Topology in Condensed Matter Physics

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Goals

The goal of this writeup is to familiarize the reader with common terms used in the study of topological systems in condensed matter physics. Some bases I want to cover are on topology, what it means for a state to be topologically protected, and the idea of a topological invariant. Such is the pedagogical sequence of rigorously teaching the subject, but I think I should disregard pedagogy in order to properly motivate the ideas behind the mathematics.

What is a topological state?

What is meant by the words "topological state" is a state which is topologically protected. In order to explain what it means for a state to be topologically protected, I would like to make an analogy.

If you've ever read the comic series Calvin and Hobbes, there was a running gag of strips which involve Calvin looking at clouds in the sky and realizing that they look like dinosaurs, or ufos, or some other cool thing that a cloud wouldn't normally be.

Figure 1: That cloud kind of looks like Wario's nose.

In this figure, Mother Nature has conjured up some cloud that, at this region of space and point in time, looks like a dinosaur. But not for long. Part of the beauty of finding familiar visages in the clouds is that they are ephemeral.

Finding topological states can be a lot like finding shapes in the clouds. As an example, in modern physics research, people are studying skyrmions. The skyrmion has a mathematical formulation but for our purposes it is adequate to supply an image.

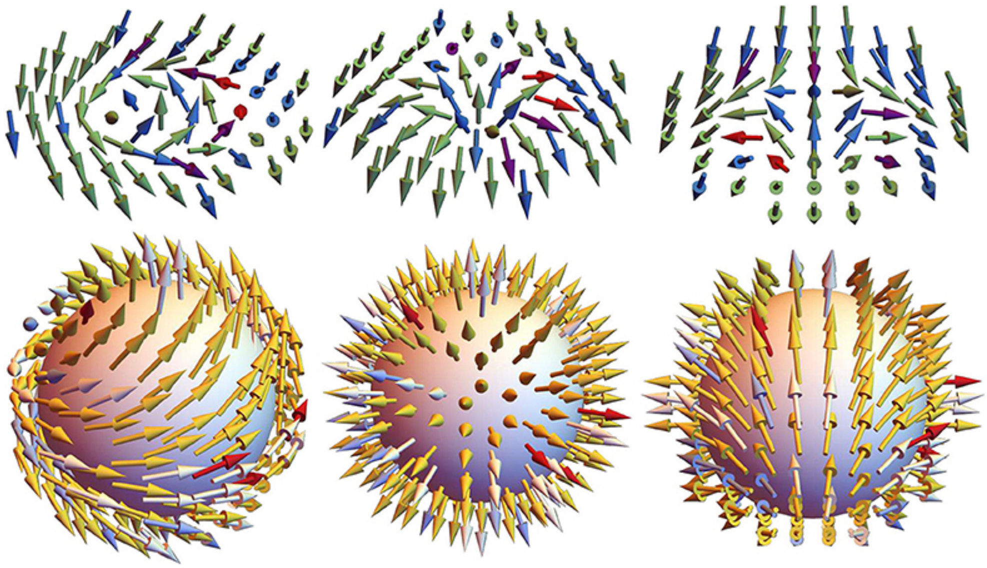

Figure 2: Three different kinds of skyrmion.

We see that a skyrmion can be described by a vector field defined on a circular region of space, where we pick a way to comb the vectors (smoothly, such that to our best effort, neighboring vectors only differ by a small angle). Note that this description is just visual, and if that were all a skyrmion was, there would be no reason to study it extensively. For all that we have done is just to find a particular dinosaur in the clouds—similar to how there are infinitely many ways a clouds can look, and only some clouds look like dinosaurs, there are infinite spatial configurations of vectors in such a vector field, and we have just dubbed some of them to be a skyrmion.

The reason that we can use skyrmions in physics research is that, unlike the dinosaur clouds, the skyrmions are not ephemeral. Alas, the dinosaur cloud will shortly be affected by a gust of wind, making the cloud not look like a dinosaur anymore. But the vector field which we have called a skyrmion is here to stay, for as long as the system which supports it exists. Why? Imagine an alternate universe where, once mother nature happens to form a cloud that looks like a dinosaur, that cloud will stay looking like a dinosaur (even if a hurricane passes through). That is the skyrmion in certain condensed matter systems of our universe. It turns out that this specific kind of spatial configuration of vectors is really hard to get rid of once you have it; it is protected by laws of nature. Thus we are motivated to classify the collection of such spatial configurations under one name, the skyrmion. Thus we have promoted the skyrmion from an arbitrary coincidence to something physically meaningful.

The laws of nature behind how exactly the skyrmion preserves itself are described using the mathematical theory of topology. That is what we mean by a state being topologically protected, and thus a topological state. [1]

[1] In another universe, when a skyrmion is born it becomes super insecure about existing and kills itself. This is because Topology was supposed to provide love to their child but traveled to our universe to buy milk. [2]

[2] Finding an absence of milk but an abundance of land, Topology settled in our universe and started a dairy farm, free-ranging spherical cows on Grassmanian Manifolds.

Protected State (Example)

$$H = -J(s_i \cdot s_{i+1})$$

$$\uparrow\uparrow\uparrow \mid \downarrow\downarrow\downarrow$$

Figure 3: 1D Ising Model. In the figure, we see there is a domain wall (marked in blue) between the domain of down spins and the domain of up spins.

Just to hammer the point home, here is a physics example of a protected state. In the one dimensional Ising model, we can distinguish between two domains: of spins aligned upwards, and of spins aligned downwards. It turns out that thermodynamically you will always have a domain wall because the free energy is minimized that way. So in analogous terms to our study of topological systems, we could say that the domain wall in the 1D Ising model is a "thermodynamically protected state": according to the laws of thermodynamics, when you have a 1D Ising model, you always have a domain wall.

Topology (ToDo)

Topology is the study of continuity.

Algebraic Topology (ToDo)

Algebraic Topology is the study of describing topological spaces in the language of abstract algebra.

Surface States (ToDo)

Topological Invariants (ToDo)

One way people like to characterize topological states is by referring to an associated topological invariant. A topological invariant is a quantity describing a topological space which is preserved when the space is mapped to another topological space under homeomorphism.

This method of characterization is nice in part because it is compact: it encodes information about the amount of physical quantity associated with that invariant, and the topological invariant itself has some algebraic structure, which can tell you about the kind of physical quantity you are dealing with.

Afterword

This writeup spawned because I felt that the topological physics literature I was exposed to did a poor job of motivating the use of topology (or even explaining how topology was being used, for that matter). But the actual thing that pushed me to write was when I thought of the cloud shapes analogy for protected topological states.

Works Cited

Skyrmion Image
https://www.frontiersin.org/journals/physics/articles/10.3389/fphy.2018.00098/full

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Graphene is an example of a state which is topologically protected in space—such are edge (surface) states. I wonder if in the future people will find states which are topologically protected in time.