

## 1 | Lie Groups

A Lie Group is a group whose elements are organized continuously and smoothly, making it a smooth manifold.

| Special Orthogonal group  $\text{SO}(3)$ 

Group of 3D rotation matrix:

$$\text{SO}(3) = \{C \in \text{GL}(3, \mathbb{R}) \mid \det(C) = 1, C^T C = I\}$$

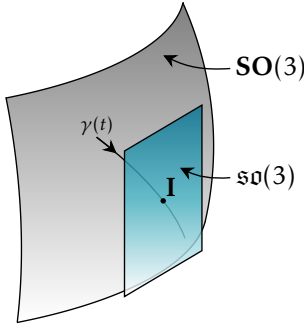
| Special Euclidian group  $\text{SE}(3)$ 

Group of 3D transformation matrix:

$$\text{SE}(3) = \left\{ T = \begin{bmatrix} C & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \text{GL}(4, \mathbb{R}) \mid C \in \text{SO}(3), \mathbf{r} \in \mathbb{R}^3 \right\}$$

## 2 | Lie algebra

A Lie algebra is the tangent space of the Lie group at the identity element. The tangent space is defined as  $\{\gamma'(1)\}$  where  $\gamma(t)$  such that  $\gamma(1) = I$

| Special Orthogonal Group  $\mathfrak{so}(3)$ 

$$\mathfrak{so}(3) = \left\{ \Phi = \phi^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \mid \phi \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in  $\mathfrak{so}(3)$  leads to an element in  $\text{SO}(3)$ :  $\exp(\Phi) \in \text{SO}(3)$ .

$$\Phi = \phi^\wedge \Rightarrow \phi = \Phi^\vee$$

| Special Euclidian Group  $\mathfrak{se}(3)$ 

$$\mathfrak{se}(3) = \left\{ \Xi = \xi^\wedge = \begin{bmatrix} \rho \\ \phi \end{bmatrix}^\wedge = \begin{bmatrix} \phi^\wedge & \rho \\ \mathbf{0}^T & 0 \end{bmatrix} \mid \rho, \phi \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in  $\mathfrak{se}(3)$  leads to an element in  $\text{SE}(3)$ :  $\exp(\Xi) \in \text{SE}(3)$ .

$$\Xi = \xi^\wedge \Rightarrow \xi = \Xi^\vee$$

## 3 | Exponential Map

For every square matrix  $A$ , we have

$$\exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

$$\ln(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (A - I)^n$$

|  $\ln \text{SO}(3)$ 

$$\exp(\phi^\wedge) = \cos(\phi)I + (1 - \cos(\phi))\mathbf{a}\mathbf{a}^T + \sin(\phi)\mathbf{a}^\wedge$$

$$\log(C) = \frac{1}{2} \frac{\theta(C)}{\sin(\theta(C))} (C - C^T)$$

$$\text{with } \begin{cases} C &= \exp(\phi^\wedge) = \exp(\phi \mathbf{a}) \\ \theta(C) &= \cos^{-1}\left(\frac{1}{2}(\text{tr}(C) - 1)\right). \end{cases}$$

## | Baker-Campbell-Hausdorff (BCH) formula

Most of the time,  $\exp(A + B) \neq \exp(A)\exp(B)$

$$\ln(C_1 C_2)^\vee = \phi_1 + \phi_2 + \frac{1}{2} \phi_1^\wedge \phi_2 + \dots$$

$$\approx \begin{cases} J(\phi_2)^{-1} \phi_1 + \phi_2 & \text{if } \phi_1 \text{ small} \\ \phi_1 J(-\phi_1)^{-1} \phi_2 & \text{if } \phi_2 \text{ small} \end{cases}$$

$$\ln(T_1 T_2)^\vee = \xi_1 + \xi_2 + \frac{1}{2} \xi_1^\wedge \xi_2 + \dots$$

$$\approx \begin{cases} \mathcal{J}(\xi_2)^{-1} \xi_1 + \xi_2 & \text{if } \xi_1 \text{ small} \\ \xi_1 \mathcal{J}(-\xi_1)^{-1} \xi_2 & \text{if } \xi_2 \text{ small} \end{cases}$$

## 4 | Adjoints

The adjoint of an element of  $\mathfrak{se}(3)$  is

$$\text{ad}(\Xi) = \text{ad}(\xi^\wedge) = \begin{bmatrix} \phi^\wedge & \rho^\wedge \\ \mathbf{0} & \phi^\wedge \end{bmatrix} = \xi^\wedge$$

The adjoint of an element of  $\text{SE}(3)$  is

$$\mathcal{T} = \text{Ad}(T) = \begin{bmatrix} C & \mathbf{r} \\ \mathbf{0} & C \end{bmatrix}$$

## 5 | Relation between spaces

$$\phi \in \mathfrak{so}(3) \xrightarrow{\exp} C \in \text{SO}(3)$$

$$\xi^\wedge \in \mathfrak{se}(3) \xrightarrow{\exp} T \in \text{SE}(3)$$

$$\begin{array}{ccc} \xi^\wedge \in \mathfrak{se}(3) & \xrightarrow{\exp} & T \in \text{SE}(3) \\ \downarrow \text{ad} & & \downarrow \text{Ad} \\ \xi^\wedge \in \text{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \mathcal{T} \in \text{Ad}(\text{SE}(3)) \end{array}$$

## 6 | (left) Jacobians

$$J(\phi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\phi^\wedge)^n \quad \mathcal{J}(\xi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\xi^\wedge)^n$$

The Jacobians have singularities (i.e., the inverse does not exist) at  $|\phi| = 2\pi m$  with  $m$  a nonzero integer.

## 7 | Interpolation

$$C = (C_2 C_1^T)^\alpha C_1 \quad T = (T_2 T_1^{-1})^\alpha T_1$$

with  $\alpha \in [0, 1]$

## 8 | Perturb Rotations and Poses

The left perturbation avoids the singularities as we stay near the identity:

$$C = \exp(\epsilon^\wedge) \bar{C} \quad T = \exp(\epsilon^\wedge) \bar{T}$$

with  $\epsilon \in \mathbb{R}^3 \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon)$ ,  $\epsilon \in \mathbb{R}^6 \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon)$

### 8.1 Example: Compounding poses

We want to find the mean and covariance of  $T = T_1 T_2$  where  $T_1 = \exp(\varepsilon_1) \bar{T}_1$ ,  $T_2 = \exp(\varepsilon_2) \bar{T}_2$  and the errors  $\varepsilon_1, \varepsilon_2$  have zero mean and covariances  $\Sigma_1, \Sigma_2$ .

$$\exp(\varepsilon^\wedge) \bar{T} = \exp(\varepsilon_1) \bar{T}_1 \exp(\varepsilon_2) \bar{T}_2$$

$$\Leftrightarrow \exp(\varepsilon^\wedge) = \exp(\varepsilon_1) \exp(\bar{T}_1 \varepsilon_2^\wedge) \quad \bar{T}_1 = \text{Ad}(\bar{T}_1)$$

$$\Leftrightarrow \varepsilon = \varepsilon_1 + \varepsilon_2' + \frac{1}{2} \varepsilon_1^\wedge \varepsilon_2' + \frac{1}{12} \varepsilon_1^\wedge \varepsilon_1^\wedge \varepsilon_2' + \dots \quad \varepsilon_2' = \bar{T}_1 \varepsilon_2$$

It is then possible to find  $\mathbb{E}[\varepsilon]$  and  $\mathbb{E}[\varepsilon \varepsilon^T]$