

Technical Appendix: Consistent Estimation, Variable Selection, and Forecasting in Factor-Augmented VAR Models*

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Abstract

Proofs to lemmas and theorems used in Consistent Estimation, Variable Selection, and Forecasting in Factor-Augmented VAR Models by Chao and Swanson (2022) are gathered in this paper.

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1 Appendix A: Proofs of the Main Theorems of the Paper

Proof of Theorem 1:

The proof of Theorem 1 requires a long series of calculations. Hence, we have divided this proof into six different steps.

Step 1:

In step 1, we shall transform the simple factor model

$$\begin{matrix} Z_t &= & \gamma & f_t & + & u_t, & t = 1, \dots, T \\ N \times 1 & & N \times 1 \times 1 & & & N \times 1 \end{matrix} \quad (1)$$

into a more convenient form. Let Π denote an $N \times N$ orthogonal matrix whose columns are the eigenvectors of the covariance matrix $\Sigma_Z = E[Z_t Z_t']$. Without loss of generality, we can partition Π as

$$\Pi_{N \times N} = \begin{bmatrix} \pi_1 & \Pi_2 \\ N \times 1 & N \times (N-1) \end{bmatrix}$$

where π_1 is the eigenvector associated with the largest eigenvalue of $\Sigma_Z = E[Z_t Z_t']$, i.e., $\lambda_{(1)}(\Sigma_Z)$. By the result of Lemma B-8, we know that

$$\pi_1 = \frac{\gamma}{\|\gamma\|_2} \text{ and } \lambda_{(1)}(\Sigma_Z) = \|\gamma\|_2^2 + 1.$$

Next, we define

$$\begin{aligned} W_t &= \Pi' Z_t \\ &= \Pi' (\gamma f_t + u_t) \\ &= \|\gamma\|_2 \Pi' \frac{\gamma}{\|\gamma\|_2} f_t + \Pi' u_t \\ &= \|\gamma\|_2 f_t \Pi' \pi_1 + \Pi' u_t \quad \left(\text{since } \pi_1 = \frac{\gamma}{\|\gamma\|_2} \right) \\ &= \|\gamma\|_2 f_t \begin{pmatrix} \pi'_1 \\ \Pi'_2 \end{pmatrix} \pi_1 + \Pi' u_t \\ &= \|\gamma\|_2 f_t \mathbf{e}_{1,N} + \eta_t \end{aligned} \quad (2)$$

where $\mathbf{e}_{1,N}$ is an elementary vector whose first component is 1 and all remaining components are

0 and where $\eta_t = \Pi' u_t$. Moreover, note that $\{\eta_t\} \equiv i.i.d.N(0, I_N)$ since Π is an orthogonal matrix and $\eta_t = \Pi' u_t$ with $\{u_t\} \equiv i.i.d.N(0, I_N)$. We can write out the covariance matrix of W_t as

$$\begin{aligned}
& \Sigma_W \\
&= E[W_t W_t'] \\
&= E[(\|\gamma\|_2 f_t \mathbf{e}_{1,N} + \eta_t)(\|\gamma\|_2 f_t \mathbf{e}_{1,N} + \eta_t)'] \\
&= \|\gamma\|_2^2 E[f_t^2] \mathbf{e}_{1,N} \mathbf{e}_{1,N}' + \|\gamma\|_2 E[\eta_t f_t] \mathbf{e}_{1,N}' + \|\gamma\|_2 \mathbf{e}_{1,N} E[f_t \eta_t'] + E[\eta_t \eta_t'] \\
&= \|\gamma\|_2^2 \mathbf{e}_{1,N} \mathbf{e}_{1,N}' + I_N \\
&= \begin{pmatrix} \|\gamma\|_2^2 + 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\end{aligned}$$

from which it is easily seen that $\lambda_{(1)}(\Sigma_W) = \|\gamma\|_2^2 + 1$ and $\lambda_{(2)}(\Sigma_W) = \lambda_{(3)}(\Sigma_W) = \cdots = \lambda_{(N)}(\Sigma_W) = 1$, where we let $\lambda_{(j)}(\Sigma_W)$ denote the j^{th} largest eigenvalue of Σ_W . In addition, the eigenvector associated with $\lambda_{(j)}(\Sigma_W)$ is $\mathbf{e}_{j,N}$, an elementary vector whose j^{th} component is 1 and all other components are 0.

Note further that we can also write W_t in the alternative form

$$\begin{aligned}
W_t &= \begin{pmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{N,t} \end{pmatrix} \\
&= \begin{pmatrix} \|\gamma\|_2 f_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2,t} \\ \vdots \\ \eta_{N,t} \end{pmatrix} \\
&= \begin{pmatrix} \|\gamma\|_2 \zeta_{1,t} \\ \zeta_{2,t} \\ \vdots \\ \zeta_{N,t} \end{pmatrix} \\
&= \sum_{j=1}^N \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N}
\end{aligned} \tag{3}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1} \eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, \dots, N$ and where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, \dots, N$. In fact, this is the representation of W_t that is given in Lemma B-10. (See Appendix B below).

Step 2:

Define $\mathbf{W}_{N \times T} = (W_1, \dots, W_T)$, where W_t is as defined in expression (2) in step 1 above. Partition \mathbf{W} as follows

$$\mathbf{W}_{N \times T} = \begin{bmatrix} \mathbf{W}'_1 \\ \mathbf{W}'_2 \end{bmatrix}_{\substack{1 \times T \\ (N-1) \times T}} = \begin{bmatrix} \pi'_1 \mathbf{Z} \\ \Pi'_2 \mathbf{Z} \end{bmatrix}_{\substack{1 \times T \\ (N-1) \times T}},$$

where $\mathbf{Z}_{N \times T} = (Z_1, \dots, Z_T)$ with Z_t as defined in expression (1). Note that the first row of \mathbf{W} , i.e., \mathbf{W}'_1 , contains the "signal" observations with the elevated variance $\lambda_1 = \|\gamma\|_2^2 + 1$ and where the remaining $N - 1$ rows contain the elements of the $(N - 1) \times T$ matrix \mathbf{W}'_2 which contain only the noise variables. Now, define the sample covariance matrix

$$\hat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \mathbf{W} \mathbf{W}' = \begin{pmatrix} T^{-1} \mathbf{W}'_1 \mathbf{W}_1 & T^{-1} \mathbf{W}'_1 \mathbf{W}_2 \\ T^{-1} \mathbf{W}'_2 \mathbf{W}_1 & T^{-1} \mathbf{W}'_2 \mathbf{W}_2 \end{pmatrix}$$

In this step, we shall further transform $\widehat{\Sigma}_{\mathbf{W}}$ into the so-called arrowhead matrix. To proceed, consider the spectral decomposition

$$\frac{\mathbf{W}_2' \mathbf{W}_2}{T} = \widetilde{\mathbf{B}}_2 \widetilde{\Lambda} \widetilde{\mathbf{B}}_2'$$

where $\widetilde{\Lambda} = \text{diag}(\widetilde{\lambda}_{(2)}, \dots, \widetilde{\lambda}_{(N)})$ with $\widetilde{\lambda}_{(2)}, \dots, \widetilde{\lambda}_{(N)}$ being the $N - 1$ eigenvalues of $\mathbf{W}_2' \mathbf{W}_2 / T$ and $\widetilde{\mathbf{B}}_2$ is an $(N - 1) \times (N - 1)$ orthogonal matrix whose columns are the eigenvectors of $\mathbf{W}_2' \mathbf{W}_2 / T$. Note that, without loss of generality, we can assume that the eigenvalues are ordered so that $\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \dots \geq \widetilde{\lambda}_{(N)}$. Next, create the modified data matrix

$$\widetilde{\mathbf{W}}_{N \times T} = \begin{bmatrix} \mathbf{W}_1' \\ 1 \times T \\ \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \\ (N-1) \times T \end{bmatrix}$$

The sample covariance matrix based on the modified data matrix is then given by

$$\begin{aligned} \widetilde{\Sigma}_{\mathbf{W}}_{N \times N} &= \frac{\widetilde{\mathbf{W}} \widetilde{\mathbf{W}}'}{T} \\ &= \begin{pmatrix} T^{-1} \mathbf{W}_1' \mathbf{W}_1 & T^{-1} \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \\ T^{-1} \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 & T^{-1} \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \end{pmatrix} \\ &= \begin{pmatrix} s & v' \\ v & \widetilde{\Lambda} \end{pmatrix} \\ &= \begin{pmatrix} s & v_2 & v_3 & \dots & v_N \\ v_2 & \widetilde{\lambda}_{(2)} & 0 & \dots & 0 \\ v_3 & 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ v_N & 0 & \dots & 0 & \widetilde{\lambda}_{(N)} \end{pmatrix} \end{aligned}$$

where $s_{1 \times 1} = \mathbf{W}_1' \mathbf{W}_1 / T$ and

$$v_{(N-1) \times 1} = \begin{pmatrix} v_2 \\ \vdots \\ v_N \end{pmatrix} = \frac{\widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1}{T}. \quad (4)$$

Note that the non-zero entries of $\widetilde{\Sigma}_{\mathbf{W}}$ form the shape of an arrow, and so such matrices have been

referred to in the linear algebra literature as an “arrowhead matrix”.

An advantage of this arrowhead form is that it allows us to obtain a useful representation for the top eigenvalue of $\tilde{\Sigma}_{\mathbf{W}}$. This part of step 2 comes from Johnstone and Paul (2018) following an approach originally due to Nadler (2008), but for completeness we provide some details of the argument here. To proceed, let $\hat{\lambda}_{(1)}$ denote the largest eigenvalue of $\tilde{\Sigma}_{\mathbf{W}}$ and let $\tilde{\mathbf{v}}_{(1)}$ be the associated eigenvector, where, following Johnstone and Paul (2018), we will normalize $\tilde{\mathbf{v}}_{(1)}$ to have the form $\tilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 & \tilde{v}_{(1),2} & \cdots & \tilde{v}_{(1),N} \end{pmatrix}'$, i.e., we normalize $\tilde{\mathbf{v}}_{(1)}$ so that its first component is 1. The eigen-equation $\tilde{\Sigma}_{\mathbf{W}}\tilde{\mathbf{v}}_{(1)} = \hat{\lambda}_{(1)}\tilde{\mathbf{v}}_{(1)}$ can then be written out more explicitly as

$$\begin{pmatrix} s & v_2 & v_3 & \cdots & v_N \\ v_2 & \tilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ v_3 & 0 & \tilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ v_N & 0 & \cdots & 0 & \tilde{\lambda}_{(N)} \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{v}_{(1),2} \\ \tilde{v}_{(1),3} \\ \vdots \\ \tilde{v}_{(1),N} \end{pmatrix} = \hat{\lambda}_{(1)} \begin{pmatrix} 1 \\ \tilde{v}_{(1),2} \\ \tilde{v}_{(1),3} \\ \vdots \\ \tilde{v}_{(1),N} \end{pmatrix} \quad (5)$$

Solving this system of equations, we see that

$$\tilde{v}_{(1),j} = \frac{v_j}{\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)}} \text{ for } j = 2, \dots, N; \quad (6)$$

where v_j is the j^{th} component of v as defined in expression (4). Hence,

$$\tilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \tilde{v}_{(1),2} \\ \vdots \\ \tilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 / (\hat{\lambda}_{(1)} - \tilde{\lambda}_{(2)}) \\ \vdots \\ v_N / (\hat{\lambda}_{(1)} - \tilde{\lambda}_{(N)}) \end{pmatrix} \quad (7)$$

Moreover, since expression (5) implies that

$$\hat{\lambda}_{(1)} = s + v_2 \tilde{v}_{(1),2} + \cdots + v_N \tilde{v}_{(1),N}$$

It follows from substituting the right-hand side of equation (6) for $j = 2, \dots, N$ into the above expression that

$$\hat{\lambda}_{(1)} = s + \sum_{j=2}^N \frac{v_j}{\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)}} = \frac{\mathbf{W}'_1 \mathbf{W}_1}{T} + \sum_{j=2}^N \frac{v_j}{\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)}}. \quad (8)$$

Finally, in this step, we shall relate the eigenvalues and eigenvectors of $\tilde{\Sigma}_{\mathbf{W}}$ to that of the

pre-transformed sample covariance matrix of our simple factor model, i.e.,

$$\hat{\Sigma}_Z = \frac{\mathbf{Z}\mathbf{Z}'}{T} = \frac{1}{T} \sum_{t=1}^T Z_t Z_t' \text{ where } \mathbf{Z}_{N \times T} = (Z_1, \dots, Z_T).$$

Understanding this relationship then allows us to derive asymptotic properties of quantities involving the leading eigenvector of $\hat{\Sigma}_Z$ using the explicit representation of $\tilde{\mathbf{v}}_1$ and $\hat{\lambda}_1$ given in expressions (7) and (8), respectively. To proceed, we first relate the eigenvalues and eigenvectors of $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{W}}\tilde{\mathbf{W}}'/T$ to that of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$. Define

$$\tilde{\mathbf{B}}_{N \times N} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix}$$

Now, since $\tilde{\mathbf{B}}_2$ is an orthogonal matrix, it follows that

$$\tilde{\mathbf{B}}'\tilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2'\tilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

and

$$\tilde{\mathbf{B}}\tilde{\mathbf{B}}' = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2\tilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

so that $\tilde{\mathbf{B}}$ is an orthogonal matrix as well. Next, note that

$$\begin{aligned} & \frac{\tilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\tilde{\mathbf{B}}}{T} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2' \end{pmatrix} \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2 \\ T^{-1}\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\mathbf{W}_2'\mathbf{W}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix} \\ &= \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2\tilde{\mathbf{B}}_2 \\ T^{-1}\tilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\tilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\tilde{\mathbf{B}}_2 \end{pmatrix} \\ &= \frac{\tilde{\mathbf{W}}\tilde{\mathbf{W}}'}{T} \\ &= \tilde{\Sigma}_{\mathbf{W}} \end{aligned}$$

Hence, to relate the eigenvalues and eigenvectors of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ to those of $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\tilde{\mathbf{B}}/T$,

we note that the eigenvalues of the $\tilde{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$\begin{aligned}
0 &= \det \left\{ \frac{\tilde{\mathbf{B}}' \mathbf{W} \mathbf{W}' \tilde{\mathbf{B}}}{T} - \lambda I_N \right\} \\
&= \det \left\{ \tilde{\mathbf{B}}' \right\} \det \left\{ \frac{\mathbf{W} \mathbf{W}'}{T} - \lambda \tilde{\mathbf{B}} \tilde{\mathbf{B}}' \right\} \det \left\{ \tilde{\mathbf{B}} \right\} \\
&= \det \left\{ \tilde{\mathbf{B}}' \right\} \det \left\{ \frac{\mathbf{W} \mathbf{W}'}{T} - \lambda I_N \right\} \det \left\{ \tilde{\mathbf{B}} \right\} \quad (\text{since } \tilde{\mathbf{B}} \text{ is an orthogonal matrix}) \\
&= \det \left\{ \frac{\mathbf{W} \mathbf{W}'}{T} - \lambda I_N \right\}
\end{aligned}$$

where the last equality holds because $\det \left\{ \tilde{\mathbf{B}}' \right\} = \det \left\{ \tilde{\mathbf{B}} \right\} = \pm 1$ given that $\tilde{\mathbf{B}}$ is an orthogonal matrix. It follows that $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W} \mathbf{W}' / T$ and $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}' \mathbf{W} \mathbf{W}' \tilde{\mathbf{B}} / T$ have the same set of eigenvalues. Moreover, let $\hat{\lambda}_{(j)}$ be the j^{th} largest eigenvalue of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W} \mathbf{W}' / T$, which is of course also the j^{th} largest eigenvalue of $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}' \mathbf{W} \mathbf{W}' \tilde{\mathbf{B}} / T$. Also, let $\tilde{\mathbf{v}}_{(j)}$ be an eigenvector of $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}' \mathbf{W} \mathbf{W}' \tilde{\mathbf{B}} / T$ associated with $\hat{\lambda}_{(j)}$. Define $\mathbf{v}_{(j)} \equiv \tilde{\mathbf{B}} \tilde{\mathbf{v}}_{(j)}$ for $j = 1, \dots, N$, and note that, since $\tilde{\Sigma}_{\mathbf{W}} \tilde{\mathbf{v}}_{(j)} = \hat{\lambda}_{(j)} \tilde{\mathbf{v}}_{(j)}$, we have

$$\begin{aligned}
\tilde{\mathbf{B}}' \hat{\Sigma}_{\mathbf{W}} \tilde{\mathbf{B}} \tilde{\mathbf{v}}_{(j)} &= \left(\frac{\tilde{\mathbf{B}}' \mathbf{W} \mathbf{W}' \tilde{\mathbf{B}}}{T} \right) \tilde{\mathbf{v}}_{(j)} \\
&= \tilde{\Sigma}_{\mathbf{W}} \tilde{\mathbf{v}}_{(j)} \\
&= \hat{\lambda}_{(j)} \tilde{\mathbf{v}}_{(j)}
\end{aligned}$$

which implies that

$$\hat{\Sigma}_{\mathbf{W}} \mathbf{v}_{(j)} = \hat{\Sigma}_{\mathbf{W}} \tilde{\mathbf{B}} \tilde{\mathbf{v}}_{(j)} = \hat{\lambda}_{(j)} \tilde{\mathbf{B}} \tilde{\mathbf{v}}_{(j)} = \hat{\lambda}_{(j)} \mathbf{v}_{(j)}$$

so that $\mathbf{v}_{(j)} = \tilde{\mathbf{B}} \tilde{\mathbf{v}}_{(j)}$ is an eigenvector of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W} \mathbf{W}' / T$ associated with $\hat{\lambda}_{(j)}$. Note, further that, previously, we have normalized the first element of $\tilde{\mathbf{v}}_{(1)}$ to be 1. This, in turn, implies that the first

element of $\mathbf{v}_{(1)}$ will be 1 as well since

$$\begin{aligned}
\mathbf{v}_{(1)} &= \tilde{\mathbf{B}}\tilde{\mathbf{v}}_{(1)} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \tilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \tilde{\mathbf{B}}_2\tilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \mathbf{v}_{(1)}^{(2)} \end{pmatrix} \tag{9}
\end{aligned}$$

where we let $\tilde{\mathbf{v}}_{(1)}^{(2)} = \begin{pmatrix} \tilde{v}_{(1),2} & \tilde{v}_{(1),3} & \cdots & \tilde{v}_{(1),N} \end{pmatrix}'$ and $\mathbf{v}_{(1)}^{(2)} = \tilde{\mathbf{B}}_2\tilde{\mathbf{v}}_{(1)}^{(2)} = \begin{pmatrix} v_{(1),2} & v_{(1),3} & \cdots & v_{(1),N} \end{pmatrix}'$.

In a similar manner, we can relate the eigenvalues and eigenvectors of $\hat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ to those of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also to those of $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\tilde{\mathbf{B}}/T$. In this case, note that the eigenvalues of the $\hat{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$\begin{aligned}
0 &= \det \left\{ \frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N \right\} \\
&= \det \left\{ \frac{\mathbf{\Pi}'\mathbf{Z}\mathbf{Z}'\mathbf{\Pi}}{T} - \lambda I_N \right\} \quad (\text{since } \mathbf{W} = \mathbf{\Pi}'\mathbf{Z}) \\
&= \det \{ \mathbf{\Pi}' \} \det \left\{ \frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda \mathbf{\Pi}\mathbf{\Pi}' \right\} \det \{ \mathbf{\Pi} \} \\
&= \det \{ \mathbf{\Pi}' \} \det \left\{ \frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N \right\} \det \{ \mathbf{\Pi} \} \\
&\quad (\text{since } \mathbf{\Pi} \text{ is an orthogonal matrix whose columns are the eigenvectors of } \Sigma_Z = E[Z_t Z_t']) \\
&= \det \left\{ \frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N \right\}
\end{aligned}$$

where the last equality holds because $\det \{ \mathbf{\Pi}' \} = \det \{ \mathbf{\Pi} \} = \pm 1$ given that $\mathbf{\Pi}$ is an orthogonal matrix. It follows that $\hat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ has the same set of eigenvalues as $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also the same set of eigenvalues as $\tilde{\Sigma}_{\mathbf{W}} = \tilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\tilde{\mathbf{B}}/T$. Using the same notation as above, we will then also let $\hat{\lambda}_{(j)}$ to denote the j^{th} largest eigenvalue of $\hat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$. Moreover, as before, let \mathbf{v}_j denote an eigenvector of $\hat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ associated with $\hat{\lambda}_{(j)}$. Now, define $\hat{\pi}_{(j)} \equiv \mathbf{\Pi}\mathbf{v}_{(j)}$, and note

that since $\widehat{\Sigma} \mathbf{w} \mathbf{v}_{(j)} = \widehat{\lambda}_{(j)} \mathbf{v}_{(j)}$, we have, for $j = 1, \dots, N$,

$$\begin{aligned} \Pi' \widehat{\Sigma}_Z \Pi \mathbf{v}_{(j)} &= \left(\frac{\Pi' \mathbf{Z} \mathbf{Z}' \Pi}{T} \right) \mathbf{v}_{(j)} \\ &= \widehat{\Sigma} \mathbf{w} \mathbf{v}_{(j)} \\ &= \widehat{\lambda}_{(j)} \mathbf{v}_{(j)} \end{aligned}$$

which implies that

$$\widehat{\Sigma}_Z \widehat{\pi}_{(j)} = \widehat{\Sigma}_Z \Pi \mathbf{v}_{(j)} = \widehat{\lambda}_{(j)} \Pi \mathbf{v}_{(j)} = \widehat{\lambda}_{(j)} \widehat{\pi}_{(j)}$$

so that

$$\widehat{\pi}_{(j)} = \Pi \mathbf{v}_{(j)} \tag{10}$$

is an eigenvector of $\widehat{\Sigma}_Z$ associated with the eigenvalue $\widehat{\lambda}_{(j)}$.

Step 3:

For the simple factor model given in expression (1), i.e.,

$$\begin{aligned} Z_t &= \gamma f_t + u_t \\ &= \|\gamma\|_2 \pi_{(1)} f_t + u_t \text{ for } t = 1, \dots, T; \end{aligned}$$

with $\pi_1 = \gamma / \|\gamma\|_2$; the principal-component estimator of the latent factor f_t can be written as

$$\begin{aligned}
\hat{f}_t &= \frac{1}{\sqrt{N}} \left\langle \frac{\hat{\pi}_{(1)}}{\|\hat{\pi}_{(1)}\|_2}, Z_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\hat{\pi}_{(1)}}{\|\hat{\pi}_{(1)}\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\hat{\pi}_{(1)}}{\|\hat{\pi}_{(1)}\|_2}, u_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\|\Pi \mathbf{v}_{(1)}\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\|\Pi \mathbf{v}_{(1)}\|_2}, u_t \right\rangle \quad (\text{making use of expression (10) in step 2}) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}'_{(1)} \Pi' \pi_1}{\|\Pi \mathbf{v}_{(1)}\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}'_{(1)} \Pi' u_t}{\|\Pi \mathbf{v}_{(1)}\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}'_1 \mathbf{e}_{1,N}}{\|\Pi \mathbf{v}_{(1)}\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}'_{(1)} \Pi' u_t}{\|\Pi \mathbf{v}_{(1)}\|_2} \\
&\quad \left(\text{since } \Pi' \pi_{(1)} = \begin{pmatrix} \pi'_{(1)} \\ \Pi'_{(2)} \end{pmatrix} \pi_{(1)} = \begin{pmatrix} 1 \\ 0 \\ (N-1) \times 1 \end{pmatrix} = \mathbf{e}_{1,N} \text{ given that } \Pi \text{ is an orthogonal matrix} \right) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}'_{(1)} \mathbf{e}_{1,N}}{\|\Pi \mathbf{v}_{(1)}\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}'_{(1)} \eta_t}{\|\Pi \mathbf{v}_{(1)}\|_2} \quad (\text{since, by definition, } \eta_t = \Pi' u_t) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \eta_t \right\rangle
\end{aligned}$$

where the notation $\langle y, x \rangle = y'x$ denotes the dot product of the vectors y and x and where the last equality above follows from the fact that

$$\|\Pi \mathbf{v}_{(1)}\|_2 = \sqrt{\mathbf{v}'_{(1)} \Pi' \Pi \mathbf{v}_{(1)}} = \sqrt{\mathbf{v}'_{(1)} \mathbf{v}_{(1)}} = \|\mathbf{v}_{(1)}\|_2.$$

Next, given expression (9) in step 2, we see that

$$\begin{aligned}
\left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle &= \frac{\mathbf{v}'_{(1)} \tilde{\mathbf{B}} \mathbf{e}_{1,N}}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \quad \left(\text{since } \tilde{\mathbf{v}}_{(1)} = \tilde{\mathbf{B}}' \mathbf{v}_{(1)} \right) \\
&= \frac{1}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \begin{pmatrix} 1 & \mathbf{v}_{(1)}^{(2)'} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \mathbf{0}_{(N-1) \times 1} \end{pmatrix} \\
&= \frac{1}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \begin{pmatrix} 1 & \mathbf{v}_{(1)}^{(2)'} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \mathbf{0}_{(N-1) \times 1} \end{pmatrix} \\
&= \frac{1}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \langle \mathbf{v}_{(1)}, \mathbf{e}_{1,N} \rangle \\
&= \left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle,
\end{aligned}$$

where the last line follows from the fact that

$$\|\tilde{\mathbf{v}}_{(1)}\|_2 = \sqrt{\tilde{\mathbf{v}}'_{(1)} \tilde{\mathbf{v}}_{(1)}} = \sqrt{\mathbf{v}'_{(1)} \tilde{\mathbf{B}} \tilde{\mathbf{B}}' \mathbf{v}_{(1)}} = \sqrt{\mathbf{v}'_{(1)} \mathbf{v}_{(1)}} = \|\mathbf{v}_{(1)}\|_2$$

since $\tilde{\mathbf{B}} \tilde{\mathbf{B}}' = I_N$. In addition, let $\tilde{\eta}_t = \tilde{\mathbf{B}}' \eta_t$, and note that

$$\begin{aligned}
\left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \eta_t \right\rangle &= \frac{1}{\|\mathbf{v}_{(1)}\|_2} \mathbf{v}'_{(1)} \eta_t \\
&= \frac{1}{\|\mathbf{v}_{(1)}\|_2} \mathbf{v}'_{(1)} \tilde{\mathbf{B}} \tilde{\mathbf{B}}' \eta_t \\
&= \frac{1}{\|\mathbf{v}_{(1)}\|_2} \tilde{\mathbf{v}}'_{(1)} \tilde{\eta}_t \\
&= \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle \quad \left(\text{given that } \|\tilde{\mathbf{v}}_{(1)}\|_2 = \|\mathbf{v}_{(1)}\|_2 \right).
\end{aligned}$$

Since

$$\{\eta_t\} \equiv i.i.d.N(0, I_N)$$

and $\tilde{\mathbf{B}}$ is an orthogonal matrix, we also have

$$\{\tilde{\eta}_t\} \equiv i.i.d.N(0, I_N).$$

Using these calculations, we can then rewrite the expression for \hat{f}_t in terms of $\tilde{\mathbf{v}}_{(1)}$ and $\tilde{\eta}_t$ as follows.

$$\begin{aligned}
\hat{f}_t &= \frac{\langle \hat{\pi}_{(1)}, Z_t \rangle}{\sqrt{N} \|\hat{\pi}_{(1)}\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\|\mathbf{v}_{(1)}\|_2}, \eta_t \right\rangle \\
&= \frac{\|\gamma\|_2}{\sqrt{N}} \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle f_t + \frac{1}{\sqrt{N}} \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle.
\end{aligned} \tag{11}$$

Given the requirement in Assumption 2-2 that

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ as } N, T \rightarrow \infty,$$

for constants c and κ such that $0 < c < \infty$ and $0 < \kappa < 1$; it is easily seen that

$$\frac{\|\gamma\|_2}{\sqrt{N}} = O\left(\left(\frac{1}{TN^\kappa}\right)^{\frac{1}{2(1+\kappa)}}\right) = o(1). \tag{12}$$

In the next two steps of this proof, we will show that

$$\left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0 \text{ and } \frac{1}{\sqrt{N}} \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle \xrightarrow{p} 0.$$

Step 4:

We will first show that

$$\left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0.$$

. To proceed, note that, from expression (7) in step 2, $\tilde{\mathbf{v}}_1$ has the explicit form

$$\tilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \tilde{v}_{(1),2} \\ \vdots \\ \tilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 / (\hat{\lambda}_{(1)} - \tilde{\lambda}_{(2)}) \\ \vdots \\ v_N / (\hat{\lambda}_{(1)} - \tilde{\lambda}_{(N)}) \end{pmatrix}$$

It follows that

$$\begin{aligned}
& \frac{\langle \tilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \rangle^2}{\|\tilde{\mathbf{v}}_{(1)}\|^2} \\
&= \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1} \\
& \left(\text{since } \langle \tilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \rangle = \begin{bmatrix} 1 & v_2/(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(2)}) & \cdots & v_N/(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(N)}) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1 \right) \\
&= \frac{1}{1 + \tau^2}
\end{aligned}$$

where

$$\tau^2 = \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2}.$$

Next, write

$$\begin{aligned}
\tau^2 &= \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \\
&= \frac{N \|\gamma\|_2^2}{T} \frac{1}{\|\gamma\|_2^{4(1+\kappa)}} \frac{1}{\hat{\lambda}_{(1)}^2 / \|\gamma\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2 / (N \|\gamma\|_2^2)}{(1 - \tilde{\lambda}_{(j)}/\hat{\lambda}_{(1)})^2} \\
&= \frac{N}{T \|\gamma\|_2^{2(1+2\kappa)}} \frac{1}{\hat{\lambda}_{(1)}^2 / \|\gamma\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2 / (N \|\gamma\|_2^2)}{(1 - \tilde{\lambda}_{(j)}/\hat{\lambda}_{(1)})^2}
\end{aligned}$$

Recall from step 2 that $\hat{\lambda}_{(1)}$ is the largest eigenvalue of the sample covariance matrix

$$\hat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \mathbf{W} \mathbf{W}' = \begin{pmatrix} T^{-1} \mathbf{W}'_1 \mathbf{W}_1 & T^{-1} \mathbf{W}'_1 \mathbf{W}_2 \\ T^{-1} \mathbf{W}'_2 \mathbf{W}_1 & T^{-1} \mathbf{W}'_2 \mathbf{W}_2 \end{pmatrix}$$

while $\tilde{\lambda}_{(j)}$ (for $j = 2, \dots, N$) is the $(j-1)^{th}$ largest eigenvalue of the submatrix $T^{-1} \mathbf{W}'_2 \mathbf{W}_2$. Applying Lemma B-9 and noting that $\hat{\Sigma}_{\mathbf{W}}$ and $T^{-1} \mathbf{W}'_2 \mathbf{W}_2$ are positive semidefinite matrices whose elements

are continuous random variables, we see that

$$0 \leq \frac{\tilde{\lambda}_{(j)}}{\tilde{\lambda}_{(1)}} < 1 \text{ a.s. for } j = 2, \dots, N.$$

Note also that, by part (a) of Lemma B-5, $\tilde{\lambda}_{(j)} = 0$ for $j = T + 2, \dots, N$. Hence, we can further write

$$\tau^2 \leq \frac{N}{T \|\gamma\|_2^{2(1+2\kappa)}} \frac{1}{\hat{\lambda}_{(1)}^2 / \|\gamma\|_2^{4(1+\kappa)}} \left(1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\hat{\lambda}_{(1)}} \right)^{-2} \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} \quad (13)$$

To analyze the asymptotic behavior of τ^2 , note first that we can apply the result of Lemma B-10 in Appendix B below to obtain

$$\begin{aligned} \frac{\hat{\lambda}_{(1)}^2}{\|\gamma\|_2^{4(1+\kappa)}} &= \left[\frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} \right]^2 \\ &= \left[c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p \left(\frac{1}{\|\gamma\|_2^{2\kappa}} \right) \right]^2 \\ &= c^2 \left[1 + O_p \left(\frac{1}{\|\gamma\|_2^{2\kappa}} \right) \right]. \end{aligned}$$

from which it follows that

$$\frac{1}{\hat{\lambda}_{(1)}^2 / \|\gamma\|_2^{4(1+\kappa)}} = \frac{1}{c^2} \left[1 + O_p \left(\frac{1}{\|\gamma\|_2^{2\kappa}} \right) \right] \quad (14)$$

where $0 < 1/c^2 < \infty$ given that $0 < c < \infty$.

Next, consider $\left(1 - \max_{2 \leq j \leq T+1} \left[\tilde{\lambda}_{(j)} / \hat{\lambda}_{(1)} \right] \right)^{-2}$. To analyze its asymptotic behavior, we make

use of Assumption 2-2, part (b) of Lemma B-5, and Lemma B-10 to obtain

$$\begin{aligned}
& \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \\
&= \frac{N-1}{T \|\gamma\|_2^{2(1+\kappa)}} \frac{1}{\widehat{\lambda}_{(1)} / \|\gamma\|_2^{2(1+\kappa)}} \frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \\
&= \left[c + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left[c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right]^{-1} \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[c + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left[c + \frac{1}{\|\gamma\|_2^{2\kappa}} \right]^{-1} \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[c + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \frac{1}{c} \left[1 + \frac{1}{c \|\gamma\|_2^{2\kappa}} \right]^{-1} \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[1 + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left[1 - \frac{1}{c \|\gamma\|_2^{2\kappa}} + O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right) \right] \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[1 - \frac{1}{c \|\gamma\|_2^{2\kappa}} + O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[1 - \frac{1}{c \|\gamma\|_2^{2\kappa}} \right] \left[1 + O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \left[1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right] \\
&= \left[1 - \frac{1}{c \|\gamma\|_2^{2\kappa}} \right] \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right]
\end{aligned}$$

so that

$$\begin{aligned}
1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} &= 1 - \left[1 - \frac{1}{c \|\gamma\|_2^{2\kappa}} \right] \left[1 + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \right] \\
&= 1 - 1 + \frac{1}{c \|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
&= \frac{1}{c \|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
&= \frac{1}{c \|\gamma\|_2^{2\kappa}} [1 + o_p(1)]
\end{aligned}$$

and, thus,

$$\left(1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \right)^{-2} = c^2 \|\gamma\|_2^{4\kappa} [1 + o_p(1)]. \quad (15)$$

Now, consider $T^{-1} \sum_{j=2}^N T^2 v_j^2 / (N \|\gamma\|_2^2)$. To proceed, note first that

$$W_{1,t} = \|\gamma\|_2 f_t + \eta_{1t} = \|\gamma\|_2 f_t + \mathbf{e}'_{1,N} \Pi' u_t$$

so that, given Assumption 2-1 and given the fact that Π is an orthogonal matrix, we have that

$$\{W_{1,t}\} \equiv i.i.d. N(0, \|\gamma\|_2^2 + 1)$$

from which we further deduce that

$$\frac{\mathbf{W}_1}{\|\gamma\|_2} = \begin{pmatrix} W_{1,1}/\|\gamma\|_2 \\ W_{1,2}/\|\gamma\|_2 \\ \vdots \\ W_{1,T}/\|\gamma\|_2 \end{pmatrix} \sim N\left(0, \left\{1 + \frac{1}{\|\gamma\|_2^2}\right\} I_T\right)$$

Moreover, note that

$$\mathbf{W}_{2,t} = \begin{pmatrix} \eta_{2t} \\ \vdots \\ \eta_{Nt} \end{pmatrix} = \begin{pmatrix} \mathbf{e}'_{2,N} \Pi' u_t \\ \vdots \\ \mathbf{e}'_{N,N} \Pi' u_t \end{pmatrix}$$

so that, under Assumption 2-1,

$$\{\mathbf{W}_{2,t}\} \equiv i.i.d. N(0, I_{N-1})$$

By direct calculation, we have for $j = 2, \dots, T + 1$

$$\begin{aligned}
E \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2 \right] &= \frac{T^2 \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}_2' \mathbf{W}_2' E [\mathbf{W}_1 \mathbf{W}_1' | \mathbf{W}_2] \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{N \|\gamma\|_2^2 T^2} \\
&\quad \left(\text{since } v_{(N-1) \times 1} = \frac{\tilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1}{T} \right) \\
&= \frac{T \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}_2' \mathbf{W}_2' E [\mathbf{W}_1 \mathbf{W}_1'] \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{N \|\gamma\|_2^2 T} \\
&\quad (\text{by independence of } \mathbf{W}_1 \text{ and } \mathbf{W}_2) \\
&= \frac{T}{N} \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{T} \\
&= \frac{T}{N} \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}_2' \tilde{\mathbf{B}}_2 \tilde{\Lambda} \tilde{\mathbf{B}}_2' \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \\
&\quad \left(\text{since } \frac{\mathbf{W}_2' \mathbf{W}_2}{T} = \tilde{\mathbf{B}}_2 \tilde{\Lambda} \tilde{\mathbf{B}}_2' \right) \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N}
\end{aligned}$$

In addition, by straightforward calculation, we also get for $j = 2, \dots, T + 1$

$$\begin{aligned}
& E \left[\frac{T^4 v_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2 \right] \\
&= \frac{T^4}{N^2} E \left\{ \left(\frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 \mathbf{W}_1 \mathbf{W}'_1 \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{\|\gamma\|_2^2 T^2} \right)^2 | \mathbf{W}_2 \right\} \\
&= \frac{T^4}{N^2 T^4} \sum_{r=1}^T \sum_{s=1}^T \sum_{t=1}^T \sum_{v=1}^T \left\{ E \left[\frac{W_{1,r}}{\|\gamma\|_2} \frac{W_{1,s}}{\|\gamma\|_2} \frac{W_{1,t}}{\|\gamma\|_2} \frac{W_{1,v}}{\|\gamma\|_2} | \mathbf{W}_2 \right] \left(\mathbf{W}'_{2,r} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \times \left(\mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \left(\mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \left. \left(\mathbf{W}'_{2,v} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right\} \\
&= \frac{T^4}{N^2 T^4} \sum_{t=1}^T E \left[\frac{W_{1,t}^4}{\|\gamma\|_2^4} \right] \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right)^2 \\
&\quad + \frac{3T^4}{N^2 T^4} \left\{ \sum_{t=1}^T E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \times \sum_{s \neq t} E \left[\frac{W_{1,s}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,s} \mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \left. \right\} \\
&= 3 \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \frac{T^4}{N^2 T^2} \left(\sum_{t=1}^T \frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{T} \right)^2 \\
&\quad \left(\text{since } \frac{W_{1,t}}{\|\gamma\|_2} = f_t + \|\gamma\|_2^{-1} \eta_{1t} \sim N \left(0, 1 + \frac{1}{\|\gamma\|_2^2} \right) \right) \\
&= 3 \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{T}{N} \tilde{\lambda}_{(j)} \right)^2
\end{aligned}$$

On the other hand, for $j = T + 2, \dots, N - 1$, we have

$$E \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2 \right] = \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} = 0$$

and

$$E \left[\frac{T^4 v_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2 \right] = 3 \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{T}{N} \tilde{\lambda}_{(j)} \right)^2 = 0$$

since $\tilde{\lambda}_{(j)} = 0$ for $j > T + 1$ by part (a) of Lemma B-5.

Next, we show that

$$E \left\{ \left(\frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \frac{1}{T} \sum_{j=2}^N E \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2 \right] \right)^2 | \mathbf{W}_2 \right\} = O_{a.s.} \left(\frac{1}{T} \right)$$

To proceed, write

$$\begin{aligned} & E \left\{ \left(\frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \frac{1}{T} \sum_{j=2}^N E \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2 \right] \right)^2 | \mathbf{W}_2 \right\} \\ &= E \left\{ \left(\frac{1}{T} \sum_{j=2}^N \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right] \right)^2 | \mathbf{W}_2 \right\} \\ &= \frac{1}{T^2} \sum_{j=2}^N E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right]^2 | \mathbf{W}_2 \right\} \\ &\quad + \frac{1}{T^2} \sum_{j \neq k} E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right] \right. \\ &\quad \quad \times \left. \left[\frac{T^2 v_k^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(k)}}{N} \right] | \mathbf{W}_2 \right\} \end{aligned} \tag{16}$$

Consider the second term on the right-hand side of expression (16)

$$\begin{aligned} & \frac{1}{T^2} \sum_{j \neq k} E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right] \left[\frac{T^2 v_k^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(k)}}{N} \right] | \mathbf{W}_2 \right\} \\ &= \frac{1}{T^2} \sum_{j \neq k} E \left[\frac{T^4 v_j^2 v_k^2}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2 \right] - \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j \neq k} \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) \end{aligned} \tag{17}$$

For the first term in expression (17), note that

$$\begin{aligned}
& E \left[\frac{T^4 v_j^2 v_k^2}{N^2 \|\gamma\|_2^4} |\mathbf{W}_2| \right] \\
&= \frac{T^4}{N^2} E \left\{ \left(\frac{\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 \mathbf{W}_1 \mathbf{W}'_1 \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{\|\gamma\|_2^2 T^2} \right) \right. \\
&\quad \times \left. \left(\frac{\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 \mathbf{W}_1 \mathbf{W}'_1 \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}}{\|\gamma\|_2^2 T^2} \right) |\mathbf{W}_2| \right\} \\
&= \frac{T^4}{N^2 T^4} \sum_{r=1}^T \sum_{s=1}^T \sum_{t=1}^T \sum_{v=1}^T \left\{ E \left[\frac{W_{1,r}}{\|\gamma\|_2} \frac{W_{1,s}}{\|\gamma\|_2} \frac{W_{1,t}}{\|\gamma\|_2} \frac{W_{1,v}}{\|\gamma\|_2} |\mathbf{W}_2| \right] \left(\mathbf{W}'_{2,r} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
&\quad \times \left(\mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \left(\mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \left(\mathbf{W}'_{2,v} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \Big\} \\
&= \frac{T^4}{N^2 T^4} \sum_{t=1}^T E \left\{ \left[\frac{W_{1,t}^4}{\|\gamma\|_2^4} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
&\quad \times \left. \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&\quad + \frac{T^4}{N^2 T^4} \left\{ \sum_{s=1}^T E \left[\frac{W_{1,s}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,s} \mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
&\quad \times \sum_{t \neq s} E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \Big\} \\
&\quad + \frac{T^4}{N^2 T^4} \left\{ \sum_{t=1}^T E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right. \\
&\quad \times \sum_{s \neq t} E \left[\frac{W_{1,s}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,s} \mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \Big\} \\
&\quad + \frac{T^4}{N^2 T^4} \left\{ \sum_{r=1}^T E \left[\frac{W_{1,r}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,r} \mathbf{W}'_{2,r} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right. \\
&\quad \times \sum_{t \neq r} E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \Big\} \tag{18}
\end{aligned}$$

Calculating the expectation for the first term on the right-hand side of expression (18) above, we

have

$$\begin{aligned}
& \frac{T^4}{N^2 T^4} \sum_{t=1}^T \left\{ E \left[\frac{W_{1,t}^4}{\|\gamma\|_2^4} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= \frac{3T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\}
\end{aligned}$$

Moreover, using the fact that

$$\begin{aligned}
\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{s=1}^T \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} &= \mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \tilde{\Lambda} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1} \tilde{\Lambda} \mathbf{e}_{j-1,N-1} \\
&= \tilde{\lambda}_{(j)}
\end{aligned}$$

and, for $j \neq k$,

$$\begin{aligned}
\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{V}}'_2 \sum_{s=1}^T \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \tilde{\mathbf{V}}_2 \mathbf{e}_{k-1,N-1} &= \mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \tilde{\Lambda} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1} \tilde{\Lambda} \mathbf{e}_{k-1,N-1} \\
&= 0
\end{aligned}$$

we further obtain

$$\begin{aligned}
& \frac{T^4}{N^2 T^4} \left\{ \sum_{s=1}^T E \left[\frac{W_{1,s}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,s} \mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \sum_{t \neq s} E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= \frac{T^4}{N^2 T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{s=1}^T \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{t=1}^T \frac{\mathbf{W}_{2,t} \mathbf{W}'_{2,t}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
& \quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) \\
& \quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\},
\end{aligned}$$

$$\begin{aligned}
& \frac{T^4}{N^2 T^4} \left\{ \sum_{t=1}^T E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right. \\
& \quad \left. \times \sum_{s \neq t} E \left[\frac{W_{1,s}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,s} \mathbf{W}'_{2,s} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= \frac{T^4}{N^2 T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{t=1}^T \frac{\mathbf{W}_{2,t} \mathbf{W}'_{2,t}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \\
& \quad \times \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{s=1}^T \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \\
& \quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\},
\end{aligned}$$

and

$$\begin{aligned}
& \frac{T^4}{N^2 T^4} \left\{ \sum_{r=1}^T E \left[\frac{W_{1,r}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,r} \mathbf{W}'_{2,r} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right. \\
& \quad \left. \times \sum_{t \neq r} E \left[\frac{W_{1,t}^2}{\|\gamma\|_2^2} \right] \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= \frac{T^4}{N^2 T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \sum_{r=1}^T \frac{\mathbf{W}_{2,r} \mathbf{W}'_{2,r}}{T} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \\
& \quad \times \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{V}}'_2 \sum_{t=1}^T \frac{\mathbf{W}_{2,t} \mathbf{W}'_{2,t}}{T} \tilde{\mathbf{V}}_2 \mathbf{e}_{k-1,N-1} \right) \\
& \quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
&= - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \quad \left. \times \left(\mathbf{e}'_{k-1,N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\}
\end{aligned}$$

It follows from these calculations that, for $j \neq k$

$$\begin{aligned}
& E \left[\frac{T^4 v_j^2 v_k^2}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2 \right] \\
&= \frac{3T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \left. \times \left(\mathbf{e}'_{k-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \right) \right\} \\
&\quad + \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) \\
&\quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \left. \times \left(\mathbf{e}'_{k-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \right) \right\} \\
&\quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \left. \times \left(\mathbf{e}'_{k-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \right) \right\} \\
&\quad - \frac{T^4}{N^2 T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^T \left\{ \left(\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1} \right) \right. \\
&\quad \left. \times \left(\mathbf{e}'_{k-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \right) \right\} \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right)
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{1}{T^2} \sum_{j \neq k} E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right] \left[\frac{T^2 v_k^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(k)}}{N} \right] | \mathbf{W}_2 \right\} \\
&= \frac{1}{T^2} \sum_{j \neq k} E \left[\frac{T^4 v_j^2 v_k^2}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2 \right] - \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j \neq k} \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) \\
&= \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j \neq k} \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) - \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j \neq k} \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right) \left(\frac{T \tilde{\lambda}_{(k)}}{N} \right) \\
&= 0
\end{aligned}$$

Hence,

$$\begin{aligned}
& E \left\{ \left(\frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \middle| \mathbf{W}_2 \right\} \\
&= \frac{1}{T^2} \sum_{j=2}^N E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right]^2 \middle| \mathbf{W}_2 \right\} \\
&\quad + \frac{1}{T^2} \sum_{j \neq k} E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right] \left[\frac{T^2 v_k^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(k)}}{N} \right] \middle| \mathbf{W}_2 \right\} \\
&= \frac{1}{T^2} \sum_{j=2}^N E \left\{ \left[\frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{T \tilde{\lambda}_{(j)}}{N} \right]^2 \middle| \mathbf{W}_2 \right\} \\
&= \frac{1}{T^2} \sum_{j=2}^N E \left[\frac{T^4 v_j^4}{N^2 \|\gamma\|_2^4} \middle| \mathbf{W}_2 \right] - \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j=2}^N \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \\
&= \frac{3}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j=2}^N \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 - \frac{1}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j=2}^N \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \\
&= \frac{2}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j=2}^N \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \\
&= \frac{2}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{j=2}^{T+1} \left(\frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \quad \left(\text{since } \tilde{\lambda}_{(j)} = 0 \text{ for } j > T+1 \right) \\
&\leq \frac{2}{T^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{N-1}{N} \right)^2 T \left(\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \right)^2 \\
&\quad \left(\text{since } \tilde{\lambda}_{(j)} \geq 0 \text{ for } j = 2, \dots, T+1 \right) \\
&= \frac{2}{T} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \left(\frac{N-1}{N} \right)^2 \left(\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \right)^2 \\
&= O_{a.s.} \left(\frac{1}{T} \right) \quad \left(\text{by Lemma B-7 and by the fact that } \|\gamma\|_2^2 \rightarrow \infty \text{ under Assumption 2-2} \right) \\
&= o_{a.s.}(1)
\end{aligned}$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995),

we see that there exists a constant $\bar{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned}
& E \left\{ T \left(\frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \right\} \\
&= E \left\{ \left(\frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \right\} \\
&= E_{\mathbf{W}_2} \left[E \left\{ \left(\frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \mid \mathbf{W}_2 \right\} \right] \\
&\leq \bar{C}.
\end{aligned}$$

Now, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{\bar{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$\begin{aligned}
& \Pr \left\{ \sqrt{T} \left| \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right| \geq C_\epsilon \right\} \\
&= \Pr \left\{ \left(\frac{\sqrt{T}}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{\sqrt{T}}{T} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \geq C_\epsilon^2 \right\} \\
&\leq \frac{1}{C_\epsilon^2} E \left\{ \left(\frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \right\} \\
&= \frac{\epsilon}{\bar{C}} E \left\{ \left(\frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{\sqrt{T}} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \right)^2 \right\} \\
&\leq \epsilon
\end{aligned}$$

which shows that

$$\begin{aligned}
& \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^N \frac{T \tilde{\lambda}_{(j)}}{N} \\
&= \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} - \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} \quad \left(\text{since } \tilde{\lambda}_{(j)} = 0 \text{ for } j > T+1 \right) \\
&= O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1)
\end{aligned} \tag{19}$$

In addition, note that

$$\begin{aligned}
\left| \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \left(\frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right) \right| &\leq \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \left| \frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right| \\
&\leq \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \max_{2 \leq j \leq T+1} \left| \frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right| \xrightarrow{a.s.} 0 \\
&\quad (\text{by Lemma B-7})
\end{aligned}$$

Making use of this result and the Slutsky's theorem, we obtain

$$\begin{aligned}
&\left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \left[\frac{T \tilde{\lambda}_{(j)}}{N} - 1 + 1 \right] \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right) + \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \left[\frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} - 1 \right] \\
&= \left(1 + \frac{1}{\|\gamma\|_2^2} \right) + \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \left(\frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right) \xrightarrow{a.s.} 1 \\
&\quad (\text{since } \|\gamma\|_2 \rightarrow \infty)
\end{aligned} \tag{20}$$

from which we further deduce, in light of expression (19), that

$$\frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} = \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} + O_p \left(\frac{1}{\sqrt{T}} \right) \xrightarrow{p} 1 \text{ as } N, T \rightarrow \infty. \tag{21}$$

Putting together the results given in expressions (13), (14), (15), and (21); we see that as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$

$$\begin{aligned}
& \tau^2 \\
& \leq \frac{N}{T \|\gamma\|_2^{2(1+2\kappa)}} \frac{1}{\hat{\lambda}_{(1)}^2 / \|\gamma\|_2^{4(1+\kappa)}} \left(1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\hat{\lambda}_{(1)}} \right)^{-2} \frac{1}{T} \sum_{j=2}^N \frac{T^2 v_j^2}{N \|\gamma\|_2^2} \\
& = \frac{1}{\|\gamma\|_2^{2\kappa}} \frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} \frac{1}{c^2} \|\gamma\|_2^{4\kappa} \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} [1 + o_p(1)] \\
& = \frac{N}{T \|\gamma\|_2^2} \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} [1 + o_p(1)] \\
& = O_p \left(\frac{N}{T \|\gamma\|_2^2} \right) \\
& \quad \left(\text{since } \left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{j=2}^{T+1} \frac{T \tilde{\lambda}_{(j)}}{N} \xrightarrow{p} 1 \text{ by expression (20)} \right)
\end{aligned} \tag{22}$$

Moreover, since Assumption 2-2 implies that $N / (T \|\gamma\|_2^2) \rightarrow \infty$ as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$, we further deduce that

$$\tau^2 \rightarrow \infty \text{ w.p.a.1.} \tag{23}$$

Finally, we note that expression (23) further implies that

$$\frac{\langle \tilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \rangle^2}{\|\tilde{\mathbf{v}}_{(1)}\|^2} = \frac{1}{1 + \tau^2} \xrightarrow{p} 0 \tag{24}$$

as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$.

Step 5:

In this step, we will show that

$$\frac{1}{\sqrt{N}} \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle \xrightarrow{p} 0.$$

To proceed, write

$$\begin{aligned}
\frac{1}{\sqrt{N}} \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle &= \frac{1}{\sqrt{N}} \frac{\langle \tilde{\mathbf{v}}_{(1)}, \tilde{\eta}_t \rangle}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \\
&= \frac{1}{\sqrt{N}} \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \left[\tilde{\eta}_{1t} + \sum_{j=2}^N \frac{v_j \tilde{\eta}_{jt}}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})} \right]
\end{aligned}$$

From the result given in expression (22) of Step 4 above, we have

$$\sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} = \tau^2 = O_p \left(\frac{N}{T \|\gamma\|_2^2} \right)$$

where $N / (T \|\gamma\|_2^2) \rightarrow \infty$ under our Assumption 2-2. This implies that

$$\frac{T \|\gamma\|_2^2}{N} \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} = O_p(1). \quad (25)$$

Next, note that

$$\begin{aligned}
\sum_{j=2}^N v_j \tilde{\eta}_{jt} &= \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} + \sum_{j=T+2}^N v_j \tilde{\eta}_{jt} \\
&= \sum_{j=2}^{T+1} \frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 \mathbf{W}_1 \tilde{\eta}_{jt}}{T} + \sum_{j=T+2}^N \frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 \mathbf{W}_1 \tilde{\eta}_{jt}}{T}
\end{aligned} \quad (26)$$

Recall that $\{\tilde{\eta}_t\} \equiv i.i.d.N(0, I_N)$ so that $\{\tilde{\eta}_{j,t}\} \equiv i.i.d.N(0, 1)$ across both j and t . Recall also that $\{f_t\} \equiv i.i.d.N(0, 1)$ and f_t and $\tilde{\eta}_s$ are independent for all s and t . In addition, since

$$\mathbf{W}_1 = \begin{pmatrix} \|\gamma\|_2 \left(f_1 + \|\gamma\|_2^{-1} \eta_{1,1} \right) \\ \|\gamma\|_2 \left(f_2 + \|\gamma\|_2^{-1} \eta_{1,2} \right) \\ \vdots \\ \|\gamma\|_2 \left(f_T + \|\gamma\|_2^{-1} \eta_{1,T} \right) \end{pmatrix} \quad \text{and} \quad \mathbf{W}_2 = \begin{pmatrix} \eta_{2,1} & \eta_{3,1} & \cdots & \eta_{N-1,1} \\ \eta_{2,2} & \eta_{3,2} & \cdots & \eta_{N-1,2} \\ \vdots & \vdots & & \vdots \\ \eta_{2,T} & \eta_{3,T} & \cdots & \eta_{N-1,T} \end{pmatrix},$$

it follows that \mathbf{W}_1 and \mathbf{W}_2 are independent. Now, focusing first on the term $\sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt}$ on the

right-hand side of expression (26) above, note that

$$\begin{aligned}
& E \left[\left(\sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \mid \mathbf{W}_2 \right] \\
&= \frac{1}{T^2} \sum_{j=2}^{T+1} \sum_{k=2}^{T+1} \tilde{\eta}_{jt} \tilde{\eta}_{kt} \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 E \left[\mathbf{W}_1 \mathbf{W}'_1 \mid \mathbf{W}_2 \right] \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \\
&= \frac{1}{T^2} \sum_{j=2}^{T+1} \sum_{k=2}^{T+1} \tilde{\eta}_{jt} \tilde{\eta}_{kt} \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \mathbf{W}'_2 E \left[\mathbf{W}_1 \mathbf{W}'_1 \right] \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \\
&= \frac{(\|\gamma\|_2^2 + 1)}{T} \sum_{j=2}^{T+1} \sum_{k=2}^{T+1} \tilde{\eta}_{jt} \tilde{\eta}_{kt} \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \left(\frac{\mathbf{W}'_2 \mathbf{W}_2}{T} \right) \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \\
&= \frac{(\|\gamma\|_2^2 + 1)}{T} \sum_{j=2}^{T+1} \sum_{k=2}^{T+1} \tilde{\eta}_{jt} \tilde{\eta}_{kt} \mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \tilde{\Lambda} \tilde{\mathbf{B}}'_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{k-1, N-1} \\
&= \frac{(\|\gamma\|_2^2 + 1)}{T} \sum_{j=2}^{T+1} \sum_{k=2}^{T+1} \tilde{\eta}_{jt} \tilde{\eta}_{kt} \mathbf{e}'_{j-1, N-1} \tilde{\Lambda} \mathbf{e}_{k-1, N-1} \\
&= \frac{(\|\gamma\|_2^2 + 1)}{T} \sum_{j=2}^{T+1} \tilde{\eta}_{jt}^2 \tilde{\lambda}_{(j)}
\end{aligned}$$

This implies that

$$\begin{aligned}
& E \left[\left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \mid \mathbf{W}_2 \right] \\
&= \frac{(\|\gamma\|_2^2 + 1)}{T \|\gamma\|_2^2} \sum_{j=2}^{T+1} \tilde{\eta}_{jt}^2 \frac{T}{N-1} \tilde{\lambda}_{(j)} \\
&\leq \frac{(\|\gamma\|_2^2 + 1)}{\|\gamma\|_2^2} \sqrt{\frac{1}{T} \sum_{j=2}^{T+1} \tilde{\eta}_{jt}^4} \sqrt{\frac{1}{T} \sum_{j=2}^{T+1} \left(\frac{T}{N-1} \tilde{\lambda}_{(j)} \right)^2} \\
&= O_{a.s.}(1)
\end{aligned}$$

given that, as $N, T \rightarrow \infty$,

$$\frac{1}{T} \sum_{j=2}^{T+1} \tilde{\eta}_{jt}^4 \xrightarrow{a.s.} 3$$

and, by Lemma B-7,

$$\begin{aligned} \frac{1}{T} \sum_{j=2}^{T+1} \left(\frac{T}{N-1} \tilde{\lambda}_{(j)} \right)^2 &\leq \left(\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \right)^2 \quad (\text{since } \tilde{\lambda}_{(j)} \geq 0 \text{ for } j = 2, \dots, T+1) \\ &= \left(\frac{T}{N-1} \tilde{\lambda}_{(2)} \right)^2 \xrightarrow{a.s.} 1. \end{aligned}$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995), we see that there exists a constant $\bar{C} < \infty$ such that for all n sufficiently large

$$\begin{aligned} E \left\{ \left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \right\} &= E_{\mathbf{W}_2} \left[E \left\{ \left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \mid \mathbf{W}_2 \right\} \right] \\ &\leq \bar{C}. \end{aligned}$$

Now, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{\bar{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$\begin{aligned} \Pr \left\{ \left\| \frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right\| \geq C_\epsilon \right\} &= \Pr \left\{ \left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \geq C_\epsilon^2 \right\} \\ &\leq \frac{1}{C_\epsilon^2} E \left\{ \left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \right\} \\ &= \frac{\epsilon}{\bar{C}} E \left\{ \left(\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right)^2 \right\} \\ &\leq \epsilon \end{aligned}$$

which shows that

$$\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| = O_p(1). \quad (27)$$

Next, consider the second term on the right-hand side of expression (26). Define

$$\tilde{D}_{T \times (N-1)} = \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ T \times T & T \times (N-T-1) \end{bmatrix}$$

where

$$\tilde{\Lambda}_1 = \begin{pmatrix} \tilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ 0 & \tilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{\lambda}_{(T+1)} \end{pmatrix}$$

Given that $N - 1 > T$ for N, T sufficiently large and given that $\tilde{\lambda}_{(j)} = 0$ for $j > T + 1$, we have the following singular-value decomposition of \mathbf{W}_2 :

$$\mathbf{W}_2 = \mathbb{O} \tilde{D} \tilde{\mathbf{B}}_2'$$

where \mathbb{O} is a $T \times T$ orthogonal matrix and $\tilde{\mathbf{B}}_2$ is as defined previously. Making use of this decomposition, we see that

$$\begin{aligned} \sum_{j=T+2}^N v_j \tilde{\eta}_{jt} &= \sum_{j=T+2}^N \frac{\mathbf{e}'_{j-1, N-1} \tilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \tilde{\eta}_{jt}}{T} \\ &= \sum_{j=T+2}^N \frac{\tilde{\eta}_{jt} \mathbf{W}_1' \mathbf{W}_2 \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{T} \\ &= \sum_{j=T+2}^N \frac{\tilde{\eta}_{jt} \mathbf{W}_1' \mathbb{O} \tilde{D} \tilde{\mathbf{B}}_2' \tilde{\mathbf{B}}_2 \mathbf{e}_{j-1, N-1}}{T} \\ &= \sum_{j=T+2}^N \frac{\tilde{\eta}_{jt} \mathbf{W}_1' \mathbb{O} \tilde{D} \mathbf{e}_{j-1, N-1}}{T} \\ &= 0 \end{aligned}$$

Putting things together, we have

$$\begin{aligned}
& \frac{1}{\sqrt{N}} \left| \left\langle \frac{\tilde{\mathbf{v}}_{(1)}}{\|\tilde{\mathbf{v}}_{(1)}\|_2}, \tilde{\eta}_t \right\rangle \right| \\
&= \frac{1}{\sqrt{N}} \left| \frac{\langle \tilde{\mathbf{v}}_{(1)}, \tilde{\eta}_t \rangle}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \right| \\
&= \frac{1}{\sqrt{N}} \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \left| \tilde{\eta}_{1t} + \sum_{j=2}^N \frac{v_j \tilde{\eta}_{jt}}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})} \right| \\
&= \frac{1}{\sqrt{N}} \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \left| \tilde{\eta}_{1t} + \frac{1}{\hat{\lambda}_{(1)}} \sum_{j=2}^{T+1} \frac{v_j \tilde{\eta}_{jt}}{(1 - \tilde{\lambda}_{(j)}/\hat{\lambda}_{(1)})} + \frac{1}{\hat{\lambda}_{(1)}} \sum_{j=T+2}^N v_j \tilde{\eta}_{jt} \right| \\
&\quad \left(\text{noting that } \tilde{\lambda}_j = 0 \text{ for } j > T+1 \right) \\
&= \frac{1}{\sqrt{N}} \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \left| \tilde{\eta}_{1t} + \frac{1}{\hat{\lambda}_{(1)}} \sum_{j=2}^{T+1} \frac{v_j \tilde{\eta}_{jt}}{(1 - \tilde{\lambda}_{(j)}/\hat{\lambda}_{(1)})} \right| \\
&\quad \left(\text{since } \sum_{j=T+2}^N v_j \tilde{\eta}_{jt} = 0 \right) \\
&= \frac{1}{\sqrt{N}} \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \left| \tilde{\eta}_{1t} + \frac{1}{\hat{\lambda}_{(1)}/\|\gamma\|_2^{2(1+\kappa)}} \frac{1}{\|\gamma\|_2^{2(1+\kappa)}} \sum_{j=2}^{T+1} \frac{v_j \tilde{\eta}_{jt}}{(1 - \tilde{\lambda}_{(j)}/\hat{\lambda}_{(1)})} \right| \\
&\leq \frac{1}{\sqrt{N}} \left\{ \left[1 + \sum_{j=2}^N \frac{v_j^2}{(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)})^2} \right]^{-1/2} \right. \\
&\quad \times \left. \left[|\tilde{\eta}_{1t}| + \frac{1}{\hat{\lambda}_{(1)}/\|\gamma\|_2^{2(1+\kappa)}} \frac{1}{\|\gamma\|_2^{2(1+\kappa)}} \left(1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\hat{\lambda}_{(1)}} \right)^{-1} \left\| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right\| \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left[1 + \sum_{j=2}^N \frac{v_j^2}{\left(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)} \right)^2} \right]^{-1/2} \\
&\quad \times \left[\frac{|\tilde{\eta}_{1t}|}{\sqrt{N}} + \left(c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p \left(\frac{1}{\|\gamma\|_2^{2\kappa}} \right) \right)^{-1} \frac{\|\gamma\|_2}{\sqrt{N}} \frac{c \|\gamma\|_2^{2\kappa}}{\|\gamma\|_2 \|\gamma\|_2^{2(1+\kappa)}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| (1 + o_p(1)) \right] \\
&\quad \left(\text{given that } \frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} = c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p \left(\frac{1}{\|\gamma\|_2^{2\kappa}} \right) \text{ for } 0 < \kappa < 1, \right. \\
&\quad \left. \text{and } \left(1 - \max_{2 \leq j \leq T+1} \frac{\tilde{\lambda}_{(j)}}{\hat{\lambda}_{(1)}} \right)^{-1} = c \|\gamma\|_2^{2\kappa} [1 + o_p(1)] \right) \\
&= \left[1 + \sum_{j=2}^N \frac{v_j^2}{\left(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)} \right)^2} \right]^{-1/2} \\
&\quad \times \left[\frac{|\tilde{\eta}_{1t}|}{\sqrt{N}} + \sqrt{\frac{N-1}{T}} \frac{\|\gamma\|_2^{2\kappa}}{\|\gamma\|_2^{2(1+\kappa)}} \frac{\|\gamma\|_2}{\sqrt{N}} \frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| (1 + o_p(1)) \right] \\
&= \left[1 + \sum_{j=2}^N \frac{v_j^2}{\left(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)} \right)^2} \right]^{-1/2} \\
&\quad \times \left[\frac{|\tilde{\eta}_{1t}|}{\sqrt{N}} + \sqrt{\frac{N-1}{T}} \frac{\|\gamma\|_2}{\|\gamma\|_2^2 \sqrt{N}} \frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| (1 + o_p(1)) \right] \\
&= \left[\frac{T \|\gamma\|_2^2}{N} + \frac{T \|\gamma\|_2^2}{N} \sum_{j=2}^N \frac{v_j^2}{\left(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)} \right)^2} \right]^{-1/2} \\
&\quad \times \left[\sqrt{\frac{T \|\gamma\|_2^2}{N}} \frac{|\tilde{\eta}_{1t}|}{N} + \sqrt{\frac{N-1}{N}} \frac{1}{\sqrt{N}} \frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| (1 + o_p(1)) \right] \\
&= o_p(1), \tag{28}
\end{aligned}$$

where the last line follows from the fact that

$$\begin{aligned}
\frac{1}{\|\gamma\|_2} \sqrt{\frac{T}{N-1}} \left| \sum_{j=2}^{T+1} v_j \tilde{\eta}_{jt} \right| &= O_p(1) \quad (\text{by expression (27)}) \\
\frac{T \|\gamma\|_2^2}{N} \sum_{j=2}^N \frac{v_j^2}{\left(\hat{\lambda}_{(1)} - \tilde{\lambda}_{(j)} \right)^2} &= O_p(1) \quad (\text{by expression (25)})
\end{aligned}$$

and the fact that

$$\|\gamma\|_2^2 \rightarrow \infty \text{ and } \frac{T \|\gamma\|_2^2}{N} \rightarrow 0 \text{ (by Assumption 2-2).}$$

Step 6:

Finally, in this last step, we bring everything together. Combining the results given in expressions (12) of step 3, (24) of step 4, and (28) of step 5 and noting the fact that $f_t = O_p(1)$, we can apply the Slutsky's theorem to deduce that

$$\hat{f}_t = \frac{\|\gamma\|_2}{\sqrt{N}} \frac{\langle \tilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \rangle}{\|\tilde{\mathbf{v}}_{(1)}\|_2} f_t + \frac{1}{\sqrt{N}} \frac{\langle \tilde{\mathbf{v}}_{(1)}, \tilde{\eta}_t \rangle}{\|\tilde{\mathbf{v}}_{(1)}\|_2} \xrightarrow{p} 0 \text{ as } N, T \rightarrow \infty$$

which is the required result. \square

Proof of Theorem 2:

To show part (a), first set

$$z = \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right).$$

where $N = N_1 + N_2$. Note that by part (b) of Lemma C-16, we have

$$\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \leq \sqrt{2(1+a)} \sqrt{\ln N} \text{ for all } N_1, N_2 \text{ sufficiently large.} \quad (29)$$

By part (a) of Assumption 3-10,

$$\frac{\sqrt{\ln N}}{T^{\min\{\frac{1-\alpha_1}{6}, \frac{\alpha_2}{2}\}}} \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty;$$

this, in turn, implies that, for some positive constant c_0

$$0 \leq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \leq c_0 \min \left\{ T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}} \right\} \text{ for all } N_1, N_2, T \text{ sufficiently large,}$$

so that $\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)$ lies within the range of values of z for which the moderate deviation inequality given in Lemma C-17 holds. Thus, plugging $\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)$ into the moderate deviation inequality

(71) given Lemma C-17, we see that there exists a positive constant A such that

$$\begin{aligned}
& P \left(|S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \leq 2 \left[1 - \Phi \left(\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \right] \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-\frac{1-\alpha_1}{2}} \right\} \\
& = 2 \left[1 - \left(1 - \frac{\varphi}{2N} \right) \right] \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-\frac{1-\alpha_1}{2}} \right\} \\
& = \frac{\varphi}{N} \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-\frac{1-\alpha_1}{2}} \right\}
\end{aligned}$$

for $\ell \in \{1, \dots, d\}$, for

$$i \in H = \{k \in \{1, \dots, N\} : \gamma_k = 0\},$$

and for all N_1, N_2, T sufficiently large. Next, note that

$$\begin{aligned}
& P \left(\max_{i \in H} \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \leq P \left(\bigcup_{i \in H} \bigcup_{1 \leq \ell \leq d} \left\{ |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right) \left(\text{since } 0 \leq \varpi_\ell \leq 1 \text{ and } \sum_{\ell=1}^d \varpi_\ell = 1 \right) \\
& \leq \sum_{i \in H} \sum_{\ell=1}^d P \left(|S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \quad (\text{by union bound}) \\
& \leq \sum_{i \in H} \sum_{\ell=1}^d \frac{\varphi}{N} \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-(1-\alpha_1)\frac{1}{2}} \right\} \\
& = d \frac{N_2 \varphi}{N} \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-(1-\alpha_1)\frac{1}{2}} \right\}
\end{aligned}$$

Using the inequality bound given in expression (29) above, we further obtain, for all N_1, N_2, T

sufficiently large,

$$\begin{aligned}
& P \left(\max_{i \in H} \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \leq \frac{dN_2\varphi}{N} \left\{ 1 + A \left[1 + \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-\frac{(1-\alpha_1)}{2}} \right\} \\
& \leq \frac{dN_2\varphi}{N} \left\{ 1 + 2^2 AT^{-\frac{(1-\alpha_1)}{2}} + 2^2 A \left[\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right]^3 T^{-\frac{(1-\alpha_1)}{2}} \right\} \\
& \quad \left(\text{by the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq c_r \sum_{i=1}^m |a_i|^r \text{ where } c_r = m^{r-1} \text{ for } r \geq 1 \right) \\
& \leq \frac{dN_2\varphi}{N} \left\{ 1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 4A \left[\sqrt{2(1+a)} \sqrt{\ln N} \right]^3 T^{-\frac{(1-\alpha_1)}{2}} \right\} \\
& = \frac{dN_2\varphi}{N} \left\{ 1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 2^{\frac{7}{2}} A (1+a)^{\frac{3}{2}} \frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}} \right\}
\end{aligned}$$

Finally, note that rate condition given in part (a) of Assumption 3-10, i.e.,

$$\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6}, \frac{\alpha_2}{2}\right\}}} \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty$$

implies that

$$\frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}} \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty,$$

from which it follows that

$$\begin{aligned}
& P \left(\max_{i \in H} \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \leq \frac{dN_2\varphi}{N} \left\{ 1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 2^{\frac{7}{2}} A (1+a)^{\frac{3}{2}} \frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}} \right\} \\
& = \frac{dN_2\varphi}{N} [1 + o(1)] \\
& = O \left(\frac{N_2\varphi}{N} \right) \\
& = o(1).
\end{aligned}$$

Next, to show part (b), note that, by a similar argument as that given for part (a) above, we

have

$$\begin{aligned}
& P \left(\max_{i \in H} \max_{1 \leq \ell \leq d} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&= P \left(\bigcup_{i \in H} \bigcup_{1 \leq \ell \leq d} \left\{ |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right) \\
&\leq \frac{dN_2\varphi}{N} \left\{ 1 + 4AT^{-\frac{(1-\alpha_1)}{2}} + 2^{\frac{7}{2}} A (1+a)^{\frac{3}{2}} \frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}} \right\} \\
&= \frac{dN_2\varphi}{N} [1 + o(1)] \\
&= O \left(\frac{N_2\varphi}{N} \right) \\
&= o(1). \quad \square
\end{aligned}$$

Proof of Theorem 3:

To show part (a), note that

$$\begin{aligned}
& P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&= P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} + \frac{\mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&\geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| - \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&= P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \left[1 - \left| \frac{\sqrt{V_{i,\ell,T}}}{\mu_{i,\ell,T}} \right| \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&= P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \left[1 - \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right),
\end{aligned}$$

where

$$\mu_{i,\ell,T} = \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \}.$$

Next, let

$$\pi_{i,\ell,T} = \sum_{r=1}^q \left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2,$$

and note that, under Assumption 3-9, there exists a positive constant \underline{c} such that for every $\ell \in \{1, \dots, d\}$ and for all N_1, N_2 , and T sufficiently large

$$\begin{aligned} & \min_{i \in H^c} \frac{\pi_{i,\ell,T}}{q\tau_1^2} \\ &= \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\ &= \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[\gamma'_i \underline{F}_t y_{\ell,t+1}] \right)^2 \\ &\geq \min_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[\gamma'_i \underline{F}_t y_{\ell,t+1}] \right)^2 \quad (\text{by Jensen's inequality}) \\ &= \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[\gamma'_i \underline{F}_t y_{\ell,t+1}] \right|^2 \\ &= \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right|^2 \\ &\geq \underline{c}^2 > 0 \quad (\text{in light of Assumption 3-9}) \end{aligned}$$

It follows that we can multiply and divide by $\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}$ to obtain for all N_1, N_2 , and T suffi-

ciently large

$$\begin{aligned}
& P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\mu_{i,\ell,T}}{\sqrt{V_{i,\ell,T}}} \right| \left[1 - \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& = P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)} + \sqrt{V_{i,\ell,T} / (q\tau_1^2)} - \sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \right. \right. \\
& \quad \left. \left. \times \left[1 - \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& = P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{1}{1 + \left(\sqrt{V_{i,\ell,T}} - \sqrt{\pi_{i,\ell,T}} \right) / \sqrt{\pi_{i,\ell,T}}} \right| \right. \right. \\
& \quad \left. \left. \times \left[1 - \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} (\mu_{i,\ell,T} / (q\tau_1))}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{1}{1 + \max_{k \in H^c} \left| \sqrt{V_{k,\ell,T}} - \sqrt{\pi_{k,\ell,T}} \right| / \sqrt{\pi_{k,\ell,T}}} \right| \right. \right. \\
& \quad \left. \left. \times \left[1 - \max_{k \in H^c} \left| \frac{\bar{S}_{k,\ell,T} - \mu_{k,\ell,T}}{\mu_{k,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{1}{1 + \max_{k \in H^c} \sqrt{|\bar{V}_{k,\ell,T} - \pi_{k,\ell,T}|} / \pi_{k,\ell,T}} \right| \right. \right. \\
& \quad \left. \left. \times \left[1 - \max_{k \in H^c} \left| \frac{\bar{S}_{k,\ell,T} - \mu_{k,\ell,T}}{\mu_{k,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \quad \left(\text{making use of the inequality } \left| \sqrt{a} - \sqrt{b} \right| \leq \sqrt{|a - b|} \text{ for } a \geq 0 \text{ and } b \geq 0 \text{ by Lemma C-14} \right) \\
& = P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{1}{1 + \max_{k \in H^c} \sqrt{|\bar{V}_{k,\ell,T} - \pi_{k,\ell,T}|} / \pi_{k,\ell,T}} \right| \right. \right. \\
& \quad \left. \left. \times \left[1 - \max_{k \in H^c} \left| \frac{\bar{S}_{k,\ell,T} - \mu_{k,\ell,T}}{\mu_{k,\ell,T}} \right| \right] \right\} \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right)
\end{aligned}$$

Now, let

$$\mathcal{E}_{k,\ell} = \frac{\bar{S}_{k,\ell,T} - \mu_{k,\ell,T}}{\mu_{k,\ell,T}} \text{ and } \mathcal{V}_{k,\ell} = \frac{\bar{V}_{k,\ell,T} - \pi_{k,\ell,T}}{\pi_{k,\ell,T}}.$$

By the result of part (a) of Lemma C-13, there exists a sequence of positive numbers $\{\epsilon_T\}$ such that, as $T \rightarrow \infty$, $\epsilon_T \rightarrow 0$ and

$$P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) \rightarrow 0$$

In addition, by the result of part (b) of Lemma C-13, there exists a sequence of positive numbers $\{\epsilon_T^*\}$ such that, as $T \rightarrow \infty$, $\epsilon_T^* \rightarrow 0$ and

$$P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| \geq \epsilon_T^* \right) \rightarrow 0.$$

Hence, for all N_1, N_2 , and T sufficiently large,

$$\begin{aligned}
& P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left\{ \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \left| \frac{1}{1 + \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \right| \left[1 - \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \right] \right\} \right. \\
& \quad \left. \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\left| \frac{1}{1 + \max_{1 \leq \ell \leq d} \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \right| \left[1 - \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \right] \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\sqrt{q} [\mu_{i,\ell,T} / (q\tau_1)]}{\sqrt{\pi_{i,\ell,T} / (q\tau_1^2)}} \right| \right. \\
& \quad \left. \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& = P \left(\left| \frac{1}{1 + \max_{1 \leq \ell \leq d} \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \right| \left[1 - \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \right] \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \right. \\
& \quad \left. \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \geq P \left(\left| \frac{1 - \epsilon_T}{1 + \epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \right. \\
& \quad \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) \\
& + P \left(\left\{ \frac{1 - \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}|}{1 + \max_{1 \leq \ell \leq d} \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \min_{i \in H} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right. \\
& \quad \left. \cap \left\{ \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \cup \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| \geq \epsilon_T^* \right\} \right) \\
& \geq P \left(\left| \frac{1 - \epsilon_T}{1 + \epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \right. \\
& \quad \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) \\
& + P \left(\frac{1 - \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}|}{1 + \max_{1 \leq \ell \leq d} \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \min_{i \in H} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right. \\
& \quad \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) \\
& = P \left(\left| \frac{1 - \epsilon_T}{1 + \epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \right. \\
& \quad \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) + o(1)
\end{aligned}$$

where the last equality above follows from the fact that

$$\begin{aligned}
& P \left(\frac{1 - \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}|}{1 + \max_{1 \leq \ell \leq d} \max_{k \in H^c} \sqrt{|\mathcal{V}_{k,\ell}|}} \min_{i \in H} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right. \\
& \quad \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) \\
& \leq P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) = o(1).
\end{aligned}$$

Moreover, making use of Assumption 3-9, the result given in part (e) of Lemma C-12, and the fact that $q = \lfloor T_0/\tau \rfloor \sim T^{1-\alpha_1}$, we see that, there exists positive constants \underline{c} and \overline{C} such that

$$\begin{aligned}
\min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| &= \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \frac{\sqrt{q} |\mu_{i,\ell,T}/(q\tau_1)|}{\sqrt{\pi_{i,\ell,T}/(q\tau_1^2)}} \\
&\geq \sqrt{q} \sum_{\ell=1}^d \varpi_{\ell} \frac{\min_{i \in H^c} |\mu_{i,\ell,T}/(q\tau_1)|}{\sqrt{\max_{i \in H^c} \pi_{i,\ell,T}/(q\tau_1^2)}} \\
&\geq \sqrt{q} \sum_{\ell=1}^d \varpi_{\ell} \frac{\underline{c}}{\sqrt{\overline{C}}} \\
&= \sqrt{q} \frac{\underline{c}}{\sqrt{\overline{C}}} \\
&\sim \sqrt{q} \sim \sqrt{\frac{T_0}{\tau}} \sim T^{(1-\alpha_1)/2}
\end{aligned}$$

On the other hand, applying part (b) of Lemma C-16, we have, for all N_1, N_2 sufficiently large,

$$\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \leq \sqrt{2(1+a)} \sqrt{\ln N} \sim \sqrt{\ln N},$$

where $N = N_1 + N_2$, from which we further deduce that

$$\begin{aligned}
\frac{1}{\Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)} \min_{i \in H^c} \sum_{\ell=1}^d \varpi_{\ell} \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| &\geq \frac{\underline{c}}{\sqrt{\overline{C}}} \sqrt{\frac{q}{2(1+a) \ln N}} \\
&\sim \sqrt{\frac{T^{(1-\alpha_1)}}{\ln N}} \rightarrow \infty \text{ as } N_1, N_2, T \rightarrow \infty.
\end{aligned}$$

This is true because the condition

$$\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6}, \frac{\alpha_2}{2}\right\}}} \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty \text{ (given in Assumption 3-10 part (a))}$$

implies that

$$\frac{\ln N}{T(1-\alpha_1)} \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty.$$

Hence, there exists a natural number M such that, for all $N_1 \geq M, N_2 \geq M$, and $T \geq M$, we have

$$\left| \frac{1 - \epsilon_T}{1 + \epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^d \varpi_\ell \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)$$

so that

$$\begin{aligned} & P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\ \geq & P \left(\left| \frac{1 - \epsilon_T}{1 + \epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^d \varpi_\ell \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \right. \\ & \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) + o(1) \\ = & P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) + o(1) \\ & (\text{for all } N_1 \geq M, N_2 \geq M, \text{ and } T \geq M) \\ \geq & P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| < \epsilon_T \right) + P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| < \epsilon_T^* \right) - 1 + o(1) \\ & \left(\text{using the inequality } P \left\{ \bigcap_{i=1}^m A_i \right\} \geq \sum_{i=1}^m P(A_i) - (m-1) \text{ given in Lemma C-15} \right) \\ = & 1 - P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) + 1 - P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| \geq \epsilon_T^* \right) - 1 + o(1) \\ = & 1 - P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{E}_{k,\ell}| \geq \epsilon_T \right) - P \left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} |\mathcal{V}_{k,\ell}| \geq \epsilon_T^* \right) + o(1) \\ = & 1 + o(1) + o(1) + o(1) \\ = & 1 + o(1). \end{aligned}$$

Next, to show part (b), note that, by applying the result in part (a), we have

$$\begin{aligned} P \left(\min_{i \in H^c} \max_{1 \leq \ell \leq d} |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) & \geq P \left(\min_{i \in H^c} \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}| \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\ & = 1 + o(1). \quad \square \end{aligned}$$

Proof of Theorem 4:

To proceed, note first that the principal component estimator of \underline{F}_t can be written as

$$\hat{\underline{F}}_t = \frac{\hat{\Gamma}' Z_{t,N}(\widehat{H}^c)}{\hat{N}_1}$$

where $\hat{\Gamma} = \sqrt{\hat{N}_1} \hat{B}$ and where the columns of the matrix \hat{B} are the eigenvectors associated with the Kp largest eigenvalues of the (post-variable-selection) sample covariance matrix

$$\hat{\Sigma}(\widehat{H}^c) = \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{\hat{N}_1 T_0}.$$

Moreover, by the result of part (d) of Lemma D-14, the matrix \hat{B} has the representation

$$\hat{B} = \hat{G}_1' \hat{V}$$

where \hat{G}_1 is an $N \times Kp$ matrix, whose columns define an orthonormal basis for an invariant subspace of $\hat{\Sigma}(\widehat{H}^c)$ and where \hat{V} is a $Kp \times Kp$ orthogonal matrix as defined in expression (129) in part (c) of Lemma D-14. (See Lemma D-14 and also Lemma D-13 for additional discussion on the origin of this representation). Making use of this representation, we can further write

$$\begin{aligned} \hat{\underline{F}}_t - Q' \underline{F}_t &= \frac{\sqrt{\hat{N}_1} \hat{V}' \hat{G}_1' Z_{t,N}(\widehat{H}^c)}{\hat{N}_1} - Q' \underline{F}_t \\ &= \frac{\hat{V}' \hat{G}_1' \Gamma(\widehat{H}^c) \underline{F}_t}{\sqrt{\hat{N}_1}} + \frac{\hat{V}' \hat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \underline{F}_t \\ &= \frac{\hat{V}' \hat{G}_1' \Gamma(\widehat{H}^c) \underline{F}_t}{\sqrt{\hat{N}_1}} - Q' \underline{F}_t + \frac{\hat{V}' \hat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{\hat{N}_1}} \\ &= \left(\frac{\hat{V}' \hat{G}_1' \Gamma(\widehat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right) \underline{F}_t + \frac{\hat{V}' \hat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{\hat{N}_1}} \end{aligned}$$

Next, note that

$$\begin{aligned}
\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{\widehat{N}_1}} - Q' &= \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}\sqrt{(\widehat{N}_1 - N_1 + N_1)/N_1}} - Q' \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} + \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q'
\end{aligned}$$

and

$$\begin{aligned}
\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{\widehat{N}_1}} &= \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}\sqrt{(\widehat{N}_1 - N_1 + N_1)/N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}}\right)
\end{aligned}$$

so that

$$\begin{aligned}
&\frac{\widehat{V}'\widehat{G}'_1\Gamma(\widehat{H}^c)\underline{E}_t}{\sqrt{\widehat{N}_1}} \\
&= Q'\underline{E}_t + \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{E}_t + \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{\widehat{N}_1}}\right)\underline{E}_t \\
&= Q'\underline{E}_t + \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q'\right)\underline{E}_t + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}}\underline{E}_t \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right] \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}}\right)\underline{E}_t
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{E}_t - Q'E_t &= \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} - Q' \right) E_t + \frac{\widehat{V}'\widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q' \right) E_t + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} E_t \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right] \widehat{V}'\widehat{G}'_1 \left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right) E_t + \frac{\widehat{V}'\widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Hence, applying the triangle inequality as well as parts (a)-(c), (g), and (i) of Lemma D-15 along with the Slutsky's theorem, we obtain

$$\begin{aligned}
&\left\| \widehat{E}_t - Q'E_t \right\|_2 \\
&\leq \left\| \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|E_t\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} \right\|_2 \|E_t\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \widehat{V}'\widehat{G}'_1 \right\|_2 \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \|E_t\|_2 + \left\| \frac{\widehat{V}'\widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= \left\| \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|E_t\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} \right\|_2 \|E_t\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \|E_t\|_2 + \left\| \frac{\widehat{V}'\widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\quad \left(\text{since } \left\| \widehat{V}'\widehat{G}'_1 \right\|_2 = \lambda_{\max} \left(\widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}'_1 \right) = \lambda_{\max} \left(\widehat{V}' \widehat{G}'_1 \widehat{G}_1 \widehat{V} \right) = \lambda_{\max} (I_{Kp}) = 1 \right) \\
&= o_p(1) O_p(1) + o_p(1) O_p(1) O_p(1) + O_p(1) o_p(1) O_p(1) + o_p(1) \\
&= o_p(1). \quad \square
\end{aligned}$$

Proof of Theorem 5:

To proceed, note that for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we have

$$\begin{aligned}
& \left| a' \widehat{Y}_{T+h} - a' (\beta_0 + B_1' \underline{Y}_T + B_2' \underline{E}_T) \right| \\
&= \left| a' (\widehat{\beta}_0 + \widehat{B}_1' \underline{Y}_T + \widehat{B}_2' \widehat{\underline{E}}_T) - a' (\beta_0 + B_1' \underline{Y}_T + B_2' \underline{E}_T) \right| \\
&= \left| a' (\widehat{\beta}_0 - \beta_0) + a' (\widehat{B}_1 - B_1)' \underline{Y}_T \right. \\
&\quad \left. + a' (\widehat{B}_2 - Q^{-1} B_2 + Q^{-1} B_2)' (\widehat{\underline{E}}_T - Q' \underline{E}_T + Q' \underline{E}_T) - a' B_2' \underline{E}_T \right| \\
&\leq \left| a' (\widehat{\beta}_0 - \beta_0) \right| + \left| a' (\widehat{B}_1 - B_1)' \underline{Y}_T \right| + \left| a' (\widehat{B}_2 - Q^{-1} B_2)' (\widehat{\underline{E}}_T - Q' \underline{E}_T) \right| \\
&\quad + \left| a' B_2' Q^{-1'} (\widehat{\underline{E}}_T - Q' \underline{E}_T) \right| + \left| a' (\widehat{B}_2 - Q^{-1} B_2)' Q' \underline{E}_T \right| + \left| a' B_2' Q^{-1'} Q' \underline{E}_T - a' B_2' \underline{E}_T \right| \\
&= \left| a' (\widehat{\beta}_0 - \beta_0) \right| + \left| a' (\widehat{B}_1 - B_1)' \underline{Y}_T \right| + \left| a' (\widehat{B}_2 - Q^{-1} B_2)' (\widehat{\underline{E}}_T - Q' \underline{E}_T) \right| \\
&\quad + \left| a' B_2' Q^{-1'} (\widehat{\underline{E}}_T - Q' \underline{E}_T) \right| + \left| a' (\widehat{B}_2 - Q^{-1} B_2)' Q' \underline{E}_T \right|
\end{aligned}$$

Lemma D-18 and Slutsky's theorem directly imply that

$$\left| a' (\widehat{\beta}_0 - \beta_0) \right| = o_p(1)$$

Now, applying the CS inequality, we obtain

$$\begin{aligned}
\left| a' (\widehat{B}_1 - B_1)' \underline{Y}_T \right| &\leq \sqrt{a' (\widehat{B}_1 - B_1)' (\widehat{B}_1 - B_1)} a \sqrt{\underline{Y}_T' \underline{Y}_T} \\
&= \sqrt{a' (\widehat{B}_1 - B_1)' (\widehat{B}_1 - B_1)} a \|\underline{Y}_T\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
& \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{E}_T \right| \\
& \leq \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{E}_T' Q Q' \underline{E}_T} \\
& = \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{E}_T' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} \underline{E}_T} \\
& = \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{E}_T' \left(\frac{\Gamma' \Gamma}{N_1} \right) \underline{E}_T} \\
& \leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \|\underline{E}_T\|_2^2 \\
& \leq \overline{C} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \|\underline{E}_T\|_2^2
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
E \left[\|\underline{Y}_T\|_2^2 \right] & \leq \left(E \|\underline{Y}_T\|_2^6 \right)^{\frac{1}{3}} \quad (\text{by Liapunov's inequality}) \\
& \leq \overline{C}^{\frac{1}{3}} = C < \infty \quad (\text{by Lemma C-5})
\end{aligned}$$

and

$$\begin{aligned}
E \left[\|\underline{E}_T\|_2^2 \right] & \leq \left(E \|\underline{E}_T\|_2^6 \right)^{\frac{1}{3}} \quad (\text{by Liapunov's inequality}) \\
& \leq \overline{C}^{\frac{1}{3}} = C < \infty \quad (\text{by Lemma C-5})
\end{aligned}$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{C/\epsilon}$, and Markov's inequality then implies that, for all $T > p - 1$,

$$\Pr \{ \|\underline{Y}_T\|_2 \geq C_\epsilon \} = \Pr \left\{ \|\underline{Y}_T\|_2^2 \geq C_\epsilon^2 \right\} \leq \frac{E \left[\|\underline{Y}_T\|_2^2 \right]}{C_\epsilon^2} = \frac{\epsilon E \left[\|\underline{Y}_T\|_2^2 \right]}{C} \leq \epsilon$$

from which it follows that

$$\|\underline{Y}_T\|_2 = O_p(1).$$

In a similar way, we can also show that

$$\|\underline{E}_T\|_2 = O_p(1).$$

Application of the result given in Lemma D-18 then allows us to deduce that

$$\left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| \leq \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \|\underline{Y}_T\|_2^2 = o_p(1)$$

and

$$\begin{aligned} & \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{E}_T \right| \\ & \leq \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{E}_T' Q Q' \underline{E}_T} \\ & \leq \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max}(Q Q')} \|\underline{E}_T\|_2 \\ & = \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right)^{\frac{1}{2}} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{\frac{1}{2}} \right\}} \|\underline{E}_T\|_2 \\ & = \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right) \right\}} \|\underline{E}_T\|_2 \\ & \quad \left(\text{since } \widehat{V} \widehat{V}' = I_{K_p} \text{ and } \Xi \Xi' = I_{K_p} \right) \\ & \leq \sqrt{\overline{C}} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \|\underline{E}_T\|_2 \quad (\text{by Assumption 3-6}) \\ & = o_p(1) \end{aligned}$$

In addition, we can apply the CS inequality to get

$$\begin{aligned} & \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{E}}_T - Q' \underline{E}_T \right) \right| \\ & \leq \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \sqrt{\left(\widehat{\underline{E}}_T - Q' \underline{E}_T \right)' \left(\widehat{\underline{E}}_T - Q' \underline{E}_T \right)} \\ & \leq \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \left\| \widehat{\underline{E}}_T - Q' \underline{E}_T \right\|_2 \\ & = o_p(1) \quad (\text{by Lemma D-18 and part (j) of Lemma D-15 in Appendix D}) \end{aligned}$$

and

$$\begin{aligned}
& \left| a' B_2' Q^{-1'} (\hat{\underline{E}}_T - Q' \underline{E}_T) \right| \\
& \leq \sqrt{a' B_2' Q^{-1'} Q^{-1} B_2 a} \sqrt{(\hat{\underline{E}}_T - Q' \underline{E}_T)' (\hat{\underline{E}}_T - Q' \underline{E}_T)} \\
& = \sqrt{a' B_2' Q^{-1'} Q^{-1} B_2 a} \left\| \hat{\underline{E}}_T - Q' \underline{E}_T \right\|_2 \\
& \leq \sqrt{\left[\lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \right]^{-1} \lambda_{\max} (B_2' B_2)} \left\| \hat{\underline{E}}_T - Q' \underline{E}_T \right\|_2 \\
& \leq \sqrt{C^*} \left\| \hat{\underline{E}}_T - Q' \underline{E}_T \right\|_2 \quad (\text{for some positive constant } C^* \text{ as shown in} \\
& \quad \text{expression (148) in Appendix D. See the proof of part (d) of Lemma D-17}) \\
& = o_p(1) \quad (\text{by part (j) of Lemma D-15})
\end{aligned}$$

Putting everything together and applying Slutsky's theorem, we then obtain

$$\begin{aligned}
& \left| a' \hat{Y}_{T+h} - a' (\beta_0 + B_1' \underline{Y}_T + B_2' \underline{E}_T) \right| \\
& \leq \left| a' (\hat{\beta}_0 - \beta_0) \right| + \left| a' (\hat{B}_1 - B_1)' \underline{Y}_T \right| + \left| a' (\hat{B}_2 - Q^{-1} B_2)' (\hat{\underline{E}}_T - Q' \underline{E}_T) \right| \\
& \quad + \left| a' B_2' Q^{-1'} (\hat{\underline{E}}_T - Q' \underline{E}_T) \right| + \left| a' (\hat{B}_2 - Q^{-1} B_2)' Q' \underline{E}_T \right| \\
& = o_p(1).
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that

$$\hat{Y}_{T+h} - (\beta_0 + B_1' \underline{Y}_T + B_2' \underline{E}_T) = o_p(1).$$

as required. \square

2 Appendix B: Supporting Lemmas Used in the Proof of Theorem 1

In this appendix, we first state and prove a number of lemmas which are used in the proof of Theorem 1.

Lemma B-1 (Weyl's inequality): Let A, B be real, symmetric $T \times T$ matrices and let the eigenvalues $\lambda_{(i)}(A)$, $\lambda_{(i)}(B)$, and $\lambda_{(i)}(A + B)$ be arranged in decreasing (or, more generally, non-

increasing) order, so that

$$\begin{aligned}\lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(T)}(A), \\ \lambda_{(1)}(B) &\geq \lambda_{(2)}(B) \geq \cdots \geq \lambda_{(T)}(B), \\ \lambda_{(1)}(A+B) &\geq \lambda_{(2)}(A+B) \geq \cdots \geq \lambda_{(T)}(A+B).\end{aligned}$$

Then, for each $j = 1, 2, \dots, T$, we have

$$\lambda_{(j)}(A) + \lambda_{(T)}(B) \leq \lambda_{(j)}(A+B) \leq \lambda_{(j)}(A) + \lambda_{(1)}(B).$$

Proof of Lemma B-1: This inequality is well-known, and its proof can be found in many linear algebra textbooks. See, for example, Theorem 4.3.1 and its proof on pages 181-182 of Horn and Johnson (1985). Hence, we shall not provide an explicit proof here. \square

Lemma B-2: Suppose that $\|\gamma\|_2^2 \rightarrow \infty$ as $N \rightarrow \infty$, and suppose that, given N ,

$$\{\zeta_{1,t,N}\} \equiv i.i.d.N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right) \text{ for } t = 1, \dots, T.$$

Let $\zeta_{1,N} = \begin{pmatrix} \zeta_{1,1,N} & \zeta_{1,2,N} & \cdots & \zeta_{1,T,N} \end{pmatrix}'$ and $\frac{A}{T \times T} = T^{-1} \|\gamma\|_2^2 \zeta_{1,N} \zeta_{1,N}'$. Then, as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$, we have

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

where $\lambda_{(1)}(A)$ denotes the largest eigenvalue of the matrix A .

Proof of Lemma B-2:

Note that, since $A = \|\gamma\|_2^2 \zeta_{1,N} \zeta_{1,N}' / T$, we can write its dual a_D as

$$a_D = \frac{1}{T} \|\gamma\|_2^2 \zeta_{1,N}' \zeta_{1,N}$$

Next, write

$$\frac{1}{T} \zeta_{1,N}' \zeta_{1,N} = \frac{1}{T} \sum_{t=1}^T \zeta_{1,t,N}^2 = \left(1 + \frac{1}{\|\gamma\|_2^2}\right) \frac{1}{T} \sum_{t=1}^T \left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1} \zeta_{1,t,N}^2$$

where, by assumption,

$$\{\zeta_{1,t,N}\} \equiv i.i.d.N \left(0, 1 + \frac{1}{\|\gamma\|_2^2} \right) \text{ for each } N.$$

This implies that

$$\begin{aligned} \left\{ \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1/2} \zeta_{1,t,N} \right\} &\equiv i.i.d.N(0, 1) \text{ and} \\ \{\mathcal{X}_{t,N}^*\} &\equiv i.i.d.\chi_1^2 \end{aligned}$$

where

$$\mathcal{X}_{t,N}^* = \left[\left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1/2} \zeta_{1,t,N} \right]^2 = \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1} \zeta_{1,t,N}^2$$

and where χ_1^2 denotes a chi-square random variable with one degree of freedom. Hence, by direct calculation, we get

$$\begin{aligned} &E \left(\frac{1}{T} \zeta'_{1,N} \zeta_{1,N} - \left[1 + \frac{1}{\|\gamma\|_2^2} \right] \right)^2 \\ &= E \left[\left(1 + \frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \sum_{t=1}^T \left(\left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1} \zeta_{1,t,N}^2 - 1 \right) \right]^2 \\ &= \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T E \left\{ \left[\left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1} \zeta_{1,t,N}^2 - 1 \right] \left[\left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1} \zeta_{1,s,N}^2 - 1 \right] \right\} \\ &= \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \frac{1}{T^2} \sum_{t=1}^T E \left\{ \left[\left(1 + \frac{1}{\|\gamma\|_2^2} \right)^{-1} \zeta_{1,t,N}^2 - 1 \right]^2 \right\} \\ &= \frac{2}{T} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \text{ (since } E[\chi_1^2] = 1 \text{ and } Var(\chi_1^2) = 2) \\ &= O\left(\frac{1}{T}\right) \end{aligned}$$

Applying Markov's inequality, we then obtain

$$\frac{1}{T} \zeta'_{1,N} \zeta_{1,N} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

Hence, as $N, T \rightarrow \infty$

$$\begin{aligned}
\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} &= \frac{a_D}{\|\gamma\|_2^2} \\
&= \left(\frac{1}{\|\gamma\|_2^2} \right) \frac{1}{T} \|\gamma\|_2^2 \zeta'_{1,N} \zeta_{1,N} \\
&= \frac{1}{T} \zeta'_{1,N} \zeta_{1,N} \\
&= 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

where the first equality above follows from the fact that $\lambda_{(1)}(A) = \lambda_{\max}(A) = \lambda_{\max}(a_D) = a_D$ given that a_D is a scalar. This proves Lemma B-2. \square

Lemma B-3: Let X_1, X_2, \dots, X_N be N independent T dimensional sub-Gaussian random vectors with zero mean vector and identity covariance matrix and the sub-Gaussian norms bounded by a constant C_0 . Then, for every $\tau \geq 0$, with probability at least

$$1 - 2 \exp\{-c\tau^2\},$$

one has

$$\begin{aligned}
\bar{w} - \max\{\delta, \delta^2\} &\leq \lambda_{(T)} \left(\frac{1}{N} \sum_{i=1}^N w_i X_i X_i' \right) \\
&\leq \lambda_{(1)} \left(\frac{1}{N} \sum_{i=1}^N w_i X_i X_i' \right) \\
&= \bar{w} + \max\{\delta, \delta^2\}
\end{aligned}$$

where

$$\delta = C \sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}}$$

for constants $C, c > 0$, depending on C_0 . Here, $|w_i|$ is bounded for all i and

$$\bar{w} = \frac{1}{N} \sum_{i=1}^N w_i.$$

Remark: Lemma B-3 is Lemma A.1 given in Appendix A of Wang and Fan (2017), and so we state this result here without proof. As discussed there, this lemma is an extension of the classical Davidson-Szarek bound. See Davidson and Szarek (2001) and Vershynin (2010) for additional

discussion.

Lemma B-4: Suppose that

$$\{\zeta_{i,t}\} \equiv i.i.d.N(0, 1) \text{ for } i = 2, \dots, N; t = 1, \dots, T$$

Let $\zeta_i = \begin{pmatrix} \zeta_{i,1} & \zeta_{i,2} & \dots & \zeta_{i,T} \end{pmatrix}'$. Also, let

$$B_{T \times T} = \frac{1}{T} \sum_{i=2}^N \zeta_i \zeta_i'$$

and let

$$\lambda_{(1)}(B) \geq \lambda_{(2)}(B) \geq \dots \geq \lambda_{(T)}(B)$$

denote the eigenvalues of B . Then, for $k = 1, \dots, T$;

$$\frac{T}{N-1} \lambda_{(k)}(B) = 1 + O_p\left(\sqrt{\frac{T}{N}}\right) = 1 + o_p(1),$$

as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$.

Proof of Lemma B-4:

Applying Lemma B-3 above for the case where $\tau = \sqrt{T}$ and where $w_i = 1$ for all i , we see that, with probability at least

$$1 - 2 \exp\{-c\tau^2\} = 1 - 2 \exp\{-cT\},$$

the following inequality holds for any $k \in \{1, \dots, T\}$

$$\begin{aligned} 1 - \max\{\delta, \delta^2\} &\leq \lambda_{(T)} \left(\frac{1}{N-1} \sum_{j=2}^N \zeta_j \zeta_j' \right) \\ &\leq \lambda_{(k)} \left(\frac{1}{N-1} \sum_{j=2}^N \zeta_j \zeta_j' \right) \\ &\leq \lambda_{(1)} \left(\frac{1}{N-1} \sum_{j=2}^N \zeta_j \zeta_j' \right) \\ &= 1 + \max\{\delta, \delta^2\}. \end{aligned}$$

Since in this case

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = (1 + C)\sqrt{\frac{T}{N}},$$

the above inequality relationship simplifies to

$$1 - (1 + C) \sqrt{\frac{T}{N}} \leq \lambda_{(k)} \left(\frac{1}{N-1} \sum_{j=2}^N \zeta_j \zeta'_j \right) \leq 1 + (1 + C) \sqrt{\frac{T}{N}}$$

or

$$1 - (1 + C) \sqrt{\frac{T}{N}} \leq \frac{T}{N-1} \lambda_{(k)} \left(\frac{1}{T} \sum_{j=2}^N \zeta_j \cdot \zeta'_j \right) = \frac{T}{N-1} \lambda_{(k)} (B) \leq 1 + (1 + C) \sqrt{\frac{T}{N}}$$

This shows that, as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$,

$$\frac{T}{N-1} \lambda_{(k)} (B) = 1 + O_p \left(\sqrt{\frac{T}{N}} \right) = 1 + o_p(1)$$

for $k = 1, \dots, T$. \square

Lemma B-5: Suppose that $\{\mathbf{W}_{2,t}\} \equiv i.i.d. N(0, I_{N-1})$. Now, let

$$\mathbf{W}'_2 = \begin{pmatrix} \mathbf{W}_{2,1} & \mathbf{W}_{2,2} & \cdots & \mathbf{W}_{2,T} \\ (N-1) \times 1 & (N-1) \times 1 & & (N-1) \times 1 \end{pmatrix}$$

and let

$$\tilde{\lambda}_{(2)} \geq \tilde{\lambda}_{(3)} \geq \cdots \geq \tilde{\lambda}_{(N)}$$

be the $N - 1$ eigenvalues of

$$\hat{\Sigma}_{\mathbf{W}_2} = \frac{\mathbf{W}'_2 \mathbf{W}_2}{T} = \frac{1}{T} \sum_{t=1}^T \mathbf{W}_{2,t} \mathbf{W}'_{2,t}.$$

Then, the following results hold as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$.

(a)

$$\tilde{\lambda}_{(j)} = 0 \text{ for } j = T + 2, \dots, N$$

(b)

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} = 1 + O_p \left(\sqrt{\frac{T}{N}} \right) = 1 + o_p(1).$$

Proof of Lemma B-5:

To show part (a), note that, by assumption, for N, T sufficiently large, we have $N - 1 > T$, so that $\widehat{\Sigma}_{\mathbf{W}_2} = \mathbf{W}_2' \mathbf{W}_2 / T$ is a $(N - 1) \times (N - 1)$ matrix with rank less than or equal to T , from which it follows trivially that

$$\widetilde{\lambda}_{(j)} = 0 \text{ for } j = T + 2, \dots, N.$$

Next, to show part (b), first write

$$\mathbf{W}_2 = \begin{pmatrix} \underline{W}_{2,1} & \underline{W}_{2,2} & \cdots & \underline{W}_{2,N-1} \end{pmatrix}_{T \times (N-1)}$$

so that $\underline{W}_{2,i}$ denotes the i^{th} column of \mathbf{W}_2 for $i = 1, \dots, N - 1$. Note that, by Sylvester's determinantal identity, the non-zero eigenvalues of $\widehat{\Sigma}_{\mathbf{W}_2} = \mathbf{W}_2' \mathbf{W}_2 / T$ (i.e., $\widetilde{\lambda}_{(2)}, \dots, \widetilde{\lambda}_{(T+1)}$) are the same as those of the dual matrix

$$\widehat{\Sigma}_{\mathbf{W}_2, D} = \frac{\mathbf{W}_2 \mathbf{W}_2'}{T} = \frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}'$$

Now, under our assumptions, $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\{0, 1\}$ for $t = 1, \dots, T$ and $i = 1, \dots, N - 1$ where $\mathbf{W}_{2,t,i}$ denotes the $(t, i)^{th}$ element of \mathbf{W}_2 . Applying Lemma B-3 above with $\tau = \sqrt{T}$, we see that, with probability at least

$$1 - 2 \exp\{-c\tau^2\} = 1 - 2 \exp\{-cT\},$$

the following inequality holds for any $j \in \{2, \dots, T + 1\}$

$$\begin{aligned} 1 - \max\{\delta, \delta^2\} &\leq \lambda_{(T)} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) \\ &\leq \lambda_{(j-1)} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) \\ &\leq \lambda_{(1)} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) \\ &= 1 + \max\{\delta, \delta^2\} \end{aligned}$$

where

$$\delta = C \sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = (1 + C) \sqrt{\frac{T}{N}}$$

Moreover, by our definition,

$$\tilde{\lambda}_{(j)} = \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}'_{2,i} \right),$$

so that, by multiplying and dividing by T , we see that

$$\begin{aligned} 1 - (1 + C) \sqrt{\frac{T}{N}} &\leq \frac{T}{N-1} \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}'_{2,i} \right) = \frac{T}{N-1} \tilde{\lambda}_{(j)} \\ &\leq 1 + (1 + C) \sqrt{\frac{T}{N}} \end{aligned}$$

Furthermore, since the above inequality relationship above holds for any $j \in \{2, \dots, T+1\}$, it must be that

$$1 - (1 + C) \sqrt{\frac{T}{N}} \leq \frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \leq 1 + (1 + C) \sqrt{\frac{T}{N}}$$

It follows that, as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$,

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} = 1 + O_p \left(\sqrt{\frac{T}{N}} \right) = 1 + o_p(1). \quad \square$$

Lemma B-6: Let X be a $N \times T$ random matrix, and let X_{it} be the $(i, t)^{th}$ element of X . Suppose that

$$\{X_{it}\} \equiv i.i.d. (0, 1)$$

and suppose that $E[X_{it}^4] < \infty$. Moreover, let

$$B = \frac{1}{N} X' X.$$

Then, as $N, T \rightarrow \infty$ such that $T/N \rightarrow c \in [0, 1)$,

$$\begin{aligned} \lambda_{\min}(B) &\xrightarrow{a.s.} (1 - \sqrt{c})^2, \\ \lambda_{\max}(B) &\xrightarrow{a.s.} (1 + \sqrt{c})^2. \end{aligned}$$

Remark: Lemma B-6 is a special case of Lemma 1 given in Shen, Shen, Zhu, and Marron (2016) and is a slightly extended version of Theorem 2 of Bai and Yin (1993). Hence, we state this result here without proof.

Lemma B-7: Suppose that $\{\mathbf{W}_{2,t}\} \equiv i.i.d.N(0, I_{N-1})$. Let

$$\tilde{\lambda}_{(2)} \geq \tilde{\lambda}_{(3)} \geq \cdots \geq \tilde{\lambda}_{(N)}$$

be the $N - 1$ eigenvalues of

$$\hat{\Sigma}_{\mathbf{W}_2} = \frac{\mathbf{W}_2' \mathbf{W}_2}{T} = \frac{1}{T} \sum_{t=1}^T \mathbf{W}_{2,t} \mathbf{W}_{2,t}'.$$

where $\mathbf{W}_2 = \begin{pmatrix} \mathbf{W}_{2,1} & \mathbf{W}_{2,2} & \cdots & \mathbf{W}_{2,T} \\ (N-1) \times 1 & (N-1) \times 1 & & (N-1) \times 1 \end{pmatrix}'$. Then, as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$,

$$\frac{T}{N-1} \tilde{\lambda}_{(j)} \xrightarrow{a.s.} 1 \text{ for any } j \in \{2, \dots, T+1\}.$$

In particular,

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} \xrightarrow{a.s.} 1$$

and

$$\max_{2 \leq j \leq T+1} \left| \frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right| \xrightarrow{a.s.} 0.$$

Proof of Lemma B-7:

To proceed, first define the dual matrix of $\hat{\Sigma}_{\mathbf{W}_2}$ given by

$$\hat{\Sigma}_{\mathbf{W}_2, D} = \frac{\mathbf{W}_2 \mathbf{W}_2'}{T} = \frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}'$$

where $\underline{W}_{2,i}$ denotes the i^{th} column of \mathbf{W}_2 for $i = 1, \dots, N-1$. Now, since $T/(N-1) \rightarrow 0$ and since $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\{0, 1\}$ for $t = 1, \dots, T$ and $i = 1, \dots, N-1$; it follows from applying Lemma B-6 that

$$\begin{aligned} \frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} &= \frac{T}{N-1} \max_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\hat{\Sigma}_{\mathbf{W}_2, D} \right) \\ &= \frac{T}{N-1} \max_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) \\ &= \max_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) \xrightarrow{a.s.} 1 \text{ as } N, T \rightarrow \infty \end{aligned} \quad (30)$$

and

$$\begin{aligned}
\frac{T}{N-1} \min_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} &= \frac{T}{N-1} \min_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\hat{\Sigma} \mathbf{w}_{2,D} \right) \\
&= \frac{T}{N-1} \min_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} W_{2,i} W'_{2,i} \right) \\
&= \min_{2 \leq j \leq T+1} \lambda_{(j-1)} \left(\frac{1}{N-1} \sum_{i=1}^{N-1} W_{2,i} W'_{2,i} \right) \xrightarrow{a.s.} 1 \text{ as } N, T \rightarrow \infty. \quad (31)
\end{aligned}$$

Expressions (30) and (31) then imply that, for any $j \in \{2, \dots, T+1\}$,

$$\frac{T}{N-1} \tilde{\lambda}_{(j)} \xrightarrow{a.s.} 1 \text{ as } N, T \rightarrow \infty,$$

so that

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \tilde{\lambda}_{(j)} = \frac{T}{N-1} \tilde{\lambda}_{(2)} \xrightarrow{a.s.} 1 \text{ as } N, T \rightarrow \infty.$$

In addition, note that, for any $j \in \{2, \dots, T+1\}$,

$$\frac{T}{N} \tilde{\lambda}_{(j)} = \frac{N-1}{N} \frac{T}{N-1} \tilde{\lambda}_{(j)} \xrightarrow{a.s.} 1 \text{ as } N, T \rightarrow \infty$$

from which it further follows that

$$\max_{2 \leq j \leq T+1} \left| \frac{T}{N} \tilde{\lambda}_{(j)} - 1 \right| \leq \left| \frac{T}{N} \tilde{\lambda}_{(1)} - 1 \right| + \left| \frac{T}{N} \tilde{\lambda}_{(T+1)} - 1 \right| \xrightarrow{a.s.} 0. \quad \square$$

Lemma B-8: Consider the simple factor model

$$\underset{N \times 1}{Z_t} = \underset{N \times 1}{\gamma} \underset{N \times 1}{f_t} + \underset{N \times 1}{u_t}, \quad t = 1, \dots, T;$$

where we assume that $\{u_t\} \equiv i.i.d.N(0, I_N)$, $\{f_t\} \equiv i.i.d.N(0, 1)$, and u_s and f_t are independent for all t, s . Let $\Sigma_Z = E[Z_t Z'_t]$; then, the eigenvalues of Σ_Z are given by

$$\lambda_{(1)} = \|\gamma\|_2^2 + 1 \text{ and } \lambda_{(j)} = 1 \text{ for } j = 2, \dots, N.$$

Moreover, let $\pi_{(1)} (N \times 1)$ be the eigenvector associated with the top eigenvalue $\lambda_{(1)}$; then,

$$\pi_{(1)} = \frac{\gamma}{\|\gamma\|_2}.$$

Proof of Lemma B-8: To show part (a), note first that

$$\begin{aligned}\Sigma_Z &= E [Z_t Z_t'] \\ &= E [(\gamma f_t + u_t) (\gamma' f_t + u_t')] \\ &= \gamma \gamma' + I_N\end{aligned}$$

Consider the determinantal equation

$$\begin{aligned}0 &= \det \{ \lambda I_N - (\gamma \gamma' + I_N) \} \\ &= \det \{ (\lambda - 1) I_N - \gamma \gamma' \} \\ &= \det \{ \kappa I_N - \gamma \gamma' \} \quad (\text{where } \kappa = \lambda - 1) \\ &= \kappa^N \det \{ I_N - \kappa^{-1} \gamma \gamma' \} \\ &= \kappa^N (1 - \kappa^{-1} \gamma' \gamma) \quad (\text{by Sylvester's determinantal theorem}) \\ &= \kappa^{N-1} (\kappa - \gamma' \gamma)\end{aligned}$$

so the roots of this equation are

$$\kappa_{(1)} = \gamma' \gamma = \|\gamma\|_2^2, \quad \kappa_{(2)} = 0, \dots, \kappa_{(N)} = 0$$

and, thus,

$$\lambda_{(1)} = \gamma' \gamma + 1 = \|\gamma\|_2^2 + 1, \quad \lambda_{(2)} = 1, \dots, \lambda_{(N)} = 1.$$

Next, note that

$$\begin{aligned}(\gamma \gamma' + I_N) \gamma &= \|\gamma\|_2^2 \gamma + \gamma \\ &= (\|\gamma\|_2^2 + 1) \gamma\end{aligned}$$

so that γ is an (unnormalized) eigenvector of the matrix $\gamma \gamma' + I_N$ associated with the eigenvalue $\lambda_{(1)} = \|\gamma\|_2^2 + 1$. It follows that we can take

$$\pi_{(1)} = \gamma / \|\gamma\|_2$$

to be the (normalized) eigenvector of $\Sigma_Z = E [Z_t Z_t'] = \gamma \gamma' + I_N$ associated with the eigenvalue

$$\lambda_{(1)} = \|\gamma\|_2^2 + 1. \quad \square$$

Lemma B-9: Let $A \in M_n$ be a Hermetian matrix, let r be an integer with $1 \leq r \leq n$, and let A_r denote any $r \times r$ principal submatrix of A (obtained by deleting $n - r$ rows and the corresponding columns of A). Let the eigenvalues of A and A_r be ordered as follows

$$\begin{aligned}\lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(n)}(A), \\ \lambda_{(1)}(A_r) &\geq \lambda_{(2)}(A_r) \geq \cdots \geq \lambda_{(r)}(A_r).\end{aligned}$$

Then, for each integer k such that $1 \leq k \leq r$, we have

$$\lambda_{(k)}(A) \geq \lambda_{(k)}(A_r) \geq \lambda_{(n-[r-k])}(A)$$

so that for $r = n - 1$, we have

$$\lambda_{(1)}(A) \geq \lambda_{(1)}(A_{n-1}) \geq \lambda_{(2)}(A) \geq \lambda_{(2)}(A_{n-1}) \geq \cdots \geq \lambda_{(n-1)}(A) \geq \lambda_{(n-1)}(A_{n-1}) \geq \lambda_{(n)}(A)$$

Proof of Lemma B-9: This result is essentially Theorem 4.3.15 in Horn and Johnson (1985), except that we use different notations here. A proof of this lemma can be obtained by a slight adaptation of the proof given in Horn and Johnson (1985) for Theorem 4.3.15 using our notations here.

Lemma B-10: Let

$$W_t = \sum_{j=1}^N \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1} \eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, \dots, N$; where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, \dots, N$; and where $\mathbf{e}_{j,N}$ is an $N \times 1$ elementary vector whose j^{th} component is 1 and all remaining components are 0. Suppose that $\{\eta_t\} \equiv i.i.d.N(0, I_N)$, $\{f_t\} \equiv i.i.d.N(0, 1)$, and f_t and η_s are independent for all t, s . In addition, suppose that the following assumptions hold.

(i) As $N \rightarrow \infty$

$$\|\gamma\|_2 \rightarrow \infty.$$

(ii) As $N, T \rightarrow \infty$

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty$$

for some κ such that $0 < \kappa < 1$.

Moreover, let $\hat{\lambda}_{(1)}$ denote the largest eigenvalue of the sample covariance matrix

$$\hat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \sum_{t=1}^T W_t W_t',$$

where $\mathbf{W}_{N \times T} = (W_1, \dots, W_T)$. Then, as $N, T \rightarrow \infty$ such that $T/N \rightarrow 0$; the largest sample eigenvalue $\hat{\lambda}_{(1)}$ satisfy

$$\frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} = c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \text{ for } 0 < \kappa < 1.$$

Proof of Lemma B-10:

Following Shen, Shen, Zhu, and Marron (2016), we shall study the sample eigenvalue properties via the dual matrix

$$\hat{\Sigma}_{\mathbf{W}, D}_{T \times T} = \frac{1}{T} \mathbf{W}' \mathbf{W}$$

which shares the same nonzero eigenvalues with the sample covariance matrix

$$\hat{\Sigma}_{\mathbf{W}}_{N \times N} = \frac{1}{T} \mathbf{W} \mathbf{W}'.$$

Define $\zeta_j = \left(\zeta_{j,1} \quad \zeta_{j,2} \quad \dots \quad \zeta_{j,T} \right)'$. Since $W_t = \sum_{j=1}^N \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N}$, we can write

$$\frac{1}{T} W_t' W_s = \sum_{k=1}^N \sum_{\ell=1}^N \ell_k^{1/2} \ell_\ell^{1/2} \zeta_{k,t} \zeta_{\ell,s} e_{k,N}^T e_{\ell,N} = \sum_{k=1}^N \ell_k \zeta_{k,t} \zeta_{k,s}$$

where

$$\begin{aligned} \ell_1 &= \|\gamma\|_2^2, \ell_2 = \dots = \ell_N = 1 \\ \zeta_{1,t} &= f_t + \frac{1}{\|\gamma\|_2} \eta_{1t}, \zeta_{2,t} = \eta_{2t}, \dots, \zeta_{N,t} = \eta_{Nt}. \end{aligned}$$

so that

$$\begin{aligned}
& \widehat{\Sigma}_{\mathbf{W},D} \\
&= \frac{1}{T} \mathbf{W}' \mathbf{W} = \frac{1}{T} \begin{pmatrix} W'_1 \\ W'_2 \\ \vdots \\ W'_T \end{pmatrix} \begin{pmatrix} W_1 & W_2 & \cdots & W_T \end{pmatrix} \\
&= \frac{1}{T} \begin{pmatrix} W'_1 W_1 & W'_1 W_2 & \cdots & W'_1 W_T \\ W'_2 W_1 & W'_2 W_2 & \cdots & W'_2 W_T \\ \vdots & \vdots & & \vdots \\ W'_T W_1 & W'_T W_2 & \cdots & W'_T W_T \end{pmatrix} = \frac{1}{T} \sum_{k=1}^N \ell_k \begin{pmatrix} \zeta_{k,1}^2 & \zeta_{k,1} \zeta_{k,2} & \cdots & \zeta_{k,1} \zeta_{k,T} \\ \zeta_{k,2} \zeta_{k,1} & \zeta_{k,2}^2 & \cdots & \zeta_{k,2} \zeta_{k,T} \\ \vdots & \vdots & & \vdots \\ \zeta_{k,T} \zeta_{k,1} & \zeta_{k,T} \zeta_{k,2} & \cdots & \zeta_{k,T}^2 \end{pmatrix} \\
&= \frac{1}{T} \sum_{k=1}^N \ell_k \begin{pmatrix} \zeta_{k,1} \\ \zeta_{k,2} \\ \vdots \\ \zeta_{k,T} \end{pmatrix} \begin{pmatrix} \zeta_{k,1} & \zeta_{k,2} & \cdots & \zeta_{k,T} \end{pmatrix} = \frac{1}{T} \sum_{k=1}^N \ell_k \zeta_k \zeta_k'.
\end{aligned}$$

which can be decomposed into sum of two matrices as follows

$$\widehat{\Sigma}_{\mathbf{W},D} = A + B$$

where

$$A_{T \times T} = \frac{1}{T} \ell_1 \zeta_1 \zeta_1' = \frac{1}{T} \|\gamma\|_2^2 \zeta_1 \zeta_1' \text{ and } B = \frac{1}{T} \sum_{k=2}^N \zeta_k \zeta_k'.$$

Next, we apply Weyl's inequality (given in Lemma B-1 above) to obtain

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} \leq \frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^2} = \frac{\lambda_{(1)}(A+B)}{\|\gamma\|_2^2} \leq \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2}$$

Moreover, as $N, T \rightarrow \infty$, $\|\gamma\|_2^2 \rightarrow \infty$ under Assumption (i); whereas Assumption (ii) states that

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty$$

from which it follows that

$$\begin{aligned}
\frac{N-1}{T \|\gamma\|_2^{2(1+\kappa)}} &= \frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} + O\left(\frac{1}{T \|\gamma\|_2^{2(1+\kappa)}}\right) \\
&= c + o\left(\frac{1}{\|\gamma\|_2^2}\right) + O\left(\frac{1}{T \|\gamma\|_2^{2(1+\kappa)}}\right) \\
&= c + o\left(\frac{1}{\|\gamma\|_2^2}\right)
\end{aligned} \tag{32}$$

In addition, recall that the result of Lemma B-4 shows that, as $N, T \rightarrow \infty$,

$$\frac{T\lambda_{(1)}(B)}{(N-1)} = 1 + O_p\left(\sqrt{\frac{T}{N}}\right) \text{ and } \frac{T\lambda_{(T)}(B)}{N-1} = 1 + O_p\left(\sqrt{\frac{T}{N}}\right)$$

Hence, applying Lemma B-4 and Assumption (ii); we obtain, as $N, T \rightarrow \infty$

$$\begin{aligned}
\frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2} &= \frac{(N-1)}{T \|\gamma\|_2^{2(1+\kappa)}} \frac{T\lambda_{(1)}(B)}{(N-1)} \\
&= \left[c + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left(1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right) \\
&= c + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
\frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(T)}(B)}{\lambda_{(1)} \|\gamma\|_2^2} &= \frac{(N-1)}{T \|\gamma\|_2^{2(1+\kappa)}} \frac{T\lambda_{(T)}(B)}{(N-1)} \\
&= \left[c + o\left(\frac{1}{\|\gamma\|_2^2}\right) \right] \left(1 + O_p\left(\sqrt{\frac{T}{N}}\right) \right) \\
&= c + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right)
\end{aligned}$$

which, together with the inequality relationship

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} \leq \frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^2} \leq \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2}$$

and the fact that, by Lemma B-2,

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

imply that

$$\begin{aligned}
\frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} &= \frac{1}{\|\gamma\|_2^{2\kappa}} + O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right) + O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa} \sqrt{T}}\right) + c \\
&\quad + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
\frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\lambda_{(1)}(B)}{\lambda_{(1)} \|\gamma\|_2^{2\kappa}} &= \frac{1}{\|\gamma\|_2^{2\kappa}} + O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right) + O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa} \sqrt{T}}\right) + c \\
&\quad + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)
\end{aligned}$$

so that

$$\frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} = \frac{1}{\|\gamma\|_2^{2\kappa}} \frac{\hat{\lambda}_{(1)}}{\|\gamma\|_2^2} = c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right). \quad \square$$

3 Appendix C: Supporting Lemmas Used in the Proofs of Theorems 2 and 3

Lemma C-1: Let a and θ be real numbers such that $a > 0$ and $\theta \geq 1$. Also, let G be a finite non-negative integer. Then,

$$\sum_{m=1}^{\infty} m^G \exp\{-am^\theta\} < \infty$$

Proof of Lemma C-1: By the integral test,

$$\sum_{m=1}^{\infty} m^G \exp\{-am^\theta\} < \infty \text{ for finite non-negative integer } G$$

if

$$\int_1^{\infty} x^G \exp\{-ax^\theta\} dx < \infty \text{ for finite non-negative integer } G$$

In addition, note that since, by assumption, $a > 0$ and $\theta \geq 1$, we have

$$\int_1^{\infty} x^G \exp\{-ax^\theta\} dx \leq \int_1^{\infty} x^G \exp\{-ax\} dx$$

We will first consider the case where $G = 0$. In this case, note that

$$\int_1^\infty x^0 \exp \{-ax\} dx = \int_1^\infty \exp \{-ax\} dx$$

Let $u = -ax$, so that $-\frac{du}{a} = dx$; and we have

$$\begin{aligned} \int_1^\infty \exp \{-ax\} dx &= -\frac{1}{a} \int_{-a}^{-\infty} \exp \{u\} du \\ &= \frac{1}{a} \int_{-\infty}^{-a} \exp \{u\} du \\ &= \frac{\exp \{-a\}}{a} \\ &< \infty \text{ for any } a > 0. \end{aligned} \tag{33}$$

Next, consider the case where G is an integer such that $G \geq 1$. Here, we will show that

$$\int_1^\infty x^G \exp \{-ax\} dx = \left[\frac{1}{a} + \sum_{k=1}^G \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{G-j}{a} \right) \right] \exp \{-a\} < \infty$$

using mathematical induction. To proceed, first consider the case where $G = 1$. Let

$$\begin{aligned} u &= x, \quad du = dx \\ dv &= \exp \{-ax\} dx, \quad v = -\frac{1}{a} \exp \{-ax\}; \end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned} \int_1^\infty x \exp \{-ax\} dx &= -\frac{x}{a} \exp \{-ax\} \Big|_1^\infty + \int_1^\infty \frac{1}{a} \exp \{-ax\} dx \\ &= \frac{1}{a} \exp \{-a\} - \frac{1}{a^2} \exp \{-ax\} \Big|_1^\infty \\ &= \frac{1}{a} \exp \{-a\} + \frac{1}{a^2} \exp \{-a\} \\ &= \left(\frac{1}{a} + \frac{1}{a^2} \right) \exp \{-a\} \\ &= \left\{ \frac{1}{a} + \sum_{k=1}^1 \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{1-j}{a} \right) \right\} \exp \{-a\} < \infty \end{aligned}$$

Next, for $G = 2$, let

$$\begin{aligned} u &= x^2, \quad du = 2x dx \\ dv &= \exp\{-ax\} dx, \quad v = -\frac{1}{a} \exp\{-ax\}; \end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned} \int_1^\infty x^2 \exp\{-ax\} dx &= -\frac{x^2}{a} \exp\{-ax\} \Big|_1^\infty + \frac{2}{a} \int_1^\infty x \exp\{-ax\} dx \\ &= \frac{1}{a} \exp\{-a\} + \frac{2}{a} \left(\frac{1}{a} + \frac{1}{a^2} \right) \exp\{-a\} \\ &= \frac{1}{a} \exp\{-a\} + 2 \left(\frac{1}{a^2} + \frac{1}{a^3} \right) \exp\{-a\} \\ &= \left(\frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} \right) \exp\{-a\} \\ &= \left[\frac{1}{a} + \sum_{k=1}^2 \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{2-j}{a} \right) \right] \exp\{-a\} \\ &< \infty \end{aligned}$$

Now, suppose that, for some $G \geq 2$,

$$\int_1^\infty x^{G-1} \exp\{-ax\} dx = \left[\frac{1}{a} + \sum_{k=1}^{G-1} \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{G-1-j}{a} \right) \right] \exp\{-a\};$$

then, let

$$\begin{aligned} u &= x^G, \quad du = Gx^{G-1} dx \\ dv &= \exp\{-ax\} dx, \quad v = -\frac{1}{a} \exp\{-ax\}; \end{aligned}$$

and, using integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x^G \exp\{-ax\} dx &= -\frac{x^G}{a} \exp\{-ax\} \Big|_1^\infty + \frac{G}{a} \int_1^\infty x^{G-1} \exp\{-ax\} dx \\
&= \frac{1}{a} \exp\{-a\} + \frac{G}{a} \left[\frac{1}{a} + \sum_{k=1}^{G-1} \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{G-1-j}{a} \right) \right] \exp\{-a\} \\
&= \frac{1}{a} \exp\{-a\} + \left[\frac{G}{a^2} + \sum_{k=1}^{G-1} \frac{1}{a} \frac{G}{a} \left(\prod_{j=0}^{k-1} \frac{G-(j+1)}{a} \right) \right] \exp\{-a\} \\
&= \left\{ \frac{1}{a} + \frac{G}{a^2} + \frac{1}{a} \frac{G}{a} \left(\frac{G-1}{a} \right) + \frac{1}{a} \frac{G}{a} \left(\frac{G-1}{a} \right) \left(\frac{G-2}{a} \right) \right. \\
&\quad \left. + \cdots + \frac{1}{a} \frac{G}{a} \left(\frac{G-1}{a} \right) \left(\frac{G-2}{a} \right) \times \cdots \times \left(\frac{1}{a} \right) \right\} \exp\{-a\} \\
&= \left\{ \frac{1}{a} + \sum_{k=1}^G \frac{1}{a} \left(\prod_{j=0}^{k-1} \frac{G-j}{a} \right) \right\} \exp\{-a\} \\
&< \infty.
\end{aligned} \tag{34}$$

In view of expressions (33) and (34), it then follows by the integral test for series convergence that

$$\sum_{m=1}^{\infty} m^G \exp\{-am^\theta\} < \infty$$

for any finite non-negative integer G and for any constants a and θ such that $a > 0$ and $\theta \geq 1$. \square

Lemma C-2: Let $\{V_t\}$ be a sequence of random variables (or random vectors) defined on some probability space (Ω, \mathcal{F}, P) , and let

$$X_t = g(V_t, V_{t-1}, \dots, V_{t-\varkappa})$$

be a measurable function for some finite positive integer \varkappa . In addition, define $\mathcal{G}_{-\infty}^t = \sigma(\dots, X_{t-1}, X_t)$, $\mathcal{G}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, \dots)$, $\mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-1}, V_t)$, and $\mathcal{F}_{t+m-\varkappa}^\infty = \sigma(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, \dots)$. Under this setting, the following results hold.

(a) Let

$$\begin{aligned}
\beta_{V, m-\varkappa} &= \sup_t \beta(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty) = \sup_t E \left[\sup \{ |P(B|\mathcal{F}_{-\infty}^t) - P(B)| : B \in \mathcal{F}_{t+m-\varkappa}^\infty \} \right], \\
\beta_{X, m} &= \sup_t \beta(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty) = \sup_t E \left[\sup \{ |P(H|\mathcal{G}_{-\infty}^t) - P(H)| : H \in \mathcal{G}_{t+m}^\infty \} \right].
\end{aligned}$$

If $\{V_t\}$ is β -mixing with

$$\beta_{V,m-\varkappa} \leq \overline{C}_1 \exp \{-C_2 (m - \varkappa)\}$$

for all $m \geq \varkappa$ and for some positive constants \overline{C}_1 and C_2 ; then X_t is also β -mixing with β -mixing coefficient satisfying

$$\beta_{X,m} \leq C_1 \exp \{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp \{C_2 \varkappa\}$.

(b) Let

$$\begin{aligned} \alpha_{V,m-\varkappa} &= \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty) = \sup_t \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m-\varkappa}^\infty} |P(G \cap H) - P(G)P(H)|, \\ \alpha_{X,m} &= \sup_t \alpha(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty) = \sup_t \sup_{G \in \mathcal{G}_{-\infty}^t, H \in \mathcal{G}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)| \end{aligned}$$

If $\{V_t\}$ is α -mixing with

$$\alpha_{V,m-\varkappa} \leq \overline{C}_1 \exp \{-C_2 (m - \varkappa)\}$$

for all $m \geq \varkappa$ and for some positive constants \overline{C}_1 and C_2 ; then X_t is also α -mixing with α -mixing coefficient satisfying

$$\alpha_{X,m} \leq C_1 \exp \{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp \{C_2 \varkappa\}$.

Proof of Lemma C-2:

To show part (a), note first that it is well known that

$$\begin{aligned} \beta_{X,m} &= \sup_t E \left[\sup \{ |P(H|\mathcal{G}_{-\infty}^t) - P(H)| : H \in \mathcal{G}_{t+m}^\infty \} \right] \\ &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(G_i \cap H_j) - P(G_i)P(H_j)| \right\} \end{aligned}$$

where the second supremum on the last line above is taken over all pairs of finite partitions $\{G_1, \dots, G_I\}$ and $\{H_1, \dots, H_J\}$ of Ω such that $G_i \in \mathcal{G}_{-\infty}^t$ for $i = 1, \dots, I$ and $H_j \in \mathcal{G}_{t+m}^\infty$ for

$j = 1, \dots, J$. See, for example, Borovkova, Burton, and Dehling (2001). Similarly,

$$\begin{aligned}\beta_{V,m-\varkappa} &= \sup_t E \left[\sup \left\{ |P(B|\mathcal{F}_{-\infty}^t) - P(B)| : B \in \mathcal{F}_{t+m-\varkappa}^\infty \right\} \right] \\ &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^L \sum_{j=1}^M |P(A_i \cap B_j) - P(A_i)P(B_j)| \right\}\end{aligned}$$

where, similar to the definition of $\beta_{X,m}$, the second supremum on the last line above is taken over all pairs of finite partitions $\{A_1, \dots, A_L\}$ and $\{B_1, \dots, B_M\}$ of Ω such that $A_i \in \mathcal{F}_{-\infty}^t$ for $i = 1, \dots, L$ and $B_j \in \mathcal{F}_{t+m-\varkappa}^\infty$ for $j = 1, \dots, M$. Moreover, since X_t is measurable on any σ -field on which $V_t, V_{t-1}, \dots, V_{t-\varkappa}$ are measurable, we also have

$$\mathcal{G}_{-\infty}^t = \sigma(\dots, X_{t-1}, X_t) \subseteq \sigma(\dots, V_{t-1}, V_t) = \mathcal{F}_{-\infty}^t$$

and

$$\mathcal{G}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, \dots) \subseteq \sigma(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, \dots) = \mathcal{F}_{t+m-\varkappa}^\infty.$$

It, thus, follows that, for all $m \geq \varkappa$,

$$\begin{aligned}\beta_{X,m} &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(G_i \cap H_j) - P(G_i)P(H_j)| \right\} \\ &\leq \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^L \sum_{j=1}^M |P(A_i \cap B_j) - P(A_i)P(B_j)| \right\} \\ &= \beta_{V,m-\varkappa} \\ &\leq \overline{C}_1 \exp\{-C_2(m-\varkappa)\} \\ &= \overline{C}_1 \exp\{C_2\varkappa\} \exp\{-C_2m\} \\ &\leq C_1 \exp\{-C_2m\}\end{aligned}$$

for some positive constant $C_1 \geq \overline{C}_1 \exp\{C_2\varkappa\}$ which exists given that \varkappa is fixed. Moreover, we have

$$\beta_{X,m} \leq C_1 \exp\{-C_2m\} \rightarrow 0 \text{ as } m \rightarrow \infty,$$

which establishes the required result for part (a).

Part (b) can be shown in a manner similar to part (a), so to avoid redundancy, we do not include an explicit proof here. \square

Remark: Note that part (b) of Lemma C-2 is similar to a result given in Theorem 14.1 of Davidson

(1994) but adapted to suit our situation and our notations here. Indeed, parts (a) and (b) of this lemma are both well-known results in the probability literature. We have chosen to state these results explicitly here only so that we can more easily refer to them in the proofs of some of our other results.

Lemma C-3: Let $\{X_t\}$ be a sequence of random variables that is α -mixing. Let $p > 1$ and $r \geq p/(p-1)$, and let $q = \max\{p, r\}$. Suppose that, for all t ,

$$\|X_t\|_q = (E|X_t|^q)^{\frac{1}{q}} < \infty$$

Then,

$$|Cov(X_t, X_{t+m})| \leq 2 \left(2^{1-1/p} + 1\right) \alpha_m^{1-1/p-1/r} \|X_t\|_p \|X_{t+m}\|_r$$

where

$$\alpha_m = \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) = \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)|.$$

Remark: This is Corollary 14.3 of Davidson (1994). For a proof, see pages 212-213 of Davidson (1994).

Lemma C-4: Suppose that Assumption 3-3 hold. Let $\tau_1 = \lfloor T_0^{\alpha_1} \rfloor$, where $1 > \alpha_1 > 0$ and $T_0 = T - p + 1$. Then,

(a)

$$\frac{1}{\tau_1^2} \sum_{\substack{g, h = (r-1)\tau + p \\ g \leq h}}^{(r-1)\tau + \tau_1 + p - 1} |E[u_{ig}u_{ih}]| = O\left(\frac{1}{\tau_1}\right)$$

(b)

$$\frac{1}{\tau_1^3} \sum_{\substack{h, v, w = (r-1)\tau + p \\ h \leq v \leq w}}^{(r-1)\tau + \tau_1 + p - 1} |E(u_{ih}u_{iv}u_{iw})| = O\left(\frac{1}{\tau_1^2}\right)$$

(c)

$$\frac{1}{\tau_1^4} \sum_{\substack{g, h, v, w = (r-1)\tau + p \\ g \leq h \leq v \leq w}}^{(r-1)\tau + \tau_1 + p - 1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| = O\left(\frac{1}{\tau_1^2}\right)$$

Proof of Lemma C-4:

To show part (a), first write

$$\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| = \frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] + \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \quad (35)$$

Consider now the first term on the right-hand side of expression (35). Note that, trivially, by Assumption 3-3(b), there exists a positive constant C such that

$$\frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] \leq \frac{C}{\tau_1} = O\left(\frac{1}{\tau_1}\right) \quad (36)$$

For the second term on the right-hand side of expression (35), note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}.$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned} & \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\ & \leq \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(h-g)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \end{aligned}$$

Next, by application of Liapunov's inequality, we have that there exists some positive constant \overline{C} such that

$$\begin{aligned} \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}} & \leq \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \\ & \leq \left(\sup_t E|u_{it}|^6\right)^{\frac{1}{3}} \\ & = \overline{C}^{\frac{1}{3}} < \infty \quad (\text{by Assumption 3-3(b)}) \end{aligned}$$

Moreover, let $\varrho = h - g$, so that $h = g + \varrho$. Using these notations and the boundedness of $\left(E |u_{ig}|^6\right)^{\frac{1}{6}} \left(E |u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(h-g)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} \left(E |u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
& \leq \frac{\overline{C}^{\frac{1}{3}}}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp\{-a_2(h-g)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{30}\varrho\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) \overline{C}^{\frac{1}{3}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
& \leq \frac{C^*}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho=1}^{\infty} \exp\left\{-\frac{a_2}{30}\varrho\right\} \\
& = \frac{C^*}{\tau_1} \sum_{\varrho_1=1}^{\infty} \exp\left\{-\frac{a_2}{30}\varrho\right\} \\
& = O\left(\frac{1}{\tau_1}\right) \quad (\text{given Lemma C-1}) \tag{37}
\end{aligned}$$

It follows from expressions (35), (36), and (37) that

$$\begin{aligned}
\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| &= \frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] + \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\
&= O\left(\frac{1}{\tau_1}\right) + O\left(\frac{1}{\tau_1}\right) \\
&= O\left(\frac{1}{\tau_1}\right).
\end{aligned}$$

To show part (b), first write

$$\begin{aligned}
\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| &= \frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E|u_{ih}|^3 + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&+ \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \tag{38}
\end{aligned}$$

For the first term on the right-hand side of expression (38) above, note that, trivially, we can apply Assumption 3-3(b) to obtain

$$\frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E|u_{ih}|^3 \leq \frac{C}{\tau_1^2} = O\left(\frac{1}{\tau_1^2}\right). \tag{39}$$

Next, for the second term on the right-hand side of expression (38) above, we can apply Lemma C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&\leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) \left[a_1 \exp\left\{-a_2(v-h)^\theta\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}}
\end{aligned}$$

Next, by application of Hölder's inequality, we have

$$\begin{aligned}
\left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(\left(E|u_{iv}|^{\frac{5}{2}}\right)^{\frac{1}{2}} \left(E|u_{iw}|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \\
&= \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E|u_{iw}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \\
&\leq \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}|^6\right)^{\frac{1}{6}} \left(E|u_{iw}|^6\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= \overline{C}^{\frac{1}{2}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$, so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using

these notations and the boundedness of $\left(E |u_{ih}|^6\right)^{\frac{1}{6}} \left(E |u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
& \leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E |u_{ih}|^6\right)^{\frac{1}{6}} \left(E |u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
& \leq \frac{\overline{C}_2^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp\{-a_2(v-h)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) \overline{C}_2^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
& \leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = \frac{C^*}{\tau_1^2} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma C-1}) \tag{40}
\end{aligned}$$

Similarly, for the third term on the right-hand side of expression (38), we can apply Lemma

C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
& \leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1 \right) [a_1 \exp \{-a_2(w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \left(E|u_{ih}u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}}
\end{aligned}$$

Next, by applying Hölder's inequality, we have

$$\begin{aligned}
\left(E|u_{ih}u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} & \leq \left(\left(E|u_{ih}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \left(E|u_{iv}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \right)^{\frac{4}{5}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} \\
& = \left(E|u_{ih}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \left(E|u_{iv}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} \\
& \leq \left(E|u_{ih}|^6 \right)^{\frac{1}{6}} \left(E|u_{iv}|^6 \right)^{\frac{1}{6}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} \\
& \quad \text{(by Liapunov's inequality)} \\
& = \overline{C}^{\frac{1}{2}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$, so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E|u_{ih}u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}}$ as shown above, we can further

write

$$\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{(r-1)\tau+\tau_1+p-1 \\ h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}} |E(u_{ih}u_{iv}u_{iw})| \\
& \leq \frac{1}{\tau_1^3} \sum_{\substack{(r-1)\tau+\tau_1+p-1 \\ h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}} 2 \left(2^{1-\frac{1}{6}} + 1 \right) [a_1 \exp \{-a_2(w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \left(E |u_{ih}u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \\
& \leq \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{(r-1)\tau+\tau_1+p-1 \\ h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}} 2 \left(2^{\frac{5}{6}} + 1 \right) [a_1 \exp \{-a_2(w-v)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{\tau_1^3} \sum_{\substack{(r-1)\tau+\tau_1+p-1 \\ h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1 \right) \overline{C}^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
& = \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} (\varrho_2 + 1) \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
& = \frac{C^*}{\tau_1^2} \left[\sum_{\varrho_2=1}^{\infty} \varrho_2 \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} + \sum_{\varrho_2=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \right] \\
& = O \left(\frac{1}{\tau_1^2} \right) \quad (\text{given Lemma C-1})
\end{aligned} \tag{41}$$

It follows from expressions (38), (39), (40), and (41) that

$$\begin{aligned}
\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| &= \frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E|u_{ih}|^3 + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&\quad + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&= O\left(\frac{1}{\tau_1^2}\right) + O\left(\frac{1}{\tau_1^2}\right) + O\left(\frac{1}{\tau_1^2}\right) \\
&= O\left(\frac{1}{\tau_1^2}\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih}) + E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih}) + E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
\leq & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \tag{42}
\end{aligned}$$

For the first term on the right-hand side of expression (42) above, note that, trivially, by Jensen's

inequality and Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| &\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} E[|u_{ig}u_{ih}^3|] \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} \sqrt{E|u_{ig}|^2} \sqrt{E|u_{ih}|^6} \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \sqrt{E|u_{ih}|^6} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{\overline{C}^{\frac{2}{3}} \tau_1^2}{\tau_1^4} \quad \text{(by Assumption 3-3(b))} \\
&= O\left(\frac{1}{\tau_1^2}\right) \tag{43}
\end{aligned}$$

Next, for the second term on the right-hand side of expression (42), we can apply Lemma C-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
&\quad \left. \times \left(E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$\begin{aligned}
E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}|^{\frac{4}{3}} &\leq \left[E(u_{ig}u_{ih} - E(u_{ig}u_{ih}))^{\frac{12}{7}}\right]^{\frac{7}{9}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}} \left(E|u_{ig}u_{ih}|^{\frac{12}{7}} + |E[u_{ig}u_{ih}]|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}} \left(E|u_{ig}u_{ih}|^{\frac{12}{7}} + E|u_{ig}u_{ih}|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}} E|u_{ig}u_{ih}|^{\frac{12}{7}}\right]^{\frac{7}{9}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[\left(E|u_{ig}|^{\frac{24}{7}}\right)^{\frac{1}{2}} \left(E|u_{ih}|^{\frac{24}{7}}\right)^{\frac{1}{2}}\right]^{\frac{7}{9}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}} \left[\left(E|u_{ig}|^{\frac{24}{7}}\right)^{\frac{7}{24}} \left(E|u_{ih}|^{\frac{24}{7}}\right)^{\frac{7}{24}}\right]^{\frac{4}{3}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[\left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}}\right]^{\frac{4}{3}} \left[E|u_{iv}|^6\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} (\overline{C})^{\frac{2}{9}} (\overline{C})^{\frac{2}{9}} (\overline{C})^{\frac{2}{9}} \quad \text{(by Assumption 3-3(b))} \\
&= 2^{\frac{4}{3}} \overline{C}^{\frac{2}{3}}
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}|^{\frac{4}{3}}$ as shown above, we

can further write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \quad \left. \times \left(E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}]^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(E|u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{\frac{1}{12}} \left(2^{\frac{4}{3}} \overline{C}^{\frac{2}{3}} \right)^{\frac{3}{4}} (\overline{C})^{\frac{1}{6}} \\
& \leq \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g < h < v < w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{4}} + 1 \right) \overline{C}^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2-1} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& \leq \frac{C^*}{\tau_1^2} \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma C-1}) \tag{44}
\end{aligned}$$

Similarly, for the third term on the right-hand side of expression (42) above, we can apply

Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \left. \times \left(E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{iv}u_{iw}|^3 \right)^{\frac{1}{3}} \right\}
\end{aligned}$$

Next, applications of Hölder's inequality yield

$$\begin{aligned}
E|u_{iv}u_{iw}|^3 & \leq \left(E|u_{iv}|^6 \right)^{\frac{1}{2}} \left(E|u_{iw}|^6 \right)^{\frac{1}{2}} \\
& \leq (\overline{C})^{\frac{1}{2}} (\overline{C})^{\frac{1}{2}} \quad (\text{by Assumption 3-3(b)}) \\
& = \overline{C} < \infty
\end{aligned}$$

and

$$\begin{aligned}
E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 & \leq 2 \left(E|u_{ig}u_{ih}|^2 + E|u_{ig}u_{ih}|^2 \right) \\
& \quad (\text{by Loève's } c_r \text{ inequality and Jensen's inequality}) \\
& = 4E|u_{ig}u_{ih}|^2 \\
& \leq 4 \left[\left(E|u_{ig}|^4 \right)^{\frac{1}{4}} \left(E|u_{ih}|^4 \right)^{\frac{1}{4}} \right]^2 \\
& \leq 4 \left[\left(E|u_{ig}|^6 \right)^{\frac{1}{6}} \left(E|u_{ih}|^6 \right)^{\frac{1}{6}} \right]^2 \quad (\text{by Liapunov's inequality}) \\
& \leq 4 \left(\sup_{i,t} E|u_{it}|^6 \right)^{\frac{2}{3}} \\
& \leq 4(\overline{C})^{\frac{2}{3}} < \infty \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E|u_{iv}u_{iw}|^3$ and $E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2$ as shown above,

we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \left. \times \left(E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{iv}u_{iw}|^3 \right)^{\frac{1}{3}} \right\} \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{\frac{1}{6}} \left(4\overline{C}^{\frac{2}{3}} \right)^{\frac{1}{2}} (\overline{C})^{\frac{1}{3}} \\
& \leq \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{2}} + 1 \right) \overline{C}^{\frac{2}{3}} a_1^{\frac{1}{6}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& = \frac{C^*}{\tau_1^2} \sum_{\rho_1=1}^{\infty} (\varrho_1 + 1) \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma C-1}) \tag{45}
\end{aligned}$$

Finally, consider the fourth term on the right-hand side of expression (42) above. For this term,

we apply the result given in part (a) to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
& \leq \left(\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| \right) \left(\frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \right) \\
& = O\left(\frac{1}{\tau_1^2}\right). \tag{46}
\end{aligned}$$

It follows from expressions (42)-(46) that

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \quad + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \quad + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
& = O\left(\frac{1}{\tau_1^2}\right). \quad \square
\end{aligned}$$

Lemma C-5: Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, there exists a positive constant \overline{C} such that

$$E \|\underline{W}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t$$

and, thus,

$$E \|\underline{Y}_t\|_2^6 \leq \overline{C} < \infty \text{ and } E \|\underline{E}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t,$$

where

$$\underline{Y}_t = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \text{ and } \underline{F}_t = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}.$$

Proof of Lemma C-5:

To proceed, note that, given Assumption 3-1, we can write the vector moving-average (VMA) representation of the companion form of the FAVAR model as

$$\begin{aligned} \underline{W}_t &= (I_{(d+K)p} - A)^{-1} \alpha + \sum_{j=0}^{\infty} A^j E_{t-j} \\ &= (I_{(d+K)p} - A)^{-1} J'_{d+K} J_{d+K} \alpha + \sum_{j=0}^{\infty} A^j J'_{d+K} J_{d+K} E_{t-j} \\ &= (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j}, \end{aligned} \quad (47)$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \quad E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

$$J_{d+K} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.$$

By the triangle inequality,

$$\|\underline{W}_t\|_2 \leq \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2$$

Moreover, using the inequality $\left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r$ for $r \geq 1$, we get

$$\|\underline{W}_t\|_2^6 \leq 2^5 \left(\left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \right)$$

so that

$$E \|\underline{W}_t\|_2^6 \leq 32 \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \quad (48)$$

Focusing first on the first term on the right-hand side of the inequality (48), we note that

$$\begin{aligned} \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 &= \left(\mu' J_{d+K} (I_{(d+K)p} - A)^{-1'} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right)^3 \\ &= \left(\mu' J_{d+K} \left[(I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right]^{-1} J'_{d+K} \mu \right)^3 \\ &\leq \left(\frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' J_{d+K} J'_{d+K} \mu)^3 \\ &= \left(\frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' \mu)^3 \end{aligned}$$

Now, by Assumption 3-7, there exists a constant $\underline{C} > 0$ such that

$$\begin{aligned} \lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\} &= \lambda_{\min} \left\{ (I_{(d+K)p} - A)' (I_{(d+K)p} - A) \right\} \\ &= \sigma_{\min}^2 (I_{(d+K)p} - A) \\ &\geq \underline{C} \lambda_{\min}^2 (I_{(d+K)p} - A) \\ &\geq \underline{C} [1 - \phi_{\max}]^2 \\ &> 0 \end{aligned}$$

where $\phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. It follows by Assumption 3-5 that, there exists a positive

constant \overline{C}_1 such that

$$\begin{aligned} \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 &\leq \left(\frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' \mu)^3 \\ &\leq \frac{\|\mu\|_2^6}{\underline{C}^3 [1 - \phi_{\max}]^6} \leq \overline{C}_1 < \infty. \end{aligned}$$

To show the boundedness of the second term on the right-hand side of the inequality (48), let $e_{g,(d+K)p}$ be a $(d+K)p \times 1$ elementary vector whose g^{th} component is 1 and all other components are 0 for $g \in \{1, 2, \dots, (d+K)p\}$, and note that

$$\begin{aligned} \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^2 &= \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^2 \\ &= \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} (A')^k e_{g,(d+K)p} \end{aligned}$$

from which we obtain, by applying the inequality $\left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r$ for $r \geq 1$

$$\begin{aligned} &\left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\ &= \left[\sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right) \right]^{2 \cdot 3} \\ &\leq [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^6 \\ &= [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} (A')^k e_{g,(d+K)p} \right. \\ &\quad \times e'_{g,(d+K)p} A^i J'_{d+K} \varepsilon_{t-i} \varepsilon'_{t-\ell} J_{d+K} (A')^{\ell} e_{g,(d+K)p} e'_{g,(d+K)p} A^r J'_{d+K} \varepsilon_{t-r} \varepsilon'_{t-s} J_{d+K} (A')^s e_{g,(d+K)p} \left. \right\} \end{aligned}$$

Hence,

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
\leq & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^6 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{3} \left(\sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^3 \right)^2 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{2} \binom{4}{2} \left(\sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^2 \right)^3 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{4} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^4 \sum_{k=0}^{\infty} E \left| e'_{g,(d+K)p} A^k J'_{d+K} \varepsilon_{t-k} \right|^2 \\
= & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^6 \\
& + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^3 \right)^2 \\
& + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^2 \right)^3 \\
& + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^4 \sum_{k=0}^{\infty} E \left| e'_{g,(d+K)p} A^k J'_{d+K} \varepsilon_{t-k} \right|^2
\end{aligned}$$

Next, applying the Cauchy-Schwarz inequality, we further obtain

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
& \leq [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& \quad + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& \quad + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& \quad + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \quad \quad \left. \times \sum_{k=0}^{\infty} \left[e'_{g,(d+K)p} A^k J'_{d+K} J_{d+K} (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\} \\
& = [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& \quad + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& \quad + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& \quad + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \quad \quad \left. \times \sum_{k=0}^{\infty} \left[e'_{g,(d+K)p} A^k (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\}
\end{aligned}$$

In addition, observe that, for every $g \in \{1, 2, \dots, (d+K)p\}$

$$\begin{aligned}
& e_{g,(d+K)p}' A^j (A^j)' e_{g,(d+K)p} \\
& \leq \lambda_{\max} \left\{ A^j (A^j)' \right\} \\
& = \lambda_{\max} \left\{ (A^j)' A^j \right\} \\
& = \sigma_{\max}^2 (A^j) \\
& \leq C \max \left\{ |\lambda_{\max} (A^j)|^2, |\lambda_{\min} (A^j)|^2 \right\} \quad (\text{by Assumption 3-7}) \\
& = C \max \left\{ |\lambda_{\max} (A)|^{2j}, |\lambda_{\min} (A)|^{2j} \right\} \\
& = C \phi_{\max}^{2j}
\end{aligned}$$

where $\phi_{\max} = \max \{|\lambda_{\max} (A)|, |\lambda_{\min} (A)|\}$ and where $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. Now, in light of Assumption 3-2(b), we can let $C \geq 1$ be a constant such that $E \|\varepsilon_{t-j}\|_2^6 \leq C < \infty$, so that, by Liapunov's inequality,

$$\begin{aligned}
E \|\varepsilon_{t-j}\|_2^2 & \leq \left(E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{1}{3}} \leq C^{\frac{1}{3}}, \quad E \|\varepsilon_{t-j}\|_2^3 \leq \left(E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}, \\
E \|\varepsilon_{t-j}\|_2^4 & \leq \left(E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{2}{3}} \leq C^{\frac{2}{3}},
\end{aligned}$$

and, thus,

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
& \leq [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& \quad + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& \quad + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& \quad + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \quad \left. \times \sum_{k=0}^{\infty} \left[e'_{g,(d+K)p} A^k (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\} \\
& \leq C [(d+K)p]^2 \left\{ \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \phi_{\max}^{6j} + 20 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \phi_{\max}^{3j} \right)^2 + 90 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \phi_{\max}^{2j} \right)^3 \right. \\
& \quad \left. + 15 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \phi_{\max}^{4j} \right) \left(\sum_{k=0}^{\infty} \phi_{\max}^{2k} \right) \right\} \\
& \leq C [(d+K)p]^3 \\
& \quad \times \left\{ \frac{1}{1 - \phi_{\max}^6} + 20 \left(\frac{1}{1 - \phi_{\max}^3} \right)^2 + 90 \left(\frac{1}{1 - \phi_{\max}^2} \right)^3 + 15 \left(\frac{1}{1 - \phi_{\max}^4} \right) \left(\frac{1}{1 - \phi_{\max}^2} \right) \right\} \\
& \leq \overline{C}_2 < \infty
\end{aligned}$$

for some constant such that

$$\begin{aligned}
& \overline{C}_2 \\
& \geq C [(d+K)p]^3 \\
& \quad \times \left\{ \frac{1}{1 - \phi_{\max}^6} + 20 \left(\frac{1}{1 - \phi_{\max}^3} \right)^2 + 90 \left(\frac{1}{1 - \phi_{\max}^2} \right)^3 + 15 \left(\frac{1}{1 - \phi_{\max}^4} \right) \left(\frac{1}{1 - \phi_{\max}^2} \right) \right\}.
\end{aligned}$$

Putting everything together, we see that

$$\begin{aligned}
E \|\underline{W}_t\|_2^6 &\leq 32 \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
&\leq 32 (\overline{C}_1 + \overline{C}_2) \\
&\leq \overline{C} < \infty
\end{aligned}$$

for a constant \overline{C} such that $0 < 32 (\overline{C}_1 + \overline{C}_2) \leq \overline{C} < \infty$.

In addition, define $\mathcal{P}_{(d+K)p}$ to be the $(d+K)p \times (d+K)p$ permutation matrix such that

$$\mathcal{P}_{(d+K)p} \underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}; \tag{49}$$

and let $S'_d = \begin{pmatrix} I_{dp} & 0 \\ 0 & I_{Kp} \end{pmatrix}$ and $S'_K = \begin{pmatrix} 0 & I_{Kp} \\ I_{dp} & 0 \end{pmatrix}$. Note that

$$\begin{aligned}
S'_d \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & 0 \\ 0 & I_{Kp} \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{Y}_t, \\
S'_K \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} 0 & I_{Kp} \\ I_{dp} & 0 \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{F}_t.
\end{aligned}$$

so that

$$\begin{aligned}
\|\underline{Y}_t\|_2 &\leq \|S'_d\|_2 \|\mathcal{P}_{(d+K)p}\|_2 \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(S'_d S'_d)} \sqrt{\lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(S'_d S_d)} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(I_{dp})} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \|\underline{W}_t\|_2
\end{aligned}$$

and

$$\begin{aligned}
\|\underline{E}_t\|_2 &\leq \|S'_K\|_2 \|\mathcal{P}_{(d+K)p}\|_2 \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(S_K S'_K)} \sqrt{\lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(S'_K S_K)} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}(I_{Kp})} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\
&= \|\underline{W}_t\|_2
\end{aligned}$$

It further follows that

$$E \|\underline{Y}_t\|_2^6 \leq E \|\underline{W}_t\|_2^6 \leq \overline{C} < \infty \text{ and } E \|\underline{E}_t\|_2^6 \leq E \|\underline{W}_t\|_2^6 \leq \overline{C} < \infty. \quad \square$$

Lemma C-6: Suppose that Assumptions 3-1, 3-2(a)-(b), 3-3(a)-(c), 3-5, 3-7, and 3-10(b) hold. Then, the following statements are true as $N_1, T \rightarrow \infty$

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_{i\underline{E}_t} \varepsilon_{\ell,t+1} \right| \xrightarrow{p} 0.$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_{i\underline{E}_t} \varepsilon_{\ell,t+1} \right)^2 \xrightarrow{p} 0$$

(c)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \xrightarrow{p} 0.$$

(d)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \xrightarrow{p} 0$$

(e)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_{i\underline{E}_t} \varepsilon_{\ell,t+1} \right) \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \xrightarrow{p} 0$$

Proof of Lemma C-6.

To show part (a), first write

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\
&\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \\
& \leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^6 \\
& \quad + \frac{20}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [|\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}|]^3 E [|\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}|]^3 \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^4 E [\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}]^2 \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} \left\{ E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^2 E [\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}]^2 \right. \\
& \quad \left. \times E [\gamma'_i \underline{F}_s \varepsilon_{\ell,r+1}]^2 \right\} \\
& \leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [(\gamma'_i \underline{F}_t)^6] E [\varepsilon_{\ell,t+1}^6] \\
& \quad + \frac{20}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \frac{1}{64} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i + \varepsilon_{\ell,t+1}^2]^3 E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i + \varepsilon_{\ell,s+1}^2]^3 \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i]^2 E [\varepsilon_{\ell,t+1}^4] E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i] E [\varepsilon_{\ell,s+1}^2] \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \left\{ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i] E [\varepsilon_{\ell,t+1}^2] E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i] E [\varepsilon_{\ell,s+1}^2] \right. \\
& \quad \left. \times \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_r \underline{F}'_r \gamma_i] E [\varepsilon_{\ell,r+1}^2] \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^6 E \left[\|\underline{F}_t\|_2^6 \right] E \left[\varepsilon_{\ell,t+1}^6 \right] \\
&\quad + \frac{(20 \cdot 16)}{64q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \left\{ \left(E \left[(\gamma'_i \underline{F}_t)^6 \right] + E \left[\varepsilon_{\ell,t+1}^6 \right] \right) \right. \\
&\quad \quad \quad \left. \times \left(E \left[(\gamma'_i \underline{F}_s)^6 \right] + E \left[\varepsilon_{\ell,s+1}^6 \right] \right) \right\} \\
&\quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \left\{ \|\gamma_i\|_2^4 E \left[\|\underline{F}_t\|_2^4 \right] E \left[\varepsilon_{\ell,t+1}^4 \right] \right. \\
&\quad \quad \quad \left. \times \|\gamma_i\|_2^2 E \left[\|\underline{F}_s\|_2^2 \right] E \left[\varepsilon_{\ell,s+1}^2 \right] \right\} \\
&\quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \left\{ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E \left[\|\underline{F}_t\|_2^2 \right] E \left[\varepsilon_{\ell,t+1}^2 \right] \right. \\
&\quad \quad \times \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E \left[\|\underline{F}_s\|_2^2 \right] E \left[\varepsilon_{\ell,s+1}^2 \right] \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E \left[\|\underline{F}_r\|_2^2 \right] E \left[\varepsilon_{\ell,r+1}^2 \right] \left. \right\} \\
&\leq C \left(\frac{N_1}{\tau_1^5} + 5 \frac{N_1}{\tau_1^4} + 15 \frac{N_1}{\tau_1^4} + 15 \frac{N_1}{\tau_1^3} \right) \\
&\quad \text{(applying Assumptions 3-2(b), Assumption 3-5, and Lemma C-5)} \\
&= O \left(\frac{N_1}{\tau_1^3} \right).
\end{aligned}$$

It follows that

$$P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \geq \epsilon \right\} = O \left(\frac{N_1}{\tau_1^3} \right) = o(1).$$

To show part (b), note that, for any $\epsilon > 0$

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \right|^3 \geq \epsilon^3 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^3 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^3 \right\} \\
&\leq \frac{1}{\epsilon^3} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6
\end{aligned}$$

The rest of the proof for part (b) then follows in a manner similar to the argument given for part (a) above.

To show part (c), first note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\quad (\text{by convexity or Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \tag{50}
\end{aligned}$$

Now, there exists a constant $C_1 > 1$ such that

$$\begin{aligned}
& \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\
& \leq \frac{C_1}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \left\{ \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}]| \right. \\
& \quad \left. \times \sum_{\ell=1}^d |E[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}]| \right\}
\end{aligned}$$

Next, note that, by repeated application of Hölder's inequality, we have by Lemma C-5 that there exists a positive constant \overline{C} such that

$$\begin{aligned}
& \sum_{\ell=1}^d |E[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}]| \\
& \leq \sum_{\ell=1}^d (E[y_{\ell,t+1}^2 y_{\ell,s+1}^2 y_{\ell,g+1}^2])^{\frac{1}{2}} (E[y_{\ell,h+1}^2 y_{\ell,v+1}^2 y_{\ell,w+1}^2])^{\frac{1}{2}} \\
& \leq \sum_{\ell=1}^d \left(\{E[y_{\ell,t+1}^6]\}^{\frac{1}{3}} \left(E[|y_{\ell,s+1}y_{\ell,g+1}|^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \left(\{E[y_{\ell,h+1}^6]\}^{\frac{1}{3}} \left(E[|y_{\ell,v+1}y_{\ell,w+1}|^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\
& \leq \sum_{\ell=1}^d \left[\left(\{E[y_{\ell,t+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,s+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,g+1}^6]\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \right. \\
& \quad \left. \times \left(\{E[y_{\ell,h+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,v+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,w+1}^6]\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \right] \\
& \leq \sum_{\ell=1}^d \{E[y_{\ell,t+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,s+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,g+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,h+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,v+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,w+1}^6]\}^{\frac{1}{6}} \\
& \leq d \max_{1 \leq \ell \leq d} \sup_t E[y_{\ell,t}^6] \\
& \leq \overline{C} < \infty \\
& \quad \left(\text{since, given that } y_{\ell,t} = \mathbf{e}'_{\ell,dp} \mathbf{Y}_t; E[y_{\ell,t}^6] \leq E\|\mathbf{Y}_t\|_2^6 \leq \overline{C} \text{ by Lemma C-5} \right. \\
& \quad \left. \text{where } \overline{C} \text{ is a constant not depending on } \ell \text{ or } t \right)
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
& \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig}^4]| \\
& \quad + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig}^4]| \\
&\quad + \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
&\quad + \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it} u_{is} u_{ig} u_{ih} - E(u_{it} u_{is} u_{ig} u_{ih})\} u_{iv} u_{iw}]| \\
&\quad + \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it} u_{is} u_{ig} u_{ih})| |E(u_{iv} u_{iw})| \\
&\quad + \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it} u_{is} u_{ig} - E(u_{it} u_{is} u_{ig})\} u_{ih} u_{iv} u_{iw}]| \\
&\quad + \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it} u_{is} u_{ig})| |E(u_{ih} u_{iv} u_{iw})| \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6, \quad (say). \tag{51}
\end{aligned}$$

Consider first \mathcal{T}_1 . Note that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} |E[u_{it} u_{is} u_{ig}^4]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} E[|u_{it} u_{is} u_{ig}^4|] \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} \left(E[|u_{it} u_{is}|^3]\right)^{\frac{1}{3}} \left(E[|u_{ig}|^6]\right)^{\frac{2}{3}} \quad (\text{by Hölder's inequality}) \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} \left(\left[E\{|u_{it}|^6\}\right]^{\frac{1}{2}} \left[E\{|u_{is}|^6\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}} \left(E[|u_{ig}|^6]\right)^{\frac{2}{3}} \\
&\quad (\text{by further application of Hölder's inequality}) \\
&= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} \left(E\{|u_{it}|^6\}\right)^{\frac{1}{6}} \left(E\{|u_{is}|^6\}\right)^{\frac{1}{6}} \left(E[|u_{ig}|^6]\right)^{\frac{1}{6}} \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g = (r-1)\tau + p \\ t \leq s \leq g}}^{(r-1)\tau + \tau_1 + p - 1} \bar{C}^{\frac{1}{2}} \quad (\text{by Assumption 3-3(b)}) \\
&\leq C_1 \bar{C}^{\frac{3}{2}} \frac{N_1}{\tau_1^5} \\
&= O\left(\frac{N_1}{\tau_1^5}\right). \tag{52}
\end{aligned}$$

Next, consider \mathcal{T}_2 . For this term, note first that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, we apply Lemma C-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned}
& \mathcal{T}_2 \\
&= \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{(r-1)\tau + \tau_1 + p - 1 \\ t, s, g, h, v, w = (r-1)\tau + p \\ t \leq s \leq g \leq h \leq v \leq w \\ w - v \geq \max\{v - h, h - g\}, w - v > 0}} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
&\leq \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{(r-1)\tau + \tau_1 + p - 1 \\ t, s, g, h, v, w = (r-1)\tau + p \\ t \leq s \leq g \leq h \leq v \leq w \\ w - v \geq \max\{v - h, h - g\}, w - v > 0}} \left\{ 2 \left(2^{1 - \frac{4}{5}} + 1 \right) [a_1 \exp\{-a_2(w - v)\}]^{1 - \frac{4}{5} - \frac{1}{6}} \right. \\
&\quad \left. \times \left(E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by Liapunov's inequality and Assumption 3-3(b), we obtain

$$\left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \leq \left(E |u_{iw}|^7 \right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}$$

Making use of this bound and by repeated application of Hölder's inequality, we have

$$\begin{aligned}
& E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \\
&\leq \left[E |u_{it} u_{is} u_{ig}|^{\frac{25}{12}} \right]^{\frac{3}{5}} \left[E |u_{ih} u_{iv}|^{\frac{25}{8}} \right]^{\frac{2}{5}} \\
&\leq \left[\left(E |u_{it} u_{is}|^{\frac{150}{47}} \right)^{\frac{47}{72}} \left(E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[\left(E |u_{ih}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
&\leq \left[\left(\sqrt{E |u_{it}|^{\frac{300}{47}}} \sqrt{E |u_{is}|^{\frac{300}{47}}} \right)^{\frac{47}{72}} \left(E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[\left(E |u_{ih}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
&= \left(E |u_{it}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left(E |u_{is}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left(E |u_{ih}|^6 \right)^{\frac{15}{72}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \left(E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \\
&= \left[\left(E |u_{it}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \left(E |u_{is}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \right]^{\frac{5}{4}} \left[\left(E |u_{ih}|^6 \right)^{\frac{1}{6}} \right]^{\frac{5}{4}} \left[\left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \left[\left(E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \\
&\leq \left[\left(E |u_{it}|^7 \right)^{\frac{1}{7}} \left(E |u_{is}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{ih}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{iv}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \\
&\leq (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} \quad (\text{by Assumption 3-3(b)}) \\
&= \overline{C}^{\frac{25}{28}}
\end{aligned}$$

Moreover, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}}$

as shown above, we can further write

$$\begin{aligned}
& \mathcal{T}_2 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{4}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\
& \quad \left. \times \left(E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \bar{C}^{\frac{25}{28}} \bar{C}^{\frac{1}{7}} \\
& \leq \frac{C_1 \bar{C}^{\frac{57}{28}}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{1}{5}} + 1 \right) C_1 \bar{C}^{\frac{57}{28}} a_1^{\frac{1}{30}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\rho_3=1}^{\infty} \sum_{\rho_1=0}^{\rho_3} \sum_{\rho_2=0}^{\rho_3} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\rho_3=1}^{\infty} (\rho_3 + 1)^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& = C^* \frac{N_1}{\tau_1^3} \left[\sum_{\rho_3=1}^{\infty} \rho_3^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + 2 \sum_{\rho_3=1}^{\infty} \rho_3 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + \sum_{\rho_3=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \right] \\
& = O \left(\frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma C-1}). \tag{53}
\end{aligned}$$

Now, consider \mathcal{T}_3 . Here, we can apply Lemma C-3 with $p = 3/2$ and $r = 7/2$ to obtain

$$\begin{aligned}
\mathcal{T}_3 &= \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau + \tau_1 + p - 1} |E[\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
&\leq \frac{C_1 \overline{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau + \tau_1 + p - 1} \left\{ 2 \left(2^{1-\frac{2}{3}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\quad \left. \times \left(E[\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\}^{\frac{3}{2}}] \right)^{\frac{2}{3}} \left(E|u_{iv}u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\}
\end{aligned}$$

Next, note that applications of Hölder's inequality yield

$$\begin{aligned}
E|u_{iv}u_{iw}|^{\frac{7}{2}} &\leq \left(E|u_{iv}|^7 \right)^{\frac{1}{2}} \left(E|u_{iw}|^7 \right)^{\frac{1}{2}} \\
&\leq (\overline{C})^{\frac{1}{2}} (\overline{C})^{\frac{1}{2}} \quad (\text{by Assumption 3-3(b)}) \\
&= \overline{C} < \infty
\end{aligned}$$

and

$$\begin{aligned}
E|\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\}|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}} \left(E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} + E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} \right) \\
&\quad (\text{by Loève's } c_r \text{ inequality}) \\
&\leq 2^{\frac{3}{2}} E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left(E|u_{it}u_{is}|^3 \right)^{\frac{1}{2}} \left(E|u_{ig}u_{ih}|^3 \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}} \left(\left(E|u_{it}|^6 \right)^{\frac{1}{2}} \left(E|u_{is}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left(\left(E|u_{ig}|^6 \right)^{\frac{1}{2}} \left(E|u_{ih}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}} \left[\left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}|^6 \right)^{\frac{1}{6}} \left(E|u_{ig}|^6 \right)^{\frac{1}{6}} \left(E|u_{ih}|^6 \right)^{\frac{1}{6}} \right]^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left[\left(E|u_{it}|^7 \right)^{\frac{1}{7}} \left(E|u_{is}|^7 \right)^{\frac{1}{7}} \left(E|u_{ig}|^7 \right)^{\frac{1}{7}} \left(E|u_{ih}|^7 \right)^{\frac{1}{7}} \right]^{\frac{3}{2}} \\
&\quad (\text{by Liapunov's inequality}) \\
&\leq 2^{\frac{3}{2}} \left[\left(\sup_{i,t} E|u_{it}|^7 \right)^{\frac{4}{7}} \right]^{\frac{3}{2}} \\
&= 2^{\frac{3}{2}} \overline{C}^{\frac{6}{7}} \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Again, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{iv} u_{iw}|^{\frac{7}{2}}$ and $E |\{u_{it} u_{is} u_{ig} u_{ih} - E(u_{it} u_{is} u_{ig} u_{ih})\}|^{\frac{3}{2}}$ as shown above, we can further write

$$\begin{aligned}
\mathcal{T}_3 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it} u_{is} u_{ig} u_{ih} - E(u_{it} u_{is} u_{ig} u_{ih})\} u_{iv} u_{iw}]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{2}{3}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\quad \times \left(E |\{u_{it} u_{is} u_{ig} u_{ih} - E(u_{it} u_{is} u_{ig} u_{ih})\}|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(E |u_{iv} u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \Big\} \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{3}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{\frac{1}{21}} \left(2^{\frac{3}{2}} \bar{C}^{\frac{6}{7}} \right)^{\frac{2}{3}} (\bar{C})^{\frac{2}{7}} \\
&\leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} \\
&\quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{3}} + 1 \right) C_1 \bar{C}^{\frac{13}{7}} a_1^{\frac{1}{21}} \leq C^* < \infty \right) \\
&\leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\rho_2=1}^{\infty} \sum_{\rho_1=0}^{\rho_2} \sum_{\rho_3=0}^{\rho_2} \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} \\
&\leq C^* \frac{N_1}{\tau_1^3} \sum_{\rho_2=1}^{\infty} (\rho_2 + 1)^2 \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} \\
&= C^* \frac{N_1}{\tau_1^3} \left[\sum_{\rho_2=1}^{\infty} \rho_2^2 \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} + 2 \sum_{\rho_2=1}^{\infty} \rho_2 \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} + \sum_{\rho_2=1}^{\infty} \exp \left\{ -\frac{a_2}{21} \rho_2 \right\} \right] \\
&= O \left(\frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma C-1}). \tag{54}
\end{aligned}$$

Turning our attention to the term \mathcal{T}_4 , note that, from the upper bounds given in the proofs of

parts (a) and (c) of Lemma C-4, it is clear that there exists a positive constant C such that

$$\frac{1}{\tau_1^4} \sum_{\substack{t,s,g,h=(r-1)\tau+p \\ t \leq s \leq g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| \leq \frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \leq \frac{C}{\tau_1}$$

from which it follows that

$$\begin{aligned} \mathcal{T}_4 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\ &\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left(\frac{1}{\tau_1^4} \sum_{\substack{t,s,g,h=(r-1)\tau+p \\ t \leq s \leq g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| \right) \left(\frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \right) \\ &\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left(\frac{C}{\tau_1^2} \right) \left(\frac{C}{\tau_1} \right) \\ &= C_1 \bar{C} C^2 \frac{N_1}{\tau_1^3} \\ &= O\left(\frac{N_1}{\tau_1^3}\right). \end{aligned} \tag{55}$$

Consider now \mathcal{T}_5 . In this case, we apply Lemma C-3 with $p = 2$ and $r = 9/4$ to obtain

$$\begin{aligned} \mathcal{T}_5 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\} u_{ih}u_{iv}u_{iw}]| \\ &\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(h-g)\}]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\ &\quad \left. \times \left(E|\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\} \end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
& E |u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}} \\
& \leq \left[E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[E |u_{iv}u_{iw}|^{\frac{63}{19}} \right]^{\frac{19}{28}} \\
& \leq \left[E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[\left(E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \left(E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \right]^{\frac{19}{28}} \\
& = \left[E |u_{ih}|^7 \right]^{\frac{9}{28}} \left(E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \left(E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \\
& = \left[E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[\left(E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \left(E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \right]^{\frac{9}{4}} \\
& \leq \left[E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[\left(E |u_{iv}|^7 \right)^{\frac{1}{7}} \left(E |u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{9}{4}} \quad (\text{by Liapunov's inequality}) \\
& \leq \left(\sup_{i,t} E |u_{it}|^7 \right)^{\frac{27}{28}} \\
& \leq \overline{C}^{\frac{27}{28}} \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

and

$$\begin{aligned}
E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 & \leq 2 \left(E |u_{it}u_{is}u_{ig}|^2 + E |u_{it}u_{is}u_{ig}|^2 \right) \\
& \quad (\text{by Loève's } c_r \text{ inequality}) \\
& \leq 4 E |u_{it}u_{is}u_{ig}|^2 \\
& \leq 4 \left(E |u_{it}|^6 \right)^{\frac{1}{3}} \left(E |u_{is}u_{ig}|^3 \right)^{\frac{2}{3}} \\
& \leq 4 \left(E |u_{it}|^6 \right)^{\frac{1}{3}} \left(\sqrt{E |u_{is}|^6} \sqrt{E |u_{ig}|^6} \right)^{\frac{2}{3}} \\
& = 4 \left[\left(E |u_{it}|^6 \right)^{\frac{1}{6}} \right]^2 \left[\left(E |u_{is}|^6 \right)^{\frac{1}{6}} \left(E |u_{ig}|^6 \right)^{\frac{1}{6}} \right]^2 \\
& \leq 4 \left[\left(E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^2 \left[\left(E |u_{is}|^7 \right)^{\frac{1}{7}} \left(E |u_{ig}|^7 \right)^{\frac{1}{7}} \right]^2 \\
& \quad (\text{by Liapunov's inequality}) \\
& \leq 4 \left[\left(\sup_{i,t} E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^6 \\
& \leq 4 \overline{C}^{\frac{6}{7}} \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Define again $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$,

$w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}}$ and $E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2$ as shown above, we can further write

$$\begin{aligned}
& \mathcal{T}_5 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) \left[a_1 \exp \left\{ -a_2 (h-g)^\theta \right\} \right]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
& \quad \left. \times \left(E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left(E |u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\} \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp \{-a_2 (h-g)\}]^{\frac{1}{18}} \left(4 \bar{C}^{\frac{6}{7}} \right)^{\frac{1}{2}} \left(\bar{C}^{\frac{27}{28}} \right)^{\frac{4}{9}} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{2}} + 1 \right) C_1 \bar{C}^{\frac{13}{7}} a_1^{\frac{1}{18}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \sum_{\varrho_3=0}^{\varrho_2} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \leq C^* \frac{N_1}{\tau_1^3} \sum_{\varrho_1=1}^{\infty} (\varrho_1 + 1)^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& = C^* \frac{N_1}{\tau_1^3} \left[\sum_{\varrho_1=1}^{\infty} \varrho_1^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + 2 \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + \sum_{\varrho_1=1}^{\infty} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \right] \\
& = O \left(\frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma C-1}) \tag{56}
\end{aligned}$$

Finally, consider \mathcal{T}_6 . Note that, from the upper bounds given in the proofs of part (b) of Lemma C-4, it is clear that there exists a positive constant C such that

$$\frac{1}{\tau_1^3} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| \leq \frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \leq \frac{C}{\tau_1^2}$$

from which it follows that

$$\begin{aligned} \mathcal{T}_6 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| |E(u_{ih}u_{iv}u_{iw})| \\ &\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left(\frac{1}{\tau_1^3} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| \right) \left(\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \right) \\ &\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left(\frac{C}{\tau_1^2} \right) \left(\frac{C}{\tau_1^2} \right) \\ &= C_1 C \bar{C}^2 \frac{N_1}{\tau_1^4} \\ &= O\left(\frac{N_1}{\tau_1^4}\right). \end{aligned} \tag{57}$$

It follows from expressions (50)-(57) that, for any $\epsilon > 0$,

$$\begin{aligned} &P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \geq \epsilon \right\} \\ &\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\ &\leq \frac{1}{\epsilon^4} (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6) \\ &= O\left(\frac{N_1}{\tau_1^5}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^4}\right) \\ &= O\left(\frac{N_1}{\tau_1^3}\right) \\ &= o(1) \quad \left(\text{by Assumption 3-10(b) which stipulates that } \frac{N_1}{\tau_1^3} \sim \frac{N_1}{T^{3\alpha_1}} \rightarrow 0 \right) \end{aligned}$$

which proves the required result.

Turning our attention to part (d), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \right|^3 \geq \epsilon^3 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^3 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^3 \right\} \\
&\leq \frac{1}{\epsilon^3} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6.
\end{aligned}$$

The rest of the proof for part (d) then follows in a manner similar to the argument given for part (c) above.

For part (e), note that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
&\leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \sqrt{\frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \sqrt{\frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \\
&\leq \left\{ \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \right. \\
&\quad \left. \times \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \right\} \\
&= o_p(1),
\end{aligned}$$

where the convergence in probability to zero in the last line above follows from applying the results in parts (b) and (c) of this lemma. \square

Lemma C-7: Suppose that Assumptions 3-1 and 3-7 hold. Then, the following statements are true.

(a) There exists a positive constant C^\dagger such that

$$\|A_{YY}\|_2 \leq C^\dagger \phi_{\max}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ with $0 < \phi_{\max} < 1$.

(b) There exists a positive constant C^\dagger such that

$$\|A_{YF}\|_2 \leq C^\dagger \phi_{\max}$$

where ϕ_{\max} is as defined in part (a).

Proof of Lemma C-7:

To proceed, recall first that the FAVAR model, i.e.,

$$\begin{aligned} Y_t &= \mu_Y + A_{YY}Y_{t-1} + A_{YF}F_{t-1} + \varepsilon_t^Y \\ F_t &= \mu_F + A_{FY}Y_{t-1} + A_{FF}F_{t-1} + \varepsilon_t^F, \end{aligned}$$

can be written in the companion form

$$\underline{W}_t = \alpha + A\underline{W}_{t-1} + E_t$$

where $\underline{W}_t = \begin{pmatrix} W'_t & W'_{t-1} & \cdots & W'_{t-p+2} & W'_{t-p+1} \end{pmatrix}'$ with $W_t = \begin{pmatrix} Y'_t & F'_t \end{pmatrix}'$ and where

$$\alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & 0 & 0 \\ 0 & I_{d+K} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}, \text{ and } E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

with $\mu = \begin{pmatrix} \mu'_Y & \mu'_F \end{pmatrix}'$, $\varepsilon_t = \begin{pmatrix} \varepsilon_t^{Y'} & \varepsilon_t^{F'} \end{pmatrix}'$, and

$$A_\ell = \begin{pmatrix} A_{YY,\ell} & A_{YF,\ell} \\ A_{FY,\ell} & A_{FF,\ell} \end{pmatrix} \text{ for } \ell = 1, \dots, p.$$

Let $\mathcal{P}_{(d+K)p}$ be the $(d+K)p \times (d+K)p$ permutation matrix defined by expression (49) in the proof of Lemma C-5; and it is easy to see that $\bar{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p}$ has the partitioned form

$$\bar{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \\ \bar{A}_{31} & \bar{A}_{32} \\ \bar{A}_{41} & \bar{A}_{42} \end{pmatrix}$$

$\begin{matrix} d \times dp & d \times Kp \\ d(p-1) \times dp & d(p-1) \times Kp \\ K \times dp & K \times Kp \\ K(p-1) \times dp & K(p-1) \times Kp \end{matrix}$

where $\bar{A}_{11} = A_{YY}$ and $\bar{A}_{12} = A_{YF}$, i.e., the first d rows of the matrix \bar{A} as given by the submatrix $\begin{bmatrix} A_{YY} & A_{YF} \end{bmatrix}$.

Now, to show part (a), let $\bar{v} \in \mathbb{R}^{dp}$ such that $\|\bar{v}\|_2 = 1$ and such that

$$\|A_{YY}\|_2 = \bar{v}' A'_{YY} A_{YY} \bar{v} = \max_{\|v\|_2=1} v' A'_{YY} A_{YY} v = \bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v}$$

and let $S_d = \begin{pmatrix} I_{dp} & 0 \\ 0 & dp \times Kp \end{pmatrix}'$. It follows that

$$\begin{aligned} \|A_{YY}\|_2 &= \sqrt{\bar{v}' A'_{YY} A_{YY} \bar{v}} \\ &= \sqrt{\bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v}} \\ &\leq \sqrt{\bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v} + \bar{v}' \bar{A}'_{21} \bar{A}_{21} \bar{v} + \bar{v}' \bar{A}'_{31} \bar{A}_{31} \bar{v} + \bar{v}' \bar{A}'_{41} \bar{A}_{41} \bar{v}} \\ &= \sqrt{\bar{v}' S'_d \bar{A}' \bar{A} S_d \bar{v}} \\ &= \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} A' \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} S_d \bar{v}} \\ &= \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} A' A \mathcal{P}'_{(d+K)p} S_d \bar{v}} \quad (\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}) \\ &\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \quad \left(\text{noting that } \left\| \mathcal{P}'_{(d+K)p} S_d \bar{v} \right\|_2 = \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p} S_d \bar{v}} = 1 \right) \\ &= \|A\|_2 \\ &= \sigma_{\max}(A) \\ &\leq C^\dagger \phi_{\max} \quad (\text{by Assumption 3-7}) \end{aligned}$$

where $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. Note further that $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1.

To show part (b), let $\tilde{v} \in \mathbb{R}^{Kp}$ such that $\|\tilde{v}\|_2 = 1$ and such that

$$\|A_{YF}\|_2 = \tilde{v}' A'_{YF} A_{YF} \tilde{v} = \max_{\|v\|_2=1} v' A'_{YF} A_{YF} v = \tilde{v}' \overline{A}'_{12} \overline{A}_{12} \tilde{v}$$

and let

$$S_K = \begin{pmatrix} 0 \\ I_{Kp} \end{pmatrix}.$$

It follows that

$$\begin{aligned} \|A_{YF}\|_2 &= \sqrt{\tilde{v}' A'_{YF} A_{YF} \tilde{v}} \\ &= \sqrt{\tilde{v}' \overline{A}'_{12} \overline{A}_{12} \tilde{v}} \\ &\leq \sqrt{\tilde{v}' \overline{A}'_{12} \overline{A}_{12} \tilde{v} + \tilde{v}' \overline{A}'_{22} \overline{A}_{22} \tilde{v} + \tilde{v}' \overline{A}'_{32} \overline{A}_{32} \tilde{v} + \tilde{v}' \overline{A}'_{42} \overline{A}_{42} \tilde{v}} \\ &= \sqrt{\tilde{v}' S'_K \overline{A}' \overline{A} S_K \tilde{v}} \\ &= \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} A' \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} S_K \tilde{v}} \\ &= \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} A' A \mathcal{P}'_{(d+K)p} S_K \tilde{v}} \quad (\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}) \\ &\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \quad (\text{noting that } \|\mathcal{P}'_{(d+K)p} S_K \tilde{v}\|_2 = \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p} S_K \tilde{v}} = 1) \\ &= \|A\|_2 \\ &= \sigma_{\max}(A) \\ &\leq C^\dagger \phi_{\max} \quad (\text{by Assumption 3-7}) \end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. As noted in the proof for part (a), $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. \square

Lemma C-8: Consider the linear process

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}$$

Suppose the process satisfies the following assumptions

- (i) Let $\{\varepsilon_t\}$ is an independent sequence of random vectors with $E[\varepsilon_t] = 0$ for all t . For some $\delta > 0$, suppose that there exists a positive constant K such that

$$E\|\varepsilon_t\|_2^{1+\delta} \leq K < \infty \text{ for all } t.$$

(ii) Suppose that ε_t has p.d.f. g_{ε_t} such that, for some positive constant $M < \infty$,

$$\sup_t \int |g_{\varepsilon_t}(v - u) - g_{\varepsilon_t}(v)| d\varepsilon \leq M |u|$$

whenever $|u| \leq \bar{\kappa}$ for some constant $\bar{\kappa} > 0$.

(iii) Suppose that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 < \infty$$

and

$$\det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} \neq 0 \text{ for all } z \text{ with } |z| \leq 1$$

Under these conditions, suppose further that

$$\sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}} < \infty;$$

then, for some positive constant \bar{K} ,

$$\beta_{\xi}(m) \leq \bar{K} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}}$$

where

$$\beta_{\xi}(m) = \sup_t E \left[\sup \left\{ |P(B|\mathcal{F}_{\xi, -\infty}^t) - P(B)| : B \in \mathcal{F}_{\xi, t+m}^{\infty} \right\} \right].$$

with $\mathcal{F}_{\xi, -\infty}^t = \sigma(\dots, \xi_{t-2}, \xi_{t-1}, \xi_t)$ and $\mathcal{F}_{\xi, t+m}^{\infty} = \sigma(\xi_{t+m}, \xi_{t+m+1}, \xi_{t+m+2}, \dots)$.

Remark: This is Theorem 2.1 of Pham and Tran (1985) restated here in our notation. For a proof, see Pham and Tran (1985).

Lemma C-9: Let A be an $n \times n$ square matrix with (ordered) singular values given by

$$\sigma_{(1)}(A) \geq \sigma_{(2)}(A) \geq \dots \geq \sigma_{(n)}(A) \geq 0.$$

Suppose that A is diagonalizable, i.e.,

$$A = S\Lambda S^{-1}$$

where Λ is diagonal matrix whose diagonal elements are the eigenvalues of A . Let the modulus of

these eigenvalues be ordered as follows:

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \cdots \geq |\lambda_{(n)}(A)|.$$

Then, for $k \in \{1, \dots, n\}$ and for any positive integer j , we have

$$\chi(S)^{-1} |\lambda_{(k)}(A^j)| \leq \sigma_{(k)}(A^j) \leq \chi(S) |\lambda_{(k)}(A^j)|$$

where

$$\chi(S) = \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}).$$

Proof of Lemma C-9: Observe first that we can assume, without loss of generality, that the decomposition

$$A = S\Lambda S^{-1} = S \cdot \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \cdot S^{-1}$$

is such that

$$\lambda_i = \lambda_{(i)}(A) \text{ for } i = 1, \dots, n$$

with

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \cdots \geq |\lambda_{(n)}(A)|.$$

This is because suppose we have the alternative representation where

$$A = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} = \tilde{S} \cdot \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n) \cdot \tilde{S}^{-1}$$

and where $\tilde{\lambda}_i \neq \lambda_{(i)}(A)$ for at least some of the i 's. Then, we can always define a permutation matrix \mathcal{P} such that

$$\mathcal{P}'\tilde{\Lambda}\mathcal{P} = \Lambda$$

so that, given that \mathcal{P} is an orthogonal matrix, we have

$$A = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} = \tilde{S}\mathcal{P}\mathcal{P}'\tilde{\Lambda}\mathcal{P}\mathcal{P}'\tilde{S}^{-1} = S\Lambda S^{-1}$$

where $S = \tilde{S}\mathcal{P}$ and, thus, $S^{-1} = (\tilde{S}\mathcal{P})^{-1} = \mathcal{P}'\tilde{S}^{-1}$.

Next, note that, for any positive integer j ,

$$A^j = S\Lambda S^{-1} \times S\Lambda S^{-1} \times \cdots \times S\Lambda S^{-1} = S\Lambda^j S^{-1}$$

where

$$\Lambda^j = \text{diag} \left(\lambda_1^j, \lambda_2^j, \dots, \lambda_n^j \right) = \text{diag} \left(\lambda_{(1)}^j(A), \lambda_{(2)}^j(A), \dots, \lambda_{(n)}^j(A) \right).$$

Moreover, since $\lambda_{(k)}(A^j) = \lambda_{(k)}^j(A)$ for any $k \in \{1, \dots, m\}$, we also have

$$\Lambda^j = \text{diag} \left(\lambda_1^j, \lambda_2^j, \dots, \lambda_n^j \right) = \text{diag} \left(\lambda_{(1)}(A^j), \lambda_{(2)}(A^j), \dots, \lambda_{(n)}(A^j) \right).$$

In addition, let $\overline{\lambda_{(k)}(A^j)}$ denote the complex conjugate of $\lambda_{(k)}(A^j)$ for $k \in \{1, \dots, m\}$, and note that, by definition,

$$\sigma_{(k)}(\Lambda^j) = \sqrt{\overline{\lambda_{(k)}(A^j)} \lambda_{(k)}(A^j)} = |\lambda_{(k)}(A^j)|$$

Since $|\lambda_{(k)}(A^j)| = |\lambda_{(k)}^j(A)| = |\lambda_{(k)}(A)|^j$, the ordering

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \dots \geq |\lambda_{(n)}(A)|$$

implies that

$$|\lambda_{(1)}(A^j)| \geq |\lambda_{(2)}(A^j)| \geq \dots \geq |\lambda_{(n)}(A^j)|$$

and, thus,

$$\sigma_{(1)}(\Lambda^j) \geq \sigma_{(2)}(\Lambda^j) \geq \dots \geq \sigma_{(n)}(\Lambda^j)$$

for any positive integer j .

Now, apply the inequality

$$\sigma_{(i+\ell-1)}(BC) \leq \sigma_{(i)}(B) \sigma_{(\ell)}(C)$$

for $i, \ell \in \{1, \dots, n\}$ and $i + \ell \leq n + 1$; we have

$$\begin{aligned} \sigma_{(k)}(A^j) &= \sigma_{(k)}(S\Lambda^j S^{-1}) \\ &\leq \sigma_{(k)}(S\Lambda^j) \sigma_{(1)}(S^{-1}) \\ &\leq \sigma_{(k)}(\Lambda^j) \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}) \\ &= \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}) |\lambda_{(k)}(A^j)| \\ &= \chi(S) |\lambda_{(k)}(A^j)| \text{ for any } k \in \{1, \dots, n\} \end{aligned}$$

Moreover, for any $k \in \{1, \dots, n\}$,

$$\begin{aligned}
|\lambda_{(k)}(A^j)| &= \sigma_{(k)}(\Lambda^j) \\
&= \sigma_{(k)}(S^{-1}S\Lambda^jS^{-1}S) \\
&= \sigma_{(k)}(S^{-1}A^jS) \\
&\leq \sigma_{(1)}(S^{-1})\sigma_{(k)}(A^j)\sigma_{(1)}(S)
\end{aligned}$$

or

$$\frac{|\lambda_{(k)}(A^j)|}{\chi(S)} = \frac{|\lambda_{(k)}(A^j)|}{\sigma_{(1)}(S)\sigma_{(1)}(S^{-1})} \leq \sigma_{(k)}(A^j)$$

Putting these two inequalities together, we have, for any $k \in \{1, \dots, n\}$ and for all positive integer j ,

$$\chi(S)^{-1}|\lambda_{(k)}(A^j)| \leq \sigma_{(k)}(A^j) \leq \chi(S)|\lambda_{(k)}(A^j)|. \quad \square$$

Remark: Note that the case where $j = 1$ in Lemma C-9 has previously been obtained in Theorem 1 of Ruhe (1975). Hence, Lemma C-9 can be viewed as providing an extension to the first part of that theorem.

Lemma C-10: Let ρ be such that $|\rho| < 1$. Then,

$$\sum_{j=0}^{\infty} (j+1)\rho^j = \frac{1}{(1-\rho)^2} < \infty$$

Proof of Lemma C-10: Define

$$S_n(\rho) = 1 + \rho + \rho^2 + \dots + \rho^n = \frac{1 - \rho^{n+1}}{1 - \rho}$$

Note that

$$\begin{aligned}
S'_n(\rho) &= 1 + 2\rho + 3\rho^2 + \cdots + n\rho^{n-1} \\
&= -\frac{(n+1)\rho^n}{1-\rho} + \frac{1-\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1} - (n+1)\rho^n(1-\rho)}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1} - (n+1)\rho^n + (n+1)\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1 - (n+1)\rho^n + n\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1 - \rho^n - n\rho^n(1-\rho)}{(1-\rho)^2}
\end{aligned}$$

It follows that

$$S'_n(\rho) = \sum_{j=0}^{n-1} (j+1)\rho^j = \frac{1 - \rho^n - n\rho^n(1-\rho)}{(1-\rho)^2} \rightarrow \frac{1}{(1-\rho)^2} \text{ as } n \rightarrow \infty. \quad \square$$

Lemma C-11: Let $W_t = (Y'_t, F'_t)'$ be generated by the factor-augmented VAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}$$

described in section 3 of the main paper. Under Assumptions 3-1, 3-2(a)-(c), and 3-7; $\{W_t\}$ is a β -mixing process with β -mixing coefficient $\beta_W(m)$ such that

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\}$$

for some positive constants C_1 and C_2 . Here,

$$\beta_W(m) = \sup_t E \left[\sup \left\{ \left| P(B|\mathcal{A}_{-\infty}^t) - P(B) \right| : B \in \mathcal{A}_{t+m}^\infty \right\} \right]$$

with $\mathcal{A}_{-\infty}^t = \sigma(\dots, W_{t-2}, W_{t-1}, W_t)$ and $\mathcal{A}_{t+m}^\infty = \sigma(W_{t+m}, W_{t+m+1}, W_{t+m+2}, \dots)$.

Proof of Lemma C-11:

To prove this lemma, we shall verify the conditions of Lemma C-8 given above for the vector

moving-average representation of W_t , i.e.,

$$W_t = J_{d+K} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} J_{d+K} A^j J'_{d+K} \varepsilon_{t-j} = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j},$$

where

$$\begin{aligned} \mu_* &= J_{d+K} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu, \quad \Psi_j = J_{d+K} A^j J'_{d+K}, \\ \begin{matrix} J_{d+K} \\ (d+K) \times (d+K)p \end{matrix} &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix} \end{aligned}$$

To proceed, set

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \quad (58)$$

and note first that, setting $\delta = 5$ in Lemma C-8, and we see that Assumptions (i) and (ii) of Lemma C-8 are the same as the conditions specified in Assumption 3-2 (a)-(c). Next, note that, since in this case $\Psi_j = J_{d+K} A^j J'_{d+K}$, we have

$$\begin{aligned} \|\Psi_j\|_2 &\leq \|J_{d+K}\|_2 \|A^j\|_2 \|J'_{d+K}\|_2 \\ &\leq \sqrt{\lambda_{\max}(J'_{d+K} J_{d+K})} \left(\sqrt{\lambda_{\max}\{(A^j)' A^j\}} \right) \sqrt{\lambda_{\max}(J_{d+K} J'_{d+K})} \\ &= \lambda_{\max}(J_{d+K} J'_{d+K}) \left(\sqrt{\lambda_{\max}\{(A^j)' A^j\}} \right) \\ &= \sqrt{\lambda_{\max}\{(A^j)' A^j\}} \\ &= \sigma_{\max}(A^j) \\ &\leq C [\max\{|\lambda_{\max}(A^j)|, |\lambda_{\min}(A^j)|\}] \quad (\text{by Assumption 3-7}) \\ &= C [\max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}]^j \\ &= C \phi_{\max}^j \end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all

eigenvalues of A have modulus less than 1. It follows that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 \leq C \sum_{j=0}^{\infty} \phi_{\max}^j = \frac{C}{1 - \phi_{\max}} < \infty.$$

Moreover, by Assumption 3-1,

$$\det \{I_{(d+K)p} - A_1 z - \dots - A_p z^p\} \neq 0 \text{ for all } z \text{ such that } |z| \leq 1$$

and, by definition,

$$\sum_{j=0}^{\infty} \Psi_j z^j = \Psi(z) = (I_{(d+K)p} - A_1 z - \dots - A_p z^p)^{-1} \text{ for all } z \text{ such that } |z| \leq 1$$

so that

$$\Psi(z) (I_{(d+K)p} - A_1 z - \dots - A_p z^p) = I_{(d+K)p} \text{ for all } z \text{ such that } |z| \leq 1$$

In addition, since

$$\begin{aligned} & \det \{\Psi(z)\} \det \{I_{(d+K)p} - A_1 z - \dots - A_p z^p\} \\ &= \det \{\Psi(z) (I_{(d+K)p} - A_1 z - \dots - A_p z^p)\} \\ &= \det \{I_{(d+K)p}\} \\ &= 1, \end{aligned}$$

it follows that

$$\begin{aligned} \det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} &= \det \{\Psi(z)\} \\ &= \frac{1}{\det \{I_{(d+K)p} - A_1 z - \dots - A_p z^p\}} \\ &\neq 0 \text{ for all } z \text{ such that } |z| \leq 1. \end{aligned}$$

Finally, note that, setting $\delta = 5$,

$$\begin{aligned}
\sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}} &= \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{5}{6}} \\
&\leq \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} C \phi_{\max}^k \right)^{\frac{5}{6}} \\
&= C^{\frac{5}{6}} \sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \phi_{\max}^k \right)^{\frac{5}{6}} \\
&\leq C^{\frac{5}{6}} \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \left(\phi_{\max}^{\frac{5}{6}} \right)^k \\
&\quad \left(\text{by the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq c_r \sum_{i=1}^m |a_i|^r \text{ where } c_r = 1 \text{ for } r \leq 1 \right) \\
&= C^{\frac{5}{6}} \sum_{j=0}^{\infty} (j+1) \left(\phi_{\max}^{\frac{5}{6}} \right)^j \\
&= C^{\frac{5}{6}} \left[1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \quad (\text{by Lemma C-10}) \\
&< \infty \quad \left(\text{since } 0 < \phi_{\max}^{\frac{5}{6}} < 1 \text{ given that } 0 < \phi_{\max} < 1 \right).
\end{aligned}$$

Hence, all conditions of Lemma C-8 are fulfilled. Applying Lemma C-8, we then obtain that

there exists a constant \overline{C} such that

$$\begin{aligned}
\beta_\xi(m) &\leq \overline{C} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{5}{6}} \\
&\leq \overline{C} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} C \phi_{\max}^k \right)^{\frac{5}{6}} \\
&= \overline{C} C^{\frac{5}{6}} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} \phi_{\max}^k \right)^{\frac{5}{6}} \\
&\leq \overline{C} C^{\frac{5}{6}} \sum_{j=m}^{\infty} \sum_{k=j}^{\infty} \left(\phi_{\max}^{\frac{5}{6}} \right)^k \\
&= \overline{C} C^{\frac{5}{6}} \left(\phi_{\max}^{\frac{5}{6}} \right)^m \sum_{j=0}^{\infty} (j+1) \left(\phi_{\max}^{\frac{5}{6}} \right)^j \\
&= \overline{C} C^{\frac{5}{6}} \left(\phi_{\max}^{\frac{5}{6}} \right)^m \left[1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \\
&= \overline{C} C^{\frac{5}{6}} \left[1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \exp \left\{ - \left[\frac{5}{6} |\ln \phi_{\max}| \right] m \right\} \quad (\text{since } 0 < \phi_{\max} < 1) \\
&\leq C_1 \exp \{-C_2 m\} \rightarrow 0 \text{ as } m \rightarrow \infty.
\end{aligned}$$

for some positive constants C_1 and C_2 such that

$$C_1 \geq \overline{C} C^{\frac{5}{6}} \left[1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \text{ and } C_2 \leq \frac{5}{6} |\ln \phi_{\max}|$$

It follows that the process $\{\xi_t\}$ (as defined in expression (58)) is β mixing with beta coefficient $\beta_\xi(m)$ satisfying

$$\beta_\xi(m) \leq C_1 \exp \{-C_2 m\}.$$

Since

$$W_t = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} = \mu_* + \xi_t$$

and since μ_* is a nonrandom parameter, we can then apply part (a) of Lemma C-2 to deduce that $\{W_t\}$ is a β mixing process with β coefficient $\beta_W(m)$ satisfying the inequality

$$\beta_W(m) \leq C_1 \exp \{-C_2 m\}. \quad \square$$

Lemma C-12: Let $\underline{Y}_t = \begin{pmatrix} Y'_t & Y'_{t-1} & \cdots & Y'_{t-p+2} & Y'_{t-p+1} \end{pmatrix}'$ and $\underline{F}_t = \begin{pmatrix} F'_t & F'_{t-1} & \cdots & F'_{t-p+2} & F'_{t-p+1} \end{pmatrix}'$. Under Assumptions 3-1, 3-2(a)-(c), 3-5, 3-7, and 3-10(b); the following statements are true as $N, T \rightarrow \infty$

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \xrightarrow{p} 0$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \xrightarrow{p} 0$$

(c)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E [\underline{F}_t]) \mu_{Y,\ell} \right| \xrightarrow{p} 0$$

(d)

$$\begin{aligned} \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E [\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\ \left. + (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\ \xrightarrow{p} 0 \end{aligned}$$

(e) There exists a positive constant \overline{C} such that

$$\begin{aligned} & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{\pi_{i,\ell,T}}{q\tau_1^2} \right) \\ &= \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E [\underline{F}_t] \mu_{Y,\ell} + E [\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E [\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\ &= \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i E [\underline{F}_t y_{\ell,t+1}] \right)^2 \\ &\leq \overline{C} < \infty \end{aligned}$$

(f)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2 = O_p(1).$$

(g)

$$\begin{aligned} & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left\{ \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right. \right. \right. \\ & \quad \left. \left. + \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} + \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right\} \right. \\ & \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right) \right\} \Big| \\ & \xrightarrow{p} 0 \end{aligned}$$

(h)

$$\begin{aligned} & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\ & \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\ & \xrightarrow{p} 0 \end{aligned}$$

(i)

$$\begin{aligned} & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\ & \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\ & \xrightarrow{p} 0 \end{aligned}$$

Proof of Lemma C-12:

To show part (a), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{i \in H^c} \max_{1 \leq \ell \leq d} \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right]' \right. \right. \\
&\quad \left. \left. \times \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' \right. \\
&\quad \left. \times (\underline{F}_s \underline{Y}'_s - E [\underline{F}_s \underline{Y}'_s]) \alpha_{YY,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \alpha'_{YY,\ell} \\
&\quad \times E [(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E [\underline{F}_s \underline{Y}'_s])] \alpha_{YY,\ell} \} \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \alpha'_{YY,\ell} \\
&\quad \times E [(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E [\underline{F}_s \underline{Y}'_s])] \alpha_{YY,\ell} \} \quad (59) \\
&\quad (\text{by Assumption 3-5})
\end{aligned}$$

Next, write

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \alpha'_{YY,\ell} \right. \right. \\
& \quad \left. \left. \times E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \right\} \right) \\
&= \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \left(\frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left\{ \alpha'_{YY,\ell} \right. \right. \\
& \quad \left. \left. \times E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \right\} \right) \\
&\leq \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \left(\frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left| \alpha'_{YY,\ell} \right. \right. \\
& \quad \left. \left. \times E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \right| \right) \quad (60)
\end{aligned}$$

Let $e_{\ell,d}$ be a $d \times 1$ elementary vector whose ℓ^{th} component is 1 and all other components are 0, and

note that

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[(\underline{E}_t \underline{Y}'_t - E[\underline{E}_t \underline{Y}'_t])' (\underline{E}_t \underline{Y}'_t - E[\underline{E}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
&= \sum_{\ell=1}^d \left(\frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E \left[(\underline{E}_t \underline{Y}'_t - E[\underline{E}_t \underline{Y}'_t])' (\underline{E}_t \underline{Y}'_t - E[\underline{E}_t \underline{Y}'_t]) \right] A'_{YY} e_{\ell,d} \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E [\underline{Y}_t \underline{F}'_t \underline{E}_t \underline{Y}'_t] A'_{YY} e_{\ell,d} \right. \\
&\quad \left. - \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E [\underline{Y}_t \underline{F}'_t] E [\underline{E}_t \underline{Y}'_t] A'_{YY} e_{\ell,d} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E \left[\|\underline{E}_t\|_2^2 (e'_{\ell,d} A_{YY} \underline{Y}_t)^2 \right] \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[\|\underline{E}_t\|_2^4 \right]} \sqrt{E \left(e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d} \right)^2} \quad (\text{by CS inequality}) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[\|\underline{E}_t\|_2^4 \right]} \sqrt{E \left[\|\underline{Y}_t\|_2^4 \right]} \sqrt{\left(e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d} \right)^2} \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[\|\underline{E}_t\|_2^4 \right]} \sqrt{E \left[\|\underline{Y}_t\|_2^4 \right]} \|A_{YY}\|_2^2 \sqrt{\left(e'_{\ell,d} e_{\ell,d} \right)^2} \\
&\leq \frac{d(C^\dagger)^2}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[\|\underline{E}_t\|_2^4 \right]} \sqrt{E \left[\|\underline{Y}_t\|_2^4 \right]} \phi_{\max}^2 \\
&\quad (\text{by part (a) of Lemma C-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector}) \\
&\leq \frac{\overline{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \tag{61}
\end{aligned}$$

for some positive constant $\overline{C} \geq d(C^\dagger)^2 \sqrt{E \left[\|\underline{E}_t\|_2^4 \right]} \sqrt{E \left[\|\underline{Y}_t\|_2^4 \right]} \phi_{\max}^2$, which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1.

To analyze the second term on the right-hand side of expression (60), note first that by Lemma C-11, $\{(Y'_t, F'_t)'\}$ is β -mixing with β mixing coefficient satisfying

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\} \quad \text{for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{W,m} \leq \beta_W(m)$, it follows that $W_t = (Y'_t, F'_t)'$ is α -mixing as well, with α mixing coefficient

satisfying

$$\alpha_{W,m} \leq C_1 \exp \{-C_2 m\}$$

Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = \underline{F}_t \underline{Y}'_t A'_{YY} e_{\ell,d}$ is also α -mixing with α mixing coefficient satisfying

$$\begin{aligned} \alpha_{X,m} &\leq C_1 \exp \{-C_2 (m - p + 1)\} \\ &\leq C_1^* \exp \{-C_2 m\} \end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp \{C_2 (p - 1)\}$. Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned} & \left| \alpha'_{YY,\ell} E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \right| \\ &= \left| e'_{\ell,d} A_{YY} E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} \right| \\ &= \left| \sum_{h=1}^{Kp} e'_{\ell,d} A_{YY} E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' e_{h,Kp} e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} \right| \\ &\leq \sum_{h=1}^{Kp} \left\{ 2 \left(2^{\frac{2}{3}} + 1 \right) \alpha_{X,m}^{\frac{1}{3}} \left(E \left| e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\ &\quad \left. \times \left(E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) A'_{YY} e_{\ell,d} \right|^3 \right)^{1/3} \right\} \end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $\{X_t\}$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ &\leq (C_1^*)^{\frac{1}{3}} \left[1 - \exp \left\{ -\frac{C_2}{3} \right\} \right]^{-1} \\ &\leq C_3 \end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp \{-(C_2 m/3)\}$ is a convergent geometric

series given that $0 < \exp \{-(C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$\begin{aligned}
& E \left| e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' e_{h,Kp} \right|^3 \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|^3 + |E [e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}]|^3 \right\} \text{ (by Loève's } c_r \text{ inequality)} \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|^3 + (E [|e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \text{ (by Jensen's inequality)} \\
& \leq 2^2 \left\{ E \left| \frac{e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}}{2} + \frac{e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}}{2} \right|^3 + (E [|e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \\
& \leq 4 \frac{2^2}{8} \left[E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}|^3 + E |e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}|^3 \right] \\
& \quad + 4 \left(\sqrt{E [e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}]} \sqrt{E [e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}]} \right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
& \leq 2 |e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d}|^3 E \|\underline{Y}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d})^{\frac{3}{2}} \left(E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& \leq 2 \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \left(E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& = 2 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (C^\dagger)^3 \phi_{\max}^3 \left(E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& \quad \text{(since } \|e_{\ell,d}\|_2 = 1 \text{ for every } \ell \in \{1, \dots, d\} \text{ given that } e_{\ell,d} \text{'s are elementary vectors)} \\
& \leq C_4
\end{aligned}$$

for some positive constant $C_4 \geq 2 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (C^\dagger)^3 \phi_{\max}^3 \left(E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}}$ which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1. In a similar way, we can also show that there exists a positive constant C_5 such that

$$\begin{aligned}
& E |e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) A'_{YY} e_{\ell,d}|^3 \\
& \leq 2 \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_{t+m}\|_2^6 + 2E \|\underline{F}_{t+m}\|_2^6 \\
& \quad + 4 \left(E \|\underline{Y}_{t+m}\|_2^2 \right)^{\frac{3}{2}} \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \left(E \|\underline{F}_{t+m}\|_2^2 \right)^{\frac{3}{2}} \\
& \leq C_5 < \infty
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{2}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YY} \\
& \quad \times E \left[(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} \Big| \\
& \leq \frac{4 \left(2^{\frac{2}{3}} + 1 \right)}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \sum_{h=1}^{Kp} \alpha_{W,m}^{\frac{1}{3}} \left(E \left| e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \\
& \quad \times \left(E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) A'_{YY} e_{\ell,d} \right|^3 \right)^{1/3} \\
& \leq \frac{4dKp \left(2^{\frac{2}{3}} + 1 \right) C_4^{\frac{1}{3}} C_5^{\frac{1}{3}}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{\infty} (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& \leq \frac{C^*}{\tau_1} \left(\frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left(\text{where } C^* \geq 4dKp \left(2^{\frac{2}{3}} + 1 \right) (C_1^*)^{\frac{1}{3}} C_4^{\frac{1}{3}} C_5^{\frac{1}{3}} \right) \\
& \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left(\frac{1}{\tau_1} \right)
\end{aligned} \tag{62}$$

It then follows from expressions (59), (60), (61), and (62) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} \right. \\
& \quad \left. \times E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E [\underline{F}_s \underline{Y}'_s]) \right] A'_{YY} e_{\ell,d} \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \right] A'_{YY} e_{\ell,d} \right) \\
& \quad + \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YY} \\
& \quad \times E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E [\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} | \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\overline{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C \overline{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{C C^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left(\frac{1}{\tau_1} \right) + O \left(\frac{1}{\tau_1} \right) \\
& = O \left(\frac{1}{\tau_1} \right) = o(1).
\end{aligned}$$

Next, to show part (b), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right]^2 \right. \right. \\
&\quad \left. \left. \times \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E [\underline{F}_s \underline{F}'_s]) \alpha_{YF,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E [\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E [\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \quad (63) \\
&\quad (\text{by Assumption 3-5})
\end{aligned}$$

Note first that

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
& \quad \left. \times E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
&= \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \left(\frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \alpha'_{YF,\ell} \right. \\
& \quad \left. \times E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \right) \\
&\leq \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |\alpha'_{YF,\ell}| \\
& \quad \times E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \Big| \quad (64)
\end{aligned}$$

Consider the first term on the majorant side of expression (64), whose order of magnitude we can

analyze as follows

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
&= \sum_{\ell=1}^d \left(\frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YF} E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \right] A'_{YF} e_{\ell,d} \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ e'_{\ell,d} A_{YF} E [\underline{F}_t \underline{F}'_t \underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d} - e'_{\ell,d} A_{YF} E [\underline{F}_t \underline{F}'_t] E [\underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d} \} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E \left[\|\underline{F}_t\|_2^2 (e'_{\ell,d} A_{YF} \underline{F}_t)^2 \right] \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E [\|\underline{F}_t\|_2^4]} \sqrt{E (e'_{\ell,d} A_{YF} \underline{F}_t A'_{YF} e_{\ell,d})^2} \quad (\text{by CS inequality}) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E [\|\underline{F}_t\|_2^4]} \sqrt{E [\|\underline{F}_t\|_2^4]} \sqrt{(e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d})^2} \\
&\leq \frac{1}{q\tau_1^2} \sum_{\ell=1}^d \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E [\|\underline{F}_t\|_2^4]} \sqrt{E [\|\underline{F}_t\|_2^4]} \|A_{YF}\|_2^2 \sqrt{(e'_{\ell,d} e_{\ell,d})^2} \\
&\leq \frac{(C^\dagger)^2}{q\tau_1^2} \sum_{\ell=1}^d \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [\|\underline{F}_t\|_2^4] \phi_{\max}^2 \\
&\quad (\text{by part (b) of Lemma C-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector}) \\
&\leq \frac{\overline{C}}{\tau_1} = O \left(\frac{1}{\tau_1} \right). \tag{65}
\end{aligned}$$

for some positive constant $\overline{C} \geq d (C^\dagger)^2 E [\|\underline{F}_t\|_2^4] \phi_{\max}^2$, which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1.

To analyze the second term on the right-hand side of expression (64), note first that by Lemma C-11, $\{F_t\}$ is β -mixing with β mixing coefficient satisfying

$$\beta_F(m) \leq C_1 \exp \{-C_2 m\} \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{F,m} \leq \beta_F(m)$, it follows that F_t is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp \{-C_2 m\}$$

Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}$ is also α -mixing with α mixing coefficient satisfying

$$\begin{aligned}\alpha_{X,m} &\leq C_1 \exp\{-C_2(m-p+1)\} \\ &\leq C_1^* \exp\{-C_2 m\}\end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2(p-1)\}$. Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned}& \left| \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \right| \\&= \left| e'_{\ell,d} A_{YF} E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \right| \\&= \left| \sum_{h=1}^{Kp} e'_{\ell,d} A_{YF} E \left[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' e_{h,Kp} e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \right| \\&\leq \sum_{h=1}^{Kp} \left\{ 2 \left(2^{\frac{2}{3}} + 1 \right) \alpha_{X,m}^{\frac{1}{3}} \left(E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\&\quad \left. \times \left(E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YF} e_{\ell,d} \right|^3 \right)^{1/3} \right\}\end{aligned}$$

where $\alpha_{X,m}$ denotes the alpha mixing coefficient for the process $\{X_t\}$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp\left\{-\frac{C_2 m}{3}\right\} \text{ for all } m \text{ sufficiently large,}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned}\sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\&\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\&\leq (C_1^*)^{\frac{1}{3}} \left[1 - \exp\left\{-\frac{C_2}{3}\right\} \right]^{-1} \\&\leq C_3\end{aligned}$$

Next, note that

$$\begin{aligned}
& E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|^3 + |E [e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}]|^3 \right\} \text{ (by Loève's } c_r \text{ inequality)} \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|^3 + (E [|e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \text{ (by Jensen's inequality)} \\
& \leq 2^2 \left\{ E \left| \frac{e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}}{2} + \frac{e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}}{2} \right|^3 + (E [|e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \\
& \leq 4 \frac{2^2}{8} \left[E |e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}|^3 + E |e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}|^3 \right] \\
& \quad + 4 \left(\sqrt{E [e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}]} \sqrt{E [e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}]} \right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
& \leq 2 |e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d}|^3 E \|\underline{F}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{F}_t\|_2^2 \right)^3 (e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d})^{\frac{3}{2}} \\
& \leq 2 \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{F}_t\|_2^2 \right)^3 \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \\
& = 2 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{F}_t\|_2^2 \right)^3 (C^\dagger)^3 \phi_{\max}^3 \\
& \quad \text{(since } \|e_{\ell,d}\|_2 = 1 \text{ for every } \ell \in \{1, \dots, d\} \text{ given that } e_{\ell,d} \text{'s are elementary vectors)} \\
& \leq C_6
\end{aligned}$$

for some positive constant $C_6 \geq 2 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + 2E \|\underline{F}_t\|_2^6 + 4 \left(E \|\underline{F}_t\|_2^2 \right)^3 (C^\dagger)^3 \phi_{\max}^3$ which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1. In a similar way, we can also show that there exists a positive constant C_7 such that

$$\begin{aligned}
E |e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E [\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YF} e_{\ell,d}|^3 & \leq 2 \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_{t+m}\|_2^6 + 2E \|\underline{F}_{t+m}\|_2^6 \\
& \quad + 4 \left(E \|\underline{F}_{t+m}\|_2^2 \right)^3 \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \\
& \leq C_7 < \infty
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{2}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YF} \\
& \quad \times E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E [\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \Big| \\
& \leq \frac{4 \left(2^{\frac{2}{3}} + 1 \right)}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \sum_{h=1}^{Kp} \left\{ \alpha_{F,m}^{\frac{1}{3}} \left(E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \left. \times \left(E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E [\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YF} e_{\ell,d} \right|^3 \right)^{1/3} \right\} \\
& \leq \frac{4dKp \left(2^{\frac{2}{3}} + 1 \right) C_6^{\frac{1}{3}} C_7^{\frac{1}{3}}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{\infty} (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& \leq \frac{C^*}{\tau_1} \left(\frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left(\text{where } C^* \geq 4dKp \left(2^{\frac{2}{3}} + 1 \right) (C_1^*)^{\frac{1}{3}} C_6^{\frac{1}{3}} C_7^{\frac{1}{3}} \right) \\
& \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left(\frac{1}{\tau_1} \right)
\end{aligned} \tag{66}$$

It then follows from expressions (63), (64), (65), and (66) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E [\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& \quad + \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left| \alpha'_{YF,\ell} \right. \\
& \quad \quad \quad \left. \times E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E [\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \right| \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\overline{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C \overline{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{C C^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left(\frac{1}{\tau_1} \right) + O \left(\frac{1}{\tau_1} \right) \\
& = O \left(\frac{1}{\tau_1} \right) = o(1).
\end{aligned}$$

Now, to show part (c), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon^2 \right\} \quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left(\gamma'_i \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)' \right. \\
&\quad \left. \times \left[\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell} (\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s]) \mu_{Y,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \quad (67) \\
&\quad (\text{by Assumption 3-5})
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_s - E[\underline{F}_s])] \right) \\
&= \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_t - E[\underline{F}_t])] \right) \\
&\quad + \sum_{\ell=1}^d \left(\frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_{t+m} - E[\underline{F}_{t+m}])] \right) \\
&\leq \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_t - E[\underline{F}_t])] \right) \\
&\quad + \frac{2}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \quad (68)
\end{aligned}$$

Consider the first term on the majorant side of expression (68), whose order of magnitude we can analyze as follows

$$\begin{aligned}
& \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_t - E[\underline{F}_t])] \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 \{E[\underline{F}_t' \underline{F}_t] - E[\underline{F}_t]' E[\underline{F}_t]\} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[\|\underline{F}_t\|_2^2] \\
&= \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[\|\underline{F}_t\|_2^2] \sum_{\ell=1}^d (\mu_{Y,\ell}^2) \\
&\leq \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[\|\underline{F}_t\|_2^2] \|\mu_Y\|_2^2 \\
&\leq \frac{\overline{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \quad (69)
\end{aligned}$$

for some positive constant $\overline{C} \geq \|\mu_Y\|_2^2 E[\|\underline{F}_t\|_2^2]$, which exists in light of Assumption 3-5 and Lemma C-5.

To analyze the second term on the right-hand side of expression (68), note first that by the same

argument as given for part (b) above, we can apply Lemma C-11 to deduce that $\{F_t\}$ is β -mixing and, thus, also α -mixing with α mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp \{-C_2 m\}$$

Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned} & |E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\ &= \left| \sum_{h=1}^{Kp} e'_{\ell,d} A_{YF} E[(\underline{F}_t - E[\underline{F}_t])'(\underline{F}_{t+m} - E[\underline{F}_{t+m}])] A'_{YF} e_{\ell,d} \right| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\ &\leq \sum_{h=1}^{Kp} 2 \left(2^{\frac{2}{3}} + 1 \right) \alpha_{F,m}^{\frac{1}{3}} \left(E |(\underline{F}_t - E[\underline{F}_t])' e_{h,Kp}|^3 \right)^{\frac{1}{3}} \left(E |e'_{h,Kp}(\underline{F}_{t+m} - E[\underline{F}_{t+m}])|^3 \right)^{1/3} \sum_{\ell=1}^d \mu_{Y,\ell}^2 \end{aligned}$$

Moreover, there exists a positive constant C_3 such that

$$\sum_{m=1}^{\infty} \alpha_{F,m}^{\frac{1}{3}} \leq C_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \leq C_3$$

where again the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp \{- (C_2 m/3)\}$ is a convergent geometric series given that $0 < \exp \{- (C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$\begin{aligned} & E |(\underline{F}_t - E[\underline{F}_t])' e_{h,Kp}|^3 \\ &\leq 2^2 \left\{ E |\underline{F}'_t e_{h,Kp}|^3 + |E[\underline{F}'_t e_{h,Kp}]|^3 \right\} \text{ (by Loève's } c_r \text{ inequality)} \\ &\leq 2^2 \left\{ E |\underline{F}'_t e_{h,Kp}|^3 + (E[|\underline{F}'_t e_{h,Kp}|])^3 \right\} \text{ (by Jensen's inequality)} \\ &\leq 2^2 \left\{ E \left[(\underline{F}'_t \underline{F}_t)^{\frac{3}{2}} (e'_{h,Kp} e_{h,Kp})^{\frac{3}{2}} \right] + \left(\sqrt{E[\underline{F}'_t \underline{F}_t]} \sqrt{e'_{h,Kp} e_{h,Kp}} \right)^3 \right\} \text{ (by CS inequality)} \\ &\leq 4 \left\{ E [\|\underline{F}_t\|_2^3] + \left(E [\|\underline{F}_t\|_2^2] \right)^{\frac{3}{2}} \right\} \\ &\leq C_8 \end{aligned}$$

for some positive constant $C_8 \geq 4 \left\{ E [\|\underline{F}_t\|_2^3] + \left(E [\|\underline{F}_t\|_2^2] \right)^{\frac{3}{2}} \right\}$ which exists in light of the result given in Lemma C-5. In a similar way, we can also show that there exists a positive constant C_9

such that

$$\begin{aligned} E |e'_\ell (\underline{F}_{t+m} - E [\underline{F}_{t+m}])|^3 &\leq 4 \left\{ E [\|\underline{F}_{t+m}\|_2^3] + \left(E [\|\underline{F}_{t+m}\|_2^2] \right)^{\frac{3}{2}} \right\} \\ &\leq C_9 < \infty \end{aligned}$$

Finally, by Assumption 3-5, there exists a positive constant C_{10} such that $\max_{1 \leq \ell \leq d} \mu_{Y,\ell}^2 \leq \|\mu_Y\|_2^2 \leq C_{10} < \infty$. Hence,

$$\begin{aligned} & \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E [(\underline{F}_t - E [\underline{F}_t])' (\underline{F}_{t+m} - E [\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\ & \leq \sum_{h=1}^{Kp} \frac{4 \left(2^{\frac{2}{3}} + 1 \right)}{\tau_1^2} \|\mu_Y\|_2^2 \\ & \quad \times \frac{1}{q} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left\{ \alpha_{F,m}^{\frac{1}{3}} \left(E |(\underline{F}_t - E [\underline{F}_t])' e_{h,Kp}|^3 \right)^{\frac{1}{3}} \right. \\ & \quad \left. \times \left(E |e'_{h,Kp} (\underline{F}_{t+m} - E [\underline{F}_{t+m}])|^3 \right)^{1/3} \right\} \\ & \leq \frac{4Kp \left(2^{\frac{2}{3}} + 1 \right) C_8^{\frac{1}{3}} C_9^{\frac{1}{3}} C_{10}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{\infty} C_1^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ & \leq \frac{C^*}{\tau_1} \left(\frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left(\text{where } C^* \geq 4Kp \left(2^{\frac{2}{3}} + 1 \right) C_1^{\frac{1}{3}} C_8^{\frac{1}{3}} C_9^{\frac{1}{3}} C_{10} \right) \\ & \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ & = O \left(\frac{1}{\tau_1} \right) \end{aligned} \tag{70}$$

It then follows from expressions (67), (68), (69), and (70) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_t - E[\underline{F}_t])] \right) \\
& \quad + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\bar{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C \bar{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{C C^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left(\frac{1}{\tau_1} \right) + O \left(\frac{1}{\tau_1} \right) \\
& = O \left(\frac{1}{\tau_1} \right) = o(1).
\end{aligned}$$

Turning our attention to part (d), note that, by apply Loève's c_r inequality, we obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& \leq 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \\
& \quad + 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \\
& \quad + 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2
\end{aligned}$$

It follows from the arguments given in the proofs of parts (a)-(c) above that, for any $\epsilon > 0$,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left(\sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[(\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E [\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \right) \\
& = o(1),
\end{aligned}$$

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left(\sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[(\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E [\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
& = o(1)
\end{aligned}$$

and

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E [\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left(\sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E \left[(\underline{F}_t - E [\underline{F}_t])' (\underline{F}_s - E [\underline{F}_s]) \right] \right) \\
& = o(1),
\end{aligned}$$

from which we deduce via the Slutsky's theorem that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E [\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E [\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& = o_p(1)
\end{aligned}$$

as required.

For part (e), note that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{\pi_{i,\ell,T}}{q\tau_1^2} \right) \\
&= \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&= \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] A'_{YY} e_{\ell,d} + \gamma'_i E[\underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d} \} \right)^2 \\
&\leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ E[|\gamma'_i \underline{F}_t|] |\mu_{Y,\ell}| + E[|\gamma'_i \underline{F}_t \underline{Y}'_t A'_{YY} e_{\ell,d}|] + E[|\gamma'_i \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}|] \} \right)^2 \\
&\quad (\text{by triangle and Jensen's inequalities}) \\
&\leq \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \sqrt{\|\gamma_i\|_2^2} \sqrt{E\|\underline{F}_t\|_2^2} \max_{1 \leq \ell \leq d} |\mu_{Y,\ell}| \right. \right. \\
&\quad \left. \left. + \sqrt{\gamma'_i E[\underline{F}_t \underline{F}'_t]} \gamma_i \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} A_{YY} E[\underline{Y}_t \underline{Y}'_t] A'_{YY} e_{\ell,d}} \right. \right. \\
&\quad \left. \left. + \sqrt{\gamma'_i E[\underline{F}_t \underline{F}'_t]} \gamma_i \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} A_{YF} E[\underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d}} \right\} \right)^2 \\
&\leq \left(\max_{i \in H^c} \|\gamma_i\|_2^2 \right) \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \sqrt{E\|\underline{F}_t\|_2^2} \max_{1 \leq \ell \leq d} |\mu_{Y,\ell}| \right. \right. \\
&\quad \left. \left. + \sqrt{E\|\underline{F}_t\|_2^2} \sqrt{E\|\underline{Y}_t\|_2^2} \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d}} + E\|\underline{F}_t\|_2^2 \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d}} \right\} \right)^2 \\
&\leq \left(\max_{i \in H^c} \|\gamma_i\|_2^2 \right) \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \sqrt{E\|\underline{F}_t\|_2^2} \max_{1 \leq \ell \leq d} |\mu_{Y,\ell}| \right. \right. \\
&\quad \left. \left. + \sqrt{E\|\underline{F}_t\|_2^2} \sqrt{E\|\underline{Y}_t\|_2^2} C^\dagger \phi_{\max} \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} e_{\ell,d}} + E\|\underline{F}_t\|_2^2 C^\dagger \phi_{\max} \sqrt{\max_{1 \leq \ell \leq d} e'_{\ell,d} e_{\ell,d}} \right\} \right)^2 \\
&\quad (\text{by Lemma C-7}) \\
&= \left(\max_{i \in H^c} \|\gamma_i\|_2^2 \right) \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \sqrt{E\|\underline{F}_t\|_2^2} \max_{1 \leq \ell \leq d} |\mu_{Y,\ell}| \right. \right. \\
&\quad \left. \left. + \sqrt{E\|\underline{F}_t\|_2^2} \sqrt{E\|\underline{Y}_t\|_2^2} C^\dagger \phi_{\max} + E\|\underline{F}_t\|_2^2 C^\dagger \phi_{\max} \right\} \right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\max_{i \in H^c} \|\gamma_i\|_2^2 \right) \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left(\sqrt{E \|\underline{E}_t\|_2^2} \max_{1 \leq \ell \leq d} |\mu_{Y,\ell}| \right. \\
&\quad \left. + \sqrt{E \|\underline{E}_t\|_2^2} \sqrt{E \|\underline{Y}_t\|_2^2} C^\dagger \phi_{\max} + E \|\underline{E}_t\|_2^2 C^\dagger \phi_{\max} \right)^2 \\
&\leq \left(\max_{i \in H^c} \|\gamma_i\|_2^2 \right) \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left(\sqrt{E \|\underline{E}_t\|_2^2} \|\mu_Y\|_2^2 \right. \\
&\quad \left. + \sqrt{E \|\underline{E}_t\|_2^2} \sqrt{E \|\underline{Y}_t\|_2^2} C^\dagger \phi_{\max} + E \|\underline{E}_t\|_2^2 C^\dagger \phi_{\max} \right)^2 \\
&\leq \bar{C} < \infty
\end{aligned}$$

for some positive constant \bar{C} such that

$$\bar{C} \geq \left(\sup_{i \in H^c} \|\gamma_i\|_2^2 \right) E \|\underline{E}_t\|_2^2 \left(\|\mu_Y\|_2^2 + \sqrt{E \|\underline{Y}_t\|_2^2} C^\dagger \phi_{\max} + \sqrt{E \|\underline{E}_t\|_2^2} C^\dagger \phi_{\max} \right)^2$$

where such a constant exists in light of Assumption 3-5, Lemma C-5, and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1.

To show part (f), note that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2 \\
\leq & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right\} \\
& \quad \left. + \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
\leq & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{2}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& \quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{2}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
& \quad \text{(by Loève's } c_r \text{ inequality)} \\
= & o_p(1) + O(1) \quad \text{(applying the results given in parts (d) and (e) of this lemma)} \\
= & O_p(1).
\end{aligned}$$

To show part (g), we apply the Cauchy-Schwarz inequality as well as parts (d) and (e) of this

lemma to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left\{ \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \gamma'_i(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right\} \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \right\} \right) \right\} \Bigg| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \gamma'_i(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right\} \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \right\} \right) \right| \\
& \leq \left[\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i(\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i(\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \right. \\
& \quad \left. \left. \left. + \gamma'_i(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right\} \right)^2 \right]^{1/2} \\
& \quad \times \left[\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \right\} \right)^2 \right]^{1/2} \\
& = o_p(1) O(1) \\
& = o_p(1).
\end{aligned}$$

For part (h), we apply the Cauchy-Schwarz inequality as well as part (d) of Lemma C-6 and

part (f) of this lemma to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& \leq \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2} \\
& \quad \times \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \\
& = O_p(1) o_p(1) \\
& = o_p(1)
\end{aligned}$$

Finally, for part (i), we apply the Cauchy-Schwarz inequality as well as part (b) of Lemma C-6

and part (f) of this lemma to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
& \leq \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2} \\
& \quad \times \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \\
& = O_p(1) o_p(1) \\
& = o_p(1). \quad \square
\end{aligned}$$

Lemma C-13: Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-5, 3-7, and 3-9 hold and suppose that $N_1, N_2, T \rightarrow \infty$ such that $N_1/\tau_1^3 = N_1/\lfloor T_0^{\alpha_1} \rfloor^3 \rightarrow 0$. Then, the following statements are true.

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \xrightarrow{p} 0$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{\pi_{i,\ell,T}} \right| \xrightarrow{p} 0$$

Proof of Lemma C-13:

To show part (a), note first that by applying parts (a) and (c) of Lemma C-6, parts (a)-(c) of

Lemma C-12, and the Slutsky theorem; we obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{q\tau_1} \right| \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right. \\
& + \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} + \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \\
& \left. - \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right| \\
\leq & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \\
= & o_p(1)
\end{aligned}$$

Moreover, by Assumption 3-9, there exist a positive constant \underline{c} such that for all N and T sufficiently large

$$\begin{aligned}
& \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{\mu_{i,\ell,T}}{q\tau_1} \right| \\
= & \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right| \\
\geq & \underline{c} > 0
\end{aligned}$$

It follows that

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{q\tau_1} \right| / \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{\mu_{i,\ell,T}}{q\tau_1} \right| = o_p(1).$$

Now, for part (b), note that, applying parts (d), (g), (h), and (i) of Lemma C-12, parts (b), (d), and (e) of Lemma C-6, and the Slutsky theorem; we have

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{q\tau_1^2} \right| \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{2}{q} \sum_{r=1}^q \left\{ \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \\
& \quad \left. \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right) \right\} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
= & o_p(1)
\end{aligned}$$

Moreover, note that, for all N and T sufficiently large,

$$\begin{aligned}
& \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{\pi_{i,\ell,T}}{q\tau_1^2} \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&\geq \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&\quad \text{(by Jensen's inequality)} \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right|^2 \\
&= \left(\min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right| \right)^2 \\
&\geq \underline{c}^2 > 0 \quad \text{(by Assumption 3-9).}
\end{aligned}$$

It follows that

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{\pi_{i,\ell,T}} \right| \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{q\tau_1^2} \right| / \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left(\frac{\pi_{i,\ell,T}}{q\tau_1^2} \right) = o_p(1). \quad \square$$

Lemma C-14: Let $a, b \in \mathbb{R}$ such that $a \geq 0$ and $b \geq 0$. Then,

$$\left| \sqrt{a} - \sqrt{b} \right| \leq \sqrt{|a - b|}$$

Proof of Lemma C-14: Note that

$$\begin{aligned}
\left(\sqrt{a} - \sqrt{b}\right)^2 &= a - 2\sqrt{a}\sqrt{b} + b \\
&= \sqrt{a}(\sqrt{a} - \sqrt{b}) + \sqrt{b}(\sqrt{b} - \sqrt{a}) \\
&\leq \sqrt{a}|\sqrt{a} - \sqrt{b}| + \sqrt{b}|\sqrt{b} - \sqrt{a}| \\
&= (\sqrt{a} + \sqrt{b})|\sqrt{a} - \sqrt{b}| \\
&= |(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})| \\
&= |a - b|
\end{aligned}$$

Taking principal square root on both sides, we obtain

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}. \quad \square$$

Lemma C-15:

$$P\left\{\bigcap_{i=1}^m A_i\right\} \geq \sum_{i=1}^m P(A_i) - (m - 1)$$

Proof of Lemma C-15:

$$\begin{aligned}
P\left\{\bigcap_{i=1}^m A_i\right\} &= 1 - P\left\{\left(\bigcap_{i=1}^m A_i\right)^c\right\} \\
&= 1 - P\left\{\bigcup_{i=1}^m A_i^c\right\} \quad (\text{by DeMorgan's Law}) \\
&\geq 1 - \sum_{i=1}^m P(A_i^c) \\
&= 1 - \sum_{i=1}^m [1 - P(A_i)] \\
&= \sum_{i=1}^m P(A_i) - m + 1 \\
&= \sum_{i=1}^m P(A_i) - (m - 1). \quad \square
\end{aligned}$$

Lemma C-16:

(a) For $t > 0$,

$$\bar{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t},$$

where $\phi(t)$ and $\Phi(t)$ denote, respectively, the pdf and the cdf of a standard normal random variable.

- (b) Let $N = N_1 + N_2$. Specify φ such that $\varphi \rightarrow 0$ as $N_1, N_2 \rightarrow \infty$ and such that, for some constant $a > 0$,

$$\varphi \geq \frac{1}{N^a}$$

for all N_1, N_2 sufficiently large. Then, for all N_1, N_2 sufficiently large such that

$$1 - \frac{\varphi}{2N} \geq \Phi(2)$$

we have

$$\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2(1+a)}\sqrt{\ln N}.$$

Proof of Lemma C-16:

- (a)

$$\begin{aligned} 1 - \Phi(t) &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &= \int_t^\infty \frac{1}{z} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &\leq \frac{1}{t} \int_t^\infty \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \end{aligned}$$

Let

$$u = -\frac{z^2}{2} \text{ and } du = -z dz$$

so that

$$\begin{aligned} \int_t^\infty \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz &= -\int_{-\frac{t^2}{2}}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\ &= \int_{-\infty}^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \\ &= \phi(t) \end{aligned}$$

It follows that

$$\bar{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t}.$$

(b) Let $t > 0$ and let

$$\Phi(t) = \Pr(Z \leq t) = 1 - \frac{\varphi}{2N}$$

Note that

$$\Phi^{-1}(\Phi(t)) = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) = t$$

and, by the result given in part (a) above,

$$1 - \Phi(t) = 1 - \left(1 - \frac{\varphi}{2N}\right) = \frac{\varphi}{2N} \leq \frac{\phi(t)}{t}$$

The latter inequality implies that

$$t \leq \phi(t) \frac{2N}{\varphi}$$

so that

$$\begin{aligned} \ln t &\leq \ln \phi(t) + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\ &= -\frac{1}{2}t^2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln \pi + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\ &= -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln \pi + \ln\left(\frac{N}{\varphi}\right) \\ &< -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 + \ln\left(\frac{N}{\varphi}\right) \\ &< -\frac{1}{2}t^2 + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \end{aligned}$$

or

$$\begin{aligned} t^2 &\leq 2(\ln 2 - \ln t) + 2\ln\left(\frac{N}{\varphi}\right) \\ &= 2\ln\left(\frac{2}{t}\right) + 2\ln\left(\frac{N}{\varphi}\right) \\ &\leq 2\ln\left(\frac{N}{\varphi}\right) \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2 \end{aligned}$$

so that

$$t \leq \sqrt{2} \sqrt{\ln\left(\frac{N}{\varphi}\right)} \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2$$

Hence, for N_1, N_2 sufficiently large so that

$$1 - \frac{\varphi}{2N} \geq \Phi(2) \text{ or, equivalently, } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2,$$

we have

$$\begin{aligned}
\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) &= t \\
&\leq \sqrt{2}\sqrt{\ln\left(\frac{N}{\varphi}\right)} \\
&= \sqrt{2}\sqrt{\ln N - \ln \varphi} \\
&= \sqrt{2}\sqrt{\ln N}\sqrt{1 - \frac{\ln \varphi}{\ln N}} \\
&\leq \sqrt{2}\sqrt{\ln N}\sqrt{1 - \frac{\ln N^{-a}}{\ln N}} \\
&= \sqrt{2(1+a)}\sqrt{\ln N}. \quad \square
\end{aligned}$$

Lemma C-17: Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c) 3-4, 3-5, 3-7, and 3-8 hold. Let $\Phi(\cdot)$ denote the cumulative distribution function of the standard normal random variable. Then, there exists a positive constant A such that

$$P(|S_{i,\ell,T}| \geq z) \leq 2[1 - \Phi(z)] \left\{1 + A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\} \quad (71)$$

for

$$i \in H = \{k \in \{1, \dots, N\} : \gamma_k = 0\},$$

for $\ell \in \{1, \dots, d\}$, for T sufficiently large, and for all z such that

$$0 \leq z \leq c_0 \min \left\{ T^{(1-\alpha_1)\frac{1}{6}}, T^{\frac{\alpha_2}{2}} \right\}$$

with c_0 being a positive constant.

Proof of Lemma C-17:

Note first that, for any i such that

$$i \in H = \{k \in \{1, \dots, N\} : \gamma_k = 0\},$$

the formula for $S_{i,\ell,T}$ reduces to

$$S_{i,\ell,T} = \left(\sum_{r=1}^q \left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right]^2 \right)^{-\frac{1}{2}} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it}.$$

Hence, to verify the conditions of Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we set $X_{it} = u_{it}y_{\ell,t+1}$, and note that

$$\begin{aligned}
E[X_{it}] &= E[u_{it}y_{\ell,t+1}] \\
&= E_Y[E[u_{it}]y_{\ell,t+1}] \quad (\text{by the law of iterated expectations}) \\
&\quad \text{given the independence of } u_{it} \text{ and } y_{\ell,t+1} \text{ in light of Assumption 3-4)} \\
&= 0 \quad (\text{by Assumption 3-3(a)})
\end{aligned}$$

so that the first part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is fulfilled. Moreover, in light of Assumptions 3-3(b) and Lemma C-5, we see that there exists some positive constant c_1 such that, for $\ell \in \{1, \dots, d\}$,

$$\begin{aligned}
E[|X_{it}|^{\frac{31}{10}}] &= E[|u_{it}y_{\ell,t+1}|^{\frac{31}{10}}] \\
&\leq \left(E|u_{it}|^{\frac{186}{29}}\right)^{\frac{29}{60}} \left(E|y_{\ell,t+1}|^6\right)^{\frac{31}{60}} \quad (\text{by Hölder's inequality}) \\
&\leq \left[\left(E|u_{it}|^{\frac{186}{29}}\right)^{\frac{29}{186}}\right]^{\frac{31}{10}} \left[E\left(\sum_{k=1}^d \sum_{j=0}^{p-1} y_{k,t+1-j}^2\right)\right]^{\frac{31}{60}} \\
&\leq \left[\left(E|u_{it}|^7\right)^{\frac{1}{7}}\right]^{\frac{31}{10}} \left[\left(E\|\underline{Y}_{t+1}\|_2^6\right)^{\frac{1}{6}}\right]^{\frac{31}{10}} \quad (\text{by Liapunov's inequality}) \\
&\leq c_1^{\frac{31}{10}}
\end{aligned}$$

Hence, the second part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is also fulfilled with $r = \frac{31}{10} > 2$. Moreover, note that, by Assumption 3-8, for all $r \geq 1$, $\tau_1 \geq 1$, and $\tau_2 = \tau - \tau_1 \geq 1$

$$\begin{aligned}
E\left\{\left[\frac{1}{\sqrt{\tau_1}} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} &= E\left\{\left[\frac{1}{\sqrt{\tau_1}} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right]^2\right\} \\
&= \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\{y_{\ell,t+1}u_{it}u_{is}y_{\ell,s+1}\} \\
&\geq \underline{c}
\end{aligned}$$

or

$$E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} = E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right]^2\right\} \geq \underline{c}\tau_1,$$

so that condition (4.2) of Chen, Shao, Wu, and Xu (2016) is also satisfied. Finally, by Lemma C-11, Assumption 3-3(c), and Assumption 3-4; $\{(y_{\ell,t+1}, u_{it})'\}$ is β mixing with β mixing coefficient satisfying

$$\beta(m) \leq a_1 \exp\{-a_2 m\}$$

for some constants $a_1 > 0$ and $a_2 > 0$. It follows by part (a) of Lemma C-2 that $\{X_{it}\}$ (with $X_{it} = u_{it}y_{\ell,t+1}$) satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for all $i \in H$. Hence, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016) for the case where $\delta = 1^1$, we obtain the Cramér-type moderate deviation result

$$\frac{P\left\{\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\}}{1 - \Phi(z)} = 1 + O(1)(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \quad (72)$$

for all $0 \leq z \leq c_0 \min\left\{T^{(1-\alpha_1)\frac{1}{6}}, T^{\frac{\alpha_2}{2}}\right\}$ and for $|O(1)| \leq A$ with A being an absolute constant.

In addition, note that

$$\begin{aligned} \frac{P\left\{\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right\}}{\Phi(-z)} &= \frac{P\left\{-\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\}}{\Phi(-z)} \\ &= \frac{P\left\{-\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\}}{1 - \Phi(z)} \end{aligned}$$

Since

$$-\bar{S}_{i,\ell,T} = \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (-u_{it}y_{\ell,t+1}),$$

for this case, we can take $X_{it} = -u_{it}y_{\ell,t+1}$, and note that, trivially, by calculations similar to those

¹Note that Theorem 4.1 of Chen, Shao, Wu, and Xu (2016) requires that $0 < \delta \leq 1$ and $\delta < r-2$. These conditions are satisfied here given that we choose $\delta = 1$ and $r = 31/10$.

given above, we have

$$\begin{aligned}
E[X_{it}] &= E[-u_{it}y_{\ell,t+1}] = 0, \\
E\left[|X_{it}|^{\frac{31}{10}}\right] &= E\left[|-u_{it}y_{\ell,t+1}|^{\frac{31}{10}}\right] = E\left[|u_{it}y_{\ell,t+1}|^{\frac{31}{10}}\right] \leq c_1^{\frac{31}{10}}, \text{ and} \\
E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} &= E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (-u_{it}y_{\ell,t+1})\right]^2\right\} \\
&= E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} u_{it}y_{\ell,t+1}\right]^2\right\} \geq \underline{c}T_1.
\end{aligned}$$

Moreover, it is easily seen that $\{X_{it}\}$ (with $X_{it} = -u_{it}y_{\ell,t+1}$) also satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for every i . Thus, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we also obtain the Cramér-type moderate deviation result

$$\frac{P\left\{\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right\}}{\Phi(-z)} = 1 + O(1)(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \quad (73)$$

for all $0 \leq z \leq c_0 \min\left\{T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}}\right\}$ and for $|O(1)| \leq A$ with A being an absolute constant.

Next, note that

$$\begin{aligned}
& \left| \frac{P(|S_{i,\ell,T}| \geq z)}{2[1 - \Phi(z)]} - 1 \right| \\
= & \left| \frac{P\left(\left|\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}}\right| \geq z\right)}{2[1 - \Phi(z)]} - 1 \right| \\
= & \left| \frac{P\left(\left\{\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\} \cup \left\{-\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\}\right)}{2[1 - \Phi(z)]} - 1 \right| \\
= & \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right) + P\left(-\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right)}{2[1 - \Phi(z)]} - 1 \right| \\
& \left(\text{since } \left\{\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\} \cap \left\{-\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right\} = \emptyset\right) \\
= & \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right)}{2[1 - \Phi(z)]} - \frac{1}{2} + \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right)}{2\Phi(-z)} - \frac{1}{2} \right| \\
= & \left| \frac{1}{2} \left[\frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right)}{[1 - \Phi(z)]} - 1 \right] + \frac{1}{2} \left[\frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right)}{\Phi(-z)} - 1 \right] \right| \\
\leq & \frac{1}{2} \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right)}{[1 - \Phi(z)]} - 1 \right| + \frac{1}{2} \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right)}{\Phi(-z)} - 1 \right|
\end{aligned}$$

so that in light of expressions (72) and (73), we have

$$\begin{aligned}
\left| \frac{P(|S_{i,\ell,T}| \geq z)}{2[1 - \Phi(z)]} - 1 \right| &= \left| \frac{P\left(\left|\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}}\right| \geq z\right)}{2[1 - \Phi(z)]} - 1 \right| \\
&\leq \frac{1}{2} \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \geq z\right)}{[1 - \Phi(z)]} - 1 \right| + \frac{1}{2} \left| \frac{P\left(\bar{S}_{i,\ell,T}/\sqrt{\bar{V}_{i,\ell,T}} \leq -z\right)}{\Phi(-z)} - 1 \right| \\
&\leq \frac{A}{2} (1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} + \frac{A}{2} (1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \\
&= A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}}
\end{aligned}$$

It then follows that

$$-A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \leq \frac{P(|S_{i,\ell,T}| \geq z)}{2[1 - \Phi(z)]} - 1 \leq A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \quad (74)$$

Focusing on the right-hand part of the inequality in (74), we have

$$\frac{P(|S_{i,\ell,T}| \geq z)}{2[1 - \Phi(z)]} - 1 \leq A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}}$$

Simple rearrangement of the inequality above then leads to the desired result

$$P(|S_{i,\ell,T}| \geq z) \leq 2[1 - \Phi(z)] \left\{ 1 + A(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \right\},$$

which holds for

$$i \in H = \{k \in \{1, \dots, N\} : \gamma_k = 0\},$$

for $\ell \in \{1, \dots, d\}$, for all T sufficiently large, and for all z such that

$$0 \leq z \leq c_0 \min \left\{ T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}} \right\}. \quad \square$$

4 Appendix D: Supporting Lemmas Used in the Proofs of Theorems 4 and 5

Derivation of the h -step Ahead Forecasting Equation Given in Expression (23) of the Main Paper:

Consider the FAVAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}, \quad (75)$$

where $W_t = (Y'_t, F'_t)'$. Suppose that equation (75) satisfies Assumptions 3-1 and 3-2 of the main paper. Then, similar to a VAR process, we can rewrite this model in the companion form

$$\underline{W}_t = \alpha + A \underline{W}_{t-1} + E_t$$

where

$$\begin{aligned} \underline{W}_t &= \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \quad W_t = \begin{pmatrix} Y_t \\ F_t \end{pmatrix}, \quad E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \text{ and} \\ A &= \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}. \end{aligned} \quad (76)$$

Successive substitution for the lagged \underline{W}_t 's gives

$$\underline{W}_{t+h} = \sum_{j=0}^{h-1} A^j \alpha + A^h \underline{W}_t + \sum_{j=0}^{h-1} A^j E_{t+h-j}$$

Let

$$J_d = \begin{bmatrix} I_d & 0 & \cdots & 0 \end{bmatrix} \text{ and } J_{d+K} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}$$

and note that

$$J_d \underline{W}_{t+h} = Y_{t+h}, \quad J_{d+K} E_{t+h-j} = \varepsilon_{t+h-j},$$

and

$$J'_{d+K} J_{d+K} E_{t+h-j} = \begin{pmatrix} I_{d+K} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{aligned} Y_{t+h} &= J_d \underline{W}_{t+h} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} J_{d+K} E_{t+h-j} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \end{aligned} \quad (77)$$

Furthermore, let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p} \underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}, \text{ where } \underline{Y}_t = \begin{pmatrix} Y_t \\ \vdots \\ Y_{t-p+1} \end{pmatrix} \text{ and } \underline{F}_t = \begin{pmatrix} F_t \\ \vdots \\ F_{t-p+1} \end{pmatrix}. \quad (78)$$

and note that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix, so that $\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} = I_{(d+K)p} = \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p}$. Next, for $g = 1, \dots, p$, let $e_{g,p}$ be a $p \times 1$ elementary vector whose g^{th} component is 1 and all other

components are 0; and define

$$\begin{aligned}
S_{d,g} &= \begin{pmatrix} e_{g,p} \otimes I_d \\ 0 \\ Kp \times d \end{pmatrix}, \quad S_{K,g} = \begin{pmatrix} 0 \\ dp \times K \\ e_{g,p} \otimes I_K \end{pmatrix}, \\
S_d &= \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} \\
&= \begin{pmatrix} e_{1,p} \otimes I_d & e_{2,p} \otimes I_d & \cdots & e_{p,p} \otimes I_d \\ 0 & 0 & \cdots & 0 \\ Kp \times d & Kp \times d & \cdots & Kp \times d \end{pmatrix} \\
&= \begin{pmatrix} I_p \otimes I_d \\ 0 \\ Kp \times dp \end{pmatrix} = \begin{pmatrix} I_{dp} \\ 0 \\ Kp \times dp \end{pmatrix} \\
S_K &= \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ dp \times K & dp \times K & \cdots & dp \times K \\ e_{1,p} \otimes I_K & e_{2,p} \otimes I_K & \cdots & e_{p,p} \otimes I_K \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ dp \times Kp \\ I_p \otimes I_K \end{pmatrix} = \begin{pmatrix} 0 \\ dp \times Kp \\ I_{Kp} \end{pmatrix}
\end{aligned}$$

It follows that

$$S_{(d+K)p \times (d+K)p} = \begin{pmatrix} S_d & S_K \\ (d+K)p \times dp & (d+K)p \times Kp \end{pmatrix} = \begin{pmatrix} I_{dp} & 0 \\ 0 & I_{Kp} \\ Kp \times dp & \end{pmatrix} = I_{(d+K)p} \quad (79)$$

In addition, using these notations, it is easy to see that

$$S'_{d,g} \mathcal{P}_{(d+K)p} \underline{W}_t = Y_{t-g+1} \text{ for } g = 1, \dots, p \quad (80)$$

and, similarly,

$$S'_{K,g} \mathcal{P}_{(d+K)p} \underline{W}_t = F_{t-g+1} \text{ for } g = 1, \dots, p. \quad (81)$$

Hence, making use of expressions (77) and (79) and the fact that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix,

we can write

$$\begin{aligned}
Y_{t+h} &= J_d \underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} S S' \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} (S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g}) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}
\end{aligned}$$

so that, in light of expressions (80) and (81), we further deduce that

$$\begin{aligned}
Y_{t+h} &= J_d \underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} (S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g}) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} S'_{d,g} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} S'_{K,g} \mathcal{P}_{(d+K)p} \underline{W}_t \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} Y_{t-g+1} + \sum_{g=1}^p J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} F_{t-g+1} \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \beta_0 + \sum_{g=1}^p B'_{1,g} Y_{t-g+1} + \sum_{g=1}^p B'_{2,g} F_{t-g+1} + \eta_{t+h}
\end{aligned}$$

where

$$\begin{aligned}
\beta_0 &= \sum_{j=0}^{h-1} J_d A^j \alpha, \quad \eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}, \\
B'_{1,g} &= J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} \text{ and } B'_{2,g} = J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} \text{ for } g = 1, \dots, p.
\end{aligned} \tag{82}$$

Next, define $B'_1 = \begin{pmatrix} B'_{1,1} & B'_{1,2} & \dots & B'_{1,p} \end{pmatrix}$ and $B'_2 = \begin{pmatrix} B'_{2,1} & B'_{2,2} & \dots & B'_{2,p} \end{pmatrix}$, and note that,

by expression (82) above,

$$\begin{aligned} B'_1 &= J_d A^h \mathcal{P}'_{(d+K)p} \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} = J_d A^h \mathcal{P}'_{(d+K)p} S_d \\ B'_2 &= J_d A^h \mathcal{P}'_{(d+K)p} \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} = J_d A^h \mathcal{P}'_{(d+K)p} S_K. \end{aligned}$$

Finally, let \underline{Y}_t and \underline{F}_t be as defined in expression (78), and we can write the h -step ahead forecast equation more succinctly as

$$\begin{aligned} Y_{t+h} &= \beta_0 + \sum_{g=1}^p B'_{1,g} Y_{t-g+1} + \sum_{g=1}^p B'_{2,g} F_{t-g+1} + \eta_{t+h} \\ &= \beta_0 + B'_1 \underline{Y}_t + B'_2 \underline{F}_t + \eta_{t+h}. \quad \square \end{aligned}$$

Lemma D-1: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7 hold. Then, the following statements are true.

(a) There exists a positive constant \underline{c} such that

$$\lambda_{\min} \left\{ \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \geq \underline{c} > 0,$$

where A is the coefficient matrix of the companion form given in expression (76) and where

$$J_{d+K} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}. \quad (83)$$

(b) The matrix

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E[\underline{Y}'_t] & E[\underline{F}'_t] \\ E[\underline{Y}_t] & E[\underline{Y}_t \underline{Y}'_t] & E[\underline{Y}_t \underline{F}'_t] \\ E[\underline{F}_t] & E[\underline{F}_t \underline{Y}'_t] & E[\underline{F}_t \underline{F}'_t] \end{pmatrix}$$

is non-singular for all $T > h + p - 1$.

Proof of Lemma D-1:

For part (a), we prove by contradiction. To proceed, let

$$J_{d+K,r} = e'_{r,p} \otimes I_{d+K} \text{ for } r \in \{1, \dots, p\}$$

where $e_{r,p}$ is a $p \times 1$ elementary vector whose r^{th} component is equal to 1 and all other components are equal to 0. Note that, under this definition, $J_{d+K,1} = J_{d+K}$, where J_{d+K} is as defined previously in expression (83). Suppose that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} (A^j)'$$

is singular; then, there exists $b \in \mathbb{R}^{(d+K)p} \setminus \{0\}$ such that

$$\sum_{j=0}^{\infty} b' A^j J'_{d+K,1} J_{d+K,1} (A^j)' b = 0$$

This, in turn, implies that $J_{d+K,1} (A^j)' b = 0$ for all j . Now, partition

$$b = \begin{pmatrix} b_1 \\ (d+K) \times 1 \\ b_2 \\ (d+K) \times 1 \\ \vdots \\ b_p \\ (d+K) \times 1 \end{pmatrix}$$

Note that, for $j = 0$, let $L_0 = I_{d+K}$, and it is easily seen that

$$\begin{aligned} 0 &= J_{d+K,1} (A^0)' b \\ &= J_{d+K,1} b \\ &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\ &= b_1 \quad (= L_0 b_1) \end{aligned}$$

Now, for $j = 1$, define $\overline{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \end{bmatrix}$, and note that

$$\begin{aligned}
0 &= J_{d+K,1} A' b \\
&= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{pmatrix} A'_1 & I_{d+K} & 0 & \cdots & 0 \\ A'_2 & 0 & I_{d+K} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A'_{p-1} & \vdots & 0 & \ddots & I_{d+K} \\ A'_p & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\
&= J_{d+K,1} \begin{bmatrix} \overline{A}' & J'_{d+K,1} & J'_{d+K,2} & \cdots & J'_{d+K,p-1} \end{bmatrix} b \\
&= \begin{bmatrix} J_{d+K,1} \overline{A}' J_{d+K,1} + J_{d+K,2} \end{bmatrix} b \\
&= [L_1 J_{d+K,1} + L_0 J_{d+K,2}] b \\
&= L_1 b_1 + L_0 b_2
\end{aligned}$$

where $L_1 = J_{d+K,1} \overline{A}' = A'_1$. Since previously we have shown that $b_1 = 0$, it follows that

$$b_2 = L_1 b_1 + L_0 b_2 = 0.$$

Moreover, for $j = 2$, using the fact that $J_{d+K,r} J'_{d+K,r} = I_{d+K}$ and $J_{d+K,r} J'_{d+K,s} = 0$ for $r \neq s$, we obtain

$$\begin{aligned}
0 &= J_{d+K,1} (A')^2 b \\
&= J_{d+K,1} \begin{bmatrix} \overline{A}' & J'_{d+K,1} & J'_{d+K,2} & \cdots & J'_{d+K,p-1} & J'_{d+K,p} \end{bmatrix}^2 b \\
&= [L_1 J_{d+K,1} + L_0 J_{d+K,2}] \begin{bmatrix} \overline{A}' & J'_{d+K,1} & J'_{d+K,2} & \cdots & J'_{d+K,p-1} & J'_{d+K,p} \end{bmatrix} b \\
&= \left([L_1 J_{d+K,1} + L_0 J_{d+K,2}] \overline{A}' J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3} \right) b \\
&= (L_2 J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3}) b \\
&= L_2 b_1 + L_1 b_2 + L_0 b_3
\end{aligned}$$

where

$$L_2 = [L_1 J_{d+K,1} + L_0 J_{d+K,2}] \overline{A}'$$

Given that $b_1 = 0$ and $b_2 = 0$, as we have previously shown, it then follows that

$$b_3 = L_2 b_1 + L_1 b_2 + L_0 b_3 = 0 \quad (\text{since } L_0 = I_{d+K})$$

We will show by mathematical induction that, in fact, $b_r = 0$ for every $r \in \{1, \dots, p\}$. To proceed, suppose that $b_1 = b_2 = \dots = b_j = 0$ and $0 = J_{d+K,1} (A')^j b$. By straightforward calculations, one can show (in a manner similar to the case where $j = 0, 1$, or 2 given earlier) that $J_{d+K,1} (A')^j b$ has the representation

$$J_{d+K,1} (A')^j b = L_j b_1 + L_{j-1} b_2 + \dots + L_1 b_j + L_0 b_{j+1}$$

for coefficients L_j, L_{j-1}, \dots, L_1 , and L_0 where $L_0 = I_{d+K}$. It follows from the induction hypotheses that

$$\begin{aligned} b_{j+1} &= L_j b_1 + L_{j-1} b_2 + \dots + L_1 b_j + L_0 b_{j+1} \\ &= J_{d+K,1} (A')^j b \\ &= 0. \end{aligned}$$

Hence, by mathematical induction, we conclude that $b_r = 0$ for every $r \in \{1, \dots, p\}$, but this implies that

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}_{(d+K)p \times 1}$$

which contradicts our initial assumption that $b \neq 0$. It then follows that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} (A^j)'$$

is positive definite and, thus, also non-singular, so that there exists a positive constant C_* such that

$$\lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} (A^j)' \right\} \geq C_* > 0$$

Moreover, in light of Assumption 3-2(d), this further implies that

$$\begin{aligned}
& \lambda_{\min} \left\{ \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \\
&= \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} \frac{1}{T_h} \sum_{t=p}^{T-h} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K,1} (A^j)' \right\} \quad (\text{since } J_{d+K,1} = J_{d+K}) \\
&\geq \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} (A^j)' \right\} \lambda_{\min} \left\{ \frac{1}{T_h} \sum_{t=p}^{T-h} E [\varepsilon_{t-j} \varepsilon'_{t-j}] \right\} \\
&\geq \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} (A^j)' \right\} \inf_t \lambda_{\min} \{ E [\varepsilon_{t-j} \varepsilon'_{t-j}] \} \\
&\geq C_* \underline{C} \\
&\geq \underline{c} > 0 \quad (\text{by choosing } \underline{c} \leq C_* \underline{C}).
\end{aligned}$$

where the second inequality above follows from the fact that

$$\begin{aligned}
\lambda_{\min} \left\{ \sum_{t=p}^{T-h} \frac{E [\varepsilon_{t-j} \varepsilon'_{t-j}]}{T_h} \right\} &\geq \sum_{t=p}^{T-h} \lambda_{\min} \left\{ \frac{E [\varepsilon_{t-j} \varepsilon'_{t-j}]}{T_h} \right\} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \lambda_{\min} \{ E [\varepsilon_{t-j} \varepsilon'_{t-j}] \} \\
&\geq \inf_t \lambda_{\min} \{ E [\varepsilon_{t-j} \varepsilon'_{t-j}] \}.
\end{aligned}$$

Now, to show part (b), note first that expression (47) in the proof of Lemma C-5 gives a vector moving-average representation for \underline{W}_t of the form

$$\underline{W}_t = (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j},$$

where $J_{d+K} = J_{d+K,1} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}$. Now, let

$$S_d = \begin{pmatrix} I_{dp} \\ 0 \\ Kp \times dp \end{pmatrix} \quad \text{and} \quad S_K = \begin{pmatrix} 0 \\ dp \times Kp \\ I_{Kp} \end{pmatrix},$$

and let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}.$$

It follows that

$$\begin{aligned} \underline{Y}_t &= S'_d \mathcal{P}_{(d+K)p} \underline{W}_t \\ &= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \end{aligned}$$

and

$$\begin{aligned} \underline{F}_t &= S'_K \mathcal{P}_{(d+K)p} \underline{W}_t \\ &= S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} S'_K \mathcal{P}_{(d+K)p} A^j J'_{d+K} \varepsilon_{t-j}. \end{aligned}$$

Moreover,

$$\begin{aligned} &E [\underline{Y}_t \underline{Y}'_t] \\ &= E \left\{ \left(S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right) \right. \\ &\quad \times \left. \left(\mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_d + \sum_{k=0}^{\infty} \varepsilon'_{t-k} J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_d \right) \right\} \\ &= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_d \\ &\quad + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-k}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_d \\ &= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_d \\ &\quad + \sum_{j=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_d, \end{aligned}$$

$$\begin{aligned}
& E [\underline{F}_t \underline{F}_t'] \\
= & E \left\{ \left(S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} S'_K \mathcal{P}_{(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right) \right. \\
& \times \left. \left(\mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K + \sum_{k=0}^{\infty} \varepsilon'_{t-k} J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K \right) \right\} \\
= & S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S'_K \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-k}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K \\
= & S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K \\
& + \sum_{j=0}^{\infty} S'_K \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K,
\end{aligned}$$

and

$$\begin{aligned}
& E [\underline{Y}_t \underline{F}_t'] \\
= & E \left\{ \left(S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right) \right. \\
& \times \left. \left(\mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K + \sum_{k=0}^{\infty} \varepsilon'_{t-k} J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K \right) \right\} \\
= & S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-k}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K \\
= & S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K \\
& + \sum_{j=0}^{\infty} S'_d \mathcal{P}_{(d+K)p} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} S_K,
\end{aligned}$$

In addition, since

$$\begin{aligned}
E [\underline{W}_t] &= (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \text{ and} \\
E [\underline{W}_t \underline{W}_t'] &= (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \\
&\quad + \sum_{j=0}^{\infty} A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)'
\end{aligned}$$

and since

$$\begin{bmatrix} S_d & S_K \end{bmatrix} = \begin{pmatrix} I_{dp} & 0 \\ 0 & I_{Kp} \end{pmatrix} = I_{(d+K)p}$$

it is easy to see that

$$\begin{aligned} & \begin{pmatrix} E[\underline{Y}_t \underline{Y}'_t] & E[\underline{Y}_t \underline{F}'_t] \\ E[\underline{F}_t \underline{Y}'_t] & E[\underline{F}_t \underline{F}'_t] \end{pmatrix} \\ &= \begin{pmatrix} S'_d \\ S'_K \end{pmatrix} \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} \begin{pmatrix} S_d & S_K \end{pmatrix} \\ &+ \begin{pmatrix} S'_d \\ S'_K \end{pmatrix} \sum_{j=0}^{\infty} \mathcal{P}_{(d+K)p} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} \begin{pmatrix} S_d & S_K \end{pmatrix} \\ &= \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} \\ &+ \sum_{j=0}^{\infty} \mathcal{P}_{(d+K)p} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \mathcal{P}'_{(d+K)p} \\ &= \mathcal{P}_{(d+K)p} E[\underline{W}_t \underline{W}'_t] \mathcal{P}'_{(d+K)p} \end{aligned}$$

and

$$\begin{aligned} & \begin{pmatrix} E[\underline{Y}'_t] & E[\underline{F}'_t] \end{pmatrix} \\ &= \begin{pmatrix} \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_d & \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} S_K \end{pmatrix} \\ &= \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} \begin{pmatrix} S_d & S_K \end{pmatrix} \\ &= \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \mathcal{P}'_{(d+K)p} \\ &= E[\underline{W}'_t] \mathcal{P}'_{(d+K)p} \end{aligned}$$

Making use of these expressions, we can then write

$$\begin{aligned} \begin{pmatrix} 1 & E[\underline{Y}'_t] & E[\underline{F}'_t] \\ E[\underline{Y}_t] & E[\underline{Y}_t \underline{Y}'_t] & E[\underline{Y}_t \underline{F}'_t] \\ E[\underline{F}_t] & E[\underline{F}_t \underline{Y}'_t] & E[\underline{F}_t \underline{F}'_t] \end{pmatrix} &= \begin{pmatrix} 1 & E[\underline{W}'_t] \mathcal{P}'_{(d+K)p} \\ \mathcal{P}_{(d+K)p} E[\underline{W}_t] & \mathcal{P}_{(d+K)p} E[\underline{W}_t \underline{W}'_t] \mathcal{P}'_{(d+K)p} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix} \begin{pmatrix} 1 & E[\underline{W}'_t] \\ E[\underline{W}_t] & E[\underline{W}_t \underline{W}'_t] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}'_{(d+K)p} \end{pmatrix}. \end{aligned}$$

Next, note that

$$\begin{aligned}
& \det \begin{pmatrix} 1 & E[\underline{W}'_t] \\ E[\underline{W}_t] & E[\underline{W}_t \underline{W}'_t] \end{pmatrix} \\
&= \det(1) \det \{ E[\underline{W}_t \underline{W}'_t] - E[\underline{W}_t] E[\underline{W}'_t] \} \\
&= \det \{ E[\underline{W}_t \underline{W}'_t] - E[\underline{W}_t] E[\underline{W}'_t] \} \\
&= \det \left\{ (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \right. \\
&\quad \left. + \sum_{j=0}^{\infty} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right. \\
&\quad \left. - (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \right\} \\
&= \det \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\}
\end{aligned}$$

Now, by Assumption 3-2(d) and by the same argument as that used to prove part (a) above, we see that there exists a constant \underline{c} such that

$$\begin{aligned}
& \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \\
&\geq \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} J_{d+K} (A^j)' \right\} \inf_j \lambda_{\min} \{ E[\varepsilon_{t-j} \varepsilon'_{t-j}] \} \\
&\geq \underline{c} > 0
\end{aligned}$$

for all t , which, in turn, implies that in this case

$$\begin{aligned}
\det \begin{pmatrix} 1 & E[\underline{W}'_t] \\ E[\underline{W}_t] & E[\underline{W}_t \underline{W}'_t] \end{pmatrix} &= \det \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \\
&\geq \underline{c}^{(d+K)p} > 0
\end{aligned}$$

for all t . Furthermore, since the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix}$$

is nonsingular, it follows that the matrix

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E[\underline{Y}_t'] & E[\underline{F}_t'] \\ E[\underline{Y}_t] & E[\underline{Y}_t \underline{Y}_t'] & E[\underline{Y}_t \underline{F}_t'] \\ E[\underline{F}_t] & E[\underline{F}_t \underline{Y}_t'] & E[\underline{F}_t \underline{F}_t'] \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix} \frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E[\underline{W}_t'] \\ E[\underline{W}_t] & E[\underline{W}_t \underline{W}_t'] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}'_{(d+K)p} \end{pmatrix} \end{aligned}$$

will be nonsingular and, thus, positive definite as required. \square

Lemma D-2: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-5, and 3-7 hold. Then, the following statements are true.

(a)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \underline{W}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{W}_t \underline{W}_t'] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ \vdots \\ W_{t-p+1} \end{pmatrix} \text{ and } W_t = \begin{bmatrix} Y_t \\ F_t \end{bmatrix}.$$

(b)

$$\begin{aligned} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{Y}_t'] &= O_p\left(\frac{1}{\sqrt{T}}\right) \\ \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{F}_t \underline{F}_t'] &= O_p\left(\frac{1}{\sqrt{T}}\right), \text{ and} \\ \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{F}_t'] &= O_p\left(\frac{1}{\sqrt{T}}\right) \end{aligned}$$

where \underline{Y}_t and \underline{F}_t are as defined in expression (78).

(c)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t = (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(d)

$$\begin{aligned}\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t &= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p \left(\frac{1}{\sqrt{T}} \right), \\ \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t &= S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p \left(\frac{1}{\sqrt{T}} \right).\end{aligned}$$

(e)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right), \text{ where } \eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}$$

$$\text{with } \underset{d \times (d+K)p}{J_d} = \begin{bmatrix} I_d & 0 & \cdots & 0 \end{bmatrix} \text{ and } \underset{(d+K) \times (d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}.$$

(f)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right) \text{ and } \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right),$$

where η_{t+h} is as defined in part (e) above.

(g)

$$\frac{\mathfrak{H}' \iota_{T_h}}{T_h} = \frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1).$$

(h)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\eta_{t+h} \eta'_{t+h}] = O_p \left(\frac{1}{\sqrt{T}} \right),$$

where η_{t+h} is as defined in part (e) above.

Proof of Lemma D-2:

To show part (a), we note that for $a, b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we can write

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) \right]^2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b])^2 \right] \\
&\quad + \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \{ (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) (a' \underline{W}_{t+m} \underline{W}'_{t+m} b - E[a' \underline{W}_{t+m} \underline{W}'_{t+m} b]) \}
\end{aligned}$$

Note first that

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b])^2 \right] &= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E (a' \underline{W}_t \underline{W}'_t b)^2 - \frac{1}{T_h^2} \sum_{t=p}^{T-h} (E[a' \underline{W}_t \underline{W}'_t b])^2 \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} E [(a' \underline{W}_t \underline{W}'_t a) (b' \underline{W}_t \underline{W}'_t b)] \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E (a' \underline{W}_t \underline{W}'_t a)^2} \sqrt{E (b' \underline{W}_t \underline{W}'_t b)^2} \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \|\underline{W}_t\|_2^4 \\
&\leq \frac{C}{T_h} = O\left(\frac{1}{T}\right)
\end{aligned}$$

where the fourth inequality above follows from applying Liapunov's inequality and the result given in Lemma C-5.

Next, note that by Lemma C-11, $\{W_t\}$ is β -mixing with β mixing coefficient satisfying $\beta_W(m) \leq C_1 \exp\{-C_2 m\}$. Since $\alpha_{W,m} \leq \beta_W(m)$, it follows that W_t is α -mixing as well, with α mixing coefficient satisfying $\alpha_{W,m} \leq C_1 \exp\{-C_2 m\}$. Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = a' \underline{W}_t \underline{W}'_t b$ is also α -mixing with α mixing coefficient satisfying

$$\begin{aligned}
\alpha_{X,m} &\leq C_1 \exp\{-C_2(m-p+1)\} \\
&\leq C_1^* \exp\{-C_2 m\}
\end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2(p-1)\}$. Hence, we can apply Lemma C-3 with $p = 2$

and $r = 3$ to obtain

$$\begin{aligned} & \left| E \left\{ \left(a' \underline{W}_t \underline{W}'_t b - E \left[a' \underline{W}_t \underline{W}'_t b \right] \right) \left(a' \underline{W}_{t+m} \underline{W}'_{t+m} b - E \left[a' \underline{W}_{t+m} \underline{W}'_{t+m} b \right] \right) \right\} \right| \\ & \leq 2 \left(2^{\frac{1}{2}} + 1 \right) \alpha_{X,m}^{\frac{1}{6}} \sqrt{E \left(a' \underline{W}_t \underline{W}'_t b \right)^2} \left(E \left| a' \underline{W}_{t+m} \underline{W}'_{t+m} b \right|^3 \right)^{1/3} \end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $X_t = a' \underline{W}_t \underline{W}'_t b$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{6}} \leq (C_1^*)^{\frac{1}{6}} \exp \left\{ -\frac{C_2 m}{6} \right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{6}} & \leq (C_1^*)^{\frac{1}{6}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{6} \right\} \\ & \leq (C_1^*)^{\frac{1}{6}} \sum_{m=0}^{\infty} \exp \left\{ -\frac{C_2 m}{6} \right\} \\ & \leq (C_1^*)^{\frac{1}{6}} \left[1 - \exp \left\{ -\frac{C_2}{6} \right\} \right]^{-1} \\ & \leq C_3 \end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp \{ - (C_2 m / 6) \}$ is a convergent geometric

series given that $0 < \exp \{-(C_2/6)\} < 1$ for $C_2 > 0$. Hence,

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \{ (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) (a' \underline{W}_{t+m} \underline{W}'_{t+m} b - E[a' \underline{W}_{t+m} \underline{W}'_{t+m} b]) \} \right| \\
& \leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E \{ (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) (a' \underline{W}_{t+m} \underline{W}'_{t+m} b - E[a' \underline{W}_{t+m} \underline{W}'_{t+m} b]) \}| \\
& \leq \frac{4}{T_h^2} (2^{\frac{1}{2}} + 1) \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \alpha_{X,m}^{\frac{1}{6}} \sqrt{E(a' \underline{W}_t \underline{W}'_t b)^2} (E|a' \underline{W}_{t+m} \underline{W}'_{t+m} b|^3)^{1/3} \\
& \leq 4(\sqrt{2} + 1) \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{ \alpha_{X,m}^{\frac{1}{6}} [E(a' \underline{W}_t)^4]^{1/4} [E(b' \underline{W}_t)^4]^{1/4} [E(a' \underline{W}_{t+m})^6]^{\frac{1}{6}} \right. \\
& \quad \left. \times [E(b' \underline{W}_{t+m})^6]^{\frac{1}{6}} \right\} \\
& \leq 4(\sqrt{2} + 1) \left(\sup_t E[\|\underline{W}_t\|_2^4] \right)^{\frac{1}{2}} \left(\sup_t E[\|\underline{W}_t\|_2^6] \right)^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{6}} \\
& \leq 4(\sqrt{2} + 1) \left(\sup_t E[\|\underline{W}_t\|_2^4] \right)^{\frac{1}{2}} \left(\sup_t E[\|\underline{W}_t\|_2^6] \right)^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} C_3 \\
& \leq \frac{\overline{C}}{T_h} = O\left(\frac{1}{T}\right) \left(\text{where } \overline{C} \geq 4(\sqrt{2} + 1) \left(\sup_t E[\|\underline{W}_t\|_2^4] \right)^{\frac{1}{2}} \left(\sup_t E[\|\underline{W}_t\|_2^6] \right)^{\frac{1}{3}} C_3 \right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) \right]^2 \\
& \leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b])^2 \right] \\
& \quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E \{ (a' \underline{W}_t \underline{W}'_t b - E[a' \underline{W}_t \underline{W}'_t b]) (a' \underline{W}_{t+m} \underline{W}'_{t+m} b - E[a' \underline{W}_{t+m} \underline{W}'_{t+m} b]) \}| \\
& = O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t \underline{W}'_t b - \frac{1}{T_h} \sum_{t=p}^{T-h} E[a' \underline{W}_t \underline{W}'_t b] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \underline{W}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{W}_t \underline{W}'_t] = O_p \left(\frac{1}{\sqrt{T}} \right).$$

To show part (b), note first that

$$\begin{aligned} S'_d \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & 0 \\ & dp \times Kp \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{Y}_t, \\ S'_K \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} 0 & I_{Kp} \\ Kp \times dp & \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{F}_t \end{aligned}$$

By the result given in part (a) above, it follows from applying Slutsky's theorem that

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{Y}'_t] \\ &= S'_d \mathcal{P}_{(d+K)p} \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \underline{W}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{W}_t \underline{W}'_t] \right) \mathcal{P}_{(d+K)p} S_d \\ &= O_p \left(\frac{1}{\sqrt{T}} \right), \\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{F}_t \underline{F}'_t] \\ &= S'_K \mathcal{P}_{(d+K)p} \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \underline{W}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{W}_t \underline{W}'_t] \right) \mathcal{P}_{(d+K)p} S_K \\ &= O_p \left(\frac{1}{\sqrt{T}} \right), \end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{F}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{F}'_t] \\
&= S'_d \mathcal{P}_{(d+K)p} \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \underline{W}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{W}_t \underline{W}'_t] \right) \mathcal{P}_{(d+K)p} S_K \\
&= O_p \left(\frac{1}{\sqrt{T}} \right).
\end{aligned}$$

To show part (c), let $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, and write

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t &= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\{ a' (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j} \right\} \\
&= a' (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j}
\end{aligned}$$

Next, note that

$$\begin{aligned}
E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j} \right]^2 &= \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{s-k}] J_{d+K} (A^k)' a \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a \\
&\quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^{m+j})' a \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a \\
&\quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' a
\end{aligned}$$

Now,

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a \\
& \leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} E \|\varepsilon_{t-j}\|_2^2 a' A^j J'_{d+K} J_{d+K} (A^j)' a \\
& \leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \left(E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{1}{3}} a' A^j (A^j)' a \\
& \quad \text{(by Liapunov's inequality and } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \bar{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j (A^j)' a \\
& \quad \left(\text{where } \bar{C} \geq 1 \text{ is a constant such that } E \|\varepsilon_{t-j}\|_2^6 \leq \bar{C} < \infty \text{ by Assumption 3-2(b)} \right) \\
& \leq \bar{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \lambda_{\max} \left\{ A^j (A^j)' \right\} a' a \\
& = \bar{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \lambda_{\max} \left\{ (A^j)' A^j \right\} \\
& \quad \left(\text{since } \lambda_{\max} \left\{ A^j (A^j)' \right\} = \lambda_{\max} \left\{ (A^j)' A^j \right\} \text{ and } a' a = 1 \right) \\
& = \bar{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \sigma_{\max}^2 (A^j) \\
& \leq \bar{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} C \max \left\{ |\lambda_{\max} (A^j)|^2, |\lambda_{\min} (A^j)|^2 \right\} \quad (\text{by Assumption 3-7}) \\
& = \bar{C}^{\frac{1}{3}} C \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \max \left\{ |\lambda_{\max} (A)|^{2j}, |\lambda_{\min} (A)|^{2j} \right\} \\
& = \bar{C}^{\frac{1}{3}} C \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \phi_{\max}^{2j}
\end{aligned}$$

where $\phi_{\max} = \max \{ |\lambda_{\max} (A)|, |\lambda_{\min} (A)| \}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies

that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a &\leq \overline{C}^{\frac{1}{3}} C \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \phi_{\max}^{2j} \\
&= \overline{C}^{\frac{1}{3}} C \frac{T-h-p+1}{T_h^2} \frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \frac{1}{1-\phi_{\max}^2} \\
&\quad (\text{since } T_h = T-h-p+1) \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' a \right| \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \left| \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' a \right| \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \left\{ \sqrt{\sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a} \right. \\
&\quad \left. \times \sqrt{\sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' \sum_{m_2=1}^{T-h-t} (A^{m_2})' a} \right\}
\end{aligned}$$

Observe that

$$\begin{aligned}
& \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a \\
& \leq \sum_{j=0}^{\infty} \lambda_{\max} (E [\varepsilon_{t-j} \varepsilon'_{t-j}] E [\varepsilon_{t-j} \varepsilon'_{t-j}]) a' A^j J'_{d+K} J_{d+K} (A^j)' a \\
& \leq \sum_{j=0}^{\infty} \lambda_{\max} (E [\varepsilon_{t-j} \varepsilon'_{t-j}] E [\varepsilon_{t-j} \varepsilon'_{t-j}]) C \phi_{\max}^{2j} \\
& = C \sum_{j=0}^{\infty} \lambda_{\max}^2 (E [\varepsilon_{t-j} \varepsilon'_{t-j}]) \phi_{\max}^{2j} \\
& \leq C \sum_{j=0}^{\infty} (\text{tr} \{E [\varepsilon_{t-j} \varepsilon'_{t-j}]\})^2 \phi_{\max}^{2j} \\
& = C \sum_{j=0}^{\infty} (E \|\varepsilon_{t-j}\|_2^2)^2 \phi_{\max}^{2j} \\
& \leq C \sum_{j=0}^{\infty} (E \|\varepsilon_{t-j}\|_2^6)^{\frac{2}{3}} \phi_{\max}^{2j} \quad (\text{by Liapunov's inequality}) \\
& \leq \overline{C}^{\frac{2}{3}} C \frac{1}{1 - \phi_{\max}^2}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' \sum_{m_2=1}^{T-h-t} (A^{m_2})' a \\
& \leq \sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} a' A^{m_1} A^j (A^j)' (A^{m_2})' a \\
& \leq C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} |a' A^{m_1} (A^{m_2})' a| \\
& \leq C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{a' A^{m_1} (A^{m_1})' a} \sqrt{a' A^{m_2} (A^{m_2})' a} \\
& \leq C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{C \phi_{\max}^{2m_1}} \sqrt{C \phi_{\max}^{2m_2}} \\
& \leq C^2 \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \phi_{\max}^{m_1} \sum_{m_2=1}^{T-h-t} \phi_{\max}^{m_2} \\
& \leq C^2 \frac{1}{1 - \phi_{\max}^2} \left(\frac{1}{1 - \phi_{\max}} \right)^2
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' a \right| \\
& \leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \left\{ \sqrt{\sum_{j=0}^{\infty} a' A^j J'_{d+K} E[\varepsilon_{t-j} \varepsilon'_{t-j}] E[\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a} \right. \\
& \quad \times \left. \sqrt{\sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' \sum_{m_2=1}^{T-h-t} (A^{m_2})' a} \right\} \\
& \leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sqrt{\bar{C}^{\frac{2}{3}} C \frac{1}{1 - \phi_{\max}^2}} \sqrt{C^2 \frac{1}{1 - \phi_{\max}^2} \left(\frac{1}{1 - \phi_{\max}} \right)^2} \\
& \leq 2 \bar{C}^{\frac{1}{3}} C^{\frac{3}{2}} \frac{1}{T_h} \left(\frac{1}{1 - \phi_{\max}^2} \right) \left(\frac{1}{1 - \phi_{\max}} \right) \\
& = O\left(\frac{1}{T}\right)
\end{aligned}$$

Putting these results together, we obtain

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j} \right]^2 \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' a \\
&\quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\infty} a' A^j J'_{d+K} E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' a \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, upon applying Markov's inequality, we get

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

from which we further deduce, upon applying Slutsky's theorem, that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t &= a' (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J'_{d+K} \varepsilon_{t-j} \\
&= a' (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Since the above result holds for all $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, we further deduce that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t = (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

To show part (d), note again that

$$\begin{aligned}
S'_d \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & 0 \\ & I_{dp \times Kp} \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{Y}_t, \\
S'_K \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} 0 & I_{Kp} \\ I_{Kp \times dp} & \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{F}_t
\end{aligned}$$

By the result given in part (c) above, it follows by Slutsky's theorem that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t &= S'_d \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \\
&= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + S'_d \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \\
&= S'_d \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p \left(\frac{1}{\sqrt{T}} \right), \\
\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t &= S'_K \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \\
&= S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + S'_K \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \\
&= S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + O_p \left(\frac{1}{\sqrt{T}} \right).
\end{aligned}$$

Turning our attention to part (e), let $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; and, by direct calculation, we obtain

$$\begin{aligned}
&E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t \eta'_{t+h} b \right]^2 \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t)^2 (\eta'_{t+h} b)^2 \right] + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \left\{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \right\}
\end{aligned}$$

Let $\sigma_{\max}(A^j)$ denotes the max singular value of A^j and let $\phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}$,

and note first that

$$\begin{aligned}
E (b' \eta_{t+h})^4 &= E \left(\sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} \right)^4 \\
&\leq h^3 \sum_{j=0}^{h-1} E \left[(b' J_d A^j J'_{d+K} \varepsilon_{t+h-j})^4 \right] \quad (\text{by Loève's } c_r \text{ inequality}) \\
&\leq h^3 \sum_{j=0}^{h-1} E \left[\left(b' J_d A^j J'_{d+K} J_{d+K} (A^j)^j J'_d b \right)^2 (\varepsilon'_{t+h-j} \varepsilon_{t+h-j})^2 \right] \\
&= h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d b \right)^2 E \|\varepsilon_{t+h-j}\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j (A^j)' J'_d b \right)^2 E \|\varepsilon_{t+h-j}\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4 (A^j) (b' J_d J'_d b)^2 E \|\varepsilon_{t+h-j}\|_2^4 \\
&= h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4 (A^j) E \|\varepsilon_{t+h-j}\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \overline{C} \left[\max \{ |\lambda_{\max} (A^j)|, |\lambda_{\min} (A^j)| \} \right]^4 E \|\varepsilon_{t+h-j}\|_2^4 \quad (\text{by Assumption 3-7}) \\
&= h^3 \sum_{j=0}^{h-1} \overline{C} \phi_{\max}^{4j} E \|\varepsilon_{t+h-j}\|_2^4 \\
&\leq C^{\frac{2}{3}} \overline{C} h^3 \sum_{j=0}^{h-1} \phi_{\max}^{4j} \\
&\leq C^*
\end{aligned} \tag{84}$$

where the next to last inequality follows from the fact that $E \|\varepsilon_{t+h-j}\|_2^4 \leq \left(\sup_t E \|\varepsilon_t\|^6 \right)^{\frac{2}{3}} \leq C^{\frac{2}{3}}$ by Liapunov's inequality and by application of Assumption 3-2(b) and where the last inequality follows from the fact that h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. Applying the Cauchy-Schwarz inequality and the existence of moment result given in Lemma C-5, it then

follows that

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t)^2 (\eta'_{t+h} b)^2 \right] &\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E (a' \underline{W}_t \underline{W}'_t a)^2} \sqrt{E (b' \eta_{t+h})^4} \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E \|\underline{W}_t\|_2^4} \sqrt{E (b' \eta_{t+h})^4} \\
&\leq \frac{C}{T_h} = \frac{C}{T-h-p+1} = O\left(\frac{1}{T}\right)
\end{aligned}$$

Next, observe that

$$\begin{aligned}
&E \left\{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \right\} \\
&= E \left\{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \right\} \\
&= E \left\{ a' \underline{W}_t \underline{W}'_{t+m} a \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+m+h-k} \right\} \\
&= E \left\{ a' \underline{W}_t \underline{W}'_{t+m} a \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} \varepsilon'_{t+m+h-k} J_{d+K} (A^j)' J'_d b \right\},
\end{aligned}$$

so that, for $m \geq h$, we have

$$\begin{aligned}
&E \left\{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \right\} \\
&= E \left\{ a' \underline{W}_t \underline{W}'_{t+m} a \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} \varepsilon'_{t+m+h-k} J_{d+K} (A^j)' J'_d b \right\} \\
&= E \left\{ a' \underline{W}_t \underline{W}'_{t+m} a \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} E \left[\varepsilon'_{t+m+h-k} | \mathcal{F}_{-\infty}^{t+m} \right] J_{d+K} (A^j)' J'_d b \right\} \\
&= E \left\{ a' \underline{W}_t \underline{W}'_{t+m} a \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} \varepsilon_{t+h-j} E \left[\varepsilon'_{t+m+h-k} \right] J_{d+K} (A^j)' J'_d b \right\} \\
&= 0
\end{aligned}$$

Hence, defining $\sum_{m=1}^0 E |(a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b)| = 0$, we have

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \} \right| \\
&= \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} E \{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \} \right| \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} E |(a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b)| \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \sqrt{E (a' \underline{W}_t \underline{W}'_{t+m} a)^2} \sqrt{E (b' \eta_{t+h} \eta'_{t+m+h} b)^2} \\
&= \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \sqrt{E (a' \underline{W}_t \underline{W}'_t a a' \underline{W}_{t+m} \underline{W}'_{t+m} a)} \sqrt{E \{ (b' \eta_{t+h})^2 (b' \eta_{t+m+h})^2 \}} \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \sqrt{E (\|\underline{W}_t\|_2^2 \|\underline{W}_{t+m}\|_2^2)} \sqrt{E \{ (b' \eta_{t+h})^2 (b' \eta_{t+m+h})^2 \}} \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \left(E \|\underline{W}_t\|_2^4 \right)^{\frac{1}{4}} \left(E \|\underline{W}_{t+m}\|_2^4 \right)^{\frac{1}{4}} \left(E (b' \eta_{t+h})^4 \right)^{\frac{1}{4}} \left(E (b' \eta_{t+m+h})^4 \right)^{\frac{1}{4}} \\
&\leq \frac{2(T-h-p)(h-1)}{T_h^2} \overline{C} \quad (\text{applying Lemma C-5 and expression (84) above}) \\
&< \frac{2(h-1)\overline{C}}{T_h} \quad (\text{since } T_h = T-h-p+1) \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t \eta'_{t+h} b \right]^2 \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \underline{W}_t)^2 (\eta'_{t+h} b)^2 \right] + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \{ (a' \underline{W}_t \eta'_{t+h} b) (a' \underline{W}_{t+m} \eta'_{t+m+h} b) \} \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{W}_t \eta'_{t+h} b = O_p \left(\frac{1}{\sqrt{T}} \right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right).$$

Now, for part (f), note that

$$\begin{aligned} S'_d \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & 0 \\ & dp \times Kp \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{Y}_t, \\ S'_K \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} 0 & I_{Kp} \\ Kp \times dp & \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix} = \underline{F}_t \end{aligned}$$

Hence, it follows by applying the result given in part (e) above and the Slutsky's theorem that

$$\begin{aligned} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \eta'_{t+h} &= S'_d \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right) \text{ and} \\ \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \eta'_{t+h} &= S'_K \mathcal{P}_{(d+K)p} \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{W}_t \eta'_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right) \end{aligned}$$

To show part (g), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$ and write

$$\begin{aligned}
E \left(\frac{b' \mathfrak{H}' \iota_{T_h}}{\sqrt{T_h}} \right)^2 &= E \left(\frac{1}{\sqrt{T_h}} \sum_{t=p}^{T-h} b' \eta_{t+h} \right)^2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{s+h-k}] J_{d+K} (A^k)' J'_d b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b \\
&\quad + \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^{m+j})' J'_d b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b \\
&\quad + \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' J'_d b
\end{aligned}$$

Now,

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b \\
& \leq \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} E \|\varepsilon_{t+h-j}\|_2^2 b' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d b \\
& \leq \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \left(E \|\varepsilon_{t+h-j}\|_2^6 \right)^{\frac{1}{3}} b' J_d A^j (A^j)' J'_d b \\
& \quad \text{(by Liapunov's inequality and the fact that } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j (A^j)' J'_d b \\
& \quad \left(\text{where } \overline{C} \geq 1 \text{ is a constant such that } E \|\varepsilon_{t-j}\|_2^6 \leq \overline{C} < \infty \text{ by Assumption 3-2(b)} \right) \\
& \leq \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \lambda_{\max} \left\{ A^j (A^j)' \right\} b' J_d J'_d b \\
& = \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \lambda_{\max} \left\{ (A^j)' A^j \right\} \\
& \quad \left(\text{since } \lambda_{\max} \left\{ A^j (A^j)' \right\} = \lambda_{\max} \left\{ (A^j)' A^j \right\}, \lambda_{\max}(J_d J'_d) = 1, \text{ and } b'b = 1 \right) \\
& = \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \sigma_{\max}^2(A^j) \\
& \leq \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} C \max \left\{ |\lambda_{\max}(A^j)|^2, |\lambda_{\min}(A^j)|^2 \right\} \quad (\text{by Assumption 3-7}) \\
& = \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \max \left\{ |\lambda_{\max}(A)|^{2j}, |\lambda_{\min}(A)|^{2j} \right\} \\
& = \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \phi_{\max}^{2j}
\end{aligned}$$

where $\phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies

that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E \left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right] J_{d+K} (A^j)' J'_d b &\leq \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \phi_{\max}^{2j} \\
&\leq \overline{C}^{\frac{1}{3}} C \frac{T-h-p+1}{T_h} \frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}} C \frac{1}{1-\phi_{\max}^2} \\
&\quad (\text{since } T_h = T-h-p+1) \\
&= O(1)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left| \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E \left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' J'_d b \right| \\
&\leq \frac{2}{T_h} \sum_{t=p}^{T-h-1} \left| \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E \left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' J'_d b \right| \\
&\leq \frac{2}{T_h} \sum_{t=p}^{T-h-1} \left\{ \sqrt{\sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E \left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right] E \left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right] J_{d+K} (A^j)' J'_d b} \right. \\
&\quad \left. \times \sqrt{\sum_{j=0}^{\max\{0, h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' (A^{m_2})' J'_d b} \right\}
\end{aligned}$$

Similar to the argument given previously, we have

$$\begin{aligned}
& \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b \\
& \leq \sum_{j=0}^{\max\{0, h-2\}} \lambda_{\max} (E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}]) b' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d b \\
& \leq \sum_{j=0}^{\max\{0, h-2\}} \lambda_{\max} (E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}]) C \phi_{\max}^{2j} \\
& = C \sum_{j=0}^{\max\{0, h-2\}} \lambda_{\max}^2 (E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}]) \phi_{\max}^{2j} \\
& \leq C \sum_{j=0}^{\max\{0, h-2\}} (tr \{E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}]\})^2 \phi_{\max}^{2j} \\
& = C \sum_{j=0}^{\max\{0, h-2\}} (E \|\varepsilon_{t+h-j}\|_2^2)^2 \phi_{\max}^{2j} \\
& \leq C \sum_{j=0}^{\max\{0, h-2\}} (E \|\varepsilon_{t+h-j}\|_2^6)^{\frac{2}{3}} \phi_{\max}^{2j} \\
& \leq \bar{C}^{\frac{2}{3}} C \frac{1}{1 - \phi_{\max}^2}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=0}^{\max\{0, h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' (A^{m_2})' J'_d b \\
& \leq \sum_{j=0}^{\max\{0, h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j (A^j)' (A^{m_2})' J'_d b \\
& \leq C \sum_{j=0}^{\max\{0, h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} |b' J_d A^{m_1} (A^{m_2})' J'_d b| \\
& \leq C \sum_{j=0}^{\max\{0, h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{b' J_d A^{m_1} (A^{m_1})' J'_d b} \sqrt{b' J_d A^{m_2} (A^{m_2})' J'_d b} \\
& \leq C \sum_{j=0}^{\max\{0, h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{C \phi_{\max}^{2m_1}} \sqrt{C \phi_{\max}^{2m_2}} \\
& \leq C^2 \sum_{j=0}^{\max\{0, h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \phi_{\max}^{m_1} \sum_{m_2=1}^{T-h-t} \phi_{\max}^{m_2} \\
& \leq C^2 \frac{1}{1 - \phi_{\max}^2} \left(\frac{1}{1 - \phi_{\max}} \right)^2
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left| \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{h-2} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' J'_d b \right| \\
& \leq \frac{2}{T_h} \sum_{t=p}^{T-h-1} \left\{ \sqrt{\sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b} \right. \\
& \quad \times \left. \sqrt{\sum_{j=0}^{\max\{0, h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} (A^j)' (A^{m_2})' J'_d b} \right\} \\
& \leq \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sqrt{\bar{C}^{\frac{2}{3}} C \frac{1}{1 - \phi_{\max}^2}} \sqrt{C^2 \frac{1}{1 - \phi_{\max}^2} \left(\frac{1}{1 - \phi_{\max}} \right)^2} \\
& = 2 \bar{C}^{\frac{1}{3}} C^{\frac{3}{2}} \frac{T-h-p+1}{T_h} \left(\frac{1}{1 - \phi_{\max}^2} \right) \left(\frac{1}{1 - \phi_{\max}} \right) \\
& = O(1)
\end{aligned}$$

Putting these results together, we obtain

$$\begin{aligned}
& E \left(\frac{1}{\sqrt{T_h}} \sum_{t=p}^{T-h} b' \eta_{t+h} \right)^2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' J'_d b \\
&\quad + \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\max\{0, h-2\}} b' J_d A^j J'_{d+K} E [\varepsilon_{t+h-j} \varepsilon'_{t+h-j}] J_{d+K} (A^j)' \sum_{m=1}^{T-h-t} (A^m)' J'_d b \\
&= O(1)
\end{aligned}$$

so that, upon applying Markov's inequality, we get

$$\frac{1}{\sqrt{T_h}} \sum_{t=p}^{T-h} b' \eta_{t+h} = O_p(1).$$

Since the above result holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$\frac{1}{\sqrt{T_h}} \sum_{t=p}^{T-h} \eta_{t+h} = O_p(1)$$

and that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} = \frac{1}{\sqrt{T_h}} \left(\frac{1}{\sqrt{T_h}} \sum_{t=p}^{T-h} \eta_{t+h} \right) = O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1).$$

Lastly, to show part (h), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and write

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) \right]^2 \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b])^2 \right] \\
&\quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \{ (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) \\
&\quad \quad \quad \times (a' \eta_{t+m+h} \eta'_{t+m+h} b - E[a' \eta_{t+m+h} \eta'_{t+m+h} b]) \}
\end{aligned}$$

Making use of the Cauchy-Schwarz inequality, we then have

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \eta_{t+h} \eta'_{t+h} b - E [a' \eta_{t+h} \eta'_{t+h} b])^2 \right] \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E (a' \eta_{t+h} \eta'_{t+h} b)^2 - \frac{1}{T_h^2} \sum_{t=p}^{T-h} (E [a' \eta_{t+h} \eta'_{t+h} b])^2 \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} E (a' \eta_{t+h} \eta'_{t+h} b)^2 \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E (a' \eta_{t+h})^4} \sqrt{E (b' \eta_{t+h})^4}
\end{aligned}$$

In the proof of part (e) of this lemma, we have shown that, given Assumptions 3-2(b) and 3-7, there exists positive constants C and \overline{C} such that

$$E (b' \eta_{t+h})^4 \leq h^3 \sum_{j=0}^{h-1} C \phi_{\max}^{4j} E \|\varepsilon_{t+h-j}\|_2^4 \leq \overline{C} < \infty.$$

where $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. In a similar manner, we can also show that

$$E (a' \eta_{t+h})^4 \leq h^3 \sum_{j=0}^{h-1} C \phi_{\max}^{4j} E \|\varepsilon_{t+h-j}\|_2^4 \leq \overline{C} < \infty.$$

It follows that

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \eta_{t+h} \eta'_{t+h} b - E [a' \eta_{t+h} \eta'_{t+h} b])^2 \right] &\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E (a' \eta_{t+h})^4} \sqrt{E (b' \eta_{t+h})^4} \\
&\leq \frac{1}{T_h^2} \sum_{t=p}^{T-h} \overline{C} \\
&= \frac{\overline{C} (T - h - p + 1)}{T_h^2} \\
&= \frac{\overline{C}}{T_h} \quad (\text{since } T_h = T - h - p + 1) \\
&= O\left(\frac{1}{T}\right)
\end{aligned} \tag{85}$$

Next, observe that

$$\begin{aligned}
& a' \eta_{t+h} \eta'_{t+h} b - E [a' \eta_{t+h} \eta'_{t+h} b] \\
&= \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} a' J_d A^j J'_{d+K} (\varepsilon_{t+h-j} \varepsilon'_{t+h-k} - E [\varepsilon_{t+h-j} \varepsilon'_{t+h-k}]) J_{d+K} (A^k)' J'_d b \\
&= \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) \{ \text{vec} (\varepsilon_{t+h-j} \varepsilon'_{t+h-k}) - \text{vec} (E [\varepsilon_{t+h-j} \varepsilon'_{t+h-k}]) \} \\
&= \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) \{ (\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j}) - E [\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j}] \}
\end{aligned}$$

and

$$\begin{aligned}
& a' \eta_{t+m+h} \eta'_{t+m+h} b - E [a' \eta_{t+m+h} \eta'_{t+m+h} b] \\
&= \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} a' J_d A^\ell J'_{d+K} (\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-r} - E [\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-r}]) J_{d+K} (A^r)' J'_d b \\
&= \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left(b' J_d A^r J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) \{ \text{vec} (\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-r}) - \text{vec} (E [\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-r}]) \} \\
&= \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left(b' J_d A^r J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) \{ (\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r}) - E [\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r}] \}
\end{aligned}$$

Moreover, note that, for $m \geq h$

$$\begin{aligned}
& E \{ (a' \eta_{t+h} \eta'_{t+h} b - E [a' \eta_{t+h} \eta'_{t+h} b]) (a' \eta_{t+m+h} \eta'_{t+m+h} b - E [a' \eta_{t+m+h} \eta'_{t+m+h} b]) \} \\
&= \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left\{ \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) \right. \\
&\quad \times E [(\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j}) - E (\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j})] \\
&\quad \times [(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r}) - E (\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r})]' \\
&\quad \left. \times \left(J_{d+K} (A^r)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right\} \\
&= 0
\end{aligned}$$

Note further that, when $h = 1$, we will always have $m \geq h$, given that by definition m is an integer ≥ 1 . This implies we need to distinguish between the case where $h = 1$ from the case where $h \geq 2$.

Consider first the case where $h = 1$. In this case, we have, for all $m \geq 1$

$$\begin{aligned}
& E \{ (a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b]) (a' \eta_{t+m+1} \eta'_{t+m+1} b - E [a' \eta_{t+m+1} \eta'_{t+m+1} b]) \} \\
= & (b' J_d A^0 J'_{d+K} \otimes a' J_d A^0 J'_{d+K}) \\
& \times E \{ [(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) - E (\varepsilon_{t+1} \otimes \varepsilon_{t+1})] [(\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1}) - E (\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1})]' \} \\
& \times \left(J_{d+K} (A^0)' J'_d b \otimes J_{d+K} (A^0)' J'_d a \right) \\
= & 0
\end{aligned}$$

so that, in this case, we have

$$\begin{aligned}
& E \left[\frac{1}{T_1} \sum_{t=p}^{T-1} (a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b]) \right]^2 \\
= & \frac{1}{T_1^2} \sum_{t=p}^{T-1} E \left[(a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b])^2 \right] \\
& + \frac{2}{T_1^2} \sum_{t=p}^{T-1} \sum_{m=1}^{T-1-t} E \{ (a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b]) \\
& \quad \times (a' \eta_{t+m+1} \eta'_{t+m+1} b - E [a' \eta_{t+m+1} \eta'_{t+m+1} b]) \} \\
= & \frac{1}{T_1^2} \sum_{t=p}^{T-1} E \left[(a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b])^2 \right] \\
= & O \left(\frac{1}{T} \right) \quad (\text{as shown previously in expression (85)}) \tag{86}
\end{aligned}$$

Consider next the case where $h \geq 2$. In this case,

$$E \{ (a' \eta_{t+1} \eta'_{t+1} b - E [a' \eta_{t+1} \eta'_{t+1} b]) (a' \eta_{t+m+1} \eta'_{t+m+1} b - E [a' \eta_{t+m+1} \eta'_{t+m+1} b]) \} = 0$$

for all $m \geq h$ as previously shown; however, for $1 \leq m \leq h-1$, we have

$$\begin{aligned}
& \left| E \left\{ \left(a' \eta_{t+h} \eta'_{t+h} b - E \left[a' \eta_{t+h} \eta'_{t+h} b \right] \right) \left(a' \eta_{t+m+h} \eta'_{t+m+h} b - E \left[a' \eta_{t+m+h} \eta'_{t+m+h} b \right] \right) \right\} \right| \\
&= \left| \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left\{ \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) \right. \right. \\
&\quad \times E \left(\left(\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j} \right) - E \left(\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j} \right) \right) \\
&\quad \times \left(\left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r} \right) - E \left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r} \right) \right)' \\
&\quad \times \left(J_{d+K} (A^r)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \left. \right\} \Big| \\
&= \left| \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left\{ \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+m+h-r} \right) \right. \right. \\
&\quad \times \left(J_{d+K} (A^r)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \left. \right\} \\
&\quad - \sum_{j=0}^{h-1} \sum_{k=0}^{h-1} \sum_{\ell=0}^{h-1} \sum_{r=0}^{h-1} \left\{ \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \otimes \varepsilon_{t+h-j} \right) E \left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-r} \right)' \right. \\
&\quad \times \left. \left(J_{d+K} (A^r)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right\} \Big| \\
&\leq \sum_{j=0}^{h-1} \left| \left(b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right) \right. \\
&\quad \times \left. \left(J_{d+K} (A^j)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \right| \\
&\quad + \sum_{j=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right) \right. \\
&\quad \times \left. \left(J_{d+K} (A^k)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \right| \\
&\quad + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \right) \right. \\
&\quad \times \left. \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right| \\
&\quad + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+h-k} \right) \right. \\
&\quad \times \left. \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^k)' J'_d a \right) \right|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left| (b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K}) E(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j}) E(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell})' \right. \\
& \quad \left. \times \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right|.
\end{aligned}$$

Analyzing each term on the majorant side of the function above, we have

$$\begin{aligned}
& \sum_{j=0}^{h-1} \left| (b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K}) E(\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j}) \right. \\
& \quad \left. \times \left(J_{d+K} (A^j)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \right| \\
& \leq \overline{C} \sum_{j=0}^{h-1} \left(b' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d b \right) \left(a' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d a \right) \\
& = \overline{C} \sum_{j=0}^{h-1} \left[b' J_d A^j (A^j)' J'_d b \right] \left[a' J_d A^j (A^j)' J'_d a \right] \quad (\text{since } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C} \sum_{j=0}^{h-1} \left[\lambda_{\max} \left\{ A^j (A^j)' \right\} \right]^2 [b' J_d J'_d b] [a' J_d J'_d a] \\
& = \overline{C} \sum_{j=0}^{h-1} \left[\lambda_{\max} \left\{ A^j (A^j)' \right\} \right]^2 \quad (\text{since } J_d J'_d = I_d \text{ and } a' a = b' b = 1) \\
& = \overline{C} \sum_{j=0}^{h-1} \left[\lambda_{\max} \left\{ (A^j)' A^j \right\} \right]^2 \\
& = \overline{C} \sum_{j=0}^{h-1} \sigma_{\max}^4(A^j) \\
& \leq \overline{C} \sum_{j=0}^{h-1} C^* \max \left\{ |\lambda_{\max}(A^j)|^4, |\lambda_{\min}(A^j)|^4 \right\} \quad (\text{by Assumption 3-7}) \\
& = \overline{C} \sum_{j=0}^{h-1} C^* \max \left\{ |\lambda_{\max}(A)|^{4j}, |\lambda_{\min}(A)|^{4j} \right\} \\
& = \overline{C} \sum_{j=0}^{h-1} C^* \phi_{\max}^{4j} \quad (\text{where } 0 < \phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \} < 1) \\
& \leq \overline{C} h C^* \quad (\text{since } 0 < \phi_{\max} < 1 \text{ and } \phi_{\max}^0 = 1) \\
& \leq C \quad (\text{for } \overline{C} h C^* \leq C < \infty),
\end{aligned}$$

$$\begin{aligned}
& \sum_{j=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right) \right. \\
& \quad \left. \times \left(J_{d+K} \left(A^k \right)' J'_d b \otimes J_{d+K} \left(A^j \right)' J'_d a \right) \right| \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left(b' J_d A^k J'_{d+K} J_{d+K} \left(A^k \right)' J'_d b \right) \left(a' J_d A^j J'_{d+K} J_{d+K} \left(A^j \right)' J'_d a \right) \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left[b' J_d A^k \left(A^k \right)' J'_d b \right] \left[a' J_d A^j \left(A^j \right)' J'_d a \right] \quad (\text{since } \lambda_{\max} (J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left[\lambda_{\max} \left\{ A^j \left(A^j \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[b' J_d J'_d b \right] \left[a' J_d J'_d a \right] \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left[\lambda_{\max} \left\{ A^j \left(A^j \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \quad (\text{since } J_d J'_d = I_d \text{ and } a' a = b' b = 1) \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left[\lambda_{\max} \left\{ \left(A^j \right)' A^j \right\} \right] \left[\lambda_{\max} \left\{ \left(A^k \right)' A^k \right\} \right] \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \sigma_{\max}^2 (A^j) \sigma_{\max}^2 (A^k) \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max} (A^j) \right|^2, \left| \lambda_{\min} (A^j) \right|^2 \right\} \max \left\{ \left| \lambda_{\max} (A^k) \right|^2, \left| \lambda_{\min} (A^k) \right|^2 \right\} \\
& \quad (\text{by Assumption 3-7}) \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max} (A) \right|^{2j}, \left| \lambda_{\min} (A) \right|^{2j} \right\} \max \left\{ \left| \lambda_{\max} (A) \right|^{2k}, \left| \lambda_{\min} (A) \right|^{2k} \right\} \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} (C^*)^2 \phi_{\max}^{2j} \phi_{\max}^{2k} \quad (\text{where } \phi_{\max} = \max \{ \left| \lambda_{\max} (A) \right|, \left| \lambda_{\min} (A) \right| \}) \\
& = \overline{C} h^2 (C^*)^2 \quad (\text{since } 0 < \phi_{\max} < 1 \text{ given Assumption 3-1}) \\
& \leq C \quad \left(\text{for } \overline{C} h^2 (C^*)^2 \leq C < \infty \right),
\end{aligned}$$

$$\begin{aligned}
& \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \quad \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right| \\
& \leq \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left\{ \left[\left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+h-k} \right) \right. \right. \\
& \quad \times \left(J_{d+K} \left(A^k \right)' J'_d b \otimes J_{d+K} \left(A^k \right)' J'_d a \right) \Big]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \quad \left. \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right]^{1/2} \right\} \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{(b' J_d A^k J'_{d+K} J_{d+K} (A^k)' J'_d b) (a' J_d A^k J'_{d+K} J_{d+K} (A^k)' J'_d a)} \\
& \quad \times \sqrt{(b' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d b) (a' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d a)} \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{[b' J_d A^k (A^k)' J'_d b] [a' J_d A^k (A^k)' J'_d a]} \sqrt{[b' J_d A^\ell (A^\ell)' J'_d b] [a' J_d A^\ell (A^\ell)' J'_d a]} \\
& \quad (\text{since } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k (A^k)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] [b' J_d J'_d b] [a' J_d J'_d a] \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k (A^k)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] \quad (\text{since } J_d J'_d = I_d \text{ and } a'a = b'b = 1) \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ (A^k)' A^k \right\} \right] \left[\lambda_{\max} \left\{ (A^\ell)' A^\ell \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sigma_{\max}^2(A^k) \sigma_{\max}^2(A^\ell) \\
&\leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A^k) \right|^2, \left| \lambda_{\min}(A^k) \right|^2 \right\} \max \left\{ \left| \lambda_{\max}(A^\ell) \right|^2, \left| \lambda_{\min}(A^\ell) \right|^2 \right\} \\
&\quad (\text{by Assumption 3-7}) \\
&= \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A) \right|^{2k}, \left| \lambda_{\min}(A) \right|^{2k} \right\} \max \left\{ \left| \lambda_{\max}(A) \right|^{2\ell}, \left| \lambda_{\min}(A) \right|^{2\ell} \right\} \\
&\leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \phi_{\max}^{2k} \phi_{\max}^{2\ell} \quad (\text{since } \phi_{\max} = \max \{ \left| \lambda_{\max}(A) \right|, \left| \lambda_{\min}(A) \right| \}) \\
&= \overline{C} h^2 (C^*)^2 \quad (\text{since } 0 < \phi_{\max} < 1 \text{ and } \phi_{\max}^0 = 1) \\
&\leq C \quad \left(\text{for } \overline{C} h^2 (C^*)^2 \leq C < \infty \right),
\end{aligned}$$

$$\begin{aligned}
& \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+h-k} \right) \right. \\
& \quad \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^k \right)' J'_d a \right) \right| \\
& \leq \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left\{ \left[\left(b' J_d A^k J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \right) \right. \right. \\
& \quad \times \left. \left(J_{d+K} \left(A^k \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+h-k} \right) \right. \\
& \quad \times \left. \left. \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^k \right)' J'_d a \right) \right]^{1/2} \right\} \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{(b' J_d A^k J'_{d+K} J_{d+K} (A^k)' J'_d b) (a' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d a)} \\
& \quad \times \sqrt{(b' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d b) (a' J_d A^k J'_{d+K} J_{d+K} (A^k)' J'_d a)} \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{[b' J_d A^k (A^k)' J'_d b] [a' J_d A^\ell (A^\ell)' J'_d a]} \sqrt{[b' J_d A^\ell (A^\ell)' J'_d b] [a' J_d A^k (A^k)' J'_d a]} \\
& \quad (\text{since } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k (A^k)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] [b' J_d J'_d b] [a' J_d J'_d a] \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k (A^k)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] \quad (\text{since } J_d J'_d = I_d \text{ and } a' a = b' b = 1) \\
& = \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ (A^k)' A^k \right\} \right] \left[\lambda_{\max} \left\{ (A^\ell)' A^\ell \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sigma_{\max}^2(A^k) \sigma_{\max}^2(A^\ell) \\
&\leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A^k) \right|^2, \left| \lambda_{\min}(A^k) \right|^2 \right\} \max \left\{ \left| \lambda_{\max}(A^\ell) \right|^2, \left| \lambda_{\min}(A^\ell) \right|^2 \right\} \\
&\quad (\text{by Assumption 3-7}) \\
&= \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A) \right|^{2k}, \left| \lambda_{\min}(A) \right|^{2k} \right\} \max \left\{ \left| \lambda_{\max}(A) \right|^{2\ell}, \left| \lambda_{\min}(A) \right|^{2\ell} \right\} \\
&\leq \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} (C^*)^2 \phi_{\max}^{2k} \phi_{\max}^{2\ell} \quad (\text{since } \phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}) \\
&= \overline{C} h^2 (C^*)^2 \quad (\text{since } 0 < \phi_{\max} < 1 \text{ and } \phi_{\max}^0 = 1) \\
&\leq C \quad \left(\text{for } \overline{C} h^2 (C^*)^2 \leq C < \infty \right),
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left| (b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K}) E(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j}) E(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell})' \right. \\
& \quad \left. \times \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right| \\
& \leq \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left\{ \left[(b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K}) E(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j}) E(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j})' \right. \right. \\
& \quad \times \left(J_{d+K} (A^j)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \left. \right]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell}) E(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell})' \right. \\
& \quad \times \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \left. \right]^{1/2} \Big\} \\
& \leq \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \sqrt{(b' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d b) (a' J_d A^j J'_{d+K} J_{d+K} (A^j)' J'_d a)} \\
& \quad \times \sqrt{(b' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d b) (a' J_d A^\ell J'_{d+K} J_{d+K} (A^\ell)' J'_d a)} \\
& = \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \sqrt{[b' J_d A^j (A^j)' J'_d b] [a' J_d A^j (A^j)' J'_d a]} \sqrt{[b' J_d A^\ell (A^\ell)' J'_d b] [a' J_d A^\ell (A^\ell)' J'_d a]} \\
& \quad (\text{since } \lambda_{\max}(J'_{d+K} J_{d+K}) = 1) \\
& \leq \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left[\lambda_{\max} \left\{ A^j (A^j)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] [b' J_d J'_d b] [a' J_d J'_d a] \\
& = \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left[\lambda_{\max} \left\{ A^j (A^j)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell (A^\ell)' \right\} \right] \quad (\text{since } J_d J'_d = I_d \text{ and } a'a = b'b = 1) \\
& = \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left[\lambda_{\max} \left\{ (A^j)' A^j \right\} \right] \left[\lambda_{\max} \left\{ (A^\ell)' A^\ell \right\} \right] \\
& = \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \sigma_{\max}^2(A^j) \sigma_{\max}^2(A^\ell)
\end{aligned}$$

$$\begin{aligned}
&\leq \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A^j) \right|^2, \left| \lambda_{\min}(A^j) \right|^2 \right\} \max \left\{ \left| \lambda_{\max}(A^\ell) \right|^2, \left| \lambda_{\min}(A^\ell) \right|^2 \right\} \\
&\quad \text{(by Assumption 3-7)} \\
&= \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} (C^*)^2 \max \left\{ \left| \lambda_{\max}(A) \right|^{2j}, \left| \lambda_{\min}(A) \right|^{2j} \right\} \max \left\{ \left| \lambda_{\max}(A) \right|^{2\ell}, \left| \lambda_{\min}(A) \right|^{2\ell} \right\} \\
&\leq \overline{C} \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} (C^*)^2 \phi_{\max}^{2j} \phi_{\max}^{2\ell} \quad (\text{since } \phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}) \\
&= \overline{C} h^2 (C^*)^2 \quad (\text{since } 0 < \phi_{\max} < 1 \text{ and } \phi_{\max}^0 = 1) \\
&\leq C \quad \left(\text{for } \overline{C} h^2 (C^*)^2 \leq C < \infty \right),
\end{aligned}$$

where upper bounds given above have made use of the fact that for all t and s

$$\begin{aligned}
&E \left[\varepsilon_t \varepsilon_t' \otimes \varepsilon_s \varepsilon_s' \right] \\
&= E \left[(\varepsilon_t \otimes \varepsilon_s) (\varepsilon_t \otimes \varepsilon_s)' \right] \\
&\leq \text{tr} \left\{ E \left[(\varepsilon_t \otimes \varepsilon_s) (\varepsilon_t \otimes \varepsilon_s)' \right] \right\} \cdot I_{(d+K)^2} \\
&\quad \text{(where the inequality holds in positive semi-definite sense)} \\
&= E \left[\text{tr} \left\{ (\varepsilon_t \otimes \varepsilon_s) (\varepsilon_t \otimes \varepsilon_s)' \right\} \right] \cdot I_{(d+K)^2} \\
&= E \left[\text{tr} \left\{ (\varepsilon_t \otimes \varepsilon_s)' (\varepsilon_t \otimes \varepsilon_s) \right\} \right] \cdot I_{(d+K)^2} \\
&= E \left[\varepsilon_t' \varepsilon_t \varepsilon_s' \varepsilon_s \right] \cdot I_{(d+K)^2} \\
&= E \left[\|\varepsilon_t\|_2^2 \|\varepsilon_s\|_2^2 \right] \cdot I_{(d+K)^2} \\
&\leq \sup_t E \left[\|\varepsilon_t\|_2^4 \right] \cdot I_{(d+K)^2} \\
&\leq \overline{C} \cdot I_{d^2} \quad (\text{by Assumption 3-2(b)})
\end{aligned}$$

and

$$\begin{aligned}
E(\varepsilon_t \otimes \varepsilon_t) E(\varepsilon_t \otimes \varepsilon_t)' &\leq \text{tr} \{E(\varepsilon_t \otimes \varepsilon_t) E(\varepsilon_t \otimes \varepsilon_t)'\} \cdot I_{(d+K)^2} \\
&\quad (\text{where the inequality holds in positive semi-definite sense}) \\
&= E(\varepsilon_t \otimes \varepsilon_t)' E(\varepsilon_t \otimes \varepsilon_t) \cdot I_{(d+K)^2} \\
&= \sum_{g=1}^d \sum_{\ell=1}^d (E[\varepsilon_{gt} \varepsilon_{\ell t}])^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^d \sum_{\ell=1}^d (E|\varepsilon_{gt} \varepsilon_{\ell t}|)^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^d \sum_{\ell=1}^d E[\varepsilon_{gt}^2] E[\varepsilon_{\ell t}^2] \cdot I_{(d+K)^2} \\
&= E \left[\sum_{g=1}^d \varepsilon_{gt}^2 \right] E \left[\sum_{\ell=1}^d \varepsilon_{\ell t}^2 \right] \cdot I_{(d+K)^2} \\
&= \left(E \|\varepsilon_t\|_2^2 \right)^2 \cdot I_{(d+K)^2} \\
&\leq \overline{C} \cdot I_{(d+K)^2} \quad (\text{by Assumption 3-2(b)})
\end{aligned}$$

for some positive constant \overline{C} . It follows from these calculations that, for $1 \leq m \leq h-1$ where

$h \geq 2$, we have

$$\begin{aligned}
& \left| E \left\{ \left(a' \eta_{t+h} \eta'_{t+h} b - E \left[a' \eta_{t+h} \eta'_{t+h} b \right] \right) \left(a' \eta_{t+m+h} \eta'_{t+m+h} b - E \left[a' \eta_{t+m+h} \eta'_{t+m+h} b \right] \right) \right\} \right| \\
& \leq \sum_{j=0}^{h-1} \left| \left(b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-j} \varepsilon'_{t+h-j} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right) \right. \\
& \quad \times \left. \left(J_{d+K} (A^j)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \right| \\
& \quad + \sum_{j=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-j} \varepsilon'_{t+h-j} \right) \right. \\
& \quad \times \left. \left(J_{d+K} (A^k)' J'_d b \otimes J_{d+K} (A^j)' J'_d a \right) \right| \\
& \quad + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \quad \times \left. \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right| \\
& \quad + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+h-k} \right) \right. \\
& \quad \times \left. \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^k)' J'_d a \right) \right| \\
& \quad + \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left| \left(b' J_d A^j J'_{d+K} \otimes a' J_d A^j J'_{d+K} \right) E \left(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j} \right) E \left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \right)' \right. \\
& \quad \times \left. \left(J_{d+K} (A^\ell)' J'_d b \otimes J_{d+K} (A^\ell)' J'_d a \right) \right| \\
& \leq 5C
\end{aligned}$$

so that, when $h \geq 2$,

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \left\{ (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) (a' \eta_{t+m+h} \eta'_{t+m+h} b - E[a' \eta_{t+m+h} \eta'_{t+m+h} b]) \right\} \right| \\
&= \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} E \left\{ (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) \right. \right. \\
&\quad \left. \left. \times (a' \eta_{t+m+h} \eta'_{t+m+h} b - E[a' \eta_{t+m+h} \eta'_{t+m+h} b]) \right\} \right| \\
&\leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} |E \left\{ (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) \right. \\
&\quad \left. \times (a' \eta_{t+m+h} \eta'_{t+m+h} b - E[a' \eta_{t+m+h} \eta'_{t+m+h} b]) \right\}| \\
&\leq \frac{2}{T_h} \frac{T-h-p}{T_h} (h-1) 5C \\
&< \frac{10(h-1)C}{T_h} \quad (\text{since } T_h = T-h-p+1) \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

Putting everything together for the case where $h \geq 2$, we see that

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) \right]^2 \\
&= \frac{1}{T_h^2} \sum_{t=p}^{T-h} E \left[(a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b])^2 \right] \\
&\quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E \left\{ (a' \eta_{t+h} \eta'_{t+h} b - E[a' \eta_{t+h} \eta'_{t+h} b]) (a' \eta_{t+m+h} \eta'_{t+m+h} b - E[a' \eta_{t+m+h} \eta'_{t+m+h} b]) \right\} \\
&= O\left(\frac{1}{T}\right) + O\left(\frac{1}{T}\right) \\
&= O\left(\frac{1}{T}\right) \tag{87}
\end{aligned}$$

In light of the results given in expressions (86) and (87), we can apply Markov's inequality to show that regardless of whether $h = 1$ or $h \geq 2$

$$\frac{1}{T_h} \sum_{t=p}^{T-h} a' \eta_{t+h} \eta'_{t+h} b - \frac{1}{T_h} \sum_{t=p}^{T-h} E[a' \eta_{t+h} \eta'_{t+h} b] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further

deduce that for all (fixed) positive integer h

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\eta_{t+h} \eta'_{t+h}] = O_p \left(\frac{1}{\sqrt{T}} \right). \quad \square$$

Lemma D-3: Suppose that A is an $N \times N$ symmetric matrix which we can partition as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} r \times r & r \times (N-r) \\ (N-r) \times r & (N-r) \times (N-r) \end{matrix}$$

Then,

$$\|A_{21}\|_2 \leq \|A\|_2.$$

Proof of Lemma D-3: Define

$$B_1 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}_{N \times r}.$$

Let $\bar{v} \in \mathbb{R}^r$ be such that $\|\bar{v}\|_2 = 1$ and

$$\bar{v}' A'_{21} A_{21} \bar{v} = \max_{\|v\|_2=1} v' A'_{21} A_{21} v$$

It follows that

$$\begin{aligned} \|A_{21}\|_2 &= \sqrt{\bar{v}' A'_{21} A_{21} \bar{v}} \\ &\leq \sqrt{\bar{v}' A'_{11} A_{11} \bar{v} + \bar{v}' A'_{21} A_{21} \bar{v}} \\ &= \sqrt{\bar{v}' B'_1 A' A B_1 \bar{v}} \\ &\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \left(\text{noting that } \|B_1 \bar{v}\|_2 = \sqrt{\bar{v}' B'_1 B_1 \bar{v}} = \sqrt{\bar{v}' \bar{v}} = 1 \right) \\ &= \|A\|_2. \quad \square \end{aligned}$$

Remark: This is a well-known linear algebraic result. A similar result has also been given in the beginning of section 6 of Johnstone and Lu (2009).

Lemma D-4: Let

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^T E [\underline{F}_t \underline{F}'_t] \quad (88)$$

where $T_0 = T - p + 1$. Then, under Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7; there exists

a positive constant \underline{C} such that

$$\lambda_{\min} \{M_{FF}\} \geq \underline{C} > 0$$

for all $T > p - 1$.

Proof of Lemma D-4:

To proceed, note that we can write

$$\frac{1}{T_0} \sum_{t=p}^T \begin{pmatrix} E[Y_t Y_t'] & E[Y_t F_t'] \\ E[F_t Y_t'] & E[F_t F_t'] \end{pmatrix} = \mathcal{P}_{(d+K)p} \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] \mathcal{P}_{(d+K)p}'$$

from which it follows that

$$\begin{aligned} \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T \begin{pmatrix} E[Y_t Y_t'] & E[Y_t F_t'] \\ E[F_t Y_t'] & E[F_t F_t'] \end{pmatrix} \right\} &= \lambda_{\min} \left\{ \mathcal{P}_{(d+K)p} \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] \mathcal{P}_{(d+K)p}' \right\} \\ &\geq \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] \right\} \lambda_{\min} \left\{ \mathcal{P}_{(d+K)p} \mathcal{P}_{(d+K)p}' \right\} \\ &= \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] \right\} \lambda_{\min} \{I_{(d+K)p}\} \\ &\quad \text{(since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix)} \\ &= \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] \right\} \end{aligned}$$

Next, note that

$$\begin{aligned} \frac{1}{T_0} \sum_{t=p}^T E[W_t W_t'] &= \frac{1}{T_0} \sum_{t=p}^T (I_{(d+K)p} - A)^{-1} J_{d+K}' \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \\ &\quad + \frac{1}{T_0} \sum_{t=p}^T \sum_{j=0}^{\infty} A^j J_{d+K}' E[\varepsilon_{t-j} \varepsilon_{t-j}'] J_{d+K} (A^j)' \\ &= (I_{(d+K)p} - A)^{-1} J_{d+K}' \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \\ &\quad + \sum_{j=0}^{\infty} A^j J_{d+K}' \frac{1}{T_0} \sum_{t=p}^T E[\varepsilon_{t-j} \varepsilon_{t-j}'] J_{d+K} (A^j)' \end{aligned}$$

so that there exists a positive constant \underline{C} such that

$$\begin{aligned}
& \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E [\underline{W}_t \underline{W}_t'] \right\} \\
& \geq \lambda_{\min} \left\{ (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \mu' J_{d+K} (I_{(d+K)p} - A')^{-1} \right\} \\
& \quad + \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} \frac{1}{T_0} \sum_{t=p}^T E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \\
& \quad \text{(by Weyl's Theorem (see Theorem 4.3.1 of Horn and Johnson, 1985))} \\
& \geq \lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K} \frac{1}{T_0} \sum_{t=p}^T E [\varepsilon_{t-j} \varepsilon'_{t-j}] J_{d+K} (A^j)' \right\} \\
& \geq \underline{C} > 0 \text{ for all } T > p-1 \text{ (by the result given in part (a) of Lemma D-1)}
\end{aligned}$$

It then follows that

$$\begin{aligned}
& \lambda_{\min} \{M_{FF}\} \\
& = \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E [\underline{F}_t \underline{F}_t'] \right\} \\
& \geq \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T \begin{pmatrix} E [\underline{Y}_t \underline{Y}_t'] & E [\underline{Y}_t \underline{F}_t'] \\ E [\underline{F}_t \underline{Y}_t'] & E [\underline{F}_t \underline{F}_t'] \end{pmatrix} \right\} \\
& \quad \text{(by the Poincaré separation theorem (see Corollary 4.3.16 of Horn and Johnson, 1985))} \\
& \geq \lambda_{\min} \left\{ \frac{1}{T_0} \sum_{t=p}^T E [\underline{W}_t \underline{W}_t'] \right\} \\
& \geq \underline{C} > 0 \text{ for all } T > p-1,
\end{aligned}$$

as required. \square

Lemma D-5: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumption 3-3 hold. Then,

(a)

$$\frac{1}{T_h} \sum_{\substack{v, w=p \\ v \leq w}}^{T-h} |E [u_{iv} u_{iw}]| = O(1)$$

(b)

$$\frac{1}{T_h} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} |E(u_{it}u_{is}u_{ig})| = O(1)$$

(c)

$$\frac{1}{T_h^2} \sum_{\substack{t, s, g, v = p \\ t \leq s \leq g \leq v}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{iv})| = O(1)$$

Proof of Lemma D-5:

To show part (a), first write

$$\frac{1}{T_h} \sum_{\substack{v, w = p \\ v \leq w}}^{T-h} |E[u_{iv}u_{iw}]| = \frac{1}{T_h} \sum_{v=p}^{T-h} E[u_{iv}^2] + \frac{1}{T_h} \sum_{\substack{v, w = p \\ v < w}}^{T-h} |E[u_{iv}u_{iw}]| \quad (89)$$

Consider now the first term on the right-hand side of expression (89). Note that, trivially, by Assumption 3-3(b),

$$\frac{1}{T_h} \sum_{v=p}^{T-h} E[u_{iv}^2] \leq C = O(1) \quad (90)$$

For the second term on the right-hand side of expression (89), note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}.$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned} & \frac{1}{T_h} \sum_{\substack{v, w = p \\ v < w}}^{T-h} |E[u_{iv}u_{iw}]| \\ & \leq \frac{1}{T_h} \sum_{\substack{v, w = p \\ v < w}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{iv}|^6 \right)^{\frac{1}{6}} \left(E|u_{iw}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \end{aligned}$$

Application of Liapunov's inequality then gives us

$$\begin{aligned}
\left(E |u_{iv}|^6\right)^{\frac{1}{6}} \left(E |u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E |u_{iv}|^6\right)^{\frac{1}{6}} \left(E |u_{iw}|^6\right)^{\frac{1}{6}} \\
&\leq \left(\sup_t E |u_{it}|^6\right)^{\frac{1}{3}} \\
&= C^{\frac{1}{3}} < \infty \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Moreover, let $\varrho = w - v$, so that $w = v + \varrho$. Using these notations and the boundedness of $\left(E |u_{iv}|^6\right)^{\frac{1}{6}} \left(E |u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
&\frac{1}{T_h} \sum_{\substack{v,w=p \\ v < w}}^{T-h} |E [u_{iv} u_{iw}]| \\
&\frac{1}{T_h} \sum_{\substack{v,w=p \\ v < w}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp \{-a_2 (w - v)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E |u_{iv}|^6\right)^{\frac{1}{6}} \left(E |u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
&\leq \frac{C^{\frac{1}{3}}}{T_h} \sum_{\substack{v,w=p \\ v < w}}^{T-h} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp \{-a_2 (w - v)\}]^{\frac{1}{30}} \\
&\leq \frac{C^*}{T_h} \sum_{\substack{v,w=p \\ v < w}}^{T-h} \exp \left\{-\frac{a_2}{30} \varrho\right\} \\
&\quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) C^{\frac{1}{3}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
&\leq \frac{C^*}{T_h} \sum_{v=p}^{T-h} \sum_{\varrho=1}^{\infty} \exp \left\{-\frac{a_2}{30} \varrho\right\} \\
&= C^* \sum_{\varrho_1=1}^{\infty} \exp \left\{-\frac{a_2}{30} \varrho\right\} \\
&= O(1) \quad (\text{given Lemma C-1})
\end{aligned} \tag{91}$$

It follows from expressions (89), (90), and (91) that

$$\begin{aligned}
\frac{1}{T_h} \sum_{\substack{v,w=p \\ v \leq w}}^{T-h} |E [u_{iv} u_{iw}]| &= \frac{1}{T_h} \sum_{v=p}^{T-h} E [u_{iv}^2] + \frac{1}{T_h} \sum_{\substack{v,w=p \\ v < w}}^{T-h} |E [u_{ig} u_{ih}]| \\
&= O(1) + O(1) \\
&= O(1).
\end{aligned}$$

To show part (b), first write

$$\begin{aligned}
\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E(u_{it}u_{is}u_{ig})| &= \frac{1}{T_h} \sum_{t=p}^{T-h} E|u_{it}|^3 + \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
&\quad + \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})|
\end{aligned} \tag{92}$$

For the first term on the right-hand side of expression (92) above, note that, trivially, we can apply Assumption 3-3(b) to obtain

$$\frac{1}{T_h} \sum_{t=p}^{T-h} E|u_{it}|^3 \leq C = O(1). \tag{93}$$

Next, note that, for the second term on the right-hand side of expression (92) above, we can apply Lemma C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned}
&\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
&\leq \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1 \right) [a_1 \exp \{-a_2(s-t)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}u_{ig}|^{\frac{5}{4}} \right)^{\frac{4}{5}}
\end{aligned}$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}
\left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}u_{ig}|^{\frac{5}{4}} \right)^{\frac{4}{5}} &\leq \left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(\left(E|u_{is}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \left(E|u_{ig}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \right)^{\frac{4}{5}} \\
&= \left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \left(E|u_{ig}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \\
&\leq \left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}|^6 \right)^{\frac{1}{6}} \left(E|u_{ig}|^6 \right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= C^{\frac{1}{2}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = s - t$ and $\varrho_2 = g - s$, so that $s = t + \varrho_1$ and $g = s + \varrho_2 = t + \varrho_1 + \varrho_2$. Using these

notations and the boundedness of $\left(E|u_{it}|^6\right)^{\frac{1}{6}}\left(E|u_{is}u_{ig}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
& \leq \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(s-t)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{it}|^6\right)^{\frac{1}{6}} \left(E|u_{is}u_{ig}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
& \leq \frac{C^{\frac{1}{2}}}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp\{-a_2(s-t)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) C^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
& \leq \frac{C^*}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \leq \frac{C^*}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = C^* \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = O(1) \quad (\text{given Lemma C-1})
\end{aligned} \tag{94}$$

Similarly, for the third term on the right-hand side of expression (92), we can apply Lemma C-3 with $p = 6$ and $r = 5/4$, we have

$$\begin{aligned}
& \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
& \leq \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(g-s)\}]^{1-\frac{4}{5}-\frac{1}{6}} \left(E|u_{it}u_{is}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E|u_{ig}|^6\right)^{\frac{1}{6}}
\end{aligned}$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}
\left(E |u_{it}u_{is}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} &\leq \left(\left(E |u_{it}|^{\frac{5}{2}}\right)^{\frac{1}{2}} \left(E |u_{is}|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} \\
&= \left(E |u_{it}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E |u_{is}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} \\
&\leq \left(E |u_{it}|^6\right)^{\frac{1}{6}} \left(E |u_{is}|^6\right)^{\frac{1}{6}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= C^{\frac{1}{2}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = s - t$ and $\varrho_2 = g - s$, so that $s = t + \varrho_1$ and $g = s + \varrho_2 = t + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E |u_{it}u_{is}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}}$ as shown above, we can further write

$$\begin{aligned}
&\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
&\leq \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(g-s)\}]^{1-\frac{4}{5}-\frac{1}{6}} \left(E |u_{it}u_{is}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E |u_{ig}|^6\right)^{\frac{1}{6}} \\
&\leq \frac{C^{\frac{1}{2}}}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp\{-a_2(g-s)\}]^{\frac{1}{30}} \\
&\leq \frac{C^*}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
&\quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) C^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
&\leq \frac{C^*}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
&= \frac{C^*}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_2=1}^{\infty} (\varrho_2 + 1) \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
&= C^* \left[\sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{30}\varrho_2\right\} + \sum_{\varrho_2=1}^{\infty} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \right] \\
&= O(1) \quad \text{(given Lemma C-1)}
\end{aligned} \tag{95}$$

It follows from expressions (38), (39), (40), and (41) that

$$\begin{aligned}
\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E(u_{it}u_{is}u_{ig})| &= \frac{1}{T_h} \sum_{t=p}^{T-h} E|u_{it}|^3 + \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ s-t > g-s, s-t > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
&\quad + \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g \\ g-s \geq s-t, g-s > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| \\
&= O(1) + O(1) + O(1) \\
&= O(1).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
&\frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{iv})| \\
&= \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E[u_{it}u_{is}^3]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{iv})| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{iv})| \\
&= \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E[u_{it}u_{is}^3]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is}) + E(u_{it}u_{is})\}u_{ig}u_{iv}]| \\
&\quad + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is}) + E(u_{it}u_{is})\}u_{ig}u_{iv}]| \\
&\leq \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E[u_{it}u_{is}^3]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\}u_{ig}u_{iv}]| \\
&\quad + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\}u_{ig}u_{iv}]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-s > 0}}^{T-h} |E(u_{it}u_{is})||E(u_{ig}u_{iv})| \quad (96)
\end{aligned}$$

For the first term on the right-hand side of expression (96) above, note that, by Jensen's inequality,

the Cauchy-Schwarz inequality, and Assumption 3-3(b); we have

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E[u_{it}u_{is}^3]| &\leq \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} E[|u_{it}u_{is}^3|] \\
&\leq \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} \sqrt{E|u_{it}|^2} \sqrt{E|u_{is}|^6} \\
&\leq \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} \left(E|u_{it}|^6\right)^{\frac{1}{6}} \sqrt{E|u_{is}|^6} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{C_3^{\frac{2}{3}} T_h^2}{T_h^2} \text{ (by Assumption 3-3(b))} \\
&= O(1)
\end{aligned} \tag{97}$$

Next, for the second term on the right-hand side of expression (96), we can apply Lemma C-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}u_{iv}]| \\
&\leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(v-g)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
&\quad \left. \times \left(E|\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(E|u_{iv}|^6 \right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality,

$$\begin{aligned}
E |\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}|^{\frac{4}{3}} &\leq \left[E(u_{it}u_{is} - E(u_{it}u_{is}))^{\frac{12}{7}} \right]^{\frac{7}{9}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}} \left(E|u_{it}u_{is}|^{\frac{12}{7}} + |E[u_{it}u_{is}]|^{\frac{12}{7}} \right) \right]^{\frac{7}{9}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}} \left(E|u_{it}u_{is}|^{\frac{12}{7}} + E|u_{it}u_{is}|^{\frac{12}{7}} \right) \right]^{\frac{7}{9}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}} E|u_{it}u_{is}|^{\frac{12}{7}} \right]^{\frac{7}{9}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[\left(E|u_{it}|^{\frac{24}{7}} \right)^{\frac{1}{2}} \left(E|u_{is}|^{\frac{24}{7}} \right)^{\frac{1}{2}} \right]^{\frac{7}{9}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}} \left[\left(E|u_{it}|^{\frac{24}{7}} \right)^{\frac{7}{24}} \left(E|u_{is}|^{\frac{24}{7}} \right)^{\frac{7}{24}} \right]^{\frac{4}{3}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[\left(E|u_{it}|^6 \right)^{\frac{1}{6}} \left(E|u_{is}|^6 \right)^{\frac{1}{6}} \right]^{\frac{4}{3}} \left[E|u_{ig}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} (C)^{\frac{2}{9}} (C)^{\frac{2}{9}} (C)^{\frac{2}{9}} \quad \text{(by Assumption 3-3(b))} \\
&= 2^{\frac{4}{3}} C^{\frac{2}{3}}
\end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s + \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E |\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}|^{\frac{4}{3}}$ as shown above, we can

further write

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}u_{iv}]| \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(v-g)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \quad \left. \times \left(E|\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(E|u_{iv}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} 2 \left(2^{\frac{1}{4}} + 1 \right) [a_1 \exp\{-a_2(v-g)\}]^{\frac{1}{12}} \left(2^{\frac{4}{3}} C^{\frac{2}{3}} \right)^{\frac{3}{4}} (C)^{\frac{1}{6}} \\
& \leq \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} \exp\left\{-\frac{a_2}{12} \varrho_2\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{4}} + 1 \right) C^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2-1} \exp\left\{-\frac{a_2}{12} \varrho_2\right\} \\
& = C^* \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{12} \varrho_2\right\} \\
& = O(1) \quad (\text{given Lemma C-1}) \tag{98}
\end{aligned}$$

Similarly, for the third term on the right-hand side of expression (96) above, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\} u_{ig}u_{iv}]| \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(g-s)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \left. \times \left(E|\{u_{it}u_{is} - E(u_{it}u_{is})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{ig}u_{iv}|^3 \right)^{\frac{1}{3}} \right\}
\end{aligned}$$

Next, applications of Hölder's inequality yield

$$\begin{aligned}
E |u_{ig}u_{iv}|^3 &\leq \left(E |u_{ig}|^6\right)^{\frac{1}{2}} \left(E |u_{iv}|^6\right)^{\frac{1}{2}} \\
&\leq (C)^{\frac{1}{2}} (C)^{\frac{1}{2}} \quad (\text{by Assumption 3-3(b)}) \\
&= C < \infty
\end{aligned}$$

and

$$\begin{aligned}
E |\{u_{it}u_{is} - E(u_{it}u_{is})\}|^2 &\leq 2 \left(E |u_{it}u_{is}|^2 + |E[u_{it}u_{is}]|^2\right) \quad (\text{by Loève's } c_r \text{ inequality}) \\
&\leq 2 \left(E |u_{it}u_{is}|^2 + E |u_{it}u_{is}|^2\right) \quad (\text{by Jensen's inequality}) \\
&= 4E |u_{it}u_{is}|^2 \\
&\leq 4 \left[\left(E |u_{it}|^4\right)^{\frac{1}{4}} \left(E |u_{is}|^4\right)^{\frac{1}{4}} \right]^2 \\
&\leq 4 \left[\left(E |u_{it}|^6\right)^{\frac{1}{6}} \left(E |u_{is}|^6\right)^{\frac{1}{6}} \right]^2 \quad (\text{by Liapunov's inequality}) \\
&\leq 4 \left(\sup_t E |u_{it}|^6 \right)^{\frac{2}{3}} \\
&\leq 4 (C)^{\frac{2}{3}} < \infty \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s + \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E |u_{ig}u_{iv}|^3$ and $E |\{u_{it}u_{is} - E(u_{it}u_{is})\}|^2$ as shown above,

we can further write

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\}u_{ig}u_{iv}]| \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(g-s)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \left. \times \left(E|\{u_{it}u_{is} - E(u_{it}u_{is})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{ig}u_{iv}|^3 \right)^{\frac{1}{3}} \right\} \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(g-s)\}]^{\frac{1}{6}} \left(4C^{\frac{2}{3}} \right)^{\frac{1}{2}} (C)^{\frac{1}{3}} \\
& \leq \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 4 \left(2^{\frac{1}{2}} + 1 \right) C^{\frac{2}{3}} a_1^{\frac{1}{6}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& = C^* \sum_{\varrho_1=1}^{\infty} (\varrho_1 + 1) \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& = O(1) \quad (\text{given Lemma C-1}) \tag{99}
\end{aligned}$$

Finally, consider the fourth term on the right-hand side of expression (96) above. For this term, we apply the result given in part (a) to obtain

$$\begin{aligned}
\frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-s > 0}}^{T-h} |E(u_{it}u_{is})| |E(u_{ig}u_{iv})| & \leq \left(\frac{1}{T_h} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E(u_{it}u_{is})| \right) \left(\frac{1}{T_h} \sum_{\substack{g,v=p \\ g \leq v}}^{T-h} |E(u_{ig}u_{iv})| \right) \\
& = O(1). \tag{100}
\end{aligned}$$

It follows from expressions (96)-(100) that

$$\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{iv})| \\
& \leq \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} |E[u_{it}u_{is}^3]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g > g-s, v-g > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\}u_{ig}u_{iv}]| \\
& \quad + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g \leq g-s, g-s > 0}}^{T-h} |E[\{u_{it}u_{is} - E(u_{it}u_{is})\}u_{ig}u_{iv}]| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-s > 0}}^{T-h} |E(u_{it}u_{is})| |E(u_{ig}u_{iv})| \\
& = O(1). \quad \square
\end{aligned}$$

Lemma D-6: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, as $N_1, N_2, T \rightarrow \infty$,

$$\max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k \underline{F}_t u_{i,t}}{\sqrt{N_1 T_h}} \right)^2 = O_p \left(\frac{N_2^{\frac{1}{3}}}{N_1 T} \right).$$

Proof of Lemma D-6:

To proceed, we first show the boundedness of the quantity

$$\frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E \left(\sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^6$$

Note first that there exist a constant $C_1 > 1$ such that

$$\begin{aligned}
& \frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E \left(\sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^6 \\
& \leq \frac{C_1}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w}}^{T-h} \{ |E[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}]| \\
& \quad \times |E[(\gamma'_k \underline{F}_t)(\gamma'_k \underline{F}_s)(\gamma'_k \underline{F}_g)(\gamma'_k \underline{F}_\ell)(\gamma'_k \underline{F}_v)(\gamma'_k \underline{F}_w)]| \}
\end{aligned}$$

Next, note that, by repeated application of Hölder's inequality, we have by Assumption 3-5 and

Lemma C-5 that there exists a positive constant C such that

$$\begin{aligned}
& |E [(\gamma'_k \underline{E}_t) (\gamma'_k \underline{E}_s) (\gamma'_k \underline{E}_g) (\gamma'_k \underline{E}_\ell) (\gamma'_k \underline{E}_v) (\gamma'_k \underline{E}_w)]| \\
& \leq E [|\gamma'_k \underline{E}_t| |\gamma'_k \underline{E}_s| |\gamma'_k \underline{E}_g| |\gamma'_k \underline{E}_\ell| |\gamma'_k \underline{E}_v| |\gamma'_k \underline{E}_w|] \\
& \leq \|\gamma_k\|_2^6 E [\|\underline{E}_t\|_2 \|\underline{E}_s\|_2 \|\underline{E}_g\|_2 \|\underline{E}_\ell\|_2 \|\underline{E}_v\|_2 \|\underline{E}_w\|_2] \\
& \leq \|\gamma_k\|_2^6 \left(E [\|\underline{E}_t\|_2^2 \|\underline{E}_s\|_2^2 \|\underline{E}_g\|_2^2] \right)^{\frac{1}{2}} \left(E [\|\underline{E}_\ell\|_2^2 \|\underline{E}_v\|_2^2 \|\underline{E}_w\|_2^2] \right)^{\frac{1}{2}} \\
& \leq \|\gamma_k\|_2^6 \left(\left\{ E [\|\underline{E}_t\|_2^6] \right\}^{\frac{1}{3}} \left(E [\|\underline{E}_s\|_2^3 \|\underline{E}_g\|_2^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\
& \quad \times \left(\left\{ E [\|\underline{E}_\ell\|_2^6] \right\}^{\frac{1}{3}} \left(E [\|\underline{E}_v\|_2^3 \|\underline{E}_w\|_2^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\
& \leq \|\gamma_k\|_2^6 \left(\left\{ E [\|\underline{E}_t\|_2^6] \right\}^{\frac{1}{3}} \left\{ E [\|\underline{E}_s\|_2^6] \right\}^{\frac{1}{3}} \left\{ E [\|\underline{E}_g\|_2^6] \right\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \\
& \quad \times \left(\left\{ E [\|\underline{E}_\ell\|_2^6] \right\}^{\frac{1}{3}} \left\{ E [\|\underline{E}_v\|_2^6] \right\}^{\frac{1}{3}} \left\{ E [\|\underline{E}_w\|_2^6] \right\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \\
& \leq \|\gamma_k\|_2^6 \left\{ E [\|\underline{E}_t\|_2^6] \right\}^{\frac{1}{6}} \left\{ E [\|\underline{E}_s\|_2^6] \right\}^{\frac{1}{6}} \left\{ E [\|\underline{E}_g\|_2^6] \right\}^{\frac{1}{6}} \\
& \quad \times \left\{ E [\|\underline{E}_\ell\|_2^6] \right\}^{\frac{1}{6}} \left\{ E [\|\underline{E}_v\|_2^6] \right\}^{\frac{1}{6}} \left\{ E [\|\underline{E}_w\|_2^6] \right\}^{\frac{1}{6}} \\
& \leq \|\gamma_k\|_2^6 \sup_t E [\|\underline{E}_t\|_2^6] \\
& \leq C < \infty
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
& \frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E \left(\sum_{t=p}^{T-h} \gamma'_k E_t u_{i,t} \right)^6 \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E[u_{it} u_{is} u_{ig}^4]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v, \ell-g\}, v-\ell > 0}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell-g \geq \max\{w-v, v-\ell\}, \ell-g > 0}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E[u_{it} u_{is} u_{ig}^4]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v, \ell-g\}, v-\ell > 0}}^{T-h} |E[\{u_{it} u_{is} u_{ig} u_{i\ell} - E(u_{it} u_{is} u_{ig} u_{i\ell})\} u_{iv} u_{iw}]| \\
& \quad + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v, \ell-g\}, v-\ell > 0}}^{T-h} |E(u_{it} u_{is} u_{ig} u_{i\ell})| |E(u_{iv} u_{iw})|
\end{aligned}$$

$$\begin{aligned}
& + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} |E[\{u_{it} u_{is} u_{ig} - E(u_{it} u_{is} u_{ig})\} u_{i\ell} u_{iv} u_{iw}]| \\
& + \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} |E(u_{it} u_{is} u_{ig})| |E(u_{i\ell} u_{iv} u_{iw})| \\
& = \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6, \quad (\text{say}).
\end{aligned}$$

Consider first $\mathcal{T}\mathcal{T}_1$. Note that

$$\begin{aligned}
\mathcal{T}\mathcal{T}_1 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} |E[u_{it} u_{is} u_{ig}^4]| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} E[|u_{it} u_{is} u_{ig}^4|] \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} \left(E[|u_{it} u_{is}|^3]\right)^{\frac{1}{3}} \left(E[|u_{ig}|^6]\right)^{\frac{2}{3}} \quad (\text{by Hölder's inequality}) \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} \left(\left[E\{|u_{it}|^6\}\right]^{\frac{1}{2}} \left[E\{|u_{is}|^6\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}} \left(E[|u_{ig}|^6]\right)^{\frac{2}{3}} \\
&\quad (\text{by further application of Hölder's inequality}) \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} \left(E\{|u_{it}|^6\}\right)^{\frac{1}{6}} \left(E\{|u_{is}|^6\}\right)^{\frac{1}{6}} \left(E[|u_{ig}|^6]\right)^{\frac{2}{3}} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} \left(\sup_t E\{|u_{it}|^7\}\right)^{\frac{6}{7}} \\
&\quad (\text{using Liapunov's inequality}) \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g = p \\ t \leq s \leq g}}^{T-h} \overline{C}^{\frac{6}{7}} \quad (\text{by Assumption 3-3(b)}) \\
&\leq C_1 \overline{C}^{\frac{6}{7}} \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \\
&= C_1 \overline{C}^{\frac{6}{7}} = O(1)
\end{aligned} \tag{101}$$

Next, consider \mathcal{TT}_2 . For this term, note first that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned} & \mathcal{TT}_2 \\ &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w - v \geq \max\{v - \ell, \ell - g\}, w - v > 0}}^{T-h} |E[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}]| \\ &\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w - v \geq \max\{v - \ell, \ell - g\}, w - v > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{4}{5}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\ &\quad \left. \times \left(E |u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\ &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w - v \geq \max\{v - \ell, \ell - g\}, w - v > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{\frac{1}{30}} \right. \\ &\quad \left. \times \left(E |u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$\begin{aligned}
& E |u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}|^{\frac{5}{4}} \\
& \leq \left[E |u_{it} u_{is} u_{ig}|^{\frac{25}{12}} \right]^{\frac{3}{5}} \left[E |u_{i\ell} u_{iv}|^{\frac{25}{8}} \right]^{\frac{2}{5}} \\
& \leq \left[\left(E |u_{it} u_{is}|^{\frac{150}{47}} \right)^{\frac{47}{72}} \left(E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[\left(E |u_{i\ell}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
& \leq \left[\left(\sqrt{E |u_{it}|^{\frac{300}{47}}} \sqrt{E |u_{is}|^{\frac{300}{47}}} \right)^{\frac{47}{72}} \left(E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[\left(E |u_{i\ell}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
& \leq \left(E |u_{it}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left(E |u_{is}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left(E |u_{ig}|^6 \right)^{\frac{15}{72}} \left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \left(E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \\
& = \left[\left(E |u_{it}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \left(E |u_{is}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \right]^{\frac{5}{4}} \left[\left(E |u_{ig}|^6 \right)^{\frac{1}{6}} \right]^{\frac{5}{4}} \left[\left(E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \left[\left(E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \\
& \leq \left[\left(E |u_{it}|^7 \right)^{\frac{1}{7}} \left(E |u_{is}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{ig}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{iv}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[\left(E |u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \\
& \quad \text{(by Liapunov's inequality)} \\
& \leq (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} (\overline{C})^{\frac{5}{28}} \quad \text{(by Assumption 3-3(b))} \\
& = \overline{C}^{\frac{25}{28}}
\end{aligned}$$

By Liapunov's inequality and Assumption 3-3(b), we also obtain

$$\left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \leq \left(E |u_{iw}|^7 \right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}.$$

Moreover, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}|^{\frac{5}{4}}$

as shown above, we can further write

$$\begin{aligned}
& \mathcal{T}\mathcal{T}_2 \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \right. \\
& \quad \left. \times \left(E |u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left(E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \left(\overline{C}^{\frac{25}{28}} \right)^{\frac{4}{5}} \overline{C}^{\frac{1}{7}} \\
& \leq \frac{C_1 C \overline{C}^{\frac{6}{7}}}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} 2 \left(2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \\
& \leq \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell, \ell-g\}, w-v > 0}}^{T-h} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \quad \left(\text{for some constant } C_1^* \text{ such that } 2 \left(2^{\frac{1}{5}} + 1 \right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{30}} \leq C_1^* < \infty \right) \\
& \leq \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} \sum_{\rho_1=0}^{\rho_3} \sum_{\rho_2=0}^{\rho_3} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \leq \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} (\rho_3 + 1)^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& = C_1^* \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \left[\sum_{\rho_3=1}^{\infty} \rho_3^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + 2 \sum_{\rho_3=1}^{\infty} \rho_3 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + \sum_{\rho_3=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \right] \\
& \leq C_1^* \overline{C}_1
\end{aligned} \tag{102}$$

for some positive constant

$$\overline{C}_1 \geq \sum_{\rho_3=1}^{\infty} \rho_3^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + 2 \sum_{\rho_3=1}^{\infty} \rho_3 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + \sum_{\rho_3=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\}.$$

which exists in light of Lemma C-1.

Now, consider \mathcal{TT}_3 . Here, we apply Lemma C-3 with $p = 3/2$ and $r = 7/2$ to obtain

$$\begin{aligned}
\mathcal{TT}_3 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} |E[\{u_{it} u_{is} u_{ig} u_{i\ell} - E(u_{it} u_{is} u_{ig} u_{i\ell})\} u_{iv} u_{iw}]| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} \left\{ 2 \left(2^{1 - \frac{2}{3}} + 1 \right) [a_1 \exp\{-a_2(v - \ell)\}]^{1 - \frac{2}{3} - \frac{2}{7}} \right. \\
&\quad \times \left(E[\{u_{it} u_{is} u_{ig} u_{i\ell} - E(u_{it} u_{is} u_{ig} u_{i\ell})\}^{\frac{3}{2}}] \right)^{\frac{2}{3}} \left(E|u_{iv} u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \Big\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{3}} + 1 \right) [a_1 \exp\{-a_2(v - \ell)\}]^{\frac{1}{21}} \right. \\
&\quad \times \left(E[\{u_{it} u_{is} u_{ig} u_{i\ell} - E(u_{it} u_{is} u_{ig} u_{i\ell})\}^{\frac{3}{2}}] \right)^{\frac{2}{3}} \left(E|u_{iv} u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \Big\}
\end{aligned}$$

Next, observe that by applying of Hölder's inequality, we have

$$\begin{aligned}
E|u_{iv} u_{iw}|^{\frac{7}{2}} &\leq \left(E|u_{iv}|^7 \right)^{\frac{1}{2}} \left(E|u_{iw}|^7 \right)^{\frac{1}{2}} \\
&\leq (\overline{C})^{\frac{1}{2}} (\overline{C})^{\frac{1}{2}} \quad (\text{by Assumption 3-3(b)}) \\
&= \overline{C} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
E |\{u_{it}u_{is}u_{ig}u_{il} - E(u_{it}u_{is}u_{ig}u_{il})\}|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}} \left(E |u_{it}u_{is}u_{ig}u_{il}|^{\frac{3}{2}} + |E[u_{it}u_{is}u_{ig}u_{il}]|^{\frac{3}{2}} \right) \\
&\quad (\text{by Loève's } c_r \text{ inequality}) \\
&\leq 2^{\frac{1}{2}} \left(E |u_{it}u_{is}u_{ig}u_{il}|^{\frac{3}{2}} + E |u_{it}u_{is}u_{ig}u_{il}|^{\frac{3}{2}} \right) \\
&\quad (\text{by Jensen's inequality}) \\
&\leq 2^{\frac{3}{2}} E |u_{it}u_{is}u_{ig}u_{il}|^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left(E |u_{it}u_{is}|^3 \right)^{\frac{1}{2}} \left(E |u_{ig}u_{il}|^3 \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}} \left(\left(E |u_{it}|^6 \right)^{\frac{1}{2}} \left(E |u_{is}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left(\left(E |u_{ig}|^6 \right)^{\frac{1}{2}} \left(E |u_{il}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&= 2^{\frac{3}{2}} \left[\left(E |u_{it}|^6 \right)^{\frac{1}{6}} \left(E |u_{is}|^6 \right)^{\frac{1}{6}} \left(E |u_{ig}|^6 \right)^{\frac{1}{6}} \left(E |u_{il}|^6 \right)^{\frac{1}{6}} \right]^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left[\left(E |u_{it}|^7 \right)^{\frac{1}{7}} \left(E |u_{is}|^7 \right)^{\frac{1}{7}} \left(E |u_{ig}|^7 \right)^{\frac{1}{7}} \left(E |u_{il}|^7 \right)^{\frac{1}{7}} \right]^{\frac{3}{2}} \\
&\quad (\text{by Liapunov's inequality}) \\
&\leq 2^{\frac{3}{2}} \left[\left(\sup_t E |u_{it}|^7 \right)^{\frac{4}{7}} \right]^{\frac{3}{2}} \\
&= 2^{\frac{3}{2}} \overline{C}^{\frac{6}{7}} \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

Again, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{iv}u_{iw}|^{\frac{7}{2}}$ and

$E |\{u_{it}u_{is}u_{ig}u_{il} - E(u_{it}u_{is}u_{ig}u_{il})\}|^{\frac{3}{2}}$ as shown above, we can further write

$$\begin{aligned}
& \mathcal{TT}_3 \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} |E[\{u_{it}u_{is}u_{ig}u_{il} - E(u_{it}u_{is}u_{ig}u_{il})\} u_{iv}u_{iw}]| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{3}} + 1 \right) [a_1 \exp\{-a_2(v - \ell)\}]^{\frac{1}{21}} \right. \\
&\quad \times \left(E |\{u_{it}u_{is}u_{ig}u_{il} - E(u_{it}u_{is}u_{ig}u_{il})\}|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(E |u_{iv}u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \Big\} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} 2 \left(2^{\frac{1}{3}} + 1 \right) [a_1 \exp\{-a_2(v - \ell)\}]^{\frac{1}{21}} \left(2^{\frac{3}{2}} \overline{C}^{\frac{6}{7}} \right)^{\frac{2}{3}} (\overline{C})^{\frac{2}{7}} \\
&\leq \frac{C_2^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v - \ell \geq \max\{w - v, \ell - g\}, v - \ell > 0}}^{T-h} \exp\left\{-\frac{a_2}{21} \varrho_2\right\} \\
&\quad \left(\text{for some constant } C_2^* \text{ such that } 4 \left(2^{\frac{1}{3}} + 1 \right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{21}} \leq C_2^* < \infty \right) \\
&\leq \frac{C_2^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \sum_{\varrho_3=0}^{\varrho_2} \exp\left\{-\frac{a_2}{21} \varrho_2\right\} \\
&= C_2^* \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \sum_{\varrho_2=1}^{\infty} (\varrho_2 + 1)^2 \exp\left\{-\frac{a_2}{21} \varrho_2\right\} \\
&= C_2^* \left[\sum_{\varrho_2=1}^{\infty} \varrho_2^2 \exp\left\{-\frac{a_2}{21} \varrho_2\right\} + 2 \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{21} \varrho_2\right\} + \sum_{\varrho_2=1}^{\infty} \exp\left\{-\frac{a_2}{21} \varrho_2\right\} \right] \\
&\leq C_2^* \overline{\overline{C}}_2
\end{aligned} \tag{103}$$

for some positive constant

$$\overline{\overline{C}}_2 \geq \sum_{\varrho_2=1}^{\infty} \varrho_2^2 \exp\left\{-\frac{a_2}{21} \varrho_2\right\} + 2 \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{21} \varrho_2\right\} + \sum_{\varrho_2=1}^{\infty} \exp\left\{-\frac{a_2}{21} \varrho_2\right\}$$

which exists in light of Lemma C-1.

Turning our attention to the term \mathcal{TT}_4 , note that, from the upper bounds given in the proofs

of parts (a) and (c) of Lemma D-5, it is clear that there exists a positive constant C^{**} such that, for all i and for all T sufficiently large,

$$\frac{1}{T_h} \sum_{\substack{v,w=p \\ v \leq w}}^{T-h} |E(u_{iv}u_{iw})| \leq C_1^{**}$$

and

$$\frac{1}{T_h^2} \sum_{\substack{t,s,g,\ell=p \\ t \leq s \leq g \leq \ell}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{i\ell})| \leq C_1^{**}$$

from which it follows that

$$\begin{aligned} \mathcal{T}\mathcal{T}_4 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v, \ell-g\}, v-\ell > 0}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{i\ell})| |E(u_{iv}u_{iw})| \\ &\leq \frac{C_1 C}{N_1 N_2} \sum_{k \in H^c} \sum_{i \in H} \left(\frac{1}{T_h^2} \sum_{\substack{t,s,g,\ell=p \\ t \leq s \leq g \leq \ell}}^{T-h} |E(u_{it}u_{is}u_{ig}u_{i\ell})| \right) \left(\frac{1}{T_h} \sum_{\substack{v,w=p \\ v \leq w}}^{T-h} |E(u_{iv}u_{iw})| \right) \\ &\leq \frac{C_1 C}{N_1 N_2} \sum_{k \in H^c} \sum_{i \in H} (C_1^{**})^2 \\ &= C_1 C (C_1^{**})^2 \frac{N_1 N_2}{N_1 N_2} \\ &= C_1 C (C_1^{**})^2 \end{aligned} \tag{104}$$

Consider now $\mathcal{T}\mathcal{T}_5$. In this case, we apply Lemma C-3 with $p = 2$ and $r = 9/4$ to obtain

$$\begin{aligned}
\mathcal{T}\mathcal{T}_5 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} |E[\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}u_{i\ell}u_{iv}u_{iw}]| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} \left\{ 2 \left(2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(\ell - g)\}]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
&\quad \left. \times \left(E|\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{i\ell}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(\ell - g)\}]^{\frac{1}{18}} \right. \\
&\quad \left. \times \left(E|\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left(E|u_{i\ell}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
&E|u_{i\ell}u_{iv}u_{iw}|^{\frac{9}{4}} \\
&\leq \left[E|u_{i\ell}|^7 \right]^{\frac{9}{28}} \left[E|u_{iv}u_{iw}|^{\frac{63}{19}} \right]^{\frac{19}{28}} \\
&\leq \left[E|u_{i\ell}|^7 \right]^{\frac{9}{28}} \left[\left(E|u_{iv}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \left(E|u_{iw}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \right]^{\frac{19}{28}} \\
&= \left[E|u_{i\ell}|^7 \right]^{\frac{9}{28}} \left(E|u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \left(E|u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \\
&= \left[E|u_{i\ell}|^7 \right]^{\frac{9}{28}} \left[\left(E|u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \left(E|u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \right]^{\frac{9}{4}} \\
&\leq \left[E|u_{i\ell}|^7 \right]^{\frac{9}{28}} \left[\left(E|u_{iv}|^7 \right)^{\frac{1}{7}} \left(E|u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{9}{4}} \quad (\text{by Liapunov's inequality}) \\
&\leq \left(\sup_t E|u_{it}|^7 \right)^{\frac{27}{28}} \\
&\leq \overline{C}^{\frac{27}{28}} \quad (\text{by Assumption 3-3(b)})
\end{aligned}$$

and

$$\begin{aligned}
E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 &\leq 2 \left(E |u_{it}u_{is}u_{ig}|^2 + |E[u_{it}u_{is}u_{ig}]|^2 \right) \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq 2 \left(E |u_{it}u_{is}u_{ig}|^2 + E |u_{it}u_{is}u_{ig}|^2 \right) \\
&\quad \text{(by Jensen's inequality)} \\
&\leq 4E |u_{it}u_{is}u_{ig}|^2 \\
&\leq 4 \left(E |u_{it}|^6 \right)^{\frac{1}{3}} \left(E |u_{is}u_{ig}|^3 \right)^{\frac{2}{3}} \\
&\leq 4 \left(E |u_{it}|^6 \right)^{\frac{1}{3}} \left(\sqrt{E |u_{is}|^6} \sqrt{E |u_{ig}|^6} \right)^{\frac{2}{3}} \\
&= 4 \left[\left(E |u_{it}|^6 \right)^{\frac{1}{6}} \right]^2 \left[\left(E |u_{is}|^6 \right)^{\frac{1}{6}} \left(E |u_{ig}|^6 \right)^{\frac{1}{6}} \right]^2 \\
&\leq 4 \left[\left(E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^2 \left[\left(E |u_{is}|^7 \right)^{\frac{1}{7}} \left(E |u_{ig}|^7 \right)^{\frac{1}{7}} \right]^2 \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq 4 \left[\left(\sup_t E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^6 \\
&\leq 4 \overline{C}^{\frac{6}{7}} \text{ (by Assumption 3-3(b))}
\end{aligned}$$

Define again $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E |u_{i\ell}u_{iv}u_{iw}|^{\frac{9}{4}}$ and

$E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2$ as shown above, we can further write

$$\begin{aligned}
& \mathcal{T}\mathcal{T}_5 \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} \left\{ 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp \{-a_2 (\ell - g)\}]^{\frac{1}{18}} \right. \\
& \quad \left. \times \left(E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left(E |u_{i\ell}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\} \\
& \leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} 2 \left(2^{\frac{1}{2}} + 1 \right) [a_1 \exp \{-a_2 (\ell - g)\}]^{\frac{1}{18}} \left(4\overline{C}^{\frac{6}{7}} \right)^{\frac{1}{2}} \left(\overline{C}^{\frac{27}{28}} \right)^{\frac{4}{9}} \\
& \leq \frac{C_3^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t, s, g, \ell, v, w = p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w - v, v - \ell\}, \ell - g > 0}}^{T-h} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \quad \left(\text{for some constant } C_3^* \text{ such that } 4 \left(2^{\frac{1}{2}} + 1 \right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{18}} \leq C_3^* < \infty \right) \\
& \leq \frac{C_3^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \sum_{\varrho_3=0}^{\varrho_1} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \leq \frac{C_3^* N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \sum_{\varrho_1=1}^{\infty} (\varrho_1 + 1)^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \leq C_3^* \left[\sum_{\varrho_1=1}^{\infty} \varrho_1^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + 2 \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + \sum_{\varrho_1=1}^{\infty} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \right] \\
& \leq C_3^* \overline{\overline{C}}_3
\end{aligned} \tag{105}$$

for some positive constant

$$\overline{\overline{C}}_3 \geq \sum_{\varrho_1=1}^{\infty} \varrho_1^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + 2 \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + \sum_{\varrho_1=1}^{\infty} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\}$$

which exists in light of Lemma C-1.

Finally, consider $\mathcal{T}\mathcal{T}_6$. Note that, from the upper bounds given in the proofs of part (b) of Lemma D-5, it is clear that there exists a positive constant C_2^{**} such that, for all i and for all T

sufficiently large,

$$\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E(u_{it}u_{is}u_{ig})| \leq C_2^{**}$$

and

$$\frac{1}{T_h} \sum_{\substack{\ell,v,w=p \\ \ell \leq v \leq w}}^{T-h} |E(u_{i\ell}u_{iv}u_{iw})| \leq C_2^{**}$$

from which it follows that

$$\begin{aligned} \mathcal{T}\mathcal{T}_6 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell-g \geq \max\{w-v, v-\ell\}, \ell-g > 0}}^{T-h} |E(u_{it}u_{is}u_{ig})| |E(u_{i\ell}u_{iv}u_{iw})| \\ &\leq \frac{C_1 C}{N_1 N_2 T_h} \sum_{k \in H^c} \sum_{i \in H} \left(\frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t \leq s \leq g}}^{T-h} |E(u_{it}u_{is}u_{ig})| \right) \left(\frac{1}{T_h} \sum_{\substack{\ell,v,w=p \\ \ell \leq v \leq w}}^{T-h} |E(u_{i\ell}u_{iv}u_{iw})| \right) \\ &\leq \frac{C_1 C}{N_1 N_2 T_h} \sum_{k \in H^c} \sum_{i \in H} (C_2^{**})^2 \\ &= C_1 C (C_2^{**})^2 \frac{N_1 N_2}{N_1 N_2 T_h} \\ &= \frac{C_1 C (C_2^{**})^2}{T_h} = O\left(\frac{1}{T}\right). \end{aligned} \tag{106}$$

It follows from expressions (101)-(106) that, for all N_1, N_2 , and T sufficiently large,

$$\begin{aligned} &\frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_1^3 T_h^3} E \left(\sum_{t=p}^{T-h} \gamma'_k \underline{E}_t u_{i,t} \right)^6 \\ &\leq \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6 \\ &\leq C_1 C \bar{C}^{\frac{6}{7}} + C_1^* \bar{C}_1 + C_2^* \bar{C}_2 + C_1 C (C_1^{**})^2 + C_3^* \bar{C}_3 + \frac{C_1 C (C_2^{**})^2}{T_h} \\ &\leq \tilde{C} \end{aligned}$$

for some positive constant \tilde{C} such that

$$\tilde{C} \geq C_1 C \bar{C}^{\frac{6}{7}} + C_1^* \bar{C}_1 + C_2^* \bar{C}_2 + C_1 C (C_1^{**})^2 + C_3^* \bar{C}_3 + \frac{C_1 C (C_2^{**})^2}{T_h}.$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \left(\tilde{C}/\epsilon\right)^{\frac{1}{3}}$, and note that

$$\begin{aligned}
& \Pr \left\{ \frac{N_1 T_h}{N_2^{\frac{1}{3}}} \max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k \underline{F}' u_{i \cdot}}{\sqrt{N_1 T_h}} \right)^2 \geq C_\epsilon \right\} \\
&= \Pr \left\{ \max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{1}{N_2^{\frac{1}{6}} \sqrt{T_h}} \sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^2 \geq C_\epsilon \right\} \\
&= \Pr \left\{ \max_{i \in H} \left[\frac{1}{N_1} \sum_{k \in H^c} \left(\frac{1}{N_2^{\frac{1}{6}} \sqrt{T_h}} \sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^2 \right]^3 \geq C_\epsilon^3 \right\} \\
&\leq \Pr \left\{ \max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{1}{N_2^{\frac{1}{6}} \sqrt{T_h}} \sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^6 \geq C_\epsilon^3 \right\} \quad (\text{by Jensen's inequality}) \\
&\leq \Pr \left\{ \frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \left(\frac{1}{N_2^{\frac{1}{6}} \sqrt{T_h}} \sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^6 \geq C_\epsilon^3 \right\} \\
&\leq \frac{\epsilon}{\tilde{C}} \frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E \left(\sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^6 \\
&\leq \frac{\epsilon}{\tilde{C}} \tilde{C} \\
&= \epsilon
\end{aligned}$$

This shows that

$$\max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k \underline{F}' u_{i \cdot}}{\sqrt{N_1 T_h}} \right)^2 = O_p \left(\frac{N_2^{\frac{1}{3}}}{N_1 T_h} \right) = O_p \left(\frac{N_2^{\frac{1}{3}}}{N_1 T} \right). \quad \square$$

Before stating the next lemma, we first introduce some more notations. Let $\mathbb{S}_{i,T}^+$ denote either

the statistic $\sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}|$ or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, and define

$$\widehat{H}^c = \left\{ i \in \{1, \dots, N\} : \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\}, \quad (107)$$

$$\widehat{H} = \left\{ i \in \{1, \dots, N\} : \mathbb{S}_{i,T}^+ < \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\}, \quad (108)$$

$$\widehat{N}_1 = \#(\widehat{H}^c), \text{ i.e., the cardinality of the set } \widehat{H}^c, \quad (109)$$

$$\begin{aligned} \Gamma(\widehat{H}^c) &= \begin{pmatrix} \gamma_1 (\widehat{H}^c)' \\ \gamma_2 (\widehat{H}^c)' \\ \vdots \\ \gamma_N (\widehat{H}^c)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\{1 \in \widehat{H}^c\} \gamma'_1 \\ \mathbb{I}\{2 \in \widehat{H}^c\} \gamma'_2 \\ \vdots \\ \mathbb{I}\{N \in \widehat{H}^c\} \gamma'_N \end{pmatrix}, \text{ and} \\ U(\widehat{H}^c) &= \begin{pmatrix} u_{1\cdot} (\widehat{H}^c)' \\ u_{2\cdot} (\widehat{H}^c)' \\ \vdots \\ u_{N\cdot} (\widehat{H}^c)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\{1 \in \widehat{H}^c\} u'_{1\cdot} \\ \mathbb{I}\{2 \in \widehat{H}^c\} u'_{2\cdot} \\ \vdots \\ \mathbb{I}\{N \in \widehat{H}^c\} u'_{N\cdot} \end{pmatrix}, \end{aligned} \quad (110)$$

where $u_{i\cdot} = (u_{i,p}, u_{i,p+1}, \dots, u_{i,T-h})'$ for $i = 1, \dots, N$.

Lemma D-7: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11 hold. Then, as $N_1, N_2, T \rightarrow \infty$, the following statements are true.

(a)

$$\sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} = O_p(\varphi)$$

(b)

$$\sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i\cdot}}{\sqrt{N_1 T_h}} \right)^2 = O_p \left(\frac{N_2^{\frac{1}{3}} \varphi}{N_1 T} \right).$$

(c)

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i\cdot}}{\sqrt{N_1 T_h}} \right)^2 = O_p \left(\frac{1}{T} \right)$$

Proof of Lemma D-7:

To show part (a), let $\mathbb{S}_{i,T}^+$ denote either the statistic $\sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}|$ or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$. Following arguments similar to that given in the proof of part (a) of Theorem 2, we see that there

exists a constant $C > 2d$ such that

$$\begin{aligned}
\sum_{i \in H} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right] &= \sum_{i \in H} \Pr \left(i \in \widehat{H}^c \right) \\
&= \sum_{i \in H} \Pr \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \\
&\leq C \frac{N_2 \varphi}{N} \\
&\leq C \varphi
\end{aligned}$$

for all N_1, N_2 , and T sufficiently large. Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}
\Pr \left\{ \frac{1}{\varphi} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \geq C_\epsilon \right\} &\leq \frac{1}{C_\epsilon \varphi} \sum_{i \in H} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right] \quad (\text{by Markov's inequality}) \\
&\leq \frac{\epsilon}{C \varphi} C \varphi \\
&= \epsilon
\end{aligned}$$

which shows that

$$\sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} = O_p(\varphi)$$

Next, to show part (b), we combine the result given in part (a) of this lemma with the result of Lemma D-6 to obtain

$$\begin{aligned}
&\sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i \cdot}}{\sqrt{N_1 T_h}} \right)^2 \\
&\leq \max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i \cdot}}{\sqrt{N_1 T_h}} \right)^2 \left[\sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right] \quad (\text{by Hölder's inequality}) \\
&= O_p \left(\frac{N_2^{\frac{1}{3}}}{N_1 T} \right) O_p(\varphi) \\
&= O_p \left(\frac{N_2^{\frac{1}{3}} \varphi}{N_1 T} \right).
\end{aligned}$$

Finally, to show part (c), note first that

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i \cdot}}{\sqrt{N_1 T_h}} \right)^2 \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i \cdot}}{T_h} \right)^2$$

Moreover, write

$$\begin{aligned}
0 &\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left(\frac{\gamma'_k \underline{F}' u_i}{T_h} \right)^2 \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^2 \\
&= \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left(\sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^2 \\
&= \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} E \{ \gamma'_k \underline{F}_s u_{i,s} u_{i,t} \underline{F}'_t \gamma_k \} \\
&= \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma'_k E_F [\underline{F}_t E(u_{i,t}^2) \underline{F}'_t] \gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E_F [\gamma'_k \underline{F}_t E(u_{i,t} u_{i,t+m}) \underline{F}'_{t+m} \gamma_k] \\
&= \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma'_k E_F [\underline{F}_t E(u_{i,t}^2) \underline{F}'_t] \gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E(u_{i,t} u_{i,t+m}) E_F [\gamma'_k \underline{F}_t \underline{F}'_{t+m} \gamma_k] \\
&\leq \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma'_k E_F [\underline{F}_t E(u_{i,t}^2) \underline{F}'_t] \gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E(u_{i,t} u_{i,t+m})| |\gamma'_k E_F [\underline{F}_t \underline{F}'_{t+m}] \gamma_k|
\end{aligned}$$

Note that by Assumption 3-3(c), $\{u_{i,t}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp \{-a_2 m\}$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp \{-a_2 m\} \text{ for every } i.$$

Hence, applying Lemma C-3 with $p = 3$ and $r = 3$ as well as Assumptions 3-3(b) and 3-5 and

Lemma C-5; we get

$$\begin{aligned}
& \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left[\left(\sum_{t=p}^{T-h} \gamma'_k \underline{F}_t u_{i,t} \right)^2 \right] \\
& \leq \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma'_k E_F [\underline{F}_t E(u_{i,t}^2) \underline{F}_t'] \gamma_k \\
& \quad + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E(u_{i,t} u_{i,t+m})| |\gamma'_k E_F [\underline{F}_t \underline{F}_{t+m}'] \gamma_k| \\
& \leq \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma'_k E_F [\underline{F}_t E(u_{i,t}^2) \underline{F}_t'] \gamma_k \\
& \quad + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E(u_{i,t} u_{i,t+m})| E |\gamma'_k \underline{F}_t \underline{F}_{t+m}' \gamma_k| \\
& \leq \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} E(u_{i,t}^2) \|\gamma_k\|_2^2 E[\|\underline{F}_t\|_2^2] \\
& \quad + \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} 2 \left(2^{\frac{2}{3}} + 1 \right) 2 \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{ \alpha_m^{\frac{1}{3}} \left(E |u_{i,t}|^3 \right)^{\frac{1}{3}} \left(E |u_{i,t+m}|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \quad \quad \left. \times \sqrt{\gamma'_k E [\underline{F}_t \underline{F}_t'] \gamma_k} \sqrt{\gamma'_k E [\underline{F}_{t+m} \underline{F}_{t+m}'] \gamma_k} \right\} \\
& \leq \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} E(u_{i,t}^2) \|\gamma_k\|_2^2 E[\|\underline{F}_t\|_2^2] \\
& \quad + \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} 4 \left(2^{\frac{2}{3}} + 1 \right) \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{ \alpha_m^{\frac{1}{3}} \left(E |u_{i,t}|^3 \right)^{\frac{1}{3}} \left(E |u_{i,t+m}|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \quad \quad \left. \times \|\gamma_k\|_2^2 \sqrt{E \|\underline{F}_t\|_2^2} \sqrt{E \|\underline{F}_{t+m}\|_2^2} \right\} \\
& \leq \frac{C_1}{T_h} + C_2 \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} a_1^{\frac{1}{3}} \exp \left\{ -\frac{a_2}{3} m \right\} \\
& \leq \frac{C_1}{T_h} + C_2 a_1^{\frac{1}{3}} \frac{1}{T_h} \sum_{m=1}^{\infty} \exp \left\{ -\frac{a_2}{3} m \right\} \\
& \leq \frac{\overline{C}}{T_h}
\end{aligned}$$

for some positive constant

$$\overline{C} \geq C_1 + C_2 a_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{a_2}{3} m \right\}$$

which exists in light of Lemma C-1. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and note that

$$\begin{aligned}
& \Pr \left\{ \frac{T_h}{N_1} \sum_{i \in H^c} \mathbb{I} \{ i \in \widehat{H}^c \} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i.}}{\sqrt{N_1} T_h} \right)^2 \geq C_\epsilon \right\} \\
& \leq \Pr \left\{ \frac{T_h}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i.}}{T_h} \right)^2 \geq C_\epsilon \right\} \\
& \leq \frac{T_h}{C_\epsilon} \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left[\left(\sum_{t=p}^{T-h} \gamma'_k F_t u_{i,t} \right)^2 \right] \\
& \leq \frac{\epsilon}{\overline{C}} T_h \frac{\overline{C}}{T_h} \\
& = \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{ i \in \widehat{H}^c \} \sum_{k \in H^c} \left(\frac{\gamma'_k F' u_{i.}}{\sqrt{N_1} T_h} \right)^2 = O_p \left(\frac{1}{T_h} \right) = O_p \left(\frac{1}{T} \right). \quad \square$$

Lemma D-8: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{ i \in \widehat{H}^c \} \left(\frac{u'_{i.} u_{i.}}{T_h} \right) = O_p(1).$$

(b)

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I} \{ i \in \widehat{H}^c \} \left(\frac{u'_{i.} u_{i.}}{T_h} \right) = O_p \left(\frac{N_1^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

Proof of Lemma D-8:

To show part (a), note first that

$$\begin{aligned}
\frac{1}{N_1} \sum_{i \in H^c} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \right] &\leq \frac{1}{N_1} \sum_{i \in H^c} E \left[\left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \right] \\
&\leq \frac{1}{N_1} \sum_{i \in H^c} \frac{1}{T_h} \sum_{t=p}^{T-h} \sup_{i,t} E \left[u_{i,t}^2 \right] \\
&\leq \frac{1}{N_1} \sum_{i \in H^c} \frac{1}{T_h} \sum_{t=p}^{T-h} C \\
&= C
\end{aligned}$$

for some positive constant $C \geq \sup_{i,t} E \left[u_{i,t}^2 \right]$ which exists in light of Assumption 3-3(b). Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}
&\Pr \left\{ \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \geq C_\epsilon \right\} \\
&\leq \frac{1}{C_\epsilon} \frac{1}{N_1} \sum_{i \in H^c} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \right] \quad (\text{by Markov's inequality}) \\
&\leq \frac{\epsilon}{C} C \\
&= \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) = O_p(1).$$

Next, to show part (b), note that

$$\begin{aligned}
& \frac{1}{N_1} \sum_{i \in H} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \right] \\
& \leq \frac{1}{N_1} \sum_{i \in H} \left(\Pr \left\{ i \in \widehat{H}^c \right\} \right)^{\frac{5}{7}} \left(E \left[\left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right)^{\frac{7}{2}} \right] \right)^{\frac{2}{7}} \quad (\text{by Hölder's inequality}) \\
& = \frac{1}{N_1} \sum_{i \in H} \left(\Pr \left\{ i \in \widehat{H}^c \right\} \right)^{\frac{5}{7}} \left(E \left[\left(\frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \right)^{\frac{7}{2}} \right] \right)^{\frac{2}{7}} \\
& \leq \frac{1}{N_1} \sum_{i \in H} \left(\Pr \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right)^{\frac{5}{7}} \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \sup_{i,t} E \left[|u_{i,t}|^7 \right] \right)^{\frac{2}{7}} \\
& \leq C_1^{\frac{2}{7}} \frac{1}{N_1} \sum_{i \in H} \left(\Pr \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right)^{\frac{5}{7}}
\end{aligned}$$

for some positive constant $C_1 \geq \sup_{i,t} E \left[|u_{i,t}|^7 \right]$ which exists in light of Assumption 3-3(b). Now, let $\mathbb{S}_{i,T}^+$ denote either the statistic $\sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}|$ or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$; and, following arguments similar to that given in the proof of part (a) of Theorem 2, we see that, for any $i \in H$, there exists a constant $C_2 > 2d$ such that

$$\Pr \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \leq C_2 \frac{\varphi}{N}$$

for all N_1, N_2 , and T sufficiently large, from which it follows that

$$\begin{aligned}
\frac{1}{N_1} \sum_{i \in H} E \left[\mathbb{I} \left\{ i \in \widehat{H}^c \right\} \left(\frac{u'_{i \cdot} u_{i \cdot}}{T_h} \right) \right] & \leq C_1^{\frac{2}{7}} \frac{1}{N_1} \sum_{i \in H} \left(\Pr \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right)^{\frac{5}{7}} \\
& \leq C_1^{\frac{2}{7}} \frac{1}{N_1} \sum_{i \in H} C_2^{\frac{5}{7}} \left(\frac{\varphi}{N} \right)^{\frac{5}{7}} \\
& = C_1^{\frac{2}{7}} C_2^{\frac{5}{7}} \frac{N_2 \varphi^{\frac{5}{7}}}{N_1 N^{\frac{5}{7}}} \\
& \leq C_3 \frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1}
\end{aligned}$$

for all N_1, N_2 , and T sufficiently large and for some positive constant $C_3 \geq C_1^{\frac{2}{7}} C_2^{\frac{5}{7}}$. Hence, for any

$\epsilon > 0$, set $C_\epsilon = C_3/\epsilon$, and note that

$$\begin{aligned}
& \Pr \left\{ \frac{N_1}{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}} \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \{i \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{i\cdot}}{T_h} \right) \geq C_\epsilon \right\} \\
& \leq \frac{N_1}{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}} \frac{1}{C_\epsilon} \frac{1}{N_1} \sum_{i \in H} E \left[\mathbb{I} \{i \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{i\cdot}}{T_h} \right) \right] \quad (\text{by Markov's inequality}) \\
& \leq \frac{N_1}{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}} \frac{\epsilon}{C_3} C_3 \frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \\
& = \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I} \{i \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{i\cdot}}{T_h} \right) = O_p \left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1). \quad \square$$

Lemma D-9: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)

$$\mathcal{T}_1 = \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I} \{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h} \right)^2 = O_p \left(\max \left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) = o_p(1).$$

(b)

$$\mathcal{T}_2 = \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I} \{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h} \right)^2 = O_p \left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(c)

$$\mathcal{T}_3 = \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I} \{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h} \right)^2 = O_p \left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(d)

$$\mathcal{T}_4 = \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I} \{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h} \right)^2 = O_p \left(\frac{N^{\frac{4}{7}} \varphi^{\frac{10}{7}}}{N_1^2} \right) = o_p(1).$$

Proof of Lemma D-9:

To show part (a), note that

$$\begin{aligned}
0 &\leq \mathcal{T}_1 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_k}{T_h} \right)^2 \\
&\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left(\frac{u'_i u_k}{T_h} \right)^2 \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} u_{i,t} u_{k,t} u_{i,s} u_{k,s} \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} u_{i,t}^2 u_{k,t}^2 \\
&\quad + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}
\end{aligned}$$

From the non-negativity of \mathcal{T}_1 , we get

$$\begin{aligned}
E |\mathcal{T}_1| &= E [\mathcal{T}_1] \\
&\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E [u_{i,t}^2 u_{k,t}^2] \\
&\quad + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E [u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}]
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E [u_{i,t}^2 u_{k,t}^2] &\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E [u_{i,t}^4]} \sqrt{E [u_{k,t}^4]} \\
&\leq \left(\sup_{i,t} E [u_{i,t}^4] \right) \frac{1}{T_h} \\
&\leq \frac{C_1}{T_h}
\end{aligned}$$

for some positive constant $C_1 \geq \sup_{i,t} E[u_{i,t}^4]$ which exists in light of Assumption 3-3(b). Moreover,

$$\begin{aligned}
& \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E[u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}] \\
= & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E[(u_{i,t} u_{k,t} - E[u_{i,t} u_{k,t}])(u_{i,t+m} u_{k,t+m} - E[u_{i,t+m} u_{k,t+m}])] \\
& + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E[u_{i,t} u_{k,t}] E[u_{i,t+m} u_{k,t+m}] \\
\leq & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E[(u_{i,t} u_{k,t} - E[u_{i,t} u_{k,t}])(u_{i,t+m} u_{k,t+m} - E[u_{i,t+m} u_{k,t+m}])]| \\
& + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| |E[u_{i,t+m} u_{k,t+m}]|
\end{aligned}$$

Consider the first term on the right-hand side above. Note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β -mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i . Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α -mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$\begin{aligned}
& \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E[(u_{i,t} u_{k,t} - E[u_{i,t} u_{k,t}]) (u_{i,t+m} u_{k,t+m} - E[u_{i,t+m} u_{k,t+m}])]| \\
& \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} 2(\sqrt{2}+1) \alpha_m^{\frac{1}{6}} \sqrt{E[u_{i,t}^2 u_{k,t}^2]} \left(E|u_{i,t+m} u_{k,t+m}|^3\right)^{\frac{1}{3}} \\
& \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{4(\sqrt{2}+1)}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} a_1^{\frac{1}{6}} \exp\left\{-\frac{a_2}{6}m\right\} \sqrt{E[u_{i,t}^2 u_{k,t}^2]} \left(E|u_{i,t+m} u_{k,t+m}|^3\right)^{\frac{1}{3}} \\
& \leq \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{-\frac{a_2}{6}m\right\} \frac{4a_1^{\frac{1}{6}} (\sqrt{2}+1) \left(E[u_{i,t}^4]\right)^{\frac{1}{4}} \left(E[u_{k,t}^4]\right)^{\frac{1}{4}} \left(E[u_{i,t+m}^6] E[u_{k,t+m}^6]\right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
& \leq \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{-\frac{a_2}{6}m\right\} \frac{4a_1^{\frac{1}{6}} (\sqrt{2}+1) \left(E[u_{i,t}^6]\right)^{\frac{1}{6}} \left(E[u_{k,t}^6]\right)^{\frac{1}{6}} \left(E[u_{i,t+m}^6] E[u_{k,t+m}^6]\right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
& \leq \frac{4\overline{C} (\sqrt{2}+1) a_1^{\frac{1}{6}} \left(\sup_{i,t} E[u_{i,t}^6]\right)^{\frac{2}{3}}}{T_h} \\
& \quad \left(\text{for some positive constant } \overline{C} \text{ such that } \overline{C} \geq \sum_{m=1}^{\infty} \exp\left\{-\frac{a_2}{6}m\right\}\right) \\
& \leq \frac{4\overline{C} (\sqrt{2}+1) a_1^{\frac{1}{6}} C^{\frac{2}{3}}}{T_h} \\
& \quad \left(\text{by Assumption 3-3(b), there exists positive constant } C \text{ such that } \sup_{i,t} E|u_{i,t}|^6 \leq C < \infty\right) \\
& \leq \frac{C_2}{T_h} \left(\text{setting } C_2 \geq 4\overline{C} (\sqrt{2}+1) a_1^{\frac{1}{6}} C^{\frac{2}{3}}\right) \\
& = O\left(\frac{1}{T}\right)
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| |E[u_{i,t+m} u_{k,t+m}]| \\
& \leq \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| \sqrt{E[u_{i,t+m}^2]} \sqrt{E[u_{k,t+m}^2]} \\
& \leq \left(\sup_{i,t} E[u_{i,t}^2] \right) \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| \\
& \leq \frac{2}{N_1} \left(\sup_{i,t} E[u_{i,t}^2] \right) \sup_t \left(\frac{1}{N_1} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| \right) \\
& \leq \frac{C_3}{N_1}.
\end{aligned}$$

for some positive constant C_3 such that

$$2 \left(\sup_{i,t} E[u_{i,t}^2] \right) \sup_t \left(\frac{1}{N_1} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| \right) \leq C_3 < \infty$$

which exists in light of Assumptions 3-3(b) and 3-3(d). It follows from these results that

$$\begin{aligned}
& E|\mathcal{T}_1| \\
& = E \left[\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i \cdot} u_{k \cdot}}{T_h} \right)^2 \right] \\
& \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E[u_{i,t}^2 u_{k,t}^2] + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E[u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}] \\
& \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E[u_{i,t}^2 u_{k,t}^2] \\
& \quad + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} |E[(u_{i,t} u_{k,t} - E[u_{i,t} u_{k,t}]) (u_{i,t+m} u_{k,t+m} - E[u_{i,t+m} u_{k,t+m}])]| \\
& \quad + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| |E[u_{i,t+m} u_{k,t+m}]| \\
& \leq \frac{C_1}{T_h} + \frac{C_2}{T_h} + \frac{C_3}{N_1} \\
& \leq \frac{\overline{C}}{\min\{N_1, T_h\}}
\end{aligned}$$

for some positive constant $\overline{C} \geq C_1 + C_2 + C_3$. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and applying Markov's inequality, we obtain

$$\begin{aligned}
& \Pr(\min\{N_1, T_h\} | \mathcal{T}_1| \geq C_\epsilon) \\
&= \Pr\left(\min\{N_1, T_h\} \left| \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h}\right)^2 \right| \geq C_\epsilon\right) \\
&\leq \frac{\min\{N_1, T_h\}}{C_\epsilon} E\left\{ \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h}\right)^2 \right\} \\
&\leq \min\{N_1, T_h\} \frac{\epsilon}{\overline{C}} \frac{\overline{C}}{\min\{N_1, T_h\}} \\
&= \epsilon
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h}\right)^2 \\
&= O_p\left(\frac{1}{\min\{N_1, T_h\}}\right) = O_p\left(\frac{1}{\min\{N_1, T\}}\right) = O_p\left(\max\left\{\frac{1}{N_1}, \frac{1}{T}\right\}\right).
\end{aligned}$$

Next, to show part (b), we apply parts (a) and (b) of Lemma D-8 to obtain

$$\begin{aligned}
\mathcal{T}_2 &= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{i\cdot}}{T_h}\right) \left(\frac{u'_{k\cdot} u_{k\cdot}}{T_h}\right) \quad (\text{by CS inequality}) \\
&= \left[\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \left(\frac{u'_{i\cdot} u_{i\cdot}}{T_h}\right) \right] \left[\frac{1}{N_1} \sum_{k \in H} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{k\cdot} u_{k\cdot}}{T_h}\right) \right] \\
&= O_p(1) O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1}\right) \\
&= O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1}\right) = o_p(1)
\end{aligned}$$

Part (c) can be shown in the same way as part (b) above. Hence, to avoid redundancy, we do not give an explicit proof here.

Finally, to show part (d), we apply part (b) of Lemma D-8 to obtain

$$\begin{aligned}
\mathcal{T}_4 &= \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i.} u_{k.}}{T_h} \right)^2 \\
&= \frac{1}{N_1^2} \sum_{i \in H} \sum_{k \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i.} u_{k.}}{T_h} \right)^2 \\
&\leq \frac{1}{N_1^2} \sum_{i \in H} \sum_{k \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_{i.} u_{i.}}{T_h} \right) \left(\frac{u'_{k.} u_{k.}}{T_h} \right) \quad (\text{by CS inequality}) \\
&= \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \left(\frac{u'_{i.} u_{i.}}{T_h} \right) \right]^2 \\
&= O_p \left(\frac{N^{\frac{4}{7}} \varphi^{\frac{10}{7}}}{N_1^2} \right) = o_p(1). \quad \square
\end{aligned}$$

Lemma D-10: Let

$$\widehat{\Sigma}(\widehat{H}^c) = \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{\widehat{N}_1 T_0} \tag{111}$$

where $T_0 = T - p + 1$, where \widehat{H}^c and \widehat{N}_1 are as defined, respectively, in expressions (107) and (109) above, and where

$$Z(\widehat{H}^c)_{T_0 \times N} = \begin{bmatrix} Z_1 \mathbb{I}\{1 \in \widehat{H}^c\} & Z_2 \mathbb{I}\{2 \in \widehat{H}^c\} & \cdots & Z_N \mathbb{I}\{N \in \widehat{H}^c\} \end{bmatrix} \tag{112}$$

with $Z_{i.} = (Z_{i,p}, Z_{i,p+1}, \dots, Z_{i,T})'$ for $i = 1, \dots, N$. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10, and 3-11* hold.

Under the assumed conditions,

$$\left\| \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 = o_p(1) \text{ as } N_1, N_2, T \rightarrow \infty,$$

where

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^T E[E_t E_t'].$$

Proof of Lemma D-10:

To proceed, note that we can write

$$Z(\widehat{H}^c) = \underline{F} \Gamma(\widehat{H}^c)' + U(\widehat{H}^c),$$

so that

$$\begin{aligned}
& \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
&= \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{\widehat{N}_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{N_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \left\{ \frac{\Gamma(\widehat{H}^c) \underline{E}' \underline{E} \Gamma(\widehat{H}^c)'}{N_1 T_0} + \frac{U(\widehat{H}^c)' \underline{E} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right. \\
&\quad \left. + \frac{\Gamma(\widehat{H}^c) \underline{E}' U(\widehat{H}^c)}{N_1 T_0} + \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
&= -\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right) \frac{\Gamma M_{FF} \Gamma'}{N_1} + \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \left\{ \frac{1}{N_1} \Gamma(\widehat{H}^c) \left[\frac{\underline{E}' \underline{E}}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c) \right. \\
&\quad + \frac{1}{N_1} \left(\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)' - \Gamma M_{FF} \Gamma' \right) + \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \\
&\quad \left. + \frac{U(\widehat{H}^c)' \underline{E} \Gamma(\widehat{H}^c)'}{N_1 T_0} + \frac{\Gamma(\widehat{H}^c) \underline{E}' U(\widehat{H}^c)}{N_1 T_0} \right\} \tag{113}
\end{aligned}$$

where M_{FF} is as defined in (88), where $\Gamma(\widehat{H}^c)$ and $U(\widehat{H}^c)$ are as defined in (110), and where $Z(\widehat{H}^c)$ is as defined in expression (112).

Consider first the term $-\left[\left(\widehat{N}_1 - N_1\right)/\widehat{N}_1\right] (\Gamma M_{FF} \Gamma'/N_1)$. Note that, for some positive con-

stant \overline{C} such that

$$\begin{aligned}
\|M_{FF}\|_F &= \left\| \frac{1}{T_0} \sum_{t=p}^T E [\underline{F}_t \underline{F}_t'] \right\|_F \\
&\leq \frac{1}{T_0} \sum_{t=p}^T \|E [\underline{F}_t \underline{F}_t']\|_F \\
&\quad \text{(by the homogeneity of matrix norm and the triangle inequality)} \\
&\leq \frac{1}{T_0} \sum_{t=p}^T E \|\underline{F}_t \underline{F}_t'\|_F \quad \text{(by the Jensen's inequality)} \\
&= \frac{1}{T_0} \sum_{t=p}^T E \left[\sqrt{\text{tr} \{ \underline{F}_t \underline{F}_t' \underline{F}_t \underline{F}_t' \}} \right] \\
&= \frac{1}{T_0} \sum_{t=p}^T E \sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_0} \sum_{t=p}^T E \left[\|\underline{F}_t\|_2^2 \right] \\
&\leq \frac{1}{T_0} \sum_{t=p}^T \left(E \left[\|\underline{F}_t\|_2^6 \right] \right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
&\leq \overline{C}^{\frac{1}{3}} \quad \text{(by Lemma C-5)} \\
&< \infty
\end{aligned} \tag{114}$$

from which it follows that

$$\begin{aligned}
\left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F &= \sqrt{\text{tr} \left\{ \frac{\Gamma M_{FF} \Gamma'}{N_1} \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right) \text{tr} \left\{ \frac{\Gamma M_{FF}^2 \Gamma'}{N_1} \right\}} \\
&= \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right) \text{tr} \left\{ \frac{M_{FF} \Gamma' \Gamma M_{FF}}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max}^2 \left(\frac{\Gamma' \Gamma}{N_1} \right) \text{tr} \{ M_{FF}^2 \}} \\
&= \lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right) \|M_{FF}\|_F \\
&\leq C^* \overline{C}^{\frac{1}{3}} < \infty \text{ for all } N_1, N_2 \text{ sufficiently large,}
\end{aligned}$$

since, by Assumption 3-6, there exists some positive constant C^* such that $\lambda_{\max}(\Gamma'\Gamma/N_1) \leq C^* < \infty$ for all N_1, N_2 sufficiently large. Moreover, applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left| - \left(\frac{\hat{N}_1 - N_1}{\hat{N}_1} \right) \right| = \left| \frac{\hat{N}_1 - N_1}{\hat{N}_1} \right| = \left| \frac{\hat{N}_1 - N_1}{N_1} \right| \left| \frac{1}{(\hat{N}_1 - N_1)/N_1 + 1} \right| \xrightarrow{p} 0$$

so that by a further application of the Slutsky's theorem, we can deduce that

$$\left\| - \left(\frac{\hat{N}_1 - N_1}{\hat{N}_1} \right) \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F = \left| \frac{\hat{N}_1 - N_1}{\hat{N}_1} \right| \left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \xrightarrow{p} 0. \quad (115)$$

Consider now the other terms on the right-hand side of expression (113). To proceed, we first note that, by applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left| \left(1 + \frac{\hat{N}_1 - N_1}{N_1} \right)^{-1} \right| = \left| 1 + \frac{\hat{N}_1 - N_1}{N_1} \right|^{-1} \xrightarrow{p} 1.$$

Next, note that

$$\begin{aligned} & \left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F^2 \\ &= \sum_{i=1}^N \sum_{k=1}^N \left(\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \gamma'_i M_{FF} \gamma_k - \gamma'_i M_{FF} \gamma_k \right)^2 \\ &= \sum_{i \in H^c} \sum_{k \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \gamma'_i M_{FF} \gamma_k - \gamma'_i M_{FF} \gamma_k \right)^2 \end{aligned}$$

where $H^c = \{k \in \{1, \dots, N\} : \gamma_k \neq 0\}$, where $\widehat{H}^c = \left\{ i \in \{1, \dots, N\} : \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\}$, and where $\mathbb{S}_{i,T}^+$ denotes either the statistic $\sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$. Note that

$$\sum_{i \in H^c} \sum_{k \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \gamma'_i M_{FF} \gamma_k - \gamma'_i M_{FF} \gamma_k \right)^2 = 0 \text{ if } \mathbb{I}\{i \in \widehat{H}^c\} = 1 \text{ for every } i \in H^c,$$

so that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{ \left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \geq \epsilon \right\} &\subseteq \left\{ i \notin \widehat{H}^c \text{ for at least one } i \in H^c \right\} \\
&= \bigcup_{i \in H^c} \{i \notin \widehat{H}^c\} \\
&= \bigcup_{i \in H^c} \left\{ \mathbb{S}_{i,T}^+ < \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \\
&= \left\{ \bigcap_{i \in H^c} \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right\}^c
\end{aligned}$$

Hence, applying either part (a) or part (b) of Theorem 3 depending on whether $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we obtain

$$\begin{aligned}
&\Pr \left(\left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \geq \epsilon \right) \\
&\leq 1 - \Pr \left(\bigcap_{i \in H^c} \left\{ \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right) \\
&= 1 - \Pr \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&\rightarrow 1 - 1 = 0 \text{ as } N_1, N_2, T \rightarrow \infty,
\end{aligned}$$

so that

$$\left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F = o_p(1) \quad (116)$$

Now, consider the term $\Gamma(\widehat{H}^c) \left[\frac{F'F}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)' / N_1$. For this term, note first that, by sub-multiplicativity of matrix norms, we have that

$$\begin{aligned}
\left\| \frac{\Gamma(\widehat{H}^c) \left[\frac{F'F}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)'}{N_1} \right\|_F &\leq \left\| \frac{\Gamma(\widehat{H}^c)}{\sqrt{N_1}} \right\|_F \left\| \frac{F'F}{T_0} - M_{FF} \right\|_F \left\| \frac{\Gamma(\widehat{H}^c)'}{\sqrt{N_1}} \right\|_F \\
&= \left\| \frac{\Gamma(\widehat{H}^c)}{\sqrt{N_1}} \right\|_F^2 \left\| \frac{F'F}{T_0} - M_{FF} \right\|_F
\end{aligned}$$

Note that

$$\begin{aligned}
\left\| \frac{\Gamma(\widehat{H}^c)}{\sqrt{N_1}} \right\|_F^2 &= \text{tr} \left\{ \frac{\Gamma(\widehat{H}^c)' \Gamma(\widehat{H}^c)}{N_1} \right\} \\
&= \text{tr} \left\{ \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \gamma_i \gamma_i' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \text{tr} \{ \gamma_i \gamma_i' \} \\
&= \frac{1}{N_1} \sum_{i=1}^N \|\gamma_i\|_2^2 \mathbb{I}\{i \in \widehat{H}^c\} \\
&\leq \sup_i \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \\
&= \sup_{i \in H^c} \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \\
&\quad (\text{since } \gamma_i = 0 \text{ for all } \gamma_i \in H) \\
&\leq C_1 \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\}
\end{aligned}$$

for some positive constant $C_1 \geq \sup_i \|\gamma_i\|_2^2 = \sup_{i \in H^c} \|\gamma_i\|_2^2$ which exists in light of Assumption 3-5. Moreover, write

$$\frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} = \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \quad (117)$$

For the first term on the right-hand side of expression (117) above, we can apply part (a) of Lemma D-7 to obtain

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} = O_p \left(\frac{\varphi}{N_1} \right) = o_p(1).$$

With regard to the second term on the right-hand side of expression (117), note that

$$\frac{1}{N_1} \sum_{i \in H^c} E \left[\mathbb{I}\{i \in \widehat{H}^c\} \right] \leq 1$$

since, by definition, N_1 is the cardinality of the set $\{i \in \{1, \dots, N\} : i \in H^c\}$. Hence, for any $\epsilon > 0$,

set $C_\epsilon = C/\epsilon$ for any positive constant $C \geq 1$, and note that

$$\begin{aligned} \Pr \left\{ \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} \geq C_\epsilon \right\} &\leq \frac{1}{C_\epsilon} \frac{1}{N_1} \sum_{i \in H^c} E \left[\mathbb{I} \{i \in \widehat{H}^c\} \right] \quad (\text{by Markov's inequality}) \\ &\leq \frac{\epsilon}{C} C \\ &= \epsilon \end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} = O_p(1).$$

It follows that

$$\begin{aligned} \left\| \frac{\Gamma(\widehat{H}^c)}{\sqrt{N_1}} \right\|_F^2 &\leq C_1 \frac{1}{N_1} \sum_{i=1}^N \mathbb{I} \{i \in \widehat{H}^c\} \\ &= \frac{C_1}{N_1} \sum_{i \in H} \mathbb{I} \{i \in \widehat{H}^c\} + \frac{C_1}{N_1} \sum_{i \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} \\ &= O_p \left(\frac{\varphi}{N_1} \right) + O_p(1) \\ &= O_p(1). \end{aligned}$$

In addition, applying the result of part (b) of Lemma D-2, we have that

$$\left\| \frac{\underline{F}' \underline{F}}{T_0} - M_{FF} \right\|_F = O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1)$$

from which we further deduce that

$$\begin{aligned} \left\| \frac{\Gamma(\widehat{H}^c) \left[\frac{\underline{F}' \underline{F}}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)'}{N_1} \right\|_F &\leq \left\| \frac{\Gamma(\widehat{H}^c)}{\sqrt{N_1}} \right\|_F^2 \left\| \frac{\underline{F}' \underline{F}}{T_0} - M_{FF} \right\|_F \\ &= O_p(1) O_p \left(\frac{1}{\sqrt{T}} \right) \\ &= O_p \left(\frac{1}{\sqrt{T}} \right). \end{aligned} \tag{118}$$

Turning our attention to the term $U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)' / (N_1 T_0)$, we first write

$$\begin{aligned}
& \left\| \frac{U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F^2 \\
&= \sum_{i=1}^N \sum_{k=1}^N \left(\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \frac{u_i' \underline{F} \gamma_k}{N_1 T_0} \right)^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \gamma_k' \underline{F}' u_i \cdot u_i' \underline{F} \gamma_k \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \gamma_k' \underline{F}' u_i \cdot u_i' \underline{F} \gamma_k \\
&= \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \frac{1}{N_1 T_0^2} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} (\gamma_k' \underline{F}' u_i)^2 \\
&\leq \frac{1}{N_1} \sum_{k \in H^c} \frac{1}{N_1 T_0^2} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} (\gamma_k' \underline{F}' u_i)^2 \\
&= \frac{1}{N_1} \sum_{k \in H^c} \frac{1}{N_1 T_0^2} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} (\gamma_k' \underline{F}' u_i)^2 + \frac{1}{N_1} \sum_{k \in H^c} \frac{1}{N_1 T_0^2} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} (\gamma_k' \underline{F}' u_i)^2 \\
&= \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_i}{\sqrt{N_1 T_0}} \right)^2 + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_i}{\sqrt{N_1 T_0}} \right)^2
\end{aligned}$$

Applying parts (b) and (c) of Lemma D-7, we obtain

$$\begin{aligned}
& \left\| \frac{U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F^2 \\
&\leq \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_i}{\sqrt{N_1 T_0}} \right)^2 + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_i}{\sqrt{N_1 T_0}} \right)^2 \\
&= O_p \left(\frac{N_2^{\frac{1}{3}} \varphi}{N_1 T} \right) + O_p \left(\frac{1}{T} \right) \\
&= O_p \left(\max \left\{ \frac{N_2^{\frac{1}{3}} \varphi}{N_1 T}, \frac{1}{T} \right\} \right) \\
&= o_p(1) \quad (\text{by Assumption 3-11*})
\end{aligned}$$

so that

$$\left\| \frac{U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F = O_p \left(\max \left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\} \right) = o_p(1). \quad (119)$$

Since

$$\left\| \frac{\Gamma(\widehat{H}^c) \underline{F}' U(\widehat{H}^c)}{N_1 T_0} \right\|_F = \left\| \frac{U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F$$

it follows immediately also that

$$\left\| \frac{U(\widehat{H}^c)' \underline{F} \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F = O_p \left(\max \left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\} \right) = o_p(1). \quad (120)$$

Finally, consider the term $\left\| U(\widehat{H}^c)' U(\widehat{H}^c) / N_1 T_0 \right\|_F^2$, where

$$U(\widehat{H}^c) = \begin{bmatrix} u_{1.} \mathbb{I} \{1 \in \widehat{H}^c\} & u_{2.} \mathbb{I} \{2 \in \widehat{H}^c\} & \cdots & u_{N.} \mathbb{I} \{N \in \widehat{H}^c\} \end{bmatrix}.$$

Given that

$$\begin{aligned} & U(\widehat{H}^c)' U(\widehat{H}^c) \\ = & \begin{pmatrix} u'_{1.} u_{1.} \mathbb{I} \{1 \in \widehat{H}^c\} & \cdots & u'_{1.} u_{N.} \mathbb{I} \{1 \in \widehat{H}^c\} \mathbb{I} \{N \in \widehat{H}^c\} \\ u'_{1.} u_{2.} \mathbb{I} \{1 \in \widehat{H}^c\} \mathbb{I} \{2 \in \widehat{H}^c\} & \cdots & u'_{2.} u_{N.} \mathbb{I} \{2 \in \widehat{H}^c\} \mathbb{I} \{N \in \widehat{H}^c\} \\ \vdots & & \vdots \\ u'_{1.} u_{N.} \mathbb{I} \{1 \in \widehat{H}^c\} \mathbb{I} \{N \in \widehat{H}^c\} & \cdots & u'_{N.} u_{N.} \mathbb{I} \{N \in \widehat{H}^c\} \end{pmatrix}, \end{aligned}$$

we can write

$$\begin{aligned}
\left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_F^2 &= \sum_{i=1}^N \sum_{k=1}^N \left(\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} \frac{u'_i u_{k\cdot}}{N_1 T_0} \right)^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} (u'_i u_{k\cdot})^2 \\
&= \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k=1}^N \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_{k\cdot}}{T_0} \right)^2 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\{k \in \widehat{H}^c\} \left(\frac{u'_i u_{k\cdot}}{T_0} \right)^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \text{ (say)},
\end{aligned}$$

where the order of magnitude in probability of the terms \mathcal{T}_1 , \mathcal{T}_2 , \mathcal{T}_3 , and \mathcal{T}_4 are given in parts (a)-(d) of Lemma D-9. It, thus, follows by applying parts (a)-(d) of Lemma D-9 with $h = 0$ that

$$\begin{aligned}
&\left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_F^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\} (u'_i u_{k\cdot})^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \\
&= O_p \left(\max \left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) + O_p \left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p \left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p \left(\frac{N^{\frac{4}{7}} \varphi^{\frac{10}{7}}}{N_1^2} \right) \\
&= O_p \left(\max \left\{ \frac{1}{N_1}, \frac{1}{T}, \frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right\} \right) \\
&= o_p(1).
\end{aligned}$$

from which we further deduce that

$$\left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_F = O_p \left(\max \left\{ \frac{1}{\sqrt{N_1}}, \frac{1}{\sqrt{T}}, \frac{N^{\frac{1}{7}} \varphi^{\frac{5}{14}}}{\sqrt{N_1}} \right\} \right) = o_p(1) \text{ as } N_1, N_2, T \rightarrow \infty. \quad (121)$$

Expressions (115)-(121) together imply that

$$\begin{aligned} & \left\| \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \\ &= \left\| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{N_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \\ &\leq \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \\ &\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1} \Gamma(\widehat{H}^c) \left[\frac{F' F}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)' \right\|_F \\ &\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' F \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F \\ &\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) F' U(\widehat{H}^c)}{N_1 T_0} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_F \\ &= o_p(1) \text{ as } N_1, N_2, T \rightarrow \infty. \end{aligned}$$

Since $\|A\|_2 \leq \|A\|_F$, we also have

$$\begin{aligned}
& \|E\|_2 \\
&= \left\| \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 \\
&= \left\| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \frac{Z(\widehat{H}^c)' Z(\widehat{H}^c)}{\widehat{N}_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 \\
&\leq \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1} \Gamma(\widehat{H}^c) \left[\frac{F' F}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)' \right\|_2 \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' F \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_2 \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) F' U(\widehat{H}^c)}{N_1 T_0} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_2 \\
&\leq \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right|^{-1} \left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) M_{FF} \Gamma(\widehat{H}^c)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1} \Gamma(\widehat{H}^c) \left[\frac{F' F}{T_0} - M_{FF} \right] \Gamma(\widehat{H}^c)' \right\|_F \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' F \Gamma(\widehat{H}^c)'}{N_1 T_0} \right\|_F \\
&\quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma(\widehat{H}^c) F' U(\widehat{H}^c)}{N_1 T_0} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0} \right\|_F \\
&= o_p(1) \text{ as } N_1, N_2, T \rightarrow \infty. \quad \square
\end{aligned}$$

Lemma D-11: Let

$$A_{N \times N} = \frac{\Gamma M_{FF} \Gamma'}{N_1}$$

where

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^T E [\underline{E}_t \underline{E}_t'] \text{ with } T_0 = T - p + 1.$$

Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, 3-6 and 3-7 hold; and let G be an $N \times N$ orthogonal matrix whose columns are the eigenvectors of A . Under the assumed conditions, the following statements are true.

(a) $\text{Rank}(A) = Kp$ for all N_1, N_2 sufficiently large, and, hence, 0 is an eigenvalue of A with algebraic multiplicity equaling $N - Kp$.

(b) Partition G as follows:

$$G_{N \times N} = \begin{bmatrix} G_1 & G_2 \\ N \times Kp & N \times (N-Kp) \end{bmatrix}$$

Without loss of generality, suppose that the columns of G_1 are eigenvectors associated with the non-zero eigenvalues of A , whereas G_2 contains the eigenvectors associated with the zero eigenvalue. Then, the matrix $G'AG$ can be partitioned as follows:

$$G'AG = \begin{pmatrix} \Lambda_1 & 0 \\ Kp \times Kp & Kp \times (N-Kp) \\ 0 & \Lambda_2 \\ (N-Kp) \times Kp & (N-Kp) \times (N-Kp) \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ Kp \times Kp & Kp \times (N-Kp) \\ 0 & 0 \\ (N-Kp) \times Kp & (N-Kp) \times (N-Kp) \end{pmatrix}. \quad (122)$$

where Λ_1 is a diagonal matrix whose diagonal elements are the non-zero eigenvalues of A and where $\Lambda_2 = 0$.

(c) Define the separation measure

$$\text{sep}(\Lambda_1, \Lambda_2) = \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F};$$

then, there exists a positive constant \underline{c} such that

$$\text{sep}(\Lambda_1, \Lambda_2) = \text{sep}(\Lambda_1, 0) = \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \geq \lambda_{\min} \left(\frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right) \geq \underline{c} > 0.$$

Proof of Lemma D-11: To show part (a), note first that, by the result of Lemma D-4 above, there exists a positive constant \underline{C} such that

$$\lambda_{\min} \{M_{FF}\} \geq \underline{C} > 0$$

for all $T > p - 1$; and, by Assumption 3-6, we have,

$$\lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \geq \frac{1}{\overline{C}} \text{ for } N_1, N_2 \text{ sufficiently large.}$$

for some constant \overline{C} such that $0 < \overline{C} < \infty$. Combining these two inequalities, we see that

$$\begin{aligned} \lambda_{\min} \left\{ \frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right\} &\geq \lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \lambda_{\min} \{M_{FF}\} \\ &\geq \frac{\underline{C}}{\overline{C}} > 0 \text{ for all } N_1, N_2, \text{ and } T \text{ sufficiently large.} \end{aligned}$$

This implies that the $Kp \times Kp$ matrix

$$\frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1}$$

is a positive definite (and, therefore, also non-singular) for N_1, N_2 , and T sufficiently large. Moreover, observe that

$$\begin{aligned} &\det \left\{ \lambda I_N - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\} \\ &= \lambda^N \det \left\{ I_N - \lambda^{-1} \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\} \\ &= \lambda^N \det \left\{ I_{Kp} - \lambda^{-1} \frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right\} \quad (\text{by Sylvester's determinantal theorem}) \\ &= \lambda^{N-Kp} \det \left\{ \lambda I_{Kp} - \frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right\} \end{aligned} \tag{123}$$

Hence, the non-zero eigenvalues of the matrix $\Gamma M_{FF} \Gamma' / N_1$ correspond exactly to the eigenvalues of the positive definite matrix $M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2} / N_1$, from which we further deduce that the matrix

$$A = \frac{\Gamma M_{FF} \Gamma'}{N_1}$$

must be of rank Kp for N_1, N_2, T sufficiently large. Since A is an $N \times N$ matrix with $N = N_1 + N_2$, it follows immediately that 0 is an eigenvalue of A with algebraic multiplicity equaling $N - Kp$ for N_1, N_2, T sufficiently large.

To show part (b), let $\Lambda_1 = \text{diag}(\lambda_{1,1}, \dots, \lambda_{1,Kp})$, whose diagonal elements $\lambda_{1,i} > 0$, for $i \in \{1, \dots, Kp\}$, denote the non-zero eigenvalues of A (which must all be positive given that they correspond to the eigenvalues of the positive definite matrix $M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2} / N_1$ as shown in the

proof of part (a)). Moreover, let

$$\Lambda_2 = \begin{matrix} 0 \\ (N-Kp) \times (N-Kp) \end{matrix}$$

whose diagonal elements are the $N - Kp$ zero eigenvalues of A . Since A is a symmetric matrix, the representation given in expression (122) follows immediately from the usual spectral decomposition.

Finally, to show part (c), note that for any $Kp \times (N - Kp)$ matrix $X \neq 0$, we have

$$\begin{aligned} \|\Lambda_1 X - X \Lambda_2\|_F &= \|\Lambda_1 X\|_F \quad (\text{since } \Lambda_2 = 0) \\ &= \sqrt{\text{tr} \{X' \Lambda_1' \Lambda_1 X\}} \\ &\geq \lambda_{\min}(\Lambda_1) \sqrt{\text{tr} \{X' X\}} \\ &= \lambda_{\min}(\Lambda_1) \|X\|_F \end{aligned}$$

It follows that

$$\begin{aligned} \text{sep}(\Lambda_1, \Lambda_2) &= \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F} \\ &= \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \quad (\text{since } \Lambda_2 = 0 \text{ in this case}) \\ &\geq \frac{\lambda_{\min}(\Lambda_1) \|X\|_F}{\|X\|_F} \\ &= \lambda_{\min}(\Lambda_1) \end{aligned}$$

Furthermore, in light of expression (123), the diagonal elements of Λ_1 , being the non-zero eigenvalues of A , must all be the solutions of the determinantal equation

$$\det \left\{ \lambda I_{Kp} - \frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right\} = 0$$

so that, as noted in the proof of part (a) above, they are also the eigenvalues of the dual matrix $M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2} / N_1$. It follows from the proof of part (a) that there exists a positive constant \underline{c} such that for all N_1 , N_2 , and T sufficiently large.

$$\begin{aligned} \text{sep}(\Lambda_1, \Lambda_2) &= \text{sep}(\Lambda_1, 0) \\ &\geq \lambda_{\min}(\Lambda_1) \\ &= \lambda_{\min} \left(\frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right) \\ &\geq \underline{c} > 0. \quad \square \end{aligned}$$

Lemma D-12: Suppose that A and E are both $n \times n$ symmetric matrices and that

$$G = \begin{bmatrix} G_1 & G_2 \\ n \times r & n \times (n-r) \end{bmatrix}$$

is an orthogonal matrix such that

$$\text{ran}(G_1) = \{y \in \mathbb{R}^n : y = G_1 x \text{ for some } x \in \mathbb{R}^r\}$$

is an invariant subspace for A , i.e., for any $\tilde{q} \in \text{ran}(G_1)$ and let $q^* = A\tilde{q}$; then $q^* \in \text{ran}(G_1)$.

Partition the matrices $G'AG$ and $G'EG$ as follows:

$$G'AG = \begin{pmatrix} \Lambda_1 & 0 \\ r \times r & r \times (n-r) \\ 0 & \Lambda_2 \\ (N-r) \times r & (n-r) \times (n-r) \end{pmatrix} \text{ and } G'EG = \begin{pmatrix} E_{11} & E'_{21} \\ r \times r & r \times (n-r) \\ E_{21} & E_{22} \\ (n-r) \times r & (n-r) \times (n-r) \end{pmatrix}.$$

If

$$\text{sep}(\Lambda_1, \Lambda_2) = \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F} > 0 \quad (124)$$

and if

$$\begin{aligned} \|E\|_2 &\leq \frac{\text{sep}(\Lambda_1, \Lambda_2)}{5} \\ &= \frac{1}{5} \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F}, \end{aligned} \quad (125)$$

then, there exists a matrix $R \in \mathbb{R}^{(n-r) \times r}$ satisfying

$$\begin{aligned} \|R\|_2 &\leq \frac{4}{\text{sep}(\Lambda_1, \Lambda_2)} \|E_{21}\|_2 \\ &= 4 \left(\min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F} \right)^{-1} \|E_{21}\|_2 \end{aligned}$$

such that the columns of

$$\hat{G}_1 = (G_1 + G_2 R) (I_r + R' R)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$.

Remark: Lemma D-12 is a well-known result in linear algebra restated here in our notations. It is given in Golub and van Loan (1996) as Theorem 8.1.10. As noted in Golub and van Loan (1996), this result is also a slight adaptation of Theorem 4.11 in Stewart (1973), which could be consulted

for proof details.

Lemma D-13: Let \mathcal{X} be an invariant subspace of A , and let the columns of X form a basis for \mathcal{X} . Then, there is a unique matrix L such that

$$AX = XL.$$

The matrix L is the representation of A on \mathcal{X} with respect to the basis X . In particular, (v, λ) is an eigenpair of L if and only if (Xv, λ) is an eigenpair of A .

Proof of Lemma D-13: This is Theorem 3.9 of Stewart and Sun (1990). For a proof of this theorem, see Stewart and Sun (1990).

A straightforward application of Lemma D-12 (or Theorem 8.1.10 of Golub and van Loan, 1996) to our setting here leads to the following lemma.

Lemma D-14: Let $\widehat{\Sigma}(\widehat{H}^c)$ be the post-variable-selection sample covariance matrix as defined in expression (111) in Lemma D-10. Decompose $\widehat{\Sigma}(\widehat{H}^c)$ as follows:

$$\widehat{\Sigma}(\widehat{H}^c) = A + E,$$

where

$$A = \frac{\Gamma M_{FF} \Gamma'}{N_1} \tag{126}$$

and where

$$\begin{aligned} E &= \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\ &= \frac{1}{N_1} \left(\Gamma(\widehat{H}^c) M_{FF} \Gamma' (\widehat{H}^c)' - \Gamma M_{FF} \Gamma' \right) + \frac{1}{N_1} \Gamma(\widehat{H}^c) \left[\frac{F' F}{T_0} - M_{FF} \right] \Gamma' (\widehat{H}^c)' \\ &\quad + \frac{U(\widehat{H}^c)' \underline{F} \Gamma' (\widehat{H}^c)'}{N_1 T_0} + \frac{\Gamma(\widehat{H}^c) \underline{F}' U(\widehat{H}^c)}{N_1 T_0} + \frac{U(\widehat{H}^c)' U(\widehat{H}^c)}{N_1 T_0}, \end{aligned} \tag{127}$$

with $T_0 = T - p + 1$ and

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^T E[\underline{F}_t \underline{F}_t'].$$

Suppose that Assumptions 3-1, 3-2, 3-3, 3-4 3-5, 3-6, 3-7, 3-8, 3-10, and 3-11* hold, and define

$$G_{N \times N} = \begin{bmatrix} G_1 & G_2 \\ N \times Kp & N \times (N-Kp) \end{bmatrix}$$

to be an orthogonal matrix whose columns are the eigenvectors of the matrix A . Without loss of generality, suppose that the columns of G_1 are the eigenvectors associated with the non-zero eigenvalues of A , whereas G_2 contains the eigenvectors associated with the zero eigenvalue which has an algebraic multiplicity of $N - Kp$ in this case². Partition the matrices $G'AG$ and $G'EG$ as follows:

$$\begin{aligned} G'AG &= \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\ G'EG &= \begin{pmatrix} E_{11} & E'_{21} \\ E_{21} & E_{22} \end{pmatrix}, \end{aligned}$$

where Λ_1 is a diagonal matrix whose diagonal elements are the Kp largest eigenvalues of the matrix A ³.

Under the assumed conditions, the following statements are true.

- (a) There exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = (G_1 + G_2R) (I_{Kp} + R'R)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $\widehat{\Sigma}(\widehat{H}^c) = A + E$. Moreover,

$$\|R\|_2 = o_p(1) \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty$$

- (b) $\|\widehat{G}_1 - G_1\|_2 = o_p(1)$ as N_1, N_2 , and $T \rightarrow \infty$

- (c) There exists a unique symmetric matrix L such that

$$(A + E)\widehat{G}_1 = \widehat{G}_1L.$$

Moreover, let

$$\widehat{\Lambda} = \text{diag}(\widehat{\lambda}_1, \dots, \widehat{\lambda}_{Kp}) \tag{128}$$

²That 0 is an eigenvalue of the matrix

$$A = \frac{\Gamma M_{FF} \Gamma'}{N_1}$$

with algebraic multiplicity equaling $N - Kp$ has already been shown previously in Lemma D-11.

³We have also previously shown in Lemma D-11 that $G'AG$ can be partitioned in the manner given here.

denote a diagonal matrix whose diagonal elements are the eigenvalues of the matrix L , and let

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} \quad (129)$$

be a $Kp \times Kp$ matrix whose ℓ^{th} column (i.e., \widehat{v}_ℓ) is an eigenvector of L associated with the eigenvalue $\widehat{\lambda}_\ell$ for $\ell = 1, \dots, Kp$. Then, \widehat{V} is an orthogonal matrix and $(\widehat{G}_1 \widehat{v}_\ell, \widehat{\lambda}_\ell)$ is an eigenpair for the matrix $A + E$ for $\ell = 1, \dots, Kp$.

(d) The columns of the matrix

$$\widehat{G}_1 \widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1 \widehat{v}_1 & \widehat{G}_1 \widehat{v}_2 & \cdots & \widehat{G}_1 \widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the post-variable-selection sample covariance matrix

$$A + E = \widehat{\Sigma} \left(\widehat{H}^c \right).$$

Proof of Lemm D-14:

To show part (a), we first verify that the conditions (124) and (125) of Lemma D-12 are satisfied here. To proceed, let $\text{ran}(G_1)$ denote the range space of G_1 , i.e.,

$$\text{ran}(G_1) = \{g \in \mathbb{R}^N : g = G_1 b \text{ for some } b \in \mathbb{R}^{Kp}\}$$

and let Λ_1 be a $Kp \times Kp$ diagonal matrix whose diagonal elements are the non-zero eigenvalues of the matrix $A = \Gamma M_{FF} \Gamma' / N_1$. Now, for any $\widetilde{g} \in \text{ran}(G_1)$, note that

$$\begin{aligned} g^* &= A \widetilde{g} \\ &= \left(\frac{\Gamma M_{FF} \Gamma'}{N_1} \right) G_1 b \\ &= G_1 \Lambda_1 b \\ &= G_1 b^* \text{ where } b^* = \Lambda_1 b. \end{aligned}$$

from which it follows that $g^* \in \text{ran}(G_1)$, so that $\text{ran}(G_1)$ is an invariant subspace of A . Next, by

applying the result of Lemma D-11, we have

$$\begin{aligned}
\text{sep}(\Lambda_1, \Lambda_2) &= \text{sep}(\Lambda_1, 0) \\
&= \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \\
&\geq \lambda_{\min}(\Lambda_1) \\
&= \lambda_{\min} \left(\frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right) \\
&\geq \underline{c} > 0 \text{ for } N_1 \text{ and } N_2 \text{ sufficiently large,}
\end{aligned}$$

so that condition (124) of Lemma D-12 is fulfilled. Next, note that, from the result of Lemma D-10, we have

$$\|E\|_2 = \left\| \widehat{\Sigma}(\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_2 = o_p(1) \text{ as } N_1, N_2, \text{ and } T \rightarrow 0;$$

from which it follows that

$$\|E\|_2 \leq \frac{\text{sep}(\Lambda_1, 0)}{5} \text{ w.p.a.1 as } N_1, N_2, \text{ and } T \rightarrow 0.$$

so that condition (125) of Lemma D-12 is also satisfied here w.p.a.1. Hence, application of Lemma D-12 allows us to conclude that there exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$. In addition,

$$\begin{aligned}
\|R\|_2 &\leq \frac{4}{\text{sep}(\Lambda_1, 0)} \|E\|_2 \\
&= 4 \left[\lambda_{\min} \left(\frac{M_{FF}^{1/2} \Gamma' \Gamma M_{FF}^{1/2}}{N_1} \right) \right]^{-1} \|E\|_2 \\
&\leq \frac{4}{\underline{c}} \|E\|_2 \text{ (for some } \underline{c} > 0 \text{ by Assumption 3-6 and Lemma D-4)} \\
&= o_p(1),
\end{aligned}$$

which shows result (a).

To show that $\left\| \widehat{G}_1 - G_1 \right\|_2 = o_p(1)$, we first show that an explicit representation for G_1 can be given as

$$G_1 = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi = \Gamma (\Gamma' \Gamma)^{-1/2} \Xi$$

where Ξ is an orthogonal matrix whose columns are eigenvectors of the matrix

$$M_{Kp \times Kp}^* = \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} M_{FF} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2}$$

To see that this representation satisfies the various properties we require of G_1 , note first that

$$G_1' G_1 = \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma' \Gamma}{N_1} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi = I_{Kp};$$

hence, G_1 so represented does have orthonormal columns. Moreover, note that

$$\begin{aligned} \frac{\Gamma M_{FF} \Gamma'}{N_1} G_1 &= \frac{\Gamma}{\sqrt{N_1}} M_{FF} \frac{\Gamma' \Gamma}{\sqrt{N_1}} (\Gamma' \Gamma)^{-1/2} \Xi \\ &= \frac{\Gamma}{\sqrt{N_1}} M_{FF} \frac{\Gamma' \Gamma}{N_1} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \\ &= \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} M_{FF} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma' \Gamma}{N_1} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \\ &= \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} M_{FF}^* \Xi \\ &= \Gamma (\Gamma' \Gamma)^{-1/2} \Xi \Lambda_1 \\ &= G_1 \Lambda_1 \end{aligned} \tag{130}$$

where Λ_1 is a $Kp \times Kp$ diagonal matrix whose diagonal elements are the eigenvalues of the matrix M_{FF}^* , which also happen to be the non-zero eigenvalues of the matrix $A = \Gamma M_{FF} \Gamma' / N_1$. Premultiplying the above equation by G_1' , we obtain

$$G_1' \frac{\Gamma M_{FF} \Gamma'}{N_1} G_1 = G_1' G_1 \Lambda_1 = \Lambda_1.$$

Since equation (130) shows that the columns of $\Gamma (\Gamma' \Gamma)^{-1/2} \Xi$ are indeed the eigenvectors of the matrix $A = \Gamma M_{FF} \Gamma' / N_1$, by the argument given previously in the proof of part (a) above, we can then deduce that $\text{ran}(G_1)$, the range space of G_1 with $G_1 = \Gamma (\Gamma' \Gamma)^{-1/2} \Xi$, is an invariant subspace of A . It follows that setting

$$G_1 = \Gamma (\Gamma' \Gamma)^{-1/2} \Xi$$

fulfills all the required properties of G_1 specified in Lemma D-12 above.

Next, write

$$\begin{aligned}
\widehat{G}_1 - G_1 &= (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2} - G_1 \\
&= G_1 \left[(I_{Kp} + R' R)^{-1/2} - I_{Kp} \right] + G_2 R (I_{Kp} + R' R)^{-1/2} \\
&= \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \left[(I_{Kp} + R' R)^{-1/2} - I_{Kp} \right] + G_2 R (I_{Kp} + R' R)^{-1/2}
\end{aligned}$$

Applying the submultiplicative property of matrix norms and the triangle inequality, we obtain

$$\begin{aligned}
&\left\| \widehat{G}_1 - G_1 \right\|_2 \\
&\leq \left\| \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right\|_2 \left\| \Xi \right\|_2 \left\| (I_{Kp} + R' R)^{-1/2} - I_{Kp} \right\|_2 \\
&\quad + \|G_2\|_2 \|R\|_2 \left\| (I_{Kp} + R' R)^{-1/2} \right\|_2 \\
&= \left\| (I_{Kp} + R' R)^{-1/2} - I_{Kp} \right\|_2 + \|R\|_2 \left\| (I_{Kp} + R' R)^{-1/2} \right\|_2
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
\left\| \Xi \right\|_2 &= \sqrt{\lambda_{\max} (\Xi' \Xi)} = \sqrt{\lambda_{\max} (I_{Kp})} = 1, \\
\left\| G_2' \right\|_2 &= \sqrt{\lambda_{\max} (G_2 G_2')} = \sqrt{\lambda_{\max} (G_2' G_2)} = \sqrt{\lambda_{\max} (I_{N-Kp})} = 1, \text{ and} \\
\left\| \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right\|_2 &= \sqrt{\lambda_{\max} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma' \Gamma}{N_1} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right\}} = \sqrt{\lambda_{\max} \{I_{Kp}\}} = 1.
\end{aligned}$$

Now, if (λ, ρ) is an eigen-pair of $R' R$ so that

$$R' R \rho = \lambda \rho \text{ with } \lambda \geq 0 \text{ given that } R' R \text{ is positive semidefinite;}$$

then,

$$\begin{aligned}
(I_{Kp} + R' R) \rho &= (1 + \lambda) \rho, \\
(I_{Kp} + R' R)^{-1/2} \rho &= \frac{1}{\sqrt{1 + \lambda}} \rho, \text{ and} \\
\left[I_{Kp} - (I_{Kp} + R' R)^{-1/2} \right] \rho &= \left(I_{Kp} - \frac{1}{\sqrt{1 + \lambda}} I_{Kp} \right) \rho \\
&= \frac{\sqrt{1 + \lambda} - 1}{\sqrt{1 + \lambda}} \rho
\end{aligned}$$

since

$\frac{1}{\sqrt[3]{1+\lambda}}$ is an eigenvalue of $(I_{Kp} + R'R)^{-1/2}$ associated with the eigenvector ρ

and

$\frac{\sqrt{1+\lambda}-1}{\sqrt[3]{1+\lambda}}$ is an eigenvalue of $I_{Kp} - (I_{Kp} + R'R)^{-1/2}$ associated with the eigenvector ρ

Moreover, let

$$g(\lambda) = \frac{\sqrt{1+\lambda}-1}{\sqrt[3]{1+\lambda}}$$

and note that

$$\begin{aligned} g'(\lambda) &= \frac{1}{2} \frac{1}{1+\lambda} - \frac{1}{2} \frac{\sqrt{1+\lambda}-1}{(1+\lambda)^{3/2}} \\ &= \frac{1}{2} \frac{\sqrt{1+\lambda} - \sqrt{1+\lambda} + 1}{(1+\lambda)^{3/2}} \\ &= \frac{1}{2(1+\lambda)^{3/2}} > 0 \end{aligned}$$

so that, in particular, $g(\lambda)$ is an increasing function of λ for $\lambda \geq 0$. It follows that

$$\begin{aligned} & \left\| \widehat{G}_1 - G_1 \right\|_2 \\ & \leq \left\| (I_{Kp} + R'R)^{-1/2} - I_{Kp} \right\|_2 + \|R\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \\ & = \left\| I_{Kp} - (I_{Kp} + R'R)^{-1/2} \right\|_2 + \|R\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \\ & \leq \sqrt{\lambda_{\max} \left(\left[I_{Kp} - (I_{Kp} + R'R)^{-1/2} \right]' \left[I_{Kp} - (I_{Kp} + R'R)^{-1/2} \right] \right)} \\ & \quad + \|R\|_2 \sqrt{\lambda_{\max} \left((I_{Kp} + R'R)^{-1/2} (I_{Kp} + R'R)^{-1/2} \right)} \\ & = \lambda_{\max} \left[I_{Kp} - (I_{Kp} + R'R)^{-1/2} \right] + \|R\|_2 \lambda_{\max} \left[(I_{Kp} + R'R)^{-1/2} \right] \\ & \quad \left(\text{since } I_{Kp} - (I_{Kp} + R'R)^{-1/2} \text{ and } (I_{Kp} + R'R)^{-1/2} \text{ are both symmetric and positive semidefinite} \right) \\ & \leq \frac{\sqrt{1 + \lambda_{\max}(R'R)} - 1}{\sqrt[3]{1 + \lambda_{\min}(R'R)}} + \frac{\|R\|_2}{\sqrt[3]{1 + \lambda_{\min}(R'R)}} \\ & \leq \sqrt{1 + \|R\|_2^2} - 1 + \|R\|_2 \quad (\text{since } \lambda_{\min}(R'R) \geq 0 \text{ given that } R'R \text{ is positive semi-definite}) \\ & = o_p(1) \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty \text{ (since } \|R\|_2 = o_p(1)). \end{aligned}$$

This shows result (b).

To show part (c), note that, by the result given in part (a) above, the columns of $\widehat{G}_1 = (G_1 + G_2 R) (I_r + R' R)^{-1/2}$ form an orthonormal basis for a subspace that is invariant for $A + E$. It then follows immediately from Lemma D-13 that there exists a unique matrix L such that

$$\begin{aligned} (A + E) \widehat{G}_1 &= (A + E) (G_1 + G_2 R) (I_r + R' R)^{-1/2} \\ &= (G_1 + G_2 R) (I_r + R' R)^{-1/2} L \\ &= \widehat{G}_1 L. \end{aligned}$$

Note further that

$$\begin{aligned} \widehat{G}_1' \widehat{G}_1 &= (I_{Kp} + R' R)^{-1/2} (G_1' + R' G_2') (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2} \\ &= (I_{Kp} + R' R)^{-1/2} (G_1' G_1 + R' G_2' G_1 + G_1' G_2 R + R' G_2' G_2 R) (I_{Kp} + R' R)^{-1/2} \\ &= (I_{Kp} + R' R)^{-1/2} (I_{Kp} + R' R) (I_{Kp} + R' R)^{-1/2} \\ &\quad \left(\text{since by assumption } G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \text{ is an orthogonal matrix} \right) \\ &= I_{Kp} \end{aligned}$$

which, in turn, implies that

$$\begin{aligned} \widehat{G}_1' (A + E) \widehat{G}_1 &= \widehat{G}_1' \left(\frac{\Gamma M_{FF} \Gamma'}{N_1} + E \right) \widehat{G}_1 = \widehat{G}_1' \widehat{G}_1 L \\ &= L \end{aligned}$$

so that L must be symmetric since, in our situation here,

$$A + E = \frac{\Gamma M_{FF} \Gamma'}{N_1} + \widehat{\Sigma} (\widehat{H}^c) - \frac{\Gamma M_{FF} \Gamma'}{N_1} = \widehat{\Sigma} (\widehat{H}^c) = \frac{Z (\widehat{H}^c)' Z (\widehat{H}^c)}{N_1 T_0}$$

is a symmetric matrix. Now, let $\widehat{\Lambda} = \text{diag} (\widehat{\lambda}_1, \dots, \widehat{\lambda}_{Kp})$ and

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix}$$

be as defined in expressions (128) and (129). The fact that L is symmetric implies that \widehat{V} is an orthogonal matrix. In addition, further application of Lemma C-13 shows that $(\widehat{G}_1 \widehat{v}_g, \widehat{\lambda}_g)$ is an eigenpair for the matrix $A + E$ for $g = 1, \dots, Kp$.

Finally, to show part (d), let $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, and note that, by assumption,

$$G'AG = \begin{pmatrix} G'_1AG_1 & G'_1AG_2 \\ G'_2AG_1 & G'_2AG_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \Lambda$$

where $\Lambda_1 = \text{diag}(\lambda_{1,1}, \dots, \lambda_{1,Kp})$ contains the Kp largest eigenvalues of A . Without loss of generality, we can further assume that $\lambda_{1,1}, \dots, \lambda_{1,Kp}$ are ordered, so that $\lambda_{1,j} = \lambda_{(j)}(A)$, i.e., $\lambda_{1,j}$ is the j^{th} largest eigenvalue of A^4 . Given that, $G'G = GG' = I_N$, we have

$$\begin{pmatrix} AG_1 & AG_2 \end{pmatrix} = AG = G\Lambda = \begin{pmatrix} G_1\Lambda_1 & 0 \end{pmatrix}$$

from which it follows that

$$AG_1G'_1\widehat{G}_1\widehat{v}_\ell = G_1\Lambda_1G'_1\widehat{G}_1\widehat{v}_\ell, \text{ for } \ell \in \{1, \dots, Kp\}. \quad (131)$$

Now, the result of part (c) above shows $(\widehat{G}_1\widehat{v}_\ell, \widehat{\lambda}_\ell)$ to be an eigenpair of the matrix $A + E$ for $\ell \in \{1, \dots, Kp\}$, so that

$$(A + E)\widehat{G}_1\widehat{v}_\ell = \widehat{\lambda}_\ell\widehat{G}_1\widehat{v}_\ell \text{ for } \ell \in \{1, \dots, Kp\} \quad (132)$$

where $\widehat{G}_1 = (G_1 + G_2R)(I_{Kp} + R'R)^{-1/2}$ as given in the result for part (a). Multiplying both sides of expression (132) by $\widehat{v}'_\ell\widehat{G}'_1G_1G'_1$, we get

$$\begin{aligned} \widehat{\lambda}_\ell\widehat{v}'_\ell\widehat{G}'_1G_1G'_1\widehat{G}_1\widehat{v}_\ell &= \widehat{v}'_\ell\widehat{G}'_1G_1G'_1(A + E)\widehat{G}_1\widehat{v}_\ell \\ &= \widehat{v}'_\ell\widehat{G}'_1G_1G'_1A\widehat{G}_1\widehat{v}_\ell + \widehat{v}'_\ell\widehat{G}'_1G_1G'_1E\widehat{G}_1\widehat{v}_\ell \end{aligned} \quad (133)$$

Since $A = \Gamma M_{FF}\Gamma'/N_1$ is symmetric, it further follows by expression (131) that

$$\widehat{v}'_\ell\widehat{G}'_1G_1G'_1A = \widehat{v}'_\ell\widehat{G}'_1G_1G'_1A' = \widehat{v}'_\ell\widehat{G}'_1G_1\Lambda_1G'_1 \quad (134)$$

⁴If this is not the case; then, we can always define a permutation matrix \mathcal{P} such that

$$\Lambda^* = \mathcal{P}'\Lambda\mathcal{P}$$

results in a diagonal matrix whose diagonal elements are repermuted in such a way, so that the required ordering of the eigenvalues is satisfied. Moreover, since \mathcal{P} is an orthogonal matrix, it further follows that

$$A = G\mathcal{P}\mathcal{P}'\Lambda\mathcal{P}\mathcal{P}'G' = G\mathcal{P}\Lambda^*\mathcal{P}'G'.$$

Now, define $\widetilde{G} = G\mathcal{P}$, and note that \widetilde{G} is an orthogonal matrix whose columns are just the columns of G repermuted. Hence, we can simply proceed with our analysis using \widetilde{G} in lieu of G , and the associated eigenvalues will be in the order which we have assumed.

Moreover, note that

$$\begin{aligned}
0 &\leq \left(\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \right)^2 \\
&\leq \left(\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell \right) \left(\widehat{v}'_\ell \widehat{G}'_1 E' E \widehat{G}_1 \widehat{v}_\ell \right) \quad (\text{by CS inequality}) \\
&= \left(\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell \right) \left(\widehat{v}'_\ell \widehat{G}'_1 E' E \widehat{G}_1 \widehat{v}_\ell \right) \quad (\text{since } G'_1 G_1 = I_{Kp}) \\
&= \left[\widehat{v}'_\ell (I_{Kp} + R' R)^{-1/2} (G'_1 + R' G'_2) G_1 G'_1 (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2} \widehat{v}_\ell \right] \left(\widehat{v}'_\ell \widehat{G}'_1 E' E \widehat{G}_1 \widehat{v}_\ell \right) \\
&= \left[\widehat{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell \right] \left(\widehat{v}'_\ell \widehat{G}'_1 E' E \widehat{G}_1 \widehat{v}_\ell \right) \\
&\leq \left[\widehat{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell \right] \lambda_{\max}(E' E)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
-\sqrt{\widehat{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \|E\|_2 &= -\sqrt{\widehat{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \sqrt{\lambda_{\max}(E' E)} \\
&\leq -\sqrt{\left(\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \right)^2} \\
&\leq -\left| \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \right| \\
&\leq \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell
\end{aligned} \tag{135}$$

where the last inequality follows from the fact that

$$\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell > -\left| \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \right| \quad \text{if } \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell > 0$$

whereas

$$\widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell = -\left| \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \right| \quad \text{if } \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \leq 0$$

Combining expressions (133), (134), and (135), we see that

$$\begin{aligned}
\widehat{\lambda}_\ell \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell &= \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 A \widehat{G}_1 \widehat{v}_\ell + \widehat{v}'_\ell \widehat{G}'_1 G_1 G'_1 E \widehat{G}_1 \widehat{v}_\ell \\
&\geq \widehat{v}'_\ell \widehat{G}'_1 G_1 \Lambda_1 G'_1 \widehat{G}_1 \widehat{v}_\ell - \sqrt{\widehat{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \|E\|_2
\end{aligned} \tag{136}$$

for $\ell \in \{1, \dots, Kp\}$. In addition, note that

$$\begin{aligned}
\tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell &= \tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2} \widehat{v}_\ell \\
&= \tilde{v}'_\ell \widehat{G}'_1 G_1 (I_{Kp} + R' R)^{-1/2} \widehat{v}_\ell \\
&= \tilde{v}'_\ell (I_{Kp} + R' R)^{-1/2} (G'_1 + R' G'_2) G_1 (I_{Kp} + R' R)^{-1/2} \widehat{v}_\ell \\
&= \tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell \\
&> 0
\end{aligned}$$

Hence, dividing both sides of expression (136) by $\tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell$, we obtain

$$\begin{aligned}
\widehat{\lambda}_\ell &\geq \frac{\tilde{v}'_\ell \widehat{G}'_1 G_1 \Lambda_1 G'_1 \widehat{G}_1 \widehat{v}_\ell}{\tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell} - \frac{\sqrt{\tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \|E\|_2}{\tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell} \\
&= \tilde{v}'_\ell \Lambda_1 \tilde{v}_\ell - \frac{\sqrt{\tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \|E\|_2}{\tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell} \\
&= \tilde{v}'_\ell \Lambda_1 \tilde{v}_\ell - \frac{\|E\|_2}{\sqrt{\tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp} \tilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{\|E\|_2}{\sqrt{\tilde{v}'_\ell (I_{Kp} + R' R)^{-1} \widehat{v}_\ell}}
\end{aligned}$$

where

$$\tilde{v}_\ell = \frac{G'_1 \widehat{G}_1 \widehat{v}_\ell}{\tilde{v}'_\ell \widehat{G}'_1 G_1 G'_1 \widehat{G}_1 \widehat{v}_\ell} \text{ so that } \|\tilde{v}_\ell\|_2^2 = \sum_{\ell=1}^{Kp} \tilde{v}_{\ell,j}^2 = 1.$$

Note also that

$$\begin{aligned}
\widehat{v}'_\ell (I_{Kp} + R'R)^{-1} \widehat{v}_\ell &\geq \lambda_{\min} \left\{ (I_{Kp} + R'R)^{-1} \right\} \widehat{v}'_\ell \widehat{v}_\ell \\
&= \lambda_{\min} \left\{ (I_{Kp} + R'R)^{-1} \right\} \quad \left(\text{since } \|\widehat{v}_\ell\|_2^2 = 1 \right) \\
&= \frac{1}{\lambda_{\max} (I_{Kp} + R'R)} \\
&\geq \frac{1}{1 + \lambda_{\max} (R'R)} \\
&= \frac{1}{1 + \|R\|_2^2} \\
&\geq \left[1 + \frac{16 \|E_{21}\|_2^2}{(\text{sep}(\Lambda_1, \Lambda_2))^2} \right]^{-1} \quad (\text{by Lemma D-12}) \\
&\geq \left[1 + \frac{16 \|E\|_2^2}{(\text{sep}(\Lambda_1, \Lambda_2))^2} \right]^{-1} \quad (\text{by Lemma D-3}) \\
&\geq \left[1 + \frac{16 (\text{sep}(\Lambda_1, \Lambda_2))^2 / 25}{(\text{sep}(\Lambda_1, \Lambda_2))^2} \right]^{-1} \quad (\text{by Lemma D-12}) \\
&= \frac{25}{41}
\end{aligned}$$

Making use of this lower bound, we obtain

$$\begin{aligned}
\widehat{\lambda}_\ell &\geq \sum_{j=1}^{Kp} \widehat{v}_{\ell,j}^2 \lambda_{1,j} - \frac{\|E\|_2}{\sqrt{\widehat{v}'_\ell (I_{Kp} + R'R)^{-1} \widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp} \widehat{v}_{\ell,j}^2 \lambda_{1,j} - \frac{25}{41} \|E\|_2.
\end{aligned}$$

Next, recall the notations we have introduced previously on the ordering of the eigenvalues of the matrices $A + E$ and A , i.e.,

$$\begin{aligned}
\lambda_{(1)}(A + E) &\geq \cdots \geq \lambda_{(Kp)}(A + E) \geq \lambda_{(Kp+1)}(A + E) \geq \cdots \geq \lambda_{(N)}(A + E), \\
\lambda_{(1)}(A) &\geq \cdots \geq \lambda_{(Kp)}(A) \geq \lambda_{(Kp+1)}(A) \geq \cdots \geq \lambda_{(N)}(A)
\end{aligned}$$

Since $A = \Gamma M_{FF} \Gamma' / N_1$ and part (a) of Lemma D-11 shows that $\text{Rank}(A) = Kp$ for all N_1, N_2 , and T sufficiently large; it follows that

$$\lambda_{(Kp+1)}(A) = \cdots = \lambda_{(N)}(A) = 0. \tag{137}$$

In addition, by Corollary 8.1.6 of Golub and van Loan (1996), we have the inequality.

$$\lambda_{(Kp+1)}(A + E) \leq \lambda_{(Kp+1)}(A) + \|E\|_2. \quad (138)$$

Making use of expressions (137) and (138); we see that, for any $\ell \in \{1, \dots, Kp\}$,

$$\begin{aligned} \widehat{\lambda}_\ell - \lambda_{(Kp+1)}(A + E) &\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{25}{41} \|E\|_2 - \{\lambda_{(Kp+1)}(A) + \|E\|_2\} \\ &= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{66}{41} \|E\|_2 \quad (\text{since } \lambda_{(Kp+1)}(A) = 0 \text{ here}) \\ &= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41} \|E\|_2 \\ &\quad (\text{since } \lambda_{1,j} = \lambda_{(j)}(A) \text{ as discussed previously}) \\ &\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41} \frac{\text{sep}(\Lambda_1, \Lambda_2)}{5} \quad (\text{by Lemma D-12}) \\ &= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{205} \text{sep}(\Lambda_1, 0) \quad (\text{since } \Lambda_2 = 0 \text{ here}) \\ &\geq \lambda_{\min}(\Lambda_1) - \frac{66}{205} \text{sep}(\Lambda_1, 0) \quad (\text{since } \Lambda_1 = \text{diag}(\lambda_{(1)}(A), \dots, \lambda_{(Kp)}(A))) \\ &= \frac{139}{205} \text{sep}(\Lambda_1, 0) \\ &\quad (\text{since } \text{sep}(\Lambda_1, 0) = \lambda_{\min}(\Lambda_1) \text{ by Theorem 3.1 of Stewart and Sun (1990)}) \\ &\geq \frac{139}{205} \underline{c} > 0 \quad (\text{by part (c) of Lemma D-11}). \end{aligned}$$

This shows that the set $\{\widehat{\lambda}_1, \dots, \widehat{\lambda}_{Kp}\}$ contains the Kp largest eigenvalues of the matrix $A + E$. It further follows from the result given in part (c) that the columns of the matrix

$$\widehat{G}_1 \widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1 \widehat{v}_1 & \widehat{G}_1 \widehat{v}_2 & \cdots & \widehat{G}_1 \widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the matrix $A + E$. \square

Lemma D-15: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{\widehat{N}_1 - N_1}{N_1} \xrightarrow{p} 0$$

(b)

$$\left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0$$

(c) Let

$$\widehat{G}_1 = (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2}$$

where G_1 , G_2 , and R are as defined in Lemma C-14 above. Also, let \widehat{V} be the $Kp \times Kp$ orthogonal matrix given in expression (129) of Lemma D-14. Then, there exists some positive constant \overline{C} such that

$$\left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \leq \overline{C} < \infty$$

for N_1, N_2 , and T sufficiently large. In addition,

$$\left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0$$

where

$$Q = \left(\frac{\Gamma' \Gamma}{N_1} \right)^{\frac{1}{2}} \Xi \widehat{V},$$

with Ξ being the $Kp \times Kp$ orthogonal matrix whose columns are the eigenvectors of the matrix

$$M_{FF}^* = \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} M_{FF} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} = \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2} \frac{1}{T - p + 1} \sum_{t=p}^T E [\underline{E}_t \underline{E}_t'] \left(\frac{\Gamma' \Gamma}{N_1} \right)^{1/2}.$$

(d) For all fixed index t

$$\left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = o_p(1).$$

(e) For all fixed index t

$$\left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 = O_p(1).$$

(f) For all fixed index t ,

$$\left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

(g) Let

$$\widehat{G}_1 = (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2}$$

where G_1 , G_2 , and R are as defined in Lemma D-14 above. Also, let \widehat{V} be the $Kp \times Kp$ orthogonal matrix given in expression (129) of Lemma D-14. Then, for all fixed index t ,

$$\left\| \frac{\widehat{V}' \widehat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty.$$

(h)

$$\left\| \frac{\Gamma(\widehat{H}^c)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 = \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 = o_p(1) \text{ as } N_1, N_2, T \rightarrow \infty.$$

where Q is as defined in part (c) above.

(i)

$$\|\underline{E}_t\|_2 = O_p(1) \text{ for all } t.$$

(j)

$$\|\widehat{\underline{E}}_T - Q' \underline{E}_T\|_2 = o_p(1) \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty$$

where $\widehat{\underline{E}}_T$ denotes the principal component estimator of the factor vector \underline{E}_T obtained after the variables have been pre-screened based on the decision rule

$$i \in \begin{cases} \widehat{H}^c & \text{if } \mathbb{S}_{i,T}^+ > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \\ \widehat{H} & \text{if } \mathbb{S}_{i,T}^+ \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \end{cases},$$

as described in section 3. Here, $\mathbb{S}_{i,T}^+$ may be either the statistic $\sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$.

Proof of Lemma D-15:

To show part (a), note first that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{ \left| \frac{\widehat{N}_1 - N_1}{N_1} \right| \geq \epsilon \right\} &= \left\{ \left| \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} - 1 \right| \geq \epsilon \right\} \\
&= \left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} - 1 \right| \geq \epsilon \right\} \\
&\subseteq \left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} - 1 \right| + \left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right| \geq \epsilon \right\} \\
&\subseteq \left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} - 1 \right| \geq \frac{\epsilon}{2} \right\} \cup \left\{ \left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right| \geq \frac{\epsilon}{2} \right\} \\
&= \left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right| \geq \frac{\epsilon}{2} \right\} \cup \left\{ \left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right| \geq \frac{\epsilon}{2} \right\}
\end{aligned}$$

By Markov's inequality, we have

$$\begin{aligned}
&\Pr \left(\left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right| \geq \frac{\epsilon}{2} \right) \\
&= \Pr \left(\left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right|^2 \geq \frac{\epsilon^2}{4} \right) \\
&\leq \frac{4}{\epsilon^2} E \left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right|^2 \right\} \\
&= \frac{4}{\epsilon^2} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} E \left[(\mathbb{I}\{i \in \widehat{H}^c\} - 1) (\mathbb{I}\{k \in \widehat{H}^c\} - 1) \right] \\
&= \frac{4}{\epsilon^2} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} (E [\mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{k \in \widehat{H}^c\}] - E [\mathbb{I}\{k \in \widehat{H}^c\}] - E [\mathbb{I}\{i \in \widehat{H}^c\}] + 1) \\
&= \frac{4}{\epsilon^2} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left\{ \Pr(\{i \in \widehat{H}^c\} \cap \{k \in \widehat{H}^c\}) - \Pr(k \in \widehat{H}^c) \right\} \\
&\quad + \frac{4}{\epsilon^2} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left\{ 1 - \Pr(i \in \widehat{H}^c) \right\} \\
&\leq \frac{4}{\epsilon^2} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left\{ \Pr(k \in \widehat{H}^c) - \Pr(k \in \widehat{H}^c) \right\} + \frac{4}{\epsilon^2} \frac{1}{N_1} \sum_{i \in H^c} \left\{ 1 - \Pr(i \in \widehat{H}^c) \right\} \\
&\leq \frac{4}{\epsilon^2} \frac{1}{N_1} \sum_{i \in H^c} \left\{ 1 - \min_{i \in H^c} \Pr(i \in \widehat{H}^c) \right\} \rightarrow 0 \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty.
\end{aligned}$$

where the last line above follows from the fact that, for $i \in H^c$ and for either the case where

$\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we can apply the results of Theorem 3 to obtain

$$\begin{aligned} \min_{i \in H^c} \Pr \left(i \in \widehat{H}^c \right) &\geq \Pr \left(\bigcap_{i \in H^c} \left\{ \mathbb{S}_{i,T}^+ > \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right\} \right) \\ &= P \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ > \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\ &\rightarrow 1 \end{aligned}$$

Also, making use of Markov's inequality, we obtain, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$,

$$\begin{aligned} &\Pr \left(\left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right| \geq \frac{\epsilon}{2} \right) \\ &= \Pr \left(\frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \geq \frac{\epsilon}{2} \right) \\ &\leq \frac{2}{\epsilon} E \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right] \\ &= \frac{2}{\epsilon} \frac{1}{N_1} \sum_{i \in H} \Pr \left(i \in \widehat{H}^c \right) \\ &= \frac{2}{\epsilon} \frac{1}{N_1} \sum_{i \in H} \Pr \left(\mathbb{S}_{i,T}^+ > \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\ &\leq \frac{2}{\epsilon} \frac{dN_2\varphi}{NN_1} [1 + o(1)] \\ &\quad \text{(following an argument similar to that given in the proof of Theorem 2)} \\ &\rightarrow 0 \left(\text{since } \frac{\varphi}{N_1} \rightarrow 0 \text{ and } \frac{N_2}{N} = O(1) \right). \end{aligned}$$

Combining these results, we have that

$$\begin{aligned}
& \Pr \left(\left| \frac{\widehat{N}_1 - N_1}{N_1} \right| \geq \epsilon \right) \\
& \leq \Pr \left(\left\{ \left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right| \geq \frac{\epsilon}{2} \right\} \cup \left\{ \left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right| \geq \frac{\epsilon}{2} \right\} \right) \\
& \leq \Pr \left(\left| \frac{1}{N_1} \sum_{i \in H^c} (\mathbb{I}\{i \in \widehat{H}^c\} - 1) \right| \geq \frac{\epsilon}{2} \right) + \Pr \left(\left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right| \geq \frac{\epsilon}{2} \right) \\
& \quad \text{(by the union bound)} \\
& \rightarrow 0
\end{aligned}$$

For part (b), note that

$$\begin{aligned}
\left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_F^2 &= \frac{1}{N_1} \text{tr} \left\{ \left(\Gamma(\widehat{H}^c) - \Gamma \right)' \left(\Gamma(\widehat{H}^c) - \Gamma \right) \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^N \text{tr} \left\{ \left(\mathbb{I}\{i \in \widehat{H}^c\} \gamma_i - \gamma_i \right) \left(\mathbb{I}\{i \in \widehat{H}^c\} \gamma_i - \gamma_i \right)' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^N \left(\mathbb{I}\{i \in \widehat{H}^c\} \gamma_i - \gamma_i \right)' \left(\mathbb{I}\{i \in \widehat{H}^c\} \gamma_i - \gamma_i \right) \\
&= \frac{1}{N_1} \sum_{i=1}^N \gamma_i' \gamma_i \left[1 - \mathbb{I}\{i \in \widehat{H}^c\} \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} \gamma_i' \gamma_i \left[1 - \mathbb{I}\{i \in \widehat{H}^c\} \right] \quad (\text{since } \gamma_i = 0 \text{ for } i \in H)
\end{aligned}$$

Applying Markov's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}
\Pr \left(\left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_F^2 \geq \epsilon \right) &\leq \frac{1}{\epsilon} E \left\{ \frac{1}{N_1} \sum_{i \in H^c} \gamma'_i \gamma_i \left[1 - \mathbb{I} \{ i \in \widehat{H}^c \} \right] \right\} \\
&= \frac{1}{\epsilon} \frac{1}{N_1} \sum_{i \in H^c} \gamma'_i \gamma_i \left[1 - \Pr(i \in \widehat{H}^c) \right] \\
&\leq \frac{1}{\epsilon} \left[1 - \min_{i \in H^c} \Pr(i \in \widehat{H}^c) \right] \frac{1}{N_1} \sum_{i \in H^c} \gamma'_i \gamma_i \\
&\leq \frac{1}{\epsilon} \left[1 - \min_{i \in H^c} \Pr(i \in \widehat{H}^c) \right] \left(\sup_{i \in H^c} \|\gamma_i\|_2 \right)^2 \\
&\leq \frac{1}{\epsilon} \left[1 - \min_{i \in H^c} \Pr(i \in \widehat{H}^c) \right] \overline{C}^2 \quad (\text{by Assumption 3-5}) \\
&\rightarrow 0 \quad \left(\text{since } \min_{i \in H^c} \Pr(i \in \widehat{H}^c) \rightarrow 1 \text{ for } i \in H^c \text{ by Theorem 3} \right)
\end{aligned}$$

from which we further deduce that

$$\left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \leq \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_F \xrightarrow{p} 0.$$

Turning our attention to part (c), note that since, by definition,

$$\widehat{G}_1 = (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2}$$

where $G'_1 G_1 = I_{Kp}$, $G'_2 G_2 = I_{N-Kp}$, and $G'_1 G_2 = 0$; it follows that

$$\begin{aligned}
\widehat{G}'_1 \widehat{G}_1 &= (I_{Kp} + R' R)^{-1/2} (G'_1 + R' G'_2) (G_1 + G_2 R) (I_{Kp} + R' R)^{-1/2} \\
&= (I_{Kp} + R' R)^{-1/2} (I_{Kp} + R' R) (I_{Kp} + R' R)^{-1/2} \\
&= I_{Kp}
\end{aligned}$$

Hence, by Assumption 3-6,

$$\begin{aligned}
\left\| \frac{\hat{V}' \hat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 &\leq \left\| \hat{V}' \hat{G}_1' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&\leq \sqrt{\lambda_{\max}(\hat{G}_1 \hat{V} \hat{V}' \hat{G}_1')} \sqrt{\lambda_{\max}\left(\frac{\Gamma' \Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}(\hat{V}' \hat{G}_1' \hat{G}_1 \hat{V})} \sqrt{\lambda_{\max}\left(\frac{\Gamma' \Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}(I_{Kp})} \sqrt{\lambda_{\max}\left(\frac{\Gamma' \Gamma}{N_1}\right)} \quad (\text{since } \hat{V} \text{ is an orthogonal matrix}) \\
&= \sqrt{\lambda_{\max}\left(\frac{\Gamma' \Gamma}{N_1}\right)} \leq \bar{C} < \infty \text{ for } N_1, N_2 \text{ sufficiently large}
\end{aligned}$$

Now, to show the second result in part (c), note that, since

$$Q = \left(\frac{\Gamma' \Gamma}{N_1}\right)^{\frac{1}{2}} \Xi \hat{V} \text{ and } G_1 = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1}\right)^{-1/2} \Xi = \Gamma (\Gamma' \Gamma)^{-1/2} \Xi ,$$

we can write

$$\begin{aligned}
\frac{\hat{V}' \hat{G}_1' \Gamma}{\sqrt{N_1}} - Q' &= \frac{\hat{V}' \hat{G}_1' \Gamma}{\sqrt{N_1}} - \hat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1}\right)^{\frac{1}{2}} \\
&= \frac{\hat{V}' \hat{G}_1' \Gamma}{\sqrt{N_1}} - \hat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1}\right)^{-1/2} \frac{\Gamma' \Gamma}{N_1} \\
&= \frac{\hat{V}' \hat{G}_1' \Gamma}{\sqrt{N_1}} - \frac{\hat{V}' G_1' \Gamma}{\sqrt{N_1}} \\
&= \hat{V}' (\hat{G}_1 - G_1)' \frac{\Gamma}{\sqrt{N_1}}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 &\leq \left\| \widehat{V}' \right\|_2 \left\| (\widehat{G}_1 - G_1)' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&\leq \sqrt{\lambda_{\max}(\widehat{V} \widehat{V}')} \sqrt{\lambda_{\max} \left\{ (\widehat{G}_1 - G_1) (\widehat{G}_1 - G_1)' \right\}} \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \\
&= \sqrt{\lambda_{\max}(I_{Kp})} \sqrt{\lambda_{\max} \left\{ (\widehat{G}_1 - G_1)' (\widehat{G}_1 - G_1) \right\}} \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \\
&\quad \left(\text{since } \widehat{V} \text{ is an orthogonal matrix and since } \lambda_{\max}(AA') = \lambda_{\max}(A'A) \right) \\
&\leq \sqrt{C} \left\| \widehat{G}_1 - G_1 \right\|_2 \quad (\text{by Assumption 3-6}) \\
&= o_p(1) \quad \text{as } N_1, N_2, \text{ and } T \rightarrow \infty \quad (\text{by part (b) of Lemma D-14}).
\end{aligned}$$

Next, to show part (d), we first write

$$\begin{aligned}
\left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &= \sum_{k=1}^{Kp} \left(\sum_{i=1}^N \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} + \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&\leq 2 \sum_{k=1}^K \left(\sum_{i \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 + 2 \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \mathbb{I} \left\{ j \in \widehat{H}^c \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \mathbb{I} \left\{ j \in \widehat{H}^c \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \tag{139}
\end{aligned}$$

where $g_{1,ik}$ denotes the $(i, k)^{th}$ element of G_1 . Now, consider the first term on the right-hand side

of expression (139). Write

$$\begin{aligned}
& \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 + 1 \right) \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 + 1 \right) g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \mathcal{E}_{1,1,t} + \mathcal{E}_{1,2,t} + \mathcal{E}_{1,3,t} + \mathcal{E}_{1,4,t}
\end{aligned}$$

Focusing first on the term $\mathcal{E}_{1,1,t}$, we have

$$\begin{aligned}
& \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right)^2 \\
&\leq \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left(\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2 \right) \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 2\mathbb{I}\{i \in \widehat{H}^c\} + 1 \right) \right] \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right) \right] \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right)
\end{aligned}$$

Now, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$,

we have

$$\begin{aligned}
0 &\leq E \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \left\{ i \in \widehat{H^c} \right\} \right) \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} \left[1 - \Pr \left(i \in \widehat{H^c} \right) \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} \left[1 - P \left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \right] \\
&\leq 1 - P \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&\quad \text{(given that } N_1 = \# \{H^c\}, \text{ where } \# \{H^c\} \text{ denotes the cardinality of the set } H^c) \\
&\rightarrow 0 \left(\text{since, by Theorem 3, } P \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \rightarrow 1 \right).
\end{aligned}$$

Moreover, by part (b) of Assumption 3-3, we have

$$E \left[\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right] = \sum_{i \in H^c} g_{1,ik}^2 E[u_{i,t}^2] \leq C \sum_{i=1}^N g_{1,ik}^2 \leq C$$

It follows by Markov's inequality that

$$\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \left\{ i \in \widehat{H^c} \right\} \right) = o_p(1) \text{ and } \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
\mathcal{E}_{1,1,t} &= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \left(\mathbb{I} \left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \left\{ i \in \widehat{H^c} \right\} \right) \right] \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= o_p(1)
\end{aligned}$$

Next, consider the term $\mathcal{E}_{1,2,t}$. To proceed, write

$$\begin{aligned}
& |\mathcal{E}_{1,2,t}| \\
&= \left| \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
&= \left| \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{j \in H^c} g_{1,jk} u_{j,t} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
&\leq \frac{2}{N_1} \sum_{k=1}^{Kp} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \left| \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
&\leq \frac{2}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \sqrt{\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
&\leq \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
&\quad \left(\text{by the inequality } |XY| \leq \frac{1}{2}X^2 + \frac{1}{2}Y^2 \right) \\
&= \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right)} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right)} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2
\end{aligned}$$

Observe that

$$\begin{aligned}
& E \left[\frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \right] \\
&= \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sum_{j \in H^c} \sum_{\ell \in H^c} g_{1,jk} g_{1,\ell k} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{\sqrt{N_1}} \sum_{j \in H^c} \sum_{\ell \in H^c} \sum_{k=1}^{Kp} g_{1,jk} g_{1,\ell k} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{\sqrt{N_1}} \sum_{j \in H^c} \sum_{\ell \in H^c} \frac{e'_{j,N} \Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \sum_{k=1}^{Kp} \Xi e_{k,Kp} e'_{k,Kp} \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma' e_{\ell,N}}{\sqrt{N_1}} E[u_{j,t} u_{\ell,t}] \\
&\quad \left(\text{since } G_1 = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \right) \\
&= \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} e'_{j,N} \Gamma_* \Xi \Xi' \Gamma_*' e_{\ell,N} E[u_{j,t} u_{\ell,t}] \quad \left(\text{where } \Gamma_* = \Gamma \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right) \\
&= \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} e'_{j,N} \Gamma_* \Gamma_*' e_{\ell,N} E[u_{j,t} u_{\ell,t}] \\
&\quad (\text{since } \Xi \text{ is an orthogonal matrix}) \\
&= \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_{*,j} \gamma_{*,\ell} E[u_{j,t} u_{\ell,t}]
\end{aligned}$$

where we take

$$\gamma_{*,j} = \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_j,$$

Applying the triangle and Cauchy-Schwarz inequalities, we further obtain

$$\begin{aligned}
& E \left[\frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \right] \\
&= \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_{*,j} \gamma_{*,\ell} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_\ell E[u_{j,t} u_{\ell,t}] \\
&\leq \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \left| \gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_\ell \right| |E[u_{j,t} u_{\ell,t}]| \\
&\leq \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \sqrt{\gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_j} \sqrt{\gamma'_\ell \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_\ell} |E[u_{j,t} u_{\ell,t}]| \\
&\leq \frac{\bar{c}}{\underline{C}} \frac{1}{\sqrt{N_1}} \frac{1}{N_1} \sum_{j \in H^c} \sum_{\ell \in H^c} |E[u_{j,t} u_{\ell,t}]| \\
&\quad \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \bar{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \sup_{i \in H^c} \|\gamma_i\|_2 \leq \bar{c} < \infty \text{ and } \lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \geq \underline{C} > 0) \\
&\leq \frac{2\bar{c}}{\underline{C}} \frac{\bar{C}}{\sqrt{N_1}} \rightarrow 0 \text{ as } N_1 \rightarrow \infty. \text{ (since, under Assumption 3-3(d) that there exists a} \\
&\quad \text{positive constant } \bar{C} \text{ such that } \sup_t \frac{1}{N_1} \sum_{j \in H} \sum_{\ell \in H^c} |E[u_{j,t} u_{\ell,t}]| \leq \bar{C} < \infty)
\end{aligned}$$

from which we further deduce, upon applying Markov's inequality, that

$$\frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 = o_p(1).$$

Moreover, since we have previously shown that

$$\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \{ i \in \widehat{H}^c \} \right) = o_p(1) \text{ and } \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1),$$

it follows from these calculations that

$$\begin{aligned}
|\mathcal{E}_{1,2,t}| &\leq \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\}\right)} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\}\right)} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
&= o_p(1).
\end{aligned}$$

In a similar way, we can also show that

$$|\mathcal{E}_{1,3,t}| = o_p(1).$$

Finally, let $U_{t,N}(H^c)$ denote an $N \times 1$ vector whose i^{th} component $U_{i,t,N}(H^c)$ is given by

$$U_{i,t,N}(H^c) = \begin{cases} u_{i,t} & \text{if } i \in H^c \\ 0 & \text{if } i \in H \end{cases}.$$

and we can write

$$\begin{aligned}
\mathcal{E}_{1,4,t} &= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= 2 \left\| \frac{G_1' U_{t,N}(H^c)}{\sqrt{N_1}} \right\|_2^2 \\
&\leq 2tr \left\{ \frac{G_1' U_{t,N}(H^c) U_{t,N}(H^c)' G_1}{N_1} \right\} \\
&= 2tr \left\{ \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \right\} \\
&= 2tr \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right\} \\
&= 2tr \left\{ \frac{\Gamma_*' U_{t,N}(H^c) U_{t,N}(H^c)' \Gamma_*}{N_1^2} \right\} \left(\text{where } \Gamma_* = \Gamma \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right) \\
&= \frac{2}{N_1^2} U_{t,N}(H^c)' \Gamma_* \Gamma_*' U_{t,N}(H^c) \\
&= \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \gamma_{*,i}' \gamma_{*,j} u_{i,t} u_{j,t}
\end{aligned}$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma (\Gamma' \Gamma / N_1)^{-1/2}$. Hence,

$$\begin{aligned}
0 &\leq E[\mathcal{E}_{1,4,t}] \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} E[u_{i,t} u_{j,t}] \\
&= \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \gamma'_{*,i} \gamma_{*,j} E[u_{i,t} u_{j,t}] \\
&= \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_j E[u_{i,t} u_{j,t}] \\
&\leq \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \left| \gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_j \right| |E[u_{i,t} u_{j,t}]| \\
&\leq \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \sqrt{\gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_i} \sqrt{\gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_j} |E[u_{i,t} u_{j,t}]| \\
&\leq \frac{2\bar{c}}{\underline{C}} \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} |E[u_{i,t} u_{j,t}]| \\
&\quad (\text{since, under Assumptions 3-5 and 3-6, there exist positive constants } \bar{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \sup_{i \in H^c} \|\gamma_i\|_2 \leq \bar{c} < \infty \text{ and } \lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \geq \underline{C} > 0) \\
&\leq \frac{2\bar{c}}{\underline{C}} \frac{\bar{C}}{N_1} \rightarrow 0 \text{ as } N_1 \rightarrow \infty. \quad (\text{since, under Assumption 3-3(d), there exists a positive constant } \bar{C} \\
&\quad \text{such that } \sup_t \frac{1}{N_1} \sum_{i \in H^c} \sum_{j \in H^c} |E[u_{i,t} u_{j,t}]| \leq \bar{C} < \infty)
\end{aligned}$$

It follows by Markov's inequality that

$$\mathcal{E}_{1,4,t} = o_p(1).$$

Application of the Slutsky's theorem then allows us to deduce that

$$\begin{aligned}
\frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} &= \mathcal{E}_{1,1,t} + \mathcal{E}_{1,2,t} + \mathcal{E}_{1,3,t} + \mathcal{E}_{1,4,t} \\
&= o_p(1) + o_p(1) + o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

Next, consider the second term on the right-hand side of expression (139). In this case, write

$$\begin{aligned}
& \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} g_{1,ik} u_{i,t} \right)^2 \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} g_{1,ik} u_{i,t} \right| \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \left[\sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right]
\end{aligned}$$

Note that, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have, by applying an argument similar to that given in the proof of Theorem 2,

$$\begin{aligned}
0 &\leq E \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \\
&= \frac{1}{N_1} \sum_{i \in H} \Pr(i \in \widehat{H}^c) \\
&= \frac{1}{N_1} \sum_{i \in H} P \left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&\leq \frac{N_2 \varphi}{N N_1} \left\{ 1 + 2^3 A T_0^{-(1-\alpha_1)\frac{1}{2}} + 2^3 A \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)^3 T_0^{-(1-\alpha_1)\frac{1}{2}} \right\} \\
&= \frac{N_2 \varphi}{N_1 (N_1 + N_2)} [1 + o(1)] \\
&\rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$E \left[\sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right] = \sum_{i \in H} g_{1,ik}^2 E[u_{i,t}^2] \leq C \sum_{i=1}^N g_{1,ik}^2 \leq C.$$

It follows by Markov's inequality that

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} = o_p(1) \text{ and } \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
& \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
& \leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \left[\sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right] \\
& = o_p(1).
\end{aligned}$$

Combining these results and using the inequality $\sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2}$, we further obtain, for all t ,

$$\begin{aligned}
\left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 & \leq \sqrt{\frac{2}{N_1} \sum_{k=1}^K \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
& \quad + \sqrt{\frac{2}{N_1} \sum_{k=1}^K \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
& = o_p(1) + o_p(1) \\
& = o_p(1).
\end{aligned}$$

For part (e), write

$$\begin{aligned}
\left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 & = \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} \\
& = \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \\
& = \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \\
& \leq \frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2
\end{aligned}$$

Note that, by Assumption 3-3(b),

$$E \left[\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 \right] = \frac{1}{N_1} \sum_{i \in H^c} E[u_{i,t}^2] \leq C \quad (\text{since } N_1 = \#\{H^c\})$$

so that, by applying Markov's inequality, we obtain

$$\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i \in H} \{i \notin \widehat{H}^c\} \subseteq \left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 < \epsilon \right\}$$

so that by DeMorgan's law

$$\left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \geq \epsilon \right\} \subseteq \left\{ \bigcap_{i \in H} \{i \notin \widehat{H}^c\} \right\}^c = \bigcup_{i \in H} \{i \in \widehat{H}^c\}$$

Hence, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have, by applying an argument similar to that given in the proof of Theorem 2,

$$\begin{aligned} & \Pr \left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \geq \epsilon \right\} \\ & \leq \Pr \left\{ \bigcup_{i \in H} \{i \in \widehat{H}^c\} \right\} \\ & \leq \sum_{i \in H} \Pr \{i \in \widehat{H}^c\} \\ & = \sum_{i \in H} P \left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\ & \leq \frac{N_2 \varphi}{N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\ & = \frac{N_2 \varphi}{N_1 + N_2} [1 + o(1)] \\ & \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty \end{aligned}$$

Hence,

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 = o_p(1)$$

from which it further follows that

$$\begin{aligned}
\left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &\leq \frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

Turning our attention to part (f), note first that since $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ is an orthogonal matrix, we have $I_N = GG' = G_1G_1' + G_2G_2'$ or $G_2G_2' = I_N - G_1G_1'$. Hence, we can write

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &= \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} - \frac{U_{t,N}(\widehat{H}^c)' G_1 G_1' U_{t,N}(\widehat{H}^c)}{N_1} \\
&\leq \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} + \frac{U_{t,N}(\widehat{H}^c)' G_1 G_1' U_{t,N}(\widehat{H}^c)}{N_1} \\
&= \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Applying the results from parts (d) and (e) above, we then obtain

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &\leq \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

so that

$$\left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

Now, to show part (g), first write

$$\begin{aligned}
\frac{\widehat{V}' \widehat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} &= \frac{\widehat{V}' \widehat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1} \sqrt{(\widehat{N}_1 - N_1 + N_1) / N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \frac{\widehat{V}' \widehat{G}_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}}
\end{aligned}$$

Note that

$$\begin{aligned}
& \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
&= \left\| \frac{\widehat{V}' (I_{Kp} + R'R)^{-1/2} [G'_1 U_{t,N}(\widehat{H}^c) + R' G'_2 U_{t,N}(\widehat{H}^c)]}{\sqrt{N_1}} \right\|_2 \\
&\leq \left\| \widehat{V} \right\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left\| \widehat{V} \right\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
&= \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 + \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
&\quad \left(\text{since } \widehat{V}' \widehat{V} = I_{Kp} \text{ so that } \left\| \widehat{V} \right\|_2 = 1 \right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
&\leq \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right. \\
&\quad \left. + \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right\} \\
&\leq \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \frac{1}{\sqrt{1 + \lambda_{\min}(R'R)}} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right. \\
&\quad \left. + \frac{\|R\|_2}{\sqrt{1 + \lambda_{\min}(R'R)}} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right\} \\
&\leq \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 + \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right\} \\
&= o_p(1)
\end{aligned}$$

where the last line follows from the fact that

$$\|R\|_2 \xrightarrow{p} 0, \quad \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \xrightarrow{p} 1, \quad \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \text{ and } \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = O_p(1)$$

as shown in part (a) in Lemma D-14 and in parts (a), (d), and (f) of this lemma.

Turning our attention to part (h), we write

$$\begin{aligned}
& \frac{\widehat{V}'\widehat{G}'_1\Gamma\left(\widehat{H}^c\right)}{\sqrt{\widehat{N}_1}} \\
&= Q' + \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right) + \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{\widehat{N}_1}}\right) \\
&= Q' + \left(\left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q'\right) \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right] \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{N_1}}\right) \\
&= Q' + \left(\frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} - Q'\right) + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \frac{\widehat{V}'\widehat{G}'_1\Gamma}{\sqrt{N_1}} \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{N_1}}\right) + \widehat{V}'\widehat{G}'_1\left(\frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{N_1}}\right)
\end{aligned}$$

so that, by the triangle inequality

$$\begin{aligned}
& \left\| \frac{\Gamma(\widehat{H}^c)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 \\
&= \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
&\leq \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \widehat{V}' \widehat{G}'_1 \left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 + \left\| \widehat{V}' \widehat{G}'_1 \left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 \\
&\leq \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \widehat{V}' \widehat{G}'_1 \right\|_2 \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \widehat{V}' \widehat{G}'_1 \right\|_2 \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \\
&= \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2
\end{aligned}$$

where the last equality follows from the fact that

$$\left\| \widehat{V}' \widehat{G}'_1 \right\|_2 = \left\| \widehat{G}_1 \widehat{V} \right\|_2 = \sqrt{\lambda_{\max}(\widehat{V}' \widehat{G}'_1 \widehat{G}_1 \widehat{V})} = \sqrt{\lambda_{\max}(I_{Kp})} = 1.$$

Now, by parts (a), (b), and (c) of this lemma, we have that

$$\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \xrightarrow{p} 0, \quad \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \quad \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0, \text{ and}$$

and

$$\left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \leq \overline{C} < \infty \text{ for all } N_1, N_2 \text{ sufficiently large.}$$

It follows that

$$\begin{aligned}
\left\| \frac{\Gamma(\widehat{H}^c)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 &= \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
&\leq \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{\widehat{N}_1}} \right\|_2 + \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= o_p(1).
\end{aligned}$$

To show part (i), let \overline{C} be the positive constant given in Lemma C-5 such that

$$E \|\underline{F}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t;$$

and, for any $\epsilon > 0$, we let $C_\epsilon = \overline{C}^{\frac{1}{6}}/\sqrt{\epsilon}$. Applying Markov's inequality, we see that

$$\begin{aligned}
\Pr(\|\underline{F}_t\|_2 \geq C_\epsilon) &\leq \Pr(\|\underline{F}_t\|_2^2 \geq C_\epsilon^2) \\
&\leq \frac{1}{C_\epsilon^2} E \|\underline{F}_t\|_2^2 \\
&\leq \frac{1}{C_\epsilon^2} \left(E \|\underline{F}_t\|_2^6 \right)^{\frac{1}{3}} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{\epsilon}{\overline{C}^{\frac{1}{3}}} \\
&\leq \epsilon
\end{aligned}$$

from which it follows that $\|\underline{F}_t\|_2 = O_p(1)$ for all t .

Lastly, to show part (j), note that, similar to the derivation given in the proof of Theorem 4,

except that we replace the fixed index t with the sample size T , we can write

$$\begin{aligned}
\widehat{\underline{E}}_T - Q' \underline{E}_T &= \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} - Q' \right) \underline{E}_T + \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{\widehat{N}_1}} - Q' \right) \underline{E}_T + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{\widehat{N}_1}} \underline{E}_T \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right] \widehat{V}' \widehat{G}'_1 \left(\frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{\widehat{N}_1}} \right) \underline{E}_T + \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Next, note that, by following the same derivation as that given for the proof of part (g), we can show that

$$\begin{aligned}
&\left\| \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 + \|R\|_2 \left\| \frac{G'_2 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \right\}
\end{aligned}$$

Moreover, by argument similar to that given for parts (d) and (f) of this lemma, we can show that, as N_1 , N_2 , and $T \rightarrow \infty$;

$$\left\| \frac{G'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \tag{140}$$

and

$$\left\| \frac{G'_2 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 = O_p(1). \tag{141}$$

It follows from applying expressions (140) and (141), part (a) of this lemma, and part (a) of Lemma D-14 that

$$\left\| \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1, N_2, \text{ and } T \rightarrow \infty. \tag{142}$$

In addition, note that by applying Lemma C-5 and the Markov's inequality in a way similar to the argument given for the proof of part (i) above, we can show that

$$\|\underline{E}_T\|_2 = O_p(1). \tag{143}$$

Making use of the results given in expressions (142) and (143) and applying the triangle inequality as well as parts (a)-(c) of this lemma, expression (143), and the Slutsky's theorem; we then obtain, as N_1, N_2 , and $T \rightarrow \infty$;

$$\begin{aligned}
& \left\| \widehat{E}_T - Q' \underline{E}_T \right\|_2 \\
& \leq \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|\underline{E}_T\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{E}_T\|_2 \\
& \quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \widehat{V}' \widehat{G}'_1 \right\|_2 \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{E}_T\|_2 + \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
& = \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|\underline{E}_T\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}'_1 \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{E}_T\|_2 \\
& \quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \frac{\Gamma(\widehat{H}^c) - \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{E}_T\|_2 + \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{T,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
& \quad \left(\text{again since } \left\| \widehat{V}' \widehat{G}'_1 \right\|_2 = \lambda_{\max} \left(\widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}'_1 \right) = \lambda_{\max} \left(\widehat{V}' \widehat{G}'_1 \widehat{G}_1 \widehat{V} \right) = \lambda_{\max} (I_{Kp}) = 1 \right) \\
& = o_p(1) O_p(1) + o_p(1) O_p(1) O_p(1) + O_p(1) o_p(1) O_p(1) + o_p(1) \\
& = o_p(1). \quad \square
\end{aligned}$$

Lemma D-16: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true as $N_1, N_2, T \rightarrow \infty$.

(a)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = o_p(1), \text{ where } T_h = T - h - p + 1.$$

(b)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 = O_p(1).$$

(c)

$$, \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 = O_p(1)$$

(d)

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{E}_t\|_2^2 = O_p(1) \quad \text{and} \quad \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{E}_t' \right\|_2 = O_p(1)$$

(e)

$$\left\| \frac{\widehat{V}' \widehat{G}_1' U(\widehat{H}^c)' U(\widehat{H}^c) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 = o_p(1)$$

(f)

$$\left\| \frac{\underline{F}' U(\widehat{H}^c) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 = o_p(1)$$

(g)

$$\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{F}_t - Q' \underline{F}_t) (\widehat{F}_t - Q' \underline{F}_t)' \right\|_2 = o_p(1).$$

Proof of Lemma D-16:

For part (a), first write

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} + \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\ &\leq \frac{2}{T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 + \frac{2}{T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\ &= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\ &\quad + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \end{aligned} \tag{144}$$

where $g_{1,ik}$ denotes the $(i, k)^{th}$ element of

$$G_1 = \frac{\Gamma_* \Xi}{\sqrt{N_1}} = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi$$

Now, where

$$\begin{aligned} & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\ = & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 + 1 \right) \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 + 1 \right) g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\ = & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\ & + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \\ & + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\ & + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\ = & \underline{\mathcal{E}}_{1,1} + \underline{\mathcal{E}}_{1,2} + \underline{\mathcal{E}}_{1,3} + \underline{\mathcal{E}}_{1,4} \end{aligned}$$

Focusing first on the term $\underline{\mathcal{E}}_{1,1}$, we have

$$\begin{aligned}
& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\{j \in \widehat{H}^c\} - 1 \right) g_{1,jk} u_{j,t} \\
&= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right)^2 \\
&\leq \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left(\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2 \right) \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 2\mathbb{I}\{i \in \widehat{H}^c\} + 1 \right) \right] \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right) \right] \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right)
\end{aligned}$$

Now, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_{\ell} |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have, by applying Theorem 3,

$$\begin{aligned}
0 &\leq E \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right) \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} \left[1 - \Pr \left(i \in \widehat{H}^c \right) \right] \\
&= \frac{1}{N_1} \sum_{i \in H^c} \left[1 - P \left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \right] \\
&\leq 1 - P \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
&\quad \text{(given that } N_1 = \# \{H^c\}, \text{ where } \# \{H^c\} \text{ denotes the cardinality of the set } H^c) \\
&\rightarrow 0 \left(\text{since } P \left(\min_{i \in H^c} \mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \rightarrow 1 \right).
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right] &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 E[u_{i,t}^2] \\
&\leq C \frac{T-h-p+1}{T_h} \sum_{i=1}^N g_{1,ik}^2 \\
&\leq C \left(\text{since } \sum_{i=1}^N g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1 \right)
\end{aligned}$$

It follows by Markov's inequality that

$$\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right) = o_p(1) \text{ and } \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
\underline{\mathcal{E}}_{1,1} &= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \left(\mathbb{I} \left\{ i \in \widehat{H}^c \right\} - 1 \right) g_{1,ik} u_{i,t} \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \right) \right] \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
&= o_p(1).
\end{aligned}$$

Next, consider the term $\underline{\mathcal{E}}_{1,2}$. To proceed, write

$$\begin{aligned}
& |\underline{\mathcal{E}}_{1,2}| \\
&= \left| \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
&= \left| \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{j \in H^c} g_{1,jk} u_{j,t} \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
&\leq \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \left| \sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
&\leq \frac{2}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
&\leq \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\{i \in \widehat{H}^c\} - 1 \right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
&\quad \left(\text{by the inequality } |XY| \leq \frac{1}{2} X^2 + \frac{1}{2} Y^2 \right) \\
&= \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right)} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\{i \in \widehat{H}^c\} \right)} \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2
\end{aligned}$$

Observe that

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \right] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sum_{j \in H^c} \sum_{\ell \in H^c} g_{1,jk} g_{1,\ell k} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{j \in H^c} \sum_{\ell \in H^c} \sum_{k=1}^{Kp} g_{1,jk} g_{1,\ell k} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{j \in H^c} \sum_{\ell \in H^c} \frac{e'_{j,N} \Gamma_*}{\sqrt{N_1}} \sum_{k=1}^{Kp} \Xi e_{k,Kp} e'_{k,Kp} \Xi' \frac{\Gamma'_* e_{\ell,N}}{\sqrt{N_1}} E[u_{j,t} u_{\ell,t}] \\
&\quad \left(\text{since } G_1 = \frac{\Gamma_* \Xi}{\sqrt{N_1}} \text{ with } \Gamma_* = \Gamma \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right) \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} e'_{j,N} \Gamma_* \Xi \Xi' \Gamma'_* e_{\ell,N} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} e'_{j,N} \Gamma_* \Gamma'_* e_{\ell,N} E[u_{j,t} u_{\ell,t}] \\
&\quad (\text{since } \Xi \text{ is an orthogonal matrix}) \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_{*,j} \gamma_{*,\ell} E[u_{j,t} u_{\ell,t}]
\end{aligned}$$

where $\gamma_{*,j} = (\Gamma' \Gamma / N_1)^{-1/2} \gamma_j$. Applying the triangle and Cauchy-Schwarz inequalities, we further

obtain

$$\begin{aligned}
& E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \right] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_{*,j} \gamma_{*,\ell} E[u_{j,t} u_{\ell,t}] \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_\ell E[u_{j,t} u_{\ell,t}] \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \left| \gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_\ell \right| |E[u_{j,t} u_{\ell,t}]| \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{N_1^{\frac{3}{2}}} \sum_{j \in H^c} \sum_{\ell \in H^c} \sqrt{\gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_j} \sqrt{\gamma'_\ell \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_\ell} |E[u_{j,t} u_{\ell,t}]| \\
&\leq \frac{\bar{c}}{\underline{C}} \frac{1}{\sqrt{N_1} T_h} \sum_{t=p}^{T-h} \frac{1}{N_1} \sum_{j \in H^c} \sum_{\ell \in H^c} |E[u_{j,t} u_{\ell,t}]| \\
&\quad (\text{since, under Assumptions 3-5 and 3-6, there exist positive constants } \bar{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \sup_{i \in H^c} \|\gamma_i\|_2 \leq \bar{c} < \infty \text{ and } \lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \geq \underline{C} > 0) \\
&\leq \frac{\bar{c}}{\underline{C}} \frac{\bar{C}}{\sqrt{N_1}} \rightarrow 0 \text{ as } N_1 \rightarrow \infty. (\text{since, under Assumption 3-3(d), there exists a} \\
&\quad \text{positive constant } \bar{C} \text{ such that } \sup_t \frac{1}{N_1} \sum_{j \in H^c} \sum_{\ell \in H^c} |E[u_{j,t} u_{\ell,t}]| \leq \bar{C} < \infty)
\end{aligned}$$

from which we further deduce, upon applying Markov's inequality, that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 = o_p(1)$$

Moreover, since we have previously shown that

$$\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I} \{ i \in \widehat{H}^c \} \right) = o_p(1) \text{ and } \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1),$$

it follows from these calculations that

$$\begin{aligned}
|\underline{\mathcal{E}}_{1,2}| &\leq \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} (1 - \mathbb{I}\{i \in \widehat{H}^c\})} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} (1 - \mathbb{I}\{i \in \widehat{H}^c\})} \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
&= o_p(1).
\end{aligned}$$

In a similar way, we can also show that

$$|\underline{\mathcal{E}}_{1,3}| = o_p(1).$$

Finally, let $U_{t,N}(H^c)$ denote an $N \times 1$ vector whose i^{th} component $U_{i,t,N}(H^c)$ is given by

$$U_{i,t,N}(H^c) = \begin{cases} u_{i,t} & \text{if } i \in H^c \\ 0 & \text{if } i \in H \end{cases}.$$

and we can write

$$\begin{aligned}
\underline{\mathcal{E}}_{1,4} &= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}(H^c)}{\sqrt{N_1}} \right\|_2^2 \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \text{tr} \left\{ \frac{G_1' U_{t,N}(H^c) U_{t,N}(H^c)' G_1}{N_1} \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \text{tr} \left\{ \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \text{tr} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \text{tr} \left\{ \frac{\Gamma_*' U_{t,N}(H^c) U_{t,N}(H^c)' \Gamma_*}{N_1^2} \right\} \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} U_{t,N}(H^c)' \Gamma_* \Gamma_*' U_{t,N}(H^c) \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \gamma'_{*,i} \gamma_{*,j} u_{i,t} u_{j,t}
\end{aligned}$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma (\Gamma' \Gamma / N_1)^{-1/2}$. Taking expectation, we then obtain

$$\begin{aligned}
0 &\leq E [\underline{\mathcal{E}}_{1,4}] \\
&= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} E [u_{i,t} u_{j,t}] \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \gamma'_{*,i} \gamma_{*,j} E [u_{i,t} u_{j,t}] \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_j E [u_{i,t} u_{j,t}] \\
&\leq \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \left| \gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \gamma_j \right| |E [u_{i,t} u_{j,t}]| \\
&\leq \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \sqrt{\gamma'_i \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_i} \sqrt{\gamma'_j \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1} \gamma_j} |E [u_{i,t} u_{j,t}]| \\
&\leq \frac{2\bar{c}}{\underline{C}} \frac{1}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} |E [u_{i,t} u_{j,t}]| \\
&\quad \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \bar{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \sup_{i \in H^c} \|\gamma_i\|_2 \leq \bar{c} < \infty \text{ and } \lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \geq \underline{C} > 0) \\
&\leq \frac{2\bar{c}}{\underline{C}} \frac{\bar{C}}{N_1} \frac{T-h-p+1}{T_h} = \frac{2\bar{c}}{\underline{C}} \frac{\bar{C}}{N_1} \rightarrow 0 \text{ as } N_1, T \rightarrow \infty. \\
&\quad \text{(since, under Assumption 3-3(d), there exist a positive constant } \bar{C} \\
&\quad \text{such that } \sup_t \frac{1}{N_1} \sum_{i \in H^c} \sum_{j \in H^c} |E [u_{i,t} u_{j,t}]| \leq \bar{C} < \infty)
\end{aligned}$$

It follows by Markov's inequality that

$$\underline{\mathcal{E}}_{1,4} = o_p(1).$$

Application of the Slutsky's theorem then allows us to deduce that

$$\begin{aligned}
\frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I} \{i \in \widehat{H}^c\} \mathbb{I} \{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} &= \underline{\mathcal{E}}_{1,1} + \underline{\mathcal{E}}_{1,2} + \underline{\mathcal{E}}_{1,3} + \underline{\mathcal{E}}_{1,4} \\
&= o_p(1) + o_p(1) + o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

Consider now the second term on the extreme right-hand side of expression (144)

$$\begin{aligned}
& \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} g_{1,ik} u_{i,t} \right)^2 \\
&= \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} g_{1,ik} u_{i,t} \right| \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right]
\end{aligned}$$

Note that, for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have

$$\begin{aligned}
0 &\leq E \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \\
&= \frac{1}{N_1} \sum_{i \in H} \Pr(i \in \widehat{H}^c) \\
&= \frac{1}{N_1} \sum_{i \in H} P\left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
&\leq \frac{N_2 \varphi}{N_1 N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\
&= \frac{N_2 \varphi}{N_1 N} [1 + o(1)] \\
&\quad \text{(following an argument similar to that given in the proof of Theorem 2)} \\
&\rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right] &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 E[u_{i,t}^2] \\
&\leq C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i=1}^N g_{1,ik}^2 \\
&\leq C \frac{T-h-p+1}{T_h} \\
&\quad \left(\text{given that } \sum_{i=1}^N g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1 \right) \\
&\leq C < \infty
\end{aligned}$$

It follows by Markov's inequality that

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} = o_p(1) \text{ and } \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
&\frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \right] \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right] \\
&= o_p(1)
\end{aligned}$$

Combining these results, we further obtain

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &\leq \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&\quad + \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\{i \in \widehat{H}^c\} \mathbb{I}\{j \in \widehat{H}^c\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

To show part (b), write

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} \\
&= \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i=1}^N \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \\
&= \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 + \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} u_{i,t}^2 \\
&\leq \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2
\end{aligned}$$

Next, note that, by making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E \left[\frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} u_{i,t}^2 \right] &= \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} E[u_{i,t}^2] \\
&\leq C \frac{T-h-p+1}{T_h} \quad (\text{since } N_1 = \#\{H\}, \\
&\quad \text{where } \#\{H\} \text{ denotes the cardinality of the set } H) \\
&\leq C \quad (\text{since } T_h = T-h-p+1)
\end{aligned}$$

so that, by Markov's inequality,

$$\frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i \in H} \{i \notin \widehat{H}^c\} \subseteq \left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 < \epsilon \right\}$$

so that, applying DeMorgan's law, we obtain

$$\left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\{i \in \widehat{H}^c\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \geq \epsilon \right\} \subseteq \left\{ \bigcap_{i \in H} \{i \notin \widehat{H}^c\} \right\}^c = \bigcup_{i \in H} \{i \in \widehat{H}^c\}$$

It follows that, for any $\epsilon > 0$ and for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell |S_{i,\ell,T}|$ or the case

where $\mathbb{S}_{i,T}^+ = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have

$$\begin{aligned}
& \Pr \left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \geq \epsilon \right\} \\
& \leq \Pr \left\{ \bigcup_{i \in H} \left\{ i \in \widehat{H}^c \right\} \right\} \\
& \leq \sum_{i \in H} \Pr \left\{ i \in \widehat{H}^c \right\} \\
& = \sum_{i \in H} P \left(\mathbb{S}_{i,T}^+ \geq \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right) \right) \\
& \leq \frac{N_2 \varphi}{N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1} \left(1 - \frac{\varphi}{2N} \right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\
& = \frac{N_2 \varphi}{N} [1 + o(1)] \\
& \quad \text{(following an argument similar to that given in the proof of Theorem 2)} \\
& \rightarrow 0 \text{ as } N_1, N_2, T \rightarrow \infty
\end{aligned}$$

Hence,

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 = o_p(1)$$

from which it we further deduce that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 & \leq \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I} \left\{ i \in \widehat{H}^c \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \\
& = O_p(1) + o_p(1) \\
& = O_p(1).
\end{aligned}$$

Now, for part (c), note first that since $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ is an orthogonal matrix, we have

$I_N = GG' = G_1G_1' + G_2G_2'$ or $G_2G_2' = I_N - G_1G_1'$. Hence, we can write

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} - \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{U_{t,N}(\widehat{H}^c)' G_1 G_1' U_{t,N}(\widehat{H}^c)}{N_1} \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{U_{t,N}(\widehat{H}^c)' U_{t,N}(\widehat{H}^c)}{N_1} + \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{U_{t,N}(\widehat{H}^c)' G_1 G_1' U_{t,N}(\widehat{H}^c)}{N_1} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Applying the results from parts (a) and (b) of this lemma, we then obtain

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

Next, to show part (d), let \overline{C} be the constant given in Lemma C-5 such that

$$E \|\underline{F}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t.$$

Now, for any $\epsilon > 0$, let $C_\epsilon^* = \overline{C}^{\frac{1}{3}}/\epsilon$; then, upon application of Markov's inequality, we have

$$\begin{aligned}
\Pr \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2 \geq C_\epsilon^* \right) &\leq \frac{1}{C_\epsilon^*} \frac{1}{T_h} \sum_{t=p}^{T-h} E \|\underline{F}_t\|_2^2 \\
&\leq \frac{1}{C_\epsilon^*} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E \|\underline{F}_t\|_2^6 \right)^{\frac{1}{3}} \text{ (by Liapunov's inequality)} \\
&= \frac{\epsilon}{\overline{C}^{\frac{1}{3}}} \frac{1}{T_h} \sum_{t=p}^{T-h} \overline{C}^{\frac{1}{3}} \\
&= \frac{T-h-p+1}{T_h} \\
&\leq \epsilon \text{ (since } T_h = T-h-p+1)
\end{aligned}$$

so that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2 = O_p(1)$$

In addition, note that

$$\begin{aligned} \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t \underline{F}_t'\|_2 \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}(\underline{F}_t \underline{F}_t' \underline{F}_t \underline{F}_t')} \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\|\underline{F}_t\|_2^2 \lambda_{\max}(\underline{F}_t \underline{F}_t')} \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\|\underline{F}_t\|_2^2 \lambda_{\max}(\underline{F}_t' \underline{F}_t)} \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\|\underline{F}_t\|_2^4} \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2 \\ &= O_p(1) \end{aligned}$$

Turning our attention to part (e), write

$$\begin{aligned}
& \frac{\widehat{V}' \widehat{G}'_1 U \left(\widehat{H}^c \right)' U \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\widehat{V}' \widehat{G}'_1 U \left(\widehat{H}^c \right)' \mathbf{e}_{t,T} \mathbf{e}_{t,T}' U \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1} \sqrt{\widehat{N}_1}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1} \sqrt{\left(\widehat{N}_1 - N_1 + N_1 \right) / N_1}} \frac{U_{t,N}' \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1} \sqrt{\left(\widehat{N}_1 - N_1 + N_1 \right) / N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} \widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' \widehat{G}_1 \widehat{V} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} \left\{ \widehat{V}' \left(I_{Kp} + R'R \right)^{-1/2} \left[G'_1 + R' G'_2 \right] U_{t,N} \left(\widehat{H}^c \right) \right. \\
&\quad \left. \times U_{t,N} \left(\widehat{H}^c \right)' \left[G_1 + G_2 R \right] \left(I_{Kp} + R'R \right)^{-1/2} \widehat{V} \right\} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \left\{ \widehat{V}' \left(I_{Kp} + R'R \right)^{-1/2} G'_1 \right. \\
&\quad \times \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1 \left(I_{Kp} + R'R \right)^{-1/2} \widehat{V} \left. \right\} \\
&\quad + \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \left\{ \widehat{V}' \left(I_{Kp} + R'R \right)^{-1/2} G'_1 \right. \\
&\quad \times \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2 R \left(I_{Kp} + R'R \right)^{-1/2} \widehat{V} \left. \right\} \\
&\quad + \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \left\{ \widehat{V}' \left(I_{Kp} + R'R \right)^{-1/2} R' G'_2 \right. \\
&\quad \times \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1 \left(I_{Kp} + R'R \right)^{-1/2} \widehat{V} \left. \right\} \\
&\quad + \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \left\{ \widehat{V}' \left(I_{Kp} + R'R \right)^{-1/2} R' G'_2 \right. \\
&\quad \times \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2 R \left(I_{Kp} + R'R \right)^{-1/2} \widehat{V} \left. \right\}
\end{aligned}$$

To analyze the four terms on the right-hand side of the expression above, note first that, by the homogeneity of matrix norm and the triangle inequality,

$$\begin{aligned}
\left\| G_1' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1 \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1}{N_1} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max} \left\{ \left(\frac{G_1' U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1}{N_1} \right)^2 \right\}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}^2 \left(\frac{G_1' U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1}{N_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \lambda_{\max} \left(\frac{G_1' U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1}{N_1} \right) \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
& \left\| G'_1 \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2 \right\|_2 \\
& \leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2}{N_1} \right\|_2 \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max} \left\{ \left(\frac{G'_2 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_1 G'_1 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2}{N_1^2} \right) \right\}} \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\left\| \frac{G'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2^2 \lambda_{\max} \left\{ \left(\frac{G'_2 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2}{N_1} \right) \right\}} \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2 \sqrt{\lambda_{\max} \left(\frac{G'_2 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2}{N_1} \right)} \\
& \leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2 \left\| \frac{G'_2 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2
\end{aligned}$$

and

$$\begin{aligned}
\left\| G'_2 \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2 \right\|_2 & \leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N} \left(\widehat{H}^c \right) U_{t,N} \left(\widehat{H}^c \right)' G_2}{N_1} \right\|_2 \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{N_1}} \right\|_2^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \left\| \frac{\widehat{V}' \widehat{G}'_1 U(\widehat{H}^c)' U(\widehat{H}^c) \widehat{G}'_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
& \leq \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \widehat{V} \right\|_2^2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \widehat{V} \right\|_2^2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
& \quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \widehat{V} \right\|_2^2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& = \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \\
& \quad + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad \left(\text{since } \widehat{V}' \widehat{V} = I_{Kp} \text{ so that } \left\| \widehat{V} \right\|_2 = 1 \right) \\
& \leq 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{\widehat{V}' \widehat{G}'_1 U(\widehat{H}^c)' U(\widehat{H}^c) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
& \leq 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \leq 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left(\frac{1}{\sqrt{1 + \lambda_{\min}(R'R)}} \right)^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left(\frac{\|R\|_2}{\sqrt{1 + \lambda_{\min}(R'R)}} \right)^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& = 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \frac{1}{1 + \lambda_{\min}(R'R)} \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \quad + 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \frac{\|R\|_2^2}{1 + \lambda_{\min}(R'R)} \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
& \leq 2 \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right] \\
& = o_p(1) \quad (\text{applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
& \quad \text{parts (a) and (c) of this lemma, and Slutsky's theorem})
\end{aligned}$$

To show part (f), first write

$$\begin{aligned}
\left\| \frac{\underline{F}' U \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 &= \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\underline{F}_t U'_{t,N} \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\underline{F}_t U'_{t,N} \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max} \left(\frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right) \underline{F}'_t \underline{F}_t U'_{t,N} \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{\lambda_{\max} \left(\frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right) U'_{t,N} \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{\frac{U'_{t,N} \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\widehat{N}_1}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \right\|_2^2}
\end{aligned}$$

Next, note that

$$\begin{aligned}
& \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
&= \left(\left\| \frac{\widehat{V}' (I_{Kp} + R'R)^{-1/2} [G'_1 + R'G'_2] U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
&\leq \left(\left\| \widehat{V} \right\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right. \\
&\quad \left. + \left\| \widehat{V} \right\|_2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
&= \left(\left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 + \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
&\quad \left(\text{since } \widehat{V}' \widehat{V} = I_{Kp} \text{ so that } \left\| \widehat{V} \right\|_2 = 1 \right) \\
&\leq 2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + 2 \left\| (I_{Kp} + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

from which we obtain

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2^2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1} \sqrt{(N_1 + \widehat{N}_1 - N_1)/N_1}} \right\|_2^2 \\
&= \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \\
&\leq \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\{ 2 \left\| (I_K + R'R)^{-1/2} \right\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right. \\
&\quad \left. + 2 \left\| (I_K + R'R)^{-1/2} \right\|_2^2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right\} \\
&\leq \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\{ \frac{2}{1 + \lambda_{\min}(R'R)} \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right. \\
&\quad \left. + \frac{2 \|R\|_2^2}{\sqrt{1 + \lambda_{\min}(R'R)}} \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right\} \\
&\leq \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\{ \frac{2}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 + 2 \|R\|_2^2 \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G'_2 U_{t,N}(\widehat{H}^c)}{\sqrt{N_1}} \right\|_2^2 \right\} \\
&= o_p(1) \quad (\text{applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
&\quad \text{parts (a) and (c) of this lemma, and Slutsky's theorem})
\end{aligned}$$

It then follows from part (d) of this lemma and the Slutsky's theorem that

$$\begin{aligned}
\left\| \frac{\underline{F}' U(\widehat{H}^c) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 &\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}'_1 U_{t,N}(\widehat{H}^c)}{\sqrt{\widehat{N}_1}} \right\|_2^2} \\
&= O_p(1) o_p(1) \\
&= o_p(1)
\end{aligned}$$

Lastly, to show part (g), first write

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\{ \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t + \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \right) \right. \\
&\quad \times \left. \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t + \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \right) \right\}' \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t \right) \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t \right)' \\
&\quad + \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t \right) \frac{U_{t,N} \left(\widehat{H}^c \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \\
&\quad + \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right) \underline{F}_t}{\sqrt{\widehat{N}_1}} - Q' \underline{F}_t \right)' \\
&\quad + \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\widehat{V}' \widehat{G}'_1 U_{t,N} \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} \frac{U_{t,N} \left(\widehat{H}^c \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \left(\frac{\Gamma \left(\widehat{H}^c \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right) \\
&\quad + \left(\frac{\widehat{V}' \widehat{G}'_1 \Gamma \left(\widehat{H}^c \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{\underline{F}' U \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \\
&\quad + \frac{\widehat{V}' \widehat{G}'_1 U \left(\widehat{H}^c \right)' \underline{F}}{T_h \sqrt{\widehat{N}_1}} \left(\frac{\Gamma \left(\widehat{H}^c \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right) + \frac{\widehat{V}' \widehat{G}'_1 U \left(\widehat{H}^c \right)' U \left(\widehat{H}^c \right) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1}
\end{aligned}$$

where $U_{t,N} \left(\widehat{H}^c \right) = U \left(\widehat{H}^c \right)' \mathbf{e}_{t,T} = \left(\mathbb{I} \left\{ 1 \in \widehat{H}^c \right\} u_{1,t} \quad \mathbb{I} \left\{ 2 \in \widehat{H}^c \right\} u_{2,t} \quad \cdots \quad \mathbb{I} \left\{ N \in \widehat{H}^c \right\} u_{N,t} \right)'$. Applying part (h) of Lemma D-15 and parts (d), (e), and (f) of this lemma and the Slutsky's

theorem, we obtain

$$\begin{aligned}
& \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} (\hat{F}_t - Q' E_t) (\hat{F}_t - Q' E_t)' \right\|_2 \\
& \leq \left\| \left(\frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right) \frac{1}{T_h} \sum_{t=p}^{T-h} E_t E_t' \left(\frac{\Gamma(\hat{H}^c)' \hat{G}_1 \hat{V}}{\sqrt{\hat{N}_1}} - Q \right) \right\|_2 \\
& \quad + \left\| \left(\frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right) \frac{E' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \sqrt{\hat{N}_1}} \right\|_2 \\
& \quad + \left\| \frac{\hat{V}' \hat{G}'_1 U(\hat{H}^c)' E}{T_h \sqrt{\hat{N}_1}} \left(\frac{\Gamma(\hat{H}^c)' \hat{G}_1 \hat{V}}{\sqrt{\hat{N}_1}} - Q \right) \right\|_2 + \left\| \frac{\hat{V}' \hat{G}'_1 U(\hat{H}^c)' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \hat{N}_1} \right\|_2 \\
& = \left\| \left(\frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right) \frac{1}{T_h} \sum_{t=p}^{T-h} E_t E_t' \left(\frac{\Gamma(\hat{H}^c)' \hat{G}_1 \hat{V}}{\sqrt{\hat{N}_1}} - Q \right) \right\|_2 \\
& \quad + 2 \left\| \left(\frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right) \frac{E' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \sqrt{\hat{N}_1}} \right\|_2 + \left\| \frac{\hat{V}' \hat{G}'_1 U(\hat{H}^c)' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \hat{N}_1} \right\|_2 \\
& \leq \left\| \frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right\|_2^2 \left\| \frac{1}{T_h} \sum_{t=p}^T E_t E_t' \right\|_2 + 2 \left\| \frac{\hat{V}' \hat{G}'_1 \Gamma(\hat{H}^c)}{\sqrt{\hat{N}_1}} - Q' \right\|_2 \left\| \frac{E' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \sqrt{\hat{N}_1}} \right\|_2 \\
& \quad + \left\| \frac{\hat{V}' \hat{G}'_1 U(\hat{H}^c)' U(\hat{H}^c) \hat{G}_1 \hat{V}}{T_h \hat{N}_1} \right\|_2 \\
& = o_p(1) O_p(1) + o_p(1) o_p(1) + o_p(1) \\
& = o_p(1). \quad \square
\end{aligned}$$

Lemma D-17: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{\hat{F}' \hat{F}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [E_t E_t'] Q = o_p(1), \text{ where } T_h = T - h - p + 1.$$

(b)

$$\frac{\widehat{\underline{F}}' \underline{Y}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{Y}_t'] = o_p(1)$$

(c)

$$\frac{\widehat{\underline{F}}' \iota_{T_h}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t] = o_p(1),$$

where $\iota_{T_h} = (1, 1, \dots, 1)'$ is a $T_h \times 1$ vector.

(d)

$$\frac{\widehat{\underline{F}}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1)$$

(e)

$$\frac{\underline{Y}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1)$$

(f)

$$\frac{\iota_{T_h}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1)$$

(g)

$$\frac{\widehat{\underline{F}}' \mathfrak{H}}{T_h} = \frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \eta_{t+h}' = o_p(1)$$

Proof of Lemma D-17:

To show part (a), first write

$$\begin{aligned}
& \frac{\widehat{\underline{F}}' \widehat{\underline{F}}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \widehat{\underline{F}}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t + Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t + Q' \underline{F}_t \right)' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' + Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \\
&\quad + \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \underline{F}_t' Q + Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{F}_t \underline{F}_t'] \right) Q
\end{aligned}$$

Now, by part (g) of Lemma D-16, we have that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \xrightarrow{p} 0$$

Moreover, for any $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$

$$\begin{aligned}
& \left| a' Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' b \right| \\
&\leq \sqrt{a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \right) Q a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' b} \\
&\leq \sqrt{a' Q' Q a \lambda_{\max} \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \right)} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' b} \\
&= \sqrt{a' Q' Q a \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \right\|_2} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' b} \\
&\quad (\text{since, for a symmetric psd matrix } A, \\
&\quad \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} = \lambda_{\max}(A))
\end{aligned}$$

Now, by Assumption 3-6, there exists a positive constant C such that

$$\begin{aligned}
a'Q'Qa &= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\Xi\widehat{V}a \\
&= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)\Xi\widehat{V}a \\
&\leq \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)a'\widehat{V}'\Xi'\Xi\widehat{V}a \\
&= \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)\left(\text{since } \Xi'\Xi = I_{Kp}, \widehat{V}'\widehat{V} = I_{Kp}, \text{ and } a'a = 1\right) \\
&\leq C \text{ for all } N_1, N_2 \text{ sufficiently large.}
\end{aligned} \tag{145}$$

while, applying the triangle inequality and part (d) of Lemma D-16 allow us to show that

$$\begin{aligned}
\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{E}_t\underline{E}_t'\right\|_2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{E}_t\underline{E}_t'\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\lambda_{\max}(\underline{E}_t\underline{E}_t'\underline{E}_t\underline{E}_t')} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{[\lambda_{\max}(\underline{E}_t\underline{E}_t')]^2} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{E}_t\|_2^4} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{E}_t\|_2^2 \\
&= O_p(1)
\end{aligned}$$

Combining this result with part (g) of Lemma D-16 and the Slutsky's Theorem, we deduce that

$$\begin{aligned}
&\left|a'Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{E}_t\left(\widehat{\underline{E}}_t - Q'\underline{E}_t\right)'b\right| \\
&\leq \sqrt{a'Q'Qa\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{E}_t\underline{E}_t'\right\|_2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\widehat{\underline{E}}_t - Q'\underline{E}_t\right)\left(\widehat{\underline{E}}_t - Q'\underline{E}_t\right)'b} \\
&= o_p(1)
\end{aligned}$$

Since this argument holds for all $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further obtain

$$Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right)' = o_p(1)$$

Now, given that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right) \underline{E}_t' Q = \left[Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right)' \right]',$$

a similar argument also shows that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right) \underline{E}_t' Q = o_p(1).$$

Making use of part (b) of Lemma D-2 and the Slutsky's theorem, we also see that

$$Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{E}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{E}_t'] \right) Q \xrightarrow{p} 0$$

Putting these results together and apply Slutsky's theorem, we then obtain

$$\begin{aligned} & \frac{\widehat{\underline{E}}' \widehat{\underline{E}}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{E}_t \underline{E}_t'] Q \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right) \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right)' + Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right)' \\ & \quad + \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{E}}_t - Q' \underline{E}_t \right) \underline{E}_t' Q + Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{E}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{E}_t'] \right) Q \\ &= o_p(1) \end{aligned}$$

To show part (b), first write, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$,

$$\begin{aligned}
& \frac{a' \widehat{\underline{F}}' \underline{Y} b}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{F}_t \underline{Y}'_t] b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \widehat{\underline{F}}_t \underline{Y}'_t b - \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{F}_t \underline{Y}'_t] b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t + Q' \underline{F}_t \right) \underline{Y}'_t b - \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{F}_t \underline{Y}'_t] b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \underline{Y}'_t b + a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{F}_t \underline{Y}'_t] \right) b
\end{aligned}$$

Focusing first on the first term on last line above, we note that,

$$\begin{aligned}
& \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\hat{E}_t - Q' E_t \right) \underline{Y}_t' b \right| \\
& \leq \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \underline{Y}_t \underline{Y}_t' b} \\
& = \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \\
& \quad \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b + \frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
& \leq \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& \quad + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
& \quad (\text{since } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2}) \\
& \leq \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& \quad + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{E}_t - Q' E_t \right) \left(\hat{E}_t - Q' E_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=r+1}^{T-h} E \left\| \underline{Y}_t \right\|_2^2} \\
& \quad \left(\text{since } b' E \left[\underline{Y}_t \underline{Y}_t' \right] b = E \left[(b' \underline{Y}_t)^2 \right] \leq E \left[b' b \underline{Y}_t' \underline{Y}_t \right] = E \left[\left\| \underline{Y}_t \right\|_2^2 \right] \right) \\
& = o_p(1)
\end{aligned}$$

by part (b) of Lemma D-2 and parts (d) and (g) of Lemma D-16. In addition, note that, by making

use of part (b) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
& \left| a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right) b \right| \\
& \leq \sqrt{a' Q' Q a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right)' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right) b} \\
& \leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right)' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right) b} \\
& = o_p(1).
\end{aligned}$$

Combining these results, we then get

$$\begin{aligned}
& \left| \frac{a' \widehat{\underline{F}}' \underline{Y} b}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{E}_t \underline{Y}'_t] b \right| \\
& \leq \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' (\widehat{\underline{E}}_t - Q' \underline{E}_t) \underline{Y}'_t b \right| + \left| a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{E}_t \underline{Y}'_t] \right) b \right| \\
& = o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\widehat{\underline{F}}' \underline{Y}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{E}_t \underline{Y}'_t] = o_p(1).$$

To show part (c), first write, for any $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$,

$$\begin{aligned}
& \frac{a' \widehat{\underline{F}}' \nu_{T_h}}{T_h} - a' Q' S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} a' \widehat{\underline{E}}_t - a' Q' S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} a' (\widehat{\underline{E}}_t - Q' \underline{E}_t + Q' \underline{E}_t) - a' Q' S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} a' (\widehat{\underline{E}}_t - Q' \underline{E}_t) + a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right)
\end{aligned}$$

Focusing first on the first term on last line above, we note that,

$$\begin{aligned}
\left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{F}_t - Q' E_t \right) \right| &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left| a' \left(\widehat{F}_t - Q' E_t \right) \right| \quad (\text{by triangle inequality}) \\
&\leq \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{F}_t - Q' E_t \right) \left(\widehat{F}_t - Q' E_t \right)' a} \quad (\text{by Liapunov's inequality}) \\
&\leq \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{F}_t - Q' E_t \right) \left(\widehat{F}_t - Q' E_t \right)' \right\|_2} \\
&= o_p(1)
\end{aligned}$$

by part (g) of Lemma D-16 and Slutsky's theorem. In addition, note that, by making use of part (d) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
&\left| a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right) \right| \\
&\leq \sqrt{a' Q' Q a} \left[\left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right)' \right. \\
&\quad \times \left. \left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right) \right]^{1/2} \\
&\leq \sqrt{\lambda_{\max} \left(\frac{\Gamma' \Gamma}{N_1} \right)} \left[\left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right)' \right. \\
&\quad \times \left. \left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right) \right]^{1/2} \\
&= o_p(1).
\end{aligned}$$

Combining these results and applying Slutsky's theorem, we then get

$$\begin{aligned}
&\left| \frac{a' \widehat{F}' \iota_{T_h}}{T_h} - a' Q' S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right| \\
&\leq \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{F}_t - Q' E_t \right) \right| + \left| a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} E_t - S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right) \right| \\
&= o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$; we further deduce that

$$\frac{\widehat{F}'}{T_h} \iota_{T_h} - Q' S'_K \mathcal{P}_{(d+K)p} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu = o_p(1).$$

Turning our attention to part (d), note that for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$\begin{aligned}
& \left| \frac{a' \widehat{\underline{F}}' (\widehat{\underline{F}} - \underline{F}Q) Q^{-1} B_2 b}{T_h} \right| \\
&= \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \widehat{\underline{F}}_t (\widehat{\underline{F}}'_t - \underline{F}'_t Q) Q^{-1} B_2 b \right| \\
&\leq \sqrt{a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \widehat{\underline{F}}'_t \right) a} \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) Q^{-1} B_2 b} \\
&\leq \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \widehat{\underline{F}}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}'_t] Q \right) a \right| + \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{F}_t \underline{F}'_t] Q a} \\
&\quad \times \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) Q^{-1} B_2 b} \\
&\leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \widehat{\underline{F}}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}'_t] Q \right) a \right|} \right. \\
&\quad \times \left. \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) Q^{-1} B_2 b} \right\} \\
&\quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E [\underline{F}_t \underline{F}'_t] Q a} \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) Q^{-1} B_2 b} \\
&\quad \text{(using the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0) \\
&\leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \widehat{\underline{F}}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}'_t] Q \right) a \right|} \right. \\
&\quad \times \left. \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \right\} \\
&\quad + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E [\|\underline{F}_t\|_2^2]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} (\widehat{\underline{F}}'_t - \underline{F}'_t Q)' (\widehat{\underline{F}}'_t - \underline{F}'_t Q) \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \\
&\quad \left(\text{since for a symmetric psd matrix } A, \ \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} \right. \\
&\quad \left. = \lambda_{\max}(A) \text{ and since } a' Q' E [\underline{F}_t \underline{F}'_t] Q a = E \left[(a' Q' \underline{F}_t)^2 \right] \leq E [a' Q' Q a \underline{F}'_t \underline{F}_t] = a' Q' Q a E [\|\underline{F}_t\|_2^2] \right)
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \right) a \right|} \right. \\
&\quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{F}_t - Q' \underline{F}_t \right) \left(\widehat{F}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \Big\} \\
&\quad + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\|\underline{F}_t\|_2^2 \right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{F}_t - Q' \underline{F}_t \right) \left(\widehat{F}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b}
\end{aligned}$$

Now, by part (a) of this lemma and Slutsky's theorem, we have

$$\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \right) a \right| = o_p(1) \quad (146)$$

Note also that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\|\underline{F}_t\|_2^2 \right] &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E \left[\|\underline{F}_t\|_2^6 \right] \right)^{\frac{1}{3}} \quad (\text{by Liapunov's inequality}) \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} (\overline{C})^{\frac{1}{3}} \quad (\text{by Lemma C-5}) \\
&= (\overline{C})^{\frac{1}{3}}. \quad (147)
\end{aligned}$$

In addition, note that, by Assumption 3-7, there exists a positive constant C such that

$$\begin{aligned}
& \lambda_{\max}(B_2' B_2) \\
&= \lambda_{\max}\left(J_d A^h \mathcal{P}'_{(d+K)p} S_K S_K' \mathcal{P}_{(d+K)p} (A^h)' J_d'\right) \\
&\leq \lambda_{\max}(S_K S_K') \lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p}) \lambda_{\max}\left\{A^h (A^h)'\right\} \lambda_{\max}(J_d J_d') \\
&= \lambda_{\max}(S_K S_K') \lambda_{\max}\left\{A^h (A^h)'\right\} \quad \left(\text{since } \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} = I_{(d+K)p} \text{ and } J_d J_d' = I_d\right. \\
&\quad \left.\text{so } \lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p}) = \lambda_{\max}(J_d J_d') = 1\right) \\
&= \lambda_{\max}(S_K' S_K) \lambda_{\max}\left\{(A^h)' A^h\right\} \\
&= \lambda_{\max}\left\{(A^h)' A^h\right\} \quad (\text{since } S_K' S_K = I_{Kp} \text{ so } \lambda_{\max}(S_K' S_K) = 1) \\
&= \sigma_{\max}^2(A^h) \\
&\leq C \max\left\{\left|\lambda_{\max}(A^h)\right|^2, \left|\lambda_{\min}(A^h)\right|^2\right\} \quad (\text{by Assumption 3-7}) \\
&= C \max\left\{|\lambda_{\max}(A)|^{2h}, |\lambda_{\min}(A)|^{2h}\right\} \\
&= C \phi_{\max}^{2h} \\
&< C \text{ for integer } h \geq 1,
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where the last equality follows from the fact that $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. The boundedness of $\lambda_{\max}(B_2' B_2)$ allows us to further deduce that

$$\begin{aligned}
& b' B_2' Q'^{-1} Q^{-1} B_2 b \\
&= b' B_2' \left(\frac{\Gamma' \Gamma}{N_1}\right)^{-1/2} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1}\right)^{-1/2} B_2 b \\
&= b' B_2' \left(\frac{\Gamma' \Gamma}{N_1}\right)^{-1} B_2 b \\
&\leq \left[\lambda_{\min}\left(\frac{\Gamma' \Gamma}{N_1}\right)\right]^{-1} b' B_2' B_2 b \\
&\leq \left[\lambda_{\min}\left(\frac{\Gamma' \Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}(B_2' B_2) b' b \\
&= \left[\lambda_{\min}\left(\frac{\Gamma' \Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}(B_2' B_2) \\
&\leq C^* < \infty
\end{aligned} \tag{148}$$

for some positive constant C^* in light of Assumption 3-6. It follows by applying expression (145) in the proof for part (a), expressions (146)-(148) here, as well as the result given in part (g) of Lemma D-16 and the Slutsky' theorem that

$$\begin{aligned}
& \left| \frac{a' \widehat{\underline{F}}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
& \leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E [\underline{F}_t \underline{F}_t'] Q \right) a \right|} \right. \\
& \quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \left. \right\} \\
& \quad + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\|\underline{F}_t\|_2^2 \right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
& = o_p(1).
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{\widehat{\underline{F}}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1).$$

To show part (e), note that for any $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$\begin{aligned}
& \left| \frac{a' \underline{Y}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
&= \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{Y}_t \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b \right| \\
&\leq \sqrt{a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}'_t \right) a} \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \\
&\leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{Y}'_t] \right) a \right|} + \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' E[\underline{Y}_t \underline{Y}'_t] a \right|} \right. \\
&\quad \times \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \Big\} \\
&\leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}'_t - \frac{1}{T_h} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{Y}'_t] \right) a \right|} \right. \\
&\quad \times \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \Big\} \\
&\quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' E[\underline{Y}_t \underline{Y}'_t] a} \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \\
&\quad \text{(using the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0)
\end{aligned}$$

$$\begin{aligned}
&\leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{Y}_t'] \right) a \right|} \right. \\
&\quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\hat{\underline{F}}_t - \underline{F}_t Q \right) \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \Big\} \\
&\quad + \sqrt{a' a \frac{1}{T_h} \sum_{t=p}^{T-h} E [\|\underline{Y}_t\|_2^2]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\hat{\underline{F}}_t - \underline{F}_t Q \right) \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
&\quad \left(\text{since for a symmetric psd matrix } A, \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} \right. \\
&\quad \left. = \lambda_{\max}(A) \text{ and since } a' E [\underline{Y}_t \underline{Y}_t'] a = E [(a' \underline{Y}_t)^2] \leq E [a' a \underline{Y}_t' \underline{Y}_t] = E [\|\underline{Y}_t\|_2^2] \right) \\
&= \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{Y}_t'] \right) a \right|} \right. \\
&\quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \Big\} \\
&\quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} E [\|\underline{Y}_t\|_2^2]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
&\quad \left(\text{since } \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\hat{\underline{F}}_t - \underline{F}_t Q \right) = \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right)' \right)
\end{aligned}$$

Now, by part (b) of Lemma D-2 and Slutsky's theorem, we have

$$\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{Y}_t'] \right) a \right| = O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1) \quad (149)$$

Note also that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} E [\|\underline{Y}_t\|_2^2] &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E [\|\underline{Y}_t\|_2^6] \right)^{\frac{1}{3}} \quad (\text{by Liapunov's inequality}) \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} (\overline{C})^{\frac{1}{3}} \quad (\text{by Lemma C-5}) \\
&= (\overline{C})^{\frac{1}{3}}. \quad (150)
\end{aligned}$$

It follows by applying expressions (148), (149), and (150) as well as the result given in part (g) of

Lemma D-16 and the Slutsky' theorem that

$$\begin{aligned}
& \left| \frac{a' \underline{Y}' \left(\hat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
& \leq \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\underline{Y}_t \underline{Y}_t'] \right) a \right|} \right. \\
& \quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \left. \right\} \\
& \quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} E [\|\underline{Y}_t\|_2^2]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\hat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
& = o_p(1).
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{\underline{Y}' \left(\hat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1).$$

To show part (f), note that for any $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we can write

$$\begin{aligned}
& \left| \frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F}Q \right) Q^{-1} B_2 b}{T_h} \right| \\
&= \left| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b \right| \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left| \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b \right| \quad (\text{by triangle inequality}) \\
&\leq \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \quad (\text{by Liapunov's inequality}) \\
&\leq \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \quad (\text{since for a symmetric} \\
&\quad \text{psd matrix } A, \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} = \lambda_{\max}(A)) \\
&= \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \\
&\quad \left(\text{since } \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) = \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right)
\end{aligned}$$

It follows by applying expression (148), the result given in part (g) of Lemma D-16, and the Slutsky' theorem that

$$\begin{aligned}
\left| \frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F}Q \right) Q^{-1} B_2 b}{T_h} \right| &\leq \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \\
&= o_p(1).
\end{aligned}$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$\frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F}Q \right) Q^{-1} B_2}{T_h} = o_p(1).$$

For part (g), note that, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we have,

by direct calculation,

$$\begin{aligned}
& \frac{a' \widehat{\underline{F}}' \mathfrak{H} b}{T_h} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \widehat{\underline{F}}_t \eta'_{t+h} b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t + Q' \underline{F}_t \right) \eta'_{t+h} b \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \eta'_{t+h} b + \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' \underline{F}_t \eta'_{t+h} b
\end{aligned}$$

Focusing first on the first term on last line above, we note that

$$\begin{aligned}
& \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \eta'_{t+h} b \right| \\
&\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{b' \frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} b} \\
&\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' a} \\
&\quad \times \sqrt{\left| b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\eta_{t+h} \eta'_{t+h}] \right) b \right| + \left| \frac{1}{T_h} \sum_{t=p}^{T-h} b' E [\eta_{t+h} \eta'_{t+h}] b \right|} \\
&\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\left| b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\eta_{t+h} \eta'_{t+h}] \right) b \right|} \\
&\quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' E [\eta_{t+h} \eta'_{t+h}] b} \\
&\quad \text{(by the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0)
\end{aligned}$$

Note that, by part (g) of Lemma D-16, we have

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' = o_p(1).$$

and, by part (h) of Lemma D-2,

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\eta_{t+h} \eta'_{t+h}] = O_p \left(\frac{1}{\sqrt{T}} \right).$$

Moreover, note that

$$\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}$$

and, using expression (84) given in the proof of part (e) of Lemma D-2 and Assumption 3-2(b), we see that there exists a positive constant C^* such that

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=p}^{T-h} b' E [\eta_{t+h} \eta'_{t+h}] b \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[(b' \eta_{t+h})^2 \right] \\ &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E \left[(b' \eta_{t+h})^4 \right] \right)^{\frac{1}{2}} \\ &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} (C^*)^{\frac{1}{2}} \\ &\quad \text{(for some positive constant } C^* \text{ as shown in expression (84))} \\ &\leq (C^*)^{\frac{1}{2}} < \infty \end{aligned}$$

Making use of these calculations and applying Slutsky's theorem, we deduce that

$$\begin{aligned} & \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' (\hat{F}_t - Q' \underline{F}_t) \eta'_{t+h} b \right| \\ &\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' (\hat{F}_t - Q' \underline{F}_t) (\hat{F}_t - Q' \underline{F}_t)' a} \sqrt{\left| b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \eta_{t+h} \eta'_{t+h} - \frac{1}{T_h} \sum_{t=p}^{T-h} E [\eta_{t+h} \eta'_{t+h}] \right) b \right|} \\ &\quad + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' (\hat{F}_t - Q' \underline{F}_t) (\hat{F}_t - Q' \underline{F}_t)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' E [\eta_{t+h} \eta'_{t+h}] b} \\ &= o_p(1). \end{aligned}$$

Next, note that, by part (f) of Lemma D-2 and Slutsky's theorem, we see that

$$\begin{aligned} \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' \underline{E}_t \eta'_{t+h} b &= a' Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{E}_t \eta'_{t+h} \right) b \\ &= O_p \left(\frac{1}{\sqrt{T}} \right) = o_p(1) \end{aligned}$$

Putting everything together and applying Slutsky's theorem once more, we then obtain

$$\begin{aligned} \frac{a' \widehat{\underline{F}}' \mathfrak{H} b}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \widehat{\underline{F}}_t \eta'_{t+h} b \\ &= \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q' \underline{E}_t \right) \eta'_{t+h} b + \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' \underline{E}_t \eta'_{t+h} b \\ &= o_p(1). \end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\widehat{\underline{F}}' \mathfrak{H}}{T_h} = \frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{\underline{F}}_t \eta'_{t+h} = o_p(1). \quad \square$$

Lemma D-18: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then,

$$\begin{pmatrix} \widehat{\beta}'_0 - \beta'_0 \\ \widehat{B}_1 - B_1 \\ \widehat{B}_2 - Q^{-1} B_2 \end{pmatrix} = o_p(1).$$

Here, $\widehat{\beta}_0$, \widehat{B}_1 , and \widehat{B}_2 denote the OLS estimators of the coefficient parameters in the (feasible) h -step ahead forecast equation

$$\begin{aligned} Y_{t+h} &= \beta_0 + \sum_{g=1}^p B'_{1,g} Y_{t-g+1} + \sum_{g=1}^p B'_{2,g} \widehat{F}_{t-g+1} + \widehat{\eta}_{t+h} \\ &= \beta_0 + B'_1 \underline{Y}_t + B'_2 \widehat{\underline{F}}_t + \widehat{\eta}_{t+h}, \end{aligned}$$

for $t = p, \dots, T-h$, where the unobserved factor vector \underline{E}_t is replaced by the estimate $\widehat{\underline{F}}_t$ and where $\widehat{\eta}_{t+h} = \eta_{t+h} - B'_2 \left(\widehat{\underline{F}}_t - \underline{E}_t \right)$ with $\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}$ as previously defined.

Proof of Lemma D-18: To proceed, we first stack the observations to obtain the representation

$$\underset{T_h \times d}{Y(h)} = \underset{T_h \times 11 \times d}{\iota_{T_h} \beta'_0} + \underset{T_h \times dp \times dp \times d}{\underline{Y} B_1} + \underset{T_h \times Kp \times Kp \times d}{\widehat{\underline{F}} B_2} + \underset{T_h \times d}{\widehat{\mathfrak{H}}} \quad (151)$$

where $T_h = T - h - p + 1$ and where

$$\underset{T_h \times d}{Y(h)} = \begin{pmatrix} Y'_{h+p} \\ \vdots \\ Y'_T \end{pmatrix}, \underset{T_h \times dp}{\underline{Y}} = \begin{pmatrix} \underline{Y}'_p \\ \vdots \\ \underline{Y}'_{T-h} \end{pmatrix}, \underset{T_h \times Kp}{\widehat{\underline{F}}} = \begin{pmatrix} \widehat{\underline{F}}'_p \\ \vdots \\ \widehat{\underline{F}}'_{T-h} \end{pmatrix}, \text{ and } \underset{T_h \times d}{\widehat{\mathfrak{H}}} = \begin{pmatrix} \widehat{\eta}'_{h+p} \\ \vdots \\ \widehat{\eta}'_T \end{pmatrix}.$$

It is easily seen from expression (151) that the OLS estimators of the coefficients β_0 , B_1 , and B_2 are given by

$$\begin{pmatrix} \widehat{\beta}_0 \\ \widehat{B}_1 \\ \widehat{B}_2 \end{pmatrix} = \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \widehat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \widehat{\underline{F}} \\ \widehat{\underline{F}}' \iota_{T_h} & \widehat{\underline{F}}' \underline{Y} & \widehat{\underline{F}}' \widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} Y(h) \\ \underline{Y}' Y(h) \\ \widehat{\underline{F}}' Y(h) \end{bmatrix}.$$

Now, rewrite expression (151) as

$$\begin{aligned} Y(h) &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + \widehat{\underline{F}} B_2 + \widehat{\mathfrak{H}} \\ &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + \widehat{\underline{F}} B_2 + \mathfrak{H} - (\widehat{\underline{F}} - \underline{F}) B_2 \\ &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + \underline{F} B_2 + \mathfrak{H} \\ &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + \underline{F} Q Q^{-1} B_2 + \mathfrak{H} \\ &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + (\widehat{\underline{F}} + \underline{F} Q - \widehat{\underline{F}}) Q^{-1} B_2 + \mathfrak{H} \\ &= \iota_{T_h} \beta'_0 + \underline{Y} B_1 + \widehat{\underline{F}} Q^{-1} B_2 - (\widehat{\underline{F}} - \underline{F} Q) Q^{-1} B_2 + \mathfrak{H} \\ &= \begin{bmatrix} \iota_{T_h} & \underline{Y} & \widehat{\underline{F}} \end{bmatrix} \begin{pmatrix} \beta'_0 \\ B_1 \\ Q^{-1} B_2 \end{pmatrix} - (\widehat{\underline{F}} - \underline{F} Q) Q^{-1} B_2 + \mathfrak{H}, \end{aligned}$$

and it follows that

$$\begin{aligned}
& \begin{pmatrix} \hat{\beta}'_0 - \beta'_0 \\ \hat{B}_1 - B_1 \\ \hat{B}_2 - Q^{-1}B_2 \end{pmatrix} \\
&= \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \\ \underline{Y}' \\ \hat{\underline{F}}' \end{bmatrix} Y(h) - \begin{pmatrix} \beta'_0 \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
&= \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \\ \underline{Y}' \\ \hat{\underline{F}}' \end{bmatrix} \begin{bmatrix} \iota_{T_h} & \underline{Y} & \hat{\underline{F}} \end{bmatrix} \begin{pmatrix} \beta'_0 \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} - \begin{pmatrix} \beta'_0 \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \\ \underline{Y}' \\ \hat{\underline{F}}' \end{bmatrix} \left(\hat{\underline{F}} - \underline{F}Q \right) Q^{-1}B_2 \\
&\quad + \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \\ \underline{Y}' \\ \hat{\underline{F}}' \end{bmatrix} \mathfrak{H}. \\
&= - \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \left(\hat{\underline{F}} - \underline{F}Q \right) Q^{-1}B_2 \\ \underline{Y}' \left(\hat{\underline{F}} - \underline{F}Q \right) Q^{-1}B_2 \\ \hat{\underline{F}}' \left(\hat{\underline{F}} - \underline{F}Q \right) Q^{-1}B_2 \end{bmatrix} \\
&\quad + \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{\underline{F}} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{\underline{F}} \\ \hat{\underline{F}}' \iota_{T_h} & \hat{\underline{F}}' \underline{Y} & \hat{\underline{F}}' \hat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \mathfrak{H} \\ \underline{Y}' \mathfrak{H} \\ \hat{\underline{F}}' \mathfrak{H} \end{bmatrix}
\end{aligned}$$

Next, applying parts (b) and (d) of Lemma D-2 and parts (a), (b), (c), and (d) of Lemma D-17,

we obtain

$$\begin{aligned}
& \begin{pmatrix} 1 & \iota'_{T_h} \underline{Y}/T_h & \iota'_{T_h} \widehat{\underline{F}}/T_h \\ \underline{Y}' \iota_{T_h}/T_h & \underline{Y}' \underline{Y}/T_h & \underline{Y}' \widehat{\underline{F}}/T_h \\ \widehat{\underline{F}}' \iota_{T_h}/T_h & \widehat{\underline{F}}' \underline{Y}/T_h & \widehat{\underline{F}}' \widehat{\underline{F}}/T_h \end{pmatrix} \\
& - \begin{pmatrix} 1 & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{F}'_t] Q \\ T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{F}'_t] Q \\ T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t] & T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t \underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t \underline{F}'_t] Q \end{pmatrix} \\
& = o_p(1).
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
& \begin{pmatrix} 1 & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{F}'_t] Q \\ T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} E[\underline{Y}_t \underline{F}'_t] Q \\ T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t] & T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t \underline{Y}'_t] & T_h^{-1} \sum_{t=p}^{T-h} Q' E[\underline{F}_t \underline{F}'_t] Q \end{pmatrix} \\
& = \frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E[\underline{Y}'_t] & E[\underline{F}'_t] Q \\ E[\underline{Y}_t] & E[\underline{Y}_t \underline{Y}'_t] & E[\underline{Y}_t \underline{F}'_t] Q \\ Q' E[\underline{F}_t] & Q' E[\underline{F}_t \underline{Y}'_t] & Q' E[\underline{F}_t \underline{F}'_t] Q \end{pmatrix} \\
& = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q' \end{pmatrix} \frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E[\underline{Y}'_t] & E[\underline{F}'_t] \\ E[\underline{Y}_t] & E[\underline{Y}_t \underline{Y}'_t] & E[\underline{Y}_t \underline{F}'_t] \\ E[\underline{F}_t] & E[\underline{F}_t \underline{Y}'_t] & E[\underline{F}_t \underline{F}'_t] \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q \end{pmatrix}
\end{aligned}$$

which is non-singular and, therefore, also positive definite for all T sufficiently large in light of the result given in part (b) of Lemma D-1.

In addition, applying parts (f) and (g) of Lemma D-2 and parts (d), (e), (f), and (g) of Lemma D-17, we have

$$\begin{aligned}
\frac{\iota'_{T_h} \mathfrak{H}}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} \eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{\underline{Y}' \mathfrak{H}}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\iota'_{T_h} (\hat{F} - FQ) Q^{-1} B_2}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} (\hat{F}'_t - F'_t Q) Q^{-1} B_2 = o_p(1), \\
\frac{\underline{Y}' (\hat{F} - FQ) Q^{-1} B_2}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t (\hat{F}'_t - F'_t Q) Q^{-1} B_2 = o_p(1), \\
\frac{\hat{F}' (\hat{F} - FQ) Q^{-1} B_2}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} \hat{F}_t (\hat{F}'_t - F'_t Q) Q^{-1} B_2 = o_p(1), \\
\frac{\hat{F}' \mathfrak{H}}{T_h} &= \frac{1}{T_h} \sum_{t=p}^{T-h} \hat{F}_t \eta'_{t+h} = o_p(1)
\end{aligned}$$

Putting everything together and applying the Slutsky's theorem

$$\begin{aligned}
&\begin{pmatrix} \hat{\beta}'_0 - \beta'_0 \\ \hat{B}_1 - B_1 \\ \hat{B}_2 - Q^{-1} B_2 \end{pmatrix} \\
&= - \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{F} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{F} \\ \hat{F}' \iota_{T_h} & \hat{F}' \underline{Y} & \hat{F}' \hat{F} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} (\hat{F} - FQ) Q^{-1} B_2 \\ \underline{Y}' (\hat{F} - FQ) Q^{-1} B_2 \\ \hat{F}' (\hat{F} - FQ) Q^{-1} B_2 \end{bmatrix} \\
&\quad + \begin{pmatrix} T_h & \iota'_{T_h} \underline{Y} & \iota'_{T_h} \hat{F} \\ \underline{Y}' \iota_{T_h} & \underline{Y}' \underline{Y} & \underline{Y}' \hat{F} \\ \hat{F}' \iota_{T_h} & \hat{F}' \underline{Y} & \hat{F}' \hat{F} \end{pmatrix}^{-1} \begin{bmatrix} \iota'_{T_h} \mathfrak{H} \\ \underline{Y}' \mathfrak{H} \\ \hat{F}' \mathfrak{H} \end{bmatrix} \\
&= - \begin{pmatrix} 1 & T_h^{-1} \iota'_{T_h} \underline{Y} & T_h^{-1} \iota'_{T_h} \hat{F} \\ T_h^{-1} \underline{Y}' \iota_{T_h} & T_h^{-1} \underline{Y}' \underline{Y} & T_h^{-1} \underline{Y}' \hat{F} \\ T_h^{-1} \hat{F}' \iota_{T_h} & T_h^{-1} \hat{F}' \underline{Y} & T_h^{-1} \hat{F}' \hat{F} \end{pmatrix}^{-1} \begin{bmatrix} T_h^{-1} \iota'_{T_h} (\hat{F} - FQ) Q^{-1} B_2 \\ T_h^{-1} \underline{Y}' (\hat{F} - FQ) Q^{-1} B_2 \\ T_h^{-1} \hat{F}' (\hat{F} - FQ) Q^{-1} B_2 \end{bmatrix} \\
&\quad + \begin{pmatrix} 1 & T_h^{-1} \iota'_{T_h} \underline{Y} & T_h^{-1} \iota'_{T_h} \hat{F} \\ T_h^{-1} \underline{Y}' \iota_{T_h} & T_h^{-1} \underline{Y}' \underline{Y} & T_h^{-1} \underline{Y}' \hat{F} \\ T_h^{-1} \hat{F}' \iota_{T_h} & T_h^{-1} \hat{F}' \underline{Y} & T_h^{-1} \hat{F}' \hat{F} \end{pmatrix}^{-1} \begin{bmatrix} T_h^{-1} \iota'_{T_h} \mathfrak{H} \\ T_h^{-1} \underline{Y}' \mathfrak{H} \\ T_h^{-1} \hat{F}' \mathfrak{H} \end{bmatrix} \\
&= o_p(1). \quad \square
\end{aligned}$$

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