

# Online Appendix: Selecting the Relevant Variables for Factor Estimation in FAVAR Models, With An Application to Forecasting the Yield Curve\*

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July 18, 2023

## Abstract

This Online Appendix contains additional supporting lemmas whose results are used in the proofs of Theorems 1 and 2 of the main paper as well as Lemmas A1-A2 of that paper.

*Keywords:* Factor analysis, forecasting, variable selection.

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## Additional Supporting Lemmas and Their Proofs

In this Online Appendix, we state and prove a number of additional supporting lemmas. The results given by these lemmas are used to prove Theorems 1 and 2 as well as Lemmas A1-A2 of the main paper and, thus, help to deliver the main results of the paper.

**Lemma OA-1:** Let  $a$  and  $\theta$  be real numbers such that  $a > 0$  and  $\theta \geq 1$ . Also, let  $G$  be a finite non-negative integer. Then,

$$\sum_{m=1}^{\infty} m^G \exp\{-am^\theta\} < \infty$$

**Proof of Lemma OA-1:** By the integral test,

$$\sum_{m=1}^{\infty} m^G \exp\{-am^\theta\} < \infty \text{ for finite non-negative integer } G$$

if

$$\int_1^{\infty} x^G \exp\{-ax^\theta\} dx < \infty \text{ for finite non-negative integer } G$$

In addition, note that since, by assumption,  $a > 0$  and  $\theta \geq 1$ , we have

$$\int_1^{\infty} x^G \exp\{-ax^\theta\} dx \leq \int_1^{\infty} x^G \exp\{-ax\} dx$$

We will first consider the case where  $G = 0$ . In this case, note that

$$\int_1^{\infty} x^0 \exp\{-ax\} dx = \int_1^{\infty} \exp\{-ax\} dx$$

Let  $u = -ax$ , so that  $-\frac{du}{a} = dx$ ; and we have

$$\begin{aligned} \int_1^{\infty} \exp\{-ax\} dx &= -\frac{1}{a} \int_{-a}^{-\infty} \exp\{u\} du \\ &= \frac{1}{a} \int_{-\infty}^{-a} \exp\{u\} du \\ &= \frac{\exp\{-a\}}{a} \\ &< \infty \text{ for any } a > 0. \end{aligned} \tag{1}$$

Next, consider the case where  $G$  is an integer such that  $G \geq 1$ . Here, we will show that

$$\int_1^{\infty} x^G \exp\{-ax\} dx = \left[ \frac{1}{a} + \sum_{k=1}^G \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{G-j}{a} \right) \right] \exp\{-a\} < \infty$$

using mathematical induction. To proceed, first consider the case where  $G = 1$ . Let

$$\begin{aligned} u &= x, \quad du = dx \\ dv &= \exp\{-ax\} dx, \quad v = -\frac{1}{a} \exp\{-ax\}; \end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x \exp \{-ax\} dx &= -\frac{x}{a} \exp \{-ax\} \Big|_1^\infty + \int_1^\infty \frac{1}{a} \exp \{-ax\} dx \\
&= \frac{1}{a} \exp \{-a\} - \frac{1}{a^2} \exp \{-ax\} \Big|_1^\infty \\
&= \frac{1}{a} \exp \{-a\} + \frac{1}{a^2} \exp \{-a\} \\
&= \left( \frac{1}{a} + \frac{1}{a^2} \right) \exp \{-a\} \\
&= \left\{ \frac{1}{a} + \sum_{k=1}^1 \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{1-j}{a} \right) \right\} \exp \{-a\} < \infty
\end{aligned}$$

Next, for  $G = 2$ , let

$$\begin{aligned}
u &= x^2, \quad du = 2x dx \\
dv &= \exp \{-ax\} dx, \quad v = -\frac{1}{a} \exp \{-ax\};
\end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned}
\int_1^\infty x^2 \exp \{-ax\} dx &= -\frac{x^2}{a} \exp \{-ax\} \Big|_1^\infty + \frac{2}{a} \int_1^\infty x \exp \{-ax\} dx \\
&= \frac{1}{a} \exp \{-a\} + \frac{2}{a} \left( \frac{1}{a} + \frac{1}{a^2} \right) \exp \{-a\} \\
&= \frac{1}{a} \exp \{-a\} + 2 \left( \frac{1}{a^2} + \frac{1}{a^3} \right) \exp \{-a\} \\
&= \left( \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} \right) \exp \{-a\} \\
&= \left[ \frac{1}{a} + \sum_{k=1}^2 \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{2-j}{a} \right) \right] \exp \{-a\} \\
&< \infty
\end{aligned}$$

Now, suppose that, for some  $G \geq 2$ ,

$$\int_1^\infty x^{G-1} \exp \{-ax\} dx = \left[ \frac{1}{a} + \sum_{k=1}^{G-1} \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{G-1-j}{a} \right) \right] \exp \{-a\};$$

then, let

$$\begin{aligned}
u &= x^G, \quad du = Gx^{G-1} dx \\
dv &= \exp \{-ax\} dx, \quad v = -\frac{1}{a} \exp \{-ax\};
\end{aligned}$$

and, using integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x^G \exp \{-ax\} dx &= -\frac{x^G}{a} \exp \{-ax\} \Big|_1^\infty + \frac{G}{a} \int_1^\infty x^{G-1} \exp \{-ax\} dx \\
&= \frac{1}{a} \exp \{-a\} + \frac{G}{a} \left[ \frac{1}{a} + \sum_{k=1}^{G-1} \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{G-1-j}{a} \right) \right] \exp \{-a\} \\
&= \frac{1}{a} \exp \{-a\} + \left[ \frac{G}{a^2} + \sum_{k=1}^{G-1} \frac{1}{a} \frac{G}{a} \left( \prod_{j=0}^{k-1} \frac{G-(j+1)}{a} \right) \right] \exp \{-a\} \\
&= \left\{ \frac{1}{a} + \frac{G}{a^2} + \frac{1}{a} \frac{G}{a} \left( \frac{G-1}{a} \right) + \frac{1}{a} \frac{G}{a} \left( \frac{G-1}{a} \right) \left( \frac{G-2}{a} \right) \right. \\
&\quad \left. + \cdots + \frac{1}{a} \frac{G}{a} \left( \frac{G-1}{a} \right) \left( \frac{G-2}{a} \right) \times \cdots \times \left( \frac{1}{a} \right) \right\} \exp \{-a\} \\
&= \left\{ \frac{1}{a} + \sum_{k=1}^G \frac{1}{a} \left( \prod_{j=0}^{k-1} \frac{G-j}{a} \right) \right\} \exp \{-a\} \\
&< \infty. \tag{2}
\end{aligned}$$

In view of expressions (1) and (2), it then follows by the integral test for series convergence that

$$\sum_{m=1}^\infty m^G \exp \{-am^\theta\} < \infty$$

for any finite non-negative integer  $G$  and for any constants  $a$  and  $\theta$  such that  $a > 0$  and  $\theta \geq 1$ .  $\square$

**Lemma OA-2:** Let  $\{V_t\}$  be a sequence of random variables (or random vectors) defined on some probability space  $(\Omega, \mathcal{F}, P)$ , and let

$$X_t = g(V_t, V_{t-1}, \dots, V_{t-\varkappa})$$

be a measurable function for some finite positive integer  $\varkappa$ . In addition, define  $\mathcal{G}_{-\infty}^t = \sigma(\dots, X_{t-1}, X_t)$ ,  $\mathcal{G}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, \dots)$ ,  $\mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-1}, V_t)$ , and  $\mathcal{F}_{t+m-\varkappa}^\infty = \sigma(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, \dots)$ . Under this setting, the following results hold.

(a) Let

$$\begin{aligned}
\beta_{V,m-\varkappa} &= \sup_t \beta(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty) = \sup_t E [\sup \{|P(B|\mathcal{F}_{-\infty}^t) - P(B)| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\}], \\
\beta_{X,m} &= \sup_t \beta(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty) = \sup_t E [\sup \{|P(H|\mathcal{G}_{-\infty}^t) - P(H)| : H \in \mathcal{G}_{t+m}^\infty\}].
\end{aligned}$$

If  $\{V_t\}$  is  $\beta$ -mixing with

$$\beta_{V,m-\varkappa} \leq \bar{C}_1 \exp \{-C_2(m-\varkappa)\}$$

for all  $m \geq \varkappa$  and for some positive constants  $\bar{C}_1$  and  $C_2$ ; then  $X_t$  is also  $\beta$ -mixing with  $\beta$ -mixing coefficient satisfying

$$\beta_{X,m} \leq C_1 \exp \{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where  $C_1$  is a positive constant such that  $C_1 \geq \bar{C}_1 \exp \{C_2 \varkappa\}$ .

(b) Let

$$\begin{aligned}\alpha_{V,m-\varkappa} &= \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty) = \sup_t \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m-\varkappa}^\infty} |P(G \cap H) - P(G)P(H)|, \\ \alpha_{X,m} &= \sup_t \alpha(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty) = \sup_t \sup_{G \in \mathcal{G}_{-\infty}^t, H \in \mathcal{G}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)|\end{aligned}$$

If  $\{V_t\}$  is  $\alpha$ -mixing with

$$\alpha_{V,m-\varkappa} \leq \bar{C}_1 \exp\{-C_2(m-\varkappa)\}$$

for all  $m \geq \varkappa$  and for some positive constants  $\bar{C}_1$  and  $C_2$ ; then  $X_t$  is also  $\alpha$ -mixing with  $\alpha$ -mixing coefficient satisfying

$$\alpha_{X,m} \leq C_1 \exp\{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where  $C_1$  is a positive constant such that  $C_1 \geq \bar{C}_1 \exp\{C_2 \varkappa\}$ .

### Proof of Lemma OA-2:

To show part (a), note first that it is well known that

$$\begin{aligned}\beta_{X,m} &= \sup_t E[\sup\{|P(H|\mathcal{G}_{-\infty}^t) - P(H)| : H \in \mathcal{G}_{t+m}^\infty\}] \\ &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(G_i \cap H_j) - P(G_i)P(H_j)| \right\}\end{aligned}$$

where the second supremum on the last line above is taken over all pairs of finite partitions  $\{G_1, \dots, G_I\}$  and  $\{H_1, \dots, H_J\}$  of  $\Omega$  such that  $G_i \in \mathcal{G}_{-\infty}^t$  for  $i = 1, \dots, I$  and  $H_j \in \mathcal{G}_{t+m}^\infty$  for  $j = 1, \dots, J$ . See, for example, Borovkova, Burton, and Dehling (2001). Similarly,

$$\begin{aligned}\beta_{V,m-\varkappa} &= \sup_t E[\sup\{|P(B|\mathcal{F}_{-\infty}^t) - P(B)| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\}] \\ &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^L \sum_{j=1}^M |P(A_i \cap B_j) - P(A_i)P(B_j)| \right\}\end{aligned}$$

where, similar to the definition of  $\beta_{X,m}$ , the second supremum on the last line above is taken over all pairs of finite partitions  $\{A_1, \dots, A_L\}$  and  $\{B_1, \dots, B_M\}$  of  $\Omega$  such that  $A_i \in \mathcal{F}_{-\infty}^t$  for  $i = 1, \dots, L$  and  $B_j \in \mathcal{F}_{t+m-\varkappa}^\infty$  for  $j = 1, \dots, M$ . Moreover, since  $X_t$  is measurable on any  $\sigma$ -field on which  $V_t, V_{t-1}, \dots, V_{t-\varkappa}$  are measurable, we also have

$$\mathcal{G}_{-\infty}^t = \sigma(\dots, X_{t-1}, X_t) \subseteq \sigma(\dots, V_{t-1}, V_t) = \mathcal{F}_{-\infty}^t$$

and

$$\mathcal{G}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, \dots) \subseteq \sigma(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, \dots) = \mathcal{F}_{t+m-\varkappa}^\infty.$$

It, thus, follows that, for all  $m \geq \varkappa$ ,

$$\begin{aligned}
\beta_{X,m} &= \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^I \sum_{j=1}^J |P(G_i \cap H_j) - P(G_i)P(H_j)| \right\} \\
&\leq \sup_t \left\{ \frac{1}{2} \sup \sum_{i=1}^L \sum_{j=1}^M |P(A_i \cap B_j) - P(A_i)P(B_j)| \right\} \\
&= \beta_{V,m-\varkappa} \\
&\leq \overline{C}_1 \exp \{-C_2(m-\varkappa)\} \\
&= \overline{C}_1 \exp \{C_2\varkappa\} \exp \{-C_2m\} \\
&\leq C_1 \exp \{-C_2m\}
\end{aligned}$$

for some positive constant  $C_1 \geq \overline{C}_1 \exp \{C_2\varkappa\}$  which exists given that  $\varkappa$  is fixed. Moreover, we have

$$\beta_{X,m} \leq C_1 \exp \{-C_2m\} \rightarrow 0 \text{ as } m \rightarrow \infty,$$

which establishes the required result for part (a).

Part (b) can be shown in a manner similar to part (a), so to avoid redundancy, we do not include an explicit proof here.  $\square$

**Remark:** Note that part (b) of Lemma OA-2 is similar to a result given in Theorem 14.1 of Davidson (1994) but adapted to suit our situation and our notations here. Indeed, parts (a) and (b) of this lemma are both well-known results in the probability literature. We have chosen to state these results explicitly here only so that we can more easily refer to them in the proofs of some of our other results.

**Lemma OA-3:** Let  $\{X_t\}$  be a sequence of random variables that is  $\alpha$ -mixing. Let  $p > 1$  and  $r \geq p/(p-1)$ , and let  $q = \max\{p, r\}$ . Suppose that, for all  $t$ ,

$$\|X_t\|_q = (E|X_t|^q)^{\frac{1}{q}} < \infty$$

Then,

$$|Cov(X_t, X_{t+m})| \leq 2 \left( 2^{1-1/p} + 1 \right) \alpha_m^{1-1/p-1/r} \|X_t\|_p \|X_{t+m}\|_r$$

where

$$\alpha_m = \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) = \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)|.$$

**Remark:** This is Corollary 14.3 of Davidson (1994). For a proof, see pages 212-213 of Davidson (1994).

**Lemma OA-4:** Suppose that Assumption 2-3 hold. Let  $\tau_1 = \lfloor T_0^{\alpha_1} \rfloor$ , where  $1 > \alpha_1 > 0$  and  $T_0 = T - p + 1$ . Then,

(a)

$$\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| = O\left(\frac{1}{\tau_1}\right)$$

(b)

$$\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| = O\left(\frac{1}{\tau_1^2}\right)$$

(c)

$$\frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| = O\left(\frac{1}{\tau_1^2}\right)$$

**Proof of Lemma OA-4:**

To show part (a), first write

$$\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| = \frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] + \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \quad (3)$$

Consider now the first term on the right-hand side of expression (3). Note that, trivially, by Assumption 2-3(b), there exists a positive constant  $C$  such that

$$\frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] \leq \frac{C}{\tau_1} = O\left(\frac{1}{\tau_1}\right) \quad (4)$$

For the second term on the right-hand side of expression (3), note that by Assumption 2-3(c),  $\{u_{it}\}_{t=-\infty}^{\infty}$  is  $\beta$ -mixing with  $\beta$  mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}.$$

for every  $i$ . Since  $\alpha_{i,m} \leq \beta_i(m)$ , it follows that  $\{u_{it}\}_{t=-\infty}^{\infty}$  is  $\alpha$ -mixing as well, with  $\alpha$  mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma OA-3 with  $p = 6$  and  $r = 5/4$  to obtain

$$\begin{aligned} & \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\ & \leq \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(h-g)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \end{aligned}$$

Next, by application of Liapunov's inequality, we have that there exists some positive constant  $\bar{C}$  such that

$$\begin{aligned} \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}} & \leq \left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \\ & \leq \left(\sup_t E|u_{it}|^6\right)^{\frac{1}{3}} \\ & = \bar{C}^{\frac{1}{3}} < \infty \quad (\text{by Assumption 2-3(b)}) \end{aligned}$$

Moreover, let  $\varrho = h-g$ , so that  $h = g+\varrho$ . Using these notations and the boundedness of  $\left(E|u_{ig}|^6\right)^{\frac{1}{6}} \left(E|u_{ih}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$

as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\
& \leq \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{1-\frac{1}{6}} + 1 \right) [a_1 \exp \{-a_2(h-g)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left( E|u_{ig}|^6 \right)^{\frac{1}{6}} \left( E|u_{ih}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \\
& \leq \frac{C^{\frac{1}{3}}}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{5}{6}} + 1 \right) [a_1 \exp \{-a_2(h-g)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{30}\varrho \right\} \\
& \quad \left( \text{for some constant } C^* \text{ such that } 2 \left( 2^{\frac{5}{6}} + 1 \right) \overline{C}^{\frac{1}{3}} a_1^{\frac{1}{30}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho=1}^{\infty} \exp \left\{ -\frac{a_2}{30}\varrho \right\} \\
& = \frac{C^*}{\tau_1} \sum_{\varrho=1}^{\infty} \exp \left\{ -\frac{a_2}{30}\varrho \right\} \\
& = O\left(\frac{1}{\tau_1}\right) \quad (\text{given Lemma OA-1}) \tag{5}
\end{aligned}$$

It follows from expressions (3), (4), and (5) that

$$\begin{aligned}
\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| &= \frac{1}{\tau_1^2} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[u_{ig}^2] + \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g < h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| \\
&= O\left(\frac{1}{\tau_1}\right) + O\left(\frac{1}{\tau_1}\right) \\
&= O\left(\frac{1}{\tau_1}\right).
\end{aligned}$$

To show part (b), first write

$$\begin{aligned}
\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| &= \frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E|u_{ih}|^3 + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&\quad + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \tag{6}
\end{aligned}$$

For the first term on the right-hand side of expression (6) above, note that, trivially, we can apply Assumption

2-3(b) to obtain

$$\frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E |u_{ih}|^3 \leq \frac{C}{\tau_1^2} = O\left(\frac{1}{\tau_1^2}\right). \quad (7)$$

Next, for the second term on the right-hand side of expression (6) above, we can apply Lemma OA-3 with  $p = 6$  and  $r = 5/4$  to obtain

$$\begin{aligned} & \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h>w-v, v-h>0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\ & \leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h>w-v, v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \end{aligned}$$

Next, by application of Hölder's inequality, we have

$$\begin{aligned} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} & \leq \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(\left(E|u_{iv}|^{\frac{5}{2}}\right)^{\frac{1}{2}} \left(E|u_{iw}|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \\ & = \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E|u_{iw}|^{\frac{5}{2}}\right)^{\frac{2}{5}} \\ & \leq \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}|^6\right)^{\frac{1}{6}} \left(E|u_{iw}|^6\right)^{\frac{1}{6}} \\ & \quad (\text{by Liapunov's inequality}) \\ & = \overline{C}^{\frac{1}{2}} < \infty \quad (\text{by Assumption 2-3(b)}) \end{aligned}$$

Moreover, let  $\varrho_1 = v - h$  and  $\varrho_2 = w - v$ , so that  $v = h + \varrho_1$  and  $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$ . Using these

notations and the boundedness of  $\left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}}$  as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
& \leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{6}-\frac{4}{5}} \left(E|u_{ih}|^6\right)^{\frac{1}{6}} \left(E|u_{iv}u_{iw}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
& \leq \frac{C^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{\frac{5}{6}} + 1\right) [a_1 \exp\{-a_2(v-h)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \quad \left(\text{for some constant } C^* \text{ such that } 2 \left(2^{\frac{5}{6}} + 1\right) \overline{C}^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
& \leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = \frac{C^*}{\tau_1^2} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma OA-1}) \tag{8}
\end{aligned}$$

Similarly, for the third term on the right-hand side of expression (6), we can apply Lemma OA-3 with  $p = 6$  and  $r = 5/4$  to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
& \leq \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left(2^{1-\frac{1}{6}} + 1\right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \left(E|u_{ih}u_{iv}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E|u_{iw}|^6\right)^{\frac{1}{6}}
\end{aligned}$$

Next, by applying Hölder's inequality, we have

$$\begin{aligned}
\left( E |u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} &\leq \left( \left( E |u_{ih}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \left( E |u_{iv}|^{\frac{5}{2}} \right)^{\frac{1}{2}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \\
&= \left( E |u_{ih}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \left( E |u_{iv}|^{\frac{5}{2}} \right)^{\frac{2}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \\
&\leq \left( E |u_{ih}|^6 \right)^{\frac{1}{6}} \left( E |u_{iv}|^6 \right)^{\frac{1}{6}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \\
&\quad (\text{by Liapunov's inequality}) \\
&= \overline{C}^{\frac{1}{2}} < \infty \quad (\text{by Assumption 2-3(b)})
\end{aligned}$$

Moreover, let  $\varrho_1 = v - h$  and  $\varrho_2 = w - v$ , so that  $v = h + \varrho_1$  and  $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$ . Using these notations and the boundedness of  $\left( E |u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}}$  as shown above, we can further write

$$\begin{aligned}
&\frac{1}{\tau_1^3} \sum_{\substack{h, v, w = (r-1)\tau + p \\ h \leq v \leq w \\ w - v \geq v - h, w - v > 0}}^{(r-1)\tau + \tau_1 + p - 1} |E(u_{ih} u_{iv} u_{iw})| \\
&\leq \frac{1}{\tau_1^3} \sum_{\substack{h, v, w = (r-1)\tau + p \\ h \leq v \leq w \\ w - v \geq v - h, w - v > 0}}^{(r-1)\tau + \tau_1 + p - 1} 2 \left( 2^{1 - \frac{1}{6}} + 1 \right) [a_1 \exp \{-a_2 (w - v)\}]^{1 - \frac{4}{5} - \frac{1}{6}} \left( E |u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \\
&\leq \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{h, v, w = (r-1)\tau + p \\ h \leq v \leq w \\ w - v \geq v - h, w - v > 0}}^{(r-1)\tau + \tau_1 + p - 1} 2 \left( 2^{\frac{5}{6}} + 1 \right) [a_1 \exp \{-a_2 (w - v)\}]^{\frac{1}{30}} \\
&\leq \frac{C^*}{\tau_1^3} \sum_{\substack{h, v, w = (r-1)\tau + p \\ h \leq v \leq w \\ w - v \geq v - h, w - v > 0}}^{(r-1)\tau + \tau_1 + p - 1} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
&\quad (\text{for some constant } C^* \text{ such that } 2 \left( 2^{\frac{5}{6}} + 1 \right) \overline{C}^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty) \\
&\leq \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
&= \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} (\varrho_2 + 1) \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \\
&= \frac{C^*}{\tau_1^2} \left[ \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} + \sum_{\varrho_2=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \varrho_2 \right\} \right] \\
&= O \left( \frac{1}{\tau_1^2} \right) \quad (\text{given Lemma OA-1})
\end{aligned} \tag{9}$$

It follows from expressions (6), (7), (8), and (9) that

$$\begin{aligned}
\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| &= \frac{1}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E|u_{ih}|^3 + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ v-h > w-v, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&\quad + \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \\
&= O\left(\frac{1}{\tau_1^2}\right) + O\left(\frac{1}{\tau_1^2}\right) + O\left(\frac{1}{\tau_1^2}\right) \\
&= O\left(\frac{1}{\tau_1^2}\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v>v-h, w-v>0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h>0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
= & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v>v-h, w-v>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih}) + E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih}) + E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
\leq & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v>v-h, w-v>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})|
\end{aligned} \tag{10}$$

For the first term on the right-hand side of expression (10) above, note that, trivially, by Jensen's inequality

and Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| &\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} E[|u_{ig}u_{ih}^3|] \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} \sqrt{E|u_{ig}|^2} \sqrt{E|u_{ih}|^6} \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} (E|u_{ih}|^6)^{\frac{1}{6}} \sqrt{E|u_{ih}|^6} \\
&\quad (\text{by Liapunov's inequality}) \\
&\leq \frac{\bar{C}^{\frac{2}{3}}\tau_1^2}{\tau_1^4} \quad (\text{by Assumption 2-3(b)}) \\
&= O\left(\frac{1}{\tau_1^2}\right)
\end{aligned} \tag{11}$$

Next, for the second term on the right-hand side of expression (10), we can apply Lemma OA-3 with  $p = 4/3$  and  $r = 6$  to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v>v-h, w-v>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
&\leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v>v-h, w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
&\quad \times \left. \left( E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left( E|u_{iw}|^6 \right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$\begin{aligned}
E |\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}|^{\frac{4}{3}} &\leq \left[ E |u_{ig}u_{ih} - E(u_{ig}u_{ih})|^{\frac{12}{7}} \right]^{\frac{7}{9}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\leq \left[ 2^{\frac{5}{7}} \left( E |u_{ig}u_{ih}|^{\frac{12}{7}} + |E[u_{ig}u_{ih}]|^{\frac{12}{7}} \right) \right]^{\frac{7}{9}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\quad (\text{by Loèeve's } c_r \text{ inequality}) \\
&\leq \left[ 2^{\frac{5}{7}} \left( E |u_{ig}u_{ih}|^{\frac{12}{7}} + E |u_{ig}u_{ih}|^{\frac{12}{7}} \right) \right]^{\frac{7}{9}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\quad (\text{by Jensen's inequality}) \\
&= \left[ 2^{\frac{12}{7}} E |u_{ig}u_{ih}|^{\frac{12}{7}} \right]^{\frac{7}{9}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[ \left( E |u_{ig}|^{\frac{24}{7}} \right)^{\frac{1}{2}} \left( E |u_{ih}|^{\frac{24}{7}} \right)^{\frac{1}{2}} \right]^{\frac{7}{9}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}} \left[ \left( E |u_{ig}|^{\frac{24}{7}} \right)^{\frac{7}{24}} \left( E |u_{ih}|^{\frac{24}{7}} \right)^{\frac{7}{24}} \right]^{\frac{4}{3}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} \left[ \left( E |u_{ig}|^6 \right)^{\frac{1}{6}} \left( E |u_{ih}|^6 \right)^{\frac{1}{6}} \right]^{\frac{4}{3}} \left[ E |u_{iv}|^6 \right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}} (\bar{C})^{\frac{2}{9}} (\bar{C})^{\frac{2}{9}} (\bar{C})^{\frac{2}{9}} \quad (\text{by Assumption 2-3(b)}) \\
&= 2^{\frac{4}{3}} \bar{C}^{\frac{2}{3}}
\end{aligned}$$

Moreover, let  $\varrho_1 = v - h$  and  $\varrho_2 = w - v$  so that  $v = h + \varrho_1$  and  $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$ . Using these

notations and the boundedness of  $E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}|^{\frac{4}{3}}$  as shown above, we can further write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{3}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \quad \times \left. \left( E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left( E|u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{4}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{\frac{1}{12}} \left( 2^{\frac{4}{3}} \bar{C}^{\frac{2}{3}} \right)^{\frac{3}{4}} (\bar{C})^{\frac{1}{6}} \\
& \leq \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{12} \varrho_2 \right\} \\
& \quad \left( \text{for some constant } C^* \text{ such that } 4 \left( 2^{\frac{1}{4}} + 1 \right) \bar{C}^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2-1} \exp \left\{ -\frac{a_2}{12} \varrho_2 \right\} \\
& \leq \frac{C^*}{\tau_1^2} \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp \left\{ -\frac{a_2}{12} \varrho_2 \right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma OA-1})
\end{aligned} \tag{12}$$

Similarly, for the third term on the right-hand side of expression (10) above, we can apply Lemma OA-3 with  $p = 2$  and  $r = 3$  to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \times \left. \left( E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 \right)^{\frac{1}{2}} \left( E|u_{iv}u_{iw}|^3 \right)^{\frac{1}{3}} \right\}
\end{aligned}$$

Next, applications of Hölder's inequality yield

$$\begin{aligned} E|u_{iv}u_{iw}|^3 &\leq \left(E|u_{iv}|^6\right)^{\frac{1}{2}}\left(E|u_{iw}|^6\right)^{\frac{1}{2}} \\ &\leq (\bar{C})^{\frac{1}{2}}(\bar{C})^{\frac{1}{2}} \quad (\text{by Assumption 2-3(b)}) \\ &= \bar{C} < \infty \end{aligned}$$

and

$$\begin{aligned} E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 &\leq 2\left(E|u_{ig}u_{ih}|^2 + E|u_{ig}u_{ih}|^2\right) \\ &\quad (\text{by Loève's } c_r \text{ inequality and Jensen's inequality}) \\ &= 4E|u_{ig}u_{ih}|^2 \\ &\leq 4\left[\left(E|u_{ig}|^4\right)^{\frac{1}{4}}\left(E|u_{ih}|^4\right)^{\frac{1}{4}}\right]^2 \\ &\leq 4\left[\left(E|u_{ig}|^6\right)^{\frac{1}{6}}\left(E|u_{ih}|^6\right)^{\frac{1}{6}}\right]^2 \quad (\text{by Liapunov's inequality}) \\ &\leq 4\left(\sup_{i,t} E|u_{it}|^6\right)^{\frac{2}{3}} \\ &\leq 4(\bar{C})^{\frac{2}{3}} < \infty \quad (\text{by Assumption 2-3(b)}) \end{aligned}$$

Moreover, let  $\varrho_1 = v - h$  and  $\varrho_2 = w - v$  so that  $v = h + \varrho_1$  and  $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$ . Using these notations and the boundedness of  $E|u_{iv}u_{iw}|^3$  and  $E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2$  as shown above, we can further

write

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih} - E(u_{ig}u_{ih})]u_{iv}u_{iw}| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp \{-a_2(v-h)\}]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \quad \times \left. \left( E|\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\}|^2 \right)^{\frac{1}{2}} \left( E|u_{iv}u_{iw}|^3 \right)^{\frac{1}{3}} \right\} \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{2}} + 1 \right) [a_1 \exp \{-a_2(v-h)\}]^{\frac{1}{6}} \left( 4\bar{C}^{\frac{2}{3}} \right)^{\frac{1}{2}} (\bar{C})^{\frac{1}{3}} \\
& \leq \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{6} \varrho_1 \right\} \\
& \quad \left( \text{for some constant } C^* \text{ such that } 4 \left( 2^{\frac{1}{2}} + 1 \right) \bar{C}^{\frac{2}{3}} a_1^{\frac{1}{6}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\infty} \exp \left\{ -\frac{a_2}{6} \varrho_1 \right\} \\
& = \frac{C^*}{\tau_1^2} \sum_{\varrho_1=1}^{\infty} (\varrho_1 + 1) \exp \left\{ -\frac{a_2}{6} \varrho_1 \right\} \\
& = O\left(\frac{1}{\tau_1^2}\right) \quad (\text{given Lemma OA-1}) \tag{13}
\end{aligned}$$

Finally, consider the fourth term on the right-hand side of expression (10) above. For this term, we apply the result given in part (a) to obtain

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
& \leq \left( \frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| \right) \left( \frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \right) \\
& = O\left(\frac{1}{\tau_1^2}\right). \tag{14}
\end{aligned}$$

It follows from expressions (10)-(14) that

$$\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| \\
& \leq \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}^3]| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v > v-h, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& \quad + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-v \leq v-h, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{ig}u_{ih} - E(u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
& \quad + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \leq h \leq v \leq w \\ w-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
& = O\left(\frac{1}{\tau_1^2}\right). \square
\end{aligned}$$

**Lemma OA-5:** Suppose that Assumptions 2-1, 2-2(a)-(b), 2-5, and 2-6 hold. Then, there exists a positive constant  $\bar{C}$  such that

$$E \|\underline{W}_t\|_2^6 \leq \bar{C} < \infty \text{ for all } t$$

and, thus,

$$E \|\underline{Y}_t\|_2^6 \leq \bar{C} < \infty \text{ and } E \|\underline{F}_t\|_2^6 \leq \bar{C} < \infty \text{ for all } t,$$

where

$$\underline{Y}_t = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}_{dp \times 1}, \text{ and } \underline{F}_t = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}_{Kp \times 1}.$$

#### Proof of Lemma OA-5:

To proceed, note that, given Assumption 2-1, we can write the vector moving-average (VMA) representation of the companion form of the FAVAR model as

$$\begin{aligned}
\underline{W}_t &= (I_{(d+K)p} - A)^{-1} \alpha + \sum_{j=0}^{\infty} A^j E_{t-j} \\
&= (I_{(d+K)p} - A)^{-1} J'_{d+K} J_{d+K} \alpha + \sum_{j=0}^{\infty} A^j J'_{d+K} J_{d+K} E_{t-j} \\
&= (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j},
\end{aligned} \tag{15}$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

$$J_{d+K}^{(d+K) \times (d+K)p} = [ I_{d+K} \ 0 \ \cdots \ 0 \ 0 ], \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.$$

By the triangle inequality,

$$\|\underline{W}_t\|_2 \leq \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2$$

Moreover, using the inequality  $\left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r$  for  $r \geq 1$ , we get

$$\|\underline{W}_t\|_2^6 \leq 2^5 \left( \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \right)$$

so that

$$E \|\underline{W}_t\|_2^6 \leq 32 \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \quad (16)$$

Focusing first on the first term on the right-hand side of the inequality (16), we note that

$$\begin{aligned} \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 &= \left( \mu' J_{d+K} (I_{(d+K)p} - A)^{-1'} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right)^3 \\ &= \left( \mu' J_{d+K} \left[ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right]^{-1} J'_{d+K} \mu \right)^3 \\ &\leq \left( \frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' J_{d+K} J'_{d+K} \mu)^3 \\ &= \left( \frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' \mu)^3 \end{aligned}$$

Now, by Assumption 2-6, there exists a constant  $\underline{C} > 0$  such that

$$\begin{aligned}\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\} &= \lambda_{\min} \left\{ (I_{(d+K)p} - A)' (I_{(d+K)p} - A) \right\} \\ &= \sigma_{\min}^2 (I_{(d+K)p} - A) \\ &\geq \underline{C} \lambda_{\min}^2 (I_{(d+K)p} - A) \\ &\geq \underline{C} [1 - \phi_{\max}]^2 \\ &> 0\end{aligned}$$

where  $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$  and where  $0 < \phi_{\max} < 1$  since, by Assumption 2-1, all eigenvalues of  $A$  have modulus less than 1. It follows by Assumption 2-5 that, there exists a positive constant  $\overline{C}_1$  such that

$$\begin{aligned}\left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 &\leq \left( \frac{1}{\lambda_{\min} \left\{ (I_{(d+K)p} - A) (I_{(d+K)p} - A)' \right\}} \right)^3 (\mu' \mu)^3 \\ &\leq \frac{\|\mu\|_2^6}{\underline{C}^3 [1 - \phi_{\max}]^6} \leq \overline{C}_1 < \infty.\end{aligned}$$

To show the boundedness of the second term on the right-hand side of the inequality (16), let  $e_{g,(d+K)p}$  be a  $(d+K)p \times 1$  elementary vector whose  $g^{th}$  component is 1 and all other components are 0 for  $g \in \{1, 2, \dots, (d+K)p\}$ , and note that

$$\begin{aligned}\left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^2 &= \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^2 \\ &= \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} (A')^k e_{g,(d+K)p}\end{aligned}$$

from which we obtain, by applying the inequality  $\left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r$  for  $r \geq 1$

$$\begin{aligned}&\left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\ &= \left[ \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^2 \right]^3 \\ &\leq [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^6 \\ &= [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} (A')^k e_{g,(d+K)p} \right. \\ &\quad \left. \times e'_{g,(d+K)p} A^i J'_d \varepsilon_{t-i} \varepsilon'_{t-\ell} J_{d+K} (A')^\ell e_{g,(d+K)p} e'_{g,(d+K)p} A^r J'_{d+K} \varepsilon_{t-r} \varepsilon'_{t-s} J_d (A')^s e_{g,(d+K)p} \right\}\end{aligned}$$

Hence,

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
\leq & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^6 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{3} \left( \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^3 \right)^2 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{2} \binom{4}{2} \left( \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^2 \right)^3 \\
& + [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \binom{6}{4} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^4 \sum_{k=0}^{\infty} E \left| e'_{g,(d+K)p} A^k J'_{d+K} \varepsilon_{t-k} \right|^2 \\
= & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^6 \\
& + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^3 \right)^2 \\
& + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^2 \right)^3 \\
& + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E \left| e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right|^4 \sum_{k=0}^{\infty} E \left| e'_{g,(d+K)p} A^k J'_{d+K} \varepsilon_{t-k} \right|^2
\end{aligned}$$

Next, applying the Cauchy-Schwarz inequality, we further obtain

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
\leq & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j J'_{d+K} J_{d+K} (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \left. \times \sum_{k=0}^{\infty} \left[ e'_{g,(d+K)p} A^k J'_{d+K} J_{d+K} (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\} \\
\leq & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \left. \times \sum_{k=0}^{\infty} \left[ e'_{g,(d+K)p} A^k (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\}
\end{aligned}$$

In addition, observe that, for every  $g \in \{1, 2, \dots, (d + K)p\}$

$$\begin{aligned}
& e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \\
& \leq \lambda_{\max} \left\{ A^j (A^j)' \right\} \\
& = \lambda_{\max} \left\{ (A^j)' A^j \right\} \\
& = \sigma_{\max}^2 (A^j) \\
& \leq C \max \left\{ |\lambda_{\max}(A^j)|^2, |\lambda_{\min}(A^j)|^2 \right\} \quad (\text{by Assumption 2-6}) \\
& = C \max \left\{ |\lambda_{\max}(A)|^{2j}, |\lambda_{\min}(A)|^{2j} \right\} \\
& = C \phi_{\max}^{2j}
\end{aligned}$$

where  $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$  and where  $0 < \phi_{\max} < 1$  given that Assumption 2-1 implies that all eigenvalues of  $A$  have modulus less than 1. Now, in light of Assumption 2-2(b), we can set  $C \geq 1$  to be a constant such that  $E \|\varepsilon_{t-j}\|_2^6 \leq C < \infty$ , so that, by Liapunov's inequality,

$$\begin{aligned}
E \|\varepsilon_{t-j}\|_2^2 & \leq \left( E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{1}{3}} \leq C^{\frac{1}{3}}, \quad E \|\varepsilon_{t-j}\|_2^3 \leq \left( E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}, \\
E \|\varepsilon_{t-j}\|_2^4 & \leq \left( E \|\varepsilon_{t-j}\|_2^6 \right)^{\frac{2}{3}} \leq C^{\frac{2}{3}},
\end{aligned}$$

and, thus,

$$\begin{aligned}
& E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
& \leq [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^3 E \|\varepsilon_{t-j}\|_2^6 \\
& \quad + 20 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^{\frac{3}{2}} E \|\varepsilon_{t-j}\|_2^3 \right)^2 \\
& \quad + 90 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-j}\|_2^2 \right)^3 \\
& \quad + 15 [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \left[ e'_{g,(d+K)p} A^j (A^j)' e_{g,(d+K)p} \right]^2 E \|\varepsilon_{t-j}\|_2^4 \right. \\
& \quad \quad \left. \times \sum_{k=0}^{\infty} \left[ e'_{g,(d+K)p} A^k (A^k)' e_{g,(d+K)p} \right] E \|\varepsilon_{t-k}\|_2^2 \right\} \\
& \leq C [(d+K)p]^2 \left\{ \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \phi_{\max}^{6j} + 20 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \phi_{\max}^{3j} \right)^2 + 90 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \phi_{\max}^{2j} \right)^3 \right. \\
& \quad \left. + 15 \sum_{g=1}^{(d+K)p} \left( \sum_{j=0}^{\infty} \phi_{\max}^{4j} \right) \left( \sum_{k=0}^{\infty} \phi_{\max}^{2k} \right) \right\} \\
& \leq C [(d+K)p]^3 \\
& \quad \times \left\{ \frac{1}{1 - \phi_{\max}^6} + 20 \left( \frac{1}{1 - \phi_{\max}^3} \right)^2 + 90 \left( \frac{1}{1 - \phi_{\max}^2} \right)^3 + 15 \left( \frac{1}{1 - \phi_{\max}^4} \right) \left( \frac{1}{1 - \phi_{\max}^2} \right) \right\} \\
& \leq \bar{C}_2 < \infty
\end{aligned}$$

for some constant such that

$$\begin{aligned}
& \bar{C}_2 \\
& \geq C [(d+K)p]^3 \\
& \quad \times \left\{ \frac{1}{1 - \phi_{\max}^6} + 20 \left( \frac{1}{1 - \phi_{\max}^3} \right)^2 + 90 \left( \frac{1}{1 - \phi_{\max}^2} \right)^3 + 15 \left( \frac{1}{1 - \phi_{\max}^4} \right) \left( \frac{1}{1 - \phi_{\max}^2} \right) \right\}.
\end{aligned}$$

Putting everything together, we see that

$$\begin{aligned}
E \|\underline{W}_t\|_2^6 & \leq 32 \left\| (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2^6 \\
& \leq 32 (\bar{C}_1 + \bar{C}_2) \\
& \leq \bar{C} < \infty
\end{aligned}$$

for a constant  $\overline{C}$  such that  $0 < 32(\overline{C}_1 + \overline{C}_2) \leq \overline{C} < \infty$ .

In addition, define  $\mathcal{P}_{(d+K)p}$  to be the  $(d+K)p \times (d+K)p$  permutation matrix such that

$$\mathcal{P}_{(d+K)p} \underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \\ \vdots \\ \underline{F}_t \end{pmatrix}; \quad (17)$$

and let  $S'_d = \begin{pmatrix} I_{dp} & 0 \\ & dp \times Kp \end{pmatrix}$  and  $S'_K = \begin{pmatrix} 0 & I_{Kp} \\ Kp \times dp & \end{pmatrix}$ . Note that

$$\begin{aligned} S'_d \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & 0 \\ & dp \times Kp \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \\ \vdots \\ \underline{F}_t \end{pmatrix} = \underline{Y}_t, \\ S'_K \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} 0 & I_{Kp} \\ Kp \times dp & \end{pmatrix} \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \\ \vdots \\ \underline{F}_t \end{pmatrix} = \underline{F}_t. \end{aligned}$$

so that

$$\begin{aligned} \|\underline{Y}_t\|_2 &\leq \|S'_d\|_2 \|\mathcal{P}_{(d+K)p}\|_2 \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(S_d S'_d)} \sqrt{\lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(S'_d S_d)} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(I_{dp})} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \|\underline{W}_t\|_2 \end{aligned}$$

and

$$\begin{aligned} \|\underline{F}_t\|_2 &\leq \|S'_K\|_2 \|\mathcal{P}_{(d+K)p}\|_2 \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(S_K S'_K)} \sqrt{\lambda_{\max}(\mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(S'_K S_K)} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \sqrt{\lambda_{\max}(I_{Kp})} \sqrt{\lambda_{\max}(I_{(d+K)p})} \|\underline{W}_t\|_2 \\ &= \|\underline{W}_t\|_2 \end{aligned}$$

It further follows that

$$E \|\underline{Y}_t\|_2^6 \leq E \|\underline{W}_t\|_2^6 \leq \overline{C} < \infty \text{ and } E \|\underline{F}_t\|_2^6 \leq E \|\underline{W}_t\|_2^6 \leq \overline{C} < \infty. \quad \square$$

**Lemma OA-6:** Suppose that Assumptions 2-1, 2-2(a)-(b), 2-3, 2-5, 2-6, and 2-9(b) hold. Then, the following statements are true as  $N_1, T \rightarrow \infty$

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \xrightarrow{p} 0.$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \xrightarrow{p} 0$$

(c)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \xrightarrow{p} 0.$$

(d)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \xrightarrow{p} 0$$

(e)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \xrightarrow{p} 0$$

**Proof of Lemma OA-6.**

To show part (a), first write

$$\begin{aligned} & P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \geq \epsilon \right\} \\ &= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\ &\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\ &\quad (\text{by Jensen's inequality}) \\ &\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^6 \right\} \\ &\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \end{aligned}$$

Next, note that

$$\begin{aligned}
& \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \\
& \leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^6 \\
& \quad + \frac{20}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [|\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}|]^3 E [|\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}|]^3 \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^4 E [\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}]^2 \\
& \quad + \frac{90}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} \left\{ E [\gamma'_i \underline{F}_t \varepsilon_{\ell,t+1}]^2 E [\gamma'_i \underline{F}_s \varepsilon_{\ell,s+1}]^2 \right. \\
& \quad \left. \times E [\gamma'_i \underline{F}_s \varepsilon_{\ell,r+1}]^2 \right\} \\
& \leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [(\gamma'_i \underline{F}_t)^6] E [\varepsilon_{\ell,t+1}^6] \\
& \quad + \frac{20}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \frac{1}{64} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i + \varepsilon_{\ell,t+1}^2]^3 E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i + \varepsilon_{\ell,s+1}^2]^3 \\
& \quad + \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i]^2 E [\varepsilon_{\ell,t+1}^4] E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i] E [\varepsilon_{\ell,s+1}^2] \\
& \quad + \frac{90}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \left\{ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_t \underline{F}'_t \gamma_i] E [\varepsilon_{\ell,t+1}^2] E [\gamma'_i \underline{F}_s \underline{F}'_s \gamma_i] E [\varepsilon_{\ell,s+1}^2] \right. \\
& \quad \left. \times \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} E [\gamma'_i \underline{F}_r \underline{F}'_r \gamma_i] E [\varepsilon_{\ell,r+1}^2] \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^6 E[\|\underline{F}_t\|_2^6] E[\varepsilon_{\ell,t+1}^6] \\
&+ \frac{(20 \cdot 16)}{64q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \left( E[(\gamma'_i \underline{F}_t)^6] + E[\varepsilon_{\ell,t+1}^6] \right) \right. \\
&\quad \times \left. \left( E[(\gamma'_i \underline{F}_s)^6] + E[\varepsilon_{\ell,s+1}^6] \right) \right\} \\
&+ \frac{15}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \|\gamma_i\|_2^4 E[\|\underline{F}_t\|_2^4] E[\varepsilon_{\ell,t+1}^4] \right. \\
&\quad \times \left. \|\gamma_i\|_2^2 E[\|\underline{F}_s\|_2^2] E[\varepsilon_{\ell,s+1}^2] \right\} \\
&+ \frac{90}{q\tau_1^6} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} \left\{ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E[\|\underline{F}_t\|_2^2] E[\varepsilon_{\ell,t+1}^2] \right. \\
&\quad \times \left. \sum_{\substack{s=(r-1)\tau+p \\ s \neq t}}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E[\|\underline{F}_s\|_2^2] E[\varepsilon_{\ell,s+1}^2] \sum_{\substack{r=(r-1)\tau+p \\ r \neq t, r \neq s}}^{(r-1)\tau+\tau_1+p-1} \|\gamma_i\|_2^2 E[\|\underline{F}_r\|_2^2] E[\varepsilon_{\ell,r+1}^2] \right\} \\
&\leq C \left( \frac{N_1}{\tau_1^5} + 5 \frac{N_1}{\tau_1^4} + 15 \frac{N_1}{\tau_1^4} + 90 \frac{N_1}{\tau_1^3} \right) \\
&\quad (\text{applying Assumptions 2-2(b), Assumption 2-5, and Lemma OA-5}) \\
&= O\left(\frac{N_1}{\tau_1^3}\right).
\end{aligned}$$

It follows that

$$P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \geq \epsilon \right\} = O\left(\frac{N_1}{\tau_1^3}\right) = o(1).$$

To show part (b), note that, for any  $\epsilon > 0$

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \right|^3 \geq \epsilon^3 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^3 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6 \geq \epsilon^3 \right\} \\
&\leq \frac{1}{\epsilon^3} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^6
\end{aligned}$$

The rest of the proof for part (b) then follows in a manner similar to the argument given for part (a) above.

To show part (c), first note that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\quad (\text{by convexity or Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^6 \right\} \\
&\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6
\end{aligned} \tag{18}$$

Now, there exists a constant  $C_1 > 1$  such that

$$\begin{aligned} & \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\ & \leq \frac{C_1}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \left\{ \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \right. \\ & \quad \left. \times \sum_{\ell=1}^d |E[y_{\ell,t+1} y_{\ell,s+1} y_{\ell,g+1} y_{\ell,h+1} y_{\ell,v+1} y_{\ell,w+1}]| \right\} \end{aligned}$$

Next, note that, by repeated application of Hölder's inequality, we have by Lemma OA-5 that there exists a positive constant  $\bar{C}$  such that

$$\begin{aligned} & \sum_{\ell=1}^d |E[y_{\ell,t+1} y_{\ell,s+1} y_{\ell,g+1} y_{\ell,h+1} y_{\ell,v+1} y_{\ell,w+1}]| \\ & \leq \sum_{\ell=1}^d (E[y_{\ell,t+1}^2 y_{\ell,s+1}^2 y_{\ell,g+1}^2])^{\frac{1}{2}} (E[y_{\ell,h+1}^2 y_{\ell,v+1}^2 y_{\ell,w+1}^2])^{\frac{1}{2}} \\ & \leq \sum_{\ell=1}^d \left( \{E[y_{\ell,t+1}^6]\}^{\frac{1}{3}} \left( E[|y_{\ell,s+1} y_{\ell,g+1}|^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \left( \{E[y_{\ell,h+1}^6]\}^{\frac{1}{3}} \left( E[|y_{\ell,v+1} y_{\ell,w+1}|^3] \right)^{\frac{2}{3}} \right)^{\frac{1}{2}} \\ & \leq \sum_{\ell=1}^d \left[ \left( \{E[y_{\ell,t+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,s+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,g+1}^6]\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \right. \\ & \quad \left. \times \left( \{E[y_{\ell,h+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,v+1}^6]\}^{\frac{1}{3}} \{E[y_{\ell,w+1}^6]\}^{\frac{1}{3}} \right)^{\frac{1}{2}} \right] \\ & \leq \sum_{\ell=1}^d \{E[y_{\ell,t+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,s+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,g+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,h+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,v+1}^6]\}^{\frac{1}{6}} \{E[y_{\ell,w+1}^6]\}^{\frac{1}{6}} \\ & \leq d \max_{1 \leq \ell \leq d} \sup_t E[y_{\ell,t}^6] \\ & \leq \bar{C} < \infty \\ & \quad \left( \text{since, given that } y_{\ell,t} = \mathbf{e}'_{\ell,d,p} \underline{Y}_t; E[y_{\ell,t}^6] \leq E\|\underline{Y}_t\|_2^6 \leq \bar{C} \text{ by Lemma OA-5} \right. \\ & \quad \left. \text{where } \bar{C} \text{ is a constant not depending on } \ell \text{ or } t \right) \end{aligned}$$

Hence, we can write

$$\begin{aligned}
& \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig}^4]| \\
& + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
& + \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it}u_{is}u_{ig}^4]| \\
&\quad + \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}]| \\
&\quad + \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
&\quad + \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
&\quad + \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\} u_{ih}u_{iv}u_{iw}]| \\
&\quad + \frac{C_1 \bar{C}}{q\tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| |E(u_{ih}u_{iv}u_{iw})| \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6, \quad (\text{say}). \tag{19}
\end{aligned}$$

Consider first  $\mathcal{T}_1$ . Note that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig}^4]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E[|u_{it} u_{is} u_{ig}^4|] \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left( E[|u_{it} u_{is}|^3] \right)^{\frac{1}{3}} \left( E[|u_{ig}|^6] \right)^{\frac{2}{3}} \quad (\text{by Hölder's inequality}) \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left( \left[ E\{|u_{it}|^6\} \right]^{\frac{1}{2}} \left[ E\{|u_{is}|^6\} \right]^{\frac{1}{2}} \right)^{\frac{1}{3}} \left( E[|u_{ig}|^6] \right)^{\frac{2}{3}} \\
&\quad (\text{by further application of Hölder's inequality}) \\
&= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left( E\{|u_{it}|^6\} \right)^{\frac{1}{6}} \left( E\{|u_{it}|^6\} \right)^{\frac{1}{6}} \left( E[|u_{ig}|^6] \right)^{\frac{2}{3}} \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t,s,g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \bar{C} \quad (\text{by Assumption 2-3(b)}) \\
&\leq C_1 \bar{C}^2 \frac{N_1}{\tau_1^5} \\
&= O\left(\frac{N_1}{\tau_1^5}\right). \tag{20}
\end{aligned}$$

Next, consider  $\mathcal{T}_2$ . For this term, note first that by Assumption 2-3(c),  $\{u_{it}\}_{t=-\infty}^\infty$  is  $\beta$ -mixing with  $\beta$  mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every  $i$ . Since  $\alpha_{i,m} \leq \beta_i(m)$ , it follows that  $\{u_{it}\}_{t=-\infty}^\infty$  is  $\alpha$ -mixing as well, with  $\alpha$  mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, we apply Lemma OA-3 with  $p = 5/4$  and  $r = 6$  to obtain

$$\begin{aligned}
& \mathcal{T}_2 \\
&= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} |E[u_{it} u_{is} u_{ig} u_{ih} u_{iv} u_{iw}]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{4}{5}} + 1 \right) [a_1 \exp\{-a_2(w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\
&\quad \left. \times \left( E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by Liapunov's inequality and Assumption 2-3(b), we obtain

$$\left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \leq \left( E |u_{iw}|^7 \right)^{\frac{1}{7}} \leq \bar{C}^{\frac{1}{7}}$$

Making use of this bound and by repeated application of Hölder's inequality, we have

$$\begin{aligned}
& E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \\
&\leq \left[ E |u_{it} u_{is} u_{ig}|^{\frac{25}{12}} \right]^{\frac{3}{5}} \left[ E |u_{ih} u_{iv}|^{\frac{25}{8}} \right]^{\frac{2}{5}} \\
&\leq \left[ \left( E |u_{it} u_{is}|^{\frac{150}{47}} \right)^{\frac{47}{72}} \left( E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[ \left( E |u_{ih}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left( E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
&\leq \left[ \left( \sqrt{E |u_{it}|^{\frac{300}{47}}} \sqrt{E |u_{is}|^{\frac{300}{47}}} \right)^{\frac{47}{72}} \left( E |u_{ig}|^6 \right)^{\frac{25}{72}} \right]^{\frac{3}{5}} \left[ \left( E |u_{ih}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \left( E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{2}} \right]^{\frac{2}{5}} \\
&= \left( E |u_{it}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left( E |u_{is}|^{\frac{300}{47}} \right)^{\frac{141}{720}} \left( E |u_{ih}|^6 \right)^{\frac{15}{72}} \left( E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \left( E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{1}{5}} \\
&= \left[ \left( E |u_{it}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \left( E |u_{is}|^{\frac{300}{47}} \right)^{\frac{47}{300}} \right]^{\frac{5}{4}} \left[ \left( E |u_{ih}|^6 \right)^{\frac{1}{6}} \right]^{\frac{5}{4}} \left[ \left( E |u_{iv}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \left[ \left( E |u_{iw}|^{\frac{25}{4}} \right)^{\frac{4}{25}} \right]^{\frac{5}{4}} \\
&\leq \left[ \left( E |u_{it}|^7 \right)^{\frac{1}{7}} \left( E |u_{is}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[ \left( E |u_{ih}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[ \left( E |u_{iv}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \left[ \left( E |u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{5}{4}} \\
&\leq (\bar{C})^{\frac{5}{28}} (\bar{C})^{\frac{5}{28}} (\bar{C})^{\frac{5}{28}} (\bar{C})^{\frac{5}{28}} (\bar{C})^{\frac{5}{28}} \quad (\text{by Assumption 2-3(b)}) \\
&= \bar{C}^{\frac{25}{28}}
\end{aligned}$$

Moreover, let  $\rho_1 = h - g$ ,  $\rho_2 = v - h$ , and  $\rho_3 = w - v$ , so that  $h = g + \rho_1$ ,  $v = h + \rho_2 = g + \rho_1 + \rho_2$ ,  $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$ . Using these notations and the boundedness of  $E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}}$  as shown

above, we can further write

$$\begin{aligned}
& \mathcal{T}_2 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{4}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\
& \quad \times \left. \left( E |u_{it} u_{is} u_{ig} u_{ih} u_{iv}|^{\frac{5}{4}} \right)^{\frac{4}{5}} \left( E |u_{iw}|^6 \right)^{\frac{1}{6}} \right\} \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \bar{C}^{\frac{25}{28}} \bar{C}^{\frac{1}{7}} \\
& \leq \frac{C_1 \bar{C}^{\frac{57}{28}}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{5}} + 1 \right) [a_1 \exp \{-a_2 (w-v)\}]^{\frac{1}{30}} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ w-v \geq \max\{v-h, h-g\}, w-v > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \quad \left( \text{for some constant } C^* \text{ such that } 2 \left( 2^{\frac{1}{5}} + 1 \right) C_1 \bar{C}^{\frac{57}{28}} a_1^{\frac{1}{30}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\rho_3=1}^{\infty} \sum_{\rho_1=0}^{\rho_3} \sum_{\rho_2=0}^{\rho_3} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\rho_3=1}^{\infty} (\rho_3 + 1)^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \\
& = C^* \frac{N_1}{\tau_1^3} \left[ \sum_{\rho_3=1}^{\infty} \rho_3^2 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + 2 \sum_{\rho_3=1}^{\infty} \rho_3 \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} + \sum_{\rho_3=1}^{\infty} \exp \left\{ -\frac{a_2}{30} \rho_3 \right\} \right] \\
& = O \left( \frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma OA-1}). \tag{21}
\end{aligned}$$

Now, consider  $\mathcal{T}_3$ . Here, we can apply Lemma OA-3 with  $p = 3/2$  and  $r = 7/2$  to obtain

$$\begin{aligned}
\mathcal{T}_3 & = \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E [\{u_{it} u_{is} u_{ig} u_{ih} - E (u_{it} u_{is} u_{ig} u_{ih})\} u_{iv} u_{iw}]| \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{2}{3}} + 1 \right) [a_1 \exp \{-a_2 (v-h)\}]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
& \quad \times \left. \left( E |\{u_{it} u_{is} u_{ig} u_{ih} - E (u_{it} u_{is} u_{ig} u_{ih})\}|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left( E |u_{iv} u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\}
\end{aligned}$$

Next, note that applications of Hölder's inequality yield

$$\begin{aligned} E|u_{iv}u_{iw}|^{\frac{7}{2}} &\leq \left(E|u_{iv}|^7\right)^{\frac{1}{2}}\left(E|u_{iw}|^7\right)^{\frac{1}{2}} \\ &\leq \frac{(\bar{C})^{\frac{1}{2}}(\bar{C})^{\frac{1}{2}}}{\bar{C}} \quad (\text{by Assumption 2-3(b)}) \\ &= \bar{C} < \infty \end{aligned}$$

and

$$\begin{aligned} E|\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\}|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}} \left( E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} + E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} \right) \\ &\quad (\text{by Loèvre's } c_r \text{ inequality}) \\ &\leq 2^{\frac{3}{2}} E|u_{it}u_{is}u_{ig}u_{ih}|^{\frac{3}{2}} \\ &\leq 2^{\frac{3}{2}} \left( E|u_{it}u_{is}|^3 \right)^{\frac{1}{2}} \left( E|u_{ig}u_{ih}|^3 \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{3}{2}} \left( \left( E|u_{it}|^6 \right)^{\frac{1}{2}} \left( E|u_{is}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \left( E|u_{ig}|^6 \right)^{\frac{1}{2}} \left( E|u_{ih}|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\leq 2^{\frac{3}{2}} \left[ \left( E|u_{it}|^6 \right)^{\frac{1}{6}} \left( E|u_{is}|^6 \right)^{\frac{1}{6}} \left( E|u_{ig}|^6 \right)^{\frac{1}{6}} \left( E|u_{ih}|^6 \right)^{\frac{1}{6}} \right]^{\frac{3}{2}} \\ &\leq 2^{\frac{3}{2}} \left[ \left( E|u_{it}|^7 \right)^{\frac{1}{7}} \left( E|u_{is}|^7 \right)^{\frac{1}{7}} \left( E|u_{ig}|^7 \right)^{\frac{1}{7}} \left( E|u_{ih}|^7 \right)^{\frac{1}{7}} \right]^{\frac{3}{2}} \\ &\quad (\text{by Liapunov's inequality}) \\ &\leq 2^{\frac{3}{2}} \left[ \left( \sup_{i,t} E|u_{it}|^7 \right)^{\frac{4}{7}} \right]^{\frac{3}{2}} \\ &= 2^{\frac{3}{2}} \bar{C}^{\frac{6}{7}} \quad (\text{by Assumption 2-3(b)}) \end{aligned}$$

Again, let  $\rho_1 = h - g$ ,  $\rho_2 = v - h$ , and  $\rho_3 = w - v$ , so that  $h = g + \rho_1$ ,  $v = h + \rho_2 = g + \rho_1 + \rho_2$ ,  $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$ . Using these notations and the boundedness of  $E|u_{iv}u_{iw}|^{\frac{7}{2}}$  and

$E | \{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\}|^{\frac{3}{2}}$  as shown above, we can further write

$$\begin{aligned}
T_3 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\} u_{iv}u_{iw}]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{2}{3}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\quad \times \left. \left( E | \{u_{it}u_{is}u_{ig}u_{ih} - E(u_{it}u_{is}u_{ig}u_{ih})\}|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left( E |u_{iv}u_{iw}|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\} \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{3}} + 1 \right) [a_1 \exp\{-a_2(v-h)\}]^{\frac{1}{21}} \left( 2^{\frac{3}{2}} \bar{C}^{\frac{6}{7}} \right)^{\frac{2}{3}} (\bar{C})^{\frac{2}{7}} \\
&\leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t, s, g, h, v, w = (r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} \\
&\quad \left( \text{for some constant } C^* \text{ such that } 4 \left( 2^{\frac{1}{3}} + 1 \right) C_1 \bar{C}^{\frac{13}{7}} a_1^{\frac{1}{21}} \leq C^* < \infty \right) \\
&\leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \sum_{\varrho_3=0}^{\varrho_2} \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} \\
&\leq C^* \frac{N_1}{\tau_1^3} \sum_{\varrho_2=1}^{\infty} (\varrho_2 + 1)^2 \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} \\
&= C^* \frac{N_1}{\tau_1^3} \left[ \sum_{\varrho_2=1}^{\infty} \varrho_2^2 \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} + 2 \sum_{\varrho_2=1}^{\infty} \varrho_2 \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} + \sum_{\varrho_2=1}^{\infty} \exp \left\{ -\frac{a_2}{21} \varrho_2 \right\} \right] \\
&= O \left( \frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma OA-1}). \tag{22}
\end{aligned}$$

Turning our attention to the term  $T_4$ , note that, from the upper bounds given in the proofs of parts (a) and (c) of Lemma OA-4, it is clear that there exists a positive constant  $C$  such that

$$\frac{1}{\tau_1^4} \sum_{\substack{t, s, g, h = (r-1)\tau+p \\ t \leq s \leq g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| \leq \frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^2} \sum_{\substack{v, w = (r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \leq \frac{C}{\tau_1}$$

from which it follows that

$$\begin{aligned}
\mathcal{T}_4 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h>0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| |E(u_{iv}u_{iw})| \\
&\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left( \frac{1}{\tau_1^4} \sum_{\substack{t,s,g,h=(r-1)\tau+p \\ t \leq s \leq g \leq h}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig}u_{ih})| \right) \left( \frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{iv}u_{iw})| \right) \\
&\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left( \frac{C}{\tau_1^2} \right) \left( \frac{C}{\tau_1} \right) \\
&= C_1 \bar{C} C^2 \frac{N_1}{\tau_1^3} \\
&= O\left(\frac{N_1}{\tau_1^3}\right). \tag{23}
\end{aligned}$$

Consider now  $\mathcal{T}_5$ . In this case, we apply Lemma OA-3 with  $p = 2$  and  $r = 9/4$  to obtain

$$\begin{aligned}
\mathcal{T}_5 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g>0}}^{(r-1)\tau+\tau_1+p-1} |E[\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\} u_{ih}u_{iv}u_{iw}]| \\
&\leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g>0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{1}{2}} + 1 \right) [a_1 \exp\{-a_2(h-g)\}]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
&\quad \times \left. \left( E |\{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left( E |u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
& E |u_{ih} u_{iv} u_{iw}|^{\frac{9}{4}} \\
& \leq \left[ E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[ E |u_{iv} u_{iw}|^{\frac{63}{19}} \right]^{\frac{19}{28}} \\
& \leq \left[ E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[ \left( E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \left( E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{1}{2}} \right]^{\frac{19}{28}} \\
& = \left[ E |u_{ih}|^7 \right]^{\frac{9}{28}} \left( E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \left( E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{56}} \\
& = \left[ E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[ \left( E |u_{iv}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \left( E |u_{iw}|^{\frac{126}{19}} \right)^{\frac{19}{126}} \right]^{\frac{9}{4}} \\
& \leq \left[ E |u_{ih}|^7 \right]^{\frac{9}{28}} \left[ \left( E |u_{iv}|^7 \right)^{\frac{1}{7}} \left( E |u_{iw}|^7 \right)^{\frac{1}{7}} \right]^{\frac{9}{4}} \quad (\text{by Liapunov's inequality}) \\
& \leq \left( \sup_{i,t} E |u_{it}|^7 \right)^{\frac{27}{28}} \\
& \leq \overline{C}^{\frac{27}{28}} \quad (\text{by Assumption 2-3(b)})
\end{aligned}$$

and

$$\begin{aligned}
E | \{u_{it} u_{is} u_{ig} - E(u_{it} u_{is} u_{ig})\} |^2 & \leq 2 \left( E |u_{it} u_{is} u_{ig}|^2 + E |u_{it} u_{is} u_{ig}|^2 \right) \\
& \quad (\text{by Loèvre's } c_r \text{ inequality}) \\
& \leq 4 E |u_{it} u_{is} u_{ig}|^2 \\
& \leq 4 \left( E |u_{it}|^6 \right)^{\frac{1}{3}} \left( E |u_{is} u_{ig}|^3 \right)^{\frac{2}{3}} \\
& \leq 4 \left( E |u_{it}|^6 \right)^{\frac{1}{3}} \left( \sqrt{E |u_{is}|^6} \sqrt{E |u_{ig}|^6} \right)^{\frac{2}{3}} \\
& = 4 \left[ \left( E |u_{it}|^6 \right)^{\frac{1}{6}} \right]^2 \left[ \left( E |u_{is}|^6 \right)^{\frac{1}{6}} \left( E |u_{ig}|^6 \right)^{\frac{1}{6}} \right]^2 \\
& \leq 4 \left[ \left( E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^2 \left[ \left( E |u_{is}|^7 \right)^{\frac{1}{7}} \left( E |u_{ig}|^7 \right)^{\frac{1}{7}} \right]^2 \\
& \quad (\text{by Liapunov's inequality}) \\
& \leq 4 \left[ \left( \sup_{i,t} E |u_{it}|^7 \right)^{\frac{1}{7}} \right]^6 \\
& \leq 4 \overline{C}^{\frac{6}{7}} \quad (\text{by Assumption 2-3(b)})
\end{aligned}$$

Define again  $\rho_1 = h - g$ ,  $\rho_2 = v - h$ , and  $\rho_3 = w - v$ , so that  $h = g + \rho_1$ ,  $v = h + \rho_2 = g + \rho_1 + \rho_2$ ,  $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$ . Using these notations and the boundedness of  $E |u_{ih} u_{iv} u_{iw}|^{\frac{9}{4}}$  and

$E | \{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2$  as shown above, we can further write

$$\begin{aligned}
& \mathcal{T}_5 \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2 \left( 2^{1-\frac{1}{2}} + 1 \right) \left[ a_1 \exp \left\{ -a_2 (h-g)^\theta \right\} \right]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
& \quad \times \left. \left( E | \{u_{it}u_{is}u_{ig} - E(u_{it}u_{is}u_{ig})\}|^2 \right)^{\frac{1}{2}} \left( E |u_{ih}u_{iv}u_{iw}|^{\frac{9}{4}} \right)^{\frac{4}{9}} \right\} \\
& \leq \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} 2 \left( 2^{\frac{1}{2}} + 1 \right) [a_1 \exp \{ -a_2 (h-g) \}]^{\frac{1}{18}} \left( 4 \bar{C}^{\frac{6}{7}} \right)^{\frac{1}{2}} \left( \bar{C}^{\frac{27}{28}} \right)^{\frac{4}{9}} \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g > 0}}^{(r-1)\tau+\tau_1+p-1} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \quad \left( \text{for some constant } C^* \text{ such that } 4 \left( 2^{\frac{1}{2}} + 1 \right) C_1 \bar{C}^{\frac{13}{7}} a_1^{\frac{1}{18}} \leq C^* < \infty \right) \\
& \leq \frac{C^*}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \sum_{\varrho_3=0}^{\varrho_1} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& \leq C^* \frac{N_1}{\tau_1^3} \sum_{\varrho_1=1}^{\infty} (\varrho_1 + 1)^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \\
& = C^* \frac{N_1}{\tau_1^3} \left[ \sum_{\varrho_1=1}^{\infty} \varrho_1^2 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + 2 \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} + \sum_{\varrho_1=1}^{\infty} \exp \left\{ -\frac{a_2}{18} \varrho_1 \right\} \right] \\
& = O \left( \frac{N_1}{\tau_1^3} \right) \quad (\text{by Lemma OA-1})
\end{aligned} \tag{24}$$

Finally, consider  $\mathcal{T}_6$ . Note that, from the upper bounds given in the proofs of part (b) of Lemma OA-4, it is clear that there exists a positive constant  $C$  such that

$$\frac{1}{\tau_1^3} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| \leq \frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \leq \frac{C}{\tau_1^2}$$

from which it follows that

$$\begin{aligned}
\mathcal{T}_6 &= \frac{C_1 \bar{C}}{q \tau_1^6} \sum_{r=1}^q \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ h-g \geq \max\{w-v, v-h\}, h-g>0}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| |E(u_{ih}u_{iv}u_{iw})| \\
&\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left( \frac{1}{\tau_1^3} \sum_{\substack{t,s,g=(r-1)\tau+p \\ t \leq s \leq g}}^{(r-1)\tau+\tau_1+p-1} |E(u_{it}u_{is}u_{ig})| \right) \left( \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| \right) \\
&\leq \frac{C_1 \bar{C}}{q} \sum_{r=1}^q \sum_{i \in H^c} \left( \frac{C}{\tau_1^2} \right) \left( \frac{C}{\tau_1^2} \right) \\
&= C_1 C \bar{C}^2 \frac{N_1}{\tau_1^4} \\
&= O\left(\frac{N_1}{\tau_1^4}\right).
\end{aligned} \tag{25}$$

It follows from expressions (18)-(25) that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
&P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \geq \epsilon \right\} \\
&\leq \frac{1}{\epsilon^6} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \\
&\leq \frac{1}{\epsilon^4} (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6) \\
&= O\left(\frac{N_1}{\tau_1^5}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^3}\right) + O\left(\frac{N_1}{\tau_1^4}\right) \\
&= O\left(\frac{N_1}{\tau_1^3}\right) \\
&= o(1) \quad \left( \text{by Assumption 2-9(b) which stipulates that } \frac{N_1}{\tau_1^3} \sim \frac{N_1}{T^{3\alpha_1}} \rightarrow 0 \right)
\end{aligned}$$

which proves the required result.

Turning our attention to part (d), note that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \right|^3 \geq \epsilon^3 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^3 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&\leq P \left\{ \sum_{\ell=1}^d \sum_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6 \geq \epsilon^3 \right\} \\
&\leq \frac{1}{\epsilon^3} \frac{1}{q} \sum_{r=1}^q \sum_{\ell=1}^d \sum_{i \in H^c} E \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^6.
\end{aligned}$$

The rest of the proof for part (d) then follows in a manner similar to the argument given for part (c) above.

For part (e), note that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
&\leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \sqrt{\frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \sqrt{\frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \\
&\leq \left\{ \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \right. \\
&\quad \times \left. \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \right\} \\
&= o_p(1),
\end{aligned}$$

where the convergence in probability to zero in the last line above follows from applying the results in parts (b) and (d) of this lemma.  $\square$

**Lemma OA-7:** Suppose that Assumptions 2-1 and 2-6 hold. Then, the following statements are true.

- (a) There exists a positive constant  $C^\dagger$  such that

$$\|A_{YY}\|_2 \leq C^\dagger \phi_{\max}$$

where  $\phi_{\max} = \max \{ |\lambda_{\max}(A)|, |\lambda_{\min}(A)| \}$  with  $0 < \phi_{\max} < 1$ .

(b) There exists a positive constant  $C^\dagger$  such that

$$\|A_{YF}\|_2 \leq C^\dagger \phi_{\max}$$

where  $\phi_{\max}$  is as defined in part (a).

### Proof of Lemma OA-7:

To proceed, recall first that the FAVAR model, i.e.,

$$\begin{aligned} Y_t &= \mu_Y + A_{YY}\underline{Y}_{t-1} + A_{YF}\underline{F}_{t-1} + \varepsilon_t^Y \\ F_t &= \mu_F + A_{FY}\underline{Y}_{t-1} + A_{FF}\underline{F}_{t-1} + \varepsilon_t^F, \end{aligned}$$

can be written in the companion form

$$\underline{W}_t = \alpha + AW_{t-1} + E_t$$

where  $\underline{W}_t = ( W'_t \quad W'_{t-1} \quad \cdots \quad W'_{t-p+2} \quad W'_{t-p+1} )'$  with  $W_t = ( Y'_t \quad F'_t )'$  and where

$$\alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & 0 & 0 \\ 0 & I_{d+K} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}, \quad \text{and } E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

with  $\mu = ( \mu'_Y \quad \mu'_F )'$ ,  $\varepsilon_t = ( \varepsilon_t^{Y'} \quad \varepsilon_t^{F'} )'$ , and

$$A_\ell = \begin{pmatrix} A_{YY,\ell} & A_{YF,\ell} \\ A_{FY,\ell} & A_{FF,\ell} \end{pmatrix} \text{ for } \ell = 1, \dots, p.$$

Let  $\mathcal{P}_{(d+K)p}$  be the  $(d+K)p \times (d+K)p$  permutation matrix defined by expression (17) in the proof of Lemma OA-5; and it is easy to see that  $\bar{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p}$  has the partitioned form

$$\bar{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \\ \bar{A}_{31} & \bar{A}_{32} \\ \bar{A}_{41} & \bar{A}_{42} \end{pmatrix}$$

where  $\bar{A}_{11} = A_{YY}$  and  $\bar{A}_{12} = A_{YF}$ , i.e., the first  $d$  rows of the matrix  $\bar{A}$  as given by the submatrix  $[ A_{YY} \quad A_{YF} ]$ .

Now, to show part (a), let  $\bar{v} \in \mathbb{R}^{dp}$  such that  $\|\bar{v}\|_2 = 1$  and such that

$$\|A_{YY}\|_2 = \bar{v}' A'_{YY} A_{YY} \bar{v} = \max_{\|v\|_2=1} v' A'_{YY} A_{YY} v = \bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v}$$

and let  $S_d = \begin{pmatrix} I_{dp} & 0 \\ & dp \times Kp \end{pmatrix}'$ . It follows that

$$\begin{aligned}
\|A_{YY}\|_2 &= \sqrt{\bar{v}' A'_{YY} A_{YY} \bar{v}} \\
&= \sqrt{\bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v}} \\
&\leq \sqrt{\bar{v}' \bar{A}'_{11} \bar{A}_{11} \bar{v} + \bar{v}' \bar{A}'_{21} \bar{A}_{21} \bar{v} + \bar{v}' \bar{A}'_{31} \bar{A}_{31} \bar{v} + \bar{v}' \bar{A}'_{41} \bar{A}_{41} \bar{v}} \\
&= \sqrt{\bar{v}' S'_d \bar{A}' \bar{A} S_d \bar{v}} \\
&= \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} A' \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} S_d \bar{v}} \\
&= \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} A' A \mathcal{P}'_{(d+K)p} S_d \bar{v}} \quad (\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}) \\
&\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \quad (\text{noting that } \left\| \mathcal{P}'_{(d+K)p} S_d \bar{v} \right\|_2 = \sqrt{\bar{v}' S'_d \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p} S_d \bar{v}} = 1) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^\dagger \phi_{\max} \quad (\text{by Assumption 2-6})
\end{aligned}$$

where  $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ . Note further that  $0 < \phi_{\max} < 1$  since, by Assumption 2-1, all eigenvalues of  $A$  have modulus less than 1.

To show part (b), let  $\tilde{v} \in \mathbb{R}^{Kp}$  such that  $\|\tilde{v}\|_2 = 1$  and such that

$$\|A_{YF}\|_2 = \tilde{v}' A'_{YF} A_{YF} \tilde{v} = \max_{\|v\|_2=1} v' A'_{YF} A_{YF} v = \tilde{v}' \bar{A}'_{12} \bar{A}_{12} \tilde{v}$$

and let

$$(d+K)p \times Kp = \begin{pmatrix} 0 \\ I_{Kp} \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\|A_{YF}\|_2 &= \sqrt{\tilde{v}' A'_{YF} A_{YF} \tilde{v}} \\
&= \sqrt{\tilde{v}' \bar{A}'_{12} \bar{A}_{12} \tilde{v}} \\
&\leq \sqrt{\tilde{v}' \bar{A}'_{12} \bar{A}_{12} \tilde{v} + \tilde{v}' \bar{A}'_{22} \bar{A}_{22} \tilde{v} + \tilde{v}' \bar{A}'_{32} \bar{A}_{32} \tilde{v} + \tilde{v}' \bar{A}'_{42} \bar{A}_{42} \tilde{v}} \\
&= \sqrt{\tilde{v}' S'_K \bar{A}' \bar{A} S_K \tilde{v}} \\
&= \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} A' \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} S_K \tilde{v}} \\
&= \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} A' A \mathcal{P}'_{(d+K)p} S_K \tilde{v}} \quad (\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}) \\
&\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \quad (\text{noting that } \left\| \mathcal{P}'_{(d+K)p} S_K \tilde{v} \right\|_2 = \sqrt{\tilde{v}' S'_K \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p} S_K \tilde{v}} = 1) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^\dagger \phi_{\max} \quad (\text{by Assumption 2-6})
\end{aligned}$$

where  $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ . As noted in the proof for part (a),  $0 < \phi_{\max} < 1$  since, by

Assumption 2-1, all eigenvalues of  $A$  have modulus less than 1.  $\square$

**Lemma OA-8:** Consider the linear process

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}$$

Suppose the process satisfies the following assumptions

- (i) Let  $\{\varepsilon_t\}$  is an independent sequence of random vectors with  $E[\varepsilon_t] = 0$  for all  $t$ . For some  $\delta > 0$ , suppose that there exists a positive constant  $K$  such that

$$E \|\varepsilon_t\|_2^{1+\delta} \leq K < \infty \text{ for all } t.$$

- (ii) Suppose that  $\varepsilon_t$  has p.d.f.  $g_{\varepsilon_t}$  such that, for some positive constant  $M < \infty$ ,

$$\sup_t \int |g_{\varepsilon_t}(v-u) - g_{\varepsilon_t}(v)| d\varepsilon \leq M |u|$$

whenever  $|u| \leq \bar{\kappa}$  for some constant  $\bar{\kappa} > 0$ .

- (iii) Suppose that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 < \infty$$

and

$$\det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} \neq 0 \text{ for all } z \text{ with } |z| \leq 1$$

Under these conditions, suppose further that

$$\sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}} < \infty;$$

then, for some positive constant  $\bar{K}$ ,

$$\beta_{\xi}(m) \leq \bar{K} \sum_{j=m}^{\infty} \left( \sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}}$$

where

$$\beta_{\xi}(m) = \sup_t E \left[ \sup \left\{ |P(B|\mathcal{F}_{\xi,-\infty}^t) - P(B)| : B \in \mathcal{F}_{\xi,t+m}^{\infty} \right\} \right].$$

with  $\mathcal{F}_{\xi,-\infty}^t = \sigma(\dots, \xi_{t-2}, \xi_{t-1}, \xi_t)$  and  $\mathcal{F}_{\xi,t+m}^{\infty} = \sigma(\xi_{t+m}, \xi_{t+m+1}, \xi_{t+m+2}, \dots)$ .

**Remark:** This is Theorem 2.1 of Pham and Tran (1985) restated here in our notation. For a proof, see Pham and Tran (1985).

**Lemma OA-9:** Let  $A$  be an  $n \times n$  square matrix with (ordered) singular values given by

$$\sigma_{(1)}(A) \geq \sigma_{(2)}(A) \geq \dots \geq \sigma_{(n)}(A) \geq 0.$$

Suppose that  $A$  is diagonalizable, i.e.,

$$A = S \Lambda S^{-1}$$

where  $\Lambda$  is diagonal matrix whose diagonal elements are the eigenvalues of  $A$ . Let the modulus of these eigenvalues be ordered as follows:

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \cdots \geq |\lambda_{(n)}(A)|.$$

Then, for  $k \in \{1, \dots, n\}$  and for any positive integer  $j$ , we have

$$\chi(S)^{-1} |\lambda_{(k)}(A^j)| \leq \sigma_{(k)}(A^j) \leq \chi(S) |\lambda_{(k)}(A^j)|$$

where

$$\chi(S) = \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}).$$

**Proof of Lemma OA-9:** Observe first that we can assume, without loss of generality, that the decomposition

$$A = S\Lambda S^{-1} = S \cdot \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \cdot S^{-1}$$

is such that

$$\lambda_i = \lambda_{(i)}(A) \text{ for } i = 1, \dots, n$$

with

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \cdots \geq |\lambda_{(n)}(A)|.$$

This is because suppose we have the alternative representation where

$$A = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} = \tilde{S} \cdot \text{diag}(\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n) \cdot \tilde{S}^{-1}$$

and where  $\tilde{\lambda}_i \neq \lambda_{(i)}(A)$  for at least some of the  $i$ 's. Then, we can always define a permutation matrix  $\mathcal{P}$  such that

$$\mathcal{P}'\tilde{\Lambda}\mathcal{P} = \Lambda$$

so that, given that  $\mathcal{P}$  is an orthogonal matrix, we have

$$A = \tilde{S}\tilde{\Lambda}\tilde{S}^{-1} = \tilde{S}\mathcal{P}\mathcal{P}'\tilde{\Lambda}\mathcal{P}\mathcal{P}'\tilde{S}^{-1} = S\Lambda S^{-1}$$

where  $S = \tilde{S}\mathcal{P}$  and, thus,  $S^{-1} = (\tilde{S}\mathcal{P})^{-1} = \mathcal{P}'\tilde{S}^{-1}$ .

Next, note that, for any positive integer  $j$ ,

$$A^j = S\Lambda S^{-1} \times S\Lambda S^{-1} \times \cdots \times S\Lambda S^{-1} = S\Lambda^j S^{-1}$$

where

$$\Lambda^j = \text{diag}(\lambda_1^j, \lambda_2^j, \dots, \lambda_n^j) = \text{diag}(\lambda_{(1)}^j(A), \lambda_{(2)}^j(A), \dots, \lambda_{(n)}^j(A)).$$

Moreover, since  $\lambda_{(k)}(A^j) = \lambda_{(k)}^j(A)$  for any  $k \in \{1, \dots, m\}$ , we also have

$$\Lambda^j = \text{diag}(\lambda_1^j, \lambda_2^j, \dots, \lambda_n^j) = \text{diag}(\lambda_{(1)}(A^j), \lambda_{(2)}(A^j), \dots, \lambda_{(n)}(A^j)).$$

In addition, let  $\overline{\lambda_{(k)}(A^j)}$  denote the complex conjugate of  $\lambda_{(k)}(A^j)$  for  $k \in \{1, \dots, m\}$ , and note that, by definition,

$$\sigma_{(k)}(\Lambda^j) = \sqrt{\overline{\lambda_{(k)}(A^j)}\lambda_{(k)}(A^j)} = |\lambda_{(k)}(A^j)|$$

Since  $|\lambda_{(k)}(A^j)| = |\lambda_{(k)}^j(A)| = |\lambda_{(k)}(A)|^j$ , the ordering

$$|\lambda_{(1)}(A)| \geq |\lambda_{(2)}(A)| \geq \cdots \geq |\lambda_{(n)}(A)|$$

implies that

$$|\lambda_{(1)}(A^j)| \geq |\lambda_{(2)}(A^j)| \geq \cdots \geq |\lambda_{(n)}(A^j)|$$

and, thus,

$$\sigma_{(1)}(\Lambda^j) \geq \sigma_{(2)}(\Lambda^j) \geq \cdots \geq \sigma_{(n)}(\Lambda^j)$$

for any positive integer  $j$ .

Now, apply the inequality

$$\sigma_{(i+\ell-1)}(BC) \leq \sigma_{(i)}(B)\sigma_{(\ell)}(C)$$

for  $i, \ell \in \{1, \dots, n\}$  and  $i + \ell \leq n + 1$ ; we have

$$\begin{aligned} \sigma_{(k)}(A^j) &= \sigma_{(k)}(S\Lambda^j S^{-1}) \\ &\leq \sigma_{(k)}(S\Lambda^j) \sigma_{(1)}(S^{-1}) \\ &\leq \sigma_{(k)}(\Lambda^j) \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}) \\ &= \sigma_{(1)}(S) \sigma_{(1)}(S^{-1}) |\lambda_{(k)}(A^j)| \\ &= \chi(S) |\lambda_{(k)}(A^j)| \text{ for any } k \in \{1, \dots, n\} \end{aligned}$$

Moreover, for any  $k \in \{1, \dots, n\}$ ,

$$\begin{aligned} |\lambda_{(k)}(A^j)| &= \sigma_{(k)}(\Lambda^j) \\ &= \sigma_{(k)}(S^{-1}S\Lambda^j S^{-1}S) \\ &= \sigma_{(k)}(S^{-1}A^j S) \\ &\leq \sigma_{(1)}(S^{-1}) \sigma_{(k)}(A^j) \sigma_{(1)}(S) \end{aligned}$$

or

$$\frac{|\lambda_{(k)}(A^j)|}{\chi(S)} = \frac{|\lambda_{(k)}(A^j)|}{\sigma_{(1)}(S) \sigma_{(1)}(S^{-1})} \leq \sigma_{(k)}(A^j)$$

Putting these two inequalities together, we have, for any  $k \in \{1, \dots, n\}$  and for all positive integer  $j$ ,

$$\chi(S)^{-1} |\lambda_{(k)}(A^j)| \leq \sigma_{(k)}(A^j) \leq \chi(S) |\lambda_{(k)}(A^j)|. \quad \square$$

**Remark:** Note that the case where  $j = 1$  in Lemma OA-9 has previously been obtained in Theorem 1 of Ruhe (1975). Hence, Lemma C-9 can be viewed as providing an extension to the first part of that theorem.

**Lemma OA-10:** Let  $\rho$  be such that  $|\rho| < 1$ . Then,

$$\sum_{j=0}^{\infty} (j+1) \rho^j = \frac{1}{(1-\rho)^2} < \infty$$

**Proof of Lemma OA-10:** Define

$$S_n(\rho) = 1 + \rho + \rho^2 + \cdots + \rho^n = \frac{1 - \rho^{n+1}}{1 - \rho}$$

Note that

$$\begin{aligned}
S'_n(\rho) &= 1 + 2\rho + 3\rho^2 + \cdots + n\rho^{n-1} \\
&= -\frac{(n+1)\rho^n}{1-\rho} + \frac{1-\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1}-(n+1)\rho^n(1-\rho)}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1}-(n+1)\rho^n+(n+1)\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-(n+1)\rho^n+n\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-\rho^n-n\rho^n(1-\rho)}{(1-\rho)^2}
\end{aligned}$$

It follows that

$$S'_n(\rho) = \sum_{j=0}^{n-1} (j+1)\rho^j = \frac{1-\rho^n-n\rho^n(1-\rho)}{(1-\rho)^2} \rightarrow \frac{1}{(1-\rho)^2} \text{ as } n \rightarrow \infty. \square$$

**Lemma OA-11:** Let  $W_t = (Y'_t, F'_t)'$  be generated by the factor-augmented VAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}$$

described in section 3 of the main paper. Under Assumptions 2-1, 2-2, and 2-6;  $\{W_t\}$  is a  $\beta$ -mixing process with  $\beta_W(m)$  such that

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\}$$

for some positive constants  $C_1$  and  $C_2$ . Here,

$$\beta_W(m) = \sup_t E \left[ \sup \left\{ |P(B|\mathcal{A}_{-\infty}^t) - P(B)| : B \in \mathcal{A}_{t+m}^\infty \right\} \right]$$

with  $\mathcal{A}_{-\infty}^t = \sigma(\dots, W_{t-2}, W_{t-1}, W_t)$  and  $\mathcal{A}_{t+m}^\infty = \sigma(W_{t+m}, W_{t+m+1}, W_{t+m+2}, \dots)$ .

#### Proof of Lemma OA-11:

To prove this lemma, we shall verify the conditions of Lemma OA-8 given above for the vector moving-average representation of  $W_t$ , i.e.,

$$W_t = J_{d+K} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} J_{d+K} A^j J'_{d+K} \varepsilon_{t-j} = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j},$$

where

$$\begin{aligned}\mu_* &= J_{d+K} (I_{(d+K)p} - A)^{-1} J'_{d+K} \mu, \quad \Psi_j = J_{d+K} A^j J'_{d+K}, \\ J_{d+K} &= [ I_{d+K} \ 0 \ \cdots \ 0 \ 0 ] , \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}_{(d+K) \times (d+K)p}\end{aligned}$$

To proceed, set

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \quad (26)$$

and note first that, setting  $\delta = 5$  in Lemma OA-8, and we see that Assumptions (i) and (ii) of Lemma OA-8 are the same as the conditions specified in Assumption 2-2 (a)-(c). Next, note that, since in this case  $\Psi_j = J_{d+K} A^j J'_{d+K}$ , we have

$$\begin{aligned}\|\Psi_j\|_2 &\leq \|J_{d+K}\|_2 \|A^j\|_2 \|J'_{d+K}\|_2 \\ &\leq \sqrt{\lambda_{\max}(J'_{d+K} J_{d+K})} \left( \sqrt{\lambda_{\max}\{(A^j)' A^j\}} \right) \sqrt{\lambda_{\max}(J_{d+K} J'_{d+K})} \\ &= \lambda_{\max}(J_{d+K} J'_{d+K}) \left( \sqrt{\lambda_{\max}\{(A^j)' A^j\}} \right) \\ &= \sqrt{\lambda_{\max}\{(A^j)' A^j\}} \\ &= \sigma_{\max}(A^j) \\ &\leq C [\max\{|\lambda_{\max}(A^j)|, |\lambda_{\min}(A^j)|\}] \quad (\text{by Assumption 2-6}) \\ &= C [\max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}]^j \\ &= C \phi_{\max}^j\end{aligned}$$

where  $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$  and where  $0 < \phi_{\max} < 1$  since, by Assumption 2-1, all eigenvalues of  $A$  have modulus less than 1. It follows that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 \leq C \sum_{j=0}^{\infty} \phi_{\max}^j = \frac{C}{1 - \phi_{\max}} < \infty.$$

Moreover, by Assumption 2-1,

$$\det\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\} \neq 0 \text{ for all } z \text{ such that } |z| \leq 1$$

and, by definition,

$$\sum_{j=0}^{\infty} \Psi_j z^j = \Psi(z) = (I_{(d+K)p} - A_1 z - \cdots - A_p z^p)^{-1} \text{ for all } z \text{ such that } |z| \leq 1$$

so that

$$\Psi(z) (I_{(d+K)p} - A_1 z - \cdots - A_p z^p) = I_{(d+K)p} \text{ for all } z \text{ such that } |z| \leq 1$$

In addition, since

$$\begin{aligned}
& \det \{\Psi(z)\} \det \{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\} \\
&= \det \{\Psi(z) (I_{(d+K)p} - A_1 z - \cdots - A_p z^p)\} \\
&= \det \{I_{(d+K)p}\} \\
&= 1,
\end{aligned}$$

and since

$$|\det \{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\}| < \infty \text{ for all } z \text{ such that } |z| \leq 1,$$

it follows that

$$\begin{aligned}
\det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} &= \det \{\Psi(z)\} \\
&= \frac{1}{\det \{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\}} \\
&\neq 0 \text{ for all } z \text{ such that } |z| \leq 1.
\end{aligned}$$

Finally, note that, setting  $\delta = 5$ ,

$$\begin{aligned}
\sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}} &= \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{5}{6}} \\
&\leq \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} C \phi_{\max}^k \right)^{\frac{5}{6}} \\
&= C^{\frac{5}{6}} \sum_{j=0}^{\infty} \left( \sum_{k=j}^{\infty} \phi_{\max}^k \right)^{\frac{5}{6}} \\
&\leq C^{\frac{5}{6}} \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \left( \phi_{\max}^{\frac{5}{6}} \right)^k \\
&\quad \left( \text{by the inequality } \left| \sum_{i=1}^{\infty} a_i \right|^r \leq \sum_{i=1}^{\infty} |a_i|^r \text{ for } r \leq 1 \right) \\
&= C^{\frac{5}{6}} \sum_{j=0}^{\infty} (j+1) \left( \phi_{\max}^{\frac{5}{6}} \right)^j \\
&= C^{\frac{5}{6}} \left[ 1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \quad (\text{by Lemma OA-10}) \\
&< \infty \quad (\text{since } 0 < \phi_{\max}^{\frac{5}{6}} < 1 \text{ given that } 0 < \phi_{\max} < 1).
\end{aligned}$$

Hence, all conditions of Lemma OA-8 are fulfilled. Applying Lemma OA-8, we then obtain that there

exists a constant  $\bar{C}$  such that

$$\begin{aligned}
\beta_\xi(m) &\leq \bar{C} \sum_{j=m}^{\infty} \left( \sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{5}{6}} \\
&\leq \bar{C} \sum_{j=m}^{\infty} \left( \sum_{k=j}^{\infty} C \phi_{\max}^k \right)^{\frac{5}{6}} \\
&= \bar{C} C^{\frac{5}{6}} \sum_{j=m}^{\infty} \left( \sum_{k=j}^{\infty} \phi_{\max}^k \right)^{\frac{5}{6}} \\
&\leq \bar{C} C^{\frac{5}{6}} \sum_{j=m}^{\infty} \sum_{k=j}^{\infty} \left( \phi_{\max}^{\frac{5}{6}} \right)^k \\
&= \bar{C} C^{\frac{5}{6}} \left( \phi_{\max}^{\frac{5}{6}} \right)^m \sum_{j=0}^{\infty} (j+1) \left( \phi_{\max}^{\frac{5}{6}} \right)^j \\
&= \bar{C} C^{\frac{5}{6}} \left( \phi_{\max}^{\frac{5}{6}} \right)^m \left[ 1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \\
&= \bar{C} C^{\frac{5}{6}} \left[ 1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \exp \left\{ - \left[ \frac{5}{6} |\ln \phi_{\max}| \right] m \right\} \quad (\text{since } 0 < \phi_{\max} < 1) \\
&\leq C_1 \exp \{-C_2 m\} \rightarrow 0 \text{ as } m \rightarrow \infty.
\end{aligned}$$

for some positive constants  $C_1$  and  $C_2$  such that

$$C_1 \geq \bar{C} C^{\frac{5}{6}} \left[ 1 - \phi_{\max}^{\frac{5}{6}} \right]^{-2} \text{ and } C_2 \leq \frac{5}{6} |\ln \phi_{\max}|$$

It follows that the process  $\{\xi_t\}$  (as defined in expression (26)) is  $\beta$  mixing with beta coefficient  $\beta_\xi(m)$  satisfying

$$\beta_\xi(m) \leq C_1 \exp \{-C_2 m\}.$$

Since

$$W_t = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} = \mu_* + \xi_t$$

and since  $\mu_*$  is a nonrandom parameter, we can then apply part (a) of Lemma OA-2 to deduce that  $\{W_t\}$  is a  $\beta$  mixing process with  $\beta$  coefficient  $\beta_W(m)$  satisfying the inequality

$$\beta_W(m) \leq C_1 \exp \{-C_2 m\}. \quad \square$$

**Lemma OA-12:** Let  $\underline{Y}_t = (Y'_t \ Y'_{t-1} \ \cdots \ Y'_{t-p+2} \ Y'_{t-p+1})'$  and  $\underline{F}_t = (F'_t \ F'_{t-1} \ \cdots \ F'_{t-p+2} \ F'_{t-p+1})'$ . Under Assumptions 2-1, 2-2, 2-5, 2-6, and 2-9(b); the following statements are true as  $N, T \rightarrow \infty$

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \xrightarrow{p} 0$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \xrightarrow{p} 0$$

(c)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \xrightarrow{p} 0$$

(d)

$$\begin{aligned} \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q & \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\ & \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\ & \xrightarrow{p} 0 \end{aligned}$$

(e)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2 = O_p(1).$$

(f)

$$\begin{aligned} \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q & \left\{ \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right. \right. \right. \\ & \left. \left. \left. + \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{\gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} + \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell}\} \right) \right\} \right. \\ & \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{\gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell}\} \right) \right\} \right| \\ & \xrightarrow{p} 0 \end{aligned}$$

(g)

$$\begin{aligned} \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q & \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\ & \left. \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\ & \xrightarrow{p} 0 \end{aligned}$$

(h)

$$\begin{aligned} & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\ & \quad \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\ & \xrightarrow{p} 0 \end{aligned}$$

**Proof of Lemma OA-12:**

To show part (a), note that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{i \in H^c} \max_{1 \leq \ell \leq d} \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right]' \right. \right. \\
&\quad \times \left. \left. \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' \right. \\
&\quad \times \left. (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \alpha_{YY,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \alpha'_{YY,\ell} \right. \\
&\quad \times \left. E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \right\} \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \alpha'_{YY,\ell} \right. \\
&\quad \times \left. E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \right\} \tag{27} \\
&\quad (\text{by Assumption 2-5})
\end{aligned}$$

Next, write

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \alpha'_{YY,\ell} \right. \right. \\
& \quad \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \left. \right) \\
& = \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \left( \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left\{ \alpha'_{YY,\ell} \right. \right. \\
& \quad \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \left. \right) \\
& \leq \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
& \quad + \sum_{\ell=1}^d \left( \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |\alpha'_{YY,\ell} \right. \\
& \quad \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \left. \right| \tag{28}
\end{aligned}$$

Let  $e_{\ell,d}$  be a  $d \times 1$  elementary vector whose  $\ell^{th}$  component is 1 and all other components are 0, and note

that

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] \alpha_{YY,\ell} \right) \\
&= \sum_{\ell=1}^d \left( \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] A'_{YY} e_{\ell,d} \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left( \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E [\underline{Y}_t \underline{F}'_t \underline{F}_t \underline{Y}'_t] A'_{YY} e_{\ell,d} \right. \\
&\quad \left. - \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E [\underline{Y}_t \underline{F}'_t] E [\underline{F}_t \underline{Y}'_t] A'_{YY} e_{\ell,d} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E \left[ \|\underline{F}_t\|_2^2 (e'_{\ell,d} A_{YY} \underline{Y}_t)^2 \right] \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left( e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d} \right)^2} \quad (\text{by CS inequality}) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{Y}_t\|_2^4 \right]} \sqrt{(e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d})^2} \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{Y}_t\|_2^4 \right]} \|A_{YY}\|_2^2 \sqrt{(e'_{\ell,d} e_{\ell,d})^2} \\
&\leq \frac{d(C^\dagger)^2}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{Y}_t\|_2^4 \right]} \phi_{\max}^2 \\
&\quad (\text{by part (a) of Lemma OA-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector}) \\
&\leq \frac{\bar{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \tag{29}
\end{aligned}$$

for some positive constant  $\bar{C} \geq d(C^\dagger)^2 \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{Y}_t\|_2^4 \right]} \phi_{\max}^2$ , which exists in light of Lemma OA-5 and the fact that  $0 < \phi_{\max} < 1$  given Assumption 2-1.

To analyze the second term on the right-hand side of expression (28), note first that by Lemma OA-11,  $\{\underline{Y}'_t, \underline{F}'_t\}'$  is  $\beta$ -mixing with  $\beta$  mixing coefficient satisfying

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\} \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since  $\alpha_{W,m} \leq \beta_W(m)$ , it follows that  $W_t = (Y'_t, F'_t)'$  is  $\alpha$ -mixing as well, with  $\alpha$  mixing coefficient satisfying

$$\alpha_{W,m} \leq C_1 \exp\{-C_2 m\}$$

Moreover, by applying part (b) of Lemma OA-2, we further deduce that  $X_{1t} = \underline{F}_t \underline{Y}'_t A'_{YY} e_{\ell,d}$  is also  $\alpha$ -mixing

with  $\alpha$  mixing coefficient satisfying

$$\begin{aligned}\alpha_{X_1,m} &\leq C_1 \exp \{-C_2(m-p+1)\} \\ &\leq C_1^* \exp \{-C_2 m\}\end{aligned}$$

for some positive constant  $C_1^* \geq C_1 \exp \{C_2(p-1)\}$ . Hence, we can apply Lemma OA-3 with  $p=3$  and  $r=3$  to obtain

$$\begin{aligned}& \left| \alpha'_{YY,\ell} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] \alpha_{YY,\ell} \right| \\ &= \left| e'_{\ell,d} A_{YY} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} \right| \\ &= \left| \sum_{h=1}^{K_p} e'_{\ell,d} A_{YY} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' e_{h,Kp} e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} \right| \\ &\leq \sum_{h=1}^{K_p} \left\{ 2 \left( 2^{\frac{2}{3}} + 1 \right) \alpha_{X_1,m}^{\frac{1}{3}} \left( E \left| e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\ &\quad \times \left. \left( E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) A'_{YY} e_{\ell,d} \right|^3 \right)^{1/3} \right\}\end{aligned}$$

where  $\alpha_{X_1,m}$  denotes the  $\alpha$  mixing coefficient for the process  $\{X_{1t}\}$  and where, by our previous calculations,

$$\alpha_{X_1,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant  $C_3$  such that

$$\begin{aligned}\sum_{m=1}^{\infty} \alpha_{X_1,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\ &= (C_1^*)^{\frac{1}{3}} \left[ 1 - \exp \left\{ -\frac{C_2}{3} \right\} \right]^{-1} \\ &\leq C_3\end{aligned}$$

where the last inequality stems from the fact that  $\sum_{m=0}^{\infty} \exp \{-(C_2 m/3)\}$  is a convergent geometric series

given that  $0 < \exp\{-C_2/3\} < 1$  for  $C_2 > 0$ . Next, note that

$$\begin{aligned}
& E \left| e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' e_{h,Kp} \right|^3 \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|^3 + |E[e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}]|^3 \right\} \text{(by Loèvre's } c_r \text{ inequality)} \\
& \leq 2^2 \left\{ E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|^3 + (E [|e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \text{(by Jensen's inequality)} \\
& \leq 2^2 \left\{ E \left| \frac{e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}}{2} + \frac{e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}}{2} \right|^3 + (E [|e'_{\ell,d} A_{YY} \underline{Y}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \\
& \leq \frac{4}{8} \left[ E |e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}|^3 + E |e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}|^3 \right] \\
& \quad + 4 \left( \sqrt{E [e'_{\ell,d} A_{YY} \underline{Y}_t \underline{Y}'_t A'_{YY} e_{\ell,d}]} \sqrt{E [e_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}]} \right)^3 \\
& \quad \text{(by Loèvre's } c_r \text{ inequality and by the CS inequality)} \\
& \leq \frac{1}{2} |e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d}|^3 E \|\underline{Y}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (e'_{\ell,d} A_{YY} A'_{YY} e_{\ell,d})^{\frac{3}{2}} \left( E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& \leq \frac{1}{2} \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \left( E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& = \frac{1}{2} (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (C^\dagger)^3 \phi_{\max}^3 \left( E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}} \\
& \quad \text{(since } \|e_{\ell,d}\|_2 = 1 \text{ for every } \ell \in \{1, \dots, d\} \text{ given that } e_{\ell,d} \text{'s are elementary vectors)} \\
& \leq C_4
\end{aligned}$$

for some positive constant  $C_4 \geq (1/2) (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_t\|_2^6 + (1/2) E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{Y}_t\|_2^2 \right)^{\frac{3}{2}} (C^\dagger)^3 \phi_{\max}^3 \left( E \|\underline{F}_t\|_2^2 \right)^{\frac{3}{2}}$  which exists in light of Lemma OA-5 and the fact that  $0 < \phi_{\max} < 1$  given Assumption 2-1. In a similar way, we can also show that there exists a positive constant  $C_5$  such that

$$\begin{aligned}
& E |e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) A'_{YY} e_{\ell,d}|^3 \\
& \leq (1/2) \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{Y}_{t+m}\|_2^6 + (1/2) E \|\underline{F}_{t+m}\|_2^6 \\
& \quad + 4 \left( E \|\underline{Y}_{t+m}\|_2^2 \right)^{\frac{3}{2}} \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \left( E \|\underline{F}_{t+m}\|_2^2 \right)^{\frac{3}{2}} \\
& \leq C_5 < \infty
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{2}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YY} \\
& \quad \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d} | \\
& \leq \frac{4(2^{\frac{2}{3}} + 1)}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \sum_{h=1}^{Kp} \alpha_{X_1,m}^{\frac{1}{3}} \left( E |e'_{\ell,d} A_{YY} (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' e_{h,Kp}|^3 \right)^{\frac{1}{3}} \\
& \quad \times \left( E |e'_{h,Kp} (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}])| A'_{YY} e_{\ell,d} \right)^{1/3} \\
& \leq \frac{4dKp (2^{\frac{2}{3}} + 1) C_4^{\frac{1}{3}} C_5^{\frac{1}{3}}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{\infty} \sum_{m=1}^{\infty} (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& \leq \frac{C^*}{\tau_1} \left( \frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left( \text{where } C^* \geq 4dKp (2^{\frac{2}{3}} + 1) (C_1^*)^{\frac{1}{3}} C_4^{\frac{1}{3}} C_5^{\frac{1}{3}} \right) \\
& \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left( \frac{1}{\tau_1} \right)
\end{aligned} \tag{30}$$

It then follows from expressions (27), (28), (29), and (30) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} \right. \\
& \quad \left. \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] A'_{YY} e_{\ell,d} \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YY} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \right] A'_{YY} e_{\ell,d} \right) \\
& \quad + \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YY} \\
& \quad \times E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_{t+m} \underline{Y}'_{t+m} - E[\underline{F}_{t+m} \underline{Y}'_{t+m}]) \right] A'_{YY} e_{\ell,d}| \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\bar{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C \bar{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{C C^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left( \frac{1}{\tau_1} \right) + O \left( \frac{1}{\tau_1} \right) \\
& = O \left( \frac{1}{\tau_1} \right) = o(1).
\end{aligned}$$

Next, to show part (b), note that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right)' \times \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s]) \alpha_{YF,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times E[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s])] \alpha_{YF,\ell} \right) \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
&\quad \left. \times E[(\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s])] \alpha_{YF,\ell} \right) \tag{31} \\
&\quad (\text{by Assumption 2-5})
\end{aligned}$$

Note first that

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} \right. \\
& \quad \times E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \Big) \\
= & \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& + \sum_{\ell=1}^d \left( \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \alpha'_{YF,\ell} \right. \\
& \quad \times E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \Big) \\
\leq & \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& + \sum_{\ell=1}^d \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |\alpha'_{YF,\ell} \\
& \quad \times E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell}| \quad (32)
\end{aligned}$$

Consider the first term on the majorant side of expression (32), whose order of magnitude we can analyze

as follows

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
&= \sum_{\ell=1}^d \left( \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e'_{\ell,d} A_{YF} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] A'_{YF} e_{\ell,d} \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left( \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{e'_{\ell,d} A_{YF} E[\underline{F}_t \underline{F}'_t \underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d} - e'_{\ell,d} A_{YF} E[\underline{F}_t \underline{F}'_t] E[\underline{F}_t \underline{F}'_t] A'_{YF} e_{\ell,d}\} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E \left[ \|\underline{F}_t\|_2^2 (e'_{\ell,d} A_{YF} \underline{F}_t)^2 \right] \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left( e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d} \right)^2} \quad (\text{by CS inequality}) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{(e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d})^2} \\
&\leq \frac{1}{q\tau_1^2} \sum_{\ell=1}^d \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \sqrt{E \left[ \|\underline{F}_t\|_2^4 \right]} \|A_{YF}\|_2^2 \sqrt{(e'_{\ell,d} e_{\ell,d})^2} \\
&\leq \frac{(C^\dagger)^2}{q\tau_1^2} \sum_{\ell=1}^d \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E \left[ \|\underline{F}_t\|_2^4 \right] \phi_{\max}^2 \\
&\quad (\text{by part (b) of Lemma OA-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector}) \\
&\leq \frac{\bar{C}}{\tau_1} = O \left( \frac{1}{\tau_1} \right). \tag{33}
\end{aligned}$$

for some positive constant  $\bar{C} \geq d(C^\dagger)^2 E \left[ \|\underline{F}_t\|_2^4 \right] \phi_{\max}^2$ , which exists in light of Lemma OA-5 and the fact that  $0 < \phi_{\max} < 1$  given Assumption 2-1.

To analyze the second term on the right-hand side of expression (32), note first that by Lemma OA-11,  $\{\underline{F}_t\}$  is  $\beta$ -mixing with  $\beta$  mixing coefficient satisfying

$$\beta_F(m) \leq C_1 \exp \{-C_2 m\} \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since  $\alpha_{F,m} \leq \beta_F(m)$ , it follows that  $F_t$  is  $\alpha$ -mixing as well, with  $\alpha$  mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp \{-C_2 m\}$$

Moreover, by applying part (b) of Lemma OA-2, we further deduce that  $X_{2t} = \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}$  is also  $\alpha$ -mixing with  $\alpha$  mixing coefficient satisfying

$$\begin{aligned}
\alpha_{X_{2t},m} &\leq C_1 \exp \{-C_2(m-p+1)\} \\
&\leq C_1^* \exp \{-C_2 m\}
\end{aligned}$$

for some positive constant  $C_1^* \geq C_1 \exp \{C_2(p-1)\}$ . Hence, we can apply Lemma OA-3 with  $p=3$  and  $r=3$  to obtain

$$\begin{aligned}
& \left| \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \right| \\
&= \left| e'_{\ell,d} A_{YF} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \right| \\
&= \left| \sum_{h=1}^{Kp} e'_{\ell,d} A_{YF} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' e_{h,Kp} e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \right| \\
&\leq \sum_{h=1}^{Kp} \left\{ 2 \left( 2^{\frac{2}{3}} + 1 \right) \alpha_{X_2,m}^{\frac{1}{3}} \left( E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\
&\quad \times \left. \left( E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YF} e_{\ell,d} \right|^3 \right)^{1/3} \right\}
\end{aligned}$$

where  $\alpha_{X_2,m}$  denotes the alpha mixing coefficient for the process  $\{X_{2t}\}$  and where, by our previous calculations,

$$\alpha_{X_2,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \text{ for all } m \text{ sufficiently large,}$$

It further follows that there exists a positive constant  $C_3$  such that

$$\begin{aligned}
\sum_{m=1}^{\infty} \alpha_{X_2,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
&\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
&= (C_1^*)^{\frac{1}{3}} \left[ 1 - \exp \left\{ -\frac{C_2}{3} \right\} \right]^{-1} \\
&\leq C_3
\end{aligned}$$

Next, note that

$$\begin{aligned}
& E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E [\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \\
& \leq 2^2 \left\{ E \left| e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp} \right|^3 + \left| E [e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}] \right|^3 \right\} \text{(by Loèvre's } c_r \text{ inequality)} \\
& \leq 2^2 \left\{ E \left| e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp} \right|^3 + (E [|e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \text{ (by Jensen's inequality)} \\
& \leq 2^2 \left\{ E \left| \frac{e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d}}{2} + \frac{e'_{h,Kp} \underline{F}_t \underline{F}'_t e_{h,Kp}}{2} \right|^3 + (E [|e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t e_{h,Kp}|])^3 \right\} \\
& \leq \frac{4}{8} \left[ E \left| e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d} \right|^3 + E \left| e'_{h,Kp} \underline{F}_t \underline{F}'_t e'_{h,Kp} \right|^3 \right] \\
& \quad + 4 \left( \sqrt{E \left[ e'_{\ell,d} A_{YF} \underline{F}_t \underline{F}'_t A'_{YF} e_{\ell,d} \right]} \sqrt{E \left[ e'_{h,Kp} \underline{F}_t \underline{F}'_t e'_{h,Kp} \right]} \right)^3 \\
& \quad \text{(by Loèvre's } c_r \text{ inequality and by the CS inequality)} \\
& \leq \frac{1}{2} |e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d}|^3 E \|\underline{F}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{F}_t\|_2^2 \right)^3 (e'_{\ell,d} A_{YF} A'_{YF} e_{\ell,d})^{\frac{3}{2}} \\
& \leq \frac{1}{2} \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{F}_t\|_2^2 \right)^3 \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \\
& = \frac{1}{2} (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + \frac{1}{2} E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{F}_t\|_2^2 \right)^3 (C^\dagger)^3 \phi_{\max}^3 \\
& \quad \text{(since } \|e_{\ell,d}\|_2 = 1 \text{ for every } \ell \in \{1, \dots, d\} \text{ given that } e_{\ell,d} \text{'s are elementary vectors)} \\
& \leq C_6
\end{aligned}$$

for some positive constant  $C_6 \geq (1/2) (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_t\|_2^6 + (1/2) E \|\underline{F}_t\|_2^6 + 4 \left( E \|\underline{F}_t\|_2^2 \right)^3 (C^\dagger)^3 \phi_{\max}^3$  which exists in light of Lemma OA-5 and the fact that  $0 < \phi_{\max} < 1$  given Assumption 2-1. In a similar way, we can also show that there exists a positive constant  $C_7$  such that

$$\begin{aligned}
& E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E [\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YY} e_{\ell,d} \right|^3 \\
& \leq \frac{1}{2} \|e_{\ell,d}\|_2^6 (C^\dagger)^6 \phi_{\max}^6 E \|\underline{F}_{t+m}\|_2^6 + \frac{1}{2} E \|\underline{F}_{t+m}\|_2^6 \\
& \quad + 4 \left( E \|\underline{F}_{t+m}\|_2^2 \right)^3 \|e_{\ell,d}\|_2^3 (C^\dagger)^3 \phi_{\max}^3 \\
& \leq C_7 < \infty
\end{aligned}$$

Hence,

$$\begin{aligned}
& \frac{2}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |e'_{\ell,d} A_{YF} \\
& \quad \times E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] A'_{YF} e_{\ell,d} \Big| \\
& \leq \frac{4 \left( 2^{\frac{2}{3}} + 1 \right)}{\tau_1^2} \sum_{\ell=1}^d \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \sum_{h=1}^{Kp} \left\{ \alpha_{X_2,m}^{\frac{1}{3}} \left( E \left| e'_{\ell,d} A_{YF} (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \times \left. \left( E \left| e'_{h,Kp} (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) A'_{YF} e_{\ell,d} \right|^3 \right)^{1/3} \right\} \\
& \leq \frac{4dKp \left( 2^{\frac{2}{3}} + 1 \right) C_6^{\frac{1}{3}} C_7^{\frac{1}{3}}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{\infty} \sum_{m=1}^{\infty} (C_1^*)^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& \leq \frac{C^*}{\tau_1} \left( \frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left( \text{where } C^* \geq 4dKp \left( 2^{\frac{2}{3}} + 1 \right) (C_1^*)^{\frac{1}{3}} C_6^{\frac{1}{3}} C_7^{\frac{1}{3}} \right) \\
& \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left( \frac{1}{\tau_1} \right)
\end{aligned} \tag{34}$$

It then follows from expressions (31), (32), (33), and (34) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \right] \alpha_{YF,\ell} \right) \\
& \quad + \frac{C}{\epsilon^2} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |\alpha'_{YF,\ell} \\
& \quad \times E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_{t+m} \underline{F}'_{t+m} - E[\underline{F}_{t+m} \underline{F}'_{t+m}]) \right] \alpha_{YF,\ell} \Big| \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\bar{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C \bar{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{CC^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O \left( \frac{1}{\tau_1} \right) + O \left( \frac{1}{\tau_1} \right) \\
& = O \left( \frac{1}{\tau_1} \right) = o(1).
\end{aligned}$$

Now, to show part (c), note that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \geq \epsilon \right\} \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon^2 \right\} \quad (\text{by Jensen's inequality}) \\
&= P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \left( \gamma'_i \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)^2 \geq \epsilon^2 \right\} \\
&\leq P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right)' \right. \\
&\quad \left. \times \left[ \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right] \right) \geq \epsilon^2 \right\} \\
&= P \left\{ \max_{i \in H^c} \|\gamma_i\|_2^2 \sum_{\ell=1}^d \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell} (\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s]) \mu_{Y,\ell} \geq \epsilon^2 \right\} \\
&\leq \frac{\max_{i \in H^c} \|\gamma_i\|_2^2}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
&\quad (\text{by Markov's inequality}) \\
&\leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \tag{35} \\
&\quad (\text{by Assumption 2-5})
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
&= \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_t - E[\underline{F}_t])] \right) \\
&\quad + \sum_{\ell=1}^d \left( \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])] \right) \\
&\leq \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_t - E[\underline{F}_t])] \right) \\
&\quad + \frac{2}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \quad (36)
\end{aligned}$$

Consider the first term on the majorant side of expression (36), whose order of magnitude we can analyze as follows

$$\begin{aligned}
& \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_t - E[\underline{F}_t])] \right) \\
&= \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \left( \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 \{E[\underline{F}'_t \underline{F}_t] - E[\underline{F}_t]' E[\underline{F}_t]\} \right) \\
&\leq \sum_{\ell=1}^d \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [\|\underline{F}_t\|_2^2] \\
&= \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [\|\underline{F}_t\|_2^2] \sum_{\ell=1}^d (\mu_{Y,\ell}^2) \\
&\leq \frac{1}{q\tau_1^2} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E [\|\underline{F}_t\|_2^2] \|\mu_Y\|_2^2 \\
&\leq \frac{\bar{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \quad (37)
\end{aligned}$$

for some positive constant  $\bar{C} \geq \|\mu_Y\|_2^2 E [\|\underline{F}_t\|_2^2]$ , which exists in light of Assumption 2-5 and Lemma OA-5.

To analyze the second term on the right-hand side of expression (36), note first that by the same argument as given for part (b) above, we can apply Lemma OA-11 to deduce that  $\{\underline{F}_t\}$  is  $\beta$ -mixing and, thus, also  $\alpha$ -mixing with  $\alpha$  mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp \{-C_2 m\}$$

Hence, we can apply Lemma OA-3 with  $p = 3$  and  $r = 3$  to obtain

$$\begin{aligned}
& \left| E \left[ (\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}]) \right] \right| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\
&= \left| \sum_{h=1}^{K_p} E \left[ (\underline{F}_t - E[\underline{F}_t])' e_{h,K_p} e_{h,K_p}' (\underline{F}_{t+m} - E[\underline{F}_{t+m}]) \right] \right| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\
&\leq \sum_{h=1}^{K_p} 2 \left( 2^{\frac{2}{3}} + 1 \right) \alpha_{F,m}^{\frac{1}{3}} \left( E |(\underline{F}_t - E[\underline{F}_t])' e_{h,K_p}|^3 \right)^{\frac{1}{3}} \left( E |e_{h,K_p}' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])|^3 \right)^{1/3} \sum_{\ell=1}^d \mu_{Y,\ell}^2
\end{aligned}$$

Moreover, there exists a positive constant  $C_3$  such that

$$\sum_{m=1}^{\infty} \alpha_{F,m}^{\frac{1}{3}} \leq C_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} = C_1^{\frac{1}{3}} \left[ 1 - \exp \left\{ -\frac{C_2}{3} \right\} \right]^{-1} \leq C_3$$

where again the last inequality stems from the fact that  $\sum_{m=0}^{\infty} \exp \{-(C_2 m/3)\}$  is a convergent geometric series given that  $0 < \exp \{-(C_2/3)\} < 1$  for  $C_2 > 0$ . Next, note that

$$\begin{aligned}
& E |(\underline{F}_t - E[\underline{F}_t])' e_{h,K_p}|^3 \\
&\leq 2^2 \left\{ E |\underline{F}_t' e_{h,K_p}|^3 + E |\underline{F}_t e_{h,K_p}|^3 \right\} \text{ (by Loèvre's } c_r \text{ inequality)} \\
&\leq 2^2 \left\{ E |\underline{F}_t' e_{h,K_p}|^3 + (E |\underline{F}_t e_{h,K_p}|)^3 \right\} \text{ (by Jensen's inequality)} \\
&\leq 2^2 \left\{ E \left[ (\underline{F}_t' \underline{F}_t)^{\frac{3}{2}} (e_{h,K_p}' e_{h,K_p})^{\frac{3}{2}} \right] + \left( \sqrt{E [\underline{F}_t' \underline{F}_t]} \sqrt{e_{h,K_p}' e_{h,K_p}} \right)^3 \right\} \text{ (by CS inequality)} \\
&\leq 4 \left\{ E \left[ \|\underline{F}_t\|_2^3 \right] + \left( E \left[ \|\underline{F}_t\|_2^2 \right] \right)^{\frac{3}{2}} \right\} \\
&\leq C_8
\end{aligned}$$

for some positive constant  $C_8 \geq 4 \left\{ E \left[ \|\underline{F}_t\|_2^3 \right] + \left( E \left[ \|\underline{F}_t\|_2^2 \right] \right)^{\frac{3}{2}} \right\}$  which exists in light of the result given in Lemma OA-5. In a similar way, we can also show that there exists a positive constant  $C_9$  such that

$$\begin{aligned}
E |e_{\ell}' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])|^3 &\leq 4 \left\{ E \left[ \|\underline{F}_{t+m}\|_2^3 \right] + \left( E \left[ \|\underline{F}_{t+m}\|_2^2 \right] \right)^{\frac{3}{2}} \right\} \\
&\leq C_9 < \infty
\end{aligned}$$

Finally, by Assumption 2-5, there exists a positive constant  $C_{10}$  such that  $\max_{1 \leq \ell \leq d} \mu_{Y,\ell}^2 \leq \|\mu_Y\|_2^2 \leq C_{10} <$

$\infty$ . Hence,

$$\begin{aligned}
& \frac{2}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\
& \leq \sum_{h=1}^{Kp} \frac{4(2^{\frac{2}{3}} + 1)}{\tau_1^2} \|\mu_Y\|_2^2 \\
& \quad \times \frac{1}{q} \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} \left\{ \alpha_{F,m}^{\frac{1}{3}} \left( E |(\underline{F}_t - E[\underline{F}_t])' e_{h,Kp}|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \left. \times \left( E |e'_{h,Kp} (\underline{F}_{t+m} - E[\underline{F}_{t+m}])|^3 \right)^{1/3} \right\} \\
& \leq \frac{4Kp(2^{\frac{2}{3}} + 1) C_8^{\frac{1}{3}} C_9^{\frac{1}{3}} C_{10}}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{\infty} C_1^{\frac{1}{3}} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& \leq \frac{C^*}{\tau_1} \left( \frac{\tau_1 - 1}{\tau_1} \right) \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \quad \left( \text{where } C^* \geq 4Kp(2^{\frac{2}{3}} + 1) C_1^{\frac{1}{3}} C_8^{\frac{1}{3}} C_9^{\frac{1}{3}} C_{10} \right) \\
& \leq \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O\left(\frac{1}{\tau_1}\right)
\end{aligned} \tag{38}$$

It then follows from expressions (35), (36), (37), and (38) that

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
& \leq \frac{C}{\epsilon^2} \sum_{\ell=1}^d \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E [(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_t - E[\underline{F}_t])] \right) \\
& \quad + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{2}{\tau_1^2} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2} \sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1} |E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_{t+m} - E[\underline{F}_{t+m}])]| \sum_{\ell=1}^d \mu_{Y,\ell}^2 \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{\bar{C}}{\tau_1} + \frac{C}{\epsilon^2} \frac{1}{q} \sum_{r=1}^q \frac{C^*}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = \frac{C\bar{C}}{\epsilon^2} \frac{1}{\tau_1} + \frac{CC^*}{\epsilon^2} \frac{1}{\tau_1} \sum_{m=1}^{\infty} \exp \left\{ -\frac{C_2 m}{3} \right\} \\
& = O\left(\frac{1}{\tau_1}\right) + O\left(\frac{1}{\tau_1}\right) \\
& = O\left(\frac{1}{\tau_1}\right) = o(1).
\end{aligned}$$

Turning our attention to part (d), note that, by apply Loève's  $c_r$  inequality, we obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& \leq 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \\
& \quad + 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \\
& \quad + 3 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2
\end{aligned}$$

It follows from the arguments given in the proofs of parts (a)-(c) above that, for any  $\epsilon > 0$ ,

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left( \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YY,\ell} E \left[ (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t])' (\underline{F}_s \underline{Y}'_s - E[\underline{F}_s \underline{Y}'_s]) \right] \alpha_{YY,\ell} \right) \\
& = o(1), \\
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left( \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \alpha'_{YF,\ell} E \left[ (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t])' (\underline{F}_s \underline{F}'_s - E[\underline{F}_s \underline{F}'_s]) \right] \alpha_{YF,\ell} \right) \\
& = o(1)
\end{aligned}$$

and

$$\begin{aligned}
& P \left\{ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right)^2 \geq \epsilon \right\} \\
& \leq \frac{C}{\epsilon^2} \frac{1}{q\tau_1^2} \\
& \quad \times \sum_{\ell=1}^d \left( \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \mu_{Y,\ell}^2 E[(\underline{F}_t - E[\underline{F}_t])' (\underline{F}_s - E[\underline{F}_s])] \right) \\
& = o(1),
\end{aligned}$$

from which we deduce via the Slutsky's theorem that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& = o_p(1)
\end{aligned}$$

as required.

To show part (e), note that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2 \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right. \\
& \quad \left. + \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{2}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& \quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{2}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
& \quad \text{(by Loèeve's } c_r \text{ inequality)} \\
& = o_p(1) + O(1) \\
& \quad \text{(applying the results given in part (d) of this lemma and in Lemma A1 of the main paper)} \\
& = O_p(1).
\end{aligned}$$

To show part (f), we apply the Cauchy-Schwarz inequality as well as part (d) of this lemma and Lemma

A1 of the main paper to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left\{ \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \right. \\
& \quad \left. \left. \left. \left. + \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right) \right. \right. \\
& \quad \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right) \right\} \right| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \\
& \quad \left. \left. \left. + \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right) \right. \right. \\
& \quad \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right) \right| \\
& \leq \left[ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \\
& \quad \left. \left. \left. + \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \right]^{1/2} \\
& \quad \times \left[ \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \right]^{1/2} \\
& = o_p(1) O(1) \\
& = o_p(1).
\end{aligned}$$

For part (g), we apply the Cauchy-Schwarz inequality as well as part (d) of Lemma OA-6 and part (e)

of this lemma to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& \leq \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2} \\
& \quad \times \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2} \\
& = O_p(1) o_p(1) \\
& = o_p(1)
\end{aligned}$$

Finally, for part (h), we apply the Cauchy-Schwarz inequality as well as part (b) of Lemma OA-6 and

part (e) of this lemma to obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
& \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left| \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \times \left. \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
& \leq \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right)^2} \\
& \quad \times \sqrt{\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2} \\
& = O_p(1) o_p(1) \\
& = o_p(1). \quad \square
\end{aligned}$$

**Lemma OA-13:** Let  $a, b \in \mathbb{R}$  such that  $a \geq 0$  and  $b \geq 0$ . Then,

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}$$

**Proof of Lemma OA-13:** Note that

$$\begin{aligned}
(\sqrt{a} - \sqrt{b})^2 &= a - 2\sqrt{a}\sqrt{b} + b \\
&= \sqrt{a}(\sqrt{a} - \sqrt{b}) + \sqrt{b}(\sqrt{b} - \sqrt{a}) \\
&\leq \sqrt{a}|\sqrt{a} - \sqrt{b}| + \sqrt{b}|\sqrt{b} - \sqrt{a}| \\
&= (\sqrt{a} + \sqrt{b})|\sqrt{a} - \sqrt{b}| \\
&= |(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})| \\
&= |a - b|
\end{aligned}$$

Taking principal square root on both sides, we obtain

$$|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}. \quad \square$$

**Lemma OA-14:**

$$P \left\{ \bigcap_{i=1}^m A_i \right\} \geq \sum_{i=1}^m P(A_i) - (m-1)$$

**Proof of Lemma OA-14:**

$$\begin{aligned}
P \left\{ \bigcap_{i=1}^m A_i \right\} &= 1 - P \left\{ \left( \bigcap_{i=1}^m A_i \right)^c \right\} \\
&= 1 - P \left\{ \bigcup_{i=1}^m A_i^c \right\} \quad (\text{by DeMorgan's Law}) \\
&\geq 1 - \sum_{i=1}^m P(A_i^c) \\
&= 1 - \sum_{i=1}^m [1 - P(A_i)] \\
&= \sum_{i=1}^m P(A_i) - m + 1 \\
&= \sum_{i=1}^m P(A_i) - (m - 1). \quad \square
\end{aligned}$$

**Lemma OA-15:**

(a) For  $t > 0$ ,

$$\overline{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t},$$

where  $\phi(t)$  and  $\Phi(t)$  denote, respectively, the pdf and the cdf of a standard normal random variable.

(b) Let  $N = N_1 + N_2$ . Specify  $\varphi$  such that  $\varphi \rightarrow 0$  as  $N_1, N_2 \rightarrow \infty$  and such that, for some constant  $a > 0$ ,

$$\varphi \geq \frac{1}{N^a}$$

for all  $N_1, N_2$  sufficiently large. Then, for all  $N_1, N_2$  sufficiently large such that

$$1 - \frac{\varphi}{2N} \geq \Phi(2)$$

we have

$$\Phi^{-1} \left( 1 - \frac{\varphi}{2N} \right) \leq \sqrt{2(1+a)} \sqrt{\ln N}.$$

**Proof of Lemma OA-15:**

(a)

$$\begin{aligned}
1 - \Phi(t) &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} dz \\
&= \int_t^\infty \frac{1}{z} \frac{z}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} dz \\
&\leq \frac{1}{t} \int_t^\infty \frac{z}{\sqrt{2\pi}} \exp \left\{ -\frac{z^2}{2} \right\} dz
\end{aligned}$$

Let

$$u = -\frac{z^2}{2} \text{ and } du = -zdz$$

so that

$$\begin{aligned}
\int_t^\infty \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz &= -\int_{-\frac{t^2}{2}}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\
&= \int_{-\infty}^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\
&= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \\
&= \phi(t)
\end{aligned}$$

It follows that

$$\bar{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t}.$$

(b) Let  $t > 0$  and set

$$\Phi(t) = \Pr(Z \leq t) = 1 - \frac{\varphi}{2N}.$$

It follows that

$$\Phi^{-1}(\Phi(t)) = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) = t$$

and, by the result given in part (a) above,

$$1 - \Phi(t) = 1 - \left(1 - \frac{\varphi}{2N}\right) = \frac{\varphi}{2N} \leq \frac{\phi(t)}{t}.$$

The latter inequality implies that

$$t \leq \phi(t) \frac{2N}{\varphi}$$

so that

$$\begin{aligned}
\ln t &\leq \ln \phi(t) + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln \pi + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln \pi + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \ln 2 + \ln\left(\frac{N}{\varphi}\right)
\end{aligned}$$

or

$$\begin{aligned}
t^2 &\leq 2(\ln 2 - \ln t) + 2\ln\left(\frac{N}{\varphi}\right) \\
&= 2\ln\left(\frac{2}{t}\right) + 2\ln\left(\frac{N}{\varphi}\right) \\
&\leq 2\ln\left(\frac{N}{\varphi}\right) \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2
\end{aligned}$$

so that

$$t \leq \sqrt{2} \sqrt{\ln \left( \frac{N}{\varphi} \right)} \text{ for any } t = \Phi^{-1} \left( 1 - \frac{\varphi}{2N} \right) \geq 2$$

Hence, for  $N_1, N_2$  sufficiently large so that

$$1 - \frac{\varphi}{2N} \geq \Phi(2) \text{ or, equivalently, } t = \Phi^{-1} \left( 1 - \frac{\varphi}{2N} \right) \geq 2,$$

we have

$$\begin{aligned} \Phi^{-1} \left( 1 - \frac{\varphi}{2N} \right) &= t \\ &\leq \sqrt{2} \sqrt{\ln \left( \frac{N}{\varphi} \right)} \\ &= \sqrt{2} \sqrt{\ln N - \ln \varphi} \\ &= \sqrt{2} \sqrt{\ln N} \sqrt{1 - \frac{\ln \varphi}{\ln N}} \\ &\leq \sqrt{2} \sqrt{\ln N} \sqrt{1 - \frac{\ln N^{-a}}{\ln N}} \\ &= \sqrt{2(1+a)} \sqrt{\ln N}. \quad \square \end{aligned}$$

**Lemma QA-16:** Suppose that Assumptions 2-1, 2-2, 2-3, 2-5, 2-6, and 2-8 hold and suppose that  $N_1, N_2, T \rightarrow \infty$  such that  $N_1/\tau_1^3 = N_1/\lfloor T_0^{\alpha_1} \rfloor^3 \rightarrow 0$ . Then, the following statements are true.

(a)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \xrightarrow{p} 0$$

(b)

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{\pi_{i,\ell,T}} \right| \xrightarrow{p} 0$$

where

$$\bar{S}_{i,\ell,T} = \sum_{r=1}^q \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it} y_{\ell,t+1} \text{ and } \bar{V}_{i,\ell,T} = \sum_{r=1}^q \left[ \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it} y_{\ell,t+1} \right]^2$$

**Proof of Lemma QA-16:**

To show part (a), note first that by applying parts (a) and (c) of Lemma OA-6, parts (a)-(c) of Lemma

OA-12, and the Slutsky theorem; we obtain

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{q\tau_1} \right| \\
&= \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right. \\
&\quad + \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} + \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \\
&\quad \left. - \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{\gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell}\} \right| \\
&\leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} \right| \\
&\quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right| \\
&\quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \right| \\
&\quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right| \\
&\quad + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \\
&= o_p(1)
\end{aligned}$$

Moreover, by Assumption 2-8, there exist a positive constant  $\underline{c}$  such that for all  $N$  and  $T$  sufficiently large

$$\begin{aligned}
& \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{\mu_{i,\ell,T}}{q\tau_1} \right| \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell}\} \right| \\
&\geq \underline{c} > 0
\end{aligned}$$

It follows that

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{\mu_{i,\ell,T}} \right| \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{S}_{i,\ell,T} - \mu_{i,\ell,T}}{q\tau_1} \right| / \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{\mu_{i,\ell,T}}{q\tau_1} \right| = o_p(1).$$

Now, for part (b), note that, applying parts (d), (f), (g), and (h) of Lemma OA-12, parts (b), (d), and (e) of Lemma OA-6, and the Slutsky theorem; we have

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{q\tau_1^2} \right| \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \\
& \quad \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right)^2 \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{2}{q} \sum_{r=1}^q \left\{ \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ (\underline{F}_t - E[\underline{F}_t]) \mu_{Y,\ell} + (\underline{F}_t \underline{Y}'_t - E[\underline{F}_t \underline{Y}'_t]) \alpha_{YY,\ell} \right. \right. \right. \\
& \quad \left. \left. \left. + (\underline{F}_t \underline{F}'_t - E[\underline{F}_t \underline{F}'_t]) \alpha_{YF,\ell} \} \right) \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right) \right\} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right)^2 \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right)^2 \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right) \right| \\
& + 2 \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t [\mu_{Y,\ell} + \underline{Y}'_t \alpha_{YY,\ell} + \underline{F}'_t \alpha_{YF,\ell}] \right) \right. \\
& \quad \left. \times \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \underline{F}_t \varepsilon_{\ell,t+1} \right) \right| \\
= & o_p(1)
\end{aligned}$$

Moreover, note that, for all  $N$  and  $T$  sufficiently large,

$$\begin{aligned}
& \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{\pi_{i,\ell,T}}{q\tau_1^2} \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \{ \gamma'_i E[\underline{F}_t] \mu_{Y,\ell} + \gamma'_i E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + \gamma'_i E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^q \left( \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&\geq \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left( \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right)^2 \\
&\quad (\text{by Jensen's inequality}) \\
&= \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right|^2 \\
&= \left( \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^q \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma'_i \{ E[\underline{F}_t] \mu_{Y,\ell} + E[\underline{F}_t \underline{Y}'_t] \alpha_{YY,\ell} + E[\underline{F}_t \underline{F}'_t] \alpha_{YF,\ell} \} \right| \right)^2 \\
&\geq \underline{\varepsilon}^2 > 0 \quad (\text{by Assumption 2-8}).
\end{aligned}$$

It follows that

$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{\pi_{i,\ell,T}} \right| \leq \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\bar{V}_{i,\ell,T} - \pi_{i,\ell,T}}{q\tau_1^2} \right| / \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left( \frac{\pi_{i,\ell,T}}{q\tau_1^2} \right) = o_p(1). \quad \square$$

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