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不定积分的换元法和分部积分法

一、填空题.

1. 设$\int xf(x)dx = \arcsin x + C$，则$\int \frac{dx}{f(x)} =$ $-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$；

2. 设$f(x) = e^{-x}$，则$\int \frac{f'(\ln x)}{x}dx =$ $\frac{1}{x}+C$；

3. $F(x)$为$f(x)$的一个原函数，$f(x) = \frac{F(x)}{1+x^2}$，则$f(x) =$ $\frac{e^{\arctan x}\cdot C}{1+x^2}$.

$F'(x)=\frac{F(x)}{1+x^2} \Rightarrow \frac{F'(x)}{F(x)}=\frac{1}{1+x^2} \Rightarrow \ln F(x)=\arctan x+C$

$F(x)=e^{\arctan x+C}=e^{\arctan x}\cdot C$

二、计算下列不定积分.

1. $\int \frac{dx}{\sqrt[3]{1-2x}}$；

令$t=\sqrt[3]{1-2x} \Rightarrow x=\frac{1-t^3}{2}$

$dx=-\frac{3}{2}t^2dt$

$=\int \frac{-\frac{3}{2}t^2}{t}dt=-\frac{3}{2}\int t\,dt$

$=-\frac{3}{4}(1-2x)^{\frac{2}{3}}+C$

2. $\int \frac{dx}{\sqrt{x-x^2}}$；

$=\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$

令$t=\sqrt{1-x}$，$x=1-t^2$，$dx=-2t\,dt$

$=\int \frac{-2t\,dt}{\sqrt{1-t^2}\,t}=-2\int \frac{dt}{\sqrt{1-t^2}}$

$=-2\arcsin\sqrt{1-x}+C$

3. $\int \frac{x^3}{1+x^2}dx$；

$=\frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx^2$

$=\frac{1}{2}\int dx^2-\frac{1}{2}\int \frac{1}{1+x^2}d(1+x^2)$

$=\frac{1}{2}x^2-\frac{1}{2}\ln(1+x^2)+C$

4. $\int \frac{\arctan\sqrt{x}}{\sqrt{x}+\sqrt{x^3}}dx$；

令$t=\sqrt{x}$，则$x=t^2$，$dx=2t\,dt$

$=\int \frac{\arctan t\cdot 2t\,dt}{t+t^3}$

$=2\int \frac{\arctan t}{1+t^2}dt$

$=2\int \arctan t\, d\arctan t$

$=(\arctan\sqrt{x})^2+C$

5. $\int \frac{1+x+\arctan x}{1+x^2}dx$；

$=\int \frac{1}{1+x^2}dx+\int \frac{x}{1+x^2}dx+\int \frac{\arctan x}{1+x^2}dx$

6. $\int \frac{dx}{1+e^x}$；

$=\int \frac{de^x}{e^x+e^{2x}}$　[令$e^x=t$]

$=\int \frac{dt}{t(1+t)}$

$=\int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt$

$=x-\ln(1+e^x)+C$

7. $\int \tan^3 x \sec^3 x \, dx$；

$= \int \frac{\sin^3 x}{\cos^6 x} dx$

$= -\int \frac{\sin^2 x \cdot d\cos x}{\cos^6 x}$

$= -\int \frac{1-\cos^2 x}{\cos^6 x} d\cos x = -\int \frac{d\cos x}{\cos^6 x} + \int \frac{d\cos x}{\cos^4 x}$

8. $\int \cos^5 x \, dx$；

$\int \cos^4 x \, d\sin x$

$= \int (1-\sin^2 x)^2 d\sin x$

9. $\int \frac{1}{\sin^2 x + 2\cos^2 x} dx$；

$= \int \frac{\sec^2 x}{\tan^2 x + 2} dx$

$= \int \frac{d\tan x}{2+\tan^2 x}$

$= \frac{\sqrt{2}}{2} \int \frac{d\frac{\tan x}{\sqrt{2}}}{1+(\frac{\tan x}{\sqrt{2}})^2} = \frac{\sqrt{2}}{2} \arctan\left(\frac{\tan x}{\sqrt{2}}\right) + C$

10. $\int \frac{\ln(x+1) - \ln x}{x(x+1)} dx$.

$\int (\ln(x+1) - \ln x)\left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= -\int (\ln(x+1) - \ln x) \, d(\ln(x+1) - \ln x)$

$= -\frac{1}{2}\left(\ln\frac{x+1}{x}\right)^2 + C$

三、计算下列不定积分.

1. $\int x^2 (1-x)^{1000} dx$；

令 $t = 1-x$，则 $x = 1-t$，$dx = -dt$

$-\int (1-t)^2 t^{1000} dt$

$= -\int (1-2t+t^2) t^{1000} dt$

2. $\int \frac{\sqrt{1-x^2}}{x^2} dx$；

令 $x = \sin t$，$dx = \cos t \, dt$

$= \int \frac{\cos^2 t}{\sin^2 t} dt$

$= \int \frac{1-\sin^2 t}{\sin^2 t} dt$

3. $\int \frac{1}{x\sqrt{x^2-1}} dx$；

$\sqrt{x^2-1} = t$

$x = \sqrt{\ }$

$= \frac{1}{2} \int \frac{dx^2}{x^2\sqrt{x^2-1}}$　$(t = x^2)$

$= \frac{1}{2} \int \frac{dt}{t\sqrt{t-1}}$

令 $u = \sqrt{t-1}$，则 $t = 1+u^2$，$dt = 2u\,du$

$= \frac{1}{2} \int \frac{2u\,du}{(1+u^2)u} = \arctan u + C$

4. $\int \frac{1}{(2x^2+1)\sqrt{x^2+1}} dx$.

令 $x = \tan t$，$dx = \sec^2 t \, dt$

$= \int \frac{\sec t \, dt}{(2\tan^2 t + 1)}$

$= \int \frac{\frac{1}{\cos t}}{2\frac{\sin^2 t}{\cos^2 t} + 1} dt$

$= \int \frac{-1}{1+\sin^2 t} d\sin t$

$\arctan\left(\frac{x}{\sqrt{1+x^2}}\right) + C$

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四、计算下列不定积分.

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1. $\int (1+x^2)\sin 2x\,\mathrm{d}x$;

$=-2\int (1+x^2)\,d\cos 2x$

$=-2(1+x^2)\cos 2x-4\int \cos 2x\cdot 2x\,dx$

$=-2(1+x^2)\cos 2x-4\int x\,d\sin 2x$

$=-2(1+x^2)\cos 2x-4x\sin 2x+4\int \sin 2x\,dx$

$=-2(1+x^2)\cos 2x-4x\sin 2x-2\cos 2x+C$

2. $\int \dfrac{\ln x}{(x-2)^2}\mathrm{d}x$;

$=-\int \ln x\,d\dfrac{1}{x-2}$

$=-\dfrac{\ln x}{x-2}+\int \dfrac{1}{x(x-2)}dx$

$=-\dfrac{\ln x}{x-2}+\dfrac{1}{2}\int\left(\dfrac{1}{x-2}-\dfrac{1}{x}\right)dx$

$=-\dfrac{\ln x}{x-2}+\dfrac{1}{2}\ln|x-2|-\dfrac{1}{2}\ln|x|+C$

$=-\dfrac{\ln x}{x-2}+\dfrac{1}{2}\ln\left|\dfrac{x-2}{x}\right|+C$

3. $\int \dfrac{x}{\sin^2 x}\mathrm{d}x$;

$=\int x\csc^2 x\,dx$

$=-\int x\,d\cot x$

$=-x\cot x+\int \dfrac{\cos x}{\sin x}dx$

$=-x\cot x+\int \dfrac{1}{\sin x}d\sin x$

$=-x\cot x+\ln|\sin x|+C$

4. $\int x\tan^2 x\,\mathrm{d}x$;

$\tan^2 x+1=\sec^2 x$

$=\int (x\tan^2 x+x-x)\,dx$

$=\int x\sec^2 x\,dx-\int x\,dx$

$=\int x\,d\tan x-\dfrac{x^2}{2}+C$

$=x\tan x-\int \dfrac{\sin x}{\cos x}dx-\dfrac{x^2}{2}+C$

$=x\tan x+\ln|\cos x|-\dfrac{x^2}{2}+C$

5. $\int x(\arctan x)^2\mathrm{d}x$;

$=\dfrac{1}{2}\int (\arctan x)^2\,d x^2$

$=\dfrac{1}{2}x^2(\arctan x)^2-\int \arctan x\,\dfrac{x^2+1-1}{1+x^2}dx$

$=\dfrac{1}{2}x^2(\arctan x)^2-\int \arctan x\,dx+\int \arctan x\,d\arctan x$

$=\dfrac{1}{2}x^2(\arctan x)^2-x\arctan x+\int \dfrac{x}{1+x^2}dx+\dfrac{(\arctan x)^2}{2}+C$

$=\dfrac{1}{2}x^2(\arctan x)-x\arctan x+\dfrac{1}{2}\ln(1+x^2)+\dfrac{(\arctan x)^2}{2}+C$

6. $\int \sin(\ln x)\mathrm{d}x$;

$\int \sin(\ln x)dx=x\sin(\ln x)-\int x\cdot\cos(\ln x)\cdot\dfrac{1}{x}dx$

$=x\sin(\ln x)-\int \cos(\ln x)\,dx$

$=x\sin(\ln x)-x\cos(\ln x)-\int \sin(\ln x)dx$

提 $\int \sin(\ln x)dx$

$=\dfrac{x\sin(\ln x)-x\cos(\ln x)}{2}+C$

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7. $\int e^{2x}(\tan x+1)^2 dx$;

$= \int e^{2x}(\tan^2 x+2\tan x+1)dx$

$= \int e^{2x}(\sec^2 x+2\tan x)dx$

$= \int e^{2x} d\tan x + 2\int e^{2x}\tan x dx$

$= \tan x e^{2x} - 2\int e^{2x}\tan x dx + 2\int e^{2x}\tan x dx$

$= \tan x e^{2x} + C$

8. $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx$;

$= \int \arctan x \, d\sqrt{1+x^2}$

$= \sqrt{1+x^2}\arctan x - \int \frac{\sqrt{1+x^2}}{1+x^2}dx$

$= \sqrt{1+x^2}\arctan x - \boxed{\int \frac{1}{\sqrt{1+x^2}}dx}$

$= \sqrt{1+x^2}\arctan x - \ln(x+\sqrt{x^2+1}) + C$

9. $\int \frac{\arctan e^x}{e^{2x}}dx$;

$= \int \frac{\arctan e^x}{e^{3x}} de^x$ $[t \triangleq e^x]$

$= \int \frac{\arctan t}{t^3} dt$

$= -\frac{1}{2}\int \arctan t \, d\frac{1}{t^2}$

$= -\frac{1}{2}\frac{\arctan t}{t^2} + \frac{1}{2}\int \frac{1}{(1+t^2)t^2}dt$

$= -\frac{1}{2}\frac{\arctan t}{t^2} + \frac{1}{2}\int(\frac{1}{t^2} - \frac{1}{t^2+1})dt = \frac{-\arctan t}{2t^2} - \frac{1}{2}\frac{1}{t} - \arctan t + C$

$= -\arctan e^x(\frac{1}{2e^{2x}}+1) - \frac{1}{2e^x} + C$

10. $\int x^3 \sin\frac{x}{2}dx$.

$= -2\int x^3 \, d\cos\frac{x}{2}$

$= -2x^3\cos\frac{x}{2} + 6\int x^2\cos\frac{x}{2}dx$

$= -2x^3\cos\frac{x}{2} + 12\int x^2 \, d\sin\frac{x}{2}$

$= -2x^3\cos\frac{x}{2} + 12x^2\sin\frac{x}{2} - 24\int x\sin\frac{x}{2}dx$

$= -2x^3\cos\frac{x}{2} + 12x^2\sin\frac{x}{2} + 48\int x \, d\cos\frac{x}{2}$

$48x\cos\frac{x}{2} - 48\int\cos\frac{x}{2}dx$

$96\sin\frac{x}{2} + C$

五、设 $I_n = \int \frac{1}{\sin^n x}dx$，试建立递推公式.

当 $n\geq 2$ 时，

$I_n = \int \frac{\cos^2 x+\sin^2 x}{\sin^n x}dx$ I_{n-2}

$= \int \frac{\cos^2 x}{\sin^n x}dx + \int \frac{1}{\sin^{n-2}x}dx$ $[n\geq 2]$

$= \int \cos x \frac{1}{\sin^n x} d\sin x + I_{n-2}$

$= \frac{1}{-n+1}\int \cos x \, d\frac{1}{\sin^{n-1}} + I_{n-2}$

$= \frac{1}{-n+1}\frac{\cos x}{\sin^{n-1}x} + \frac{1}{-n+1}\int \cancel{\sin x}\frac{dx}{\sin^{n-2}} + I_{n-2}$ I_{n-2}

$= \frac{-1}{n-1}\frac{\cos x}{\sin^{n-1}x} + \frac{-1}{n-1}I_{n-2} + I_{n-2}$

$= \frac{-1}{n-1}\frac{\cos x}{\sin^{n-1}x} + \frac{n-2}{n-1}I_{n-2}$

$n=0$ 时 $I_0 = x+C$

$n=1$ 时 $I_1 = \int\frac{1}{\sin x}dx$

六. 已知函数 $f(x) = \begin{cases} 2(x-1), x<1, \\ \ln(x), x\geq 1, \end{cases}$ 求 $f(x)$ 的原函数 $F(x)$.

当 $x<1$ 时 $F(x) = \int f(x)dx = \int 2(x-1)dx = (x-1)^2+C$

当 $x>1$ 时 $F(x) = \int f(x)dx = \int \ln x dx = x\ln x - x + \tilde{C}$

$F(x)$ 要在 $x=1$ 处连续 $F(1^-) = C$. $F(1^+) = -1+\tilde{C}$

故 $C = -1+\tilde{C} \Rightarrow \tilde{C} = C+1$

$F(x) = \begin{cases} (x-1)^2+C, & x<1 \\ x\ln x - x + C+1, & x\geq 1 \end{cases}$