Packings with Large Block Size

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Packings

Definition

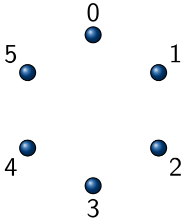
A packing design $\mathrm{PD}_\lambda(v, k, t)$, is a pair $(V, \mathcal{B})$ where:

- V is a set of v points, and
- $\mathcal{B}$ is a collection of k-subsets of V, called blocks, such that
- Each t-tuple of points occurs in at most λ blocks.

If $\lambda = 1$, or $t = 2$ we drop it from the notation.

Example (A $\mathrm{PD}(6, 3)$)

$\{0, 1, 2\}$
$\{0, 3, 4\}$
$\{1, 3, 5\}$
$\{2, 4, 5\}$



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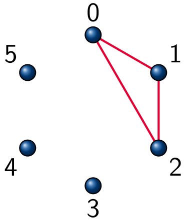

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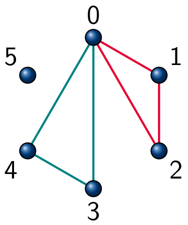

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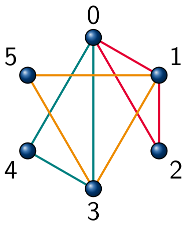

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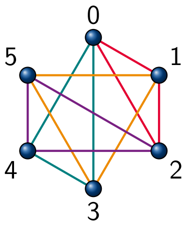

Packing Number

Definition

The size of a packing is the number of blocks.

We write $\mathrm{PD}_\lambda(n; v, k)$ to denote that the packing has size n.

Definition

The packing number $\mathrm{PDN}_\lambda(v, k, t)$ is the maximum size of a $\mathrm{PD}_\lambda(v, k, t)$.

Example (A $\mathrm{PD}(4; 6, 3)$)

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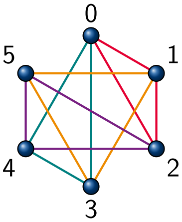

$\mathrm{PDN}(6, 3) = 4$

Bounds on the packing number: First Johnson Bound

Theorem (Johnson (1962), Schönheim (1966))

$$\begin{aligned}\mathrm{PD}_\lambda(v,k,t) &\leq U_\lambda(v,k,t) \\ &= \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \left\lfloor \cdots \left\lfloor \frac{v-t+2}{k-t+2} \left\lfloor \frac{\lambda(v-t+1)}{k-t+1} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor .\end{aligned}$$

Theorem (Hanani (1975))

If $t = 2$ *and*

$$\lambda(v-1) \equiv 0 \pmod{k-1} \text{ and } \lambda v(v-1) \equiv -1 \pmod{k}, \tag{1}$$

then

$$\mathrm{PD}_\lambda(v,k) \leq U_\lambda(v,k) - 1.$$

Let $B_\lambda(v,k) = \begin{cases} U_\lambda(v,k) - 1, & \text{if } t = 2, \text{ and } v,\ k \text{ and } \lambda \text{ satisfy (1)} \\ U_\lambda(v,k), & \text{otherwise} \end{cases}$

Bounds on the Packing Number: Second Johnson Bound

Theorem (Johnson (1962))

Let $d = \mathrm{PDN}(v, k, t)$ *and let* q *and* r *be the integers satisfying* $kd = qv + r$ *with* $0 \leq r < v$*. Then*

$$d(d-1)(t-1) \geq q(q-1)v + 2qr.$$

This result implies the following:

Theorem (Second Johnson bound – weaker form)

If $v < (t-1)k^2$*, then*

$$\mathrm{PD}_\lambda(v, k, t) \leq \left\lfloor \frac{v(k+1-t)}{k^2 - v(t-1)} \right\rfloor.$$

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Known results on packings

Most known results concern small block size with $t = 2$.

Theorem (Hanani, 1975)

For all $v \geq 3$ and $\lambda \geq 1$, $\mathrm{PD}_\lambda(v, 3) = B_\lambda(v, 3)$.

Theorem (Brouwer, 1979; Billington, Stanton and Stinson, 1984; Hartman, 1986; Assaf, 1991)

For $v \geq 20$,

$$\mathrm{PD}_\lambda(v,4) = \begin{cases} B_\lambda(v,4) - 1, & v \equiv 7, 10 \pmod{12} \text{ and } \lambda = 1, \\ B_\lambda(v,4), & \text{otherwise.} \end{cases}$$

Many values of $\mathrm{PD}_\lambda(v, 5)$ are known, and some for $\mathrm{PD}_\lambda(v, 6)$.

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Directed packings

Definition

A directed packing design $\mathrm{DPD}_\lambda(v, k, t)$, is a pair $(V, \mathcal{B})$ where:

- V is a set of v points, and
- $\mathcal{B}$ is a collection of ordered k-tuples of V, called blocks, such that
- Each ordered pair of points occurs in at most λ blocks.

If $\lambda = 1$, or $t = 2$, we drop it from the notation.

A $\mathrm{DPD}_\lambda(n; v, k, t)$ has n blocks.

The directed packing number is denoted $\mathrm{DPDN}_\lambda(v, k)$.

Example (A $\mathrm{DPD}(4; 6, 4)$)

$$\begin{gathered}(0, 1, 2, 3)\\(4, 3, 5, 0)\\(5, 3, 2, 4)\\(2, 1, 0, 5)\end{gathered}$$

Known results on directed packings

Again, most known results concern small block size with $t = 2$.

Lemma

$\mathrm{DPDN}_\lambda(v,k) \le \mathrm{PDN}_{2\lambda}(v,k) \le B_{2\lambda}(v,k)$

Theorem (Skillicorn, 1982; Shalaby, Yin, 1995; Assaf, Shalaby, Mahmoodi, Yin, 1996; Assaf, Shalaby, Yin, 1998; Assaf, Shalaby, Yin, 2001; Abel, Assaf, Bluskov, Greig, Shalaby, 2010)

For $k \in \{3,4,5\}$, $\mathrm{DPDN}_\lambda(v,k) = B_{2\lambda}(v,k)$, *except that:*

- $\mathrm{DPDN}(v,k) = B_2(v,k) - 1$ *if* $(v,k) \in \{(9,4),(13,5)\}$
- $\mathrm{DPDN}(15,5) = B_2(15,5) - 2$,

and except possibly when $k = 5$ *and* $(v,\lambda) \in \{(19,1),(27,1),(43,3)\}$.

Packings and codes

- A $\mathrm{PD}(v, k, t)$ is equivalent to a binary constant-weight code with:
  - length v
  - minimum distance at least $2(k - t + 1)$
  - weight k
- The blocks of a $\mathrm{DPD}(v, k)$ form a $(k - t)$-deletion correcting code.

Frequency

Definition

Let $(V, \mathcal{B})$ be a $\mathrm{PD}_\lambda(v, k)$ or $\mathrm{DPD}_\lambda(v, k)$.

The frequency of $x \in V$, denoted $r(x)$, is the number of blocks containing x.

Let N_i be the number of points of frequency i.

Example

In the following $\mathrm{PD}_2(4; 12, 7, 2)$:

$$\begin{array}{ll} \{0,1,2,3,4,5,6\} & \{0,1,3,4,7,8,9\} \\ \{0,2,5,6,9,10,11\} & \{1,2,7,8,9,10,11\} \end{array}$$

- $r(0) = 3$;
- $N_3 = 4$.

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- $r(0) = 3$;
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Counting

Lemma

In a $\mathrm{PD}_\lambda(n; v, k, t)$*:*

- $\displaystyle\sum_{i=0}^{n} N_i = v;$
- $\displaystyle\sum_{i=0}^{n} iN_i = nk;$
- $\displaystyle\sum_{i=1}^{n} \binom{i}{\lambda+1} N_i \leq (t-1)\binom{n}{\lambda+1}.$

Theorem (Johnson (1962))

Let $d = \mathrm{PDN}(v, k, t)$ *and let* q *and* r *be the integers satisfying* $dk = qv + r$ *with* $0 \leq r < v$. *Then*

$$(t-1)d(d-1) \geq vq(q-1) + 2rq.$$

Packing numbers with large block size: Upper Bound

Theorem (General Upper Bound)

Let $d = \mathrm{PDN}_\lambda(v, k, t)$ *and let* q *and* r *be the integers satisfying* $dk = qv + r$ *with* $0 \le r < v$. *Then*

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Note that (after some calculation) this implies the following.

Corollary

Let $v \geq k \geq t \geq 2$, $\lambda \geq 1$ *and* $n \geq 1$.

If $\lambda v < (n+1)k - (t-1)\binom{n+1}{\lambda+1}$, *then* $\mathrm{PDN}_\lambda(v, k) \leq n$.

Packing numbers with large block size: Lower bound

Theorem

Let $v \geq k \geq t \geq 2$, $\lambda \geq 1$ and $n \geq 1$. If

$$nk - (t-1)\binom{n}{\lambda+1} \leq \lambda v < (n+1)k - (t-1)\binom{n+1}{\lambda+1},$$

then $\mathrm{PDN}_\lambda(v, k, t) \geq n$.

- The given conditions imply $v > (t-1)\binom{n}{\lambda+1}$.
- Partition V into $V = U \cup W$, where $|U| = (t-1)\binom{n}{\lambda+1}$ and $|W| = v - (t-1)\binom{n}{\lambda+1}$.
- We construct a $\mathrm{PD}_\lambda(n; v-(t-1)\binom{n}{\lambda+1}, k-(t-1)\binom{n-1}{\lambda}, t)$, $(W, \{W_1, \ldots, W_n\})$ with $r(x) \leq \lambda + 1$ for each $x \in W$.
- Index the points of U by the $(\lambda+1)$-subsets of $\{1, \ldots, n\}$ $\times\{1, 2, \ldots, t-1\}$.
- Add to W_i the points indexed by the $(\lambda+1)$-subsets containing the element i.

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Example

Example (A $\mathrm{PD}(4; 14, 5)$)

$$|U| = (t-1)\binom{n}{\lambda+1} = \binom{4}{2} = 6$$
$$|W| = v - (t-1)\binom{n}{\lambda+1} = 14 - \binom{4}{2} = 8$$

A $\mathrm{PD}(4; 8, 2)$:

$$w_{11}, w_{12}$$
$$w_{21}, w_{22}$$
$$w_{31}, w_{32}$$
$$w_{41}, w_{42}$$

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A $\mathrm{PD}(4; 14, 5)$:

$$w_{11}, w_{12}, u_{12}, u_{13}, u_{14}$$
$$w_{21}, w_{22}, u_{12}, u_{23}, u_{24}$$
$$w_{31}, w_{32}, u_{13}, u_{23}, u_{34}$$
$$w_{41}, w_{42}, u_{14}, u_{24}, u_{34}$$

Putting these together gives:

Theorem

If

$$nk - \binom{n}{\lambda+1} \leq \lambda v < (n+1)k - \binom{n+1}{\lambda+1},$$

then $\mathrm{PDN}_\lambda(v,k) = n$.

What does k large mean?

Our result determines $\mathrm{PDN}_\lambda(v,k,t)$ when, for some positive integer n,

$$k > \frac{\lambda v}{n+1} + \frac{t-1}{\lambda+1}\binom{n}{\lambda}.$$

For fixed λ and t, and large v, this gives the value of $\mathrm{PDN}_\lambda(v,k,t)$ when $k > f_\lambda(v)$, where

$$f_\lambda(v) \sim c_{t,\lambda} v^{\lambda/(\lambda+1)},$$

with $c_{t,\lambda} = (\lambda+1)\left(\frac{t-1}{(\lambda+1)!}\right)^{1/(\lambda+1)}$.

In particular, $f_{2,1} \sim \sqrt{2v}$.

Asymptotic behaviour

Fix t and λ.

Suppose $k \sim cv^{\alpha}$ as v becomes large, where $c > 0$ and $0 < \alpha \leq 1$ are constant.

- If $\alpha = 1$, then $\mathrm{PDN}_{\lambda}(v, k, t) \sim \left\lfloor \frac{\lambda}{c} \right\rfloor$.
- If $\frac{\lambda}{\lambda+1} < \alpha < 1$, then $\mathrm{PDN}_{\lambda}(v, k, t) \sim \frac{\lambda}{c} v^{1-\alpha}$.
- If $\alpha = \frac{\lambda}{\lambda+1}$, then $\mathrm{PDN}_{\lambda}(v, k, t) \sim dv^{1-\alpha}$,
  where d is a constant defined in terms of t and λ.

Directing ($t = 2$)

Lemma

Let $(V, \mathcal{B})$ be a $\mathrm{PD}_2(n; v, k)$ with $r(x) \le 3$ for each $x \in V$. There is a $\mathrm{DPD}(n; v, k)$ whose blocks are permutations of the blocks in $\mathcal{B}$.

Proof.

By Induction on v. □

Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

Start with a $\mathrm{PD}_2(4; 12, 7)$

$$\{0, 1, 2, 3, 4, 5, 6\}$$
$$\{0, 1, 3, 4, 7, 8, 9\}$$
$$\{0, 2, 5, 6, 9, 10, 11\}$$
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Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

Remove a point, say 0.

$$\begin{gathered}\{0,1,2,3,4,5,6\}\\ \{0,1,3,4,7,8,9\}\\ \{0,2,5,6,9,10,11\}\\ \{1,2,7,8,9,10,11\}\end{gathered}$$

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Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

Remove a point, say 0.

$$\{1, 2, 3, 4, 5, 6\}$$
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Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

By induction, order the remaining blocks.

$$\{1, 2, 3, 4, 5, 6\}$$
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Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

By induction, order the remaining blocks.

$$\begin{array}{l} (1,2,3,4,5,6) \\ (4,3,7,8,9,1) \\ (6,5,10,11,9,2) \\ (2,1,9,11,10,8,7) \end{array}$$

Directing ($t = 2$)

Lemma

Let $(V, \mathcal{B})$ be a $\mathrm{PD}_2(n; v, k)$ with $r(x) \leq 3$ for each $x \in V$. There is a $\mathrm{DPD}(n; v, k)$ whose blocks are permutations of the blocks in $\mathcal{B}$.

Proof.

By Induction on v. □

Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

In each block from which it was removed, find an appropriate position to re-insert 0.
(Showing that this is always possible is the hard part of the proof.)

$$
\begin{array}{l}
(1, 2, 3, 4, 5, 6) \\
(4, 3, 7, 8, 9, 1) \\
(6, 5, 10, 11, 9, 2) \\
(2, 1, 9, 11, 10, 8, 7)
\end{array}
$$

Directing ($t = 2$)

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Proof.

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Example (Forming a $\mathrm{DPD}(4; 12, 7)$)

In each block from which it was removed, find an appropriate position to re-insert 0.
(Showing that this is always possible is the hard part of the proof.)

$$\begin{array}{l}(1, 0, 2, 3, 4, 5, 6)\\(4, 3, 7, 8, 0, 9, 1)\\(6, 5, 10, 11, 9, 2, 0)\\(2, 1, 9, 11, 10, 8, 7)\end{array}$$

Directed packings

The $\mathrm{PD}_2(v, k)$ that we previously constructed had all $r(x) \leq 3$, so we get:

Theorem

If $nk - \binom{n}{\lambda} \leq 2v < (n+1)k - \binom{n+1}{\lambda+1}$, *then*

$$\mathrm{DPDN}(v, k) = n = \mathrm{PD}_2(v, k).$$

Remarks on bounds

We have generalised the second Johnson bound to arbitrary λ and shown that it is tight when k is large.

- The First Johnson Bound $U_\lambda(v, k)$ performs poorly when k is large.
- When $\lambda = 1$ and $nk - \binom{n}{2} \leq v < (n+1)k - \binom{n+1}{2}$, the Second Johnson bound gives the exact packing number.

Open Problems

- Is it true that for fixed k, $\mathrm{PDN}_\lambda(v,k) \geq U_\lambda(v,k) - 1$ when v is sufficiently large?

- Prove or disprove the following conjecture:

Conjecture (Burgess, Danziger, Horsley, Javed, 2025+)

For all integers $v \geq k \geq 3$, $\mathrm{DPDN}(v,k) = \mathrm{PDN}_2(v,k)$.