On the cost of making a metric basis fault tolerant

Jelena Sedlar

University of Split, Croatia

(joint work with Martin Knor and Riste Škrekovski)

NORCOM 2025,

Reykjavik, Iceland.

16-18 June 2025.

Introduction

A **metric basis** of a graph G is a smallest set $S \subseteq V(G)$ such that for each pair of vertices $x, y \in V(G)$ there exists a vertex $s \in S$ such that $d(x, s) \neq d(y, s)$.

Introduction

A **metric basis** of a graph G is a smallest set $S \subseteq V(G)$ such that for each pair of vertices $x, y \in V(G)$ there exists a vertex $s \in S$ such that $d(x, s) \neq d(y, s)$.

The **metric dimension** $\dim(G)$ of a graph G is the cardinality of a metric basis in G.

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

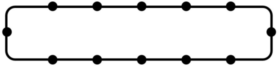
Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

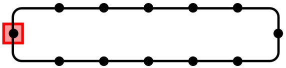
Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

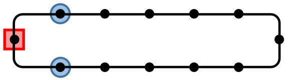
Introduction

An exaple of a **metric basis** for:

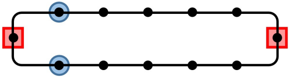
	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

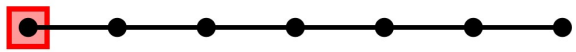
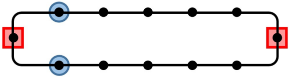
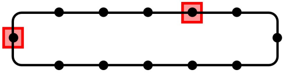
	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **metric basis** for:

	NOT metric basis	IS metric basis
path P_n		
cycle C_n		

A **motivation** for metric basis and metric dimension:

- graph G is a network;
- vertices $s \in S$ are sensor positions;
- vertices $x \in V(G)$ are posible positions of robots moving through the network G.

Introduction

A **motivation** for metric basis and metric dimension:

- graph G is a network;
- vertices $s \in S$ are sensor positions;
- vertices $x \in V(G)$ are posible positions of robots moving through the network G.

Problem: a sensor in a physical world <u>**can fail**</u>!

Introduction

A **motivation** for metric basis and metric dimension:

- graph G is a network;
- vertices $s \in S$ are sensor positions;
- vertices $x \in V(G)$ are posible positions of robots moving through the network G.

Problem: a sensor in a physical world **can fail**!

Question: how to make a metric basis **fault-tolerant**?

Introduction

A **fault-tolerant metric basis** of G is a smallest set $S \subseteq V(G)$ such that $S \backslash \{s\}$ is a metric basis of G for any $s \in S$.

Introduction

A **fault-tolerant metric basis** of G is a smallest set $S \subseteq V(G)$ such that $S \backslash \{s\}$ is a metric basis of G for any $s \in S$.

A **fault-tolerant metric dimension** $\mathrm{ftdim}(G)$ of G is the cardinality of a fault-tolerant metric basis of G.

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

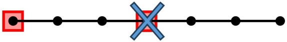
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

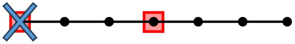
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

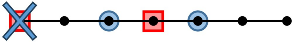
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

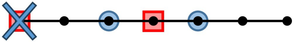
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

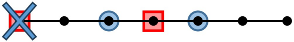
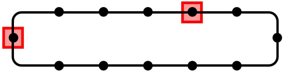
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

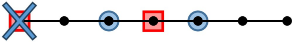
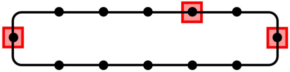
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

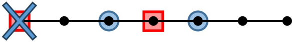
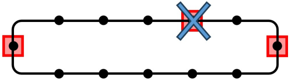
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

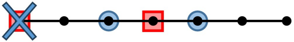
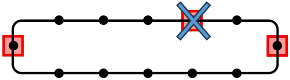
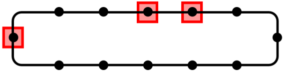
Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **fault-tolerant metric basis** for:

	NOT f-t metric basis	IS f-t metric basis
path P_n		
cycle C_n		

Introduction

A **k-metric basis** of G is a smallest set $S \subseteq V(G)$ such that $S \backslash S_0$ is a metric basis of G for any $S_0 \subseteq S$ with $|S_0| \leq k-1$.

Introduction

A **k-metric basis** of G is a smallest set $S \subseteq V(G)$ such that $S \backslash S_0$ is a metric basis of G for any $S_0 \subseteq S$ with $|S_0| \leq k - 1$.

A **k-metric dimension** $\dim_k(G)$ of G is the cardinality of a k-metric basis of G.

Introduction

A **k-metric basis** of G is a smallest set $S \subseteq V(G)$ such that $S \backslash S_0$ is a metric basis of G for any $S_0 \subseteq S$ with $|S_0| \leq k-1$.

A **k-metric dimension** $\dim_k(G)$ of G is the cardinality of a k-metric basis of G.

Remark: notice that $\mathrm{ftdim}(G) = \dim_2(G)$.

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

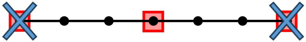
Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

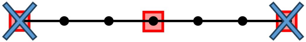
Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

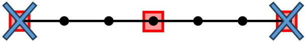
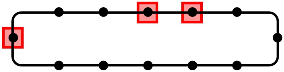
Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

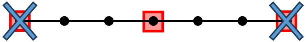
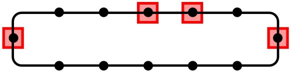
Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

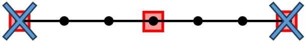
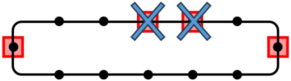
Introduction

An exaple of a **k-metric basis** for:

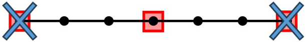
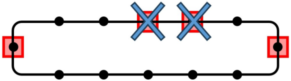
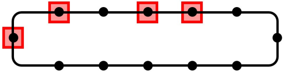
	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

Introduction

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

An exaple of a **k-metric basis** for:

	NOT 3-metric basis	IS 3-metric basis
path P_n		
cycle C_n		

Introduction

Question: how much does the fault-tolerance of a metric basis costs?

Introduction

Question: how much does the fault-tolerance of a metric basis costs?

In our examples (path and cycle) we see that $\mathrm{ftdim}(G)$ grows **linearly** in terms of $\dim(G)$.

Fault-tolerant metric dimension

Fault-tolerant metric dimension

On **some graph classes** we have:

Fault-tolerant metric dimension

On **some graph classes** we have:

Proposition. If T is a tree, then

$$\dim(T) + 1 \leq \mathrm{ftdim}(T) \leq 2\dim(T).$$

Moreover, let $n \geq 4$ and $1 \leq d \leq \lfloor n/3 \rfloor$. Then there exists a tree T on n vertices with $\mathrm{ftdim}(T) - \dim(T) = d$.

Fault-tolerant metric dimension

On **some graph classes** we have:

Proposition. If T is a tree, then

$$\dim(T) + 1 \leq \mathrm{ftdim}(T) \leq 2\dim(T).$$

Moreover, let $n \geq 4$ and $1 \leq d \leq \lfloor n/3 \rfloor$. Then there exists a tree T on n vertices with $\mathrm{ftdim}(T) - \dim(T) = d$.

Proposition. If G is a complete multipartite graph, then

$$\dim(G) + 1 \leq \mathrm{ftdim}(G) \leq 2\dim(G).$$

Moreover, let $n \geq 2$ and let $1 \leq d \leq \lfloor n/2 \rfloor$. Then there exists a complete multipartite graph G on n vertices with $\mathrm{ftdim}(G) - \dim(G) = d$.

But, the **best known upper bound** in literature is:

Fault-tolerant metric dimension

But, the **best known upper bound** in literature is:

Theorem [1]. Fault-tolerant metric dimension is bounded by a function of the metric dimension (independent of the graph). In particular, for every graph G we have

$$\mathrm{ftdim}(G) \leq \dim(G)(1 + 2 \cdot 5^{\dim(G)-1}).$$

[1] C. Hernando, M. Mora, P. J. Slater, D. R. Woods, Fault-tolrant metric dimension of graphs, in: Proc. Internat. Conf. Convexity in Discrete Structures, in: Ramanujan Math. Society Lecture Notes, vol. 5, 2008.

Fault-tolerant metric dimension

We construct the graph M_t on $n = t + 2^t$ vertices:

Fault-tolerant metric dimension

We construct the graph M_t on $n = t + 2^t$ vertices:

- $V_1 = \{u_0, \ldots, u_{2^t-1}\}$ is a set of 2^t vertices where u_j is identified with the binary code for j;
- $V_2 = \{v_1, \ldots, v_t\}$ is a set of t vertices;
- $V(M_t) = V_1 \cup V_2$;

Fault-tolerant metric dimension

We construct the graph M_t on $n = t + 2^t$ vertices:

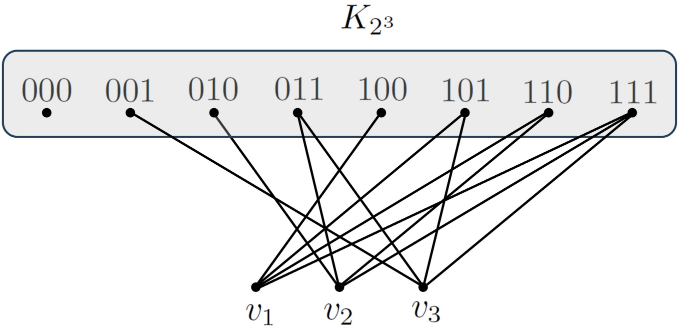
- $V_1 = \{u_0, \ldots, u_{2^t-1}\}$ is a set of 2^t vertices where u_j is identified with the binary code for j;
- $V_2 = \{v_1, \ldots, v_t\}$ is a set of t vertices;
- $V(M_t) = V_1 \cup V_2$;
- vertices of V_1 induce a complete subgraph K_{2^t} in M_t;

Fault-tolerant metric dimension

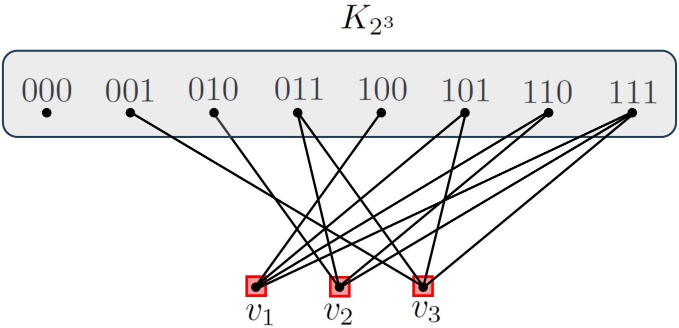
We construct the graph M_t on $n = t + 2^t$ vertices:

- $V_1 = \{u_0, \ldots, u_{2^t-1}\}$ is a set of 2^t vertices where u_j is identified with the binary code for j;
- $V_2 = \{v_1, \ldots, v_t\}$ is a set of t vertices;
- $V(M_t) = V_1 \cup V_2$;
- vertices of V_1 induce a complete subgraph K_{2^t} in M_t;
- vertex v_i from V_2 is adjacent in M_t to all vertices of V_1 which have 1 on the i-th coordinate.

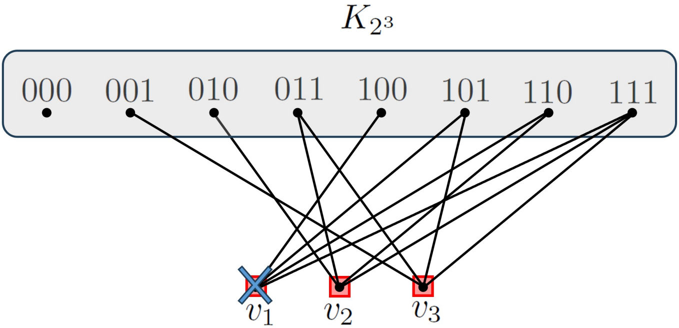
Fault-tolerant metric dimension



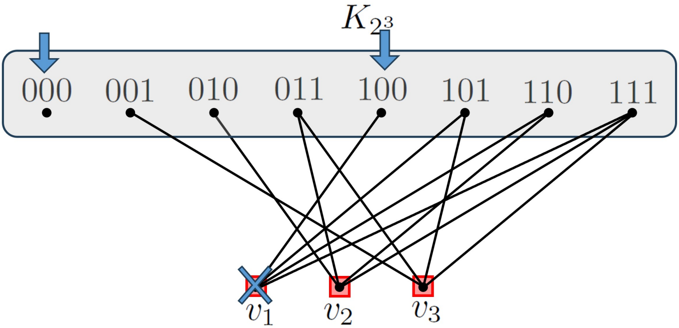
Fault-tolerant metric dimension



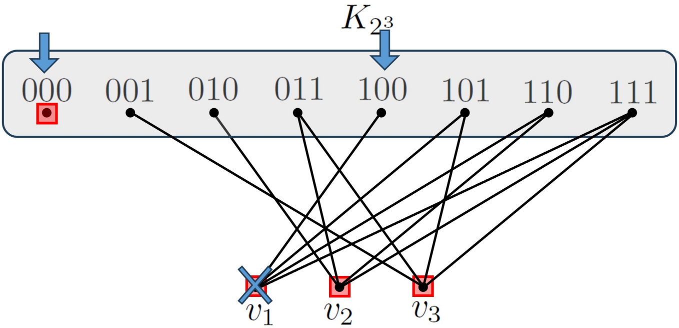
Fault-tolerant metric dimension



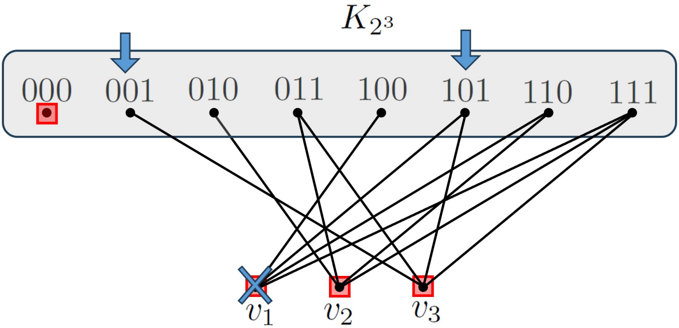
Fault-tolerant metric dimension



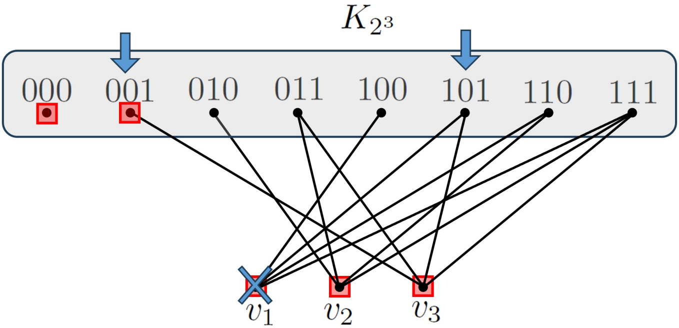
Fault-tolerant metric dimension



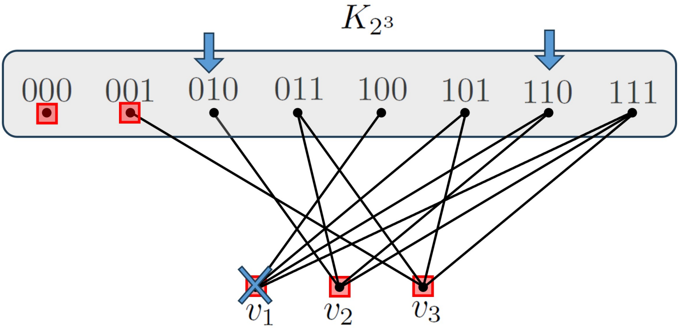
Fault-tolerant metric dimension



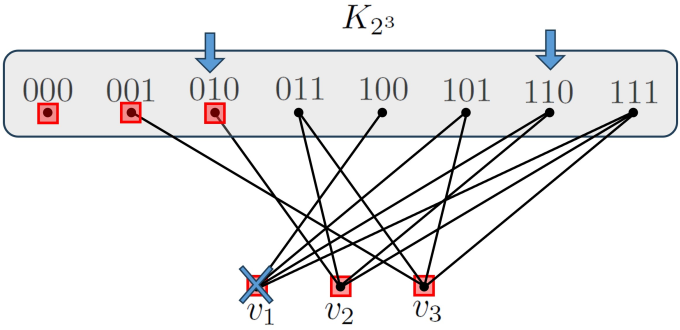
Fault-tolerant metric dimension



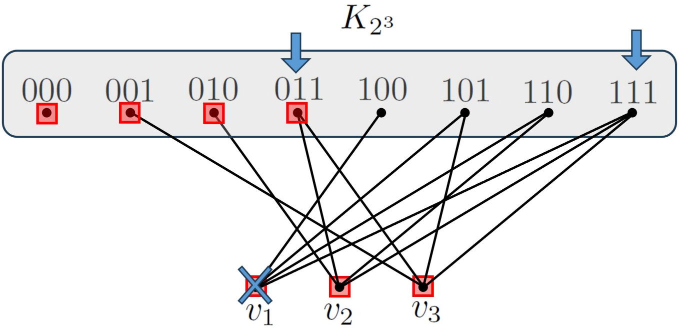
Fault-tolerant metric dimension



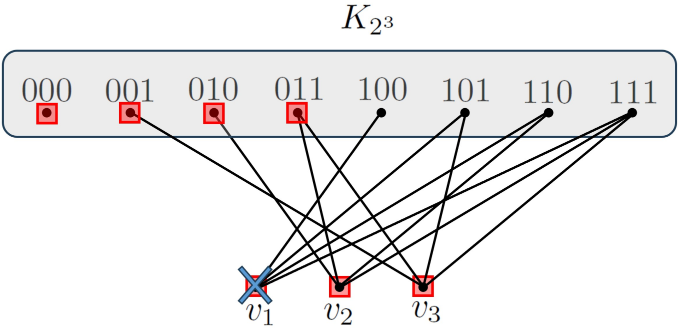
Fault-tolerant metric dimension



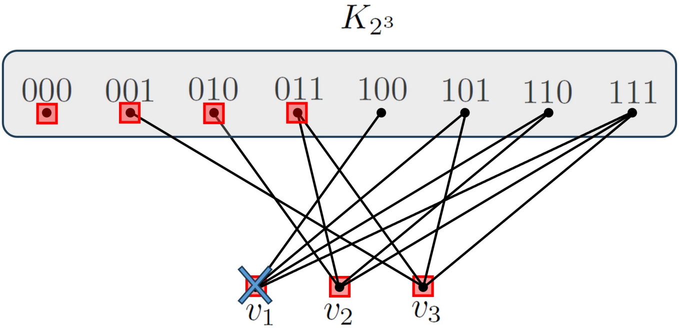
Fault-tolerant metric dimension



Fault-tolerant metric dimension



Fault-tolerant metric dimension



Theorem. For every $t \geq 1$, it holds that $\dim(M_t) = t$ and $\mathrm{ftdim}(M_t) = t + 2^{t-1}$.

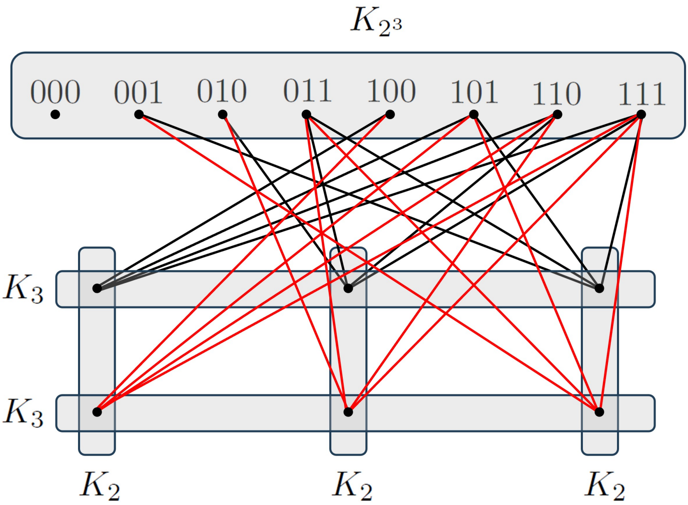
k-metric dimension

k-metric dimension

Theorem. Let $k \geq 2$ be an integer and G a graph with $\kappa(G) \geq k+1$. The $(k+1)$-metric dimension of G is bounded by a function of the k-metric dimension of G. In particular,

$$\dim_{k+1}(G) \leq \dim_k(G)(1 + 2 \cdot 5^{\dim_k(G)-1}).$$

k-metric dimension



k-metric dimension

Theorem. For every $k \geq 2$ and every big enough t, it holds that $\dim_k(M_{t,k}) = kt$ and $\dim_{k+1}(M_{t,k}) \geq 2^{t-1}$.

Concluding remarks

Concluding remarks

For a **maximal** graph G with respect to fault-tolerant metric dimension we know:

$$\dim(G) + 2^{\dim(G)-1} \leq \operatorname{ftdim}(G) \leq \dim(G)(1 + 2 \cdot 5^{\dim(G)-1})$$

Concluding remarks

For a **maximal** graph G with respect to fault-tolerant metric dimension we know:

$$\dim(G) + 2^{\dim(G)-1} \leq \operatorname{ftdim}(G) \leq \dim(G)(1 + 2 \cdot 5^{\dim(G)-1})$$

Problem: narow the gap between maximal $\operatorname{ftdim}(G)$ and upper bound.

Concluding remarks

For a **maximal** graph G with respect to k-metric dimension we know:

$$2^{\frac{1}{k}\dim_k(G)-1} \leq \dim_{k+1}(G) \leq \dim_k(G)(1 + 2 \cdot 5^{\dim_k(G)-1})$$

Concluding remarks

For a **maximal** graph G with respect to k-metric dimension we know:

$$2^{\frac{1}{k}\dim_k(G)-1} \leq \dim_{k+1}(G) \leq \dim_k(G)(1 + 2 \cdot 5^{\dim_k(G)-1})$$

Problem: narow the gap between maximal $\dim_k(G)$ and upper bound.

Concluding remarks

I gratefuly achnowledge the support of HRZZ project HRZZ-IP-2024-05-2130.

Concluding remarks

I gratefuly achnowledge the support of HRZZ project HRZZ-IP-2024-05-2130.

Thank you for the attention.