathcal{A}$. We define two relations, col and cop, on 3-element and 4-element subsets of a given point set P, respectively, as follows:

- *For every subset $S \subseteq P$ such that $|S| = 3$, $\mathrm{col}(S)$ if and only if the points in S are collinear;*
- *For every subset $K \subseteq P$ such that $|K| = 4$, $\mathrm{cop}(K)$ if and only if the points in K are coplanar.*

Then the formulas from $\mathcal{B}$ will be satisfied.

B_1: $(\exists a, b, c)\ \neg\,\mathrm{col}(\{a, b, c\})$

follows directly from

I_3: There exist at least three points that do not lie on a single line.

B_5: $$(\forall a, b, c, d, e)\ \big(\neg\,\mathrm{col}(\{a, b, c\}) \wedge \neg\,\mathrm{col}(\{a, d, e\}) \Rightarrow \mathrm{cop}(\{a, b, c, d\}) \vee \mathrm{cop}(\{a, b, c, e\}) \vee \mathrm{cop}(\{a, d, e, b\}) \vee \mathrm{cop}(\{a, d, e, f\}) \vee (\exists f)\ (\mathrm{cop}(\{a, b, c, f\}) \wedge \mathrm{cop}(\{a, d, e, f\}))\big)$$

A neat duality

Theorem

Let Mod, Mod=(P, L, Pl), be a finite model of $\mathcal{A}$. We define two relations, col and cop, on 3-element and 4-element subsets of a given point set P, respectively, as follows:

- *For every subset $S \subseteq P$ such that $|S| = 3$, $\mathrm{col}(S)$ if and only if the points in S are collinear;*
- *For every subset $K \subseteq P$ such that $|K| = 4$, $\mathrm{cop}(K)$ if and only if the points in K are coplanar.*

Then the formulas from $\mathcal{B}$ will be satisfied.

B_1: $(\exists a, b, c)\ \neg\,\mathrm{col}(\{a, b, c\})$

follows directly from

I_3: There exist at least three points that do not lie on a single line.

B_5:
$$(\forall a, b, c, d, e)\ (\neg\,\mathrm{col}(\{a, b, c\}) \wedge \neg\,\mathrm{col}(\{a, d, e\}) \Rightarrow \mathrm{cop}(\{a, b, c, d\}) \vee \mathrm{cop}(\{a, b, c, e\}) \vee \mathrm{cop}(\{a, d, e, b\}) \vee \mathrm{cop}(\{a, d, e, f\}) \vee (\exists f)\ (\mathrm{cop}(\{a, b, c, f\}) \wedge \mathrm{cop}(\{a, d, e, f\})))$$

follows from

I_4: For any three non-collinear points A, B, C, there exists a plane α that contains all three.

I_7: If two planes α, β share a point A, then they share at least one more point B.

A neat duality

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \dots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines* (L) *as the max. el. of the set* $\{D \subseteq P \mid \forall S \subseteq D, |S| = 3 \Rightarrow \mathrm{col}(S)\}$,
- *Planes* (Pl) *as the max. el. of the set* $\{D \subseteq P \mid \forall K \subseteq D, |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ *is a model of the axiom set* $\mathcal{A}$.

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines* (L) *as the max. el. of the set* $\{D \subseteq P \mid \forall S \subseteq D,\ |S| = 3 \Rightarrow \mathrm{col}(S)\}$,
- *Planes* (Pl) *as the max. el. of the set* $\{D \subseteq P \mid \forall K \subseteq D,\ |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ *is a model of the axiom set* $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines (L) as the max. el. of the set $\{D \subseteq P \mid \forall S \subseteq D, |S| = 3 \Rightarrow \mathrm{col}(S)\}$,*
- *Planes (Pl) as the max. el. of the set $\{D \subseteq P \mid \forall K \subseteq D, |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,*

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ is a model of the axiom set $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

Claim. Every plane in $\mathrm{Mod}_{\mathcal{B}}$ contains three non-collinear points.

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines (L) as the max. el. of the set $\{D \subseteq P \mid \forall S \subseteq D, |S| = 3 \Rightarrow \mathrm{col}(S)\}$,*
- *Planes (Pl) as the max. el. of the set $\{D \subseteq P \mid \forall K \subseteq D, |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,*

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ is a model of the axiom set $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

Claim. Every plane in $\mathrm{Mod}_{\mathcal{B}}$ contains three non-collinear points.

Verification of I_6:

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines (L) as the max. el. of the set $\{D \subseteq P \mid \forall S \subseteq D, |S| = 3 \Rightarrow \mathrm{col}(S)\}$,*
- *Planes (Pl) as the max. el. of the set $\{D \subseteq P \mid \forall K \subseteq D, |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,*

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ is a model of the axiom set $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

Claim. Every plane in $\mathrm{Mod}_{\mathcal{B}}$ contains three non-collinear points.

Verification of I_6: Assume $\{A, B\} \subseteq p \cap \alpha$.

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines* (L) *as the max. el. of the set* $\{D \subseteq P \mid \forall S \subseteq D, |S| = 3 \Rightarrow \mathrm{col}(S)\}$,
- *Planes* (Pl) *as the max. el. of the set* $\{D \subseteq P \mid \forall K \subseteq D, |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ *is a model of the axiom set* $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

Claim. Every plane in $\mathrm{Mod}_{\mathcal{B}}$ contains three non-collinear points.

Verification of I_6: Assume $\{A, B\} \subseteq p \cap \alpha$.

- $p = \{A, B\} \subseteq \alpha$ ✓

A neat duality

Theorem

Suppose there exists a valuation that satisfies the formulas in $\mathcal{B}$. Using this valuation, we determine which subsets of the point set $P = \{1, 2, \ldots, n\}$ satisfy the relations col and cop, where col is a relation defined on 3-element subsets of P, and cop is defined on 4-element subsets of P, respectively. If we define:

- *Lines (L) as the max. el. of the set $\{D \subseteq P \mid \forall S \subseteq D,\ |S| = 3 \Rightarrow \mathrm{col}(S)\}$,*
- *Planes (Pl) as the max. el. of the set $\{D \subseteq P \mid \forall K \subseteq D,\ |K| = 4 \Rightarrow \mathrm{cop}(K)\}$,*

then the structure $\mathrm{Mod}_{\mathcal{B}} = (P, L, \mathrm{Pl})$ is a model of the axiom set $\mathcal{A}$.

I_6: If two points A, B of a line p lie in a plane α, then every point on p lies in α.

Claim. Every plane in $\mathrm{Mod}_{\mathcal{B}}$ contains three non-collinear points.

Verification of I_6: Assume $\{A, B\} \subseteq p \cap \alpha$.

- $p = \{A, B\} \subseteq \alpha$ ✓
- Suppose, for contradiction, that there exists $C \in p \setminus \{A, B\}$ such that $C \notin \alpha$. From the claim, there exists $D \in \alpha$ such that $D \notin p$. By axiom B_4, we have:

  $$\mathrm{col}(\{A, B, C\}) \wedge \neg\,\mathrm{col}(\{A, B, D\}) \Rightarrow \mathrm{cop}(\{A, B, C, D\}).$$

  Thus, the points A, B, C, and D lie in some plane. Since A, B, and D are not collinear, it follows from axiom I_5 that this plane must be exactly α. But then $C \in \alpha$, which contradicts the assumption.

A neat duality

A neat duality

Theorem

(a) *Let* Mod *be a model of* $\mathcal{A}$. *Define the relations* col *and* cop *as in the construction, and suppose all formulas in* $\mathcal{B}$ *are satisfied. Then, defining lines and planes as presented, we obtain* $\mathrm{Mod} = \mathrm{Mod}_{\mathcal{B}}$.

(b) *Conversely, starting from a valuation that satisfies all formulas in* $\mathcal{B}$, *define points, lines, and planes accordingly to obtain a model of* $\mathcal{A}$. *Then, defining relations col and cop as presented, we recover the original valuation.*

Counting models via SAT solving

Counting models via SAT solving

- Thanks to the equivalence between systems $\mathcal{A}$ and $\mathcal{B}$, instead of counting finite models of $\mathcal{A}$, it suffices to count how many satisfying valuations exist for the formulas in $\mathcal{B}$.

Counting models via SAT solving

- Thanks to the equivalence between systems $\mathcal{A}$ and $\mathcal{B}$, instead of counting finite models of $\mathcal{A}$, it suffices to count how many satisfying valuations exist for the formulas in $\mathcal{B}$.
- Although modern SAT solvers can return the total number of satisfying assignments (model counters), this feature is computationally expensive.

Counting models via SAT solving

- Thanks to the equivalence between systems $\mathcal{A}$ and $\mathcal{B}$, instead of counting finite models of $\mathcal{A}$, it suffices to count how many satisfying valuations exist for the formulas in $\mathcal{B}$.
- Although modern SAT solvers can return the total number of satisfying assignments (model counters), this feature is computationally expensive.
- Instead, we used a standard SAT solver that returns:
  - whether the set of formulas is satisfiable,
  - and one satisfying valuation if it exists.

Counting models via SAT solving

- Thanks to the equivalence between systems $\mathcal{A}$ and $\mathcal{B}$, instead of counting finite models of $\mathcal{A}$, it suffices to count how many satisfying valuations exist for the formulas in $\mathcal{B}$.
- Although modern SAT solvers can return the total number of satisfying assignments (model counters), this feature is computationally expensive.
- Instead, we used a standard SAT solver that returns:
  - whether the set of formulas is satisfiable,
  - and one satisfying valuation if it exists.
- We iteratively search for all models as follows:
  1. Run the solver to obtain a satisfying assignment.
  2. Extract the model corresponding to this assignment.
  3. Add a formula that excludes this specific model (or its isomorphism class).
  4. Repeat until the formula set becomes unsatisfiable.

Counting models via SAT solving

- Thanks to the equivalence between systems $\mathcal{A}$ and $\mathcal{B}$, instead of counting finite models of $\mathcal{A}$, it suffices to count how many satisfying valuations exist for the formulas in $\mathcal{B}$.
- Although modern SAT solvers can return the total number of satisfying assignments (model counters), this feature is computationally expensive.
- Instead, we used a standard SAT solver that returns:
  - whether the set of formulas is satisfiable,
  - and one satisfying valuation if it exists.
- We iteratively search for all models as follows:
  1. Run the solver to obtain a satisfying assignment.
  2. Extract the model corresponding to this assignment.
  3. Add a formula that excludes this specific model (or its isomorphism class).
  4. Repeat until the formula set becomes unsatisfiable.
- In this way, we systematically enumerate all distinct valuations satisfying $\mathcal{B}$, which correspond to models of $\mathcal{A}$.

Excluding specific models

Excluding specific models

- To eliminate known models and search for genuinely new configurations, we identify a distinguishing property and add its negation as an additional constraint in the SAT input.

Excluding specific models

- To eliminate known models and search for genuinely new configurations, we identify a distinguishing property and add its negation as an additional constraint in the SAT input.
- To avoid redundant enumeration of isomorphic models, we fix the longest line to have exactly k collinear points (for $3 \leq k \leq n-3$). Specifically:
    - the first k points are required to be collinear;
    - no $k+1$ points are collinear.

Excluding specific models

- To eliminate known models and search for genuinely new configurations, we identify a distinguishing property and add its negation as an additional constraint in the SAT input.
- To avoid redundant enumeration of isomorphic models, we fix the longest line to have exactly k collinear points (for $3 \leq k \leq n-3$). Specifically:
  - the first k points are required to be collinear;
  - no $k+1$ points are collinear.
- For example, to exclude the **Tetrahedron-model**, characterized by all points lying on two disjoint lines, we additionally require that:
  - the remaining points (from $k+1$ to n) do not lie on a common line.

Counting up to 18

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$*, is given in the following table:*

n	13	14	15	16	17	18
$\mathrm{HilbInc}(n)$	5	8	7	7	7	9

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$, is given in the following table:

n	13	14	15	16	17	18
$\mathrm{HilbInc}(n)$	5	8	7	7	7	9

All these models are tetrahedron-models, with exactly four exceptions:

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$, is given in the following table:

n	13	14	15	16	17	18
$\mathrm{HilbInc}(n)$	5	8	7	7	7	9

All these models are tetrahedron-models, with exactly four exceptions:

- a projective-plane-model for $n = 14$,

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$, is given in the following table:

n	13	14	15	16	17	18
HilbInc(n)	5	8	7	7	7	9

All these models are tetrahedron-models, with exactly four exceptions:

- a projective-plane-model for $n = 14$,
- a projective-space-model for $n = 15$,

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$, is given in the following table:

n	13	14	15	16	17	18
HilbInc(n)	5	8	7	7	7	9

All these models are tetrahedron-models, with exactly four exceptions:

- a projective-plane-model for $n = 14$,
- a projective-space-model for $n = 15$,
- two new models that do not fall into any of the previously defined classes: one for $n = 14$ and one for $n = 18$.

Counting up to 18

Theorem

The exact number of nonisomorphic finite models of the first group of Hilbert's axiomatic system with n points, for $13 \leq n \leq 18$, is given in the following table:

n	13	14	15	16	17	18
HilbInc(n)	5	8	7	7	7	9

All these models are tetrahedron-models, with exactly four exceptions:

- a projective-plane-model for $n = 14$,
- a projective-space-model for $n = 15$,
- two new models that do not fall into any of the previously defined classes: one for $n = 14$ and one for $n = 18$.

Ongoing research aims to establish whether these new models can exist and constitute representatives of new classes.

Limitations of the SAT-based approach

Limitations of the SAT-based approach

- Current SAT-based method becomes infeasible beyond $n = 18$ due to extreme memory requirements (e.g., > 100GB input for $n = 18$).

Limitations of the SAT-based approach

- Current SAT-based method becomes infeasible beyond $n = 18$ due to extreme memory requirements (e.g., > 100GB input for $n = 18$).
- SAT instance for $n = 19$ estimated to require > 300GB, exceeding current hardware limits.