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Lie Algebra for Robotics

Based on “State Estimation for Robotics” by Tim Barfoot

1 | Lie Groups

A Lie Group is a group whose elements are organized continuously and smoothly, making it a smooth manifold.

| Special Orthogonal group $\mathbf{SO}(3)$

Group of 3D rotation matrix:

$$\mathbf{SO}(3) = \left\{ \boldsymbol{C} \in \mathbb{R}^{3\times 3} \,\middle|\, \det(\boldsymbol{C}) = 1, \boldsymbol{C}^T\boldsymbol{C} = \mathbf{I} \right\}$$

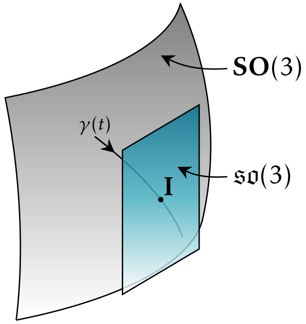
| Special Euclidian group $\mathbf{SE}(3)$

Group of 3D transformation matrix:

$$\mathbf{SE}(3) = \left\{ \begin{bmatrix} \boldsymbol{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4} \,\middle|\, \boldsymbol{C} \in \mathbf{SO}(3), \mathbf{r} \in \mathbb{R}^3 \right\}$$

2 | Lie algebra

A Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space of G at the identity element. The tangent space is defined as the set $\{\gamma'(0)\}$ where $\gamma(t) \in G, \gamma(0) = \mathbf{I}$



| Special Orthogonal Group $\mathfrak{so}(3)$

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\Phi} = \boldsymbol{\phi}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \middle| \boldsymbol{\phi} \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in $\mathfrak{so}(3)$ leads to an element in $\mathbf{SO}(3)$: $\exp(\boldsymbol{\Phi}) \in \mathbf{SO}(3)$.

$$\boldsymbol{\Phi} = \boldsymbol{\phi}^\wedge \Rightarrow \boldsymbol{\phi} = \boldsymbol{\Phi}^\vee$$

| Special Euclidian Group $\mathfrak{se}(3)$

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \middle| \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in $\mathfrak{se}(3)$ leads to an element in $\mathbf{SE}(3)$: $\exp(\boldsymbol{\Xi}) \in \mathbf{SE}(3)$.

$$\boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \Rightarrow \boldsymbol{\xi} = \boldsymbol{\Xi}^\vee$$

3 | Exponential Map

For every square matrix $\mathbf{A}$, we have

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n$$

$$\ln(\mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(\mathbf{A} - \mathbf{I})^n$$

| In $\mathbf{SO}(3)$

$$\exp(\boldsymbol{\phi}^\wedge) = \cos(\phi)\mathbf{I} + (1 - \cos(\phi))\boldsymbol{a}\boldsymbol{a}^T + \sin(\phi)\boldsymbol{a}^\wedge$$

$$\log(\boldsymbol{C}) = \frac{1}{2}\frac{\theta(\boldsymbol{C})}{\sin(\theta(\boldsymbol{C}))}(\boldsymbol{C} - \boldsymbol{C}^T)$$

$$\text{with} \begin{cases} \boldsymbol{C} & = \exp(\boldsymbol{\phi}^\wedge) = \exp(\phi\boldsymbol{a}) \\ \theta(\boldsymbol{C}) & = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(\boldsymbol{C}) - 1)\right). \end{cases}$$

| Baker-Campbell-Hausdorff (BCH) formula

Most of the time, $\exp(\mathbf{A} + \mathbf{B}) \neq \exp(\mathbf{A})\exp(\mathbf{B})$

$$\ln(\boldsymbol{C}_1\boldsymbol{C}_2)^\vee = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \frac{1}{2}\boldsymbol{\phi}_1^\wedge\boldsymbol{\phi}_2 + \cdots$$
$$\approx \begin{cases} \boldsymbol{J}(\boldsymbol{\phi}_2)^{-1}\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_1 \text{ small} \\ \boldsymbol{\phi}_1 + \boldsymbol{J}(-\boldsymbol{\phi}_1)^{-1}\boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_2 \text{ small} \end{cases}$$

$$\ln(\boldsymbol{T}_1\boldsymbol{T}_2)^\vee = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \frac{1}{2}\boldsymbol{\xi}_1^\wedge\boldsymbol{\xi}_2 + \cdots$$
$$\approx \begin{cases} \boldsymbol{\mathcal{J}}(\boldsymbol{\xi}_2)^{-1}\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ small} \\ \boldsymbol{\xi}_1 + \boldsymbol{\mathcal{J}}(-\boldsymbol{\xi}_1)^{-1}\boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ small} \end{cases}$$

4 | Adjoints

The adjoint of an element of $\mathfrak{se}(3)$ is

$$\mathrm{ad}(\boldsymbol{\Xi}) = \mathrm{ad}(\boldsymbol{\xi}^\wedge) = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho}^\wedge \\ \mathbf{0} & \boldsymbol{\phi}^\wedge \end{bmatrix} = \boldsymbol{\xi}^\curlywedge$$

The adjoint of an element of $\mathbf{SE}(3)$ is

$$\boldsymbol{\mathcal{T}} = \mathrm{Ad}(\boldsymbol{T}) = \begin{bmatrix} \boldsymbol{C} & \mathbf{r} \\ \mathbf{0} & \boldsymbol{C} \end{bmatrix}$$

5 | Relation between spaces

$$\boldsymbol{\phi} \in \mathfrak{so}(3) \xrightarrow{\exp} \boldsymbol{C} \in \mathbf{SO}(3)$$

$$\begin{array}{ccc} \boldsymbol{\xi}^\wedge \in \mathfrak{se}(3) & \xrightarrow{\exp} & \boldsymbol{T} \in \mathbf{SE}(3) \\ \downarrow \mathrm{ad} & & \downarrow \mathrm{Ad} \\ \boldsymbol{\xi}^\curlywedge \in \mathrm{ad}(\mathfrak{se}(3)) & \xrightarrow{\exp} & \boldsymbol{\mathcal{T}} \in \mathrm{Ad}(\mathbf{SE}(3)) \end{array}$$

6 | (left) Jacobians

$$\boldsymbol{J}(\boldsymbol{\phi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\boldsymbol{\phi}^\wedge\right)^n \quad \boldsymbol{\mathcal{J}}(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\boldsymbol{\xi}^\curlywedge\right)^n$$

The Jacobians have singularities (i.e., the inverse does not exist) at $|\boldsymbol{\phi}| = 2\pi m$ with m a nonzero integer.

7 | Interpolation

$$\boldsymbol{C} = (\boldsymbol{C}_2\boldsymbol{C}_1^T)^\alpha \boldsymbol{C}_1 \qquad \boldsymbol{T} = (\boldsymbol{T}_2\boldsymbol{T}_1^{-1})^\alpha \boldsymbol{T}_1$$

with $\alpha \in [0, 1]$

8 | Perturb Rotations and Poses

The left perturbation avoids the singularities as we stay near the identity:

$$\boldsymbol{C} = \exp(\boldsymbol{\epsilon}^\wedge)\bar{\boldsymbol{C}} \qquad \boldsymbol{T} = \exp(\boldsymbol{\varepsilon}^\wedge)\bar{\boldsymbol{T}}$$

with $\boldsymbol{\epsilon} \in \mathbb{R}^3 \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon)$, $\boldsymbol{\varepsilon} \in \mathbb{R}^6 \sim \mathcal{N}(\mathbf{0}, \Sigma_\varepsilon)$