Introduction to Smooth Manifolds (Lee) – Exercises

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July 25, 2025

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.6

Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.

Letting $\phi : \mathbb{RP}^{\ltimes} \to S^{n-1}$ such that $\phi([x]) = \frac{x}{|x|}$ is a homeomorphism with continuous inverse, since we can consider $\phi \circ \pi : \mathbb{R}^n \setminus \{0\} \to S^n$, which is a composition of elementary continuous functions and thus is continuous. Then by characteristic property of quotient maps, ϕ is continuous. But the topology on S^{n-1} is the usual one, which can be thought of as induced from a metric, so Hausdorff. Since S^{n-1} homeomorphic to $\mathbb{RP}^n$, $\mathbb{RP}^n$ is Hausdorff. Note that S^{n-1} should also be second-countable using the subspace (topology) construction from base of rational radii + rational center balls in $\mathbb{R}^{\ltimes}$, so by homeomorphism $\mathbb{RP}^n$ is also second-countable. $\square$

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The restriction of π, the quotient mapping for $\mathbb{RP}^n$, to S^n, is surjective, since drawing a line from arbitrary point in $\mathbb{R}^{n+1}$ to 0 always intersects with S^n. Thus π is a homeomorphism, and since S^n is compact, so is $\mathbb{RP}^n$. $\square$

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Suppose $\mathcal{X}$ is a locally finite collection of subsets of a topological space M.

a)

To prove that $\overline{\mathcal{X}} = \{\overline{X} : X \in \mathcal{X}\}$ is also locally finite, we need to show for arbitrary $a \in M$, there exists a neighborhood of a which intersects finitely many members of $\overline{\mathcal{X}}$. Note that $\mathcal{X}$ is locally finite, so let the neighborhood that works for $\mathcal{X}$ be called $\mathcal{O}_a$.

We ask if we have open $\mathcal{O}$ disjoint from set X, can $\mathcal{O} \cap \overline{X}$ be non-empty? Take a (limit) point $q \in \mathcal{O} \cap \overline{X}$; by definition limit point, any open neighborhood of q has non-empty intersection with X. But since $\mathcal{O} \cap X = \varnothing$, this means we cannot find an open neighborhood of $q \in \mathcal{O}$ which is contained in $\mathcal{O}$, so $\mathcal{O}$ not empty, a contradiction.

Thus going from $\mathcal{X}$ to $\overline{\mathcal{X}}$ should not change which members intersect with $\mathcal{O}_a$. Thus $\overline{\mathcal{X}}$ is locally finite. $\square$

b)

Note that the closure operation is monotonic by Kuratowski closure axioms. Thus for arbitrary $X \in \mathcal{X}$, we have that $\overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$ So for arbitrary X, if we have $a \in \overline{X}$, then $a \in \overline{\bigcup_{X \in \mathcal{X}} X}$. Thus we have that

$$\bigcup_{X \in \mathcal{X}} \overline{X} \subset \overline{\bigcup_{X \in \mathcal{X}} X}$$

We finish the proof by showing that $\bigcup_{X \in \mathcal{X}} \overline{X}$ is closed. Suppose we have a limit point a of $\bigcup_{X \in \mathcal{X}} \overline{X}$. By $\overline{\mathcal{X}}$ locally finite, there exists neighborhood $\mathcal{O}_a$ which intersects finitely many $\overline{X'} \in \overline{\mathcal{X}}$.
Suppose that none of the $\overline{X'}$ had non-empty intersection with arbitrary neighborhoods of a. Then, for each $\overline{X'}$, choose a neighborhood of a which is disjoint from $\overline{X'}$, and take the intersection of all chosen neighborhoods (finitely many, so intersection is still open). The resulting intersection, call it B, is a neighborhood of a which intersects none of the $\overline{X'}$. Taking $B \cap \mathcal{O}_a$ nets us a neighborhood of a which is disjoint from all members of $\overline{\mathcal{X}}$, meaning that a is not a limit point of $\bigcup_{X \in \mathcal{X}} \overline{X}$, a contradiction.

Thus one of the $\overline{X'}$, call it $\overline{X''}$, has non-empty intersection with arbitrary neighborhoods of a, so a is a limit point of $\overline{X''}$. Since $\overline{X''}$ is closed, $a \in \overline{X''}$, so $a \in \bigcup_{X\in\mathcal{X}} \overline{X}$. Thus since a arbitrary limit point, $\bigcup_{X\in\mathcal{X}} \overline{X}$ is closed. Since closure of a set A is the least closed set containing A, we have that

$$\bigcup_{X\in\mathcal{X}} \overline{X} \supset \overline{\bigcup_{X\in\mathcal{X}} X}$$

□

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Prove that two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Note that the smooth structure determined by an atlas $\mathcal{A}$ is the maximal smooth atlas for M which contains $\mathcal{A}$. Let that for M, we have smooth atlases $\mathcal{A}, \mathcal{A}'$, and corresponding smooth structures $\aleph, \aleph'$. Suppose $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas. This means all charts in $\mathcal{A}'$ are smoothly compatible with all charts in $\mathcal{A}$. Thus by definition of maximal smooth atlas, $\aleph$ contains all charts in $\mathcal{A}'$, and $\aleph'$ contains all charts in $\mathcal{A}$. Thus by maximality, $\aleph' \supset \aleph$ and $\aleph \supset \aleph'$. Thus $\aleph = \aleph'$.

Now suppose that two smooth atlases for M, call them $\mathcal{A}, \mathcal{A}'$, determine the same smooth structure. Then all charts in $\mathcal{A}$ are smoothly compatible with all charts in $\mathcal{A}'$, so by definition of union, all charts of $\mathcal{A} \cup \mathcal{A}'$ charts are smoothly compatible, and $\mathcal{A} \cup \mathcal{A}'$ still covers M. Thus $\mathcal{A} \cup \mathcal{A}'$ is a smooth atlas by definition.

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Prove that every smooth manifold M has a countable basis of regular coordinate balls.

In this paragraph we tackle the idea of centering our ball images around 0. We know from lemma 1.10 that every smooth manifold has a countable basis of precompact coordinate balls $\mathcal{B}$, where basis sets are $B \in \mathcal{B}$. In order to get regularity, we can modify the smooth coordinate maps φ corresponding to the B by translating; if $\varphi(B) = B_r(a)$, then we transform $\varphi \to \varphi - a = \varphi'$. Thus $\varphi'(B) = B_r(0)$. We can then take $r^* < r$ and consider $\varphi'^{-1}(B_{r^*}(0))$ to be our regular coordinate ball in the new topological basis we are constructing.

Now we tackle the problem of making $\varphi(\overline{B}) = \overline{B_r(0)}$. Note that when constructing the countable basis of precompact coordinate balls, we are free to obtain pairs of balls such that one is a dilation of the other (by density of rationals in reals, and $2\mathbb{N}$ being countable). Thus by construction we can make it such that any $B \in \mathcal{B}$ has a partner B^* precompact coordinate ball which is contained in B and has the same center. This is to say that we get up to the point where $\varphi'(B) = B_r(0)$ in our above construction, now we will also have that $\varphi'(B^*) = B_{r^*}(0)$, and by precompactness, $\varphi'(\overline{B^*})$ is compact since φ is a homeomorphism. Since compact subset of a Hausdorff space ($\mathbb{R}^n$) is closed, $\varphi'(\overline{B^*})$ is closed, and it contains $\varphi'(B^*)$, so by definition of closure we have that

$$\varphi'(\overline{B^*}) \supset \overline{\varphi'(B^*)} = \overline{B_{r^*}(0)}$$

Note that by definition of continuity we have

$$\begin{aligned} &\varphi'(B^*) \subset \overline{\varphi'(B^*)} \\ \Longrightarrow\ &B^* \subset \varphi'^{-1}(\overline{\varphi'(B^*)}) \\ \Longrightarrow\ &\varphi'(\overline{B^*}) \subset \overline{\varphi'(B^*)} \end{aligned}$$

so subset in both directions gives us equality.

We should also make sure that how we get the $\varphi'^{-1}(B_{r^*}(0))$ actually form a basis. Well, every member of our original basis has some $\varphi'^{-1}(B_{r^*}(0))$ contained in it by construction, so we just need to make sure that our collection covers M. Well, actually what we can do is instead of take a single $r^* < r$ as per the first above construction, we can take all sets of form $\varphi'^{-1}(B_{r^*}(0))$ such that $r^* < r$ and $r^* \in \mathbb{Q}$ to be the sets in our basis. Note our collection is still countable. Since $B_r(0)$ is an open set, now the collection of all $\varphi'^{-1}(B_{r^*}(0))$ will cover M and we have gotten a basis adequate for our solution.

□

Lemma 1.35 (Smooth Manifold Chart Lemma) Comments on Proof:

All sets of form $\varphi^{-1}(V)$ meaning any time we can obtain such a set (taking a coordinate map, and taking a valid open set in the codomain).

Condition (iii) being that the transition map is smooth implies that the transition map preimage of an open set is open in the transition map domain, which is $\varphi_\alpha(U_\alpha \cap U_\beta)$.

Condition (ii) implies open in $\mathbb{R}^n$ by construction of subspace topology on $\varphi_\alpha(U_\alpha \cap U_\beta)$, since finite intersection of open sets is open (we are promoting an open set on $\varphi_\alpha(U_\alpha \cap U_\beta)$ topology to the superset topology on $\mathbb{R}^n$).

The equation

$$\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W) = \varphi_\alpha^{-1}(V \cap (\varphi_\beta \circ \varphi_\alpha^{-1})^{-1}(W))$$

shows that $\varphi_\alpha^{-1}(V) \cap \varphi_\beta^{-1}(W)$ is in the form of a basis set as specified in the Lemma.

Condition (v) implies Hausdorff since when $p, q \in M$ are in the same U_α, image from respective charts is in $\mathbb{R}^n$, which is hausdorff, and φ_α is a bijection.
When $p, q \in M$ are in disjoint U_α, U_β, suppose that p, q are a counterexample to M being Hausdorff, then U_α, U_β cannot be disjoint, a contradiction.

Example 1.36 (on Grassmann Manifolds) Comments:

S and P are isomorphic by errata: S is assumed to be same dimension (k) as P.
To see that S is the graph of X, consider

$$X = (\pi_Q|_S) \circ (\pi_P|_S)^{-1}$$

A graph can be thought of as a collection of tuples $\{v + Xv : v \in P\}$ in our case, since $P \bigoplus Q = V$. Consider arbitrary $v \in P$. Note that $(\pi_P|_S)^{-1}$ maps v to $s \in S$ which projects onto v, then $(\pi_Q|_S)$ maps this collection onto the corresponding projections in Q. What this means is that said projection in Q plus v recoups our specific s. But the projection in Q is Xv, so we have $v + Xv = s$. Since $v \in P$ arbitrary, and P isomorphic to S, $\{v + Xv : v \in P\} = S$.

In proving condition (iii), instantiating $X' = \varphi' \circ \varphi^{-1}(X)$ implies that $X' = (\pi_{Q'}|_S) \circ (\pi_{P'}|_S)^{-1}$ since the φ are the bijective correspondence between subspaces which intersect a subspace Q trivially and $L(P, Q)$ (the vector space of linear maps from P to Q). So the implication follows just by construction of φ'.

.39

Let M be a topological n-manifold with boundary.

a)

Int M is an open subset of M and a topological n-manifold without boundary.

By definition of Int M, points of Int M are contained in domains of charts which homeomorph the chart to an open subset of $\mathbb{R}^n$, so the domains themselves are open. Thus since every point in Int M has an open neighborhood which is contained in Int M, Int M is an open subset of M. Note that by this discussion, all point in Int M are interior points of M, so by the theorem on topological invariance of the boundary, Int M has no boundary points. Equivalently, we could have just recognized that by our discussion, every point has a neighborhood homeomorphic to an open neighborhood of $\mathbb{R}^n$ by existence of our charts, and Hausdorff and second-countability come from M being a manifold with boundary.

b)

∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

Note that any point of M which is not a boundary point is an interior point, so $(\partial M)^C =$ Int M which is open. So ∂M is closed. To show that ∂M is a topological $(n-1)$-manifold without boundary, we will show that the charts from M lend themselves to charts on ∂M which map to open sets on $\mathbb{R}^{n-1}$. By definition of ∂M, every $p \in \partial M$ lies in the domain $U \subset M$ of some boundary chart $\varphi : U \to \mathcal{O}_{\mathbb{H}^n}$ (open set in the subspace topology on $\mathbb{H}^n$) for which $\varphi(p) \in \partial\mathbb{H}^n$. We claim that the restriction $\varphi|_{\partial M} : \partial M \cap U \to \varphi|_{\partial M}(\partial M \cap U)$ is our requisite homeomorphism onto an open subset of $\mathbb{R}^{n-1}$. Note that if a chart maps a boundary point to Int $\mathbb{H}^n$, then it is actually an interior point, a contradiction. Thus $\varphi|_{\partial M}(\partial M \cap U) \subset \partial\mathbb{H}^n \cap \varphi(U)$. To show subset in the other direction, suppose we had arbitrary $y \in \partial\mathbb{H}^n \cap \varphi(U)$ which did not belong to $\varphi|_{\partial M}(\partial M \cap U)$. But $\varphi^{-1}(y)$ is a boundary point of M by construction of y, and it also belongs to U. Thus $\varphi|_{\partial M}(\varphi^{-1}(y)) = y \in \varphi|_{\partial M}(\partial M \cap U)$, so subset in both directions gives us that $\varphi|_{\partial M}(\partial M \cap U) = \partial\mathbb{H}^n \cap \varphi(U)$.

Note that $\partial\mathbb{H}^n$ is homeomorphic to $\mathbb{R}^{n-1}$, so $\partial\mathbb{H}^n \cap \varphi(U)$ is an open subset of (a space homeomorphic to) $\mathbb{R}^{n-1}$. Left to prove that $\varphi|_{\partial M}$ is a homeomorphism. Note that φ is continuous, so its restriction to ∂M is continuous. Left to show that $\varphi|_{\partial M}^{-1}$ is continuous. We proceed by showing $\varphi|_{\partial M}$ maps open sets to open sets. Note open sets in the domain are open sets in the subspace topology from U. Let arbitrary open set in the domain, of form $\partial M \cap \mathcal{O}_U$, and consider $\varphi|_{\partial M}(\partial M \cap \mathcal{O}_U)$. Note this is equal to $\varphi(\partial M \cap \mathcal{O}_U)$, and by a similar argument to the above discussion, we have that $\varphi(\partial M \cap \mathcal{O}_U) = \partial\mathbb{H}^n \cap \varphi(\mathcal{O}_U)$. Since φ is a homeomorphism, $\varphi(\mathcal{O}_U)$ is an open set, so indeed we have that arbitrary open set $\partial M \cap \mathcal{O}_U$ (in subspace topology on $\partial M \cap U$) maps to open set $\partial \mathbb{H}^n \cap \varphi(\mathcal{O}_U)$ (open in subspace topology on $\mathbb{H}^n \cap \varphi(U)$) under $\varphi|_{\partial M}$. Thus $\varphi|_{\partial M}$ is a valid coordinate mapping, and we have our requisite smooth structure (our topological $(n-1)$ manifold without boundary).

c)

$$M \text{ is a topological manifold if and only if } \partial M = \varnothing.$$

If $\partial M = \varnothing$, then all points of M are interior points, so by definition M is a topological manifold.

If M is a topological manifold, all points of M are interior points by definition, so by theorem on topological invariance of the boundary, $\partial M = \varnothing$.

d)

$$\text{If } n = 0, \text{ then } \partial M = \varnothing \text{ and } M \text{ is a 0-manifold.}$$

If $n = 0$, note that $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so Int $\mathbb{H}^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \varnothing$. Note that M cannot have boundary points since $\partial \mathbb{H}^0 = \varnothing$, and having a chart map a point to nothing doesn't make sense. Thus by theorem on topological invariance of the boundary, all points in M are interior points, so M is a 0-manifold by definition.

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Let M be a topological manifold with boundary.

Definition: A subset of X is said to be **precompact in** X if its closure in X is compact.

Definition: Given a chart (U, φ), if $\varphi(U)$ is some open ball in $\mathbb{R}^n$, then U is called a **coordinate ball**.

Definition: A boundary chart whose image is a set of form $B_r(x) \cap \mathbb{H}^n$ for some $x \in \partial \mathbb{H}^n$ and $r > 0$ is called a **coordinate half-ball**.

a)

$$M \text{ has a countable basis of precompact coordinate balls and half-balls.}$$

By second countability of M and since the charts (U, φ) on M cover M, there is a countable subcover of those charts which cover M. Note that if we take rational open balls which are contained in $\varphi(U)$, these form a basis for the topology of $\varphi(U) \subset \mathbb{R}^n$, or $\varphi(U) \subset \mathbb{H}^n$ for boundary charts. So the preimage of these rational open balls using φ^{-1} are coordinate balls, and they are precompact since φ is a homeomorphism which will send closed sets to closed sets, and closed and bounded subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$ (locally compact) are compact.

b)

$$M \text{ is locally compact.}$$

M has a basis of precompact coordinate balls, so any point of M will be contained in such a precompact coordinate ball. By construction, this is an open neighborhood whose closure is compact, so we have shown that any point of M has a neighborhood (precompact coordinate ball) contained in a compact subset of M (closure of said ball).

c)

Definition: A collection $\mathcal{X}$ of subsets of M is said to be **locally finite** if every point of M admits a neighborhood which intersects with at most finitely many members of $\mathcal{X}$.

Definition: Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ of M is said to be a **refinement** of $\mathcal{U}$ if for any $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $\subset U$.

Definition: We say that M is **paracompact** if every open cover of M admits an open, locally finite refinement.

$$M \text{ is paracompact.}$$

Earlier we have shown that M is locally compact, so by **A.60**, M admits an exhaustion by compact sets, and the proof for paracompactness is identical to that of the case where M has no boundary.

d)

Definition: A space X is **locally path-connected at** $x \in X$ if for any open neighborhood of x, there is an open sub-neighborhood of x which is path-connected. X is **locally path-connected** if it is locally path-connected at every point.

M is locally path-connected.

We prove this part by showing that M has a basis of path-connected open sets. Note that $\mathbb{R}^n$ and $\mathbb{H}^n$ both have such a basis, and consider the preimage of these basis sets under charts. Charts are homeomorphisms so the preimages are also path connected. The preimage of these basis sets will cover M, and since intersection of two path-connected sets is path-connected, the collection of all open path-connected subsets of M forms a basis for M.

e)

M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.

By second countability, M has countably many components. To show that components of M are open, note that the definition of a component is a maximal connected subset of M. Let arbitrary component $V \subset M$. Since M is locally path-connected, if we consider arbitrary $x \in V$, x has an open path-connected neighborhood $\mathcal{O}_x$. Note that $\mathcal{O}_x \subset M$ by maximality of M, so since $x \in V$, V is open.

Note that each V is connected by definition component, and is a topological manifold with boundary as a subset of M (by construction of open sub-manifold).

f)

The fundamental group of M is countable.

Looking at the proof for the case where M is a topological manifold without boundary (**Prop 1.16**), it seems like coordinate half-balls and coordinate balls have the same requisite properties to prove this part of the theorem.

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Show that every smooth manifold with boundary (M) has a countable basis consisting of regular coordinate balls and half-balls.

Our proof should mirror that of **1.20**. Note that if any kind of coordinate map is smooth on U, it will be smooth on a subset of U by definition. So actually our proof is extremely similar to that in **1.20**, and I will not elaborate.

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Show that the smooth manifold chart lemma (**Lemma 1.35**) holds with "$\mathbb{R}^n$" replaced by "$\mathbb{R}^n$ or $\mathbb{H}^n$" and "smooth manifold" replaced by "smooth manifold with boundary."

For each $\varphi_\alpha(U_\alpha)$, we can consider the open ball subsets and thus their preimages under φ_α, taking this collection to be a basis for a topology on M. Note that we have to consider the boundary coordinate maps as a special case. Suppose that $\varphi_\alpha(U_\alpha)$ maps to an open subset of $\mathbb{H}^n$ with nonempty intersection with $\partial\mathbb{H}^n$. Then we can consider subsets of $\varphi_\alpha(U_\alpha)$ of form such that they are an open ball in $\mathbb{R}^n$ intersect with $\mathbb{H}^n$ (so taking the subspace topology). We use the above method to get basis elements from such open sets in the image of the boundary coordinate maps. Note that taking the topology in this way makes the φ_α homeomorphisms, which makes the $(U_\alpha, \varphi_\alpha)$ valid charts.

We need to show that our proposed charts for M are smoothly compatible (forming a smooth atlas, as the U_α already cover M). Two coordinate maps φ, ϕ are smoothly compatible if either their domains are disjoint or the transition map $\varphi \circ \phi^{-1}$ is a diffeomorphism. But we have from the assumptions of the smooth manifold chart lemma that all the transition maps are smooth, so our collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ forms a smooth atlas.

Note that we do not need to show that the charts which we have on hand, the $(U_\alpha, \varphi_\alpha)$, define a maximal smooth atlas. Instead, per the wording of the smooth manifold chart lemma, we only need to show that there exists a unique smooth structure on M for which each of our $(U_\alpha, \varphi_\alpha)$ is a smooth chart. But if we just take the maximal smooth atlas on M which supersets our previously defined $\{(U_\alpha, \varphi_\alpha)\}$ (we know such a smooth atlas must exist since $\{(U_\alpha, \varphi_\alpha)\}$ is itself a smooth atlas), we are done by maximality. Hausdorff and second-countability follow from the other assumptions of smooth manifold chart lemma.

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Suppose M is a smooth n-manifold with boundary and U is an open subset of M. Prove the following statements:

a)

U is a topological n-manifold with boundary, and the atlas consisting of all smooth charts (V, φ) for M such that $V \subset U$ defines a smooth structure on U. With this topology and smooth structure, U is called an **open submanifold with boundary**.

We prove this part by appealing to the smooth manifold chart lemma. See textbook for referenced roman numerals.

Note that an open subset of M should be second countable and Hausdorff under subset topology. Let $\mathcal{C}$ be the collection of smooth charts on M whose domains are subset of U. Coordinate maps corresponding to members of $\mathcal{C}$ should still be smooth since their domains are subset of U. Since $\mathcal{C}$ is a subset of the smooth structure on M, all members of $\mathcal{C}$ should be smoothly compatible. Furthermore, a smooth map on V such that $V \cap U^C \neq \varnothing$ can be restricted to a smooth map on $V \cap U$ by taking subspace topology on $V \cap U$, since the map was smooth on a superset of $V \cap U$. Furthermore, since the smooth atlas of M covers M, $\mathcal{C}$ must cover U by construction (countably). There's more I could mention, but note that most conditions of the smooth manifold chart lemma are inherited just based on taking the subspace topology from M onto U.

b)

If $U \subset \text{Int } M$, then U is actually a smooth manifold (without boundary); in this case we call it an **open submanifold of** M.

Note in this case, the charts of our smooth structure on U must correspond to interior coordinate maps, since the domains of our coordinate maps are contained in Int M, and boundary points and interior points of manifold U are mutually exclusive.

c)

Int M is an open submanifold of M (without boundary).

We need to show that Int M is an open subset of M. Note that Int M consists of all points in M which are contained in an interior chart's domain. But since an interior chart's domain is itself an open set, this means that all points of Int M are have a cooresponding open neighborhood which is contained in Int M. $\square$

Theorem 1.46 (Smooth Invariance of the Boundary) Comments:

We can assume that $B \subset \tau^{-1}(W)$ for sufficiently shrunk B around x_0 since τ is a homeomorphism, so $\tau^{-1}(W)$ is open. So by niceness of $\mathbb{R}$ (I dunno what property), shrinking B (a euclidean ball neighborhood of $x_0 \in \tau^{-1}(W)$ should get a sufficiently small basis set in $\tau^{-1}(W)$.

Nonsingularity of $D\tau(x)$ follows from composition of square matrices preserving determinant.

τ being an open map follows from derivative being invertible (nonsingular), smooth, and defined on an open set. Apparently there's some theorem that restricts smooth functions with this property to diffeomorphisms on smaller sets.

Chapter 1 Problems

1-4

Let M be a topological manifold, and let $\mathcal{U}$ be an open cover of M.

a)

Assuming that each set in $\mathcal{U}$ intersects only finitely many others, show that $\mathcal{U}$ is locally finite.

We want to show that every point of M admits a neighborhood which intersects with at most finitely members of $\mathcal{U}$.

Let arbitrary $x \in M$. Since $\mathcal{U}$ covers M, there exists $U \in \mathcal{U}$ such that $x \in U$. By assumption U intersects with only finitely many other sets in $\mathcal{U}$. $\square$

b)

Give an example to show that the converse to **a)** may be false.

Take product space $S^1 \times \mathbb{R}$. Take a finite open cover of S^1, call it $\mathcal{O}$, and consider the set $U = S^1 \times (0,1)$. U is open by construction of the subbase for the product topology. Note that if we consider the collection $\mathcal{U}$ containing U, and all sets of form $O \times \mathbb{R}$ such that $O \in \mathcal{O}$, $\mathcal{X}$ is an open cover of $S^1 \times \mathbb{R}$, and is locally finite since any $x \in S^1$ belongs to at most finitely many $O \in \mathcal{O}$. But U has infinite non-trivial intersections with members of $\mathcal{O}$. □

c)

Now assume that the sets in $\mathcal{U}$ are precompact in M; and prove the converse: if $\mathcal{U}$ is locally finite, then each set in $\mathcal{U}$ intersects only finitely many others.

Try by direct proof. Take arbitrary $U^* \in \mathcal{U}$. Now take $\overline{U^*}$, which is compact by assumption that all $U^* \in \mathcal{U}$ are precompact. Since $\mathcal{U}$ is locally finite, for every $x \in \overline{U^*}$, there exists a neighborhood O_x which intersects at most finitely many members of $\mathcal{U}$.Getting the collection of all O_x for all $x \in \overline{U^*}$ gets us an open cover of $\overline{U^*}$, call it $\mathcal{O}_x$, so take the corresponding finite subcover by compactness. The union of all elements in the subcover is an open superset of U^* which intersects at most finite members of $\mathcal{U}$, since the union of elements in the subcover of $\mathcal{O}_x$ is a finite union, and all $O_x \in \mathcal{O}_x$ intersect at most finitely many $U \in \mathcal{U}$.

1-5

Suppose M is a locally Euclidean Hausdorff space. Show that M is second-countable if and only if it is paracompact and has countably many connected components. [Hint: assuming M is paracompact, show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.]

Assuming that M is second-countable, it must have countably many connected components, or else for each connected component we can find a basis element contained in it, and thus any basis we have of M is uncountable. To show paracompactness of M, we need to show any open cover of M has a locally finite refinement.
Is there some easy way to distill any open cover to a locally finite refinement? I want such a distillation to be sparse, but many.
For every point $x \in M$, take its euclidean neighborhood by M being locally Euclidean. Then, by properties of $\mathbb{R}^n$, there exists a precompact coordinate ball which is a neighborhood of x, call it $\psi_x^{-1}(B_x)$. Taking all $\psi_x^{-1}(B_x)$ for $x \in M$ gives us an open cover of M. Then, suppose we have an arbitrary open cover of M. By compactness of $\overline{\psi_x^{-1}(B_x)}$, take finite subcover of $\overline{\psi_x^{-1}(B_x)}$, and for each member of the subcover, take its intersection with $\psi_x^{-1}(B_x)$. The resulting finite collection of intersections cover $\psi_x^{-1}(B_x)$.
When our arbitrary open cover of M is such that no point $x \in M$ is contained in infinitely many members of the open cover, then the collection generated by repeating the construction of the finite collection of intersections which cover $\psi_x^{-1}(B_x)$ for all $x \in M$... is still not necessarily locally finite, since for all $x \in M$ means we have so many x nearby which to take $\psi_x^{-1}(B_x)$ of...
What if we take $\psi_x^{-1}(B_x)$, and then we choose $x' \notin \psi_x^{-1}(B_x)$ and proceed from there?
Note that the open cover constituting of the $\psi_x^{-1}(B_x)$ for all $x \in M$ is countable. Is there a way to sparse the countable open cover? Can we index the countable open cover $\mathcal{C} = \{O_i\}$ and then consider the sets U_i such that

$$U_1 = O_1$$
$$U_{i+1} = O_{i+1} \setminus \bigcup_{k=1}^{i} \overline{O_k}$$

Is this still an open cover of M?

Trying from the other direction. Assume that M is paracompact. We know that it is locally Euclidean and Hausdorff, and we want to prove that M is second countable. The hint says to show that each component of M has a locally finite cover by precompact coordinate domains, and extract from this a countable subcover.

Since M is locally Euclidean, for each $x \in M$ we can instantiate a neighborhood O_x which is homeomorphic (under, say ψ_x) to an open subset of $\mathbb{R}^n$. Then by properties of $\mathbb{R}^n$, we can find (using basis argument) an epsilon ball $B_\epsilon(x)$ centered at x which is contained in $\psi_x(O_x)$. We can then cover M by the collection of all such $\psi_x^{-1}(B_\epsilon(x))$, and since M is paracompact, we know that $\{\psi_x^{-1}(B_\epsilon(x))\}$ has a locally finite refinement, call it $\mathcal{R}$. Now, the sets in $\mathcal{R}$ are by definition open subsets of sets in $\{\psi_x^{-1}(B_\epsilon(x))\}$, and since all O_x are homeomorphic to $\mathbb{R}^n$, then we will always have a homeomorphism which sends any set in $\mathcal{R}$ to an open subset of $\mathbb{R}^n$. Note that epsilon balls are precompact, so the $\psi_x^{-1}(B_\epsilon(x))$ are all precompact. By definition their closures are compact. Since $\mathbb{R}^n$ is locally compact (?), the closures of $\psi_x^{-1}(B_\epsilon(x))$ are closed and bounded. By construction of refinement, closures of sets in $\mathcal{R}$ are all bounded. Thus closures of sets in $\mathcal{R}$ are all compact, so the sets in $\mathcal{R}$ are all precompact. We have thus shown that each component of M has a locally finite open cover by precompact coordinate domains. Now, how do we extract from this a countable subcover?

Every point $x \in M$ has a neighborhood which has finitely many intersection with precompact coordinate domains. We can further construct such that the neighborhood has finitely many intersection with their closures, which are compact. If we repeat this process for all $x \in M$, we will have open-covered M and necessarily (by subset) we will have open-covered the compact closures of coordinate domains. Wait. Coordinate domains imply we have charts from the coordinate domains to subsets of $\mathbb{R}^n$, which has a countable basis.

Note that our coordinate domains cover M. Imagine taking a countable basis for $\mathbb{R}^n$, then preimaging the basis elements any valid way by the coordinate charts (for which we may have uncountably many). Can we just choose the coordinate domains which intersect some basis element? I mean, the collection of coordinate domains is supposed to be locally finite, so for any $x \in M$, there exists a neighborhood O_x which is supposed to have finitely many coordinate domains which intersect O_x.

1-7

1-10

A

.27

a)

The definition of a quotient map $\pi : X \to Y$ is a continuous surjective map such that Y has the quotient topology determined by π.

To prove one half of the characteristic property, suppose that $F : Y \to B$ is continuous. We want to prove that $F \circ \pi : X \to B$ is continuous. But by definition of quotient topology, $U \subset Y$ open if and only if $\pi^{-1}(U) \subset X$ open. Thus preimage of open sets are open under inverse of $F \circ \pi$, so $F \circ \pi$ is continuous.

Now for the converse: suppose that $F \circ \pi : X \to B$ is continuous. We want to show that $F : Y \to B$ is continuous. We can proceed with a proof by contradiction. Suppose we had an open set $U \in B$ such that $F^{-1}(U)$ was not open. By definition of quotient topology, $\pi^{-1}(F^{-1}(U))$ cannot be open. But $\pi^{-1} \circ F^{-1} \equiv (F \circ \pi)^{-1}$, which we know is continuous. Thus a contradiction. $\square$

b)

We should prove two things: that if the characteristic property holds, then we have a quotient topology, and that the quotient topology is the unique *quotient topology* for which the Characteristic Property holds. These two properties combined will give us that the quotient topology is the unique *topology* for which the Characteristic Property holds.

Suppose there are two topologies on Y for which the characteristic property holds,

e)

Let $\pi : X \to Y$ be a quotient map. Let $U \subset X$ be a saturated open or closed set (with reference to $\pi : X \to Y$). Prove that $\pi|_U : U \to \pi(U)$ is a quotient map.

We need to show that $V \subset \pi(U)$ is open if and only if $\pi^{-1}(V)$ is open. Suppose that $V \subset \pi(U)$ is open. By def subspace topology, π being quotient map, we have that $\pi^{-1}(V)$ is open in the subspace topology on U.

Now suppose that $\pi^{-1}(V)$ is open. Using a similar argument we are done. (TLDR: we promote subspace topology on U to the original X topology, taking advantage of how we already have a quotient topology on Y determined by the same π. This leads to open set in Y whose form can be coaxed into an open set for the topology on $\pi(U)$.

.60 (Notes)

(Lemma) If X is a locally compact Hausdorff space, it has a basis of precompact open subsets.

Pf: Note the definition of locally compact is that for all $x \in X$, there exists an open neighborhood U and compact set K such that $x \in U \subset K$. Note that a compact subset of Hausdorff space is closed, so $\overline{U} \subset K$ by definition closure. Note that a closed subset of compact space K is compact, so $\overline{U}$ is compact, so U is precompact. Thus we are motivated to construct a basis for X out of the U.

Well, we know they exist, so let us instantiate the collection of all precompact open subsets of X, call it $\mathcal{U}$. Is the intersection of arbitrary $U_1, U_2 \in \mathcal{U}$ itself a precompact open subset of X? Note that by monotonicity of closure, $\overline{U_1 \cap U_2} \subset \overline{U_1}$, and $\overline{U_1}$ is compact, so $\overline{U_1 \cap U_2}$ is compact. By definition of X being locally compact, we know that $\mathcal{U}$ is an open cover of X, so by theorem on basis we have that $\mathcal{U}$ is a basis of precompact open subsets of X.

B

.22

Suppose V, W, X are finite-dimensional vector spaces, and $S : V \to W$ and $T : W \to X$ are linear maps. Prove the following statements.

a)

$$\text{rank}(S) \leq \dim V, \text{ with equality if and only if S is injective.}$$

By rank nullity, we have

$$\begin{aligned}\dim V &= \text{null } S + \text{rank } S \\ \implies \text{rank } S &= \dim V - \text{null } S\end{aligned}$$

Note if S is injective then its kernel is just the singleton set containing the zero vector, so nullity is zero, thereby giving equality.

d)

$$\text{rank } (T \circ S) \leq \text{rank } S, \text{ with equality if and only if } \operatorname{Im} S \cap \ker T = \{0\}.$$

By rank nullity,

$$\begin{aligned}&\dim V = \text{rank } (T \circ S) + \text{null } (T \circ S) \\ &\dim V = \text{rank } S + \text{null } S \\ &\implies \text{rank } (T \circ S) + \text{null } (T \circ S) = \text{rank } S + \text{null } S \\ &\implies \text{rank } (T \circ S) - (\text{null } (T \circ S) - \text{null } S) = \text{rank } S\end{aligned}$$

Note that homomorphisms always map zero to zero, thus null $(T \circ S) \geq$ null S, thus rank $(T \circ S) \leq$ rank S.

Note equality when null $(T \circ S) -$ null $S = 0$. This means that when T acts on image of S, the only vector sent to zero is the zero of S image. In other words, Im $S \cap \ker T = \{0\}$.