Recursive Polynomials and Combinatorics

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Polynomial Sequences

$$F_n(x) = xF_{n-1}(x) + F_{n-a}(x),\ a \in Z^+,$$
$$\mathrm{F}_i(x) = x^i, 0 \leq i \leq a-1$$

$$C_n(x) = xC_{n-1}(x) + C_{n-a}(x),\ a \in Z^+,$$
$$\mathrm{C}_0 = a,\ C_j(x) = x^j,\ 0 \leq i \leq a-1$$

Examples

$$a = 2 : F_0 = 1, F_1 = x, F_n = xF_{n-1} + F_{n-2}$$

The first few polynomials are

- $F_0 = 1$
- $F_1 = x$
- $F_2 = x^2 + 1$
- $F_3 = x^3 + 2x$
- $F_4 = x^4 + 3x^2 + 1$

Examples

$$a = 3 : F_0 = 1, F_1 = x, F_2 = x^2, F_n = xF_{n-1} + F_{n-3}$$

The first few polynomials are

- $F_0 = 1$
- $F_1 = x$
- $F_2 = x^2$
- $F_3 = x^3 + 1$
- $F_4 = x^4 + x + 1$
- $F_5 = x^5 + 2x^2 + x$
- $F_6 = x^6 + 3x^3 + x^2 + 1$

Combinatorial Interpretations

- The $F_n's$ with $a = 2$ and $x = 1$ yield the Fibonacci numbers.
- The $F_n's$ with $a = 3$ and $x = 1$ yield the Narayana's numbers.
- The polynomials F_n count the number of ways to tile a $1 \times n$ board using 1×1 tiles ("monominos") that come in x colors and $1 \times a$ tiles ("a-ominos")
- The polynomials C_n count the number of ways to tile a bracelet of length n using monominos in x colors and a-ominos

Combinatorial Interpretation of F_n

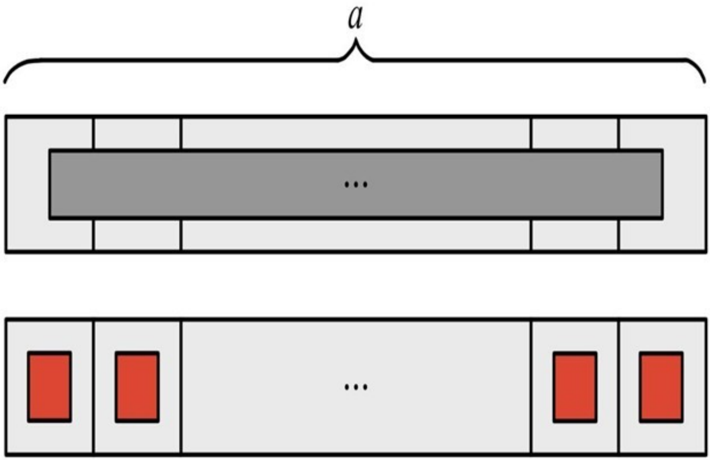
Relate F_n to tiling of a board.

- Start with the polynomials F_1 through F_a
  - F_1: Tile a 1×1 board with monominoes of x colors and a-ominoes
  - Note that an a-omino cannot fit onto this board and
  - Only one monomino can fit onto this board.
  - Monominoes come in x colors, and there are x ways to tile this board.
  - Hence $F_1 = x$

- F_2: Tile a 1×2 board with monominoes of x colors and a-ominoes
- By the same reasoning, we get x^2 to tile this board.
- Hence $F_2 = x^2$
- Similar reasoning can be applied to all initial conditions to relate F_{a-1} to tiling of $1 \times (a-1)$ board.
- Next we relate F_a to tiling a $1 \times a$ board.
- We consider two distinct cases to tiling it.
  - Case 1: Place a monominoes
  - Case 2: Put one a-omino.

- Case 1: There are x choices of colors for the monominoes,
  - Hence we have x^a ways of tiling using monominoes
- Case 2: One way of tiling using an a-omino
- A total of $x^a + 1$ ways to tile this $1 \times a$ board

The Tilings of the F_a board



The General Case

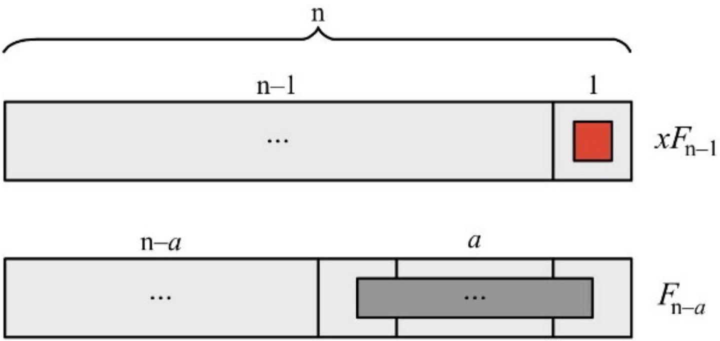
- Assume: For $k < n$, F_k can be represented as the number of ways to tile a $1 \times k$ board with monominoes in x colors and a-ominoes.

- Consider the $1 \times n$ board which can be covered by two types of tiles: by monomino or by the right end of an a-omino

- Cover by a monomino: Treat the rest of the board as a separate $1 \times (n-1)$ board, which is equivalent to F_{n-1} by the inductive hypothesis.
- Since there are x monomino colors, we know that this case has $x \cdot F_{n-1}$ different tilings.
- Place an a-omino at the right end of the board.
- Treat the rest of the board as a separate $1 \times (n-a)$ board
- This is equivalent to F_{n-a} by the inductive hypothesis
- Has F_{n-a} different tilings.
- Combining: the number of ways to tile a $1 \times n$ board is

$$xF_{n-1} + F_{n-a} = F_n$$

Combinatorial Interpretation of F_n



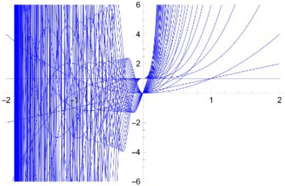
Closed Forms

$$F_n = \sum_{k=0}^{\lfloor \frac{n}{a} \rfloor} \binom{n-(a-1)k}{k} x^{n-ak}$$

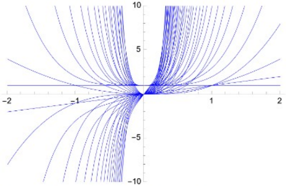
$$C_n = \sum_{k=0}^{\lfloor \frac{n}{a} \rfloor} \left(a\binom{n-1-(a-1)k}{k-1} + \binom{n-1-(a-1)k}{k} \right) x^{n-ak}$$

$$C_n = F_n + (a-1)F_{n-a}$$

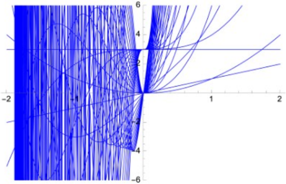
Analytic Results: The Zeros of F_n and C_n



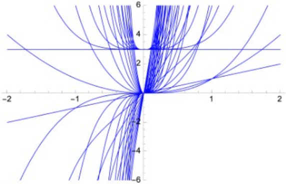
First 40 terms of $F_n = xF_{n-1} + F_{n-3}$



First 40 terms of $F_n = xF_{n-1} + F_{n-4}$



First 40 terms of $C_n = xC_{n-1} + C_{n-3}$



First 40 terms of $C_n = xC_{n-1} + C_{n-4}$

Proposition 1. *Polynomials in F_n and C_n with even a in their recurrence will have no real roots other than $x = 0$.*

Analytic Results: Minimum Real Roots

- a is odd: F_n will always have a nonzero real root
- The sequence of minimum real roots is a bounded monotonically decreasing sequence
- the sequence of minimum real roots converge to

$$L_a = \frac{-a}{\sqrt[a]{(a-1)^{(a-1)}}}$$

THANK YOU

References

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