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隐函数参数方程求导、相关变化率

一、求下列方程所确定的隐函数 $y=y(x)$ 的导数 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $x^3+y^3=6xy$；　2. $\sin(x+y)=y^2\cos x$；　3. $\ln(x^2+y^2)=\arctan\frac{y}{x}$.

$$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$(y^2-2x)\frac{dy}{dx}=2y-x^2$$

$$\frac{dy}{dx}=\frac{2y-x^2}{y^2-2x}$$

$$\left(1+\frac{dy}{dx}\right)\cos(x+y)=2y\frac{dy}{dx}\cos x-y^2\sin x$$

$$\frac{dy}{dx}=\frac{-y^2\sin x-\cos(x+y)}{\cos(x+y)-2y\cos x}$$

$$\frac{2x+2y\frac{dy}{dx}}{x^2+y^2}=\frac{-\frac{y}{x^2}+\frac{\frac{dy}{dx}}{x}}{1+\frac{y^2}{x^2}}$$

$$2x+2y\frac{dy}{dx}=-y+x\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{-y-2x}{2y-x}$$

二、求曲线 $y^2=5x^4-x^2$ 在点 (1,2) 处的切线方程和法线方程.

$$2yy'=20x^3-2x$$

$$\Rightarrow y'\Big|_{x=1,y=2}=\frac{10x^3-x}{y}\Big|_{x=1,y=2}=\frac{10-1}{2}=\frac{9}{2}$$

从而切线方程: $y-2=\frac{9}{2}(x-1)$

法线方程: $y-2=-\frac{2}{9}(x-1)$

三、证明:曲线 $\sqrt{x}+\sqrt{y}=\sqrt{a}$ 上任意点处的切线在两坐标轴上的截距之和恒为 a.

$$\frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0$$

$$\Rightarrow y'=\frac{-\sqrt{y}}{\sqrt{x}}$$

在 (x_0,y_0) 处的切线方程，其中 $\sqrt{x_0}+\sqrt{y_0}=\sqrt{a}$

$$y-y_0=-\sqrt{\frac{y_0}{x_0}}(x-x_0)$$

当 $x=0$ 时 $y=x_0\sqrt{\frac{y_0}{x_0}}+y_0$

当 $y=0$ 时 $x=y_0\sqrt{\frac{x_0}{y_0}}+x_0$

于是 $x+y=x_0+y_0+x_0\sqrt{\frac{y_0}{x_0}}+y_0\sqrt{\frac{x_0}{y_0}}$

$$=(\sqrt{x_0}+\sqrt{y_0})^2=a$$

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四、设函数 $y=y(x)$ 满足方程 $e^{xy}+\sin(x^2y)=y$，试求 $y'(0)$.

$$e^{xy}(y+xy')+\cos(x^2y)(2xy+x^2y')=y'$$

当 $x=0$ 时 $y=1$. 将 $(x,y)=(0,1)$ 代入上式得

$$y'(0)=1$$

五、求下列函数的导数.

1. $y=x^{\sqrt{x}}$；

$$\ln y=\sqrt{x}\ln x$$

$$\Rightarrow \frac{y'}{y}=\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}$$

$$\Rightarrow y'=x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}+\frac{\ln x}{2\sqrt{x}}\right)$$

2. $y=(\ln x)^x$；

$$\ln y=x\ln(\ln x)$$

$$\Rightarrow \frac{y'}{y}=\ln(\ln x)+x\frac{1}{x\ln x}$$

$$\Rightarrow y'=(\ln x)^x\left[\ln(\ln x)+\frac{1}{\ln x}\right]$$

3. $y=\sqrt{\dfrac{x^3(x^2+1)^{\ln x}}{e^x(x+1)^{x^2}}}$；

$$\ln y=\frac{1}{2}\ln\frac{x^3(x^2+1)^{\ln x}}{e^x(x+1)^{x^2}}$$

$$=\frac{1}{2}\left[3\ln x+\ln x\ln(x^2+1)-x-x^2\ln(x+1)\right]$$

$$\Rightarrow \frac{y'}{y}=\frac{1}{2}\left[\frac{3}{x}+\frac{\ln(x^2+1)}{x}+\frac{2x\ln x}{x^2+1}-1-2x\ln(x+1)-\frac{x^2}{x+1}\right]$$

$$\Rightarrow y'=\frac{1}{2}\sqrt{\frac{x^3(x^2+1)^{\ln x}}{e^x(x+1)^{x^2}}}\left[\frac{3}{x}+\frac{\ln(x^2+1)}{x}+\frac{2x\ln x}{x^2+1}-1-2x\ln(x+1)-\frac{x^2}{x+1}\right]$$

4. $y=\sqrt[3]{\dfrac{x(x^2+1)}{(x-1)^2}}$.

$$\ln y=\frac{1}{3}\left[\ln x+\ln(x^2+1)-2\ln(x-1)\right]$$

$$\Rightarrow \frac{y'}{y}=\frac{1}{3}\left[\frac{1}{x}+\frac{2x}{x^2+1}-\frac{2}{x-1}\right]$$

$$\Rightarrow y'=\frac{1}{3}\sqrt[3]{\frac{x(x^2+1)}{(x-1)^2}}\left[\frac{1}{x}+\frac{2x}{x^2+1}-\frac{2}{x-1}\right]$$

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六、设 $x^y = y^x + x^2$，求 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

两端关于 x 求导得 $(x^y)' = (y^x)' + 2x$

令 $u = x^y$，$v = y^x$，则 $\ln u = y\ln x$，$\ln v = x\ln y$

于是 $u' = u\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right) = x^y\left(\frac{dy}{dx}\ln x + \frac{y}{x}\right)$

$v' = v\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right) = y^x\left(\ln y + \frac{x}{y}\frac{dy}{dx}\right)$

从而 $\frac{dy}{dx} = \frac{2x + y^x\ln y - x^{y-1}y}{x^y\ln x - y^{x-1}x}$

七、求下列隐函数的一阶导数 $\frac{\mathrm{d}y}{\mathrm{d}x}$ 和二阶导数 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

1. $x^4 + y^4 = 16$；

$4x^3 + 4y^3\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-4x^3}{4y^3} = \frac{-x^3}{y^3}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x^3}{y^3}\right) = \frac{-3x^2y^3 + 3y^2\frac{dy}{dx}x^3}{y^6}$

$= \frac{-3x^2y^4 - 3x^6}{y^7}$

2. $\mathrm{e}^y = xy + 3$；

$e^y\frac{dy}{dx} = y + x\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{e^y - x}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y}{e^y - x}\right) = \frac{\frac{dy}{dx}(e^y - x) - y\left(e^y\frac{dy}{dx} - 1\right)}{(e^y - x)^2}$

$= \frac{y - y\frac{ye^y}{e^y - x} + y}{(e^y - x)^2}$

$= \frac{2y(e^y - x) - y^2e^y}{(e^y - x)^3}$

3. $y = \tan(x + y) - 1$；

$\frac{dy}{dx} = \sec^2(x+y)\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = \frac{\sec^2(x+y)}{1 - \sec^2(x+y)} = -\csc^2(x+y)$

$\frac{dy^2}{dx^2} = \frac{d}{dx}\left(-\csc^2(x+y)\right)$

$= -2\csc^2(x+y)\cdot\cot(x+y)\left(1 + \frac{dy}{dx}\right)$

$= 2\csc^2(x+y)\cot^3(x+y)$

4. $y = \mathrm{e}^{y^2} + x$.

$\frac{dy}{dx} = 2ye^{y^2}\frac{dy}{dx} + 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{1 - 2ye^{y^2}}$

$\frac{dy^2}{dx^2} = \frac{d}{dx}\left(\frac{1}{1 - 2ye^{y^2}}\right)$

$= \frac{2\frac{dy}{dx}e^{y^2} + 4y^2e^{y^2}\frac{dy}{dx}}{(1 - 2ye^{y^2})^2}$

$= \frac{2e^{y^2} + 4y^2e^{y^2}}{(1 - 2ye^{y^2})^3}$

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八、已知 $f(x)$ 为二阶可导的单值函数，$f(1)=0, f'(1)=5, f''(1)=7$. $y=y(x)$ 满足方程：$f(x+y)=xy+x$，求 $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}, \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$.

当 $x=0$ 时 $f(y)=0$，故由 f 的单值性知 $y=1$

方程两端关于 x 求导

$$f'(x+y)\left(1+\frac{dy}{dx}\right)=x\frac{dy}{dx}+y+1$$

$$\Rightarrow \frac{dy}{dx}=\frac{y+1-f'(x+y)}{f'(x+y)-x}$$

$$\frac{d^2y}{dx^2}=\frac{\left[\frac{dy}{dx}-f''(x+y)\left(1+\frac{dy}{dx}\right)\right]\left(f'(x+y)-x\right)-\left[f''(x+y)\left(1+\frac{dy}{dx}\right)-1\right]\left[y+1-f'(x+y)\right]}{\left[f'(x+y)-x\right]^2}$$

得 $\left.\frac{dy}{dx}\right|_{x=0,y=1}=-\frac{3}{5}$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0,y=0}=\frac{-5\times 17+9\times 3}{5^3}=\frac{-58}{125}$$

九、求下列参数方程所确定的函数 $y=y(x)$ 的导数 $\frac{\mathrm{d}y}{\mathrm{d}x}$.

1. $x=t^3+3t+1, y=3t^5+5t^3+1$；
2. $x=\mathrm{e}^{-t}\sin t, y=\mathrm{e}^{t}\cos t$；
3. $x=\arcsin\frac{t}{\sqrt{1+t^2}}, y=\arccos\frac{1}{\sqrt{1+t^2}}$.

1、$\frac{dy}{dx}=\frac{15t^4+15t^2}{3t^2+3}=5t^2$

2、$\frac{dy}{dx}=\frac{-e^t\sin t+e^t\cos t}{e^{-t}\cos t-e^{-t}\sin t}=e^{2t}$

3、$\frac{dy}{dt}=\frac{\frac{t}{(1+t^2)^{\frac{3}{2}}}}{\sqrt{1-\frac{1}{1+t^2}}}=\frac{1}{1+t^2}$，$\frac{dx}{dt}=\frac{\frac{\sqrt{1+t^2}-\frac{t^2}{\sqrt{1+t^2}}}{1+t^2}}{\sqrt{1-\frac{t^2}{1+t^2}}}=\frac{\frac{1}{(1+t^2)^{\frac{3}{2}}}}{\frac{1}{\sqrt{1+t^2}}}=\frac{1}{1+t^2}$

故 $\frac{dy}{dx}=1$

十、求 $x=\frac{3at}{1+t^2}, y=\frac{3at^2}{1+t^2}$ 在 $t=2$ 处的切线方程和法线方程.

$$\frac{dy}{dx}=\frac{\frac{6at(1+t^2)-6at^3}{(1+t^2)^2}}{\frac{3a(1+t^2)-6at^2}{(1+t^2)^2}}=\frac{2t(1+t^2)-2t^3}{1-t^2}=\frac{2t}{1-t^2}$$

得 $\left.\frac{dy}{dx}\right|_{t=2}=\frac{4}{-3}$，且 $\left.x\right|_{t=2}=\frac{6a}{5}$，$\left.y\right|_{t=2}=\frac{12a}{5}$

从而切线方程：$y-\frac{12a}{5}=-\frac{4}{3}\left(x-\frac{6}{5}a\right)$

法线方程：$y-\frac{12a}{5}=\frac{3}{4}\left(x-\frac{6}{5}a\right)$

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十一、设 $x=at^3, y=bt^2, a\neq 0$，求 $\frac{dy}{dx}, \frac{d^2y}{dx^2}$.

$$\frac{dy}{dx}=\frac{2bt}{3at^2}=\frac{2b}{3at}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{2b}{3at}\right)=\frac{d}{dt}\left(\frac{2b}{3at}\right)\frac{dt}{dx}=\frac{-6ab}{9a^2t^2}\cdot\frac{1}{3at^2}=\frac{-2b}{9a^2t^4}$$

十二、设 $\begin{cases}x=\ln(1+t),\\ y=\arctan t,\end{cases}$ 求 $\left.\frac{d^2y}{dx^2}\right|_{t=0}$.

$$\frac{dy}{dx}=\frac{\frac{1}{1+t^2}}{\frac{1}{1+t}}=\frac{1+t}{1+t^2}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1+t}{1+t^2}\right)=\frac{d}{dt}\left(\frac{1+t}{1+t^2}\right)\frac{dt}{dx}=\frac{1+t^2-2t(1+t)}{(1+t^2)^2}(1+t)$$

于是 $\left.\frac{d^2y}{dx^2}\right|_{t=0}=1$

十三、设 $\begin{cases}x=\ln(1+t^2),\\ y=t-\arctan t,\end{cases}$ 求 $\frac{d^3y}{dx^3}$.

$$\frac{dy}{dx}=\frac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\frac{t^2}{2t}=\frac{t}{2}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{t}{2}\right)\frac{dt}{dx}=\frac{1}{2}\,\frac{1+t^2}{2t}=\frac{1+t^2}{4t}$$

$$\frac{d^3y}{dx^3}=\frac{d}{dt}\left(\frac{1+t^2}{4t}\right)\frac{dt}{dx}=\frac{8t^2-4-4t^2}{16t^2}\,\frac{1+t^2}{2t}=\frac{t^2-1}{4t^2}\,\frac{t^2+1}{2t}$$

$$=\frac{t^4-1}{8t^3}$$

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十四、设 $y = y(x)$ 是由方程 $\begin{cases} x = 3t^2 + 2t + 3, \\ e^y \sin t - y + 1 = 0 \end{cases}$ 所确定的隐函数，求 $\left.\dfrac{dy}{dx}\right|_{t=0}$ 和 $\left.\dfrac{d^2 y}{dx^2}\right|_{t=0}$.

$e^y \sin t - y + 1 = 0$ 两端关于 t 求导得

$e^y \frac{dy}{dt} \sin t + e^y \cos t - \frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = \frac{-e^y \cos t}{e^y \sin t - 1}$

$\frac{dy}{dx} = \frac{\frac{-e^y \cos t}{e^y \sin t - 1}}{6t+2} = \frac{-e^y \cos t}{(6t+2)(e^y \sin t - 1)}$

故 $\left.\frac{dy}{dx}\right|_{t=0} = \frac{-e}{-2} = \frac{e}{2}$

$\left.\frac{d^2y}{dx^2}\right|_{t=0} = \left.\frac{d}{dt}\left(\frac{-e^y \cos t}{(6t+2)(e^y \sin t - 1)}\right)\right|_{t=0} \left.\frac{dt}{dx}\right|_{t=0}$

$= \frac{1}{2} \cdot \frac{(-2)\cdot(-e)(\frac{e}{2}) + e(6+2e)}{4}$

$= \frac{e^2 + 6e + 2e^2}{8}$

$= \frac{3e^2 + 6e}{8}$

十五、一个球形雪球的体积以 1 cm^3/min 的速度减少，求直径为 10 cm 时，雪球直径的减少速度.

设雪球体积为 $V\ cm^3$，直径为 R cm，于是

$V = \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{\pi}{6}R^3$

两端对 t 求导

$1 = \frac{dV}{dt} = \frac{\pi}{2}R^2 \frac{dR}{dt} \Rightarrow \frac{dR}{dt} = \frac{2}{\pi R^2}$

故 $\left.\frac{dR}{dt}\right|_{R=10} = \frac{1}{50\pi}$ cm/min

十六、将水注入深 8 m，上顶直径为 8 m 的正圆锥形容器中，注水速度为 4 m^3/min，当水深为 5 m 时，其表面上升的速度为多少？表面上升的加速度又为多少？

设表面半径为 r m，水深为 h m，水的体积为 $V\ m^3$

于是 $\frac{h}{8} = \frac{r}{4} \Rightarrow r = \frac{h}{2}$

从而 $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12}h^3$

两端关于 t 求导

$4 = \frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$

于是 $\frac{dh}{dt} = \frac{16}{\pi h^2}$，从而 $\left.\frac{dh}{dt}\right|_{h=5} = \frac{16}{25\pi}$ m/min

$\frac{d^2h}{dt^2} = \frac{d}{dh}\left(\frac{16}{\pi h^2}\right)\frac{dh}{dt}$

$= \frac{-32\pi h}{\pi^2 h^4} \cdot \frac{16}{\pi h^2} = \frac{-32\times 16}{\pi^2 h^5}$

故 $\left.\frac{d^2h}{dt^2}\right|_{h=5} = \frac{-2^9}{5^5\pi^2}$ m/min²