Exercise 1

FY8503 Advanced theoretical physics
Transport modelling with Stochastic differential equations

September 11, 2023

Problem 1

There was a question at the lecture about what happens if you construct a stochastic process with non-Gaussian increments. We will investigate this question in this exercise, by generating realisations of a random process, X_t, by adding increments from a non-Gaussian distribution with mean $\mu = 0$ and variance $\sigma^2 < \infty$. Try the different tasks below with different distributions, for example:

- Maxwell distribution
- Weibull distribution
- Laplace distribution
- Some discrete distribution (for example $X \in \{-a, 0, a\}$ with suitably selected probabilities)
- Some non-symmetric discrete distribution (for example $X \in \{-a, 0, 2a\}$ with suitably selected probabilities)

We will use the notation that X_n is the numerical value of the process at time $t_n = n\Delta t$, and that the smallest increments (which we use to construct the numerical realisation of the process) are called $\Delta X_n = X_n - X_{n-1}$.

Task a

Generate a single numerical approximation, X_t, of a stochastic process by starting at zero. Construct the process from $t = 0$ to $t = T$, by adding $N_t = T/\Delta t$ increments with $\mu = 0$ and variance $\sigma^2 = \Delta t$. Plot the resulting process. Keep T constant, but change Δt, and try observe how the process changes.

Task b

Investigate the distribution of the increments. Clearly, the increments of two neighbouring points, *e.g.*, $X_n - X_{n-1}$ are the same increments that the process was constructed from. But how about other increments, such as $X_n - X_{n-10}$ or $X_n - X_{n-100}$? Plot the distribution of the increments for a few different intervals. Use sufficiently many realisations that you get a nice and smooth histogram of the distribution. Is this distribution the same as the distribution you started out with for the smallest increments?

Investigate this using a few different distributions to construct the stochastic process. Note that you must choose the distribution such that it has $\mu = 0$ and $\sigma^2 = \Delta t$. This is possible even for non-symmetric discrete distributions with only two or three different outcomes, if you choose the values and the probabilities correctly.