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Lie Algebra for Robotics

Lab

1 | Lie Groups

A Lie Group is a group whose elements are organized continuously and smoothly, making it a smooth manifold.

| Special Orthogonal group $\mathbf{SO}(3)$

Group of 3D rotation matrix:

$$\mathbf{SO}(3) = \left\{\mathbf{C} \in \mathbf{GL}(3, \mathbb{R}) \,|\, \det(\mathbf{C}) = 1, \mathbf{C}^T\mathbf{C} = \mathbf{I}\right\}$$

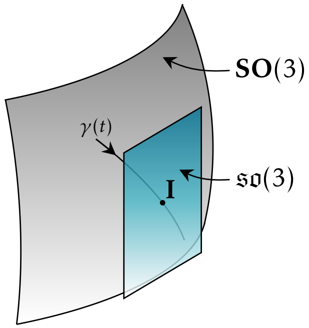
| Special Euclidian group $\mathbf{SE}(3)$

Group of 3D transformation matrix:

$$\mathbf{SE}(3) = \left\{\mathbf{T} = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbf{GL}(4, \mathbb{R}) \,|\, \mathbf{C} \in \mathbf{SO}(3), \mathbf{r} \in \mathbb{R}^3\right\}$$

2 | Lie algebra

A Lie algebra is the tangent space of the Lie group at the identity element. The tangent space is defined as $\{\gamma'(1)\}$ where $\gamma(t)$ such that $\gamma(1) = \mathbf{I}$



| Special Orthogonal Group $\mathfrak{so}(3)$

$$\mathfrak{so}(3) = \left\{\Phi = \phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \,|\, \phi \in \mathbb{R}^3\right\}$$

Taking the exponential of an element in $\mathfrak{so}(3)$ leads to an element in $\mathbf{SO}(3)$: $\exp(\Phi) \in \mathbf{SO}(3)$.

$$\Phi = \phi^\wedge \Rightarrow \phi = \Phi^\vee$$

| Special Euclidian Group $\mathfrak{se}(3)$

$$\mathfrak{se}(3) = \left\{\Xi = \xi^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \,|\, \rho, \phi \in \mathbb{R}^3\right\}$$

Taking the exponential of an element in $\mathfrak{se}(3)$ leads to an element in $\mathbf{SE}(3)$: $\exp(\Xi) \in \mathbf{SE}(3)$.

$$\Xi = \xi^\wedge \Rightarrow \xi = \Xi^\vee$$

3 | Exponential Map

For every square matrix $\mathbf{A}$, we have

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n$$

$$\ln(\mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(\mathbf{A} - \mathbf{I})^n$$

| Baker-Campbell-Hausdorff (BCH) formula

Most of the time, $\exp(\mathbf{A} + \mathbf{B}) \neq \exp(\mathbf{A})\exp(\mathbf{B})$

$$\ln(\mathbf{C}_1\mathbf{C}_2)^\vee = \phi_1 + \phi_2 + \frac{1}{2}\phi_1^\wedge\phi_2 + \cdots$$
$$\approx \begin{cases} \mathbf{J}(\phi_2)^{-1}\phi_1 + \phi_2 & \text{if } \phi_1 \text{ small} \\ \phi_1\mathbf{J}(-\phi_1)^{-1}\phi_2 & \text{if } \phi_2 \text{ small} \end{cases}$$

$$\ln(\mathbf{T}_1\mathbf{T}_2)^\vee = \xi_1 + \xi_2 + \frac{1}{2}\xi_1^\wedge\xi_2 + \cdots$$
$$\approx \begin{cases} \mathcal{J}(\xi_2)^{-1}\xi_1 + \xi_2 & \text{if } \xi_1 \text{ small} \\ \xi_1\mathcal{J}(-\xi_1)^{-1}\xi_2 & \text{if } \xi_2 \text{ small} \end{cases}$$

4 | Adjoints

The adjoint of an element of $\mathfrak{se}(3)$ is

$$\mathrm{ad}(\Xi) = \mathrm{ad}(\xi^\wedge) = \begin{bmatrix} \phi^\wedge & \rho^\wedge \\ \mathbf{0} & \phi^\wedge \end{bmatrix} = \xi^\curlywedge$$

The adjoint of an element of $\mathbf{SE}(3)$ is

$$\mathcal{T} = \mathrm{Ad}(T) = \begin{bmatrix} \mathbf{C} & \mathbf{r} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}$$

5 | Relation between spaces

$$\phi \in \mathfrak{so}(3) \xrightarrow{\exp} \mathbf{C} \in \mathbf{SO}(3)$$

$$\begin{array}{ccc} \xi^\wedge \in \mathfrak{se}(3) & \xrightarrow{\exp} & \mathbf{T} \in \mathbf{SE}(3) \\ \downarrow \mathrm{ad} & & \downarrow \mathrm{Ad} \\ \xi^\curlywedge\ \mathrm{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \mathcal{T} \in \mathrm{Ad}(\mathbf{SE}(3)) \end{array}$$

6 | (left) Jacobians

$$\mathbf{J}(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\phi^\wedge\right)^n \quad \mathcal{J}(\xi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\xi^\curlywedge\right)^n$$

The Jacobians have singularities (i.e., the inverse does not exist) at $\phi = 2\pi m$ with m a nonzero integer.

7 | Interpolation

$$\mathbf{C} = (\mathbf{C}_2\mathbf{C}_1^T)^\alpha\mathbf{C}_1 \qquad \mathbf{T} = (\mathbf{T}_2\mathbf{T}_1^{-1})^\alpha\mathbf{T}_1$$

with $\alpha \in [0, 1]$

8 | Perturb Rotations and Poses

The left perturbation avoids the singularities as we stay near the identity:

$$\mathbf{C} = \exp(\epsilon^\wedge)\bar{\mathbf{C}} \qquad \mathbf{T} = \exp(\varepsilon^\wedge)\bar{\mathbf{T}}$$

with $\epsilon \in \mathbb{R}^3 \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon)$, $\varepsilon \in \mathbb{R}^6 \sim \mathcal{N}(\mathbf{0}, \Sigma_\varepsilon)$

8.1 Example: Compounding poses

We want to find the mean and covariance of $\mathbf{T} = \mathbf{T}_1\mathbf{T}_2$ where $T_1 = \exp(\varepsilon_1)\bar{\mathbf{T}}_1, T_2 = \exp(\varepsilon_2)\bar{\mathbf{T}}_2$ and the errors $\varepsilon_1, \varepsilon_2$ have zero mean and covariances Σ_1, Σ_2.

$$\exp(\varepsilon^\wedge)\bar{\mathbf{T}} = \exp(\varepsilon_1)\bar{\mathbf{T}}_1\exp(\varepsilon_2)\bar{\mathbf{T}}_2$$
$$\Leftrightarrow \exp(\varepsilon^\wedge) = \exp(\varepsilon_1)\exp(\bar{\mathcal{T}}_1\varepsilon_2^\wedge) \qquad \bar{\mathcal{T}}_1 = \mathrm{Ad}(\bar{T}_1)$$
$$\Leftrightarrow \varepsilon = \varepsilon_1 + \varepsilon_2' + \frac{1}{2}\varepsilon_1^\curlywedge\varepsilon_2' + \frac{1}{12}\varepsilon_1^\curlywedge\varepsilon_1^\curlywedge\varepsilon_2' + \cdots \qquad \varepsilon_2' = \bar{\mathcal{T}}_1\varepsilon_2$$

It is then possible to find $\mathbb{E}[\varepsilon]$ and $\mathbb{E}[\varepsilon\varepsilon^T]$