Online Supplement: Selecting the Relevant Variables for Factor Estimation in FAVAR Models, With An Application to Forecasting the Yield Curve*

John C. Chao[1], Kaiwen Qiu[2], and Norman R. Swanson[2]

[1]University of Maryland and [2]Rutgers University

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Abstract

This Online Supplement contains the proofs of Theorems 1 and 2 of the main paper as well as all supporting Lemmas.

Keywords: Factor analysis, forecasting, variable selection.

**Corresponding Author:* Norman R. Swanson, Department of Economics, 9500 Hamilton Street, Rutgers University, nswanson@econ.rutgers.edu.

John C. Chao, Department of Economics, 7343 Preinkert Drive, University of Maryland, jcchao@umd.edu. Kaiwen Qiu, Department of Economics, 9500 Hamilton Street, Rutgers University, kq60@econ.rutgers.edu. The authors are grateful to Matteo Barigozzi, Mehmet Caner, Bin Chen, Rong Chen, Simon Freyaldenhoven, Yuan Liao, Esther Ruiz, Minchul Shin, Jim Stock, Timothy Vogelsang, Endong Wang, Xiye Yang, Peter Zadrozny, Bo Zhou and seminar participants at the University of Glasgow, the University of California Riverside, the Federal Reserve Bank of Pihadelphia as well as participants at the 2022 Summer Econometrics Society Meetings, the 2022 International Association of Applied Econometrics Association meetings, the DC-MD-VA Econometrics Workshop, the 2022 NBER-NSF Time Series Conference, the Spring 2023 Rochester Conference in Econometrics, and the 8th Annual Conference of the Society for Economic Measurement for useful comments received on earlier versions of this paper. Chao thanks the University of Maryland for research support.

Theorems, Lemmas and Their Proofs

This appendix contains the proofs of the main results of the paper: Theorems 1 and 2, as well as all supporting lemmas.

Proof of Theorem 1: To show part (a), first set $z = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$, where $N = N_1 + N_2$. Note that, under Assumption 2-10, we can easily show that
$\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2(1+a)}\sqrt{\ln N}$, for all N_1, N_2 sufficiently large.[1]

By part (a) of Assumption 2-9, $\sqrt{\ln N}/\min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\} \to 0$ as $N_1, N_2, T \to \infty$; this, in turn, implies that, for some positive constant c_0, $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$ satisfies the inequality constraint $0 \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq c_0 \min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\}$ for all N_1, N_2, T sufficiently large, so that $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$ lies within the range of values of z for which the moderate deviation inequality given in Lemma A2 holds. Thus, plugging $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$ into the moderate deviation inequality (1) given in Lemma A2 below, we see that there exists a positive constant A such that:

$$P\left(|S_{i,\ell,T}| \geq \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right) \leq 2\left[1 - \Phi\left(\Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right)\right]\left\{1 + A\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\}$$
$$= 2\left[1 - \left(1 - \tfrac{\varphi}{2N}\right)\right]\left\{1 + A\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\} = \tfrac{\varphi}{N}\left\{1 + A\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\},$$

for $\ell \in \{1, ..., d\}$, for $i \in H = \{k \in \{1, ..., N\} : \gamma_k = 0\}$, and for all N_1, N_2, T sufficiently large. Next, note that:

$$P\left(\max_{i \in H} \sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}| \geq \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right)$$
$$\leq P\left(\bigcup_{i \in H} \bigcup_{1 \leq \ell \leq d} \left\{|S_{i,\ell,T}| \geq \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right\}\right) \quad \left(\text{since } 0 \leq \varpi_\ell \leq 1 \text{ and } \sum_{\ell=1}^{d} \varpi_\ell = 1\right)$$
$$\leq \sum_{i \in H} \sum_{\ell=1}^{d} P\left(|S_{i,\ell,T}| \geq \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right) \quad \text{(by union bound)}$$
$$\leq \sum_{i \in H} \sum_{\ell=1}^{d} \tfrac{\varphi}{N}\left\{1 + A\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\}$$
$$= d\tfrac{N_2 \varphi}{N}\left\{1 + A\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\}$$

Using the inequality $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2(1+a)}\sqrt{\ln N}$ discussed above, we further obtain, for all N_1, N_2, T sufficiently large:

$$P\left(\max_{i \in H} \sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}| \geq \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right) \leq \tfrac{dN_2\varphi}{N}\left\{1 + \tfrac{A}{T^{(1-\alpha_1)/2}}\left[1 + \Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3\right\}$$
$$\leq \tfrac{dN_2\varphi}{N}\left\{1 + 2^2 A T^{-\frac{(1-\alpha_1)}{2}} + 2^2 A\left[\Phi^{-1}\left(1 - \tfrac{\varphi}{2N}\right)\right]^3 T^{-\frac{(1-\alpha_1)}{2}}\right\}$$
$$\left(\text{by the inequality } \left|\sum_{i=1}^{m} a_i\right|^r \leq c_r \sum_{i=1}^{m} |a_i|^r \text{ where } c_r = m^{r-1} \text{ for } r \geq 1\right)$$
$$\leq \tfrac{dN_2\varphi}{N}\left\{1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 4A\left[\sqrt{2(1+a)}\sqrt{\ln N}\right]^3 T^{-\frac{(1-\alpha_1)}{2}}\right\}$$
$$= \tfrac{dN_2\varphi}{N}\left\{1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 2^{\frac{7}{2}} A (1+a)^{\frac{3}{2}} \frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\right\}.$$

Finally, note that the rate condition given in part (a) of Assumption 2-9
(i.e., $\sqrt{\ln N}/\min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\} \to 0$ as $N_1, N_2, T \to \infty$) implies that
$(\ln N)^{\frac{3}{2}}/T^{\frac{1-\alpha_1}{2}} \to 0$ as $N_1, N_2, T \to \infty$, from which it follows that:

[1] An explicit proof of this result is given in Lemma OA-15 of this appendix. See, in particular, part (b) of the lemma.

$$P\left(\max_{i\in H}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$$
$$\leq\frac{dN_2\varphi}{N}\left\{1+4AT^{-\frac{(1-\alpha_1)}{2}}+2^{\frac{7}{2}}A\left(1+a\right)^{\frac{3}{2}}\frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\right\}=\frac{dN_2\varphi}{N}\left[1+o\left(1\right)\right]=O\left(\frac{N_2\varphi}{N}\right)=o\left(1\right).$$

Next, to show part (b), note that, by a similar argument as that given for part (a) above, we have:

$$P\left(\max_{i\in H}\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)=P\left(\bigcup_{i\in H}\bigcup_{1\leq\ell\leq d}\left\{\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)$$
$$\leq\frac{dN_2\varphi}{N}\left\{1+\frac{4A}{T^{(1-\alpha_1)/2}}+\frac{2^{\frac{7}{2}}A(1+a)^{\frac{3}{2}}(\ln N)^{\frac{3}{2}}}{T^{(1-\alpha_1)/2}}\right\}=\frac{dN_2\varphi}{N}\left[1+o\left(1\right)\right]=O\left(\frac{N_2\varphi}{N}\right)=o\left(1\right).\ \square$$

Proof of Theorem 2: To show part (a), let $\overline{S}_{i,\ell,T}$ and $\overline{V}_{i,\ell,T}$ be as defined in expression (12) of the main paper, and note that:

$$P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$$
$$=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\overline{S}_{i,,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}+\frac{\mu_{i,,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$$
$$\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\right\}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$$
$$=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\sqrt{\overline{V}_{i,\ell,T}}}{\mu_{i,\ell,T}}\right|\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$$
$$=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right),$$

where $\mu_{i,\ell,T}=\sum_{r=1}^{q}\sum_{t=b_1(r)}^{b_2(r)}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}$,for $b_1\left(r\right)=\left(r-1\right)\tau+p$ and $b_2\left(r\right)=b_1\left(r\right)+\tau_1-1$. Next, let $\pi_{i,\ell,T}=\sum_{r=1}^{q}\left(\sum_{t=b_1(r)}^{b_2(r)}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2$, and we see that, under Assumption 2-8, there exists a positive constant $\underline{c}$ such that for every $\ell\in\{1,...,d\}$ and for all N_1,N_2, and T sufficiently large:

$$\min_{i\in H^c}\left\{\pi_{i,\ell,T}/\left(q\tau_1^2\right)\right\}$$
$$=\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2$$
$$=\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}E\left[\gamma_i'\underline{F}_t y_{\ell,t+1}\right]\right)^2$$
$$\geq\min_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}E\left[\gamma_i'\underline{F}_t y_{\ell,t+1}\right]\right)^2\quad\text{(by Jensen's inequality)}$$
$$=\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right|^2$$
$$\geq\underline{c}^2>0\quad\text{(in light of Assumption 2-8)}.$$

It follows that for all N_1,N_2, and T sufficiently large:

$$
\begin{aligned}
&P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}+\sqrt{\overline{V}_{i,\ell,T}/\left(q\tau_1^2\right)}-\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\right.\right.\\
&\qquad\qquad\left.\left.\times\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\left(\sqrt{\overline{V}_{i,\ell,T}}-\sqrt{\pi_{i,\ell,T}}\right)/\sqrt{\pi_{i,\ell,T}}}\right|\right.\right.\\
&\qquad\qquad\left.\left.\times\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left(\mu_{i,\ell,T}/(q\tau_1)\right)}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\frac{1}{1+\max_{k\in H^c}\left|\sqrt{\overline{V}_{k,\ell,T}}-\sqrt{\pi_{k,\ell,T}}\right|/\sqrt{\pi_{k,\ell,T}}}\right.\right.\\
&\qquad\qquad\left.\left.\times\left[1-\max_{k\in H^c}\left|\frac{\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}}{\mu_{k,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\frac{1}{1+\max_{k\in H^c}\sqrt{\left|\overline{V}_{k,\ell,T}-\pi_{k,\ell,T}\right|/\pi_{k,\ell,T}}}\right.\right.\\
&\qquad\qquad\left.\left.\times\left[1-\max_{k\in H^c}\left|\frac{\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}}{\mu_{k,\ell,T}}\right|\right]\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\quad\left(\text{making use of the inequality }\left|\sqrt{x}-\sqrt{y}\right|\leq\sqrt{|x-y|}\text{ for }x\geq0\text{ and }y\geq0\right)\\
&=P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\frac{1-\max_{k\in H^c}\left|\mathcal{E}_{k,\ell,T}\right|}{1+\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell,T}\right|}}\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right),
\end{aligned}
$$

where $\mathcal{E}_{k,\ell,T}=\left(\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}\right)/\mu_{k,\ell,T}$ and $\mathcal{V}_{k,\ell,T}=\left(\overline{V}_{k,\ell,T}-\pi_{k,\ell,T}\right)/\pi_{k,\ell,T}$. By part (a) of Lemma QA-16 (see below), there exists a sequence of positive numbers $\{\epsilon_T\}$ such that, as $T\to\infty$, $\epsilon_T\to0$ and $P\left(\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell,T}\right|\geq\epsilon_T\right)\to0$. In addition, by the result of part (b) of Lemma QA-16, there exists a sequence of positive numbers $\{\epsilon_T^*\}$ such that, as $T\to\infty$, $\epsilon_T^*\to0$ and $P\left(\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell,T}\right|\geq\epsilon_T^*\right)\to0$. Further define $\overline{\mathbb{E}}_T=\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell,T}\right|$ and $\overline{\mathbb{V}}_T=\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell,T}\right|$; and note that, for all N_1,N_2, and T sufficiently large,

$$
\begin{aligned}
&P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\frac{1-\max_{k\in H^c}\left|\mathcal{E}_{k,\ell,T}\right|}{1+\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell,T}\right|}}\right\}\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\frac{1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell,T}\right|}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell,T}\right|}}\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/(q\tau_1)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&=P\left(\frac{1-\overline{\mathbb{E}}_T}{1+\sqrt{\overline{\mathbb{V}}_T}}\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right)\\
&\geq P\left(\left\{\left|\frac{1-\epsilon_T}{1+\sqrt{\epsilon_T^*}}\right|\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right\}\cap\left\{\overline{\mathbb{E}}_T<\epsilon_T\right\}\cap\left\{\overline{\mathbb{V}}_T<\epsilon_T^*\right\}\right)\\
&+P\left(\left\{\frac{1-\overline{\mathbb{E}}_T}{1+\sqrt{\overline{\mathbb{V}}_T}}\min_{i\in H}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\tfrac{\varphi}{2N}\right)\right\}\cap\left\{\overline{\mathbb{E}}_T\geq\epsilon_T\cup\overline{\mathbb{V}}_T\geq\epsilon_T^*\right\}\right)
\end{aligned}
$$

$\geq P\left(\left\{\left|\frac{1-\epsilon_T}{1+\sqrt{\epsilon_T^*}}\right|\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\cap\left\{\mathbb{E}_T<\epsilon_T\right\}\cap\left\{\mathbb{V}_T<\epsilon_T^*\right\}\right)$
$+P\left(\left\{\frac{1-\mathbb{E}_T}{1+\sqrt{\mathbb{V}_T}}\min_{i\in H}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\cap\left\{\mathbb{E}_T\geq\epsilon_T\right\}\right)$
$=P\left(\left\{\left|\frac{1-\epsilon_T}{1+\sqrt{\epsilon_T^*}}\right|\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\cap\left\{\mathbb{E}_T<\epsilon_T\right\}\cap\left\{\mathbb{V}_T<\epsilon_T^*\right\}\right)$
$+o(1)$.
where the last equality above follows from the fact that
$P\left(\left\{\frac{1-\mathbb{E}_T}{1+\sqrt{\mathbb{V}_T}}\min_{i\in H}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\cap\left\{\mathbb{E}_T\geq\epsilon_T\right\}\right)\leq P\left(\mathbb{E}_T\geq\epsilon_T\right)=o(1)$
Moreover, making use of Assumptions 2-8, the result given in Lemma A1 (see below), and the fact that $q=\lfloor T_0/\tau\rfloor\sim T^{1-\alpha_1}$, we see that, there exists positive constants $\underline{c}$ and $\overline{C}$ such that:
$\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|=\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\frac{\sqrt{q}\left|\mu_{i,\ell,T}/(q\tau_1)\right|}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}$
$\geq\sqrt{q}\sum_{\ell=1}^d\varpi_\ell\frac{\min_{i\in H^c}\left|\mu_{i,\ell,T}/(q\tau_1)\right|}{\sqrt{\max_{i\in H^c}\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\geq\sqrt{q}\sum_{\ell=1}^d\varpi_\ell\frac{\underline{c}}{\sqrt{\overline{C}}}=\sqrt{q}\frac{\underline{c}}{\sqrt{\overline{C}}}\sim\sqrt{q}\sim\sqrt{\frac{T_0}{\tau}}\sim T^{(1-\alpha_1)/2}$.
On the other hand, applying the inequality
$\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\leq\sqrt{2(1+a)}\sqrt{\ln N}\sim\sqrt{\ln N}$,[2] we further deduce that, as $N_1,N_2,T\rightarrow\infty$,

$$\frac{1}{\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)}\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\frac{\underline{c}}{\sqrt{\overline{C}}}\sqrt{\frac{q}{2(1+a)\ln N}}\sim\sqrt{\frac{T^{(1-\alpha_1)}}{\ln N}}\rightarrow\infty.$$

This is true because the condition $\sqrt{\ln N}/\min\left\{T^{(1-\alpha_1)/6},T^{\alpha_2/2}\right\}\rightarrow 0$ as $N_1,N_2,T\rightarrow\infty$ (as specified in Assumption 2-9 part (a)) implies that $\ln N/T^{(1-\alpha_1)}\rightarrow 0$ as $N_1,N_2,T\rightarrow\infty$. Hence, there exists a natural number M such that, for all $N_1\geq M,N_2\geq M$, and $T\geq M$, we have $\left|\frac{1-\epsilon_T}{1+\sqrt{\epsilon_T^*}}\right|\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)$ so that:
$P\left(\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$
$\geq P\left(\left\{\left|\frac{1-\epsilon_T}{1+\sqrt{\epsilon_T^*}}\right|\min_{i\in H^c}\sum_{\ell=1}^d\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\cap\left\{\mathbb{E}_T<\epsilon_T\right\}\cap\left\{\mathbb{V}_T<\epsilon_T^*\right\}\right)$
$+o(1)$
$=P\left(\left\{\mathbb{E}_T<\epsilon_T\right\}\cap\left\{\mathbb{V}_T<\epsilon_T^*\right\}\right)+o(1)$
(for all $N_1\geq M,N_2\geq M$, and $T\geq M$)
$\geq P\left(\mathbb{E}_T<\epsilon_T\right)+P\left(\mathbb{V}_T<\epsilon_T^*\right)-1+o(1)$ (using the inequality
$P\left\{\bigcap_{i=1}^m A_i\right\}\geq\sum_{i=1}^m P(A_i)-(m-1)$ see Lemma OA-14 below)
$=1-P\left(\mathbb{E}_T\geq\epsilon_T\right)+1-P\left(\mathbb{V}_T\geq\epsilon_T^*\right)-1+o(1)$
$=1-P\left(\mathbb{E}_T\geq\epsilon_T\right)-P\left(\mathbb{V}_T\geq\epsilon_T^*\right)+o(1)=1+o(1)$.

Next, to show part (b), note that, by applying the result in part (a), we have that:
$P\left(\min_{i\in H^c}\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)$

[2] As noted previously, an explicit proof of this result is given in part (b) of Lemma OA-15 in this appendix.

$$\geq P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)=1+o\left(1\right).\ \square$$

Lemma A1: Let $\underline{Y}_t=\begin{pmatrix}Y_t' & Y_{t-1}' & \cdots & Y_{t-p+1}'\end{pmatrix}'$ and $\underline{F}_t=\begin{pmatrix}F_t' & F_{t-1}' & \cdots & F_{t-p+1}'\end{pmatrix}'$, and define $b_1\left(r\right)=\left(r-1\right)\tau+p$ and $b_2\left(r\right)=b_1\left(r\right)+\tau_1-1$. Under Assumptions 2-1, 2-2, 2-5, 2-6, and 2-9(b); there exists a positive constant C such that:

$$\max_{1\leq\ell\leq d,\ i\in H^c}\left(\frac{\pi_{i,\ell,T}}{q\tau_1^2}\right)$$
$$=\max_{1\leq\ell\leq d,\ i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2$$

$\leq C<\infty$, for all N_1,N_2,T sufficiently large.

Proof of Lemma A1: To proceed, let $\phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}$ and, for $\ell\in\{1,...,d\}$, let $e_{\ell,d}$ denote a $d\times1$ elementary vector whose ℓ^{th} component is 1 and all other components are 0. Now, note that:

$$\max_{1\leq\ell\leq d,\ i\in H^c}\left\{\pi_{i,\ell,T}/\left(q\tau_1^2\right)\right\}$$
$$=\max_{1\leq\ell\leq d,\ i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2$$
$$\leq\max_{1\leq\ell\leq d,\ i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{E\left[\left|\gamma_i'\underline{F}_t\right|\right]\left|\mu_{Y,\ell}\right|+E\left[\left|\gamma_i'\underline{F}_t\underline{Y}_t'A_{YY}'e_{\ell,d}\right|\right]\right.\right.$$
$$\left.\left.+E\left[\left|\gamma_i'\underline{F}_t\underline{F}_t'A_{YF}'e_{\ell,d}\right|\right]\right\}\right)^2\text{ (by triangle and Jensen's inequalities)}$$
$$\leq\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\sqrt{\left\|\gamma_i\right\|_2^2}\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq\ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right.$$
$$+\sqrt{\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\gamma_i}\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YY}E\left[\underline{Y}_t\underline{Y}_t'\right]A_{YY}'e_{\ell,d}}$$
$$\left.\left.+\sqrt{\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\gamma_i}\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YF}E\left[\underline{F}_t\underline{F}_t'\right]A_{YF}'e_{\ell,d}}\right\}\right)^2$$
$$\leq\left(\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq\ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right.$$
$$\left.\left.+\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YY}A_{YY}'e_{\ell,d}}+E\left\|\underline{F}_t\right\|_2^2\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YF}A_{YF}'e_{\ell,d}}\right\}\right)^2$$
$$\leq\left(\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq\ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right.$$
$$\left.\left.+\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}C^{\dagger}\phi_{\max}+E\left\|\underline{F}_t\right\|_2^2C^{\dagger}\phi_{\max}\right\}\right)^2,$$

where the last inequality follows from the fact that, by making use of Assumption 2-6, it is easy to show that there exists a constant $C^{\dagger}>0$ such that $\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YY}A_{YY}'e_{\ell,d}}\leq\left\|A_{YY}\right\|_2\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'e_{\ell,d}}=\left\|A_{YY}\right\|_2\leq C^{\dagger}\phi_{\max}$ and, similarly, $\sqrt{\max_{1\leq\ell\leq d}e_{\ell,d}'A_{YF}A_{YF}'e_{\ell,d}}\leq\left\|A_{YF}\right\|_2\leq C^{\dagger}\phi_{\max}$.[3] Hence,

$$\max_{1\leq\ell\leq d}\max_{k\in H^c}\left\{\pi_{i,\ell,T}/\left(q\tau_1^2\right)\right\}$$
$$\leq\left(\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq\ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right.$$

[3] Explicit proofs of these two inequalities are given below in Lemma OA-7.

$$+\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}C^{\dagger}\phi_{\max}+E\left\|\underline{F}_t\right\|_2^2 C^{\dagger}\phi_{\max}\Big\}\Big)^2$$

$$\leq\left(\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=b_1(r)}^{b_2(r)}E\left\|\underline{F}_t\right\|_2^2\left(\left\|\mu_Y\right\|_2^2+\left[\sqrt{E\left\|\underline{Y}_t\right\|_2^2}+\sqrt{E\left\|\underline{F}_t\right\|_2^2}\right]C^{\dagger}\phi_{\max}\right)^2$$

$$\leq C<\infty,$$

for some positive constant C such that

$C\geq\left(\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\right)E\left\|\underline{F}_t\right\|_2^2\left(\left\|\mu_Y\right\|_2^2+\left[\sqrt{E\left\|\underline{Y}_t\right\|_2^2}+\sqrt{E\left\|\underline{F}_t\right\|_2^2}\right]C^{\dagger}\phi_{\max}\right)^2$, where such a constant exists because $\max_{i\in H^c}\left\|\gamma_i\right\|_2^2$ and $\left\|\mu_Y\right\|_2^2$ are both bounded given Assumption 2-5; because $0<\phi_{\max}<1$ given Assumption 2-1; and because, under Assumptions 2-1, 2-2(a)-(b), 2-5, and 2-6; one can easily show that there exists a constant $C^*>0$ such that $E\left\|\underline{F}_t\right\|_2^2\leq C^*$ and $E\left\|\underline{Y}_t\right\|_2^2\leq\left(E\left\|\underline{Y}_t\right\|_2^6\right)^{1/3}\leq C^*$.[4] $\square$

Lemma A2: Suppose that Assumptions 2-1, 2-2, 2-3, 2-4, 2-5, 2-6, and 2-7 hold. Let $\Phi(\cdot)$ denote the cumulative distribution function of the standard normal random variable. Then, there exists a positive constant A such that

$$P\left(\left|S_{i,\ell,T}\right|\geq z\right)\leq 2\left[1-\Phi(z)\right]\left\{1+A(1+z)^3T^{-(1-\alpha_1)\frac{1}{2}}\right\}\tag{1}$$

for $i\in H=\{k\in\{1,....,N\}:\gamma_k=0\}$, for $\ell\in\{1,...,d\}$, for T sufficiently large, and for all z such that $0\leq z\leq c_0\min\left\{T^{(1-\alpha_1)/6},T^{\alpha_2/2}\right\}$ with c_0 being a positive constant.
Proof of Lemma A2: Note first that, for any i such that
$i\in H=\{k\in\{1,....,N\}:\gamma_k=0\}$, the formula for $S_{i,\ell,T}$ reduces to:
$S_{i,\ell,T}=\left(\sum_{r=1}^{q}\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right]^2\right)^{-\frac{1}{2}}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}.$
Hence, to verify the conditions of Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we set $X_{it}=u_{it}y_{\ell,t+1}$, and note that $E\left[X_{it}\right]=E\left[u_{it}y_{\ell,t+1}\right]=E_Y\left[E\left[u_{it}\right]y_{\ell,t+1}\right]=0$, where the second equality follows by the law of iterated expectations given that Assumption 2-4 implies the independence of u_{it} and $y_{\ell,t+1}$ and where the third equality follows by Assumption 2-3(a). Hence, the first part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is fulfilled. Moreover, in light of Assumption 2-3(b) and in light of the fact that, under Assumptions 2-1, 2-2(a)-(b), 2-5, and 2-6; one can show by straightforward calculations that there exists a positive constant $\overline{C}$ such that $E\left\|\underline{Y}_t\right\|_2^6\leq\overline{C}$; we see that there exists some positive constant c_1 such that, for every $\ell\in\{1...,d\}$,

$$\begin{aligned}E\left[\left|X_{it}\right|^{\frac{31}{10}}\right] &= E\left[\left|u_{it}y_{\ell,t+1}\right|^{\frac{31}{10}}\right]\leq\left(E\left|u_{it}\right|^{\frac{186}{29}}\right)^{\frac{29}{60}}\left(E\left|y_{\ell,t+1}\right|^6\right)^{\frac{31}{60}}\\ &\leq \left[\left(E\left|u_{it}\right|^{\frac{186}{29}}\right)^{\frac{29}{186}}\right]^{\frac{31}{10}}\left[E\left(\sum_{k=1}^{d}\sum_{j=0}^{p-1}y_{k,t+1-j}^2\right)^3\right]^{\frac{31}{60}}\\ &\leq\left[\left(E\left|u_{it}\right|^7\right)^{\frac{1}{7}}\right]^{\frac{31}{10}}\left[\left(E\left\|\underline{Y}_{t+1}\right\|_2^6\right)^{\frac{1}{6}}\right]^{\frac{31}{10}}\leq c_1^{\frac{31}{10}},\end{aligned}$$

where the first and third inequalities above follow, respectively, by Hölder's and Liapunov's inequalities. Hence, the second part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is also fulfilled with $r=\frac{31}{10}>2$.

[4] An explicit proof that, under Assumptions 2-1, 2-2(a)-(b), 2-5, and 2-6; there exists some positive constant $C^{\#}$ such that $E\left\|\underline{F}_t\right\|_2^6\leq C^{\#}$ and $E\left\|\underline{Y}_t\right\|_2^6\leq C^{\#}$ is given below in Lemma OA-5.

Moreover, note that, by Assumption 2-7, for all $r \geq 1$ and $\tau_1 \geq 1$:

$$E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} = \tau_1 E\left\{\left[\frac{1}{\sqrt{\tau_1}}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right]^2\right\} \geq \tau_1 \underline{c},$$

so that condition (4.2) of Chen, Shao, Wu, and Xu (2016) is satisfied here. Now, making use of Assumption 2-3(c) and Assumption 2-4 and applying Theorem 2.1 of Pham and Tran (1985), it can be shown that $\left\{(y_{\ell,t+1}, u_{it})'\right\}$ is β mixing with β mixing coefficient satisfying $\beta(m) \leq \overline{a}_1 \exp\{-a_2 m\}$ for some constants $\overline{a}_1 > 0$ and $a_2 > 0$. Next, define $X_{it} = y_{\ell,t+1}u_{it}$, and note that $\{X_{it}\}$ is a β-mixing process with β-mixing coefficient $\beta_{X,m}$ satisfying the condition $\beta_{X,m} \leq a_1 \exp\{-a_2 m\}$ for some constant $a_1 > 0$ and for all m sufficiently large, given that measurable functions of a finite number of β-mixing random variables are also β-mixing, with β-mixing coefficients having the same order of magnitude[5]. It follows that $\{X_{it}\}$ satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for all $i \in H$. Hence, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016) for the case where $\delta = 1$[6], we obtain the Cramér-type moderate deviation result

$$\frac{P\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi(z)} = 1 + O(1)(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}}, \tag{2}$$

which holds for all $0 \leq z \leq c_0 \min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\}$ and for $|O(1)| \leq A$, where A is an absolute constant and where $\overline{S}_{i,\ell,T}$ and $\overline{V}_{i,\ell,T}$ are as defined in expression (12) in the main paper.

Next, consider obtaining a moderate deviation result for $P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}/[1-\Phi(z)]$. As $\overline{S}_{i,\ell,T} = \sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}(-u_{it}y_{\ell,t+1})$, we can take $X_{it} = -u_{it}y_{\ell,t+1}$, and note that, by calculations similar to those given above, we have $E[X_{it}] = E[-u_{it}y_{\ell,t+1}] = 0$, $E\left[|X_{it}|^{\frac{31}{10}}\right] = E\left[|-u_{it}y_{\ell,t+1}|^{\frac{31}{10}}\right] = E\left[|u_{it}y_{\ell,t+1}|^{\frac{31}{10}}\right] \leq c_1^{\frac{31}{10}}$, and

$$E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} = E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (-u_{it}y_{\ell,t+1})\right]^2\right\} \geq \underline{c}\tau_1.$$

Moreover, it is easily seen that $\{X_{it}\}$ (with $X_{it} = -u_{it}y_{\ell,t+1}$) also satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for every i. Thus, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we also obtain the Cramér-type moderate deviation result

$$\frac{P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi(z)} = 1 + O(1)(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}}, \tag{3}$$

which holds for all $0 \leq z \leq c_0 \min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\}$ and for $|O(1)| \leq A$ with A being an absolute constant. Next, note that:

$\left|\frac{P(|S_{i,\ell,T}| \geq z)}{2[1-\Phi(z)]} - 1\right| = \left|\frac{P\left(\left|\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}}\right| \geq z\right)}{2[1-\Phi(z)]} - 1\right|$

[5] For α-mixing and ϕ-mixing, this result is given in Theorem 14.1 of Davidson (1994). However, using essentially the same argument as that given in the proof of Theorem 14.1, one can also prove a similar result for β-mixing. For an explicit proof of this result, see Lemma OA-2 part (a) in this appendix.

[6] Note that Theorem 4.1 of Chen, Shao, Wu and Xu (2016) requires that $0 < \delta \leq 1$ and $\delta < r - 2$. These conditions are satisfied here given that we choose $\delta = 1$ and $r = 31/10$.

$$= \left| \frac{P\left(\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} \cup \left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}\right)}{2[1-\Phi(z)]} - 1 \right|$$
$$= \left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right) + P\left(-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{2[1-\Phi(z)]} - 1 \right|$$
$$\left(\text{since } \left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} \cap \left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} = \varnothing \text{ w.p.1}\right)$$
$$\leq \frac{1}{2} \left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{1-\Phi(z)} - 1 \right| + \frac{1}{2} \left| \frac{P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi(z)} - 1 \right|.$$

Thus, in light of expressions (2) and (3), we have that:

$$\left| \frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \right| \leq \frac{1}{2} \left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{1-\Phi\left(z\right)} - 1 \right| + \frac{1}{2} \left| \frac{P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi\left(z\right)} - 1 \right|$$
$$\leq \quad \frac{A}{2}\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}} + \frac{A}{2}\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}} = A\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}}$$

It then follows that:

$$-A\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}} \leq \frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \leq A\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}} \tag{4}$$

where $S_{i,\ell,T} = \overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}}$. Focusing on the right-hand part of the inequality in (4), we have that: $P\left(\left|S_{i,\ell,T}\right| \geq z\right)/\left(2\left[1-\Phi\left(z\right)\right]\right) - 1 \leq A\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}}$. Simple rearrangement of this inequality then leads to the desired result:

$$P\left(\left|S_{i,\ell,T}\right| \geq z\right) \leq 2\left[1-\Phi\left(z\right)\right]\left\{1 + A\left(1+z\right)^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\},$$

which holds for all $i \in H = \{k \in \{1, ..., N\} : \gamma_k = 0\}$, for every $\ell \in \{1, ..., d\}$, for all T sufficiently large, and for all z such that $0 \leq z \leq c_0 \min\left\{T^{(1-\alpha_1)/6}, T^{\alpha_2/2}\right\}$. □

Lemma OA-1: Let a and θ be real numbers such that $a > 0$ and $\theta \geq 1$. Also, let G be a finite non-negative integer. Then,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

Proof of Lemma OA-1: By the integral test,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty \text{ for finite non-negative integer } G$$

if

$$\int_1^{\infty} x^G \exp\left\{-ax^\theta\right\} dx < \infty \text{ for finite non-negative integer } G$$

In addition, note that since, by assumption, $a > 0$ and $\theta \geq 1$, we have

$$\int_1^{\infty} x^G \exp\left\{-ax^\theta\right\} dx \leq \int_1^{\infty} x^G \exp\left\{-ax\right\} dx$$

We will first consider the case where $G = 0$. In this case, note that

$$\int_1^\infty x^0 \exp\{-ax\}\, dx = \int_1^\infty \exp\{-ax\}\, dx$$

Let $u = -ax$, so that $-\frac{du}{a} = dx$; and we have

$$\begin{aligned}
\int_1^\infty \exp\{-ax\}\, dx &= -\frac{1}{a}\int_{-a}^{-\infty} \exp\{u\}\, du \\
&= \frac{1}{a}\int_{-\infty}^{-a} \exp\{u\}\, du \\
&= \frac{\exp\{-a\}}{a} \\
&< \infty \text{ for any } a > 0. \qquad (5)
\end{aligned}$$

Next, consider the case where G is an integer such that $G \geq 1$. Here, we will show that

$$\int_1^\infty x^G \exp\{-ax\}\, dx = \left[\frac{1}{a} + \sum_{k=1}^{G} \frac{1}{a}\left(\prod_{j=0}^{k-1} \frac{G-j}{a}\right)\right] \exp\{-a\} < \infty$$

using mathematical induction. To proceed, first consider the case where $G = 1$. Let

$$\begin{aligned}
u &= x,\ du = dx \\
dv &= \exp\{-ax\}\, dx,\ v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x \exp\{-ax\}\, dx &= -\frac{x}{a}\exp\{-ax\}\Big|_1^\infty + \int_1^\infty \frac{1}{a}\exp\{-ax\}\, dx \\
&= \frac{1}{a}\exp\{-a\} - \frac{1}{a^2}\exp\{-ax\}\Big|_1^\infty \\
&= \frac{1}{a}\exp\{-a\} + \frac{1}{a^2}\exp\{-a\} \\
&= \left(\frac{1}{a} + \frac{1}{a^2}\right)\exp\{-a\} \\
&= \left\{\frac{1}{a} + \sum_{k=1}^{1} \frac{1}{a}\left(\prod_{j=0}^{k-1} \frac{1-j}{a}\right)\right\}\exp\{-a\} < \infty
\end{aligned}$$

Next, for $G = 2$, let

$$\begin{aligned}
u &= x^2,\ du = 2x dx \\
dv &= \exp\{-ax\}\, dx,\ v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned}
\int_1^\infty x^2 \exp\{-ax\}\,dx &= -\frac{x^2}{a}\exp\{-ax\}\Big|_1^\infty + \frac{2}{a}\int_1^\infty x\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} + \frac{2}{a}\left(\frac{1}{a}+\frac{1}{a^2}\right)\exp\{-a\} \\
&= \frac{1}{a}\exp\{-a\} + 2\left(\frac{1}{a^2}+\frac{1}{a^3}\right)\exp\{-a\} \\
&= \left(\frac{1}{a}+\frac{2}{a^2}+\frac{2}{a^3}\right)\exp\{-a\} \\
&= \left[\frac{1}{a}+\sum_{k=1}^{2}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{2-j}{a}\right)\right]\exp\{-a\} \\
&< \infty
\end{aligned}$$

Now, suppose that, for some $G \geq 2$,

$$\int_1^\infty x^{G-1}\exp\{-ax\}\,dx = \left[\frac{1}{a}+\sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\};$$

then, let

$$\begin{aligned}
u &= x^G,\ du = Gx^{G-1}dx \\
dv &= \exp\{-ax\}\,dx,\ v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and, using integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x^G\exp\{-ax\}\,dx &= -\frac{x^G}{a}\exp\{-ax\}\Big|_1^\infty + \frac{G}{a}\int_1^\infty x^{G-1}\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} + \frac{G}{a}\left[\frac{1}{a}+\sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\} \\
&= \frac{1}{a}\exp\{-a\} + \left[\frac{G}{a^2}+\sum_{k=1}^{G-1}\frac{1}{a}\frac{G}{a}\left(\prod_{j=0}^{k-1}\frac{G-(j+1)}{a}\right)\right]\exp\{-a\} \\
&= \left\{\frac{1}{a}+\frac{G}{a^2}+\frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)+\frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\right. \\
&\quad \left. +\cdots+\frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\times\cdots\times\left(\frac{1}{a}\right)\right\}\exp\{-a\} \\
&= \left\{\frac{1}{a}+\sum_{k=1}^{G}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-j}{a}\right)\right\}\exp\{-a\} \\
&< \infty. \qquad (6)
\end{aligned}$$

In view of expressions (5) and (6), it then follows by the integral test for series convergence that

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

for any finite non-negative integer G and for any constants a and θ such that $a > 0$ and $\theta \geq 1$. $\square$

Lemma OA-2: Let $\{V_t\}$ be a sequence of random variables (or random vectors) defined on some probability space $(\Omega, \mathcal{F}, P)$, and let

$$X_t = g\left(V_t, V_{t-1}, ..., V_{t-\varkappa}\right)$$

be a measurable function for some finite positive integer $\varkappa$. In addition, defne $\mathcal{G}_{-\infty}^t = \sigma\left(...., X_{t-1}, X_t\right)$, $\mathcal{G}_{t+m}^\infty = \sigma\left(X_{t+m}, X_{t+m+1}, ...\right)$, $\mathcal{F}_{-\infty}^t = \sigma\left(...., V_{t-1}, V_t\right)$, and $\mathcal{F}_{t+m-\varkappa}^\infty = \sigma\left(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...\right)$. Under this setting, the following results hold.

(a) Let

$$\begin{aligned}
\beta_{V,m-\varkappa} &= \sup_t \beta\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^t\right) - P\left(B\right)\right| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\right\}\right], \\
\beta_{X,m} &= \sup_t \beta\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P\left(H\right)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right].
\end{aligned}$$

If $\{V_t\}$ is β-mixing with

$$\beta_{V,m-\varkappa} \leq \overline{C}_1 \exp\left\{-C_2\left(m - \varkappa\right)\right\}$$

for all $m \geq \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also β-mixing with β-mixing coefficient satisfying

$$\beta_{X,m} \leq C_1 \exp\left\{-C_2 m\right\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp\left\{C_2 \varkappa\right\}$.

(b) Let

$$\begin{aligned}
\alpha_{V,m-\varkappa} &= \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m-\varkappa}^\infty} \left|P\left(G \cap H\right) - P\left(G\right)P\left(H\right)\right|, \\
\alpha_{X,m} &= \sup_t \alpha\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t \sup_{G \in \mathcal{G}_{-\infty}^t, H \in \mathcal{G}_{t+m}^\infty} \left|P\left(G \cap H\right) - P\left(G\right)P\left(H\right)\right|
\end{aligned}$$

If $\{V_t\}$ is α-mixing with

$$\alpha_{V,m-\varkappa} \leq \overline{C}_1 \exp\left\{-C_2\left(m - \varkappa\right)\right\}$$

for all $m \geq \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also α-mixing with α-mixing coefficient satisfying

$$\alpha_{X,m} \leq C_1 \exp\left\{-C_2 m\right\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp\left\{C_2 \varkappa\right\}$.

Proof of Lemma OA-2:

To show part (a), note first that it is well known that

$$\begin{aligned}
\beta_{X,m} &= \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P\left(H\right)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right] \\
&= \sup_t \left\{\frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} \left|P\left(G_i \cap H_j\right) - P\left(G_i\right)P\left(H_j\right)\right|\right\}
\end{aligned}$$

where the second supremum on the last line above is taken over all pairs of finite partitions $\{G_1, ..., G_I\}$ and $\{H_1, ..., H_J\}$ of Ω such that $G_i \in \mathcal{G}_{-\infty}^{t}$ for $i = 1, ..., I$ and $H_j \in \mathcal{G}_{t+m}^{\infty}$ for $j = 1, ..., J$. See, for example, Borovkova, Burton, and Dehling (2001). Similarly,

$$\begin{aligned}
\beta_{V,m-\varkappa} &= \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^{t}\right) - P\left(B\right)\right| : B \in \mathcal{F}_{t+m-\varkappa}^{\infty}\right\}\right] \\
&= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P\left(A_i \cap B_j\right) - P\left(A_i\right)P\left(B_j\right)\right|\right\}
\end{aligned}$$

where, similar to the definition of $\beta_{X,m}$, the second supremum on the last line above is taken over all pairs of finite partitions $\{A_1, ..., A_L\}$ and $\{B_1, ..., B_M\}$ of Ω such that $A_i \in \mathcal{F}_{-\infty}^{t}$ for $i = 1, ..., I$ and $B_j \in \mathcal{F}_{t+m-\varkappa}^{\infty}$ for $j = 1, ..., M$. Moreover, since X_t is measurable on any σ-field on which $V_t, V_{t-1}, ..., V_{t-\varkappa}$ are measurable, we also have

$$\mathcal{G}_{-\infty}^{t} = \sigma\left(...., X_{t-1}, X_t\right) \subseteq \sigma\left(...., V_{t-1}, V_t\right) = \mathcal{F}_{-\infty}^{t}$$

and

$$\mathcal{G}_{t+m}^{\infty} = \sigma\left(X_{t+m}, X_{t+m+1}, ...\right) \subseteq \sigma\left(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...\right) = \mathcal{F}_{t+m-\varkappa}^{\infty}.$$

It, thus, follows that, for all $m \geq \varkappa$,

$$\begin{aligned}
\beta_{X,m} &= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{I}\sum_{j=1}^{J}\left|P\left(G_i \cap H_j\right) - P\left(G_i\right)P\left(H_j\right)\right|\right\} \\
&\leq \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P\left(A_i \cap B_j\right) - P\left(A_i\right)P\left(B_j\right)\right|\right\} \\
&= \beta_{V,m-\varkappa} \\
&\leq \overline{C}_1 \exp\left\{-C_2\left(m-\varkappa\right)\right\} \\
&= \overline{C}_1 \exp\left\{C_2\varkappa\right\}\exp\left\{-C_2 m\right\} \\
&\leq C_1 \exp\left\{-C_2 m\right\}
\end{aligned}$$

for some positive constant $C_1 \geq \overline{C}_1 \exp\{C_2\varkappa\}$ which exists given that $\varkappa$ is fixed. Moreover, we have

$$\beta_{X,m} \leq C_1 \exp\left\{-C_2 m\right\} \to 0 \text{ as } m \to \infty,$$

which establishes the required result for part (a).

Part (b) can be shown in a manner similar to part (a), so to avoid redundancy, we do not include an explicit proof here. $\square$

Remark: Note that part (b) of Lemma OA-2 is similar to a result given in Theorem 14.1 of Davidson (1994) but adapted to suit our situation and our notatons here. Indeed, parts (a) and (b) of this lemma are both well-known results in the probability literature. We have chosen to state these results explicitly here only so that we can more easily refer to them in the proofs of some of our other results.

Lemma OA-3: Let $\{X_t\}$ be a sequence of random variables that is α-mixing. Let $p > 1$ and $r \geq p/\left(p-1\right)$, and let $q = \max\{p, r\}$. Suppose that, for all t,

$$\left\|X_t\right\|_q = \left(E\left|X_t\right|^q\right)^{\frac{1}{q}} < \infty$$

Then,

$$\left|Cov\left(X_t, X_{t+m}\right)\right| \leq 2\left(2^{1-1/p} + 1\right)\alpha_m^{1-1/p-1/r}\left\|X_t\right\|_p\left\|X_{t+m}\right\|_r$$

where

$$\alpha_m = \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^{\infty}\right) = \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m}^{\infty}} \left| P\left(G \cap H\right) - P\left(G\right) P\left(H\right) \right|.$$

Remark: This is Corollary 14.3 of Davidson (1994). For a proof, see pages 212-213 of Davidson (1994).

Lemma OA-4: Suppose that Assumption 2-3 hold. Let $\tau_1 = \lfloor T_0^{\alpha_1} \rfloor$, where $1 > \alpha_1 > 0$ and $T_0 = T - p + 1$. Then,

(a)

$$\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}\right] \right| = O\left(\frac{1}{\tau_1}\right)$$

(b)

$$\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} \left| E\left(u_{ih} u_{iv} u_{iw}\right) \right| = O\left(\frac{1}{\tau_1^2}\right)$$

(c)

$$\frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} u_{iv} u_{iw}\right] \right| = O\left(\frac{1}{\tau_1^2}\right)$$

Proof of Lemma OA-4:

To show part (a), first write

$$\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}\right] \right| = \frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[u_{ig}^2\right] + \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}\right] \right| \quad (7)$$

Consider now the first term on the right-hand side of expression (7). Note that, trivially, by Assumption 2-3(b), there exists a positive constant C such that

$$\frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[u_{ig}^2\right] \leq \frac{C}{\tau_1} = O\left(\frac{1}{\tau_1}\right) \quad (8)$$

For the second term on the right-hand side of expression (7), note that by Assumption 2-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right) \leq a_1 \exp\left\{-a_2 m\right\}.$$

for every i. Since $\alpha_{i,m} \leq \beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\left\{-a_2 m\right\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma OA-3 with $p = 6$ and $r = 5/4$ to obtain

$$
\begin{aligned}
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} \right] \right| \\
\leq \quad & \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}} + 1 \right) \left[a_1 \exp\left\{ -a_2 (h-g) \right\} \right]^{1-\frac{1}{6}-\frac{4}{5}} \left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^{\frac{5}{4}} \right)^{\frac{4}{5}}
\end{aligned}
$$

Next, by application of Liapunov's inequality, we have that there exists some positive constant $\overline{C}$ such that

$$
\begin{aligned}
\left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^{\frac{5}{4}} \right)^{\frac{4}{5}} \quad & \leq \quad \left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^6 \right)^{\frac{1}{6}} \\
& \leq \quad \left(\sup_t E\left| u_{it} \right|^6 \right)^{\frac{1}{3}} \\
& = \quad \overline{C}^{\frac{1}{3}} < \infty \quad \text{(by Assumption 2-3(b))}
\end{aligned}
$$

Moreover, let $\varrho = h - g$, so that $h = g + \varrho$. Using these notations and the boundedness of $\left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^{\frac{5}{4}} \right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} \right] \right| \\
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}} + 1 \right) \left[a_1 \exp\left\{ -a_2 (h-g) \right\} \right]^{1-\frac{1}{6}-\frac{4}{5}} \left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^{\frac{5}{4}} \right)^{\frac{4}{5}} \\
\leq \quad & \frac{\overline{C}^{\frac{1}{3}}}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}} + 1 \right) \left[a_1 \exp\left\{ -a_2 (h-g) \right\} \right]^{\frac{1}{30}} \\
\leq \quad & \frac{C^*}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{ -\frac{a_2}{30} \varrho \right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}} + 1 \right) \overline{C}^{\frac{1}{3}} a_1^{\frac{1}{30}} \leq C^* < \infty \right) \\
\leq \quad & \frac{C^*}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho=1}^{\infty} \exp\left\{ -\frac{a_2}{30} \varrho \right\} \\
= \quad & \frac{C^*}{\tau_1} \sum_{\varrho=1}^{\infty} \exp\left\{ -\frac{a_2}{30} \varrho \right\} \\
= \quad & O\left(\frac{1}{\tau_1} \right) \quad \text{(given Lemma OA-1)} \qquad (9)
\end{aligned}
$$

It follows from expressions (7), (8), and (9) that

$$
\begin{aligned}
\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}\right]\right| &= \frac{1}{\tau_1^2}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[u_{ig}^2\right]+\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}\right]\right| \\
&= O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
&= O\left(\frac{1}{\tau_1}\right).
\end{aligned}
$$

To show part (b), first write

$$
\begin{aligned}
\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| &= \frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3+\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&\quad +\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \qquad (10)
\end{aligned}
$$

For the first term on the right-hand side of expression (10) above, note that, trivially, we can apply Assumption 2-3(b) to obtain

$$
\frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3\leq\frac{C}{\tau_1^2}=O\left(\frac{1}{\tau_1^2}\right). \qquad (11)
$$

Next, for the second term on the right-hand side of expression (10) above, we can apply Lemma OA-3 with $p=6$ and $r=5/4$ to obtain

$$
\begin{aligned}
&\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ &\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}
\end{aligned}
$$

Next, by application of Hölder's inequality, we have

$$
\begin{aligned}
\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \\
&= \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{iw}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}} \\
&\leq \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= \overline{C}^{\frac{1}{2}}<\infty\ \text{ (by Assumption 2-3(b))}
\end{aligned}
$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$, so that $v = h + \varrho_1$ and $w = v +\ \varrho_2 = h + \varrho_1 +\ \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}} \left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ & \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}} \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}} \left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\ & \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-h\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)\overline{C}^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
\leq\ & \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & \frac{C^*}{\tau_1^2} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma OA-1)}
\end{aligned} \tag{12}
$$

Similarly, for the third term on the right-hand side of expression (10), we can apply Lemma OA-3 with $p = 6$ and $r = 5/4$ to obtain

$$
\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ & \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}}
\end{aligned}
$$

Next, by applying Hölder's inequality, we have

$$\begin{aligned}
\left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} &\leq \left(\left(E\left|u_{ih}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
&= \left(E\left|u_{ih}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
&\leq \left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= \overline{C}^{\frac{1}{2}} < \infty \text{ (by Assumption 2-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$, so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ & \frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
\leq\ & \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)\overline{C}^{\frac{1}{2}}a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\ \sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^*}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\ \sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^*}{\tau_1^2}\left[\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{30}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho_2\right\}\right] \\
=\ & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma OA-1)} \qquad (13)
\end{aligned}$$

It follows from expressions (10), (11), (12), and (13) that

$$\begin{aligned}
\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| &= \frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3+\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&\quad +\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&= O\left(\frac{1}{\tau_1^2}\right)+O\left(\frac{1}{\tau_1^2}\right)+O\left(\frac{1}{\tau_1^2}\right) \\
&= O\left(\frac{1}{\tau_1^2}\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$
\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} u_{iv} u_{iw}\right] \right| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \le h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}^3\right] \right| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} u_{iv} u_{iw}\right] \right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v \le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih} u_{iv} u_{iw}\right] \right| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \le h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}^3\right] \right| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{u_{ig} u_{ih} - E\left(u_{ig} u_{ih}\right) + E\left(u_{ig} u_{ih}\right)\right\} u_{iv} u_{iw}\right] \right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v \le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{u_{ig} u_{ih} - E\left(u_{ig} u_{ih}\right) + E\left(u_{ig} u_{ih}\right)\right\} u_{iv} u_{iw}\right] \right| \\
\le & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \le h}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{ig} u_{ih}^3\right] \right| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{u_{ig} u_{ih} - E\left(u_{ig} u_{ih}\right)\right\} u_{iv} u_{iw}\right] \right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v \le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{u_{ig} u_{ih} - E\left(u_{ig} u_{ih}\right)\right\} u_{iv} u_{iw}\right] \right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left(u_{ig} u_{ih}\right) \right| \left| E\left(u_{iv} u_{iw}\right) \right| \qquad (14)
\end{aligned}
$$

For the first term on the right-hand side of expression (14) above, note that, trivially, by Jensen's inequality and Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\le h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}^3\right]\right| &\le \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\le h}}^{(r-1)\tau+\tau_1+p-1}E\left[\left|u_{ig}u_{ih}^3\right|\right]\\
&\le \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\le h}}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left|u_{ig}\right|^2}\sqrt{E\left|u_{ih}\right|^6}\\
&\le \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\le h}}^{(r-1)\tau+\tau_1+p-1}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\sqrt{E\left|u_{ih}\right|^6}\\
&\quad\text{(by Liapunov's inequality)}\\
&\le \frac{\overline{C}^{\frac{2}{3}}\tau_1^2}{\tau_1^4}\quad\text{(by Assumption 2-3(b))}\\
&= O\left(\frac{1}{\tau_1^2}\right) \qquad (15)
\end{aligned}$$

Next, for the second term on the right-hand side of expression (14), we can apply Lemma OA-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\le h\le v\le w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right|\\
\le\ &\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\le h\le v\le w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}}\right.\\
&\qquad\left.\times\left(E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}}\right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$
\begin{aligned}
E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}} &\leq \left[E\left|u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right|^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}+\left|E\left[u_{ig}u_{ih}\right]\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}+E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}}E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\left(E\left|u_{ih}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\left(E\left|u_{ih}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\right]^{\frac{4}{3}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{4}{3}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left(\overline{C}\right)^{\frac{2}{9}}\left(\overline{C}\right)^{\frac{2}{9}}\left(\overline{C}\right)^{\frac{2}{9}} \quad \text{(by Assumption 2-3(b))} \\
&= 2^{\frac{4}{3}}\overline{C}^{\frac{2}{3}}
\end{aligned}
$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h+ \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}}\right. \\
& \qquad\qquad\left.\times\left(E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}}\right\} \\
\leq\ & \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{\frac{1}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{12}}\left(2^{\frac{4}{3}}\overline{C}^{\frac{2}{3}}\right)^{\frac{3}{4}}\left(\overline{C}\right)^{\frac{1}{6}} \\
\leq\ & \frac{C^*}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{4}}+1\right)\overline{C}^{\frac{2}{3}}a_1^{\frac{1}{12}}\leq C^*<\infty\right) \\
\leq\ & \frac{C^*}{\tau_1^4}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\ \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\ \sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2-1}\exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
\leq\ & \frac{C^*}{\tau_1^2}\sum_{\rho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
=\ & O\left(\frac{1}{\tau_1^2}\right)\quad\text{(given Lemma OA-1)} \qquad (16)
\end{aligned}$$

Similarly, for the third term on the right-hand side of expression (14) above, we can apply Lemma OA-3 with $p=2$ and $r=3$ to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{1}{2}-\frac{1}{3}}\right. \\
& \qquad\qquad\left.\times\left(E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}\right|^{2}\right)^{\frac{1}{2}}\left(E\left|u_{iv}u_{iw}\right|^{3}\right)^{\frac{1}{3}}\right\}
\end{aligned}$$

Next, applications of Hölder's inequality yield

$$
\begin{aligned}
E\left|u_{iv}u_{iw}\right|^3 &\leq \left(E\left|u_{iv}\right|^6\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{2}} \\
&\leq \left(\overline{C}\right)^{\frac{1}{2}}\left(\overline{C}\right)^{\frac{1}{2}} \quad \text{(by Assumption 2-3(b))} \\
&= \overline{C} < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
E\left|\left\{u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\}\right|^2 &\leq 2\left(E\left|u_{ig}u_{ih}\right|^2 + E\left|u_{ig}u_{ih}\right|^2\right) \\
&\quad \text{(by Loève's } c_r \text{ inequality and Jensen's inequality)} \\
&= 4E\left|u_{ig}u_{ih}\right|^2 \\
&\leq 4\left[\left(E\left|u_{ig}\right|^4\right)^{\frac{1}{4}}\left(E\left|u_{ih}\right|^4\right)^{\frac{1}{4}}\right]^2 \\
&\leq 4\left[\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\right]^2 \quad \text{(by Liapunov's inequality)} \\
&\leq 4\left(\sup_{i,t} E\left|u_{it}\right|^6\right)^{\frac{2}{3}} \\
&\leq 4\left(\overline{C}\right)^{\frac{2}{3}} < \infty \quad \text{(by Assumption 2-3(b))}
\end{aligned}
$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|u_{iv}u_{iw}\right|^3$ and $E\left|\left\{u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\}\right|^2$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\} u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \Big\{ 2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{1}{2}-\frac{1}{3}} \\
& \qquad \times \left(E\left|\left\{u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{iv}u_{iw}\right|^3\right)^{\frac{1}{3}} \Big\} \\
\leq\ & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-h\right)\right\}\right]^{\frac{1}{6}} \left(4\overline{C}^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(\overline{C}\right)^{\frac{1}{3}} \\
\leq\ & \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)\overline{C}^{\frac{2}{3}} a_1^{\frac{1}{6}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
=\ & \frac{C^*}{\tau_1^2} \sum_{\rho_1=1}^{\infty} \left(\varrho_1+1\right) \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
=\ & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma OA-1)} \qquad (17)
\end{aligned}$$

Finally, consider the fourth term on the right-hand side of expression (14) above. For this term, we apply the result given in part (a) to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right| \left|E\left(u_{iv}u_{iw}\right)\right| \\
\leq\ & \left(\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right|\right) \left(\frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
=\ & O\left(\frac{1}{\tau_1^2}\right). \qquad (18)
\end{aligned}$$

It follows from expressions (14)-(18) that

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq \; & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}^3\right]\right| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
= \; & O\left(\frac{1}{\tau_1^2}\right). \;\square
\end{aligned}$$

Lemma OA-5: Suppose that Assumptions 2-1, 2-2(a)-(b), 2-5, and 2-6 hold. Then, there exists a positve constant $\overline{C}$ such that

$$E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t$$

and, thus,

$$E\left\|\underline{Y}_t\right\|_2^6 \leq \overline{C} < \infty \text{ and } E\left\|\underline{F}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t,$$

where

$$\underset{dp\times 1}{\underline{Y}_t} = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \text{ and } \underset{Kp\times 1}{\underline{F}_t} = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}.$$

Proof of Lemma OA-5:

To proceed, note that, given Assumption 2-1, we can write the vector moving-average (VMA) representation of the companion form of the FAVAR model as

$$\begin{aligned}
\underline{W}_t &= \left(I_{(d+K)p}-A\right)^{-1}\alpha + \sum_{j=0}^{\infty} A^j E_{t-j} \\
&= \left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' J_{d+K}\alpha + \sum_{j=0}^{\infty} A^j J_{d+K}' J_{d+K} E_{t-j} \\
&= \left(I_{(d+K)p}-A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} A^j J_{d+K}'\varepsilon_{t-j}, \qquad (19)
\end{aligned}$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \; E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

$$\underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.$$

By the triangle inequality,

$$\|\underline{W}_t\|_2 \leq \left\| \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu \right\|_2 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K}\varepsilon_{t-j} \right\|_2$$

Moreover, using the inequality $\left|\sum_{i=1}^m a_i\right|^r \leq m^{r-1}\sum_{i=1}^m |a_i|^r$ for $r \geq 1$, we get

$$\|\underline{W}_t\|_2^6 \leq 2^5 \left(\left\| \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu \right\|_2^6 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K}\varepsilon_{t-j} \right\|_2^6 \right)$$

so that

$$E\|\underline{W}_t\|_2^6 \leq 32 \left\| \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K}\varepsilon_{t-j} \right\|_2^6 \tag{20}$$

Focusing first on the first term on the right-hand side of the inequality (20), we note that

$$\begin{aligned} \left\| \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu \right\|_2^6 &= \left(\mu' J_{d+K} \left(I_{(d+K)p} - A\right)^{-1\prime} \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu\right)^3 \\ &= \left(\mu' J_{d+K} \left[\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right]^{-1} J'_{d+K}\mu\right)^3 \\ &\leq \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu' J_{d+K} J'_{d+K}\mu\right)^3 \\ &= \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu'\mu\right)^3 \end{aligned}$$

Now, by Assumption 2-6, there exists a constant $\underline{C} > 0$ such that

$$\begin{aligned}
\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\} &= \lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)'\left(I_{(d+K)p} - A\right)\right\} \\
&= \sigma^2_{\min}\left(I_{(d+K)p} - A\right) \\
&\geq \underline{C}\lambda^2_{\min}\left(I_{(d+K)p} - A\right) \\
&\geq \underline{C}\left[1 - \phi_{\max}\right]^2 \\
&> 0
\end{aligned}$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 2-1, all eigenvalues of A have modulus less than 1. It follows by Assumption 2-5 that, there exists a positive constant $\overline{C}_1$ such that

$$\begin{aligned}
\left\|\left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu\right\|_2^6 &\leq \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu'\mu\right)^3 \\
&\leq \frac{\left\|\mu\right\|_2^6}{\underline{C}^3\left[1 - \phi_{\max}\right]^6} \leq \overline{C}_1 < \infty.
\end{aligned}$$

To show the boundedness of the second term on the right-hand side of the inequality (20), let $e_{g,(d+K)p}$ be a $(d+K)p \times 1$ elementary vector whose g^{th} component is 1 and all other components are 0 for $g \in \{1, 2, ..., (d+K)p\}$, and note that

$$\begin{aligned}
\left\|\sum_{j=0}^{\infty} A^j J'_{d+K}\varepsilon_{t-j}\right\|_2^2 &= \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} e'_{g,(d+K)p}A^j J'_{d+K}\varepsilon_{t-j}\right)^2 \\
&= \sum_{g=1}^{(d+K)p}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} e'_{g,(d+K)p}A^j J'_{d+K}\varepsilon_{t-j}\varepsilon'_{t-k}J_{d+K}\left(A'\right)^k e_{g,(d+K)p}
\end{aligned}$$

from which we obtain, by applying the inequality $\left|\sum_{i=1}^m a_i\right|^r \leq m^{r-1}\sum_{i=1}^m \left|a_i\right|^r$ for $r \geq 1$

$$\begin{aligned}
&\left\|\sum_{j=0}^{\infty} A^j J'_{d+K}\varepsilon_{t-j}\right\|_2^6 \\
&= \left[\sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} e'_{g,(d+K)p}A^j J'_{d+K}\varepsilon_{t-j}\right)^2\right]^3 \\
&\leq \left[(d+K)p\right]^2\sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} e'_{g,(d+K)p}A^j J'_{d+K}\varepsilon_{t-j}\right)^6 \\
&= \left[(d+K)p\right]^2\sum_{g=1}^{(d+K)p}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e'_{g,(d+K)p}A^j J'_{d+K}\varepsilon_{t-j}\varepsilon'_{t-k}J_{d+K}\left(A'\right)^k e_{g,(d+K)p}\right. \\
&\qquad \left.\times e'_{g,(d+K)p}A^i J'_d\varepsilon_{t-i}\varepsilon'_{t-\ell}J_{d+K}\left(A'\right)^{\ell} e_{g,(d+K)p}e'_{g,(d+K)p}A^r J'_{d+K}\varepsilon_{t-r}\varepsilon'_{t-s}J_d\left(A'\right)^s e_{g,(d+K)p}\right\}
\end{aligned}$$

Hence,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}\right\|_2^6 \\
\leq\; & \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^6 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{3} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^3\right)^2 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{2}\binom{4}{2} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^2\right)^3 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{4} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^4 \sum_{k=0}^{\infty} E\left|e_{g,(d+K)p}' A^k J_{d+K}' \varepsilon_{t-k}\right|^2 \\
=\; & \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^6 \\
& +20\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^3\right)^2 \\
& +90\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^2\right)^3 \\
& +15\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^4 \sum_{k=0}^{\infty} E\left|e_{g,(d+K)p}' A^k J_{d+K}' \varepsilon_{t-k}\right|^2
\end{aligned}
$$

Next, applying the Cauchy-Schwarz inequality, we further obtain

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}\right\|_2^6 \\
\leq\ & [(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad\qquad \left. \times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k J_{d+K}' J_{d+K} \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\} \\
\leq\ & [(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad\qquad \left. \times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\}
\end{aligned}
$$

In addition, observe that, for every $g \in \{1, 2, ..., (d+K)p\}$

$$\begin{aligned}
& e_{g,(d+K)p}^{\prime} A^j \left(A^j\right)^{\prime} e_{g,(d+K)p} \\
\leq \;& \lambda_{\max}\left\{A^j \left(A^j\right)^{\prime}\right\} \\
= \;& \lambda_{\max}\left\{\left(A^j\right)^{\prime} A^j\right\} \\
= \;& \sigma_{\max}^2\left(A^j\right) \\
\leq \;& C \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \quad \text{(by Assumption 2-6)} \\
= \;& C \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \\
= \;& C \phi_{\max}^{2j}
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ given that Assumption 2-1 implies that all eigenvalues of A have modulus less than 1. Now, in light of Assumption 2-2(b), we can set $C \geq 1$ to be a constant such that $E\|\varepsilon_{t-j}\|_2^6 \leq C < \infty$, so that, by Liapunov's inequality,

$$\begin{aligned}
E\|\varepsilon_{t-j}\|_2^2 &\leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{3}} \leq C^{\frac{1}{3}},\; E\|\varepsilon_{t-j}\|_2^3 \leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}, \\
E\|\varepsilon_{t-j}\|_2^4 &\leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{2}{3}} \leq C^{\frac{2}{3}},
\end{aligned}$$

and, thus,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right\|_{2}^{6} \\
\leq\ & [(d+K) p]^{2} \sum_{g=1}^{(d+K) p} \sum_{j=0}^{\infty}\left[e_{g,(d+K) p}^{\prime} A^{j}\left(A^{j}\right)^{\prime} e_{g,(d+K) p}\right]^{3} E\left\|\varepsilon_{t-j}\right\|_{2}^{6} \\
& +20[(d+K) p]^{2} \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K) p}^{\prime} A^{j}\left(A^{j}\right)^{\prime} e_{g,(d+K) p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_{2}^{3}\right)^{2} \\
& +90[(d+K) p]^{2} \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K) p}^{\prime} A^{j}\left(A^{j}\right)^{\prime} e_{g,(d+K) p}\right] E\left\|\varepsilon_{t-j}\right\|_{2}^{2}\right)^{3} \\
& +15[(d+K) p]^{2} \sum_{g=1}^{(d+K) p}\left\{\sum_{j=0}^{\infty}\left[e_{g,(d+K) p}^{\prime} A^{j}\left(A^{j}\right)^{\prime} e_{g,(d+K) p}\right]^{2} E\left\|\varepsilon_{t-j}\right\|_{2}^{4}\right. \\
& \qquad\qquad\left.\times \sum_{k=0}^{\infty}\left[e_{g,(d+K) p}^{\prime} A^{k}\left(A^{k}\right)^{\prime} e_{g,(d+K) p}\right] E\left\|\varepsilon_{t-k}\right\|_{2}^{2}\right\} \\
\leq\ & C[(d+K) p]^{2}\left\{\sum_{g=1}^{(d+K) p} \sum_{j=0}^{\infty} \phi_{\max }^{6 j}+20 \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty} \phi_{\max }^{3 j}\right)^{2}+90 \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty} \phi_{\max }^{2 j}\right)^{3}\right. \\
& \qquad\left.+15 \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty} \phi_{\max }^{4 j}\right)\left(\sum_{k=0}^{\infty} \phi_{\max }^{2 k}\right)\right\} \\
\leq\ & C[(d+K) p]^{3} \\
& \times\left\{\frac{1}{1-\phi_{\max }^{6}}+20\left(\frac{1}{1-\phi_{\max }^{3}}\right)^{2}+90\left(\frac{1}{1-\phi_{\max }^{2}}\right)^{3}+15\left(\frac{1}{1-\phi_{\max }^{4}}\right)\left(\frac{1}{1-\phi_{\max }^{2}}\right)\right\} \\
\leq\ & \overline{C}_{2}<\infty
\end{aligned}
$$

for some constant such that

$$
\begin{aligned}
& \overline{C}_{2} \\
\geq\ & C[(d+K) p]^{3} \\
& \times\left\{\frac{1}{1-\phi_{\max }^{6}}+20\left(\frac{1}{1-\phi_{\max }^{3}}\right)^{2}+90\left(\frac{1}{1-\phi_{\max }^{2}}\right)^{3}+15\left(\frac{1}{1-\phi_{\max }^{4}}\right)\left(\frac{1}{1-\phi_{\max }^{2}}\right)\right\} .
\end{aligned}
$$

Putting everything together, we see that

$$
\begin{aligned}
E\left\|\underline{W}_{t}\right\|_{2}^{6} & \leq 32\left\|\left(I_{(d+K) p}-A\right)^{-1} J_{d+K}^{\prime} \mu\right\|_{2}^{6}+32 E\left\|\sum_{j=0}^{\infty} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right\|_{2}^{6} \\
& \leq 32\left(\overline{C}_{1}+\overline{C}_{2}\right) \\
& \leq \overline{C}<\infty
\end{aligned}
$$

for a constant $\overline{C}$ such that $0 < 32\left(\overline{C}_1 + \overline{C}_2\right) \leq \overline{C} < \infty$.

In addition, define$\mathcal{P}_{(d+K)p}$ to be the $(d+K)\,p \times (d+K)\,p$ permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix}; \tag{21}$$

and let $S_d' = \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}$ and $S_K' = \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix}$. Note that

$$\begin{aligned} S_d'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix} \begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\ S_K'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix} \begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t. \end{aligned}$$

so that

$$\begin{aligned} \left\|\underline{Y}_t\right\|_2 &\leq \left\|S_d'\right\|_2 \left\|\mathcal{P}_{(d+K)p}\right\|_2 \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_d S_d'\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}_{(d+K)p}'\mathcal{P}_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_d' S_d\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(I_{dp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \left\|\underline{W}_t\right\|_2 \end{aligned}$$

and

$$\begin{aligned} \left\|\underline{F}_t\right\|_2 &\leq \left\|S_K'\right\|_2 \left\|\mathcal{P}_{(d+K)p}\right\|_2 \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_K S_K'\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}_{(d+K)p}'\mathcal{P}_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_K' S_K\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(I_{Kp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\ &= \left\|\underline{W}_t\right\|_2 \end{aligned}$$

It further follows that

$$E\left\|\underline{Y}_t\right\|_2^6 \leq E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty \text{ and } E\left\|\underline{F}_t\right\|_2^6 \leq E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty. \ \square$$

Lemma OA-6: Suppose that Assumptions 2-1, 2-2(a)-(b), 2-3, 2-5, 2-6, and 2-9(b) hold. Then, the following statements are true as $N_1, T \to \infty$

(a)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right| \xrightarrow{p} 0.$$

(b)

$$\max_{1\le \ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2\xrightarrow{p}0$$

(c)

$$\max_{1\le \ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right|\xrightarrow{p}0.$$

(d)

$$\max_{1\le \ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2\xrightarrow{p}0$$

(e)

$$\max_{1\le \ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right|\xrightarrow{p}0$$

Proof of Lemma OA-6.

To show part (a), first write

$$\begin{aligned}
& P\left\{\max_{1\le \ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right|\ge\epsilon\right\}\\
= \;& P\left\{\max_{1\le \ell\le d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\ge\epsilon^6\right\}\\
\le \;& P\left\{\max_{1\le \ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\ge\epsilon^6\right\}\\
& \text{(by Jensen's inequality)}\\
\le \;& P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\ge\epsilon^6\right\}\\
\le \;& \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6
\end{aligned}$$

Next, note that

$$
\begin{aligned}
& \frac{1}{q} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} E\left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \\
\leq \quad & \frac{1}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t \varepsilon_{\ell,t+1}\right]^6 \\
& + \frac{20}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\left|\gamma_i' \underline{F}_t \varepsilon_{\ell,t+1}\right|\right]^3 E\left[\left|\gamma_i' \underline{F}_s \varepsilon_{\ell,s+1}\right|\right]^3 \\
& + \frac{15}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t \varepsilon_{\ell,t+1}\right]^4 E\left[\gamma_i' \underline{F}_s \varepsilon_{\ell,s+1}\right]^2 \\
& + \frac{90}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{r=(r-1)\tau+p \\ r \neq t,\ r \neq s}}^{(r-1)\tau+\tau_1+p-1} \Big\{ E\left[\gamma_i' \underline{F}_t \varepsilon_{\ell,t+1}\right]^2 E\left[\gamma_i' \underline{F}_s \varepsilon_{\ell,s+1}\right]^2 \\
& \qquad \times E\left[\gamma_i' \underline{F}_s \varepsilon_{\ell,r+1}\right]^2 \Big\} \\
\leq \quad & \frac{1}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\left(\gamma_i' \underline{F}_t\right)^6\right] E\left[\varepsilon_{\ell,t+1}^6\right] \\
& + \frac{20}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \frac{1}{64} E\left[\gamma_i' \underline{F}_t \underline{F}_t' \gamma_i + \varepsilon_{\ell,t+1}^2\right]^3 E\left[\gamma_i' \underline{F}_s \underline{F}_s' \gamma_i + \varepsilon_{\ell,s+1}^2\right]^3 \\
& + \frac{15}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t \underline{F}_t' \gamma_i\right]^2 E\left[\varepsilon_{\ell,t+1}^4\right] E\left[\gamma_i' \underline{F}_s \underline{F}_s' \gamma_i\right] E\left[\varepsilon_{\ell,s+1}^2\right] \\
& + \frac{90}{q\tau_1^6} \sum_{r=1}^{q} \sum_{\ell=1}^{d} \sum_{i \in H^c} \left\{ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t \underline{F}_t' \gamma_i\right] E\left[\varepsilon_{\ell,t+1}^2\right] E\left[\gamma_i' \underline{F}_s \underline{F}_s' \gamma_i\right] E\left[\varepsilon_{\ell,s+1}^2\right] \right. \\
& \qquad \left. \times \sum_{\substack{r=(r-1)\tau+p \\ r \neq t,\ r \neq s}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_r \underline{F}_r' \gamma_i\right] E\left[\varepsilon_{\ell,r+1}^2\right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
\leq\;& \frac{1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\|\gamma_i\|_2^6\,E\left[\|\underline{F}_t\|_2^6\right]E\left[\varepsilon_{\ell,t+1}^6\right] \\
&+\frac{(20\cdot 16)}{64q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\left\{\left(E\left[(\gamma_i'\underline{F}_t)^6\right]+E\left[\varepsilon_{\ell,t+1}^6\right]\right)\right. \\
&\qquad\qquad\left.\times\left(E\left[(\gamma_i'\underline{F}_s)^6\right]+E\left[\varepsilon_{\ell,s+1}^6\right]\right)\right\} \\
&+\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\left\{\|\gamma_i\|_2^4\,E\left[\|\underline{F}_t\|_2^4\right]E\left[\varepsilon_{\ell,t+1}^4\right]\right. \\
&\qquad\qquad\left.\times\|\gamma_i\|_2^2\,E\left[\|\underline{F}_s\|_2^2\right]E\left[\varepsilon_{\ell,s+1}^2\right]\right\} \\
&+\frac{90}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\left\{\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\|\gamma_i\|_2^2\,E\left[\|\underline{F}_t\|_2^2\right]E\left[\varepsilon_{\ell,t+1}^2\right]\right. \\
&\quad\left.\times\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\|\gamma_i\|_2^2\,E\left[\|\underline{F}_s\|_2^2\right]E\left[\varepsilon_{\ell,s+1}^2\right]\sum_{\substack{r=(r-1)\tau+p\\ r\neq t,\ r\neq s}}^{(r-1)\tau+\tau_1+p-1}\|\gamma_i\|_2^2\,E\left[\|\underline{F}_r\|_2^2\right]E\left[\varepsilon_{\ell,r+1}^2\right]\right\} \\
\leq\;& C\left(\frac{N_1}{\tau_1^5}+5\frac{N_1}{\tau_1^4}+15\frac{N_1}{\tau_1^4}+90\frac{N_1}{\tau_1^3}\right) \\
&\text{(applying Assumptions 2-2(b), Assumption 2-5, and Lemma OA-5)} \\
=\;& O\left(\frac{N_1}{\tau_1^3}\right).
\end{aligned}
$$

It follows that

$$
P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right|\geq\epsilon\right\}=O\left(\frac{N_1}{\tau_1^3}\right)=o\left(1\right).
$$

To show part (b), note that, for any $\epsilon > 0$

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2\geq\epsilon\right\} \\
= & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2\right|^3\geq\epsilon^3\right\} \\
\leq & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\geq\epsilon^3\right\} \\
& \text{(by Jensen't inequality)} \\
\leq & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\geq\epsilon^3\right\} \\
\leq & \frac{1}{\epsilon^3}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6
\end{aligned}
$$

The rest of the proof for part (b) then follows in a manner similar to the argument given for part (a) above.

To show part (c), first note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right|\geq\epsilon\right\} \\
= & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6\geq\epsilon^6\right\} \\
\leq & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6\geq\epsilon^6\right\} \\
& \text{(by convexity or Jensen's inequality)} \\
\leq & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6\geq\epsilon^6\right\} \\
\leq & \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \qquad (22)
\end{aligned}
$$

Now, there exists a constant $C_1 > 1$ such that

$$
\begin{aligned}
& \frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \\
\leq \quad & \frac{C_1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\left\{\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \right. \\
& \qquad \left. \times \sum_{\ell=1}^{d}\left|E\left[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}\right]\right|\right\}
\end{aligned}
$$

Next, note that, by repeated application of Hölder's inequality, we have by Lemma OA-5 that there exists a positive constant $\overline{C}$ such that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left|E\left[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}\right]\right| \\
\leq \quad & \sum_{\ell=1}^{d}\left(E\left[y_{\ell,t+1}^2y_{\ell,s+1}^2y_{\ell,g+1}^2\right]\right)^{\frac{1}{2}}\left(E\left[y_{\ell,h+1}^2y_{\ell,v+1}^2y_{\ell,w+1}^2\right]\right)^{\frac{1}{2}} \\
\leq \quad & \sum_{\ell=1}^{d}\left(\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{3}}\left(E\left[\left|y_{\ell,s+1}y_{\ell,g+1}\right|^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{3}}\left(E\left[\left|y_{\ell,v+1}y_{\ell,w+1}\right|^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
\leq \quad & \sum_{\ell=1}^{d}\left[\left(\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,s+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,g+1}^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}}\right. \\
& \qquad \left.\times\left(\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,v+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,w+1}^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}}\right] \\
\leq \quad & \sum_{\ell=1}^{d}\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,s+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,g+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,v+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,w+1}^6\right]\right\}^{\frac{1}{6}} \\
\leq \quad & d\max_{1\leq\ell\leq d}\sup_{t} E\left[y_{\ell,t}^6\right] \\
\leq \quad & \overline{C} < \infty \\
& \Big(\text{since, given that } y_{\ell,t} = \mathbf{e}_{\ell,dp}^{\prime}\underline{Y}_t;\ E\left[y_{\ell,t}^6\right] \leq E\left\|\underline{Y}_t\right\|_2^6 \leq \overline{C} \text{ by Lemma OA-5} \\
& \text{where } \overline{C} \text{ is a constant not depending on } \ell \text{ or } t\ \Big)
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
& \frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
=\ & \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5+\mathcal{T}_6,\ \ (say)\,. \qquad (23)
\end{aligned}
$$

Consider first $\mathcal{T}_1$. Note that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}E\left[\left|u_{it}u_{is}u_{ig}^4\right|\right] \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(E\left[\left|u_{it}u_{is}\right|^3\right]\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \quad \text{(by Hölder's inequality)} \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(\left[E\left\{\left|u_{it}\right|^6\right\}\right]^{\frac{1}{2}}\left[E\left\{\left|u_{is}\right|^6\right\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\qquad \text{(by further application of Hölder's inequality)} \\
&= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\overline{C} \quad \text{(by Assumption 2-3(b))} \\
&\leq C_1\overline{C}^2\frac{N_1}{\tau_1^5} \\
&= O\left(\frac{N_1}{\tau_1^5}\right). \qquad (24)
\end{aligned}$$

Next, consider $\mathcal{T}_2$. For this term, note first that by Assumption 2-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right)\leq a_1\exp\left\{-a_2m\right\}$$

for every i. Since $\alpha_{i,m}\leq\beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m}\leq a_1\exp\left\{-a_2m\right\} \text{ for every } i.$$

Hence, we apply Lemma OA-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned}
& \mathcal{T}_2 \\
= \quad & \frac{C_1 \overline{C}}{q\tau_1^6} \sum_{r=1}^{q} \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq \quad & \frac{C_1 \overline{C}}{q\tau_1^6} \sum_{r=1}^{q} \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v>0}}^{(r-1)\tau+\tau_1+p-1} \Big\{ 2\left(2^{1-\frac{4}{5}}+1\right)\left[a_1 \exp\left\{-a_2(w-v)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \\
& \qquad \times \left(E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \Big\}
\end{aligned}$$

Next, by Liapunov's inequality and Assumption 2-3(b), we obtain

$$\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \leq \left(E\left|u_{iw}\right|^7\right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}$$

Making use of this bound and by repeated application of Hölder's inequality, we have

$$\begin{aligned}
& E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}} \\
\leq \quad & \left[E\left|u_{it}u_{is}u_{ig}\right|^{\frac{25}{12}}\right]^{\frac{3}{5}} \left[E\left|u_{ih}u_{iv}\right|^{\frac{25}{8}}\right]^{\frac{2}{5}} \\
\leq \quad & \left[\left(E\left|u_{it}u_{is}\right|^{\frac{150}{47}}\right)^{\frac{47}{72}} \left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}} \left[\left(E\left|u_{ih}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}} \left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq \quad & \left[\left(\sqrt{E\left|u_{it}\right|^{\frac{300}{47}}}\sqrt{E\left|u_{is}\right|^{\frac{300}{47}}}\right)^{\frac{47}{72}} \left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}} \left[\left(E\left|u_{ih}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}} \left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
= \quad & \left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}} \left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}} \left(E\left|u_{ih}\right|^{6}\right)^{\frac{15}{72}} \left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}} \left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}} \\
= \quad & \left[\left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}} \left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}} \\
\leq \quad & \left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}} \left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{ih}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \left[\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \\
\leq \quad & \left(\overline{C}\right)^{\frac{5}{28}} \left(\overline{C}\right)^{\frac{5}{28}} \left(\overline{C}\right)^{\frac{5}{28}} \left(\overline{C}\right)^{\frac{5}{28}} \left(\overline{C}\right)^{\frac{5}{28}} \quad \text{(by Assumption 2-3(b))} \\
= \quad & \overline{C}^{\frac{25}{28}}
\end{aligned}$$

Moreover, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}$ as shown above, we can further write

$$
\begin{aligned}
& \mathcal{T}_2 \\
\leq\; & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{4}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\right. \\
& \left.\times\left(E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}}\right\} \\
\leq\; & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{\frac{1}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}}\overline{C}^{\frac{25}{28}}\overline{C}^{\frac{1}{7}} \\
\leq\; & \frac{C_1\overline{C}^{\frac{57}{28}}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{\frac{1}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\; & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{1}{5}}+1\right)C_1\overline{C}^{\frac{57}{28}}a_1^{\frac{1}{30}}\leq C^*<\infty\right) \\
\leq\; & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\rho_3=1}^{\infty}\sum_{\rho_1=0}^{\rho_3}\sum_{\rho_2=0}^{\rho_3}\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
\leq\; & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\rho_3=1}^{\infty}\left(\rho_3+1\right)^2\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
=\; & C^*\frac{N_1}{\tau_1^3}\left[\sum_{\rho_3=1}^{\infty}\rho_3^2\exp\left\{-\frac{a_2}{30}\rho_3\right\}+2\sum_{\rho_3=1}^{\infty}\rho_3\exp\left\{-\frac{a_2}{30}\rho_3\right\}+\sum_{\rho_3=1}^{\infty}\exp\left\{-\frac{a_2}{30}\rho_3\right\}\right] \\
=\; & O\left(\frac{N_1}{\tau_1^3}\right)\quad\left(\text{by Lemma OA-1}\right).
\end{aligned}
\tag{25}
$$

Now, consider $\mathcal{T}_3$. Here, we can apply Lemma OA-3 with $p=3/2$ and $r=7/2$ to obtain

$$
\begin{aligned}
\mathcal{T}_3 \;=\; & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq\; & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{2}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{2}{3}-\frac{2}{7}}\right. \\
& \left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}}\right\}
\end{aligned}
$$

Next, note that applications of Hölder's inequality yield

$$\begin{aligned} E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}} &\leq \left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{2}} \\ &\leq \left(\overline{C}\right)^{\frac{1}{2}}\left(\overline{C}\right)^{\frac{1}{2}} \quad \text{(by Assumption 2-3(b))} \\ &= \overline{C} < \infty \end{aligned}$$

and

$$\begin{aligned} E\left|\left\{u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}}\left(E\left|u_{it}u_{is}u_{ig}u_{ih}\right|^{\frac{3}{2}} + E\left|u_{it}u_{is}u_{ig}u_{ih}\right|^{\frac{3}{2}}\right) \\ &\quad \text{(by Loève's } c_r \text{ inequality)} \\ &\leq 2^{\frac{3}{2}}E\left|u_{it}u_{is}u_{ig}u_{ih}\right|^{\frac{3}{2}} \\ &\leq 2^{\frac{3}{2}}\left(E\left|u_{it}u_{is}\right|^{3}\right)^{\frac{1}{2}}\left(E\left|u_{ig}u_{ih}\right|^{3}\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{3}{2}}\left(\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{3}{2}} \\ &\leq 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ih}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{3}{2}} \\ &\quad \text{(by Liapunov's inequality)} \\ &\leq 2^{\frac{3}{2}}\left[\left(\sup_{i,t}E\left|u_{it}\right|^{7}\right)^{\frac{4}{7}}\right]^{\frac{3}{2}} \\ &= 2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 2-3(b))} \end{aligned}$$

Again, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}}$ as shown above, we can further write

$$\begin{aligned}
\mathcal{T}_3 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\le s\le g\le h\le v\le w \\ v-h\ge\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&\le \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\le s\le g\le h\le v\le w \\ v-h\ge\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{2}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{2}{3}-\frac{2}{7}}\right. \\
&\qquad\qquad \left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}}\right\} \\
&\le \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\le s\le g\le h\le v\le w \\ v-h\ge\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{\frac{1}{21}}\left(2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}}\right)^{\frac{2}{3}}\left(\overline{C}\right)^{\frac{2}{7}} \\
&\le \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\le s\le g\le h\le v\le w \\ v-h\ge\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&\quad \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{3}}+1\right)C_1\overline{C}^{\frac{13}{7}}a_1^{\frac{1}{21}}\le C^*<\infty\right) \\
&\le \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\sum_{\varrho_3=0}^{\varrho_2}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&\le C^*\frac{N_1}{\tau_1^3}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&= C^*\frac{N_1}{\tau_1^3}\left[\sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}\right] \\
&= O\left(\frac{N_1}{\tau_1^3}\right)\quad(\text{by Lemma OA-1}). \qquad (26)
\end{aligned}$$

Turning our attention to the term $\mathcal{T}_4$, note that, from the upper bounds given in the proofs of parts (a) and (c) of Lemma OA-4, it is clear that there exists a positive constant C such that

$$\frac{1}{\tau_1^4}\sum_{\substack{t,s,g,h=(r-1)\tau+p \\ t\le s\le g\le h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\le\frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^2}\sum_{\substack{v,w=(r-1)\tau+p \\ v\le w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{iv}u_{iw}\right)\right|\le\frac{C}{\tau_1}$$

from which it follows that

$$\begin{aligned}
\mathcal{T}_4 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{1}{\tau_1^4}\sum_{\substack{t,s,g,h=(r-1)\tau+p\\ t\leq s\leq g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\right)\left(\frac{1}{\tau_1^2}\sum_{\substack{v,w=(r-1)\tau+p\\ v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{C}{\tau_1^2}\right)\left(\frac{C}{\tau_1}\right) \\
&= C_1\overline{C}C^2\frac{N_1}{\tau_1^3} \\
&= O\left(\frac{N_1}{\tau_1^3}\right). \qquad (27)
\end{aligned}$$

Consider now $\mathcal{T}_5$. In this case, we apply Lemma OA-3 with $p = 2$ and $r = 9/4$ to obtain

$$\begin{aligned}
\mathcal{T}_5 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}u_{ih}u_{iv}u_{iw}\right]\right| \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\Big\{2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}} \\
&\qquad\times\left(E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}}\left(E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}}\Big\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
& E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}} \\
\leq\; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[E\left|u_{iv}u_{iw}\right|^{\frac{63}{19}}\right]^{\frac{19}{28}} \\
\leq\; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\right]^{\frac{19}{28}} \\
=\; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \\
=\; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\right]^{\frac{9}{4}} \\
\leq\; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{9}{4}} \quad \text{(by Liapunov's inequality)} \\
\leq\; & \left(\sup_{i,t} E\left|u_{it}\right|^{7}\right)^{\frac{27}{28}} \\
\leq\; & \overline{C}^{\frac{27}{28}} \quad \text{(by Assumption 2-3(b))}
\end{aligned}$$

and

$$\begin{aligned}
E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^{2} \leq\; & 2\left(E\left|u_{it}u_{is}u_{ig}\right|^{2}+E\left|u_{it}u_{is}u_{ig}\right|^{2}\right) \\
& \text{(by Loève's } c_r \text{ inequality)} \\
\leq\; & 4E\left|u_{it}u_{is}u_{ig}\right|^{2} \\
\leq\; & 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(E\left|u_{is}u_{ig}\right|^{3}\right)^{\frac{2}{3}} \\
\leq\; & 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(\sqrt{E\left|u_{is}\right|^{6}}\sqrt{E\left|u_{ig}\right|^{6}}\right)^{\frac{2}{3}} \\
=\; & 4\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\right]^{2}\left[\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\right]^{2} \\
\leq\; & 4\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{2}\left[\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\right]^{2} \\
& \text{(by Liapunov's inequality)} \\
\leq\; & 4\left[\left(\sup_{i,t} E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{6} \\
\leq\; & 4\overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 2-3(b))}
\end{aligned}$$

Define again $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^{2}$ as shown above, we can further write

$$
\begin{aligned}
& \mathcal{T}_5 \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)^{\theta}\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}}\right. \\
& \qquad\left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^{2}\right)^{\frac{1}{2}}\left(E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}}\right\} \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{\frac{1}{18}}\left(4\overline{C}^{\frac{6}{7}}\right)^{\frac{1}{2}}\left(\overline{C}^{\frac{27}{28}}\right)^{\frac{4}{9}} \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)C_1\overline{C}^{\frac{13}{7}}a_1^{\frac{1}{18}}\leq C^*<\infty\right) \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1}\sum_{\varrho_3=0}^{\varrho_1}\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq\ & C^*\frac{N_1}{\tau_1^3}\sum_{\varrho_1=1}^{\infty}\left(\varrho_1+1\right)^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
=\ & C^*\frac{N_1}{\tau_1^3}\left[\sum_{\varrho_1=1}^{\infty}\varrho_1^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+2\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{18}\varrho_1\right\}\right] \\
=\ & O\left(\frac{N_1}{\tau_1^3}\right)\quad\text{(by Lemma OA-1)}
\end{aligned}
\tag{28}
$$

Finally, consider $\mathcal{T}_6$. Note that, from the upper bounds given in the proofs of part (b) of Lemma OA-4, it is clear that there exists a positive constant C such that

$$
\frac{1}{\tau_1^3}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\leq\frac{C}{\tau_1^2}
$$

and

$$
\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\leq\frac{C}{\tau_1^2}
$$

from which it follows that

$$
\begin{aligned}
\mathcal{T}_6 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{1}{\tau_1^3}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\right)\left(\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{C}{\tau_1^2}\right)\left(\frac{C}{\tau_1^2}\right) \\
&= C_1C\overline{C}^2\frac{N_1}{\tau_1^4} \\
&= O\left(\frac{N_1}{\tau_1^4}\right). \qquad (29)
\end{aligned}
$$

It follows from expressions (22)-(29) that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right|\geq\epsilon\right\} \\
&\leq \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \\
&\leq \frac{1}{\epsilon^4}\left(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5+\mathcal{T}_6\right) \\
&= O\left(\frac{N_1}{\tau_1^5}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^4}\right) \\
&= O\left(\frac{N_1}{\tau_1^3}\right) \\
&= o(1) \quad \left(\text{by Assumption 2-9(b) which stipulates that } \frac{N_1}{\tau_1^3}\sim\frac{N_1}{T^{3\alpha_1}}\to 0\right)
\end{aligned}
$$

which proves the required result.

Turning our attention to part (d), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2 \geq \epsilon\right\} \\
= \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2\right|^3 \geq \epsilon^3\right\} \\
\leq \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^3\right\} \\
& \text{(by Jensen's inequality)} \\
\leq \; & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^3\right\} \\
\leq \; & \frac{1}{\epsilon^3}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 .
\end{aligned}$$

The rest of the proof for part (d) then follows in a manner similar to the argument given for part (c) above.

For part (e), note that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)\right| \\
\leq \; & \max_{1\leq \ell\leq d}\max_{i\in H^c}\sqrt{\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2}\sqrt{\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2} \\
\leq \; & \left\{\sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2}\right. \\
& \left.\times\sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2}\right\} \\
= \; & o_p(1),
\end{aligned}$$

where the convergence in probability to zero in the last line above follows from applying the results in parts (b) and (d) of this lemma. □

Lemma OA-7: Suppose that Assumptions 2-1 and 2-6 hold. Then, the following statements are true.

(a) There exists a positive constant $C^{\dagger}$ such that

$$\|A_{YY}\|_2 \leq C^{\dagger}\phi_{\max}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ with $0 < \phi_{\max} < 1$.

(b) There exists a positive constant $C^{\dagger}$ such that

$$\|A_{YF}\|_2 \leq C^{\dagger}\phi_{\max}$$

where $\phi_{\max}$ is as defined in part (a).

Proof of Lemma OA-7:

To proceed, recall first that the FAVAR model, i.e.,

$$\begin{aligned} Y_t &= \mu_Y + A_{YY}\underline{Y}_{t-1} + A_{YF}\underline{F}_{t-1} + \varepsilon_t^Y \\ F_t &= \mu_F + A_{FY}\underline{Y}_{t-1} + A_{FF}\underline{F}_{t-1} + \varepsilon_t^F, \end{aligned}$$

can be written in the companion form

$$\underline{W}_t = \alpha + A\underline{W}_{t-1} + E_t$$

where $\underline{W}_t = \begin{pmatrix} W_t' & W_{t-1}' & \cdots & W_{t-p+2}' & W_{t-p+1}' \end{pmatrix}'$ with $W_t = \begin{pmatrix} Y_t' & F_t' \end{pmatrix}'$ and where

$$\alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},\ A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & 0 & 0 \\ 0 & I_{d+K} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix},\ \text{and } E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

with $\mu = \begin{pmatrix} \mu_Y' & \mu_F' \end{pmatrix}'$, $\varepsilon_t = \begin{pmatrix} \varepsilon_t^{Y\prime} & \varepsilon_t^{F\prime} \end{pmatrix}'$, and

$$A_\ell = \begin{pmatrix} A_{YY,\ell} & A_{YF,\ell} \\ A_{FY,\ell} & A_{FF,\ell} \end{pmatrix} \text{ for } \ell = 1, ..., p.$$

Let $\mathcal{P}_{(d+K)p}$ be the $(d+K)\,p \times (d+K)\,p$ permutation matrix defined by expression (21) in the proof of Lemma OA-5; and it is easy to see that $\overline{A} = \mathcal{P}_{(d+K)p}A\mathcal{P}_{(d+K)p}'$ has the partitioned form

$$\overline{A} = \mathcal{P}_{(d+K)p}A\mathcal{P}_{(d+K)p}' = \begin{pmatrix} \underset{d\times dp}{\overline{A}_{11}} & \underset{d\times Kp}{\overline{A}_{12}} \\ \underset{d(p-1)\times dp}{\overline{A}_{21}} & \underset{d(p-1)\times Kp}{\overline{A}_{22}} \\ \underset{K\times dp}{\overline{A}_{31}} & \underset{K\times Kp}{\overline{A}_{32}} \\ \underset{K(p-1)\times dp}{\overline{A}_{41}} & \underset{K(p-1)\times Kp}{\overline{A}_{42}} \end{pmatrix}$$

where $\overline{A}_{11} = A_{YY}$ and $\overline{A}_{12} = A_{YF}$, i.e., the first d rows of the matrix $\overline{A}$ as given by the submatrix $\begin{bmatrix} A_{YY} & A_{YF} \end{bmatrix}$.

Now, to show part (a), let $\overline{v} \in \mathbb{R}^{dp}$ such that $\|\overline{v}\|_2 = 1$ and such that

$$\|A_{YY}\|_2 = \overline{v}'A_{YY}'A_{YY}\overline{v} = \max_{\|v\|_2=1} v'A_{YY}'A_{YY}v = \overline{v}'\overline{A}_{11}'\overline{A}_{11}\overline{v}$$

and let $S_d = \left(\begin{array}{cc} I_{dp} & \underset{dp \times Kp}{0} \end{array} \right)'$. It follows that

$$\begin{aligned}
\|A_{YY}\|_2 &= \sqrt{\overline{\upsilon}' A_{YY}' A_{YY} \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' \overline{A}_{11}' \overline{A}_{11} \overline{\upsilon}} \\
&\leq \sqrt{\overline{\upsilon}' \overline{A}_{11}' \overline{A}_{11} \overline{\upsilon} + \overline{\upsilon}' \overline{A}_{21}' \overline{A}_{21} \overline{\upsilon} + \overline{\upsilon}' \overline{A}_{31}' \overline{A}_{31} \overline{\upsilon} + \overline{\upsilon}' \overline{A}_{41}' \overline{A}_{41} \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S_d' \overline{A}' \overline{A} S_d \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S_d' \mathcal{P}_{(d+K)p} A' \mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p} A \mathcal{P}_{(d+K)p}' S_d \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S_d' \mathcal{P}_{(d+K)p} A' A \mathcal{P}_{(d+K)p}' S_d \overline{\upsilon}} \ \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&\leq \sqrt{\max_{\|\upsilon\|_2 = 1} \upsilon' A' A \upsilon} \ \left(\text{noting that } \left\|\mathcal{P}_{(d+K)p}' S_d \overline{\upsilon}\right\|_2 = \sqrt{\overline{\upsilon}' S_d' \mathcal{P}_{(d+K)p} \mathcal{P}_{(d+K)p}' S_d \overline{\upsilon}} = 1\right) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^{\dagger} \phi_{\max} \ \text{(by Assumption 2-6)}
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. Note further that $0 < \phi_{\max} < 1$ since, by Assumption 2-1, all eigenvalues of A have modulus less than 1.

To show part (b), let $\widetilde{\upsilon} \in \mathbb{R}^{Kp}$ such that $\|\widetilde{\upsilon}\|_2 = 1$ and such that

$$\|A_{YF}\|_2 = \widetilde{\upsilon}' A_{YF}' A_{YF} \widetilde{\upsilon} = \max_{\|\upsilon\|_2 = 1} \upsilon' A_{YF}' A_{YF} \upsilon = \widetilde{\upsilon}' \overline{A}_{12}' \overline{A}_{12} \widetilde{\upsilon}$$

and let

$$\underset{(d+K)p \times Kp}{S_K} = \left(\begin{array}{c} 0 \\ I_{Kp} \end{array} \right).$$

It follows that

$$\begin{aligned}
\|A_{YF}\|_2 &= \sqrt{\widetilde{\upsilon}' A_{YF}' A_{YF} \widetilde{\upsilon}} \\
&= \sqrt{\widetilde{\upsilon}' \overline{A}_{12}' \overline{A}_{12} \widetilde{\upsilon}} \\
&\leq \sqrt{\widetilde{\upsilon}' \overline{A}_{12}' \overline{A}_{12} \widetilde{\upsilon} + \widetilde{\upsilon}' \overline{A}_{22}' \overline{A}_{22} \widetilde{\upsilon} + \widetilde{\upsilon}' \overline{A}_{32}' \overline{A}_{32} \widetilde{\upsilon} + \widetilde{\upsilon}' \overline{A}_{42}' \overline{A}_{42} \widetilde{\upsilon}} \\
&= \sqrt{\widetilde{\upsilon}' S_K' \overline{A}' \overline{A} S_K \widetilde{\upsilon}} \\
&= \sqrt{\widetilde{\upsilon}' S_K' \mathcal{P}_{(d+K)p} A' \mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p} A \mathcal{P}_{(d+K)p}' S_K \widetilde{\upsilon}} \\
&= \sqrt{\widetilde{\upsilon}' S_K' \mathcal{P}_{(d+K)p} A' A \mathcal{P}_{(d+K)p}' S_K \widetilde{\upsilon}} \ \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&\leq \sqrt{\max_{\|\upsilon\|_2 = 1} \upsilon' A' A \upsilon} \ \left(\text{noting that } \left\|\mathcal{P}_{(d+K)p}' S_K \widetilde{\upsilon}\right\|_2 = \sqrt{\widetilde{\upsilon}' S_K' \mathcal{P}_{(d+K)p} \mathcal{P}_{(d+K)p}' S_K \widetilde{\upsilon}} = 1\right) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^{\dagger} \phi_{\max} \ \text{(by Assumption 2-6)}
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. As noted in the proof for part (a), $0 < \phi_{\max} < 1$ since, by

Assumption 2-1, all eigenvalues of A have modulus less than 1. $\square$

Lemma OA-8: Consider the linear process

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}$$

Suppose the process satisfies the following assumptions

(i) Let $\{\varepsilon_t\}$ is an independent sequence of random vectors with $E[\varepsilon_t] = 0$ for all t. For some $\delta > 0$, suppose that there exists a positive constant K such that

$$E \|\varepsilon_t\|_2^{1+\delta} \leq K < \infty \text{ for all } t.$$

(ii) Suppose that ε_t has p.d.f. g_{ε_t} such that, for some positive constant $M < \infty$,

$$\sup_t \int |g_{\varepsilon_t}(\upsilon - u) - g_{\varepsilon_t}(\upsilon)| \, d\varepsilon \leq M |u|$$

whenever $|u| \leq \overline{\kappa}$ for some constant $\overline{\kappa} > 0$.

(iii) Suppose that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 < \infty$$

and

$$\det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} \neq 0 \text{ for all } z \text{ with } |z| \leq 1$$

Under these conditions, suppose further that

$$\sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_j\|_2 \right)^{\frac{\delta}{1+\delta}} < \infty;$$

then, for some positive constant $\overline{K}$,

$$\beta_\xi(m) \leq \overline{K} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}}$$

where

$$\beta_\xi(m) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{\xi,-\infty}^t\right) - P(B)\right| : B \in \mathcal{F}_{\xi,t+m}^{\infty}\right\}\right].$$

with $\mathcal{F}_{\xi,-\infty}^t = \sigma\left(....,\xi_{t-2},\xi_{t-1},\xi_t\right)$ and $\mathcal{F}_{\xi,t+m}^{\infty} = \sigma\left(\xi_{t+m},\xi_{t+m+1},\xi_{t+m+2},....\right)$.

Remark: This is Theorem 2.1 of Pham and Tran (1985) restated here in our notation. For a proof, see Pham and Tran (1985).

Lemma OA-9: Let A be an $n \times n$ square matrix with (ordered) singular values given by

$$\sigma_{(1)}(A) \geq \sigma_{(2)}(A) \geq \cdots \geq \sigma_{(n)}(A) \geq 0.$$

Suppose that A is diagonalizable, i.e.,

$$A = S \Lambda S^{-1}$$

where Λ is diagonal matrix whose diagonal elements are the eigenvalues of A. Let the modulus of these eigenvalues be ordered as follows:

$$\left|\lambda_{(1)}(A)\right| \geq \left|\lambda_{(2)}(A)\right| \geq \cdots \geq \left|\lambda_{(n)}(A)\right|.$$

Then, for $k \in \{1, ..., n\}$ and for any positive integer j, we have

$$\chi(S)^{-1}\left|\lambda_{(k)}\left(A^j\right)\right| \leq \sigma_{(k)}\left(A^j\right) \leq \chi(S)\left|\lambda_{(k)}\left(A^j\right)\right|$$

where

$$\chi(S) = \sigma_{(1)}(S)\,\sigma_{(1)}\left(S^{-1}\right).$$

Proof of Lemma OA-9: Observe first that we can assume, without loss of generality, that the decomposition

$$A = S\Lambda S^{-1} = S \cdot diag(\lambda_1, \lambda_2, ..., \lambda_n) \cdot S^{-1}$$

is such that

$$\lambda_i = \lambda_{(i)}(A) \text{ for } i = 1, ..., n$$

with

$$\left|\lambda_{(1)}(A)\right| \geq \left|\lambda_{(2)}(A)\right| \geq \cdots \geq \left|\lambda_{(n)}(A)\right|.$$

This is because suppose we have the alternative representation where

$$A = \widetilde{S}\widetilde{\Lambda}\widetilde{S}^{-1} = \widetilde{S} \cdot diag\left(\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_n\right) \cdot \widetilde{S}^{-1}$$

and where $\widetilde{\lambda}_i \neq \lambda_{(i)}(A)$ for at least some of the $i's$. Then, we can always define a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}'\widetilde{\Lambda}\mathcal{P} = \Lambda$$

so that, given that $\mathcal{P}$ is an orthogonal matrix, we have

$$A = \widetilde{S}\widetilde{\Lambda}\widetilde{S}^{-1} = \widetilde{S}\mathcal{P}\mathcal{P}'\widetilde{\Lambda}\mathcal{P}\mathcal{P}'\widetilde{S}^{-1} = S\Lambda S^{-1}$$

where $S = \widetilde{S}\mathcal{P}$ and, thus, $S^{-1} = \left(\widetilde{S}\mathcal{P}\right)^{-1} = \mathcal{P}'\widetilde{S}^{-1}$.

Next, note that, for any positive integer j,

$$A^j = S\Lambda S^{-1} \times S\Lambda S^{-1} \times \cdots \times S\Lambda S^{-1} = S\Lambda^j S^{-1}$$

where

$$\Lambda^j = diag\left(\lambda_1^j, \lambda_2^j, ..., \lambda_n^j\right) = diag\left(\lambda_{(1)}^j(A), \lambda_{(2)}^j(A), ..., \lambda_{(n)}^j(A)\right).$$

Moreover, since $\lambda_{(k)}\left(A^j\right) = \lambda_{(k)}^j(A)$ for any $k \in \{1, ..., m\}$, we also have

$$\Lambda^j = diag\left(\lambda_1^j, \lambda_2^j, ..., \lambda_n^j\right) = diag\left(\lambda_{(1)}\left(A^j\right), \lambda_{(2)}\left(A^j\right), ..., \lambda_{(n)}\left(A^j\right)\right).$$

In addition, let $\overline{\lambda_{(k)}(A^j)}$ denote the complex conjugate of $\lambda_{(k)}\left(A^j\right)$ for $k \in \{1, ..., m\}$, and note that, by definition,

$$\sigma_{(k)}\left(\Lambda^j\right) = \sqrt{\overline{\lambda_{(k)}(A^j)}\lambda_{(k)}(A^j)} = \left|\lambda_{(k)}\left(A^j\right)\right|$$

Since $\left|\lambda_{(k)}\left(A^j\right)\right| = \left|\lambda_{(k)}^j(A)\right| = \left|\lambda_{(k)}(A)\right|^j$, the ordering

$$\left|\lambda_{(1)}(A)\right| \geq \left|\lambda_{(2)}(A)\right| \geq \cdots \geq \left|\lambda_{(n)}(A)\right|$$

implies that

$$\left|\lambda_{(1)}\left(A^j\right)\right| \geq \left|\lambda_{(2)}\left(A^j\right)\right| \geq \cdots \geq \left|\lambda_{(n)}\left(A^j\right)\right|$$

and, thus,

$$\sigma_{(1)}\left(\Lambda^j\right) \geq \sigma_{(2)}\left(\Lambda^j\right) \geq \cdots \geq \sigma_{(n)}\left(\Lambda^j\right)$$

for any positive integer j.

Now, apply the inequality

$$\sigma_{(i+\ell-1)}\left(BC\right) \leq \sigma_{(i)}\left(B\right)\sigma_{(\ell)}\left(C\right)$$

for $i, \ell \in \{1, ..., n\}$ and $i + \ell \leq n + 1$; we have

$$\begin{aligned}
\sigma_{(k)}\left(A^j\right) &= \sigma_{(k)}\left(S\Lambda^j S^{-1}\right) \\
&\leq \sigma_{(k)}\left(S\Lambda^j\right)\sigma_{(1)}\left(S^{-1}\right) \\
&\leq \sigma_{(k)}\left(\Lambda^j\right)\sigma_{(1)}\left(S\right)\sigma_{(1)}\left(S^{-1}\right) \\
&= \sigma_{(1)}\left(S\right)\sigma_{(1)}\left(S^{-1}\right)\left|\lambda_{(k)}\left(A^j\right)\right| \\
&= \chi\left(S\right)\left|\lambda_{(k)}\left(A^j\right)\right| \text{ for any } k \in \{1, ..., n\}
\end{aligned}$$

Moreover, for any $k \in \{1, ..., n\}$,

$$\begin{aligned}
\left|\lambda_{(k)}\left(A^j\right)\right| &= \sigma_{(k)}\left(\Lambda^j\right) \\
&= \sigma_{(k)}\left(S^{-1}S\Lambda^j S^{-1}S\right) \\
&= \sigma_{(k)}\left(S^{-1}A^j S\right) \\
&\leq \sigma_{(1)}\left(S^{-1}\right)\sigma_{(k)}\left(A^j\right)\sigma_{(1)}\left(S\right)
\end{aligned}$$

or

$$\frac{\left|\lambda_{(k)}\left(A^j\right)\right|}{\chi\left(S\right)} = \frac{\left|\lambda_{(k)}\left(A^j\right)\right|}{\sigma_{(1)}\left(S\right)\sigma_{(1)}\left(S^{-1}\right)} \leq \sigma_{(k)}\left(A^j\right)$$

Putting these two inequalities together, we have, for any $k \in \{1, ..., n\}$ and for all positive integer j,

$$\chi\left(S\right)^{-1}\left|\lambda_{(k)}\left(A^j\right)\right| \leq \sigma_{(k)}\left(A^j\right) \leq \chi\left(S\right)\left|\lambda_{(k)}\left(A^j\right)\right|. \ \square$$

Remark: Note that the case where $j = 1$ in Lemma OA-9 has previously been obtained in Theorem 1 of Ruhe (1975). Hence, Lemma C-9 can be viewed as providing an extension to the first part of that theorem.

Lemma OA-10: Let ρ be such that $|\rho| < 1$. Then,

$$\sum_{j=0}^{\infty}(j+1)\rho^j = \frac{1}{(1-\rho)^2} < \infty$$

Proof of Lemma OA-10: Define

$$S_n\left(\rho\right) = 1 + \rho + \rho^2 + \cdots + \rho^n = \frac{1-\rho^{n+1}}{1-\rho}$$

Note that

$$\begin{aligned} S_n'(\rho) &= 1 + 2\rho + 3\rho^2 + \cdots + n\rho^{n-1} \\ &= -\frac{(n+1)\rho^n}{1-\rho} + \frac{1-\rho^{n+1}}{(1-\rho)^2} \\ &= \frac{1-\rho^{n+1} - (n+1)\rho^n(1-\rho)}{(1-\rho)^2} \\ &= \frac{1-\rho^{n+1} - (n+1)\rho^n + (n+1)\rho^{n+1}}{(1-\rho)^2} \\ &= \frac{1-(n+1)\rho^n + n\rho^{n+1}}{(1-\rho)^2} \\ &= \frac{1-\rho^n - n\rho^n(1-\rho)}{(1-\rho)^2} \end{aligned}$$

It follows that

$$S_n'(\rho) = \sum_{j=0}^{n-1}(j+1)\rho^j = \frac{1-\rho^n - n\rho^n(1-\rho)}{(1-\rho)^2} \to \frac{1}{(1-\rho)^2} \text{ as } n \to \infty. \ \square$$

Lemma OA-11: Let $W_t = (Y_t', F_t')'$ be generated by the factor-augmented VAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}$$

described in section 3 of the main paper. Under Assumptions 2-1, 2-2, and 2-6; $\{W_t\}$ is a β-mixing process with β-mixing coefficient $\beta_W(m)$ such that

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\}$$

for some positive constants C_1 and C_2. Here,

$$\beta_W(m) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{A}_{-\infty}^t\right) - P(B)\right| : B \in \mathcal{A}_{t+m}^{\infty}\right\}\right]$$

with $\mathcal{A}_{-\infty}^t = \sigma(..., W_{t-2}, W_{t-1}, W_t)$ and $\mathcal{A}_{t+m}^{\infty} = \sigma(W_{t+m}, W_{t+m+1}, W_{t+m+2}, ...)$.

Proof of Lemma OA-11:

To prove this lemma, we shall verify the conditions of Lemma OA-8 given above for the vector moving-average representation of W_t, i.e.,

$$W_t = J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} J_{d+K} A^j J_{d+K}' \varepsilon_{t-j} = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j},$$

where

$$\mu_* = J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu,\ \Psi_j = J_{d+K}A^j J'_{d+K},$$

$$\underset{(d+K)\times(d+K)p}{J_{d+K}} = \left[\begin{array}{cccccc} I_{d+K} & 0 & \cdots & 0 & 0 \end{array}\right], \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}$$

To proceed, set

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \tag{30}$$

and note first that, setting $\delta = 5$ in Lemma OA-8, and we see that Assumptions (i) and (ii) of Lemma OA-8 are the same as the conditions specified in Assumption 2-2 (a)-(c). Next, note that, since in this case $\Psi_j = J_{d+K}A^j J'_{d+K}$, we have

$$\begin{aligned}
\|\Psi_j\|_2 &\leq \|J_{d+K}\|_2 \left\|A^j\right\|_2 \left\|J'_{d+K}\right\|_2 \\
&\leq \sqrt{\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)} \left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right) \sqrt{\lambda_{\max}\left(J_{d+K}J'_{d+K}\right)} \\
&= \lambda_{\max}\left(J_{d+K}J'_{d+K}\right) \left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right) \\
&= \sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}} \\
&= \sigma_{\max}\left(A^j\right) \\
&\leq C\left[\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}\right] \quad \text{(by Assumption 2-6)} \\
&= C\left[\max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}\right]^j \\
&= C\phi_{\max}^j
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 2-1, all eigenvalues of A have modulus less than 1. It follows that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 \leq C \sum_{j=0}^{\infty} \phi_{\max}^j = \frac{C}{1 - \phi_{\max}} < \infty.$$

Moreover, by Assumption 2-1,

$$\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \neq 0 \text{ for all } z \text{ such that } |z| \leq 1$$

and, by definition,

$$\sum_{j=0}^{\infty} \Psi_j z^j = \Psi(z) = \left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)^{-1} \text{ for all } z \text{ such that } |z| \leq 1$$

so that

$$\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right) = I_{(d+K)p} \text{ for all } z \text{ such that } |z| \leq 1$$

In addition, since

$$\begin{aligned}
& \det\{\Psi(z)\}\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \\
= & \det\left\{\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)\right\} \\
= & \det\left\{I_{(d+K)p}\right\} \\
= & 1,
\end{aligned}$$

and since

$$\left|\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\}\right| < \infty \text{ for all } z \text{ such that } |z| \leq 1,$$

it follows that

$$\begin{aligned}
\det\left\{\sum_{j=0}^{\infty}\Psi_j z^j\right\} &= \det\{\Psi(z)\} \\
&= \frac{1}{\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\}} \\
&\neq 0 \text{ for all } z \text{ such that } |z| \leq 1.
\end{aligned}$$

Finally, note that, setting $\delta = 5$,

$$\begin{aligned}
\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{\delta}{1+\delta}} &= \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^k\right)^{\frac{5}{6}} \\
&= C^{\frac{5}{6}}\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^k\right)^{\frac{5}{6}} \\
&\leq C^{\frac{5}{6}}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^k \\
&\quad \left(\text{by the inequality } \left|\sum_{i=1}^{\infty}a_i\right|^r \leq \sum_{i=1}^{\infty}|a_i|^r \text{ for } r \leq 1\right) \\
&= C^{\frac{5}{6}}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^j \\
&= C^{\frac{5}{6}}\left[1 - \phi_{\max}^{\frac{5}{6}}\right]^{-2} \quad \text{(by Lemma OA-10)} \\
&< \infty \quad \left(\text{since } 0 < \phi_{\max}^{\frac{5}{6}} < 1 \text{ given that } 0 < \phi_{\max} < 1\right).
\end{aligned}$$

Hence, all conditions of Lemma OA-8 are fulfilled. Applying Lemma OA-8, we then obtain that there exists a constant $\overline{C}$ such that

$$\begin{aligned}
\beta_{\xi}(m) &\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
&\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&= \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&\leq \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^{k} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^{j} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \\
&= \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2}\exp\left\{-\left[\frac{5}{6}\left|\ln\phi_{\max}\right|\right]m\right\} \quad (\text{since } 0<\phi_{\max}<1) \\
&\leq C_1\exp\{-C_2 m\}\to 0 \text{ as } m\to\infty.
\end{aligned}$$

for some positive constants C_1 and C_2 such that

$$C_1 \geq \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \text{ and } C_2 \leq \frac{5}{6}\left|\ln\phi_{\max}\right|$$

It follows that the process $\{\xi_t\}$ (as defined in expression (30)) is β mixing with beta coefficient $\beta_{\xi}(m)$ satisfying

$$\beta_{\xi}(m) \leq C_1\exp\{-C_2 m\}.$$

Since

$$W_t = \mu_* + \sum_{j=0}^{\infty}\Psi_j\varepsilon_{t-j} = \mu_* + \xi_t$$

and since μ_* is a nonrandom parameter, we can then apply part (a) of Lemma OA-2 to deduce that $\{W_t\}$ is a β mixing process with β coefficient $\beta_W(m)$ satisfying the inequality

$$\beta_W(m) \leq C_1\exp\{-C_2 m\}. \;\square$$

Lemma OA-12: Let $\underline{Y}_t = \begin{pmatrix} Y_t' & Y_{t-1}' & \cdots & Y_{t-p+2}' & Y_{t-p+1}' \end{pmatrix}'$ and $\underline{F}_t = \begin{pmatrix} F_t' & F_{t-1}' & \cdots & F_{t-p+2}' & F_{t-p+1}' \end{pmatrix}'$. Under Assumptions 2-1, 2-2, 2-5, 2-6, and 2-9(b); the following statements are true as $N,T\to\infty$

(a)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right| \xrightarrow{p} 0$$

(b)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right|\xrightarrow{p}0$$

(c)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right|\xrightarrow{p}0$$

(d)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.$$
$$\left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2$$
$$\xrightarrow{p}0$$

(e)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2=O_p\left(1\right).$$

(f)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left\{\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right.\right.\right.$$
$$\left.+\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)$$
$$\left.\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right\}\right|$$
$$\xrightarrow{p}0$$

(g)

$$\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.$$
$$\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right|$$
$$\xrightarrow{p}0$$

(h)

$$\max_{1\leq \ell \leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.$$
$$\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right|$$
$$\xrightarrow{p} 0$$

Proof of Lemma OA-12:

To show part (a), note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right|\ge\epsilon\right\}\\
= \;& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\ge\epsilon^2\right\}\\
\le \;& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\ge\epsilon^2\right\}\\
& \text{(by Jensen's inequality)}\\
= \;& P\left\{\max_{i\in H^c}\max_{1\le\ell\le d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)^2\ge\epsilon^2\right\}\\
\le \;& P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)^2\ge\epsilon^2\right\}\\
\le \;& P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]'\right.\right.\\
& \qquad\left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)\ge\epsilon^2\right\}\\
= \;& P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\right.\\
& \qquad\left.\times\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\alpha_{YY,\ell}\ge\epsilon^2\right\}\\
\le \;& \frac{\max_{i\in H^c}\|\gamma_i\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right.\\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\}\\
& \text{(by Markov's inequality)}\\
\le \;& \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right.\\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\} \qquad (31)\\
& \text{(by Assumption 2-5)}
\end{aligned}
$$

Next, write

$$\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right.\right. \\
& \qquad \left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left\{\alpha_{YY,\ell}'\right.\right. \\
& \qquad \left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]\alpha_{YY,\ell}\right\}\right) \\
\leq & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|\alpha_{YY,\ell}'\right.\right. \\
& \qquad \left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]\alpha_{YY,\ell}\right|\right) \qquad (32)
\end{aligned}$$

Let $e_{\ell,d}$ be a $d\times 1$ elementary vector whose ℓ^{th} component is 1 and all other components are 0, and note that

$$\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]A_{YY}'e_{\ell,d}\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}E\left[\underline{Y}_t\underline{F}_t'\underline{F}_t\underline{Y}_t'\right]A_{YY}'e_{\ell,d}\right. \\
& \qquad\left.-\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}E\left[\underline{Y}_t\underline{F}_t'\right]E\left[\underline{F}_t\underline{Y}_t'\right]A_{YY}'e_{\ell,d}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\left\|\underline{F}_t\right\|_2^2\left(e_{\ell,d}'A_{YY}\underline{Y}_t\right)^2\right] \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left(e_{\ell,d}'A_{YY}\underline{Y}_t\underline{Y}_t'A_{YY}'e_{\ell,d}\right)^2}\ \text{(by CS inequality)} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{Y}_t\right\|_2^4\right]}\sqrt{\left(e_{\ell,d}'A_{YY}A_{YY}'e_{\ell,d}\right)^2} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{Y}_t\right\|_2^4\right]}\left\|A_{YY}\right\|_2^2\sqrt{\left(e_{\ell,d}'e_{\ell,d}\right)^2} \\
\leq & \frac{d\left(C^{\dagger}\right)^2}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{Y}_t\right\|_2^4\right]}\phi_{\max}^2 \\
& \text{(by part (a) of Lemma OA-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector)} \\
\leq & \frac{\overline{C}}{\tau_1}=O\left(\frac{1}{\tau_1}\right). \qquad (33)
\end{aligned}$$

for some positive constant $\overline{C}\geq d\left(C^{\dagger}\right)^2\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{Y}_t\right\|_2^4\right]}\phi_{\max}^2$, which exists in light of Lemma OA-5 and the fact that $0<\phi_{\max}<1$ given Assumption 2-1.

To analyze the second term on the right-hand side of expression (32), note first that by Lemma OA-11, $\left\{\left(Y_t',F_t'\right)'\right\}$ is β-mixing with β mixing coefficient satisfying

$$\beta_W(m)\leq C_1\exp\{-C_2m\}\ \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{W,m}\leq\beta_W(m)$, it follows that $W_t=\left(Y_t',F_t'\right)'$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{W,m}\leq C_1\exp\{-C_2m\}$$

Moreover, by applying part (b) of Lemma OA-2, we further deduce that $X_{1t}=\underline{F}_t\underline{Y}_t'A_{YY}'e_{\ell,d}$ is also α-mixing with α mixing coefficient satisfying

$$\begin{aligned} \alpha_{X_1,m} &\leq C_1 \exp\left\{-C_2\left(m-p+1\right)\right\} \\ &\leq C_1^* \exp\left\{-C_2 m\right\} \end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp\left\{C_2\left(p-1\right)\right\}$. Hence, we can apply Lemma OA-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned} & \left|\alpha_{YY,\ell}' E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] \alpha_{YY,\ell}\right| \\ = & \left|e_{\ell,d}' A_{YY} E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] A_{YY}' e_{\ell,d}\right| \\ = & \left|\sum_{h=1}^{Kp} e_{\ell,d}' A_{YY} E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' e_{h,Kp} e_{h,Kp}' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] A_{YY}' e_{\ell,d}\right| \\ \leq & \sum_{h=1}^{Kp} \left\{ 2\left(2^{\frac{2}{3}}+1\right) \alpha_{X_1,m}^{\frac{1}{3}} \left(E\left|e_{\ell,d}' A_{YY}\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \right. \\ & \left. \times \left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right) A_{YY}' e_{\ell,d}\right|^3\right)^{1/3} \right\} \end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $\{X_{1t}\}$ and where, by our previous calculations,

$$\alpha_{X_1,m}^{\frac{1}{3}} \leq \left(C_1^*\right)^{\frac{1}{3}} \exp\left\{-\frac{C_2 m}{3}\right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned} \sum_{m=1}^{\infty} \alpha_{X_1,m}^{\frac{1}{3}} &\leq \left(C_1^*\right)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\ &\leq \left(C_1^*\right)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\ &= \left(C_1^*\right)^{\frac{1}{3}} \left[1 - \exp\left\{-\frac{C_2}{3}\right\}\right]^{-1} \\ &\leq C_3 \end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\left\{-\left(C_2 m/3\right)\right\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$
\begin{aligned}
& E\left|e_{\ell,d}' A_{YY}\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' e_{h,Kp}\right|^3 \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}' A_{YY} \underline{Y}_t \underline{F}_t' e_{h,Kp}\right|^3 + \left|E\left[e_{\ell,d}' A_{YY} \underline{Y}_t \underline{F}_t' e_{h,Kp}\right]\right|^3\right\} \text{(by Loève's } c_r \text{ inequality)} \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}' A_{YY} \underline{Y}_t \underline{F}_t' e_{h,Kp}\right|^3 + \left(E\left[\left|e_{\ell,d}' A_{YY} \underline{Y}_t \underline{F}_t' e_{h,Kp}\right|\right]\right)^3\right\} \text{ (by Jensen's inequality)} \\
\leq\ & 2^2\left\{E\left|\frac{e_{\ell,d}' A_{YY} \underline{Y}_t \underline{Y}_t' A_{YY}' e_{\ell,d}}{2} + \frac{e_{h,Kp}' \underline{F}_t \underline{F}_t' e_{h,Kp}}{2}\right|^3 + \left(E\left[\left|e_{\ell,d}' A_{YY} \underline{Y}_t \underline{F}_t' e_{h,Kp}\right|\right]\right)^3\right\} \\
\leq\ & \frac{4}{8}\left[E\left|e_{\ell,d}' A_{YY} \underline{Y}_t \underline{Y}_t' A_{YY}' e_{\ell,d}\right|^3 + E\left|e_{h,Kp}' \underline{F}_t \underline{F}_t' e_{h,Kp}\right|^3\right] \\
& +4\left(\sqrt{E\left[e_{\ell,d}' A_{YY} \underline{Y}_t \underline{Y}_t' A_{YY}' e_{\ell,d}\right]}\sqrt{E\left[e_{h,Kp} \underline{F}_t \underline{F}_t' e_{h,Kp}\right]}\right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
\leq\ & \frac{1}{2}\left|e_{\ell,d}' A_{YY} A_{YY}' e_{\ell,d}\right|^3 E\left\|\underline{Y}_t\right\|_2^6 + \frac{1}{2} E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(e_{\ell,d}' A_{YY} A_{YY}' e_{\ell,d}\right)^{\frac{3}{2}}\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
\leq\ & \frac{1}{2}\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6 \phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + \frac{1}{2} E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3 \phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
=\ & \frac{1}{2}\left(C^{\dagger}\right)^6 \phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + \frac{1}{2} E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(C^{\dagger}\right)^3 \phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
& \left(\text{since } \left\|e_{\ell,d}\right\|_2 = 1 \text{ for every } \ell \in \{1, ..., d\} \text{ given that } e_{\ell,d}\text{'s are elementary vectors}\right) \\
\leq\ & C_4
\end{aligned}
$$

for some positive constant $C_4 \geq (1/2)\left(C^{\dagger}\right)^6 \phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + (1/2) E\left\|\underline{F}_t\right\|_2^6$ $+4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(C^{\dagger}\right)^3 \phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}}$ which exists in light of Lemma OA-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 2-1. In a similar way, we can also show that there exists a positive constant C_5 such that

$$
\begin{aligned}
& E\left|e_{h,Kp}'\left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right) A_{YY}' e_{\ell,d}\right|^3 \\
\leq\ & (1/2)\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6 \phi_{\max}^6 E\left\|\underline{Y}_{t+m}\right\|_2^6 + (1/2) E\left\|\underline{F}_{t+m}\right\|_2^6 \\
& +4\left(E\left\|\underline{Y}_{t+m}\right\|_2^2\right)^{\frac{3}{2}}\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3 \phi_{\max}^3\left(E\left\|\underline{F}_{t+m}\right\|_2^2\right)^{\frac{3}{2}} \\
\leq\ & C_5 < \infty
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{2}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|e'_{\ell,d}A_{YY} \\
& \qquad \times E\left[\left(\underline{F}_t\underline{Y}'_t - E\left[\underline{F}_t\underline{Y}'_t\right]\right)'\left(\underline{F}_{t+m}\underline{Y}'_{t+m} - E\left[\underline{F}_{t+m}\underline{Y}'_{t+m}\right]\right)\right]A'_{YY}e_{\ell,d}\Big| \\
\leq\ & \frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\sum_{h=1}^{Kp}\alpha_{X_1,m}^{\frac{1}{3}}\left(E\left|e'_{\ell,d}A_{YY}\left(\underline{F}_t\underline{Y}'_t - E\left[\underline{F}_t\underline{Y}'_t\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \\
& \qquad \times\left(E\left|e'_{h,Kp}\left(\underline{F}_{t+m}\underline{Y}'_{t+m} - E\left[\underline{F}_{t+m}\underline{Y}'_{t+m}\right]\right)A'_{YY}e_{\ell,d}\right|^3\right)^{1/3} \\
\leq\ & \frac{4dKp\left(2^{\frac{2}{3}}+1\right)C_4^{\frac{1}{3}}C_5^{\frac{1}{3}}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}\left(C_1^*\right)^{\frac{1}{3}}\exp\left\{-\frac{C_2 m}{3}\right\} \\
\leq\ & \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\}\ \left(\text{where } C^*\geq 4dKp\left(2^{\frac{2}{3}}+1\right)\left(C_1^*\right)^{\frac{1}{3}}C_4^{\frac{1}{3}}C_5^{\frac{1}{3}}\right) \\
\leq\ & \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right) \qquad (34)
\end{aligned}
$$

It then follows from expressions (31), (32), (33), and (34) that

$$
\begin{aligned}
& P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right|\geq\epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}\right. \\
& \qquad \left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]A_{YY}'e_{\ell,d}\right) \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]A_{YY}'e_{\ell,d}\right) \\
& +\frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|e_{\ell,d}'A_{YY}\right. \\
& \qquad \left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]A_{YY}'e_{\ell,d}\right| \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
=\ & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\ & O\left(\frac{1}{\tau_1}\right)=o(1).
\end{aligned}
$$

Next, to show part (b), note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right|\geq\epsilon\right\} \\
= \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2\geq\epsilon^2\right\} \\
\leq \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2\geq\epsilon^2\right\} \\
& \text{(by Jensen's inequality)} \\
= \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)^2\geq\epsilon^2\right\} \\
\leq \; & P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)^2\geq\epsilon^2\right\} \\
\leq \; & P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]'\right.\right. \\
& \qquad\left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)\geq\epsilon^2\right\} \\
= \; & P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\left.\times\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\alpha_{YF,\ell}\geq\epsilon^2\right\} \\
\leq \; & \frac{\max_{i\in H^c}\|\gamma_i\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& \text{(by Markov's inequality)} \\
\leq \; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \qquad (35) \\
& \text{(by Assumption 2-5)}
\end{aligned}
$$

Note first that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s' - E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\right) \\
\leq & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\sum_{\ell=1}^{d}\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\right| \qquad (36)
\end{aligned}
$$

Consider the first term on the majorant side of expression (36), whose order of magnitude we can analyze as follows

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YF}E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]A_{YF}'e_{\ell,d}\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{e_{\ell,d}'A_{YF}E\left[\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right]A_{YF}'e_{\ell,d}-e_{\ell,d}'A_{YF}E\left[\underline{F}_t\underline{F}_t'\right]E\left[\underline{F}_t\underline{F}_t'\right]A_{YF}'e_{\ell,d}\right\}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\|\underline{F}_t\|_2^2\left(e_{\ell,d}'A_{YF}\underline{F}_t\right)^2\right] \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left(e_{\ell,d}'A_{YF}\underline{F}_t\underline{F}_t'A_{YF}'e_{\ell,d}\right)^2}\text{ (by CS inequality)} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{\left(e_{\ell,d}'A_{YF}A_{YF}'e_{\ell,d}\right)^2} \\
\leq & \frac{1}{q\tau_1^2}\sum_{\ell=1}^{d}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\|A_{YF}\|_2^2\sqrt{\left(e_{\ell,d}'e_{\ell,d}\right)^2} \\
\leq & \frac{\left(C^{\dagger}\right)^2}{q\tau_1^2}\sum_{\ell=1}^{d}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\|\underline{F}_t\|_2^4\right]\phi_{\max}^2 \\
& \text{(by part (b) of Lemma OA-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector)} \\
\leq & \frac{\overline{C}}{\tau_1}=O\left(\frac{1}{\tau_1}\right). \qquad (37)
\end{aligned}
$$

for some positive constant $\overline{C}\geq d\left(C^{\dagger}\right)^2E\left[\|\underline{F}_t\|_2^4\right]\phi_{\max}^2$, which exists in light of Lemma OA-5 and the fact that $0<\phi_{\max}<1$ given Assumption 2-1.

To analyze the second term on the right-hand side of expression (36), note first that by Lemma OA-11, $\{F_t\}$ is β-mixing with β mixing coefficient satisfying

$$\beta_F(m)\leq C_1\exp\{-C_2m\}\text{ for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{F,m}\leq\beta_F(m)$, it follows that F_t is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{F,m}\leq C_1\exp\{-C_2m\}$$

Moreover, by applying part (b) of Lemma OA-2, we further deduce that $X_{2t}=\underline{F}_t\underline{F}_t'A_{YF}'e_{\ell,d}$ is also α-mixing with α mixing coefficient satisfying

$$
\begin{aligned}
\alpha_{X_2,m} &\leq C_1\exp\{-C_2(m-p+1)\} \\
&\leq C_1^*\exp\{-C_2m\}
\end{aligned}
$$

for some positive constant $C_1^*\geq C_1\exp\{C_2(p-1)\}$. Hence, we can apply Lemma OA-3 with $p=3$ and $r=3$ to obtain

$$\begin{aligned}
& \left|\alpha_{YF,\ell}' E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\right| \\
= & \left|e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right] A_{YF}' e_{\ell,d}\right| \\
= & \left|\sum_{h=1}^{Kp} e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)' e_{h,Kp} e_{h,Kp}'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right] A_{YF}' e_{\ell,d}\right| \\
\leq & \sum_{h=1}^{Kp}\left\{2\left(2^{\frac{2}{3}}+1\right)\alpha_{X_2,m}^{\frac{1}{3}}\left(E\left|e_{\ell,d}' A_{YF}\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)' e_{h,Kp}\right|^3\right)^{\frac{1}{3}}\right. \\
& \left.\quad \times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}\underline{F}_{t+m}' - E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right) A_{YF}' e_{\ell,d}\right|^3\right)^{1/3}\right\}
\end{aligned}$$

where $\alpha_{X_2,m}$ denotes the alpha mixing coefficient for the process $\{X_{2t}\}$ and where, by our previous calculations,

$$\alpha_{X_2,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp\left\{-\frac{C_2 m}{3}\right\} \text{ for all } m \text{ sufficiently large,}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned}
\sum_{m=1}^{\infty}\alpha_{X_2,m}^{\frac{1}{3}} & \leq (C_1^*)^{\frac{1}{3}}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
& \leq (C_1^*)^{\frac{1}{3}}\sum_{m=0}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
& = (C_1^*)^{\frac{1}{3}}\left[1-\exp\left\{-\frac{C_2}{3}\right\}\right]^{-1} \\
& \leq C_3
\end{aligned}$$

Next, note that

$$
\begin{aligned}
& E\left|e_{\ell,d}^{\prime}A_{YF}\left(\underline{F}_t\underline{F}_t^{\prime}-E\left[\underline{F}_t\underline{F}_t^{\prime}\right]\right)^{\prime}e_{h,Kp}\right|^3 \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|^3+\left|E\left[e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right]\right|^3\right\} \text{(by Loève's } c_r \text{ inequality)} \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|^3+\left(E\left[\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|\right]\right)^3\right\}\ \text{(by Jensen's inequality)} \\
\leq\ & 2^2\left\{E\left|\frac{e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}}{2}+\frac{e_{h,Kp}^{\prime}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}}{2}\right|^3+\left(E\left[\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|\right]\right)^3\right\} \\
\leq\ & \frac{4}{8}\left[E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}\right|^3+E\left|e_{h,Kp}^{\prime}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}^{\prime}\right|^3\right] \\
& +4\left(\sqrt{E\left[e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}\right]}\sqrt{E\left[e_{h,Kp}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right]}\right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
\leq\ & \frac{1}{2}\left|e_{\ell,d}^{\prime}A_{YF}A_{YF}^{\prime}e_{\ell,d}\right|^3E\left\|\underline{F}_t\right\|_2^6+\frac{1}{2}E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(e_{\ell,d}^{\prime}A_{YF}A_{YF}^{\prime}e_{\ell,d}\right)^{\frac{3}{2}} \\
\leq\ & \frac{1}{2}\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+\frac{1}{2}E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
=\ & \frac{1}{2}\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+\frac{1}{2}E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
& \left(\text{since } \left\|e_{\ell,d}\right\|_2=1 \text{ for every } \ell\in\{1,...,d\} \text{ given that } e_{\ell,d}\text{'s are elementary vectors}\right) \\
\leq\ & C_6
\end{aligned}
$$

for some positive constant $C_6 \geq (1/2)\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+(1/2)E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(C^{\dagger}\right)^3\phi_{\max}^3$ which exists in light of Lemma OA-5 and the fact that $0<\phi_{\max}<1$ given Assumption 2-1. In a similar way, we can also show that there exists a positive constant C_7 such that

$$
\begin{aligned}
& E\left|e_{h,Kp}^{\prime}\left(\underline{F}_{t+m}\underline{F}_{t+m}^{\prime}-E\left[\underline{F}_{t+m}\underline{F}_{t+m}^{\prime}\right]\right)A_{YY}^{\prime}e_{\ell,d}\right|^3 \\
\leq\ & \frac{1}{2}\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_{t+m}\right\|_2^6+\frac{1}{2}E\left\|\underline{F}_{t+m}\right\|_2^6 \\
& +4\left(E\left\|\underline{F}_{t+m}\right\|_2^2\right)^3\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
\leq\ & C_7<\infty
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{2}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|e_{\ell,d}'A_{YF} \\
& \qquad\qquad \times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]A_{YF}'e_{\ell,d}\Big| \\
\leq\ & \frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\sum_{h=1}^{Kp}\Bigg\{\alpha_{X_2,m}^{\frac{1}{3}}\left(E\left|e_{\ell,d}'A_{YF}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \\
& \qquad\qquad \times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)A_{YF}'e_{\ell,d}\right|^3\right)^{1/3}\Bigg\} \\
\leq\ & \frac{4dKp\left(2^{\frac{2}{3}}+1\right)C_6^{\frac{1}{3}}C_7^{\frac{1}{3}}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}\left(C_1^*\right)^{\frac{1}{3}}\exp\left\{-\frac{C_2m}{3}\right\} \\
\leq\ & \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\}\ \left(\text{where } C^*\geq 4dKp\left(2^{\frac{2}{3}}+1\right)\left(C_1^*\right)^{\frac{1}{3}}C_6^{\frac{1}{3}}C_7^{\frac{1}{3}}\right) \\
\leq\ & \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right) \qquad (38)
\end{aligned}
$$

It then follows from expressions (35), (36), (37), and (38) that

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right|\geq\epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|\alpha_{YF,\ell}' \\
& \qquad\qquad \times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\Big| \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\ & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\ & O\left(\frac{1}{\tau_1}\right)=o\left(1\right).
\end{aligned}
$$

Now, to show part (c), note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right| \geq \epsilon\right\} \\
= \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \geq \epsilon^2\right\} \text{ (by Jensen's inequality)} \\
= \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]'\right.\right. \\
& \qquad\qquad \left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right) \geq \epsilon^2\right\} \\
= \;& P\left\{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\mu_{Y,\ell} \geq \epsilon^2\right\} \\
\leq \;& \frac{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \\
& \text{(by Markov's inequality)} \\
\leq \;& \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \qquad (39) \\
& \text{(by Assumption 2-5)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\ \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s-E\left[\underline{F}_s\right]\right)\right]\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\ \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right) \\
\leq & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\frac{2}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\ \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \qquad (40)
\end{aligned}
$$

Consider the first term on the majorant side of expression (40), whose order of magnitude we can analyze as follows

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\right]\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2\left\{E\left[\underline{F}_t'\underline{F}_t\right]-E\left[\underline{F}_t\right]'E\left[\underline{F}_t\right]\right\}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left\|\underline{F}_t\right\|_2^2\right] \\
= & \frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\left\|\underline{F}_t\right\|_2^2\right]\sum_{\ell=1}^{d}\left(\mu_{Y,\ell}^2\right) \\
\leq & \frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\left\|\underline{F}_t\right\|_2^2\right]\left\|\mu_Y\right\|_2^2 \\
\leq & \frac{\overline{C}}{\tau_1}=O\left(\frac{1}{\tau_1}\right). \qquad (41)
\end{aligned}
$$

for some positive constant $\overline{C}\geq\left\|\mu_Y\right\|_2^2 E\left[\left\|\underline{F}_t\right\|_2^2\right]$, which exists in light of Assumption 2-5 and Lemma OA-5.

To analyze the second term on the right-hand side of expression (40), note first that by the same argument as given for part (b) above, we can apply Lemma OA-11 to deduce that $\{F_t\}$ is β-mixing and, thus, also α-mixing with α mixing coefficient satisfying

$$\alpha_{F,m}\leq C_1\exp\left\{-C_2 m\right\}$$

Hence, we can apply Lemma OA-3 with $p = 3$ and $r = 3$ to obtain

$$
\begin{aligned}
& \left|E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right]\right| \sum_{\ell=1}^{d} \mu_{Y,\ell}^2 \\
= & \left|\sum_{h=1}^{Kp} E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)' e_{h,Kp} e_{h,Kp}' \left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right]\right| \sum_{\ell=1}^{d} \mu_{Y,\ell}^2 \\
\leq & \sum_{h=1}^{Kp} 2\left(2^{\frac{2}{3}}+1\right)\alpha_{F,m}^{\frac{1}{3}} \left(E\left|\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)' e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right|^3\right)^{1/3} \sum_{\ell=1}^{d} \mu_{Y,\ell}^2
\end{aligned}
$$

Moreover, there exists a positive constant C_3 such that

$$
\sum_{m=1}^{\infty} \alpha_{F,m}^{\frac{1}{3}} \leq C_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} = C_1^{\frac{1}{3}}\left[1 - \exp\left\{-\frac{C_2}{3}\right\}\right]^{-1} \leq C_3
$$

where again the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\{-(C_2 m/3)\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$
\begin{aligned}
& E\left|\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)' e_{h,Kp}\right|^3 \\
\leq & 2^2\left\{E\left|\underline{F}_t' e_{h,Kp}\right|^3 + \left|E\left[\underline{F}_t' e_{h,Kp}\right]\right|^3\right\} \text{(by Loève's } c_r \text{ inequality)} \\
\leq & 2^2\left\{E\left|\underline{F}_t' e_{h,Kp}\right|^3 + \left(E\left[\left|\underline{F}_t' e_{h,Kp}\right|\right]\right)^3\right\} \quad \text{(by Jensen's inequality)} \\
\leq & 2^2\left\{E\left[\left(\underline{F}_t'\underline{F}_t\right)^{\frac{3}{2}}\left(e_{h,Kp}' e_{h,Kp}\right)^{\frac{3}{2}}\right] + \left(\sqrt{E\left[\underline{F}_t'\underline{F}_t\right]}\sqrt{e_{h,Kp}' e_{h,Kp}}\right)^3\right\} \quad \text{(by CS inequality)} \\
\leq & 4\left\{E\left[\left\|\underline{F}_t\right\|_2^3\right] + \left(E\left[\left\|\underline{F}_t\right\|_2^2\right]\right)^{\frac{3}{2}}\right\} \\
\leq & C_8
\end{aligned}
$$

for some positive constant $C_8 \geq 4\left\{E\left[\left\|\underline{F}_t\right\|_2^3\right] + \left(E\left[\left\|\underline{F}_t\right\|_2^2\right]\right)^{\frac{3}{2}}\right\}$ which exists in light of the result given in Lemma OA-5. In a similar way, we can also show that there exists a positive constant C_9 such that

$$
\begin{aligned}
E\left|e_\ell'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right|^3 & \leq 4\left\{E\left[\left\|\underline{F}_{t+m}\right\|_2^3\right] + \left(E\left[\left\|\underline{F}_{t+m}\right\|_2^2\right]\right)^{\frac{3}{2}}\right\} \\
& \leq C_9 < \infty
\end{aligned}
$$

Finally, by Assumption 2-5, there exists a positive constant C_{10} such that $\max_{1\leq \ell \leq d} \mu_{Y,\ell}^2 \leq \left\|\mu_Y\right\|_2^2 \leq C_{10} <$

∞. Hence,

$$
\begin{aligned}
& \frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \\
\leq\; & \sum_{h=1}^{Kp}\frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\left\|\mu_Y\right\|_2^2 \\
& \times\frac{1}{q}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left\{\alpha_{F,m}^{\frac{1}{3}}\left(E\left|\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}}\right. \\
& \qquad\qquad\left.\times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right|^3\right)^{1/3}\right\} \\
\leq\; & \frac{4Kp\left(2^{\frac{2}{3}}+1\right)C_8^{\frac{1}{3}}C_9^{\frac{1}{3}}C_{10}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}C_1^{\frac{1}{3}}\exp\left\{-\frac{C_2m}{3}\right\} \\
\leq\; & \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\}\;\left(\text{where } C^*\geq 4Kp\left(2^{\frac{2}{3}}+1\right)C_1^{\frac{1}{3}}C_8^{\frac{1}{3}}C_9^{\frac{1}{3}}C_{10}\right) \\
\leq\; & \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\; & O\left(\frac{1}{\tau_1}\right)
\end{aligned}
\tag{42}
$$

It then follows from expressions (39), (40), (41), and (42) that

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right|\geq\epsilon\right\} \\
\leq\; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s-E\left[\underline{F}_s\right]\right)\right]\right) \\
\leq\; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \\
\leq\; & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\; & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\; & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\; & O\left(\frac{1}{\tau_1}\right)=o\left(1\right).
\end{aligned}
$$

Turning our attention to part (d), note that, by apply Loève's c_r inequality, we obtain

$$\begin{aligned}
&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\\
&\qquad\qquad\left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2\\
\leq\;& 3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2\\
&+3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\\
&+3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2
\end{aligned}$$

It follows from the arguments given in the proofs of parts (a)-(c) above that, for any $\epsilon>0$,

$$\begin{aligned}
&P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\geq\epsilon\right\}\\
\leq\;&\frac{C}{\epsilon^2}\frac{1}{q\tau_1^2}\\
&\times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right)\\
=\;&o(1),
\end{aligned}$$

$$\begin{aligned}
&P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2\geq\epsilon\right\}\\
\leq\;&\frac{C}{\epsilon^2}\frac{1}{q\tau_1^2}\\
&\times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right)\\
=\;&o(1)
\end{aligned}$$

and

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2\geq\epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q\tau_1^2} \\
& \times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s-E\left[\underline{F}_s\right]\right)\right]\right) \\
=\ & o\left(1\right),
\end{aligned}
$$

from which we deduce via the Slutsky's theorem that

$$
\begin{aligned}
& \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\qquad \left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
=\ & o_p\left(1\right)
\end{aligned}
$$

as required.

To show part (e), note that

$$
\begin{aligned}
& \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2 \\
\leq\ & \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\qquad \left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\} \\
& \qquad \left.+\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
\leq\ & \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{2}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\qquad \left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
& +\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{2}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
& \text{(by Loève's } c_r \text{ inequality)} \\
=\ & o_p\left(1\right)+O\left(1\right) \\
& \text{(applying the results given in part (d) of this lemma and in Lemma A1 of the main paper)} \\
=\ & O_p\left(1\right).
\end{aligned}
$$

To show part (f), we apply the Cauchy-Schwarz inequality as well as part (d) of this lemma and Lemma

A1 of the main paper to obtain

$$\begin{aligned}
&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left\{\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\right.\\
&\qquad\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)\\
&\qquad\left.\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right\}\right|\\
\leq\;&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\\
&\qquad\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)\\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right|\\
\leq\;&\left[\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\\
&\qquad\left.\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2\right]^{1/2}\\
&\times\left[\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2\right]^{1/2}\\
=\;&o_p(1)\,O(1)\\
=\;&o_p(1)\,.
\end{aligned}$$

For part (g), we apply the Cauchy-Schwarz inequality as well as part (d) of Lemma OA-6 and part (e) of this lemma to obtain

$$
\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
\leq\; & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
\leq\; & \sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2} \\
& \times\sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2} \\
=\; & O_p(1)\,o_p(1) \\
=\; & o_p(1)
\end{aligned}
$$

Finally, for part (h), we apply the Cauchy-Schwarz inequality as well as part (b) of Lemma OA-6 and part (e) of this lemma to obtain

$$
\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.\\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right|\\
\leq\ & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.\\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right|\\
\leq\ & \sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2}\\
&\times\sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2}\\
=\ & O_p(1)\,o_p(1)\\
=\ & o_p(1).\ \square
\end{aligned}
$$

Lemma OA-13: Let $a, b \in \mathbb{R}$ such that $a \geq 0$ and $b \geq 0$. Then,

$$\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{|a-b|}$$

Proof of Lemma OA-13: Note that

$$
\begin{aligned}
\left(\sqrt{a}-\sqrt{b}\right)^2 &= a-2\sqrt{a}\sqrt{b}+b\\
&= \sqrt{a}\left(\sqrt{a}-\sqrt{b}\right)+\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right)\\
&\leq \sqrt{a}\left|\sqrt{a}-\sqrt{b}\right|+\sqrt{b}\left|\sqrt{b}-\sqrt{a}\right|\\
&= \left(\sqrt{a}+\sqrt{b}\right)\left|\sqrt{a}-\sqrt{b}\right|\\
&= \left|\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)\right|\\
&= |a-b|
\end{aligned}
$$

Taking principal square root on both sides, we obtain

$$\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{|a-b|}.\ \square$$

Lemma OA-14:

$$P\left\{\bigcap_{i=1}^{m}A_i\right\} \geq \sum_{i=1}^{m}P(A_i)-(m-1)$$

Proof of Lemma OA-14:

$$
\begin{aligned}
P\left\{\bigcap_{i=1}^{m} A_i\right\} &= 1 - P\left\{\left(\bigcap_{i=1}^{m} A_i\right)^c\right\} \\
&= 1 - P\left\{\bigcup_{i=1}^{m} A_i^c\right\} \quad \text{(by DeMorgan's Law)} \\
&\geq 1 - \sum_{i=1}^{m} P\left(A_i^c\right) \\
&= 1 - \sum_{i=1}^{m} \left[1 - P\left(A_i\right)\right] \\
&= \sum_{i=1}^{m} P\left(A_i\right) - m + 1 \\
&= \sum_{i=1}^{m} P\left(A_i\right) - (m-1)\,. \ \square
\end{aligned}
$$

Lemma OA-15:

(a) For $t > 0$,

$$\overline{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t},$$

where $\phi(t)$ and $\Phi(t)$ denote, respectively, the pdf and the cdf of a standard normal random variable.

(b) Let $N = N_1 + N_2$. Specify φ such that $\varphi \to 0$ as $N_1, N_2 \to \infty$ and such that, for some constant $a > 0$,

$$\varphi \geq \frac{1}{N^a}$$

for all N_1, N_2 sufficiently large. Then, for all N_1, N_2 sufficiently large such that

$$1 - \frac{\varphi}{2N} \geq \Phi(2)$$

we have

$$\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2(1+a)}\sqrt{\ln N}.$$

Proof of Lemma OA-15:

(a)

$$
\begin{aligned}
1 - \Phi(t) &= \int_t^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\
&= \int_t^{\infty} \frac{1}{z} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\
&\leq \frac{1}{t} \int_t^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz
\end{aligned}
$$

Let

$$u = -\frac{z^2}{2} \text{ and } du = -z dz$$

so that

$$\begin{aligned}
\int_t^{\infty} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz &= -\int_{-\frac{t^2}{2}}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\{u\}\, du \\
&= \int_{-\infty}^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\{u\}\, du \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \\
&= \phi(t)
\end{aligned}$$

It follows that

$$\overline{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t}.$$

(b) Let $t > 0$ and set

$$\Phi(t) = \Pr(Z \leq t) = 1 - \frac{\varphi}{2N}.$$

It follows that

$$\Phi^{-1}(\Phi(t)) = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) = t$$

and, by the result given in part (a) above,

$$1 - \Phi(t) = 1 - \left(1 - \frac{\varphi}{2N}\right) = \frac{\varphi}{2N} \leq \frac{\phi(t)}{t}.$$

The latter inequality implies that

$$t \leq \phi(t)\frac{2N}{\varphi}$$

so that

$$\begin{aligned}
\ln t &\leq \ln\phi(t) + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\pi + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln\pi + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \ln 2 + \ln\left(\frac{N}{\varphi}\right)
\end{aligned}$$

or

$$\begin{aligned}
t^2 &\leq 2(\ln 2 - \ln t) + 2\ln\left(\frac{N}{\varphi}\right) \\
&= 2\ln\left(\frac{2}{t}\right) + 2\ln\left(\frac{N}{\varphi}\right) \\
&\leq 2\ln\left(\frac{N}{\varphi}\right) \quad \text{for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2
\end{aligned}$$

so that

$$t \leq \sqrt{2}\sqrt{\ln\left(\frac{N}{\varphi}\right)} \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2$$

Hence, for N_1, N_2 sufficiently large so that

$$1 - \frac{\varphi}{2N} \geq \Phi\left(2\right) \text{ or, equivalently, } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2,$$

we have

$$\begin{aligned}
\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) &= t \\
&\leq \sqrt{2}\sqrt{\ln\left(\frac{N}{\varphi}\right)} \\
&= \sqrt{2}\sqrt{\ln N - \ln\varphi} \\
&= \sqrt{2}\sqrt{\ln N}\sqrt{1 - \frac{\ln\varphi}{\ln N}} \\
&\leq \sqrt{2}\sqrt{\ln N}\sqrt{1 - \frac{\ln N^{-a}}{\ln N}} \\
&= \sqrt{2\left(1+a\right)}\sqrt{\ln N}. \; \square
\end{aligned}$$

Lemma QA-16: Suppose that Assumptions 2-1, 2-2, 2-3, 2-5, 2-6, and 2-8 hold and suppose that $N_1, N_2, T \to \infty$ such that $N_1/\tau_1^3 = N_1/\left\lfloor T_0^{\alpha_1}\right\rfloor^3 \to 0$. Then, the following statements are true.

(a)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right| \xrightarrow{p} 0$$

(b)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T} - \pi_{i,\ell,T}}{\pi_{i,\ell,T}}\right| \xrightarrow{p} 0$$

where

$$\overline{S}_{i,\ell,T} = \sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it}y_{\ell,t+1} \text{ and } \overline{V}_{i,\ell,T} = \sum_{r=1}^{q}\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it}y_{\ell,t+1}\right]^2$$

Proof of Lemma QA-16:

To show part (a), note first that by applying parts (a) and (c) of Lemma OA-6, parts (a)-(c) of Lemma

OA-12, and the Slutsky theorem; we obtain

$$
\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{q\tau_1}\right| \\
= & \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right. \\
& \quad +\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}+\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it} \\
& \quad \left.-\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right| \\
\leq & \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right| \\
& +\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right| \\
& +\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right| \\
& +\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right| \\
& +\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right| \\
= & \; o_p(1)
\end{aligned}
$$

Moreover, by Assumption 2-8, there exist a positive constant $\underline{c}$ such that for all N and T sufficiently large

$$
\begin{aligned}
& \min_{1\leq \ell\leq d}\min_{i\in H^c}\left|\frac{\mu_{i,\ell,T}}{q\tau_1}\right| \\
= & \min_{1\leq \ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right| \\
\geq & \; \underline{c}>0
\end{aligned}
$$

It follows that

$$
\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\leq \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{q\tau_1}\right|/\min_{1\leq \ell\leq d}\min_{i\in H^c}\left|\frac{\mu_{i,\ell,T}}{q\tau_1}\right|=o_p(1).
$$

Now, for part (b), note that, applying parts (d), (f), (g), and (h) of Lemma OA-12, parts (b), (d), and (e) of Lemma OA-6, and the Slutsky theorem; we have

$$
\begin{aligned}
& \max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{q\tau_1^2}\right| \\
= \;& \max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\qquad\qquad\qquad \left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{2}{q}\sum_{r=1}^{q}\left\{\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\right. \\
& \qquad\qquad\qquad\qquad \left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right) \\
& \qquad\quad \left.\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right\}\right| \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2 \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2 \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right| \\
= \;& o_p\left(1\right)
\end{aligned}
$$

Moreover, note that, for all N and T sufficiently large,

$$\begin{aligned}
& \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{\pi_{i,\ell,T}}{q\tau_1^2} \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
\geq \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
& \text{(by Jensen's inequality)} \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right|^2 \\
= \quad & \left(\min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right|\right)^2 \\
\geq \quad & \underline{c}^2>0 \quad \text{(by Assumption 2-8)}.
\end{aligned}$$

It follows that

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{\pi_{i,\ell,T}}\right|\leq\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{q\tau_1^2}\right|/\min_{1\leq\ell\leq d}\min_{i\in H^c}\left(\frac{\pi_{i,\ell,T}}{q\tau_1^2}\right)=o_p(1).\ \square$$

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