

Supplemental Appendix for “Jackknife Estimation of a Cluster-Sample IV Regression Model with Many Weak Instruments”¹

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Abstract

This Supplemental Appendix is comprised of two sub-appendices. Appendix S1 provides proofs for Theorems 2 and 3 of the main paper. Appendix S2 states additional supporting lemmas used to prove the main theorems of the paper. Proofs for these additional lemmas are reported in a separate Online Appendix which can be viewed at the URL:

http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

Appendix S1: Proof of Theorems 2 and 3

Proof of Theorem 2: Define $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$. Note that, by the result of Lemma S2-9 given in Appendix S2 below, we have that $D_\mu^{-1} \widehat{\Delta}(\delta_0) = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1) = \mathcal{Y}_n + o_p(1)$.

We now establish the asymptotic normality of \mathcal{Y}_n , upon appropriate standardization, in the case where $K_{2,n} / (\mu_n^{\min})^2 = O(1)$. To proceed, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and define $b_{1n} = \Sigma_n^{-1/2} a$ and $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1} \Sigma_n^{-1/2} a$, where $\Sigma_n = VC(\mathcal{Y}_n | \mathcal{F}_n^Z) = \Sigma_{1,n} + \Sigma_{2,n}$, with $\Sigma_{1,n} = VC(\Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)$ and $\Sigma_{2,n} = VC(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z)$. Now, let $\mathcal{L}_{(i,t),n} = b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} \varepsilon_{(i,t)} / \sqrt{n}$ and $\mathcal{N}_{(i,t),n} = K_{2,n}^{-1/2} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} [\underline{u}_{2,(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{2,(j,s),n} \varepsilon_{(i,t)}]$, where $\underline{u}_{2,(i,t),n} = b'_{2n} \underline{U}_{(i,t)}$, with $\underline{u}_{2,(j,s),n}$ similarly defined, and where $e_{(i,t)}$ denotes an $m_n \times 1$ elementary vector whose $(i,t)^{th}$ component is 1 and all other components are 0. Using these notations, note that we can write $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{L}_{(1,1),n} + \sum_{(i,t)=2}^{m_n} \{\mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}\}$. Next, observe that,

$$\begin{aligned} E[\mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z] &= E[\varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z] \frac{[a' \Sigma_n^{-1/2} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(1,1)}]^2}{n} \\ &\leq E[\varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z] a' \Sigma_n^{-1} a \left(\frac{\|\Upsilon' Z'_2 M^{(Z_1, Q)} e_{(1,1)}\|_2}{\sqrt{n}} \right)^2 \quad (\text{by CS inequality}) \end{aligned}$$

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$$\begin{aligned}
&\leq \left(\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right) a' \Sigma_n^{-1} a \left(\frac{\max_{1 \leq (i,t) \leq m_n} \| \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \|_2}{\sqrt{n}} \right)^2 \\
&= o_p(1)
\end{aligned}$$

in light of Assumptions 2(i) and 7 and part (d) of Lemma S2-3². Moreover, under Assumptions 2 and 3(iii), there exists a positive constant C^* such that

$$\begin{aligned}
E_Z \left\{ \left(E \left[\mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\} &= \frac{E_Z \left\{ \left[a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right]^4 \left(E \left[\varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\}}{n^2} \\
&\leq \frac{C}{n^2} E \left(\left[a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right]^4 \right) \text{ (by Assumption 2(i))} \\
&\leq C E \left(\frac{a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \right)^2 \text{ (by CS inequality)} \\
&\leq C \bar{C} = C^* < \infty \text{ (by Assumption 3(iii) and Lemma S2-3(d))}
\end{aligned}$$

Since the upper bound above does not depend on n , we further deduce that

$\sup_n E_Z \left\{ \left(E \left[\mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\} < \infty$. It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that $E \left(\mathcal{L}_{(1,1),n}^2 \right) = E_Z \left(E \left[\mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right) \rightarrow 0$. Application of Markov's inequality then allows us to deduce that $\mathcal{L}_{(1,1),n} = b_{1n}' \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \varepsilon_{(1,1)} / \sqrt{n} = o_p(1)$, from which we obtain the representation $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{V}_n + o_p(1)$, where $\mathcal{V}_n = \sum_{(i,t)=2}^{m_n} \mathcal{V}_{(i,t),n}$ with $\mathcal{V}_{(i,t),n} = \mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}$. Note we can also write $\mathcal{V}_n = \mathcal{L}_n + \mathcal{N}_n$, where $\mathcal{L}_n = \sum_{(i,t)=2}^{m_n} \mathcal{L}_{(i,t),n}$ and $\mathcal{N}_n = \sum_{(i,t)=2}^{m_n} \mathcal{N}_{(i,t),n}$.

Next, define the σ -fields $\mathcal{F}_{(i,t),n} = \sigma \left(\{ \varepsilon_{(k,v)}, U_{(k,v)} \}_{(k,v)=1}^{(i,t)}, Z \right)$ for $(i,t) = 1, 2, \dots, m_n$, note that by construction $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$ for $(i,t) = 2, \dots, m_n$ and $\mathcal{V}_{(i,t),n}$ is $\mathcal{F}_{(i,t),n}$ -measurable. Note also that, under Assumption 1, it is easily seen that $E [\mathcal{V}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n}] = 0$. In addition, note that, by part (d) of Lemma S2-3 and Lemma S2-6, and Assumption 2(i);

$$\begin{aligned}
E \left[u_{2,(i,t),n}^2 | \mathcal{F}_n^Z \right] &\leq (b_{2n}' b_{2n}) \max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] \\
&\leq \frac{K_{2,n}}{(\mu_n^{\min})^2} a' \Sigma_n^{-1} a \max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] = O_{a.s.}(1) \tag{1}
\end{aligned}$$

since, for this theorem, we assume that $K_{2,n} / (\mu_n^{\min})^2 = O(1)$. It follows then from straightforward calculations, from applying the triangle and CS inequalities, as well as from expression (1), part

²Lemma S2-3 is stated in Appendix S2 below. A proof of this lemma is provided in section 1 of the Additional Online Appendix which can be viewed at the URL:

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(d) of Lemma S2-1, part (d) of Lemma S2-3, and Assumptions 2(i) and 3(iii) that

$$\begin{aligned}
& \text{Var}(\mathcal{V}_{(i,t),n} | \mathcal{F}_n^Z) \\
&= E[\mathcal{L}_{(i,t),n}^2 | \mathcal{F}_n^Z] + E[\mathcal{N}_{(i,t),n}^2 | \mathcal{F}_n^Z] \\
&\leq \left(\max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z] \right) a' \Sigma_n^{-1} a \lambda_{\max} \left(\frac{\Upsilon' Z_2' Z_2 \Upsilon}{n} \right) \\
&\quad + \frac{4}{K_{2,n}} \left(\max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z] \right) \left(\max_{1 \leq (i,t) \leq m_n} E[u_{2,(i,t),n}^2 | \mathcal{F}_n^Z] \right) \left(\max_{1 \leq (i,t) \leq m_n} \sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 \right) \\
&= O_{a.s.}(1) + O_{a.s.}\left(\frac{K_{2,n}}{(\mu_n^{\min})^2 n}\right) = O_{a.s.}(1)
\end{aligned}$$

By the law of iterated expectations and Theorem 16.1 of Billingsley (1995), there exists a constant \bar{C} such that $\text{Var}(\mathcal{V}_{(i,t),n}) = E(\mathcal{V}_{(i,t),n}^2) = E_Z[E(\mathcal{V}_{(i,t),n}^2 | \mathcal{F}_n^Z)] \leq \bar{C} < \infty$ for all n sufficiently large. These results show that $\{\mathcal{V}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n, n \geq 1\}$ forms a square-integrable martingale difference array.

To show the asymptotic normality of \mathcal{V}_n , we verify the conditions of the central limit theorem for martingale difference arrays given in Lemma S2-15. To proceed, first consider condition (22), which, as noted in the remark following Lemma S2-15, is a sufficient condition for condition (20) of Lemma S2-15. We shall verify (22) for the case where $\delta = 2$. Note first that, by applying Loève's c_r inequality, we get

$$\sum_{(i,t)=2}^{m_n} E[\mathcal{V}_{(i,t),n}^4] = \sum_{(i,t)=2}^{m_n} E[(\mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n})^4] \leq 8 \sum_{(i,t)=2}^{m_n} E[\mathcal{L}_{(i,t),n}^4] + 8 \sum_{(i,t)=2}^{m_n} E[\mathcal{N}_{(i,t),n}^4]$$

Hence, to verify condition (22), it suffices to show that $\sum_{(i,t)=2}^{m_n} E[\mathcal{L}_{(i,t),n}^4] = o(1)$ and $\sum_{(i,t)=2}^{m_n} E[\mathcal{N}_{(i,t),n}^4] = o(1)$. To do this, we first focus on a conditional expectation analogue of $\sum_{(i,t)=2}^{m_n} E[\mathcal{L}_{(i,t),n}^4]$. Note that

$$\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E[\mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z] \\
&= \frac{1}{n^2} \sum_{(i,t)=2}^{m_n} \left[a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^4 E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] \\
&\leq a' \Sigma_n^{-1} a \frac{1}{n} \sum_{(i,t)=2}^{m_n} \left[a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^2 \left(\frac{\|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] \\
&\quad (\text{by CS inequality})
\end{aligned}$$

$$\begin{aligned}
&\leq a'\Sigma_n^{-1}a \left(\max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] \right) \left(\frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \\
&\quad \times \frac{1}{n} a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e'_{(i,t)} M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \\
&\leq a'\Sigma_n^{-1}a \left(\max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] \right) \left(\frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \\
&\quad \times \frac{a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \\
&\leq (a'\Sigma_n^{-1}a)^2 \left(\max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] \right) \left(\frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \lambda_{\max} \left(\frac{\Upsilon' Z_2' Z_2 \Upsilon}{n} \right) \\
&\leq C \left(\frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 = o_p(1)
\end{aligned}$$

where the last line above follows from Assumptions 2(i), 3(iii), and 7 and by Lemma S2-3(d). Next, note that, under Assumptions 2 and 3(iii), there exists a positive constant C^* such that

$$\begin{aligned}
&E_Z \left(\sum_{(i,t)=2}^{m_n} E[\mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z] \right)^2 \\
&= \frac{1}{n^4} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=2}^{m_n} E_Z \left(\left[a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)} \right]^4 \left[a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} e_{(j,s)} \right]^4 \right. \\
&\quad \left. \times E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z] E[\varepsilon_{(j,s)}^4 | \mathcal{F}_n^Z] \right) \\
&\leq \frac{C}{n^4} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=2}^{m_n} E_Z \left(\left[a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} e_{(i,t)} \right]^4 \left[a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} e_{(j,s)} \right]^4 \right) \\
&\leq \frac{C}{n^4} E_Z \left\{ a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e'_{(i,t)} M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} \right. \\
&\quad \left. \times \sum_{(j,s)=1}^{m_n} e_{(j,s)} e'_{(j,s)} M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \left(a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \right)^2 \right\} \\
&= C E_Z \left(\frac{a'\Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \right)^4 \\
&\leq \bar{C}\bar{C} = C^* < \infty \quad (\text{by Assumption 3(iii) and Lemma S2-3(d)})
\end{aligned}$$

where the second inequality above follows from applying the CS inequality. Since the upper bound above does not depend on n , we further deduce that

$\sup_n E_Z \left(\sum_{(i,t)=2}^{m_n} E \left[\mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \right)^2 < \infty$. It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that $\sum_{(i,t)=2}^{m_n} E \left[\mathcal{L}_{(i,t),n}^4 \right] = \sum_{(i,t)=2}^{m_n} E_Z \left(E \left[\mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \right) \rightarrow 0$.

Turning our attention to the bilinear term, note that by Loèv's c_r inequality we have $\sum_{(i,t)=2}^{m_n} E \left[\mathcal{N}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq \mathcal{R}_1 + \mathcal{R}_2$, where

$$\mathcal{R}_1 = \sum_{(i,t)=2}^{m_n} \left(8/K_{2,n}^2 \right) E \left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}_n^Z \right] \text{ and}$$

$$\mathcal{R}_2 = \sum_{(i,t)=2}^{m_n} \left(8/K_{2,n}^2 \right) E \left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right)^4 | \mathcal{F}_n^Z \right]. \text{ Focusing first on the term } \mathcal{R}_1,$$

note that, by straightforward calculations as well as by making use of Assumptions 2(i) and 5(ii), parts (b) and (c) of Lemma S2-1, part (d) of Lemma S2-3, and Lemma S2-6; we deduce that, there exists a positive constant \bar{C} such that

$$\begin{aligned} \frac{(\mu_n^{\min})^4 n}{K_{2,n}^2} \mathcal{R}_1 &\leq 24n (a' \Sigma_n^{-1} a)^2 \left(\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^Z \right] \right) \left(\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right) \\ &\quad \times \left[\frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &\leq \bar{C}n \left[\frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &= O_{a.s.} \left(\frac{K_{2,n}}{n} \right) + O_{a.s.} (1) = O_{a.s.} (1). \end{aligned}$$

Applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we then have

$$\begin{aligned} \frac{(\mu_n^{\min})^4 n}{K_{2,n}^2} E_Z (\mathcal{R}_1) &\leq \bar{C}n E_Z \left[\frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &= O(1) \end{aligned}$$

from which we further deduce that

$$E_Z (\mathcal{R}_1) = \sum_{(i,t)=2}^{m_n} \frac{8}{K_{2,n}^2} E \left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 \right] = O \left(\frac{K_{2,n}^2}{(\mu_n^{\min})^4 n} \right) = o(1)$$

In a similar way, we can also show that

$$E_Z(\mathcal{R}_2) = \left(8/K_{2,n}^2\right) E \left[\sum_{(i,t)=2}^{m_n} \left(\sum_{(j,s)=1}^{(i,t)-1} A_{(j,s),(i,t)} \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right)^4 \right] = o(1). \text{ It follows that}$$

$$\sum_{(i,t)=2}^{m_n} E \left[\mathcal{N}_{(i,t),n}^4 \right] \leq E_Z(\mathcal{R}_1) + E_Z(\mathcal{R}_2) = o(1). \text{ This verifies condition (22).}$$

Next, we verify condition (21) of Lemma S2-15. To proceed, first let $s_Z^2 = \text{Var} [\mathcal{V}_n | \mathcal{F}_n^Z] = \text{Var} \left(\sum_{(i,t)=2}^{m_n} \mathcal{V}_{(i,t),n} | \mathcal{F}_n^Z \right)$, and note that

$$s_Z^2 = \text{Var} \left(\frac{b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon}{\sqrt{n}} + \frac{b'_{2n} \underline{U}' A \varepsilon}{\sqrt{K_{2,n}}} | \mathcal{F}_n^Z \right) + o_p(1) = a' \Sigma_n^{-1/2} \Sigma_n \Sigma_n^{-1/2} a + o_p(1) = 1 + o_p(1) \quad (2)$$

On the other hand, by straightforward calculation, we can write

$$\begin{aligned} s_Z^2 &= \frac{1}{n} \sum_{(i,t)=2}^{m_n} \left[b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} \right]^2 E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + \frac{1}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left\{ E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] + E \left[\underline{u}_{2,(j,s)}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right\} \\ &\quad + \frac{2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] E \left[\underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \end{aligned} \quad (3)$$

Making use of expression (3), we obtain, after some further calculations,

$$\begin{aligned} &\sum_{(i,t)=2}^{m_n} E \left[\mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - s_Z^2 \\ &= \frac{2}{\sqrt{n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} \right] \frac{A_{(i,t),(j,s)}}{\sqrt{K_{2,n}}} \left\{ \varepsilon_{(j,s)} E \left[\varepsilon_{(i,t)} \underline{u}_{2,(i,t)} | \mathcal{F}_n^Z \right] + \underline{u}_{2,(j,s)} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right\} \\ &\quad + \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left(\varepsilon_{(j,s)}^2 - E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left(\underline{u}_{2,(j,s)}^2 - E \left[\underline{u}_{2,(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + 2 \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left(\varepsilon_{(j,s)} \underline{u}_{2,(j,s)} - E \left[\underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \\ &\quad + 2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \left\{ \underline{u}_{2,(j,s)} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{2,(k,v)} \right\} \end{aligned}$$

$$\begin{aligned}
& + 2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \\
& + 2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} \underline{u}_{2,(j,s)} \underline{u}_{2,(k,v)} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\
& = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7, \quad (\text{say})
\end{aligned}$$

Note first that, by applying parts (a)-(c) of Lemma S2-14, we have $\mathcal{T}_1 \xrightarrow{p} 0$, $\mathcal{T}_2 \xrightarrow{p} 0$, and $\mathcal{T}_3 \xrightarrow{p} 0$. Consider next the term

$$\mathcal{T}_4 = 2 \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left(\underline{u}_{2,(j,s)} \varepsilon_{(j,s)} - E \left[\underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right].$$

In this case, we apply part (a) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$, $\bar{\psi}_{(j,s)} = E \left[\underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right]$, and $\phi_{(i,t)} = E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$. Note that, in this case, $\left\{ \left(\underline{u}_{2,(i,t)}, \varepsilon_{(i,t)} \right) \right\}_{(i,t)=1}^{m_n}$ is independent conditional on \mathcal{F}_n^Z , and $\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s. by Assumptions 1 and 2(i), respectively. Moreover, note that Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that $K_{2,n}/(\mu_n^{\min})^2 = O(1)$ in this case together imply that there exists a constant $C \geq 1$ such that $E \left[\underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq \left[K_{2,n}^2 / (\mu_n^{\min})^4 \right] E \left[\left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^Z \right] (a' \Sigma_n^{-1} a)^2 \leq C < \infty$ a.s. for all $(i,t) \in \{1, 2, \dots, m_n\}$ and for all n sufficiently large, so that

$\max_{1 \leq (i,t) \leq m_n} E \left[\underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s.n. Finally, using the upper bound derived in expression (29) in the proof of part (a) of Lemma S2-14³, we obtain

$$\begin{aligned}
\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t)}| & \leq \max_{1 \leq (i,t) \leq m_n} E \left[\left| \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} \right| | \mathcal{F}_n^Z \right] \leq C \text{ a.s.n. and } \max_{1 \leq (j,s) \leq m_n} |\bar{\psi}_{(j,s)}| \leq \\
\max_{1 \leq (i,t) \leq m_n} E \left[\left| \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} \right| | \mathcal{F}_n^Z \right] & \leq C \text{ a.s.n. It follows by part (a) of Lemma S2-8 that } \mathcal{T}_4 \xrightarrow{p} 0.
\end{aligned}$$

Now, consider \mathcal{T}_5 . Here, we apply part (b) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$ and $\phi_{(i,t)} = E \left[\underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$. Note again that $\left\{ \left(\underline{u}_{2,(i,t)}, \varepsilon_{(i,t)} \right) \right\}_{(i,t)=1}^{m_n}$ is independent conditional on \mathcal{F}_n^Z , and $\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s. by Assumptions 1 and 2(i), respectively. Moreover, previously, we have shown that $E \left[\underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s.n. and $\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t)}| \leq C$ a.s.n. Hence, applying part (b) of Lemma S2-8, we deduce that $\mathcal{T}_5 \xrightarrow{p} 0$.

Turning our attention to \mathcal{T}_6 , we note that, for this term, we can apply part (c) of Lemma S2-8 with $\phi_{(i,t)} = E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right]$. From (1), there exists a positive constant C such that $E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C < \infty$ a.s. for all $(i,t) \in \{1, 2, \dots, m_n\}$ and for all n sufficiently large, so that

$$\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t)}| = \max_{1 \leq (i,t) \leq m_n} E \left[\underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C \text{ a.s.n. Hence, applying part (c) of Lemma }$$

³A proof of Lemma S2-14 is given in section 1 of the Additional Online Appendix which can be viewed at the URL:

http://econweb.umd.edu/~chao/Research/research_files/Additional_Oline_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

S2-8, we obtain $\mathcal{T}_6 \xrightarrow{p} 0$.

Finally, consider \mathcal{T}_7 . In this case, we apply part (d) of Lemma S2-8 with $u_{(j,s)} = \underline{u}_{2,(j,s)}$, $u_{(k,v)} = \underline{u}_{2,(k,v)}$, and $\phi_{(i,t)} = E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z]$. Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain $E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z] \leq (E[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z])^{1/2} \leq C < \infty$ a.s. for all $(i,t) \in \{1, 2, \dots, m_n\}$ and for all n , so that $\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t)}| = \max_{1 \leq (i,t) \leq m_n} E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z] \leq C$ a.s. Moreover, as noted previously, Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that $K_{2,n}/(\mu_n^{\min})^2 = O(1)$ together imply that $\max_{1 \leq (i,t) \leq m_n} E[\underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z] \leq C$ a.s.n. It follows by applying part (d) of Lemma S2-8 that $\mathcal{T}_7 \xrightarrow{p} 0$.

The above argument shows that $\sum_{(i,t)=2}^{m_n} E[\mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n}] - s_Z^2 = \sum_{k=1}^7 \mathcal{T}_k = o_p(1)$. On the other hand, expression (2) above implies that $s_Z^2 - 1 = o_p(1)$. Putting these two results together, we obtain $\sum_{(i,t)=2}^{m_n} E[\mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n}] - 1 = o_p(1)$, which establishes condition (21) of Lemma S2-15.

It now follows from Lemma S2-15 that

$$\mathcal{V}_n = \sum_{(i,t)=2}^{m_n} \left\{ b'_{1n} \Upsilon' Z_2' M^Q e_{(i,t)} \varepsilon_{(i,t)} / \sqrt{n} + \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} [\underline{u}_{2,(i,t)} \varepsilon_{(j,s)} + \underline{u}_{2,(j,s)} \varepsilon_{(i,t)}] \right\} \xrightarrow{d} N(0, 1).$$

Since, previously, we have shown that $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{V}_n + o_p(1)$, this further implies that $a' \Sigma_n^{-1/2} \mathcal{Y}_n \xrightarrow{d} N(0, 1)$. Given that this result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we can then apply the Cramér-Wold device to obtain

$$\Sigma_n^{-1/2} \mathcal{Y}_n = \Sigma_n^{-1/2} \left(\frac{\Upsilon' Z_2' M^{(Z_1,Q)\varepsilon}}{\sqrt{n}} + D_\mu^{-1} \underline{U}' A \varepsilon \right) \xrightarrow{d} N(0, I_d) \quad (4)$$

Next, let $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon / n$, $\Lambda_{I,n} = H_n^{-1} \Sigma_n H_n^{-1}$, and $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1,Q)\varepsilon} / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$, as given above. Consider first $\widehat{\delta}_{L,n}$. Theorem 1 has already shown that $\widehat{\delta}_{L,n} \xrightarrow{p} \delta_0$. To show asymptotic normality of $\widehat{\delta}_L$, note first that, by Lemma S2-11, $\widehat{\delta}_{L,n}$ satisfies the set of (normalized) first-order conditions $\widehat{\Delta}(\widehat{\delta}_{L,n}) = 0$, where

$\widehat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1,Q)} (y - X\delta)/2] [\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta]$. Applying the mean-value theorem to each component of $\widehat{\Delta}(\delta)$ and expanding it around the point $\delta = \delta_0$, we obtain $0 = \widehat{\Delta}(\widehat{\delta}_{L,n}) = \widehat{\Delta}(\delta_0) + (\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta') (\widehat{\delta}_{L,n} - \delta_0)$, with $\bar{\delta}_n$ lying on the line segment between $\widehat{\delta}_{L,n}$ and δ_0 . Multiplying both sides of this equation by D_μ^{-1} , we further obtain

$$0 = D_\mu^{-1} \widehat{\Delta}(\delta_0) + D_\mu^{-1} \frac{\partial \widehat{\Delta}(\bar{\delta}_n)}{\partial \delta'} (\widehat{\delta}_{L,n} - \delta_0) = D_\mu^{-1} \widehat{\Delta}(\delta_0) + D_\mu^{-1} \frac{\partial \widehat{\Delta}(\bar{\delta}_n)}{\partial \delta'} D_\mu^{-1} D_\mu (\widehat{\delta}_{L,n} - \delta_0) \quad (5)$$

From the result of Lemma S2-10, we have $-D_\mu^{-1} (\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta') D_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Upsilon' Z_2' M^{(Z_1,Q)} Z_2 \Upsilon / n$ is a positive definite matrix a.s.n. by Assumption 3(iii), which, in turn, implies that $D_\mu^{-1} (\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta')$ D_μ^{-1} is nonsingular and, thus, invertible w.p.a.1. It follows that, for all n

sufficiently large, we can solve for $D_\mu(\widehat{\delta}_{L,n} - \delta_0)$ in (5) above to get

$$\begin{aligned} D_\mu(\widehat{\delta}_{L,n} - \delta_0) &= - \left[D_\mu^{-1} \left(\frac{\partial \widehat{\Delta}(\bar{\delta}_n)}{\partial \delta'} \right) D_\mu^{-1} \right]^{-1} D_\mu^{-1} \widehat{\Delta}(\delta_0) \\ &= H_n^{-1} \left(\frac{\Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} \underline{U}' A \varepsilon \right) [1 + o_p(1)], \end{aligned} \quad (6)$$

where the last equality follows by applying Lemma S2-9. By part (d) of Lemma S2-3, Σ_n is positive definite *a.s.n.*, so that Σ_n^{-1} is well-defined for all n sufficiently large, and both $\Sigma_n^{1/2}$ and $\Sigma_n^{-1/2}$ can be taken to be symmetric matrices. Since H_n is also symmetric, it further follows that $\Lambda_{I,n} = H_n^{-1} \Sigma_n H_n^{-1}$ is symmetric and positive definite *a.s.n.*, and both $\Lambda_{I,n}^{-1} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1}$ and $\Lambda_{I,n}^{-1/2} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2}$ are well-defined for all n sufficiently large. Multiplying both sides of the equation above by $\Lambda_{I,n}^{-1/2}$, we then get $\Lambda_{I,n}^{-1/2} D_\mu(\widehat{\delta}_{L,n} - \delta_0) = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \mathcal{Y}_n [1 + o_p(1)]$, where $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$. Let $R_{W,n} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \Sigma_n^{1/2}$, and note that $R_{W,n} R_{W,n}' = I_d$ for all n sufficiently large. It, thus, follows from the result given in (4) above and the continuous mapping theorem that $\Lambda_{I,n}^{-1/2} D_\mu(\widehat{\delta}_{L,n} - \delta_0) \xrightarrow{d} N(0, I_d)$, as $n \rightarrow \infty$, as required.

Turning our attention now to $\widehat{\delta}_{F,n}$, note that we can write this estimator, appropriately standardized, as

$$D_\mu(\widehat{\delta}_{F,n} - \delta_0) = \left(D_\mu^{-1} X' \left[A - \widehat{\ell}_{F,n} M^{(Z_1, Q)} \right] X D_\mu^{-1} \right)^{-1} D_\mu^{-1} X' \left[A - \widehat{\ell}_{F,n} M^{(Z_1, Q)} \right] (y - X \delta_0) \quad (7)$$

so that, multiplying by $\Lambda_{I,n}^{-1/2} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2}$ and applying Lemmas S2-12 and S2-13, we obtain $\Lambda_{I,n}^{-1/2} D_\mu(\widehat{\delta}_{F,n} - \delta_0) = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \mathcal{Y}_n [1 + o_p(1)]$. It follows from the result given in (4) above and the continuous mapping theorem that $\Lambda_{I,n}^{-1/2} D_\mu(\widehat{\delta}_{F,n} - \delta_0) \xrightarrow{d} N(0, I_d)$, as $n \rightarrow \infty$, as required. \square

Proof of Theorem 3: To proceed, note that, in this case, $(\mu_n^{\min}) / \sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$, so that, by the result given in Lemma S2-9, we have

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu^{-1} \widehat{\Delta}(\delta_0) = \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1) \quad (8)$$

where $\underline{U} = U - \varepsilon \rho'$. Again, let $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n$, and $\Sigma_{2,n} = VC(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z) = D_\mu^{-1} VC(\underline{U}' A \varepsilon | \mathcal{F}_n^Z) D_\mu^{-1}$. Now, by assumption, \widetilde{L}_n can be any sequence of bounded $(l \times d)$ non-random matrices such that $\lambda_{\min}((\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n}) \geq \underline{C}$ *a.s.n.* for some constant $\underline{C} > 0$. It follows that $(\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n}$ is positive definite *a.s.n.*, so that, with probability one, $((\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n})^{-1/2}$ is well-defined for all n sufficiently large. Hence, we can let

$$\tilde{\mathcal{N}}_n = \left((\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)^{-1/2} \widetilde{L}_n H_n^{-1} (\mu_n^{\min} / \sqrt{K_{2,n}}) D_\mu^{-1} \underline{U}' A \varepsilon \text{ and construct the}$$

linear combination $\mathcal{J}_n = a' \tilde{\mathcal{N}}_n$ for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$. Next, define $\underline{u}_{(i,t),n} = a' \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$, with $\underline{u}_{(j,s),n}$ similarly defined, and we can write $\mathcal{J}_n = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[\underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right]$
 $= \sum_{(i,t)=2}^{m_n} \mathcal{J}_{(i,t),n}$, where $\mathcal{J}_{(i,t),n} = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[\underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right]$. Again, define the σ -fields $\mathcal{F}_{(i,t),n} = \sigma \left(\{\varepsilon_{(k,v)}, U_{(k,v)}\}_{(k,v)=1}^{(i,t)}, Z \right)$ for $(i,t) = 1, 2, \dots, m_n$, noting that by construction $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$ for $(i,t) = 2, \dots, m_n$ and $\mathcal{J}_{(i,t),n}$ is $\mathcal{F}_{(i,t),n}$ -measurable. In addition, note that, using Assumption 1, it is easily seen that $E \left[\underline{u}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] = 0$ and $E \left[\varepsilon_{(i,t)} | \mathcal{F}_{(i,t)-1,n} \right] = 0$, from which it follows that $E \left[\mathcal{J}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left\{ \varepsilon_{(j,s)} E \left[\underline{u}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] + \underline{u}_{(j,s),n} E \left[\varepsilon_{(i,t)} | \mathcal{F}_{(i,t)-1,n} \right] \right\} = 0$. Moreover, applying the CS inequality and making use of the fact that

$$E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq \frac{\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] \left\| \tilde{L}_n \right\|_F^2}{\lambda_{\min} \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) [\lambda_{\min}(H_n)]^2} \left(\frac{1}{\mu_n^{\min}} \right)^2 = O_{a.s.} \left(\frac{1}{(\mu_n^{\min})^2} \right) \quad (9)$$

and that $E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq \bar{C}$ a.s. by Assumption 2(i), we see that

$$\begin{aligned} & Var \left(\mathcal{J}_{(i,t),n} | \mathcal{F}_n^Z \right) \\ & \leq \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] + E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] E \left[\underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] \right. \\ & \quad \left. + 2 \sqrt{E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]} \sqrt{E \left[\underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right]} \right) \\ & \leq \frac{4\bar{C}^2}{(\mu_n^{\min})^2} \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 = \frac{4\bar{C}^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \text{ a.s.n.} \end{aligned} \quad (10)$$

Hence, applying the law of iterated expectations, part (d) of Lemma S2-1, and Theorem 16.1 of Billingsley (1995), we further deduce that $Var \left(\mathcal{J}_{(i,t),n} \right) = E_Z \left[E \left(\mathcal{J}_{(i,t),n}^2 | \mathcal{F}_n^Z \right) \right]$
 $\leq \left(4\bar{C}^2 / K_{2,n} \right) \sum_{(j,s)=1}^{(i,t)-1} E_Z \left[A_{(i,t),(j,s)}^2 \right] \leq C$ for some positive constant C for all n sufficiently large. These results show that $\{\mathcal{J}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n, n \geq 1\}$ forms a square-integrable martingale difference array.

Next, we verify condition (22) of the central limit theorem for martingale difference arrays given

in Lemma S2-15 below. By Loève's c_r inequality we have

$$\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E \left[\left(\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} [\underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)}] \right)^4 | \mathcal{F}_n^Z \right] \\
& \leq 8 \sum_{(i,t)=2}^{m_n} \frac{(\mu_n^{\min})^4}{K_{2,n}^2} E \left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}_n^Z \right] \\
& \quad + 8 \sum_{(i,t)=2}^{m_n} \frac{(\mu_n^{\min})^4}{K_{2,n}^2} E \left[\left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right)^4 | \mathcal{F}_n^Z \right] \\
& = \mathcal{E}_1 + \mathcal{E}_2, \quad (\text{say}). \tag{11}
\end{aligned}$$

Focusing first on \mathcal{E}_1 , it is easy to see that there exists some positive constant C such that

$$\begin{aligned}
& \mathcal{E}_1 \\
& = \frac{8(\mu_n^{\min})^4}{K_{2,n}^2} E \left[\sum_{(i,t)=2}^{m_n} \left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}_n^Z \right] \\
& \leq \frac{8(\mu_n^{\min})^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s)=1 \\ (j,s) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^4 E \left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(j,s)}^4 | \mathcal{F}_n^Z \right] \\
& \quad + \frac{24(\mu_n^{\min})^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t) \\ (j,s) \neq (k,v)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 E \left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] E \left[\varepsilon_{(k,v)}^2 | \mathcal{F}_n^Z \right] \\
& \leq \frac{C}{K_{2,n}} \left[\frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (j,s) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right]
\end{aligned}$$

where the second inequality above follows from Assumption 2(i) and from an upper bound on the conditional fourth moment of

$\underline{u}_{(i,t),n} = a' \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$ given by

$$\begin{aligned}
E \left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] & \leq \frac{1}{(\mu_n^{\min})^4} \left(\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^Z \right] \right) \frac{1}{[\lambda_{\min}(H_n)]^4} \\
& \quad \times \left\| \tilde{L}_n \right\|_F^4 \left(\frac{1}{\lambda_{\min} \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)} \right)^2 \\
& \leq \frac{C^*}{(\mu_n^{\min})^4} \quad \text{a.s.n., for some constant } C^* > 0. \tag{12}
\end{aligned}$$

Note also that, in deriving the upper bound given in (12), we have applied Assumption 3(iii), Lemma S2-6, the boundedness of $\|\tilde{L}_n\|_F^2$, and the assumption that

$\lambda_{\min} \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) \geq \underline{C} > 0$ a.s.n. Moreover, by parts (b) and (c) of Lemma S2-1, we have that $K_{2,n}^{-1} \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}(K_{2,n}^2/n^2)$ and $K_{2,n}^{-1} \sum_{(i,t)=1}^{m_n} \sum_{(j,s),(k,v)=1,(j,s)\neq(i,t),(k,v)\neq(i,t)}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 = O_{a.s.}(K_{2,n}/n)$. from which it follows that $n\mathcal{E}_1 = O_{a.s.}(1)$ in light of Assumption 5(ii). Hence, by applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we obtain, for all n sufficiently large,

$$\begin{aligned} & nE_Z[\mathcal{E}_1] \\ &= \frac{8n(\mu_n^{\min})^4}{K_{2,n}^2} E_Z \left\{ E \left[\sum_{(i,t)=2}^{m_n} \left(\sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}_n^Z \right] \right\} \\ &\leq \frac{Cn}{K_{2,n}} \left\{ E_Z \left[\frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t)\neq(j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s)\neq(i,t),(k,v)\neq(i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \right\} \\ &= O(1), \end{aligned}$$

which shows that $E_Z[\mathcal{E}_1] = O(1/n) = o(1)$. In a similar way, we can also show that

$$E_Z[\mathcal{E}_2] = 8 \left[(\mu_n^{\min})^4 / K_{2,n}^2 \right] E \left[\sum_{(i,t)=2}^{m_n} \left(\sum_{(j,s)=1}^{(i,t)-1} A_{(j,s),(i,t)} \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right)^4 \right] = o(1). \quad \text{Condition (22) of Lemma S2-15 then follows from these calculations since}$$

$$\sum_{(i,t)=2}^{m_n} E \left[\left(\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} [\underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)}] \right)^4 \right] \leq E_Z[\mathcal{E}_1] + E_Z[\mathcal{E}_2] = o(1)$$

Next, we verify condition (21) of Lemma S2-15. Note first that, by construction, $Var(\mathcal{J}_n | \mathcal{F}_n^Z) = a' \left(\tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \right)^{-1/2} \tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \left(\tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \right)^{-1/2} a = 1$, with $\Lambda_{II,n} = \left[(\mu_n^{\min})^2 / K_{2,n} \right] H_n^{-1} \Sigma_{2,n} H_n^{-1}$. This, in turn, implies that $Var(\mathcal{J}_n) = E_Z[E(\mathcal{J}_n^2 | \mathcal{F}_n^Z)] = E_Z[Var(\mathcal{J}_n | \mathcal{F}_n^Z)] = 1$. On the other hand, by direct calculation, we obtain

$$\begin{aligned} 1 &= Var(\mathcal{J}_n | \mathcal{F}_n^Z) \\ &= \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z] E[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z] \\ &\quad + \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E[\underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z] E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z] \end{aligned}$$

$$+2\frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[\underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] E \left[\underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \quad (13)$$

Making use of expression (13), we obtain, after some further calculations,

$$\begin{aligned} & \sum_{(i,t)=2}^{m_n} E \left[\mathcal{J}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - 1 \\ = & \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(j,s)}^2 - E \left[\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{(j,s),n}^2 - E \left[\underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] \right) E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{(j,s),n} \varepsilon_{(j,s)} - E \left[\underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} E \left[\underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \left\{ \underline{u}_{(j,s),n} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{(k,v),n} \right\} \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \underline{u}_{(j,s),n} \underline{u}_{(k,v),n} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ = & \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6 \end{aligned} \quad (14)$$

To analyze the terms $\mathcal{T}\mathcal{T}_k$ ($k = 1, \dots, 6$), note first that, by applying parts (b) and (a) of Lemma S2-16, we obtain $\mathcal{T}\mathcal{T}_1 \xrightarrow{p} 0$ and $\mathcal{T}\mathcal{T}_2 \xrightarrow{p} 0$, respectively. Consider now the term

$$\mathcal{T}\mathcal{T}_3 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{(j,s),n} \varepsilon_{(j,s)} - E \left[\underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[\underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$$

In this case, we apply part (a) of Lemma S2-8 with $u_{(j,s),n} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$, $\bar{\psi}_{(j,s)} = E \left[(\mu_n^{\min}) \underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right]$, and $\phi_{(i,t)} = E \left[(\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$. Note that, in this case, $\{(u_{(i,t),n}, \varepsilon_{(i,t)})\}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z = \sigma(Z)$, and

$\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s. by Assumptions 1(i) and 2(i), respectively. Moreover, the upper bound given by (12) implies that there exists a constant $C^* > 0$ such that

$\max_{1 \leq (i,t) \leq m_n} E \left[u_{(i,t),n}^4 | \mathcal{F}_n^Z \right] = \max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq (\mu_n^{\min})^4 C^* / (\mu_n^{\min})^4 = C^*$ a.s.n. Finally, note that, by using the fact that

$\underline{u}_{(i,t),n} = a' \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$ and by applying Assumption

2(i), Lemma S2-6, and the assumption that

$\lambda_{\min} \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) \geq \underline{C} > 0$ a.s.n.; we can show that there exists a constant $C > 0$ such that

$$\begin{aligned}
& E \left[\left| (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right| \middle| \mathcal{F}_n^Z \right] \\
&= (\mu_n^{\min}) E \left[\left| \varepsilon_{(i,t)} \underline{U}'_{(i,t)} D_\mu^{-1} H_n^{-1} \tilde{L}'_n \left(\frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} a \right| \middle| \mathcal{F}_n^Z \right] \\
&\leq (\mu_n^{\min}) \sqrt{E \left[\varepsilon_{(i,t)}^2 \middle| \mathcal{F}_n^Z \right]} \left[a' \left(\frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \right. \\
&\quad \times E \left[\underline{U}_{(i,t)} \underline{U}'_{(i,t)} \middle| \mathcal{F}_n^Z \right] D_\mu^{-1} H_n^{-1} \tilde{L}'_n \left(\frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} a \left. \right]^{1/2} \text{ (by CS inequality)} \\
&\leq (\mu_n^{\min}) \sqrt{E \left[\varepsilon_{(i,t)}^2 \middle| \mathcal{F}_n^Z \right]} \frac{1}{(\mu_n^{\min})} \left(\sqrt{\max_{1 \leq (i,t) \leq m_n} E \left[\left\| \underline{U}_{(i,t)} \right\|_2^2 \middle| \mathcal{F}_n^Z \right]} \right) \\
&\quad \times \frac{1}{\lambda_{\min} (\Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n)} \left\| \tilde{L}_n \right\|_F \left(\frac{1}{\sqrt{\lambda_{\min} ((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n})}} \right) \\
&\leq C < \infty \text{ a.s. for all } (i, t) \in \{1, 2, \dots, m_n\} \text{ and for all } n \text{ sufficiently large} \tag{15}
\end{aligned}$$

from which we further deduce that $\max_{(i,t)} |\phi_{(i,t)}| \leq \max_{(i,t)} E \left[\left| (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right| \middle| \mathcal{F}_n^Z \right] \leq C$ a.s.n. and also that $\max_{(j,s)} |\psi_{(j,s)}| \leq \max_{(j,s)} E \left[\left| (\mu_n^{\min}) \underline{u}_{(j,s),n} \varepsilon_{(j,s)} \right| \middle| \mathcal{F}_n^Z \right] \leq C$ a.s.n. Hence, applying part (a) of Lemma S2-8, we have $\mathcal{T}\mathcal{T}_3 \xrightarrow{p} 0$.

Next, consider the term

$$\mathcal{T}\mathcal{T}_4 = \frac{2 (\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} E \left[\underline{u}_{(i,t),n} \varepsilon_{(i,t)} \middle| \mathcal{F}_n^Z \right] \left\{ \underline{u}_{(j,s),n} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{(k,v),n} \right\}$$

Here, we apply part (b) of Lemma S2-8 with $u_{(j,s),n} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$ and

$\phi_{(i,t)} = E \left[(\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \middle| \mathcal{F}_n^Z \right]$. Note that $\{ (u_{(i,t),n}, \varepsilon_{(i,t)}) \}_{(i,t)=1}^{m_n}$ is independent conditional on $\mathcal{F}_n^Z = \sigma(Z)$, and

$\max_{1 \leq (i,t) \leq m_n} E \left[\varepsilon_{(i,t)}^4 \middle| \mathcal{F}_n^Z \right] \leq C$ a.s. by Assumptions 1 and 2(i), respectively. Moreover, from

calculations given previously, we have $\max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[\underline{u}_{(i,t),n}^4 \middle| \mathcal{F}_n^Z \right] \leq C$ a.s.n. and

$\max_{(i,t)} |\phi_{(i,t)}| \leq C$ a.s.n. Hence, by applying part (b) of Lemma S2-8, we deduce that $\mathcal{T}\mathcal{T}_4 \xrightarrow{p} 0$.

Turning our attention to the term

$$\mathcal{TT}_5 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right]$$

For this term, we apply part (c) of Lemma S2-8 with $\phi_{(i,t)} = E \left[u_{(i,t),n}^2 | \mathcal{F}_n^Z \right]$ and $u_{(i,t),n} = (\mu_n^{\min}) \underline{u}_{(i,t),n}$. From (9), there exists a positive constant C such that $E \left[u_{(i,t),n}^2 | \mathcal{F}_n^Z \right] = (\mu_n^{\min})^2 E \left[\underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq C < \infty$ a.s. for all $(i,t) \in \{1, 2, \dots, m_n\}$ and for all n sufficiently large, so that $\max_{(i,t)} |\phi_{(i,t)}| = \max_{1 \leq (i,t) \leq m_n} E \left[u_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq C$ a.s.n. Hence, applying part (c) of Lemma S2-8, we obtain $\mathcal{TT}_5 \xrightarrow{p} 0$.

Finally, consider the term

$$\mathcal{TT}_6 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \underline{u}_{(j,s),n} \underline{u}_{(k,v),n} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$$

In this case, we apply part (d) of Lemma S2-8 with $u_{(j,s)} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$, $u_{(k,v)} = (\mu_n^{\min}) \underline{u}_{(k,v),n}$, and $\phi_{(i,t)} = E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$. Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain $E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq \left(E \left[\varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right)^{1/2} \leq C < \infty$ a.s. for all $(i,t) \in \{1, 2, \dots, m_n\}$ and for all n sufficiently large, so that $\max_{(i,t)} |\phi_{(i,t)}| = \max_{(i,t)} E \left[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C$ a.s.n. Moreover, the upper bound in (12) implies that $\max_{1 \leq (i,t) \leq m_n} E \left[u_{(i,t),n}^4 | \mathcal{F}_n^Z \right] = \max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[\underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq C$ a.s.n. It follows by applying part (d) of Lemma S2-8 that $\mathcal{TT}_6 \xrightarrow{p} 0$.

It follows from the above calculations that the terms $\mathcal{TT}_k \xrightarrow{p} 0$ for each $k \in \{1, \dots, 6\}$, which in light of equation (14) implies that $\sum_{(i,t)=2}^{m_n} E \left[\mathcal{J}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - 1 = o_p(1)$. This establishes condition (21) of Lemma S2-15. It now follows from Lemma S2-15 that \mathcal{J}_n

$= (\mu_n^{\min} / \sqrt{K_{2,n}}) a' \left((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0, 1)$. Since this result holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, applying the Cramér-Wold device, we further deduce that

$$\left(\mu_n^{\min} / \sqrt{K_{2,n}} \right) \left(\tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0, I_d), \quad (16)$$

where $\Lambda_{II,n} = (\mu_n^{\min})^2 H_n^{-1} \Sigma_{2,n} H_n^{-1} / K_{2,n}$ with $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n$. Next, recall that $\widehat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1, Q)} (y - X\delta) / 2] [\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta]$; and note that, by Lemma S2-10, we have $-D_\mu^{-1} (\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta') D_\mu^{-1} = H_n + o_p(1)$, with H_n being positive definite in light of Assumption 3(iii), so that upon inverting the expansion given in expression (5) above and multiplying by $(\mu_n^{\min}) / \sqrt{K_{2,n}}$, we obtain

$$\left(\mu_n^{\min} / \sqrt{K_{2,n}} \right) D_\mu (\widehat{\delta}_{L,n} - \delta_0) = \left(\mu_n^{\min} / \sqrt{K_{2,n}} \right) H_n^{-1} D_\mu^{-1} \widehat{\Delta}(\delta_0) [1 + o_p(1)]$$

$$= \left(\mu_n^{\min} / \sqrt{K_{2,n}} \right) H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon [1 + o_p(1)],$$

where the last equality comes from applying expression (8). It follows by multiplying both sides of the equation above by $(\tilde{L}_n \Lambda_{II,n} \tilde{L}_n)^{-1/2} \tilde{L}_n$ and applying the result given in expression (16) that $(\mu_n^{\min} / \sqrt{K_{2,n}}) (\tilde{L}_n \Lambda_{II,n} \tilde{L}_n)^{-1/2} \tilde{L}_n D_\mu (\hat{\delta}_{L,n} - \delta_0) \xrightarrow{d} N(0, I_d)$.

Turning our attention now to $\hat{\delta}_{F,n}$, note that, using expression (7) above, we can write

$$\begin{aligned} & \frac{(\mu_n^{\min}) D_\mu (\hat{\delta}_{F,n} - \delta_0)}{\sqrt{K_{2,n}}} \\ &= \frac{(\mu_n^{\min}) \left(D_\mu^{-1} X' [A - \hat{\ell}_{F,n} M^{(Z_1,Q)}] X D_\mu^{-1} \right)^{-1} D_\mu^{-1} X' [A - \hat{\ell}_{F,n} M^{(Z_1,Q)}] (y - X \delta_0)}{\sqrt{K_{2,n}}} \end{aligned}$$

It follows by applying Lemmas S2-12 and S2-13 that

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu (\hat{\delta}_{F,n} - \delta_0) = \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1), \quad (17)$$

noting that, in this case, $(\mu_n^{\min}) / \sqrt{K_{2,n}} = o(1)$ but $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$. It follows by multiplying both sides of equation (17) above by $(\tilde{L}_n \Lambda_{II,n} \tilde{L}_n)^{-1/2} \tilde{L}_n$ and applying the result given in expression (16) that

$$(\mu_n^{\min} / \sqrt{K_{2,n}}) (\tilde{L}_n \Lambda_{II,n} \tilde{L}_n)^{-1/2} \tilde{L}_n D_\mu (\hat{\delta}_{F,n} - \delta_0) \xrightarrow{d} N(0, I_d). \quad \square$$

Appendix S2: Key Lemmas Used in Proving the Main Theorems

In this appendix, we state a number of lemmas that are used in the proofs of the main theorems of the paper. Proofs for these lemmas are available in a separate online appendix which can be viewed at the URL: http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf

Lemma S2-1: Let $A = P^\perp - M^{(Z,Q)} D_{\hat{\vartheta}} M^{(Z,Q)}$. Then, under Assumptions 2-6, the following statements hold as $K_{2,n}, n \rightarrow \infty$.

- (a) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 = O_{a.s.}(K_{2,n})$.
- (b) $\sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}(K_{2,n}^3/n^2)$.
- (c) $\sum_{(j,s)=1}^{m_n} \sum_{(i,t),(k,v)=1,(i,t)\neq(j,s),(k,v)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 A_{(j,s),(k,v)}^2 = O_{a.s.}(K_{2,n}^2/n)$.
- (d) $\max_{1 \leq (i,t) \leq m_n} \left(\sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 \right) = O_{a.s.}(K_{2,n}/n)$.
- (e) $\sum_{i_1,i_2=1}^n \sum_{j=1,j\neq i_1,i_2}^n \sum_{t_1=1}^{T_{i_1}} \sum_{t_2=1}^{T_{i_2}} \sum_{s_1,s_2=1}^{T_j} A_{(i_1,t_1),(j,s_1)}^2 A_{(i_2,t_2),(j,s_2)}^2 = O_{a.s.}(K_{2,n}^2/n)$ and $\sum_{i=1}^n \sum_{j_1=1,j_1\neq i}^n \sum_{j_2=1,j_2\neq i}^n \sum_{t_1,t_2=1}^{T_i} \sum_{s_1=1}^{T_{j_1}} \sum_{s_2=1}^{T_{j_2}} A_{(i,t_1),(j_1,s_1)}^2 A_{(i,t_2),(j_2,s_2)}^2 = O_{a.s.}(K_{2,n}^2/n)$

$$(f) \sum_{i=1}^n \sum_{t=1}^{T_i} \sum_{s=1}^{T_i} A_{(i,t),(i,s)}^2 = O_{a.s.}(K_{2,n}^2/n).$$

$$(g) \sum_{i=1}^n \sum_{s_1,t_1=1,s_1 \neq t_1}^{T_i} \sum_{s_2,t_2=1,s_2 \neq t_2}^{T_i} A_{(i,t_1),(i,s_1)}^2 A_{(i,t_2),(i,s_2)}^2 = O_{a.s.}(K_{2,n}^4/n^3).$$

Lemma S2-2: Let Assumptions 1-6 be satisfied. Then, the following statements are true:

- (a) $D_\mu^{-1} X' M^{(Z_1, Q)} X D_\mu^{-1} = O_p(n(\mu_n^{\min})^{-2})$; (b) $D_\mu^{-1} X' A X D_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n = O_p(1)$.

Lemma S2-3: Let $\underline{U} = U - \varepsilon\rho'$ and $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$ and let $VC(X|\mathcal{F}_n^Z)$ denote the conditional covariance matrix of the random vector X given \mathcal{F}_n^Z . Under Assumptions 1-2, 5-6, and 8; there exists positive constants $0 < \underline{C} \leq \bar{C} < \infty$ such that the following statements are true.

- (a) $\lambda_{\max}[VC(\Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)] \leq \bar{C}$ a.s. and $\lambda_{\min}[VC(\Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)] \geq \underline{C}$ a.s. for all n sufficiently large.

$$(b) VC(\underline{U}' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z) \geq \underline{C} I_d > \frac{0}{d \times d} \text{ a.s., for all } n \text{ sufficiently large.}$$

$$(c) \lambda_{\max}(VC[\underline{U}' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z]) \leq \bar{C} \text{ a.s., } \lambda_{\max}(VC[\underline{U}' A \varepsilon / \sqrt{K_{2,n}}]) \leq \bar{C},$$

$\lambda_{\max}(VC[U' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z]) \leq \bar{C}$ a.s., and $\lambda_{\max}(VC[U' A \varepsilon / \sqrt{K_{2,n}}]) \leq \bar{C}$, for all n sufficiently large.

- (d) For any $a \in \mathbb{R}^d$ with $\|a\|_2 = 1$ and for all n sufficiently large, $\lambda_{\min}(\Sigma_n) \geq \underline{C} > 0$ a.s. and $a' \Sigma_n^{-1} a \leq \bar{C} < \infty$ a.s., where $\Sigma_n = VC(\mathcal{Y}_n | \mathcal{F}_n^Z) = \Sigma_{1,n} + \Sigma_{2,n}$, as defined in section 4 of the main paper, and where $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$.

Lemma S2-4: Under Assumptions 1-6, $D_\mu^{-1} X' A \varepsilon = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon = O_p(\max\{1, \sqrt{K_{2,n}} / (\mu_n^{\min})\})$

Lemma S2-5: Under Assumptions 1-6, $D_\mu^{-1} X' M^{(Z_1, Q)} \varepsilon = O_p(n/\mu_n^{\min})$.

Lemma S2-6: If Assumptions 2 and 8 are satisfied; then, for $1 \leq p \leq 8$ and for all n , there exists a positive constant C such that $\max_{1 \leq (i,t) \leq m_n} E[\|\underline{U}_{(i,t)}\|_2^p | \mathcal{F}_n^Z] \leq C < \infty$ a.s., where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho\varepsilon_{(i,t)}$.

Lemma S2-7: Under Assumptions 1-6, the following results hold: (a) $\hat{\ell}_{L,n} = o_p([\mu_n^{\min}]^2 / n)$; (b) $\hat{\ell}_{F,n} = o_p([\mu_n^{\min}]^2 / n)$.

Lemma S2-8: Let A be as defined above. Assume that i) $(u_{(1,1),n}, \varepsilon_{(1,1)})$, ..., $(u_{(1,T_1),n}, \varepsilon_{(1,T_1)})$, $(u_{(2,1),n}, \varepsilon_{(2,1),n})$, ..., $(u_{(2,T_2),n}, \varepsilon_{(2,T_2),n})$, ..., $(u_{(n,1),n}, \varepsilon_{(n,1),n})$, ..., $(u_{(n,T_n),n}, \varepsilon_{(n,T_n),n})$ are independent conditional on $\mathcal{F}_n^Z = \sigma(Z)$; ii) there exists a constant C such that, almost surely for all n sufficiently large, $\max_{1 \leq (i,t) \leq m_n} E(u_{(i,t),n}^4 | \mathcal{F}_n^Z) \leq C$, $\max_{1 \leq (i,t) \leq m_n} E(\varepsilon_{(i,t),n}^4 | \mathcal{F}_n^Z) \leq C$, and

$\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t),n}| \leq C$. In addition, define $\bar{\psi}_{(j,s),n} = E[u_{(j,s),n} \varepsilon_{(j,s),n} | \mathcal{F}_n^Z]$ for $(j, s) = 1, \dots, m_n$.

Then, under Assumptions 5 and 6, the following statements are true:

- (a) $K_{2,n}^{-1} \sum_{1 \leq (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)}^2 \phi_{(i,t),n} \{u_{(j,s),n} \varepsilon_{(j,s),n} - \bar{\psi}_{(j,s),n}\} \xrightarrow{p} 0$;
- (b) $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \{u_{(j,s),n} \varepsilon_{(k,v),n} + \varepsilon_{(j,s),n} u_{(k,v),n}\} \xrightarrow{p} 0$;
- (c) $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \varepsilon_{(j,s),n} \varepsilon_{(k,v),n} \xrightarrow{p} 0$;
- (d) $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} u_{(j,s),n} u_{(k,v),n} \xrightarrow{p} 0$.

Lemma S2-9: Let

$$\widehat{\Delta}(\delta_0) = -\frac{(y - X\delta_0)' M^{(Z_1, Q)} (y - X\delta_0)}{2} \frac{\partial}{\partial \delta} \left\{ \frac{(y - X\delta)' A (y - X\delta)}{(y - X\delta)' M^{(Z_1, Q)} (y - X\delta)} \right\} \Big|_{\delta=\delta_0}.$$

If Assumptions 1-6 and 8 are satisfied; then, $D_\mu^{-1} \widehat{\Delta}(\delta_0) = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1)$, where $\underline{U} = U - \varepsilon \rho'$ and where $\rho = \lim_{n \rightarrow \infty} E[U' M^Q \varepsilon] / E[\varepsilon' M^Q \varepsilon]$.

Lemma S2-10: Let Assumptions 1-6 be satisfied, and let $\bar{\delta}_n$ be any estimator such that, as $n \rightarrow \infty$, $D_\mu(\bar{\delta}_n - \delta_0) / \mu_n^{\min} = o_p(1)$. Then, $-D_\mu^{-1} (\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta') D_\mu^{-1} = H_n + o_p(1)$, where $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n$ and where

$$\begin{aligned} \widehat{\Delta}(\delta) &= -[(y - X\delta)' M^{(Z_1, Q)} (y - X\delta) / 2] [\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta] \\ &= X' A (y - X\delta) - \widehat{\ell}(\delta) X' M^{(Z_1, Q)} (y - X\delta), \text{ with} \\ \widehat{\ell}(\delta) &= (y - X\delta)' A (y - X\delta) / [(y - X\delta)' M^{(Z_1, Q)} (y - X\delta)]. \end{aligned}$$

In addition, we also have

$$D_\mu^{-1} X' [A - \widehat{\ell}(\bar{\delta}_n) M^{(Z_1, Q)}] X D_\mu^{-1} = H_n + o_p(1). \quad (18)$$

Lemma S2-11: Let $\widehat{\ell}_L = Q(\tilde{\beta}) = \min_{\beta \in \overline{B}} Q(\beta)$, where $Q(\beta)$ is as defined in Assumption 9. Then, $\widehat{\ell}_L$ is also the smallest root of the determinantal equation $\det[\overline{X}' A \overline{X} - \ell \overline{X}' M^{(Z_1, Q)} \overline{X}] = 0$, where $\overline{X} = [y, X]$. Assume in addition that condition (13) in Assumption 9 is satisfied; then, $\widehat{\ell}_L$ has the representation

$$\widehat{\ell}_L = \frac{(y - X\widehat{\delta}_L)' A (y - X\widehat{\delta}_L)}{(y - X\widehat{\delta}_L)' M^{(Z_1, Q)} (y - X\widehat{\delta}_L)}, \quad (19)$$

where $\widehat{\delta}_L$ denotes the FELIM estimator. Moreover, $\overline{X}' A (y - X\widehat{\delta}_L) - \widehat{\ell}_L \overline{X}' M^{(Z_1, Q)} (y - X\widehat{\delta}_L) = 0$. In particular, this implies that $\widehat{\Delta}(\widehat{\delta}_L) = 0$, where

$\widehat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1, Q)} (y - X\delta) / 2] (\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta)$, so that $\widehat{\delta}_L$ satisfies the set of (normalized) first-order conditions for minimizing the variance ratio objective function $\widehat{Q}_{FELIM}(\delta) = (y - X\delta)' A (y - X\delta) / [(y - X\delta)' M^{(Z_1, Q)} (y - X\delta)]$.

Lemma S2-12: If Assumptions 1-6 are satisfied; then,

$$D_\mu^{-1} X' [A - \widehat{\ell}_{F,n} M^{(Z_1, Q)}] X D_\mu^{-1} = H_n + o_p(1), \text{ where } H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n,$$

$\widehat{\ell}_{F,n} = [\widehat{\ell}_{L,n} - (1 - \widehat{\ell}_{L,n})(C/m_n)] / [1 - (1 - \widehat{\ell}_{L,n})(C/m_n)]$, and $\widehat{\ell}_{L,n}$ is smallest root of the determinantal equation $\det\{\overline{X}' A \overline{X} - \ell \overline{X}' M^{(Z_1, Q)} \overline{X}\} = 0$, with $\overline{X} = [y \ X]$.

Lemma S2-13: If Assumptions 1-6 and 8-9 are satisfied; then, $D_\mu^{-1} X' [A - \widehat{\ell}_{F,n} M^{(Z_1, Q)}] (y - X\delta_0) = \mathcal{Y}_n [1 + o_p(1)]$, where $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$ with $\underline{U} = U - \varepsilon \rho'$ and $\rho = \lim_{n \rightarrow \infty} E[U' M^Q \varepsilon] / E[\varepsilon' M^Q \varepsilon]$.

Lemma S2-14: For any $a \in \mathbb{R}^d$ such that $\|a\| = 1$, define $b_{1n} = \Sigma_n^{-1/2} a$, $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1} \Sigma_n^{-1/2} a$, $\underline{u}_{2,(i,t),n} = b'_{2n} \underline{U}_{(i,t)}$

$= \sqrt{K_{2,n}} a' \Sigma_n^{-1/2} D_\mu^{-1} \underline{U}_{(i,t)}, \sigma_{(i,t),n}^2 = E [\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z], \tilde{\psi}_{(i,t),n} = E [\underline{u}_{2,(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z],$ and $\tilde{\omega}_{(i,t)}^2 = E [\underline{u}_{2,(i,t),n}^2 | \mathcal{F}_n^Z]$. If Assumptions 1-2 and 5-6 are satisfied; then, the following statements are true.

(a) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} [b_{1n}' \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} / \sqrt{n}] (A_{(i,t),(j,s)} / \sqrt{K_{2,n}}) \left\{ \varepsilon_{(j,s)} \tilde{\psi}_{(i,t),n} + \underline{u}_{2,(j,s)} \sigma_{(i,t),n}^2 \right\}$
 $= O_p (K_{2,n}^{1/4} / \mu_n^{\min}) = o_p (1).$

(b) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} (A_{(i,t),(j,s)}^2 / K_{2,n}) (\varepsilon_{(j,s)}^2 - \sigma_{(j,s),n}^2) \tilde{\omega}_{(i,t),n}^2 = O_p (K_{2,n} (\mu_n^{\min})^{-2} n^{-1/2}) = o_p (1).$

(c) $\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} (A_{(i,t),(j,s)}^2 / K_{2,n}) (\underline{u}_{2,(j,s),n}^2 - \tilde{\omega}_{(j,s),n}^2) \sigma_{(i,t),n}^2 = O_p (K_{2,n} (\mu_n^{\min})^{-2} n^{-1/2}) = o_p (1).$

Lemma S2-15 (Gänsler and Stute, 1977): Let $\{X_{i,n}, \mathcal{F}_{i,n}, 1 \leq i \leq k_n, n \geq 1\}$ be a square integrable martingale difference array. Suppose that for all $\epsilon > 0$

$$\sum_{i=1}^{k_n} E [X_{i,n}^2 \mathbb{I}\{|X_{i,n}| > \epsilon\} | \mathcal{F}_{i-1,n}] \xrightarrow{p} 0 \text{ and} \quad (20)$$

$$\sum_{i=1}^{k_n} E [X_{i,n}^2 | \mathcal{F}_{i-1,n}] \xrightarrow{p} 1. \quad (21)$$

Then, $\sum_{i=1}^{k_n} X_{i,n} \xrightarrow{d} N(0, 1)$.

Remark: Note that a sufficient condition for condition (20), which we will verify in lieu of (20) in the proof of Theorems 2 and 3 in Appendix S1, is the following

$$\sum_{i=1}^{k_n} E [|X_{i,n}|^{2+\delta}] \xrightarrow{p} 0, \text{ for some } \delta > 0. \quad (22)$$

Lemma S2-16: Let \tilde{L}_n be a sequence of $l \times d$ nonrandom matrices (with $l \leq d$) such that $\|\tilde{L}_n\|_F^2 \leq \bar{C} < \infty$ for some constant \bar{C} , and let $\Sigma_{2,n} = VC(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z)$

$= D_\mu^{-1} VC(\underline{U}' A \varepsilon | \mathcal{F}_n^Z) D_\mu^{-1}$. Assume that there exists a positive constant \underline{C} such that

$\lambda_{\min} ((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n}) \geq \underline{C} > 0$ a.s.n. Furthermore, let $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$

and let $\underline{u}_{a,(i,t),n} = a' ((\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n})^{-1/2} \tilde{L}_n D_\mu^{-1} \underline{U}_{(i,t)}$. Let Assumptions 1-2 and 5-6 be satisfied and assume that $(\mu_n^{\min})^2 / K_{2,n} = o(1)$ but

$\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$. Under these conditions, the following statements are true:

- (a) $\left[(\mu_n^{\min})^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\underline{u}_{a,(j,s),n}^2 - E [\underline{u}_{a,(j,s),n}^2 | \mathcal{F}_n^Z] \right) E [\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z]$
 $= O_p (n^{-1/2}) = o_p (1);$
- (b) $\left[(\mu_n^{\min})^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left(\varepsilon_{(j,s)}^2 - E [\varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z] \right) E [\underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^Z]$
 $= O_p (n^{-1/2}) = o_p (1).$

Lemma S2-17 Under Assumptions 1-6, $D_\mu^{-1} X' A D(\varepsilon \circ \varepsilon) A X D_\mu^{-1} = \Sigma_{1,n}$
 $+ \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1} + o_p(1)$, where $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} Z_2 \Upsilon / n$,
 $\sigma_{(i,t)}^2 = E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z]$, $D_{\sigma^2} = \text{diag}(\sigma_{(1,1)}^2, \dots, \sigma_{(n,T_n)}^2)$, and $\Psi_{(j,s)} = E[U_{(j,s)} U'_{(j,s)} | \mathcal{F}_n^Z]$.

Lemma S2-18 Let Assumptions 1-6 and 8 be satisfied, and let $\{\widehat{\delta}_n\}$ be any sequence of estimators such that $\|\widehat{\delta}_n - \delta_0\|_2 \xrightarrow{p} 0$ as $n \rightarrow \infty$, as long as $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$. Also, define the following notations: let $\widehat{\varepsilon} = M^{(Z,Q)}(y - X\widehat{\delta}_n)$, $J = [M^Q \circ M^Q]^{-1}$, $S_1 = X' A D(J[\widehat{\varepsilon} \circ \widehat{\varepsilon}]) A X$, $S_2 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J(A \circ A) J(\widehat{\varepsilon} \circ \widehat{\varepsilon})$, $S_3 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J(A \circ A) J(\widehat{\varepsilon} \circ \widehat{\varepsilon})$, $S_4 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J(A \circ A) J(\widehat{\varepsilon} \circ \widehat{\varepsilon})$, $\underline{S}_4 = (\widehat{\varepsilon} \circ \widehat{\varepsilon})' J(A \circ A) J(\widehat{\varepsilon} \circ \widehat{\varepsilon})$, and $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1,Q)} D_{\sigma^2} M^{(Z_1,Q)} Z_2 \Upsilon / n$. In addition, define $\sigma_{(i,t)}^2 = E[\varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z]$, $D_{\sigma^2} = \text{diag}(\sigma_{(1,1)}^2, \dots, \sigma_{(n,T_n)}^2)$, $\phi_{(i,t)} = E[U_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z]$, $\Psi_{(i,t)} = E[U_{(i,t)} U'_{(i,t)} | \mathcal{F}_n^Z]$, $\underline{\phi}_{(i,t)} = E[\underline{U}_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z]$, and $\underline{\Psi}_{(i,t)} = E[\underline{U}_{(i,t)} \underline{U}'_{(i,t)} | \mathcal{F}_n^Z]$ where $\underline{U}_{(i,t)} = U_{(i,t)} - \rho \varepsilon_{(i,t)}$ and where for notational convenience we suppress the dependence of $\sigma_{(i,t)}^2$, $\phi_{(i,t)}$, $\Psi_{(i,t)}$, $\underline{\phi}_{(i,t)}$, and $\underline{\Psi}_{(i,t)}$ on $\mathcal{F}_n^Z = \sigma(Z)$. Then, under the above conditions, the following statements are true.

- (a) $D_\mu^{-1} S_1 D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \sigma_{(j,s)}^2 D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1}$
 $+ o_p(\max\{1, K_{2,n} (\mu_n^{\min})^{-2}\})$.
- (b) $S_3 / K_{2,n} - K_{2,n}^{-1} \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \sigma_{(j,s)}^2 = o_p(1)$.
- (c) $D_\mu^{-1} S_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 D_\mu^{-1} \phi_{(i,t)} \phi'_{(j,s)} D_\mu^{-1} = o_p(K_{2,n} (\mu_n^{\min})^{-2})$.
- (d) $(\mu_n^{\min} / K_{2,n}) S_2 D_\mu^{-1} - (\mu_n^{\min} / K_{2,n}) \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \phi'_{(j,s)} D_\mu^{-1} = o_p(1)$.
- (e) $D_\mu^{-1} \widehat{\rho}_n = O_p((\mu_n^{\min})^{-1})$ and $D_\mu^{-1} (\widehat{\rho}_n - \rho) = o_p((\mu_n^{\min})^{-1})$, where $\rho = \lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} (E[U' M^Q \varepsilon] / n) / (E[\varepsilon' M^Q \varepsilon] / n)$.
- (f) $D_\mu^{-1} \underline{S}_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 D_\mu^{-1} \underline{\phi}_{(i,t)} \underline{\phi}'_{(j,s)} D_\mu^{-1} = o_p(K_{2,n} (\mu_n^{\min})^{-2})$.
- (g) $(\mu_n^{\min} / K_{2,n}) - (\mu_n^{\min} / K_{2,n}) \sum_{(i,t),(j,s)=1,(i,t)\neq(j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \underline{\phi}'_{(j,s)} D_\mu^{-1} = o_p(1)$.

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