Exercise 2

FY8503 Advanced theoretical physics
Transport modelling with Stochastic differential equations

September 18, 2023

Problem 1

Implement the Euler-Maruyama method, and use it to solve the simplest SDE from the last lecture:

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}W_t, \tag{1}$$

where the drift and diffusion terms, a and b, are constants.

As this is just the Wiener process scaled by a constant b and shifted by an amount at, we can see that the distribution of a large number of realisations X_t, at time t, will follow a Gaussian distribution with some mean μ and variance σ^2.

Task a

Solve for a large number of realisations, and plot a histogram of values of X_t together with the expected Gaussian distribution, for a few different times, t.

Task b

A more rigorous way (than visually comparing the expected distribution to a histogram) to look at the convergence to a distribution is perhaps to look at the moments of the distribution, for example the mean (first moment) or the variance (central second moment).

Solve the same equation for N_p realisations and a timestep Δt, and compare the mean and variance of the distribution of solutions to the mean and variance of the expected Gaussian distribution. Try different values of N_p, and different values of Δt. Do you observe any trends?

Problem 2

Use the Euler-Maruyama method to solve the other SDE we looked at in the last lecture:

$$\mathrm{d}X_t = aX_t\,\mathrm{d}t + bX_t\,\mathrm{d}W_t, \tag{2}$$

where the drift and diffusion terms, aX_t and bX_t, are now proportional to X_t. We recall from the lecture (see also Øksendal, 2003, p. 67) that the expectation value of the solution is

$$E(X_t) = \langle X_t \rangle = X_0 \mathrm{e}^{at}, \tag{3}$$

where X_0 is the initial value at $t = 0$.

Task a

Solve the equation for N_p realisations and a timestep Δt, and compare the mean of the distribution of solutions to the expectation value. Try this for different values of N_p, and different values of Δt. Do you observe any trends?

Task b

If you want, you can try to compare the distribution of the numerical solutions to the expected distribution, which is log-normal (which means that $\log(X_t)$ is normally distributed). You can read about geometric Brownian motion, for example at Wikipedia, and try to compare against the distribution described there. Note that the Wikipedia page uses a little different notation.