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Lie Algebra for Robotics

1 | Lie Groups

A Lie Group is a group whose elements are organized continuously and smoothly, making it a smooth manifold.

| Special Orthogonal group $\mathbf{SO}(3)$

Group of 3D rotation matrix:

$$\mathbf{SO}(3) = \left\{ \boldsymbol{C} \in \mathrm{GL}(3, \mathbb{R}) \mid \det(\boldsymbol{C}) = 1, \boldsymbol{C}^T \boldsymbol{C} = \mathbf{I} \right\}$$

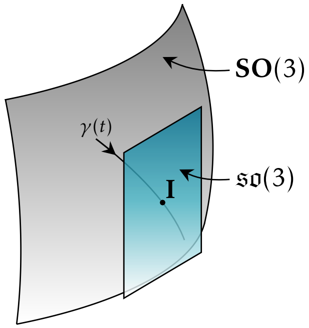
| Special Euclidian group $\mathbf{SE}(3)$

Group of 3D transformation matrix:

$$\mathbf{SE}(3) = \left\{ \boldsymbol{T} = \begin{bmatrix} \boldsymbol{C} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{bmatrix} \in \mathrm{GL}(4, \mathbb{R}) \mid \boldsymbol{C} \in \mathbf{SO}(3), \mathbf{r} \in \mathbb{R}^3 \right\}$$

2 | Lie algebra

A Lie algebra is the tangent space of the Lie group at the identity element. The tangent space is defined as $\{\gamma'(1)\}$ where $\gamma(t)$ such that $\gamma(1) = \mathbf{I}$



| Special Orthogonal Group $\mathfrak{so}(3)$

$$\mathfrak{so}(3) = \left\{ \boldsymbol{\Phi} = \boldsymbol{\phi}^\wedge = \begin{bmatrix} 0 & -\phi_3 & \phi_2 \\ \phi_3 & 0 & -\phi_1 \\ -\phi_2 & \phi_1 & 0 \end{bmatrix} \mid \boldsymbol{\phi} \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in $\mathfrak{so}(3)$ leads to an element in $\mathbf{SO}(3)$: $\exp(\boldsymbol{\Phi}) \in \mathbf{SO}(3)$.

$$\boldsymbol{\Phi} = \boldsymbol{\phi}^\wedge \Rightarrow \boldsymbol{\phi} = \boldsymbol{\Phi}^\vee$$

| Special Euclidian Group $\mathfrak{se}(3)$

$$\mathfrak{se}(3) = \left\{ \boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge = \begin{bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\phi} \end{bmatrix}^\wedge = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho} \\ \mathbf{0}^T & 0 \end{bmatrix} \mid \boldsymbol{\rho}, \boldsymbol{\phi} \in \mathbb{R}^3 \right\}$$

Taking the exponential of an element in $\mathfrak{se}(3)$ leads to an element in $\mathbf{SE}(3)$: $\exp(\boldsymbol{\Xi}) \in \mathbf{SE}(3)$.

$$\boldsymbol{\Xi} = \boldsymbol{\xi}^\wedge \Rightarrow \boldsymbol{\xi} = \boldsymbol{\Xi}^\vee$$

3 | Exponential Map

For every square matrix $\mathbf{A}$, we have

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n$$

$$\ln(\mathbf{A}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (\mathbf{A} - \mathbf{I})^n$$

| Baker-Campbell-Hausdorff (BCH) formula

Most of the time, $\exp(\mathbf{A} + \mathbf{B}) \neq \exp(\mathbf{A}) \exp(\mathbf{B})$

$$\ln(\boldsymbol{C}_1 \boldsymbol{C}_2)^\vee = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \frac{1}{2} \boldsymbol{\phi}_1^\wedge \boldsymbol{\phi}_2 + \cdots$$

$$\approx \begin{cases} \boldsymbol{J}(\boldsymbol{\phi}_2)^{-1} \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_1 \text{ small} \\ \boldsymbol{\phi}_1 \boldsymbol{J}(-\boldsymbol{\phi}_1)^{-1} \boldsymbol{\phi}_2 & \text{if } \boldsymbol{\phi}_2 \text{ small} \end{cases}$$

$$\ln(\boldsymbol{T}_1 \boldsymbol{T}_2)^\vee = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \frac{1}{2} \boldsymbol{\xi}_1^\wedge \boldsymbol{\xi}_2 + \cdots$$

$$\approx \begin{cases} \boldsymbol{\mathcal{J}}(\boldsymbol{\xi}_2)^{-1} \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_1 \text{ small} \\ \boldsymbol{\xi}_1 \boldsymbol{\mathcal{J}}(-\boldsymbol{\xi}_1)^{-1} \boldsymbol{\xi}_2 & \text{if } \boldsymbol{\xi}_2 \text{ small} \end{cases}$$

4 | Adjoints

The adjoint of an element of $\mathfrak{se}(3)$ is

$$\mathrm{ad}(\boldsymbol{\Xi}) = \mathrm{ad}(\boldsymbol{\xi}^\wedge) = \begin{bmatrix} \boldsymbol{\phi}^\wedge & \boldsymbol{\rho}^\wedge \\ \mathbf{0} & \boldsymbol{\phi}^\wedge \end{bmatrix} = \boldsymbol{\xi}^\curlywedge$$

The adjoint of an element of $\mathbf{SE}(3)$ is

$$\boldsymbol{\mathcal{T}} = \mathrm{Ad}(\boldsymbol{T}) = \begin{bmatrix} \boldsymbol{C} & \mathbf{r} \\ \mathbf{0} & \boldsymbol{C} \end{bmatrix}$$

5 | Relation between spaces

$$\boldsymbol{\phi} \in \mathfrak{so}(3) \xrightarrow{\exp} \boldsymbol{C} \in \mathbf{SO}(3)$$

$$\begin{tikzcd} \boldsymbol{\xi}^\wedge \in \mathfrak{se}(3) \arrow[r, "\exp"] \arrow[d, "\mathrm{ad}"] & \boldsymbol{T} \in \mathbf{SE}(3) \arrow[d, "\mathrm{Ad}"] \\ \boldsymbol{\xi}^\curlywedge \in \mathrm{ad}(\mathfrak{se}(3)) \arrow[r, "\exp"] & \boldsymbol{\mathcal{T}} \in \mathrm{Ad}(\mathbf{SE}(3)) \end{tikzcd}$$

6 | (left) Jacobians

$$\boldsymbol{J}(\boldsymbol{\phi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\boldsymbol{\phi}^\wedge\right)^n \quad \boldsymbol{\mathcal{J}}(\boldsymbol{\xi}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\boldsymbol{\xi}^\curlywedge\right)^n$$

The Jacobians have singularities (i.e., the inverse does not exist) at $|\boldsymbol{\phi}| = 2\pi m$ with m a nonzero integer.

7 | Interpolation

$$\boldsymbol{C} = (\boldsymbol{C}_2 \boldsymbol{C}_1^T)^\alpha \boldsymbol{C}_1 \qquad \boldsymbol{T} = (\boldsymbol{T}_2 \boldsymbol{T}_1^{-1})^\alpha \boldsymbol{T}_1$$

with $\alpha \in [0, 1]$

8 | Perturb Rotations and Poses

The left perturbation avoids the singularities as we stay near the identity:

$$\boldsymbol{C} = \exp(\boldsymbol{\epsilon}^\wedge) \bar{\boldsymbol{C}} \qquad \boldsymbol{T} = \exp(\boldsymbol{\varepsilon}^\wedge) \bar{\boldsymbol{T}}$$

with $\boldsymbol{\epsilon} \in \mathbb{R}^3 \sim \mathcal{N}(\mathbf{0}, \Sigma_\epsilon)$, $\boldsymbol{\varepsilon} \in \mathbb{R}^6 \sim \mathcal{N}(\mathbf{0}, \Sigma_\varepsilon)$

8.1 Example: Compounding poses

We want to find the mean and covariance of $\boldsymbol{T} = \boldsymbol{T}_1 \boldsymbol{T}_2$ where $T_1 = \exp(\varepsilon_1) \bar{\boldsymbol{T}}_1, T_2 = \exp(\varepsilon_2) \bar{\boldsymbol{T}}_2$ and the errors $\varepsilon_1, \varepsilon_2$ have zero mean and covariances Σ_1, Σ_2.

$$\exp(\boldsymbol{\varepsilon}^\wedge) \bar{\boldsymbol{T}} = \exp(\varepsilon_1) \bar{\boldsymbol{T}}_1 \exp(\varepsilon_2) \bar{\boldsymbol{T}}_2$$

$$\Leftrightarrow \exp(\boldsymbol{\varepsilon}^\wedge) = \exp(\varepsilon_1) \exp(\bar{\boldsymbol{\mathcal{T}}}_1 \varepsilon_2^\wedge) \qquad \bar{\boldsymbol{\mathcal{T}}}_1 = \mathrm{Ad}(\bar{\boldsymbol{T}}_1)$$

$$\Leftrightarrow \boldsymbol{\varepsilon} = \varepsilon_1 + \varepsilon_2' + \frac{1}{2} \varepsilon_1^\curlywedge \varepsilon_2' + \frac{1}{12} \varepsilon_1^\curlywedge \varepsilon_1^\curlywedge \varepsilon_2' + \cdots \qquad \varepsilon_2' = \bar{\boldsymbol{\mathcal{T}}}_1 \varepsilon_2$$

It is then possible to find $\mathbb{E}[\varepsilon]$ and $\mathbb{E}[\varepsilon \varepsilon^T]$