Regular digraphs from finite groups

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Joint work with A. E. Brouwer, D. Crnković, M. Zubović Žutolija and T. Zrinski

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The outline of the talk:

1. The method
2. Examples of dsrgs
3. Orbit matrices of directed strongly regular graphs

A t-(v, k, λ) **design** is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

1. $|\mathcal{P}| = v$,
2. every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,
3. every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

Every element of $\mathcal{P}$ is incident with exactly r elements of $\mathcal{B}$.

The number of blocks is denoted by b.

If $b = v$ (or equivalently $k = r$) then the design is called **symmetric**.

- A 2-(v, k, λ) design is called a block design.
- If $\mathcal{D}$ is a t-design, then it is also a s-design, for $1 \leq s \leq t-1$.
- An **incidence matrix** of a design $\mathcal{D}$ is a matrix $A = [a_{ij}]$ where $a_{ij} = 1$ if jth point is incident with the ith block and $a_{ij} = 0$ otherwise.

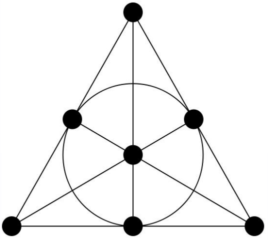


Figure: 2-$(7, 3, 1)$ design

- D. Crnković, V. Mikulić Crnković, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3,3)$, J. Statist. Plann. Inference 144 (2014), 19–40.

Theorem

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}|} \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitively on points and blocks of the design. If $\Delta_2 = \Omega_2$ then the set $\mathcal{B}$ consists of one block, and $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s)$ is a design with parameters $1-(n, n, 1)$.

- D. Crnković, V. Mikulić Crnković, A. Švob, New 3-designs and 2-designs having U(3,3) as an automorphism group, Discrete Math. 340 (2017), 2507–2515.
- D. Crnković, A. Švob, New symmetric 2-(176,50,14) designs, Discrete Math. 344 (2021), 112623, 3 pages.
- D. Crnković, N. Mostarac, A. Švob, Distance-regular graphs and new block designs obtained from the Mathieu groups, Appl. Algebra Engrg. Comm. Comput. 35 (2024), 177–194.
- . . .

If a group G acts transitively on Ω, $\alpha \in \Omega$, and Δ is an orbit of G_α, then $\Delta' = \{\alpha g \mid g \in G,\ \alpha g^{-1} \in \Delta\}$ is also an orbit of G_α. Δ' is called the orbit of G_α paired with Δ. It is obvious that $\Delta'' = \Delta$ and $|\Delta'| = |\Delta|$. If $\Delta' = \Delta$, then Δ is said to be self-paired.

Corollary

If $\Omega_1 = \Omega_2$ and Δ_2 is a union of self-paired and mutually paired orbits of G_α, then the design $\mathcal{D}(G, \alpha, \delta_1, ..., \delta_s)$ is a symmetric self-dual design and the incidence matrix of that design is the adjacency matrix of a $|\Delta_2|-$regular graph.

The construction described in Theorem gives us all simple designs on which the group G acts transitively on the points and blocks, *i.e.* if G acts transitively on the points and blocks of a simple 1-design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem.

Note that the construction from Theorem gives us 1-designs, and **the incidence matrices of some of these 1-designs may be the adjacency matrices of directed strongly regular graphs**.

Since the construction given in Theorem gives all designs having G as an automorphism group acting transitively on points and blocks, it gives us also **all directed strongly regular graphs admitting a transitive action** of the set of vertices.

Clearly, the adjacency matrix of a directed strongly regular graph with parameters (n, k, t, λ, μ) is the incidence matrix of a 1-(n, k, k) design. In that way, the neighbourhoods of a directed strongly regular graph correspond to the blocks of a design, where the neighbourhood of a vertex x is the set of all vertices y such that there is an arc $x \to y$.

dsrgs

A directed strongly regular graph with parameters (v, k, t, λ, μ) is a directed graph on v vertices without loops such that

(i) every vertex has in-degree and out-degree k,

(ii) every vertex x has t out-neighbours that are also in-neighbours of x,

(iii) the number of directed paths of length 2 from a vertex x to another vertex y is λ if there is an edge from x to y, and is μ if there is no edge from x to y.

- A. E. Brouwer, D. Crnković, A. Švob, A construction of directed strongly regular graphs with parameters (63,11,8,1,2), Discrete Math. 347 (2024), 114146, 3 pages.

Theorem

Up to isomorphism, there are exactly two directed strongly regular graphs with parameters $(63, 11, 8, 1, 2)$ on which the linear group $PSL(2, 8)$ acts transitively. These directed strongly regular graphs have $PSL(2, 8) : Z_3$ as the full automorphism group.

The linear group $PSL(2, 8)$ is the simple group of order 504 and up to conjugation it has exactly one subgroup of order 8, which is isomorphic to the elementary abelian group E_8.

By taking $G = PSL(2, 8)$ and $G_\alpha = E_8$, we constructed two non-isomorphic directed strongly regular graphs with parameters $(63, 11, 8, 1, 2)$, both having $PSL(2, 8) : Z_3$ as the full automorphism group.

Description:

Let Γ be a distance-regular graph with intersection array $\{q, q-2, 1; 1, q-2, q\}$, an antipodal $(q-1)$-cover of the complete graph K_{q+1}, with q^2-1 vertices and $\lambda = \mu = 1$.

Remark

The automorphism group G of Γ is the semilinear group $P\Sigma L(2, q)$, acting transitively and edge-transitively on Γ. The orbits of the stabilizer in $L_2(q)$ of the vertex p (which is elementary abelian of order q) are the $q-1$ singletons in the antipodal class of p together with the $q-1$ sets of size q that are the conics with equations $Q(p)Q(x) + c(x, p)^2 = 0$ for $c \in \mathbb{F}_q \setminus \{0\}$ minus the point $C \cap p^{\perp}$.

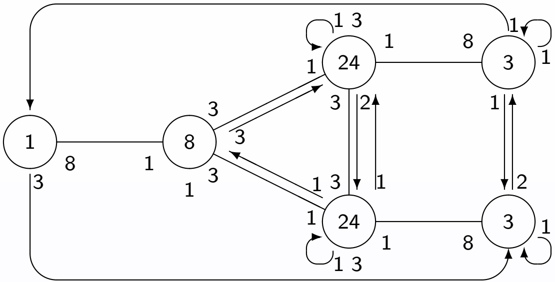
In the special case $q = 8$, with $G = L_2(8):3$, the six antipodes of a point p fall into two orbits of size 3.

Adding to Γ (viewed as a directed graph by viewing an undirected edge xy as a pair of directed edges xy and yx) arrows from p to the three antipodes in one orbit, and adding all images of these arrows under G, produces a directed strongly regular graph with parameters $(63, 11, 8, 1, 2)$.

Let Δ_1 and Δ_2 be the two dsrgs obtained for the two choices of orbit, with adjacency matrices A_1 and A_2, respectively. Then Δ_2 is not isomorphic to Δ_1, but Δ_2 is isomorphic to the dsrg with adjacency matrix $A_1^\top$.

dsrg(63,11,8,1,2)

Δ:



$v = 63$

- A. E. Brouwer, D. Crnković, A. Švob, M. Zubović Žutolija, On some directed strongly regular graphs constructed from linear groups, preprint, 2025.

We classify directed strongly regular graphs admitting a transitive action of the linear groups $L_2(q), q \leq 32$, $L_3(q), q \leq 7$, and $L_4(2)$, for which the rank of the permutation representation is at most 20 (i.e. the number of orbits of the stabilizer of a vertex is at most 20).

We have looked at other small simple groups, of order at most 10^6.

Several of the graphs found have a so far unknown parameter set.

Some examples generalize to an infinite family.

Main steps:

- The method of construction applied for obtaining dsrgs.
- Checking for each of the constructed digraphs if its parameters can be obtained from a construction described by A. E. Brouwer and S. A. Hobart database of dsrgs.
- Checking the isomorphism with the known digraphs.
- Finding the constructions for parametricaly new dsrgs.

(126,40,20,14,12), (126,45,25,16,16), (140,19,7,6,2), (165,60,36,23,21), (240,50,32,4,12), (280,39,15,14,4), (280,123,63,54,54), (288,70,28,20,16), (288,77,35,22,20), (336,89,59,22,24), (420,167,68,67,66), (465,156,56,51,53), (560,117,39,26,24), (560,234,156,100,96), (750,224,84,83,60), (930,313,111,106,105), (1360,159,39,38,16), (3192,794,206,196,198)

Table: New dsrg parameter sets

A geometric construction

Let q be a prime power. Construct a directed graph as follows. Take as vertices the ordered pairs (s, t) of distinct points in a projective plane $PG(2, q)$ (desarguesian or not). Let L_{st} be the line on s and t. Make an arrow $(s, t) \to (u, v)$ when either ($L_{st} = L_{uv}$ and $u = t$) or ($L_{st} \neq L_{uv}$ and $u = L_{st} \cap L_{uv}$).

The graph thus obtained is a directed strongly regular graph with parameters $n = q(q+1)(q^2+q+1)$, $k = q(q^2+q+1)$, $t = q^2+1$, $\lambda = q^2$, $\mu = q^2+1$. The graph is invariant under $P\Gamma L(3, q)$.

A partition $\Pi = \{C_0, C_1, ..., C_{t-1}\}$ of the vertex set of a graph Γ is called *equitable* (or regular) if for each pair of (possibly equal) indices $i, j \in \{0, 1, ..., t-1\}$ there exists a nonnegative integer b_{ij} so that every vertex $x \in C_i$ is adjacent with b_{ij} vertices in C_j, independent of the selection of x.

The $t \times t$ matrix $B = [b_{ij}]$ is called a *quotient matrix* of Γ with respect to the equitable partition Π.

The action of an automorphism group G of a dsrg induces equitable partition of its vertex set. The corresponding quotient matrix is called an orbit matrix of the dsrg with respect to the action of G.

Theorem [D. Crnković, T. Zrinski, AŠ, 202?]

Let Γ be a dsrg(v, k, t, λ, μ) and let G be an automorphism group of Γ. Further, let $O_1, O_2, \ldots, O_b$ be the G-orbits on vertex set of Γ, and let $|O_i| = n_i$, $i = 1, \ldots, b$, be the corresponding orbit lengths. If $R = [r_{ij}]$ is the row orbit matrix of Γ with respect to the action of G, then the following hold

$$0 \leq r_{ij} \leq n_j, \text{ for } 1 \leq i, j \leq b,$$

$$0 \leq r_{ii} \leq n_i - 1, \text{ for } 1 \leq i \leq b,$$

$$\sum_{j=1}^{b} r_{ij} = k, \text{ for } 1 \leq i \leq b,$$

$$\sum_{s=1}^{b} r_{is} r_{sj} = \delta_{ij}(t - \mu) + r_{ij}\lambda + (n_j - r_{ij})\mu, \text{ for } 1 \leq i, j \leq b,$$

where δ_{ij} is the Kronecker delta.

Definition

Let n_i, $i = 1, \ldots, b$, be positive integers such that $\sum_{i=1}^{b} n_i = v$. A $(b \times b)$-matrix $R = [r_{ij}]$, where r_{ij}, $1 \leq i, j \leq b$, are non-negative integers satisfying conditions given by Theorem, is **a row orbit matrix associated with a dsrg**(v, k, t, λ, μ) **and the distribution of orbit lengths** $(n_1, n_2, \ldots n_b)$.

Remark

Orbit matrices from Definition may or may not correspond to a directed strongly regular graph with parameters (v, k, t, λ, μ). Those matrices can be used for constructing directed strongly regular graphs with a presumed automorphism group in a similar way as orbit matrices of 2-designs are used for a construction of 2-designs and orbit matrices of strongly regular graphs are used for a construction of strongly regular graphs

Thank you very much for your attention!