Technical Appendix: Consistent Estimation, Variable Selection, and Forecasting in Factor-Augmented VAR Models*

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Abstract

Proofs to lemmas and theorems used in Consistent Estimation, Variable Selection, and Forecasting in Factor-Augmented VAR Models by Chao and Swanson (2022) are gathered in this paper.

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1 Appendix A: Proofs of the Main Theorems of the Paper

Proof of Theorem 1:

The proof of Theorem 1 requires a long series of calculations. Hence, we have divided this proof into six different steps.

Step 1:

In step 1, we shall transform the simple factor model

$$\underset{N\times 1}{Z_t} = \underset{N\times 1}{\gamma}\underset{1\times 1}{f_t} + \underset{N\times 1}{u_t},\ t = 1, ..., T \tag{1}$$

into a more convenient form. Let Π denote an $N \times N$ orthogonal matrix whose columns are the eigenvectors of the covariance matrix $\Sigma_Z = E\left[Z_t Z_t'\right]$. Without loss of generality, we can partition Π as

$$\underset{N\times N}{\Pi} = \left[\begin{array}{cc} \underset{N\times 1}{\pi_1} & \underset{N\times(N-1)}{\Pi_2} \end{array}\right]$$

where π_1 is the eigenvector associated with the largest eigenvalue of $\Sigma_Z = E\left[Z_t Z_t'\right]$, i.e., $\lambda_{(1)}\left(\Sigma_Z\right)$. By the result of Lemma B-8, we know that

$$\pi_1 = \frac{\gamma}{\left\|\gamma\right\|_2} \text{ and } \lambda_{(1)}\left(\Sigma_Z\right) = \left\|\gamma\right\|_2^2 + 1.$$

Next, we define

$$\begin{aligned}
W_t &= \Pi' Z_t \\
&= \Pi'\left(\gamma f_t + u_t\right) \\
&= \left\|\gamma\right\|_2 \Pi' \frac{\gamma}{\left\|\gamma\right\|_2} f_t + \Pi' u_t \\
&= \left\|\gamma\right\|_2 f_t \Pi' \pi_1 + \Pi' u_t \quad \left(\text{since } \pi_1 = \frac{\gamma}{\left\|\gamma\right\|_2}\right) \\
&= \left\|\gamma\right\|_2 f_t \left(\begin{array}{c} \pi_1' \\ \Pi_2' \end{array}\right) \pi_1 + \Pi' u_t \\
&= \left\|\gamma\right\|_2 f_t \mathbf{e}_{1,N} + \eta_t
\end{aligned} \tag{2}$$

where $\mathbf{e}_{1,N}$ is an elementary vector whose first component is 1 and all remaining components are

0 and where $\eta_t = \Pi' u_t$. Moreover, note that $\{\eta_t\} \equiv i.i.d.N(0, I_N)$ since Π is an orthogonal matrix and $\eta_t = \Pi' u_t$ with $\{u_t\} \equiv i.i.d.N(0, I_N)$. We can write out the covariance matrix of W_t as

$$
\begin{aligned}
& \Sigma_W \\
= \ & E\left[W_t W_t'\right] \\
= \ & E\left[\left(\|\gamma\|_2 f_t \mathbf{e}_{1,N} + \eta_t\right)\left(\|\gamma\|_2 f_t \mathbf{e}_{1,N} + \eta_t\right)'\right] \\
= \ & \|\gamma\|_2^2 E\left[f_t^2\right] \mathbf{e}_{1,N}\mathbf{e}_{1,N}' + \|\gamma\|_2 E\left[\eta_t f_t\right] \mathbf{e}_{1,N}' + \|\gamma\|_2 \mathbf{e}_{1,N} E\left[f_t \eta_t'\right] + E\left[\eta_t \eta_t'\right] \\
= \ & \|\gamma\|_2^2 \mathbf{e}_{1,N}\mathbf{e}_{1,N}' + I_N \\
= \ & \begin{pmatrix} \|\gamma\|_2^2 + 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\end{aligned}
$$

from which it is easily seen that $\lambda_{(1)}(\Sigma_W) = \|\gamma\|_2^2 + 1$ and $\lambda_{(2)}(\Sigma_W) = \lambda_{(3)}(\Sigma_W) = \cdots = \lambda_{(N)}(\Sigma_W) = 1$, where we let $\lambda_{(j)}(\Sigma_W)$ denote the j^{th} largest eigenvalue of Σ_W. In addition, the eigenvector associated with $\lambda_{(j)}(\Sigma_W)$ is $\mathbf{e}_{j,N}$, an elementary vector whose j^{th} component is 1 and all other components are 0.

Note further that we can also write W_t in the alternative form

$$\begin{aligned} W_t &= \begin{pmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{N,t} \end{pmatrix} \\ &= \begin{pmatrix} \|\gamma\|_2 f_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2,t} \\ \vdots \\ \eta_{N,t} \end{pmatrix} \\ &= \begin{pmatrix} \|\gamma\|_2 \zeta_{1,t} \\ \zeta_{2,t} \\ \vdots \\ \zeta_{N,t} \end{pmatrix} \\ &= \sum_{j=1}^{N} \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N} \end{aligned} \tag{3}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1} \eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, ..., N$ and where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, ..., N$. In fact, this is the representation of W_t that is given in Lemma B-10. (See Appendix B below).

Step 2:

Define $\underset{N \times T}{\mathbf{W}} = (W_1, ..., W_T)$, where W_t is as defined in expression (2) in step 1 above. Partition $\mathbf{W}$ as follows

$$\underset{N \times T}{\mathbf{W}} = \begin{bmatrix} \underset{1 \times T}{\mathbf{W}_1'} \\ \underset{(N-1) \times T}{\mathbf{W}_2'} \end{bmatrix} = \begin{bmatrix} \underset{1 \times T}{\pi_1' \mathbf{Z}} \\ \underset{(N-1) \times T}{\Pi_2' \mathbf{Z}} \end{bmatrix},$$

where $\underset{N \times T}{\mathbf{Z}} = (Z_1, ..., Z_T)$ with Z_t as defined in expression (1). Note that the first row of $\mathbf{W}$, i.e., $\mathbf{W}_1'$, contains the "signal" observations with the elevated variance $\lambda_1 = \|\gamma\|_2^2 + 1$ and where the remaining $N-1$ rows contain the elements of the $(N-1) \times T$ matrix $\mathbf{W}_2'$ which contain only the noise variables. Now, define the sample covariance matrix

$$\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \mathbf{W} \mathbf{W}' = \begin{pmatrix} T^{-1} \mathbf{W}_1' \mathbf{W}_1 & T^{-1} \mathbf{W}_1' \mathbf{W}_2 \\ T^{-1} \mathbf{W}_2' \mathbf{W}_1 & T^{-1} \mathbf{W}_2' \mathbf{W}_2 \end{pmatrix}$$

In this step, we shall further transform $\widehat{\Sigma}_{\mathbf{W}}$ into the so-called arrowhead matrix. To proceed, consider the spectral decomposition

$$\frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}_2'$$

where $\widetilde{\Lambda} = diag\left(\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(N)}\right)$ with $\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(N)}$ being the $N-1$ eigenvalues of $\mathbf{W}_2'\mathbf{W}_2/T$ and $\widetilde{\mathbf{B}}_2$ is an $(N-1)\times(N-1)$ orthogonal matrix whose columns are the eigenvectors of $\mathbf{W}_2'\mathbf{W}_2/T$. Note that, without loss of generality, we can assume that the eigenvalues are ordered so that $\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$. Next, create the modified data matrix

$$\underset{N\times T}{\widetilde{\mathbf{W}}} = \begin{bmatrix} \underset{1\times T}{\mathbf{W}_1'} \\ \underset{(N-1)\times T}{\widetilde{\mathbf{B}}_2'\mathbf{W}_2'} \end{bmatrix}$$

The sample covariance matrix based on the modified data matrix is then given by

$$\begin{aligned} \underset{N\times N}{\widetilde{\Sigma}_{\mathbf{W}}} &= \frac{\widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'}{T} \\ &= \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \\ T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \end{pmatrix} \\ &= \begin{pmatrix} s & \upsilon' \\ \upsilon & \widetilde{\Lambda} \end{pmatrix} \\ &= \begin{pmatrix} s & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_N \\ \upsilon_2 & \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ \upsilon_3 & 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \upsilon_N & 0 & \cdots & 0 & \widetilde{\lambda}_{(N)} \end{pmatrix} \end{aligned}$$

where $\underset{1\times 1}{s} = \mathbf{W}_1'\mathbf{W}_1/T$ and

$$\underset{(N-1)\times 1}{\upsilon} = \begin{pmatrix} \upsilon_2 \\ \vdots \\ \upsilon_N \end{pmatrix} = \frac{\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1}{T}. \tag{4}$$

Note that the non-zero entries of $\widetilde{\Sigma}_{\mathbf{W}}$ form the shape of an arrow, and so such matrices have been referred to in the linear algebra literature as an "arrowhead matrix".

An advantage of this arrowhead form is that it allows us to obtain a useful representation for the top eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}}$. This part of step 2 comes from Johnstone and Paul (2018) following an approach originally due to Nadler (2008), but for completeness we provide some details of the argument here. To proceed, let $\widehat{\lambda}_{(1)}$ denote the largest eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}}$ and let $\widetilde{\mathbf{v}}_{(1)}$ be the associated eigenvector, where, following Johnstone and Paul (2018), we will normalize $\widetilde{\mathbf{v}}_{(1)}$ to have the form $\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 & \widetilde{v}_{(1),2} & \cdots & \widetilde{v}_{(1),N} \end{pmatrix}'$, i.e., we normalize $\widetilde{\mathbf{v}}_{(1)}$ so that its first component is 1. The eigen-equation $\widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(1)} = \widehat{\lambda}_{(1)}\widetilde{\mathbf{v}}_{(1)}$ can then be written out more explicitly as

$$\begin{pmatrix} s & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_N \\ \upsilon_2 & \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ \upsilon_3 & 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \upsilon_N & 0 & \cdots & 0 & \widetilde{\lambda}_{(N)} \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \widetilde{v}_{(1),3} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \widehat{\lambda}_{(1)} \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \widetilde{v}_{(1),3} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} \tag{5}$$

Solving this system of equations, we see that

$$\widetilde{v}_{(1),j} = \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}} \text{ for } j = 2, ..., N; \tag{6}$$

where υ_j is the j^{th} component of υ as defined in expression (4). Hence,

$$\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ \upsilon_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)}\right) \\ \vdots \\ \upsilon_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)}\right) \end{pmatrix} \tag{7}$$

Moreover, since expression (5) implies that

$$\widehat{\lambda}_{(1)} = s + \upsilon_2 \widetilde{v}_{(1),2} + \cdots + \upsilon_N \widetilde{v}_{(1),N}$$

It follows from substituting the right-hand side of equation (6) for $j = 2, ..., N$ into the above expression that

$$\widehat{\lambda}_{(1)} = s + \sum_{j=2}^{N} \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}} = \frac{\mathbf{W}_1'\mathbf{W}_1}{T} + \sum_{j=2}^{N} \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}}. \tag{8}$$

Finally, in this step, we shall relate the eigenvalues and eigenvectors of $\widetilde{\Sigma}_{\mathbf{W}}$ to that of the pre-transformed sample covariance matrix of our simple factor model, i.e.,

$$\widehat{\Sigma}_Z = \frac{\mathbf{Z}\mathbf{Z}'}{T} = \frac{1}{T}\sum_{t=1}^{T} Z_t Z_t' \text{ where } \underset{N\times T}{\mathbf{Z}} = (Z_1, ..., Z_T).$$

Understanding this relationship then allows us to derive asymptotic properties of quantities involving the leading eigenvector of $\widehat{\Sigma}_Z$ using the explicit representation of $\widetilde{\mathbf{v}}_1$ and $\widehat{\lambda}_1$ given in expressions (7) and (8), respectively. To proceed, we first relate the eigenvalues and eigenvectors of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'/T$ to that of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$. Define

$$\underset{N\times N}{\widetilde{\mathbf{B}}} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix}$$

Now, since $\widetilde{\mathbf{B}}_2$ is an orthogonal matrix, it follows that

$$\widetilde{\mathbf{B}}'\widetilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2'\widetilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

and

$$\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}' = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2\widetilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

so that $\widetilde{\mathbf{B}}$ is an orthogonal matrix as well. Next, note that

$$\begin{aligned}
&\frac{\widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}}{T} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix}\begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2 \\ T^{-1}\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\mathbf{W}_2'\mathbf{W}_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \\
&= \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \\ T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \end{pmatrix} \\
&= \frac{\widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'}{T} \\
&= \widetilde{\Sigma}_{\mathbf{W}}
\end{aligned}$$

Hence, to relate the eigenvalues and eigenvectors of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ to those of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$, we note that the eigenvalues of the $\widetilde{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$
\begin{aligned}
0 &= \det\left\{\frac{\widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}}{T} - \lambda I_N\right\} \\
&= \det\left\{\widetilde{\mathbf{B}}'\right\}\det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\right\}\det\left\{\widetilde{\mathbf{B}}\right\} \\
&= \det\left\{\widetilde{\mathbf{B}}'\right\}\det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\}\det\left\{\widetilde{\mathbf{B}}\right\} \quad \left(\text{since } \widetilde{\mathbf{B}} \text{ is an orthogonal matrix}\right) \\
&= \det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\}
\end{aligned}
$$

where the last equality holds because $\det\left\{\widetilde{\mathbf{B}}'\right\} = \det\left\{\widetilde{\mathbf{B}}\right\} = \pm 1$ given that $\widetilde{\mathbf{B}}$ is an orthogonal matrix. It follows that $\widehat{\Sigma}_{\mathbf{W}}= \mathbf{W}\mathbf{W}'/T$ and $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$ have the same set of eigenvalues. Moreover, let $\widehat{\lambda}_{(j)}$ be the j^{th} largest eigenvalue of $\widehat{\Sigma}_{\mathbf{W}}= \mathbf{W}\mathbf{W}'/T$, which is of course also the j^{th} largest eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. Also, let $\widetilde{\mathbf{v}}_{(j)}$ be an eigenvector of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$ associated with $\widehat{\lambda}_{(j)}$. Define $\mathbf{v}_{(j)} \equiv \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)}$ for $j = 1, ..., N$, and note that, since $\widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\widetilde{\mathbf{v}}_{(j)}$, we have

$$
\begin{aligned}
\widetilde{\mathbf{B}}'\widehat{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} &= \left(\frac{\widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}}{T}\right)\widetilde{\mathbf{v}}_{(j)} \\
&= \widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(j)} \\
&= \widehat{\lambda}_{(j)}\widetilde{\mathbf{v}}_{(j)}
\end{aligned}
$$

which implies that

$$\widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)}=\widehat{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}$$

so that $\mathbf{v}_{(j)} = \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)}$ is an eigenvector of $\widehat{\Sigma}_{\mathbf{W}}= \mathbf{W}\mathbf{W}'/T$ associated with $\widehat{\lambda}_{(j)}$. Note, further that, previously, we have normalized the first element of $\widetilde{\mathbf{v}}_{(1)}$ to be 1. This, in turn, implies that the first element of $\mathbf{v}_{(1)}$ will be 1 as well since

$$\begin{aligned}
\mathbf{v}_{(1)} &= \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(1)} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \widetilde{\mathbf{B}}_2 \widetilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \mathbf{v}_{(1)}^{(2)} \end{pmatrix} \qquad (9)
\end{aligned}$$

where we let $\underset{(N-1)\times 1}{\widetilde{\mathbf{v}}_{(1)}^{(2)}} = \begin{pmatrix} \widetilde{v}_{(1),2} & \widetilde{v}_{(1),3} & \cdots & \widetilde{v}_{(1),N} \end{pmatrix}'$ and $\underset{(N-1)\times 1}{\mathbf{v}_{(1)}^{(2)}} = \widetilde{\mathbf{B}}_2 \widetilde{\mathbf{v}}_{(1)}^{(2)} = \begin{pmatrix} v_{(1),2} & v_{(1),3} & \cdots & v_{(1),N} \end{pmatrix}'$.

In a similar manner, we can relate the eigenvalues and eigenvectors of $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ to those of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also to those of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. In this case, note that the eigenvalues of the $\widehat{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$\begin{aligned}
0 &= \det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\} \\
&= \det\left\{\frac{\Pi'\mathbf{Z}\mathbf{Z}'\Pi}{T} - \lambda I_N\right\} \quad (\text{ since } \mathbf{W} = \Pi'\mathbf{Z}) \\
&= \det\left\{\mathbf{\Pi}'\right\} \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda \mathbf{\Pi}\mathbf{\Pi}'\right\} \det\left\{\mathbf{\Pi}\right\} \\
&= \det\left\{\mathbf{\Pi}'\right\} \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N\right\} \det\left\{\mathbf{\Pi}\right\} \\
&\quad (\text{since } \mathbf{\Pi} \text{ is an orthogonal matrix whose columns are the eigenvectors of } \Sigma_Z = E\left[Z_t Z_t'\right]) \\
&= \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N\right\}
\end{aligned}$$

where the last equality holds because $\det\{\mathbf{\Pi}'\} = \det\{\mathbf{\Pi}\} = \pm 1$ given that $\mathbf{\Pi}$ is an orthogonal matrix. It follows that $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ has the same set of eigenvaluses as $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also the same set of eigenvalues as $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. Using the same notation as above, we will then also let $\widehat{\lambda}_{(j)}$ to denote the j^{th} largest eigenvalue of $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$. Moreover, as before, let $\mathbf{v}_j$ denote an eigenvector of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ associated with $\widehat{\lambda}_{(j)}$. Now, define $\widehat{\pi}_{(j)} \equiv \Pi\mathbf{v}_{(j)}$, and note that since $\widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}$, we have, for $j = 1, ..., N$,

$$\begin{aligned}\Pi'\widehat{\Sigma}_Z\Pi\mathbf{v}_{(j)} &= \left(\frac{\Pi'\mathbf{Z}\mathbf{Z}'\Pi}{T}\right)\mathbf{v}_{(j)} \\ &= \widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)} \\ &= \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}\end{aligned}$$

which implies that

$$\widehat{\Sigma}_Z\widehat{\pi}_{(j)} = \widehat{\Sigma}_Z\Pi\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\Pi\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\widehat{\pi}_{(j)}$$

so that

$$\widehat{\pi}_{(j)} = \Pi\mathbf{v}_{(j)} \tag{10}$$

is an eigenvector of $\widehat{\Sigma}_Z$ associated with the eigenvalue $\widehat{\lambda}_{(j)}$.

Step 3:

For the simple factor model given in expression (1), i.e.,

$$\begin{aligned}Z_t &= \gamma f_t + u_t \\ &= \|\gamma\|_2\, \pi_{(1)} f_t + u_t \text{ for } t = 1, ..., T;\end{aligned}$$

with $\pi_1 = \gamma / \|\gamma\|_2$; the principal-component estimator of the latent factor f_t can be written as

$$\begin{aligned}
\widehat{f}_t &= \frac{1}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, Z_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, u_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2}, u_t \right\rangle \text{ (making use of expression (10) in step 2)} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' \pi_1}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' u_t}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_1' \mathbf{e}_{1,N}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' u_t}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} \\
& \left(\text{since } \Pi' \pi_{(1)} = \begin{pmatrix} \pi_{(1)}' \\ \Pi_{(2)}' \end{pmatrix} \pi_{(1)} = \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} = \mathbf{e}_{1,N} \text{ given that } \Pi \text{ is an orthogonal matrix} \right) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \mathbf{e}_{1,N}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \eta_t}{\left\|\Pi \mathbf{v}_1\right\|_2} \left(\text{since, by definition, } \eta_t = \Pi' u_t\right) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \eta_t \right\rangle
\end{aligned}$$

where the notation $\langle y, x \rangle = y'x$ denotes the dot product of the vectors y and x and where the last equality above follows from the fact that

$$\left\|\Pi \mathbf{v}_{(1)}\right\|_2 = \sqrt{\mathbf{v}_{(1)}' \Pi' \Pi \mathbf{v}_{(1)}} = \sqrt{\mathbf{v}_{(1)}' \mathbf{v}_{(1)}} = \left\|\mathbf{v}_{(1)}\right\|_2.$$

Next, given expression (9) in step 2, we see that

$$\begin{aligned}
\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle &= \frac{\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\mathbf{e}_{1,N}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \ \left(\text{since } \widetilde{\mathbf{v}}_{(1)} = \widetilde{\mathbf{B}}'\mathbf{v}_{(1)}\right) \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \begin{pmatrix} 1 & \mathbf{v}^{(2)\prime}_{(1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \begin{pmatrix} 1 & \mathbf{v}^{(2)}_{(1)} \end{pmatrix} \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \left\langle \mathbf{v}_{(1)}, \mathbf{e}_{1,N} \right\rangle \\
&= \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle,
\end{aligned}$$

where the last line follows from the fact that

$$\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2 = \sqrt{\widetilde{\mathbf{v}}'_{(1)}\widetilde{\mathbf{v}}_{(1)}} = \sqrt{\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\mathbf{v}_{(1)}} = \sqrt{\mathbf{v}'_{(1)}\mathbf{v}_{(1)}} = \left\|\mathbf{v}_{(1)}\right\|_2$$

since $\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}' = I_N$. In addition, let $\widetilde{\eta}_t = \widetilde{\mathbf{B}}'\eta_t$, and note that

$$\begin{aligned}
\left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \eta_t \right\rangle &= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\mathbf{v}'_{(1)}\eta_t \\
&= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\eta_t \\
&= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\widetilde{\mathbf{v}}'_{(1)}\widetilde{\eta}_t \\
&= \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \widetilde{\eta}_t \right\rangle \ \left(\text{given that } \left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2 = \left\|\mathbf{v}_{(1)}\right\|_2\right).
\end{aligned}$$

Since

$$\{\eta_t\} \equiv i.i.d.N\left(0, I_N\right)$$

and $\widetilde{\mathbf{B}}$ is an orthogonal matrix, we also have

$$\{\widetilde{\eta}_t\} \equiv i.i.d.N\left(0, I_N\right).$$

Using these calculations, we can then rewrite the expression for $\widehat{f}_t$ in terms of $\widetilde{\mathbf{v}}_{(1)}$ and $\widetilde{\eta}_t$ as follows.

$$\begin{aligned}
\widehat{f}_t &= \frac{\langle \widehat{\pi}_{(1)}, Z_t \rangle}{\sqrt{N} \left\| \widehat{\pi}_{(1)} \right\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\| \mathbf{v}_{(1)} \right\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\| \mathbf{v}_{(1)} \right\|_2}, \eta_t \right\rangle \\
&= \frac{\|\gamma\|_2}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2}, \mathbf{e}_{1,N} \right\rangle f_t + \frac{1}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2}, \widetilde{\eta}_t \right\rangle . \qquad (11)
\end{aligned}$$

Given the requirement in Assumption 2-2 that

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2} \right), \text{ as } N, T \to \infty,$$

for constants c and κ such that $0 < c < \infty$ and $0 < \kappa < 1$; it is easily seen that

$$\frac{\|\gamma\|_2}{\sqrt{N}} = O\left(\left(\frac{1}{T N^{\kappa}} \right)^{\frac{1}{2(1+\kappa)}} \right) = o(1) . \qquad (12)$$

In the next two steps of this proof, we will show that

$$\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0 \text{ and } \frac{1}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2}, \widetilde{\eta}_t \right\rangle \xrightarrow{p} 0.$$

Step 4:

We will first show that

$$\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\| \widetilde{\mathbf{v}}_1 \right\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0.$$

. To proceed, note that, from expression (7) in step 2, $\widetilde{\mathbf{v}}_1$ has the explicit form

$$\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)} \right) \\ \vdots \\ v_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)} \right) \end{pmatrix}$$

It follows that

$$
\begin{aligned}
& \frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle^2}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|^2} \\
= & \left[1 + \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2} \right]^{-1} \\
& \left(\text{since } \left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle = \left[\begin{array}{cccc} 1 & \upsilon_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)} \right) & \cdots & \upsilon_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)} \right) \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] = 1 \right) \\
= & \frac{1}{1 + \tau^2}
\end{aligned}
$$

where

$$
\tau^2 = \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2}.
$$

Next, write

$$
\begin{aligned}
\tau^2 &= \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2} \\
&= \frac{N \left\| \gamma \right\|_2^2}{T} \frac{1}{\left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 \upsilon_j^2 / \left(N \left\| \gamma \right\|_2^2 \right)}{\left(1 - \widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right)^2} \\
&= \frac{N}{T \left\| \gamma \right\|_2^{2(1+2\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 \upsilon_j^2 / \left(N \left\| \gamma \right\|_2^2 \right)}{\left(1 - \widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right)^2}
\end{aligned}
$$

Recall from step 2 that $\widehat{\lambda}_{(1)}$ is the largest eigenvalue of the sample covariance matrix

$$
\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \mathbf{W}\mathbf{W}' = \left(\begin{array}{cc} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2 \\ T^{-1}\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\mathbf{W}_2'\mathbf{W}_2 \end{array} \right)
$$

while $\widetilde{\lambda}_{(j)}$ (for $j = 2, ..., N$) is the $(j-1)^{th}$ largest eigenvalue of the submatrix $T^{-1}\mathbf{W}_2'\mathbf{W}_2$. Applying Lemma B-9 and noting that $\widehat{\Sigma}_{\mathbf{W}}$ and $T^{-1}\mathbf{W}_2'\mathbf{W}_2$ are positive semidefinite matrices whose elements are continuous random variables, we see that

$$0 \leq \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} < 1 \ a.s. \text{ for } j = 2, ..., N.$$

Note also that, by part (a) of Lemma B-5, $\widetilde{\lambda}_{(j)} = 0$ for $j = T+2, ..., N$. Hence, we can further write

$$\tau^2 \leq \frac{N}{T \left\| \gamma \right\|_2^{2(1+2\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \left(1 - \max_{2 \leq j \leq T+1} \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \right)^{-2} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 v_j^2}{N \left\| \gamma \right\|_2^2} \tag{13}$$

To analyze the asymptotic behavior of τ^2, note first that we can apply the result of Lemma B-10 in Appendix B below to obtain

$$\begin{aligned}
\frac{\widehat{\lambda}_{(1)}^2}{\left\| \gamma \right\|_2^{4(1+\kappa)}} &= \left[\frac{\widehat{\lambda}_{(1)}}{\left\| \gamma \right\|_2^{2(1+\kappa)}} \right]^2 \\
&= \left[c + \frac{1}{\left\| \gamma \right\|_2^{2\kappa}} + o_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right]^2 \\
&= c^2 \left[1 + O_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right].
\end{aligned}$$

from which it follows that

$$\frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} = \frac{1}{c^2} \left[1 + O_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right] \tag{14}$$

where $0 < 1/c^2 < \infty$ given that $0 < c < \infty$.

Next, consider $\left(1 - \max_{2 \leq j \leq T+1} \left[\widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right] \right)^{-2}$. To analyze its asymptotic behavior, we make use of Assumption 2-2, part (b) of Lemma B-5, and Lemma B-10 to obtain

$$
\begin{aligned}
& \max_{2\leq j\leq T+1} \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \\
= \; & \frac{N-1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}/\left\|\gamma\right\|_2^{2(1+\kappa)}} \frac{T}{N-1} \max_{2\leq j\leq T+1} \widetilde{\lambda}_{(j)} \\
= \; & \left[c+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\left[c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]^{-1}\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[c+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\left[c+\frac{1}{\|\gamma\|_2^{2\kappa}}\right]^{-1}\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[c+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\frac{1}{c}\left[1+\frac{1}{c\|\gamma\|_2^{2\kappa}}\right]^{-1}\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\left[1-\frac{1}{c\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right)\right]\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\|\gamma\|_2^{2\kappa}}\right]\left[1+O\left(\frac{1}{\|\gamma\|_2^{4\kappa}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right)\right]\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\|\gamma\|_2^{2\kappa}}\right]\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
1-\max_{2\leq j\leq T+1} \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} &= 1-\left[1-\frac{1}{c\|\gamma\|_2^{2\kappa}}\right]\left[1+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right] \\
&= 1-1+\frac{1}{c\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
&= \frac{1}{c\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
&= \frac{1}{c\|\gamma\|_2^{2\kappa}}\left[1+o_p(1)\right]
\end{aligned}
$$

and, thus,

$$
\left(1-\max_{2\leq j\leq T+1} \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-2} = c^2\left\|\gamma\right\|_2^{4\kappa}\left[1+o_p(1)\right]. \tag{15}
$$

Now, consider $T^{-1}\sum_{j=2}^{N} T^2 v_j^2 / \left(N \left\| \gamma \right\|_2^2 \right)$. To proceed, note first that

$$W_{1,t} = \left\| \gamma \right\|_2 f_t + \eta_{1t} = \left\| \gamma \right\|_2 f_t + \mathbf{e}_{1,N}' \Pi' u_t$$

so that, given Assumption 2-1 and given the fact that Π is an orthogonal matrix, we have that

$$\{W_{1,t}\} \equiv i.i.d.\ N\left(0, \left\| \gamma \right\|_2^2 + 1\right)$$

from which we further deduce that

$$\frac{\mathbf{W}_1}{\left\| \gamma \right\|_2} = \begin{pmatrix} W_{1,1} / \left\| \gamma \right\|_2 \\ W_{1,2} / \left\| \gamma \right\|_2 \\ \vdots \\ W_{1,T} / \left\| \gamma \right\|_2 \end{pmatrix} \sim N\left(0, \left\{ 1 + \frac{1}{\left\| \gamma \right\|_2^2} \right\} I_T \right)$$

Moreover, note that

$$\mathbf{W}_{2,t} = \begin{pmatrix} \eta_{2t} \\ \vdots \\ \eta_{Nt} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_{2,N}' \Pi' u_t \\ \vdots \\ \mathbf{e}_{N,N}' \Pi' u_t \end{pmatrix}$$

so that, under Assumption 2-1,

$$\{\mathbf{W}_{2,t}\} \equiv i.i.d. N\left(0, I_{N-1}\right)$$

By direct calculation, we have for $j = 2, ..., T + 1$

$$\begin{aligned}
E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right] &= T^2\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'E\left[\mathbf{W}_1\mathbf{W}_1'|\mathbf{W}_2\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{N\left\|\gamma\right\|_2^2 T^2} \\
&\quad \left(\text{since} \underset{(N-1)\times 1}{\upsilon} = \frac{\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1}{T}\right) \\
&= \frac{T}{N}\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'E\left[\mathbf{W}_1\mathbf{W}_1'\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{\left\|\gamma\right\|_2^2 T} \\
&\quad \text{(by independence of } \mathbf{W}_1 \text{ and } \mathbf{W}_2\text{)} \\
&= \frac{T}{N}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{T} \\
&= \frac{T}{N}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}_2'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1} \\
&\quad \left(\text{since } \frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}_2'\right) \\
&= \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}
\end{aligned}$$

In addition, by straightforward calculation, we also get for $j = 2, ..., T + 1$

$$
\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] \\
= \;& \frac{T^4}{N^2} E\left\{\left(\frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{\|\gamma\|_2^2 T^2}\right)^2 | \mathbf{W}_2\right\} \\
= \;& \frac{T^4}{N^2 T^4} \sum_{r=1}^T \sum_{s=1}^T \sum_{t=1}^T \sum_{\upsilon=1}^T \left\{ E\left[\frac{W_{1,r}}{\|\gamma\|_2} \frac{W_{1,s}}{\|\gamma\|_2} \frac{W_{1,t}}{\|\gamma\|_2} \frac{W_{1,\upsilon}}{\|\gamma\|_2} | \mathbf{W}_2\right] \left(\mathbf{W}_{2,r}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \left. \times \left(\mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \left(\mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \left(\mathbf{W}_{2,\upsilon}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right\} \\
= \;& \frac{T^4}{N^2 T^4} \sum_{t=1}^T E\left[\frac{W_{1,t}^4}{\|\gamma\|_2^4}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right)^2 \\
& + \frac{3T^4}{N^2 T^4} \left\{ \sum_{t=1}^T E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \qquad \left. \times \sum_{s \neq t} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,s} \mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right\} \\
= \;& 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \frac{T^4}{N^2 T^2} \left(\sum_{t=1}^T \frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{T}\right)^2 \\
& \left(\text{since } \frac{W_{1,t}}{\|\gamma\|_2} = f_t + \|\gamma\|_2^{-1} \eta_{1t} \sim N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right)\right) \\
= \;& 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T}{N} \widetilde{\lambda}_{(j)}\right)^2
\end{aligned}
$$

On the other hand, for $j = T+2, .., N-1$, we have

$$
E\left[\frac{T^2 \upsilon_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2\right] = \left(1 + \frac{1}{\|\gamma\|_2^2}\right) \frac{T \widetilde{\lambda}_{(j)}}{N} = 0
$$

and

$$
E\left[\frac{T^4 \upsilon_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] = 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T}{N} \widetilde{\lambda}_{(j)}\right)^2 = 0
$$

since $\widetilde{\lambda}_{(j)} = 0$ for $j > T+1$ by part (a) of Lemma B-5.

Next, we show that

$$E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\frac{1}{T}\sum_{j=2}^{N}E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right]\right)^2|\mathbf{W}_2\right\}=O_{a.s.}\left(\frac{1}{T}\right)$$

To proceed, write

$$\begin{aligned}
& E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\frac{1}{T}\sum_{j=2}^{N}E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right]\right)^2|\mathbf{W}_2\right\} \\
= \; & E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\right)^2|\mathbf{W}_2\right\} \\
= \; & \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
& +\frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\right. \\
& \qquad\qquad \left.\times\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \qquad (16)
\end{aligned}$$

Consider the second term on the right-hand side of expression (16)

$$\begin{aligned}
& \frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \\
= \; & \frac{1}{T^2}\sum_{j\neq k}E\left[\frac{T^4\upsilon_j^2\upsilon_k^2}{N^2\left\|\gamma\right\|_2^4}|\mathbf{W}_2\right]-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \qquad (17)
\end{aligned}$$

For the first term in expression (17), note that

$$
\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^2 \upsilon_k^2}{N^2 \left\|\gamma\right\|_2^4}|\mathbf{W}_2\right] \\
= \;& \frac{T^4}{N^2} E\left\{\left(\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{\left\|\gamma\right\|_2^2 T^2}\right)\right. \\
& \qquad \left. \times \left(\frac{\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}}{\left\|\gamma\right\|_2^2 T^2}\right)|\mathbf{W}_2\right\} \\
= \;& \frac{T^4}{N^2T^4}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{\upsilon=1}^{T}\left\{E\left[\frac{W_{1,r}}{\left\|\gamma\right\|_2}\frac{W_{1,s}}{\left\|\gamma\right\|_2}\frac{W_{1,t}}{\left\|\gamma\right\|_2}\frac{W_{1,\upsilon}}{\left\|\gamma\right\|_2}|\mathbf{W}_2\right]\left(\mathbf{W}_{2,r}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \left. \times\left(\mathbf{W}_{2,s}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\left(\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\left(\mathbf{W}_{2,\upsilon}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& \frac{T^4}{N^2T^4}\sum_{t=1}^{T}E\left\{\left[\frac{W_{1,t}^4}{\left\|\gamma\right\|_2^4}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad \left. \times\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
& + \frac{T^4}{N^2T^4}\left\{\sum_{s=1}^{T}E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,s}\mathbf{W}_{2,s}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad \left. \times\sum_{t\neq s}E\left[\frac{W_{1,t}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
& + \frac{T^4}{N^2T^4}\left\{\sum_{t=1}^{T}E\left[\frac{W_{1,t}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left. \times\sum_{s\neq t}E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,s}\mathbf{W}_{2,s}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
& + \frac{T^4}{N^2T^4}\left\{\sum_{r=1}^{T}E\left[\frac{W_{1,r}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,r}\mathbf{W}_{2,r}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left. \times\sum_{t\neq r}E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \qquad (18)
\end{aligned}
$$

Calculating the expectation for the first term on the right-hand side of expression (18) above, we have

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\sum_{t=1}^{T}\left\{E\left[\frac{W_{1,t}^4}{\|\gamma\|_2^4}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& \frac{3T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\}
\end{aligned}
$$

Moreover, using the fact that

$$
\begin{aligned}
\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}'_{2,s}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1} &= \mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1}\widetilde{\Lambda}\mathbf{e}_{j-1,N-1} \\
&= \widetilde{\lambda}_{(j)}
\end{aligned}
$$

and, for $j \neq k$,

$$
\begin{aligned}
\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{V}}'_2\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}'_{2,s}}{T}\widetilde{\mathbf{V}}_2\mathbf{e}_{k-1,N-1} &= \mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1}\widetilde{\Lambda}\mathbf{e}_{k-1,N-1} \\
&= 0
\end{aligned}
$$

we further obtain

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{s=1}^{T} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,s}\mathbf{W}'_{2,s}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times \sum_{t\neq s} E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \; & \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}'_{2,s}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}'_{2,t}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \; & \left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{t=1}^{T} E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\sum_{s\neq t} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,s}\mathbf{W}'_{2,s}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}'_{2,t}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& \times\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}'_{2,s}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{r=1}^{T} E\left[\frac{W_{1,r}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,r}\mathbf{W}'_{2,r}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\sum_{t\neq r} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{r=1}^{T}\frac{\mathbf{W}_{2,r}\mathbf{W}'_{2,r}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& \times\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{V}}'_2\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}'_{2,t}}{T}\widetilde{\mathbf{V}}_2\mathbf{e}_{k-1,N-1}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= \;& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\}
\end{aligned}
$$

It follows from these calculations that, for $j \neq k$

$$
\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^2 \upsilon_k^2}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] \\
= \; & \frac{3T^4}{N^2 T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \\
& \qquad \times \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right)\Big\} \\
& + \left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
& - \frac{T^4}{N^2 T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \\
& \qquad \times \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right)\Big\} \\
& - \frac{T^4}{N^2 T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \\
& \qquad \times \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right)\Big\} \\
& - \frac{T^4}{N^2 T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \\
& \qquad \times \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right)\Big\} \\
= \; & \left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
& \frac{1}{T^2}\sum_{j\neq k} E\left\{\left[\frac{T^2\upsilon_j^2}{N\|\gamma\|_2^2} - \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2\upsilon_k^2}{N\|\gamma\|_2^2} - \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right] | \mathbf{W}_2\right\} \\
= \; & \frac{1}{T^2}\sum_{j\neq k} E\left[\frac{T^4 \upsilon_j^2 \upsilon_k^2}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] - \frac{1}{T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
= \; & \frac{1}{T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) - \frac{1}{T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2 \sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
= \; & 0
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 \upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2|\mathbf{W}_2\right\} \\
= & \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2 \upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
& +\frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2 \upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2 \upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \\
= & \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2 \upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
= & \frac{1}{T^2}\sum_{j=2}^{N}E\left[\frac{T^4 \upsilon_j^4}{N^2\left\|\gamma\right\|_2^4}|\mathbf{W}_2\right]-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \\
= & \frac{3}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \\
= & \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2 \\
= & \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2 \quad \left(\text{since } \widetilde{\lambda}_{(j)}=0 \text{ for } j>T+1\right) \\
\leq & \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{N-1}{N}\right)^2 T\left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \\
& \left(\text{since } \widetilde{\lambda}_{(j)}\geq 0 \text{ for } j=2,..,T+1\right) \\
= & \frac{2}{T}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{N-1}{N}\right)^2\left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \\
= & O_{a.s.}\left(\frac{1}{T}\right) \quad \left(\text{by Lemma B-7 and by the fact that } \left\|\gamma\right\|_2^2\to\infty \text{ under Assumption 2-2}\right) \\
= & o_{a.s.}(1)
\end{aligned}
$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995), we see that there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left\{T\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \ & E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \ & E_{\mathbf{W}_2}\left[E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 |\mathbf{W}_2\right\}\right] \\
\leq \ & \overline{C}.
\end{aligned}
$$

Now, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$
\begin{aligned}
& \Pr\left\{\sqrt{T}\left|\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right| \geq C_\epsilon\right\} \\
= \ & \Pr\left\{\left(\frac{\sqrt{T}}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{\sqrt{T}}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \geq C_\epsilon^2\right\} \\
\leq \ & \frac{1}{C_\epsilon^2}E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \ & \frac{\epsilon}{\overline{C}}E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
\leq \ & \epsilon
\end{aligned}
$$

which shows that

$$
\begin{aligned}
& \frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N} \\
= \ & \frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} - \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} \quad \left(\text{since } \widetilde{\lambda}_{(j)} = 0 \text{ for } j > T+1\right) \\
= \ & O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1) \qquad (19)
\end{aligned}
$$

In addition, note that

$$
\begin{aligned}
\left|\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right)\right| &\leq \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right| \\
&\leq \left(1+\frac{1}{\|\gamma\|_2^2}\right)\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right| \stackrel{a.s.}{\to} 0 \\
&\quad \text{(by Lemma B-7)}
\end{aligned}
$$

Making use of this result and the Slutsky's theorem, we obtain

$$
\begin{aligned}
& \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} \\
= & \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left[\frac{T\widetilde{\lambda}_{(j)}}{N}-1+1\right] \\
= & \left(1+\frac{1}{\|\gamma\|_2^2}\right)+\left(1+\frac{1}{\|\gamma\|_2^2}\right)\left[\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}-1\right] \\
= & \left(1+\frac{1}{\|\gamma\|_2^2}\right)+\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right) \stackrel{a.s.}{\to} 1 \qquad (20) \\
& \text{(since } \|\gamma\|_2 \to \infty)
\end{aligned}
$$

from which we further deduce, in light of expression (19), that

$$
\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 \upsilon_j^2}{N\|\gamma\|_2^2} = \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}+O_p\left(\frac{1}{\sqrt{T}}\right) \stackrel{p}{\to} 1 \text{ as } N,T\to\infty. \qquad (21)
$$

Putting together the results given in expressions (13), (14), (15), and (21); we see that as $N,T\to\infty$ such that $T/N\to 0$

$$
\begin{aligned}
&\tau^2 \\
\leq\ & \frac{N}{T\left\|\gamma\right\|_2^{2(1+2\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2/\left\|\gamma\right\|_2^{4(1+\kappa)}} \left(1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-2} \frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2} \\
=\ & \frac{1}{\left\|\gamma\right\|_2^{2\kappa}} \frac{N}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} \frac{1}{c^2} c^2 \left\|\gamma\right\|_2^{4\kappa} \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right) \frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}\left[1+o_p(1)\right] \\
=\ & \frac{N}{T\left\|\gamma\right\|_2^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right) \frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}\left[1+o_p(1)\right] \\
=\ & O_p\left(\frac{N}{T\left\|\gamma\right\|_2^2}\right) \qquad (22)\\
& \left(\text{since } \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} \xrightarrow{p} 1 \text{ by expression (20)}\right)
\end{aligned}
$$

Moreover, since Assumption 2-2 implies that $N/\left(T\left\|\gamma\right\|_2^2\right)\rightarrow\infty$ as $N,T\rightarrow\infty$ such that $T/N\rightarrow 0$, we further deduce that

$$
\tau^2 \rightarrow \infty \text{ w.p.a.1.} \tag{23}
$$

Finally, we note that expression (23) further implies that

$$
\frac{\left\langle\widetilde{\mathbf{v}}_{(1)},\mathbf{e}_{1,N}\right\rangle^2}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|^2} = \frac{1}{1+\tau^2} \xrightarrow{p} 0 \tag{24}
$$

as $N,T\rightarrow\infty$ such that $T/N\rightarrow 0$.

Step 5:

In this step, we will show that

$$
\frac{1}{\sqrt{N}}\left\langle\frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2},\widetilde{\eta}_t\right\rangle \xrightarrow{p} 0.
$$

To proceed, write

$$\begin{aligned}
\frac{1}{\sqrt{N}}\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2},\widetilde{\eta}_t \right\rangle &= \frac{1}{\sqrt{N}}\frac{\left\langle \widetilde{\mathbf{v}}_{(1)},\widetilde{\eta}_t \right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \\
&= \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left[\widetilde{\eta}_{1t}+\sum_{j=2}^{N}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)}\right]
\end{aligned}$$

From the result given in expression (22) of Step 4 above, we have

$$\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}=\tau^2=O_p\left(\frac{N}{T\left\|\gamma\right\|_2^2}\right)$$

where $N/\left(T\left\|\gamma\right\|_2^2\right)\to\infty$ under our Assumption 2-2. This implies that

$$\frac{T\left\|\gamma\right\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}=O_p\left(1\right). \tag{25}$$

Next, note that

$$\begin{aligned}
\sum_{j=2}^{N}\upsilon_j\widetilde{\eta}_{jt} &= \sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}+\sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt} \\
&= \sum_{j=2}^{T+1}\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\widetilde{\eta}_{jt}}{T}+\sum_{j=T+2}^{N}\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\widetilde{\eta}_{jt}}{T}
\end{aligned} \tag{26}$$

Recall that $\{\widetilde{\eta}_t\}\equiv i.i.d.N\left(0,I_N\right)$ so that $\left\{\widetilde{\eta}_{j,t}\right\}\equiv i.i.d.N\left(0,1\right)$ across both j and t. Recall also that $\{f_t\}\equiv i.i.d.N\left(0,1\right)$ and f_t and $\widetilde{\eta}_s$ are independent for all s and t. In addition, since

$$\underset{T\times 1}{\mathbf{W}_1}=\begin{pmatrix}\left\|\gamma\right\|_2\left(f_1+\left\|\gamma\right\|_2^{-1}\eta_{1,1}\right)\\ \left\|\gamma\right\|_2\left(f_2+\left\|\gamma\right\|_2^{-1}\eta_{1,2}\right)\\ \vdots \\ \left\|\gamma\right\|_2\left(f_T+\left\|\gamma\right\|_2^{-1}\eta_{1,T}\right)\end{pmatrix} \text{ and } \underset{T\times(N-1)}{\mathbf{W}_2}=\begin{pmatrix}\eta_{2,1} & \eta_{3,1} & \cdots & \eta_{N-1,1}\\ \eta_{2,2} & \eta_{3,2} & \cdots & \eta_{N-1,2}\\ \vdots & \vdots & & \vdots \\ \eta_{2,T} & \eta_{3,T} & \cdots & \eta_{N-1,T}\end{pmatrix},$$

it follows that $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent. Now, focusing first on the term $\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}$ on the right-hand side of expression (26) above, note that

$$
\begin{aligned}
& E\left[\left(\sum_{j=2}^{T+1} \upsilon_j \widetilde{\eta}_{jt}\right)^2 |\mathbf{W}_2\right] \\
= & \frac{1}{T^2}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1} \widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2 E\left[\mathbf{W}_1\mathbf{W}'_1|\mathbf{W}_2\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= & \frac{1}{T^2}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1} \widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2 E\left[\mathbf{W}_1\mathbf{W}'_1\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= & \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1} \widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\left(\frac{\mathbf{W}'_2\mathbf{W}_2}{T}\right)\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= & \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1} \widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= & \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1} \widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\Lambda}\mathbf{e}_{k-1,N-1} \\
= & \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1} \widetilde{\eta}_{jt}^2\widetilde{\lambda}_{(j)}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& E\left[\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1} \upsilon_j \widetilde{\eta}_{jt}\right)^2 |\mathbf{W}_2\right] \\
= & \frac{\left(\|\gamma\|_2^2+1\right)}{T\,\|\gamma\|_2^2}\sum_{j=2}^{T+1} \widetilde{\eta}_{jt}^2\frac{T}{N-1}\widetilde{\lambda}_{(j)} \\
\leq & \frac{\left(\|\gamma\|_2^2+1\right)}{\|\gamma\|_2^2}\sqrt{\frac{1}{T}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^4}\sqrt{\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N-1}\widetilde{\lambda}_{(j)}\right)^2} \\
= & O_{a.s.}(1)
\end{aligned}
$$

given that, as $N, T \to \infty$,

$$\frac{1}{T}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^4 \overset{a.s.}{\to} 3$$

and, by Lemma B-7,

$$\begin{aligned}
\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N-1}\widetilde{\lambda}_{(j)}\right)^2 &\leq \left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \;\left(\text{since } \widetilde{\lambda}_{(j)}\geq 0 \text{ for } j=2,..,T+1\right)\\
&= \left(\frac{T}{N-1}\widetilde{\lambda}_{(2)}\right)^2 \overset{a.s.}{\to} 1.
\end{aligned}$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995), we see that there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large

$$\begin{aligned}
E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\} &= E_{\mathbf{W}_2}\left[E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2|\mathbf{W}_2\right\}\right]\\
&\leq \overline{C}.
\end{aligned}$$

Now, for any $\epsilon>0$, set $C_\epsilon=\sqrt{\overline{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$\begin{aligned}
\Pr\left\{\left|\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|\right|\geq C_\epsilon\right\} &= \Pr\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\geq C_\epsilon^2\right\}\\
&\leq \frac{1}{C_\epsilon^2}E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\}\\
&= \frac{\epsilon}{\overline{C}}E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\}\\
&\leq \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|=O_p(1). \tag{27}$$

Next, consider the second term on the right-hand side of expression (26). Define

$$\underset{T\times(N-1)}{\widetilde{D}}=\left[\begin{array}{cc}\underset{T\times T}{\widetilde{\Lambda}_1} & \underset{T\times(N-T-1)}{0}\end{array}\right]$$

where

$$\widetilde{\Lambda}_1 = \begin{pmatrix} \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{\lambda}_{(T+1)} \end{pmatrix}$$

Given that $N-1 > T$ for N, T sufficiently large and given that $\widetilde{\lambda}_{(j)} = 0$ for $j > T+1$, we have the following singular-value decomposition of $\mathbf{W}_2$:

$$\mathbf{W}_2 = \mathbb{O}\widetilde{D}\widetilde{\mathbf{B}}_2'$$

where $\mathbb{O}$ is a $T \times T$ orthogonal matrix and $\widetilde{\mathbf{B}}_2$ is as defined previously. Making use of this decomposition, we see that

$$\begin{aligned} \sum_{j=T+2}^{N} \upsilon_j \widetilde{\eta}_{jt} &= \sum_{j=T+2}^{N} \frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\widetilde{\eta}_{jt}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt}\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt}\mathbf{W}_1'\mathbb{O}\widetilde{D}\widetilde{\mathbf{B}}_2'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt}\mathbf{W}_1'\mathbb{O}\widetilde{D}\mathbf{e}_{j-1,N-1}}{T} \\ &= 0 \end{aligned}$$

Putting things together, we have

$$
\begin{aligned}
& \frac{1}{\sqrt{N}}\left|\left\langle\frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2},\widetilde{\eta}_t\right\rangle\right| \\
= & \frac{1}{\sqrt{N}}\left|\frac{\left\langle\widetilde{\mathbf{v}}_{(1)},\widetilde{\eta}_t\right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}\right| \\
= & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\sum_{j=2}^{N}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)}\right| \\
= & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt}\right| \\
& \left(\text{ noting that } \widetilde{\lambda}_j=0 \text{ for } j>T+1\right) \\
= & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}\right| \\
& \left(\text{since } \sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt}=0\right) \\
= & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}/\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{1}{\left\|\gamma\right\|_2^{2(1+\kappa)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}\right| \\
\leq & \frac{1}{\sqrt{N}}\left\{\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\right. \\
& \left.\times\left[\left|\widetilde{\eta}_{1t}\right|+\frac{1}{\widehat{\lambda}_{(1)}/\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{1}{\left\|\gamma\right\|_2^{2(1+\kappa)}}\left(1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-1}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad \times \left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\left(c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right)^{-1}\frac{\|\gamma\|_2}{\sqrt{N}}\frac{c\,\|\gamma\|_2^{2\kappa}}{\|\gamma\|_2\,\|\gamma\|_2^{2(1+\kappa)}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&\quad \left(\text{given that } \frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}}=c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \text{ for } 0<\kappa<1,\right. \\
&\quad \left.\text{and } \left(1-\max_{2\le j\le T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-1}=c\,\|\gamma\|_2^{2\kappa}\left[1+o_p(1)\right]\right) \\
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad \times \left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\sqrt{\frac{N-1}{T}}\frac{\|\gamma\|_2^{2\kappa}}{\|\gamma\|_2^{2(1+\kappa)}}\frac{\|\gamma\|_2}{\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad \times \left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\sqrt{\frac{N-1}{T}}\frac{\|\gamma\|_2}{\|\gamma\|_2^2\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= \left[\frac{T\,\|\gamma\|_2^2}{N}+\frac{T\,\|\gamma\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad \times \left[\sqrt{\frac{T\,\|\gamma\|_2^2}{N}}\frac{|\widetilde{\eta}_{1t}|}{N}+\sqrt{\frac{N-1}{N}}\frac{1}{\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= o_p(1), \qquad (28)
\end{aligned}
$$

where the last line follows from the fact that

$$
\begin{aligned}
\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right| &= O_p(1) \quad (\text{ by expression (27)}) \\
\frac{T\,\|\gamma\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2} &= O_p(1) \quad (\text{by expression (25)})
\end{aligned}
$$

and the fact that

$$\|\gamma\|_2^2 \to \infty \text{ and } \frac{T\,\|\gamma\|_2^2}{N} \to 0 \;\; (\text{by Assumption 2-2})\,.$$

Step 6:

Finally, in this last step, we bring everything together. Combining the results given in expressions (12) of step 3, (24) of step 4, and (28) of step 5 and noting the fact that $f_t = O_p(1)$, we can apply the Slutsky's theorem to deduce that

$$\widehat{f}_t = \frac{\|\gamma\|_2}{\sqrt{N}} \frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2} f_t + \frac{1}{\sqrt{N}} \frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \widetilde{\eta}_t \right\rangle}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|_2} \xrightarrow{p} 0 \text{ as } N, T \to \infty$$

which is the required result. □

Proof of Theorem 2:

To show part (a), first set

$$z = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right).$$

where $N = N_1 + N_2$. Note that by part (b) of Lemma C-16, we have

$$\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2\,(1+a)}\sqrt{\ln N} \text{ for all } N_1, N_2 \text{ sufficiently large.} \tag{29}$$

By part (a) of Assumption 3-10,

$$\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6}, \frac{\alpha_2}{2}\right\}}} \to 0 \text{ as } N_1, N_2, T \to \infty;$$

this, in turn, implies that, for some positive constant c_0

$$0 \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq c_0 \min\left\{T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}}\right\} \text{ for all } N_1, N_2, T \text{ sufficiently large,}$$

so that $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$ lies within the range of values of z for which the moderate deviation inequality given in Lemma C-17 holds. Thus, plugging $\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$ into the moderate deviation inequality

(71) given Lemma C-17, we see that there exists a positive constant A such that

$$\begin{aligned}
& P\left(|S_{i,\ell,T}| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq\ & 2\left[1-\Phi\left(\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\right]\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\} \\
=\ & 2\left[1-\left(1-\frac{\varphi}{2N}\right)\right]\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\} \\
=\ & \frac{\varphi}{N}\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^3 T^{-\frac{1-\alpha_1}{2}}\right\}
\end{aligned}$$

for $\ell \in \{1, ..., d\}$, for

$$i \in H = \{k \in \{1, ..., N\} : \gamma_k = 0\},$$

and for all N_1, N_2, T sufficiently large. Next, note that

$$\begin{aligned}
& P\left(\max_{i \in H} \sum_{\ell=1}^{d} \varpi_\ell \, |S_{i,\ell,T}| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq\ & P\left(\bigcup_{i \in H} \bigcup_{1 \leq \ell \leq d} \left\{|S_{i,\ell,T}| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right) \quad \left(\text{since } 0 \leq \varpi_\ell \leq 1 \text{ and } \sum_{\ell=1}^{d} \varpi_\ell = 1\right) \\
\leq\ & \sum_{i \in H} \sum_{\ell=1}^{d} P\left(|S_{i,\ell,T}| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \quad \text{(by union bound)} \\
\leq\ & \sum_{i \in H} \sum_{\ell=1}^{d} \frac{\varphi}{N}\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\} \\
=\ & d\frac{N_2\varphi}{N}\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^3 T^{-(1-\alpha_1)\frac{1}{2}}\right\}
\end{aligned}$$

Using the inequality bound given in expression (29) above, we further obtain, for all N_1, N_2, T

sufficiently large,

$$
\begin{aligned}
& P\left(\max_{i\in H}\sum_{\ell=1}^{d}\varpi_{\ell}\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq\;& \frac{dN_2\varphi}{N}\left\{1+A\left[1+\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^{3}T^{-\frac{(1-\alpha_1)}{2}}\right\} \\
\leq\;& \frac{dN_2\varphi}{N}\left\{1+2^{2}AT^{-\frac{(1-\alpha_1)}{2}}+2^{2}A\left[\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right]^{3}T^{-\frac{(1-\alpha_1)}{2}}\right\} \\
& \left(\text{by the inequality }\left|\sum_{i=1}^{m}a_i\right|^{r}\leq c_r\sum_{i=1}^{m}\left|a_i\right|^{r}\text{ where }c_r=m^{r-1}\text{ for }r\geq 1\right) \\
\leq\;& \frac{dN_2\varphi}{N}\left\{1+4AT^{-\frac{(1-\alpha_1)}{2}}+4A\left[\sqrt{2\left(1+a\right)}\sqrt{\ln N}\right]^{3}T^{-\frac{(1-\alpha_1)}{2}}\right\} \\
=\;& \frac{dN_2\varphi}{N}\left\{1+4AT^{-\frac{(1-\alpha_1)}{2}}+2^{\frac{7}{2}}A\left(1+a\right)^{\frac{3}{2}}\frac{\left(\ln N\right)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\right\}
\end{aligned}
$$

Finally, note that rate condition given in part (a) of Assumption 3-10, i.e.,

$$
\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6},\frac{\alpha_2}{2}\right\}}}\rightarrow 0\text{ as }N_1,N_2,T\rightarrow\infty
$$

implies that

$$
\frac{\left(\ln N\right)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\rightarrow 0\text{ as }N_1,N_2,T\rightarrow\infty,
$$

from which it follows that

$$
\begin{aligned}
& P\left(\max_{i\in H}\sum_{\ell=1}^{d}\varpi_{\ell}\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq\;& \frac{dN_2\varphi}{N}\left\{1+4AT^{-\frac{(1-\alpha_1)}{2}}+2^{\frac{7}{2}}A\left(1+a\right)^{\frac{3}{2}}\frac{\left(\ln N\right)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\right\} \\
=\;& \frac{dN_2\varphi}{N}\left[1+o\left(1\right)\right] \\
=\;& O\left(\frac{N_2\varphi}{N}\right) \\
=\;& o\left(1\right).
\end{aligned}
$$

Next, to show part (b), note that, by a similar argument as that given for part (a) above, we have

$$
\begin{aligned}
& P\left(\max_{i\in H}\max_{1\leq \ell\leq d}\left|S_{i,\ell,T}\right| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
= \ & P\left(\bigcup_{i\in H}\bigcup_{1\leq \ell\leq d}\left\{\left|S_{i,\ell,T}\right| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right) \\
\leq \ & \frac{dN_2\varphi}{N}\left\{1+4AT^{-\frac{(1-\alpha_1)}{2}}+2^{\frac{7}{2}}A\left(1+a\right)^{\frac{3}{2}}\frac{(\ln N)^{\frac{3}{2}}}{T^{\frac{1-\alpha_1}{2}}}\right\} \\
= \ & \frac{dN_2\varphi}{N}\left[1+o\left(1\right)\right] \\
= \ & O\left(\frac{N_2\varphi}{N}\right) \\
= \ & o\left(1\right). \ \Box
\end{aligned}
$$

Proof of Theorem 3:

To show part (a), note that

$$
\begin{aligned}
& P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
= \ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\overline{S}_{i,,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}+\frac{\mu_{i,,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq \ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
= \ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\sqrt{\overline{V}_{i,\ell,T}}}{\mu_{i,\ell,T}}\right|\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
= \ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right),
\end{aligned}
$$

where

$$
\mu_{i,\ell,T}=\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}.
$$

Next, let

$$\pi_{i,\ell,T} = \sum_{r=1}^{q} \left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma_i' E\left[\underline{F}_t\right] \mu_{Y,\ell} + \gamma_i' E\left[\underline{F}_t \underline{Y}_t'\right] \alpha_{YY,\ell} + \gamma_i' E\left[\underline{F}_t \underline{F}_t'\right] \alpha_{YF,\ell} \right\} \right)^2,$$

and note that, under Assumption 3-9, there exists a positive constant $\underline{c}$ such that for every $\ell \in \{1, ..., d\}$ and for all N_1, N_2, and T sufficiently large

$$\begin{aligned}
& \min_{i \in H^c} \frac{\pi_{i,\ell,T}}{q\tau_1^2} \\
= \quad & \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^{q} \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma_i' E\left[\underline{F}_t\right] \mu_{Y,\ell} + \gamma_i' E\left[\underline{F}_t \underline{Y}_t'\right] \alpha_{YY,\ell} + \gamma_i' E\left[\underline{F}_t \underline{F}_t'\right] \alpha_{YF,\ell} \right\} \right)^2 \\
= \quad & \min_{i \in H^c} \frac{1}{q} \sum_{r=1}^{q} \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t y_{\ell,t+1}\right] \right)^2 \\
\geq \quad & \min_{i \in H^c} \left(\frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t y_{\ell,t+1}\right] \right)^2 \quad \text{(by Jensen's inequality)} \\
= \quad & \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i' \underline{F}_t y_{\ell,t+1}\right] \right|^2 \\
= \quad & \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma_i' E\left[\underline{F}_t\right] \mu_{Y,\ell} + \gamma_i' E\left[\underline{F}_t \underline{Y}_t'\right] \alpha_{YY,\ell} + \gamma_i' E\left[\underline{F}_t \underline{F}_t'\right] \alpha_{YF,\ell} \right\} \right|^2 \\
\geq \quad & \underline{c}^2 > 0 \quad \text{(in light of Assumption 3-9)}
\end{aligned}$$

It follows that we can multiply and divide by $\sqrt{\pi_{i,\ell,T} / \left(q\tau_1^2\right)}$ to obtain for all N_1, N_2, and T sufficiently large

$$
\begin{aligned}
& P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right| \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\mu_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}}\right|\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
=\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}+\sqrt{\overline{V}_{i,\ell,T}/\left(q\tau_1^2\right)}-\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\right.\right. \\
& \left.\left.\times\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
=\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\left(\sqrt{\overline{V}_{i,\ell,T}}-\sqrt{\pi_{i,\ell,T}}\right)/\sqrt{\pi_{i,\ell,T}}}\right|\right.\right. \\
& \left.\left.\times\left[1-\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left(\mu_{i,\ell,T}/\left(q\tau_1\right)\right)}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\max_{k\in H^c}\left|\sqrt{\overline{V}_{k,\ell,T}}-\sqrt{\pi_{k,\ell,T}}\right|/\sqrt{\pi_{k,\ell,T}}}\right|\right.\right. \\
& \left.\left.\times\left[1-\max_{k\in H^c}\left|\frac{\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}}{\mu_{k,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\max_{k\in H^c}\sqrt{\left|\overline{V}_{k,\ell,T}-\pi_{k,\ell,T}\right|/\pi_{k,\ell,T}}}\right|\right.\right. \\
& \left.\left.\times\left[1-\max_{k\in H^c}\left|\frac{\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}}{\mu_{k,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
& \left(\text{making use of the inequality } \left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{|a-b|} \text{ for } a\geq 0 \text{ and } b\geq 0 \text{ by Lemma C-14}\right) \\
=\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\max_{k\in H^c}\sqrt{\left|\left(\overline{V}_{k,\ell,T}-\pi_{k,\ell,T}\right)/\pi_{k,\ell,T}\right|}}\right|\right.\right. \\
& \left.\left.\times\left[1-\max_{k\in H^c}\left|\frac{\overline{S}_{k,\ell,T}-\mu_{k,\ell,T}}{\mu_{k,\ell,T}}\right|\right]\right\} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)
\end{aligned}
$$

Now, let

$$\mathcal{E}_{k,\ell} = \frac{\overline{S}_{k,\ell,T} - \mu_{k,\ell,T}}{\mu_{k,\ell,T}} \text{ and } \mathcal{V}_{k,\ell} = \frac{\overline{V}_{k,\ell,T} - \pi_{k,\ell,T}}{\pi_{k,\ell,T}}.$$

By the result of part (a) of Lemma C-13, there exists a sequence of positive numbers $\{\epsilon_T\}$ such that, as $T \to \infty$, $\epsilon_T \to 0$ and

$$P\left(\max_{1\leq\ell\leq d}\max_{k\in H^c}|\mathcal{E}_{k,\ell}| \geq \epsilon_T\right) \to 0$$

In addition, by the result of part (b) of Lemma C-13, there exists a sequence of positive numbers $\{\epsilon_T^*\}$ such that, as $T \to \infty$, $\epsilon_T^* \to 0$ and

$$P\left(\max_{1\leq\ell\leq d}\max_{k\in H^c}|\mathcal{V}_{k,\ell}| \geq \epsilon_T^*\right) \to 0.$$

Hence, for all N_1, N_2, and T sufficiently large,

$$
\begin{aligned}
& P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left\{\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\left|\frac{1}{1+\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\right|\left[1-\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\right]\right\}\right. \\
& \left.\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\left|\frac{1}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\right|\left[1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\right]\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\sqrt{q}\left[\mu_{i,\ell,T}/\left(q\tau_1\right)\right]}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}}\right|\right. \\
& \left.\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
=\ & P\left(\left|\frac{1}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\right|\left[1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\right]\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\right. \\
& \left.\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\geq\ & P\left(\left|\frac{1-\epsilon_T}{1+\epsilon_T^*}\right|\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|<\epsilon_T\right. \\
& \left.\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell}\right|<\epsilon_T^*\right) \\
& +P\left(\left\{\frac{1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\min_{i\in H}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right. \\
& \left.\cap\left\{\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\geq\epsilon_T\cup\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell}\right|\geq\epsilon_T^*\right\}\right) \\
\geq\ & P\left(\left|\frac{1-\epsilon_T}{1+\epsilon_T^*}\right|\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|<\epsilon_T\right. \\
& \left.\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell}\right|<\epsilon_T^*\right) \\
& +P\left(\frac{1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\min_{i\in H}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right. \\
& \left.\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\geq\epsilon_T\right) \\
=\ & P\left(\left|\frac{1-\epsilon_T}{1+\epsilon_T^*}\right|\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|<\epsilon_T\right. \\
& \left.\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{V}_{k,\ell}\right|<\epsilon_T^*\right)+o\left(1\right)
\end{aligned}
$$

where the last equality above follows from the fact that

$$
\begin{aligned}
& P\left(\frac{1-\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|}{1+\max_{1\leq\ell\leq d}\max_{k\in H^c}\sqrt{\left|\mathcal{V}_{k,\ell}\right|}}\min_{i\in H}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right|\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right. \\
& \qquad \left.\cap\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\geq\epsilon_T\right) \\
\leq\ & P\left(\max_{1\leq\ell\leq d}\max_{k\in H^c}\left|\mathcal{E}_{k,\ell}\right|\geq\epsilon_T\right)=o\left(1\right).
\end{aligned}
$$

Moreover, making use of Assumption 3-9, the result given in part (e) of Lemma C-12, and the fact that $q=\lfloor T_0/\tau\rfloor\sim T^{1-\alpha_1}$, we see that, there exists positive constants $\underline{c}$ and $\overline{C}$ such that

$$
\begin{aligned}
\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right| &= \min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\frac{\sqrt{q}\left|\mu_{i,\ell,T}/\left(q\tau_1\right)\right|}{\sqrt{\pi_{i,\ell,T}/\left(q\tau_1^2\right)}} \\
&\geq \sqrt{q}\sum_{\ell=1}^{d}\varpi_\ell\frac{\min_{i\in H^c}\left|\mu_{i,\ell,T}/\left(q\tau_1\right)\right|}{\sqrt{\max_{i\in H^c}\pi_{i,\ell,T}/\left(q\tau_1^2\right)}} \\
&\geq \sqrt{q}\sum_{\ell=1}^{d}\varpi_\ell\frac{\underline{c}}{\sqrt{\overline{C}}} \\
&= \sqrt{q}\frac{\underline{c}}{\sqrt{\overline{C}}} \\
&\sim \sqrt{q}\sim\sqrt{\frac{T_0}{\tau}}\sim T^{(1-\alpha_1)/2}
\end{aligned}
$$

On the other hand, applying part (b) of Lemma C-16, we have, for all N_1, N_2 sufficiently large,

$$
\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\leq\sqrt{2\left(1+a\right)}\sqrt{\ln N}\sim\sqrt{\ln N},
$$

where $N=N_1+N_2$, from which we further deduce that

$$
\begin{aligned}
\frac{1}{\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)}\min_{i\in H^c}\sum_{\ell=1}^{d}\varpi_\ell\left|\frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}}\right| &\geq \frac{\underline{c}}{\sqrt{\overline{C}}}\sqrt{\frac{q}{2\left(1+a\right)\ln N}} \\
&\sim \sqrt{\frac{T^{(1-\alpha_1)}}{\ln N}}\to\infty \text{ as } N_1,N_2,T\to\infty.
\end{aligned}
$$

This is true because the condition

$$
\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6},\frac{\alpha_2}{2}\right\}}}\to 0 \text{ as } N_1,N_2,T\to\infty \quad \text{(given in Assumption 3-10 part (a))}
$$

implies that

$$\frac{\ln N}{T^{(1-\alpha_1)}} \to 0 \text{ as } N_1, N_2, T \to \infty.$$

Hence, there exists a natural number M such that, for all $N_1 \geq M, N_2 \geq M$, and $T \geq M$, we have

$$\left| \frac{1-\epsilon_T}{1+\epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^{d} \varpi_\ell \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)$$

so that

$$\begin{aligned}
& P\left(\min_{i \in H^c} \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right| \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right) \\
\geq \;& P\left(\left| \frac{1-\epsilon_T}{1+\epsilon_T^*} \right| \min_{i \in H^c} \sum_{\ell=1}^{d} \varpi_\ell \left| \frac{\mu_{i,\ell,T}}{\sqrt{\pi_{i,\ell,T}}} \right| \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{E}_{k,\ell} \right| < \epsilon_T \right. \\
& \left. \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{V}_{k,\ell} \right| < \epsilon_T^* \right) + o(1) \\
= \;& P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{E}_{k,\ell} \right| < \epsilon_T \cap \max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{V}_{k,\ell} \right| < \epsilon_T^* \right) + o(1) \\
& (\text{for all } N_1 \geq M, N_2 \geq M, \text{and } T \geq M) \\
\geq \;& P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{E}_{k,\ell} \right| < \epsilon_T \right) + P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{V}_{k,\ell} \right| < \epsilon_T^* \right) - 1 + o(1) \\
& \left(\text{using the inequality } P\left\{ \bigcap_{i=1}^{m} A_i \right\} \geq \sum_{i=1}^{m} P(A_i) - (m-1) \text{ given in Lemma C-15} \right) \\
= \;& 1 - P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{E}_{k,\ell} \right| \geq \epsilon_T \right) + 1 - P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{V}_{k,\ell} \right| \geq \epsilon_T^* \right) - 1 + o(1) \\
= \;& 1 - P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{E}_{k,\ell} \right| \geq \epsilon_T \right) - P\left(\max_{1 \leq \ell \leq d} \max_{k \in H^c} \left| \mathcal{V}_{k,\ell} \right| \geq \epsilon_T^* \right) + o(1) \\
= \;& 1 + o(1) + o(1) + o(1) \\
= \;& 1 + o(1).
\end{aligned}$$

Next, to show part (b), note that, by applying the result in part (a), we have

$$\begin{aligned}
P\left(\min_{i \in H^c} \max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right| \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right) \;&\geq\; P\left(\min_{i \in H^c} \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right| \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right) \\
&=\; 1 + o(1). \;\square
\end{aligned}$$

Proof of Theorem 4:

To proceed, note first that the principal component estimator of $\underline{F}_t$ can be written as

$$\widehat{\underline{F}}_t = \frac{\widehat{\Gamma}' Z_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1}$$

where $\widehat{\Gamma} = \sqrt{\widehat{N}_1}\widehat{B}$ and where the columns of the matrix $\widehat{B}$ are the eigenvectors associated with the Kp largest eigenvalues of the (post-variable-selection) sample covariance matrix

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0}.$$

Moreover, by the result of part (d) of Lemma D-14, the matrix $\widehat{B}$ has the representation

$$\widehat{B} = \widehat{G}_1 \widehat{V}$$

where $\widehat{G}_1$ is an $N \times Kp$ matrix, whose columns define an orthonormal basis for an invariant subspace of $\widehat{\Sigma}\left(\widehat{H^c}\right)$ and where $\widehat{V}$ is a $Kp \times Kp$ orthogonal matrix as defined in expression (129) in part (c) of Lemma D-14. (See Lemma D-14 and also Lemma D-13 for additional discussion on the origin of this representation). Making use of this representation, we can further write

$$\begin{aligned}
\widehat{\underline{F}}_t - Q'\underline{F}_t &= \frac{\sqrt{\widehat{N}_1}\widehat{V}'\widehat{G}_1' Z_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1} - Q'\underline{F}_t \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Next, note that

$$
\begin{aligned}
\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q' &= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} - Q' \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} + \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}} &= \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} \\
&= Q'\underline{F}_t + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}}\right)\underline{F}_t \\
&= Q'\underline{F}_t + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right)\underline{F}_t + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\underline{F}_t \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right)\underline{F}_t
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\underline{\widehat{F}}_t - Q'\underline{F}_t &= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right)\underline{F}_t + \left[\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\underline{F}_t \\
&\quad + \left[\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}
$$

Hence, applying the triangle inequality as well as parts (a)-(c), (g), and (i) of Lemma D-15 along with the Slutsky's theorem, we obtain

$$
\begin{aligned}
& \left\|\underline{\widehat{F}}_t - Q'\underline{F}_t\right\|_2 \\
\leq\ & \left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right\|_2 \|\underline{F}_t\|_2 + \left|\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right|\left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\right\|_2 \|\underline{F}_t\|_2 \\
& + \left|\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right|\left\|\widehat{V}'\widehat{G}_1'\right\|_2\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_2 \|\underline{F}_t\|_2 + \left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2 \\
=\ & \left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right\|_2 \|\underline{F}_t\|_2 + \left|\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right|\left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\right\|_2 \|\underline{F}_t\|_2 \\
& + \left|\left(1+\frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right|\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_2 \|\underline{F}_t\|_2 + \left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2 \\
& \left(\text{since } \left\|\widehat{V}'\widehat{G}_1'\right\|_2 = \lambda_{\max}\left(\widehat{G}_1\widehat{V}\widehat{V}'\widehat{G}_1'\right) = \lambda_{\max}\left(\widehat{V}'\widehat{G}_1'\widehat{G}_1\widehat{V}\right) = \lambda_{\max}\left(I_{Kp}\right) = 1\right) \\
=\ & o_p(1)\,O_p(1) + o_p(1)\,O_p(1)\,O_p(1) + O_p(1)\,o_p(1)\,O_p(1) + o_p(1) \\
=\ & o_p(1)\,. \ \square
\end{aligned}
$$

Proof of Theorem 5:

To proceed, note that for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we have

$$\begin{aligned}
& \left|a'\widehat{Y}_{T+h} - a'\left(\beta_0 + B_1'\underline{Y}_T + B_2'\underline{F}_T\right)\right| \\
= & \left|a'\left(\widehat{\beta}_0 + \widehat{B}_1'\underline{Y}_T + \widehat{B}_2'\underline{\widehat{F}}_T\right) - a'\left(\beta_0 + B_1'\underline{Y}_T + B_2'\underline{F}_T\right)\right| \\
= & \left|a'\left(\widehat{\beta}_0 - \beta_0\right) + a'\left(\widehat{B}_1 - B_1\right)'\underline{Y}_T\right. \\
& \left. + a'\left(\widehat{B}_2 - Q^{-1}B_2 + Q^{-1}B_2\right)'\left(\underline{\widehat{F}}_T - Q'\underline{F}_T + Q'\underline{F}_T\right) - a'B_2'\underline{F}_T\right| \\
\leq & \left|a'\left(\widehat{\beta}_0 - \beta_0\right)\right| + \left|a'\left(\widehat{B}_1 - B_1\right)'\underline{Y}_T\right| + \left|a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\underline{\widehat{F}}_T - Q'\underline{F}_T\right)\right| \\
& + \left|a'B_2'Q^{-1\prime}\left(\underline{\widehat{F}}_T - Q'\underline{F}_T\right)\right| + \left|a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'Q'\underline{F}_T\right| + \left|a'B_2'Q^{-1\prime}Q'\underline{F}_T - a'B_2'\underline{F}_T\right| \\
= & \left|a'\left(\widehat{\beta}_0 - \beta_0\right)\right| + \left|a'\left(\widehat{B}_1 - B_1\right)'\underline{Y}_T\right| + \left|a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\underline{\widehat{F}}_T - Q'\underline{F}_T\right)\right| \\
& + \left|a'B_2'Q^{-1\prime}\left(\underline{\widehat{F}}_T - Q'\underline{F}_T\right)\right| + \left|a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'Q'\underline{F}_T\right|
\end{aligned}$$

Lemma D-18 and Slutsky's theorem directly imply that

$$\left|a'\left(\widehat{\beta}_0 - \beta_0\right)\right| = o_p(1)$$

Now, applying the CS inequality, we obtain

$$\begin{aligned}
\left|a'\left(\widehat{B}_1 - B_1\right)'\underline{Y}_T\right| & \leq \sqrt{a'\left(\widehat{B}_1 - B_1\right)'\left(\widehat{B}_1 - B_1\right)a}\sqrt{\underline{Y}_T'\underline{Y}_T} \\
& = \sqrt{a'\left(\widehat{B}_1 - B_1\right)'\left(\widehat{B}_1 - B_1\right)a}\,\|\underline{Y}_T\|_2^2,
\end{aligned}$$

and

$$
\begin{aligned}
& \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right| \\
\leq \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{F}_T' Q Q' \underline{F}_T} \\
= \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{F}_T' \left(\frac{\Gamma'\Gamma}{N_1} \right)^{1/2} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma'\Gamma}{N_1} \right)^{1/2} \underline{F}_T} \\
= \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{F}_T' \left(\frac{\Gamma'\Gamma}{N_1} \right) \underline{F}_T} \\
\leq \; & \sqrt{\lambda_{\max} \left(\frac{\Gamma'\Gamma}{N_1} \right)} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \left\| \underline{F}_T \right\|_2^2 \\
\leq \; & \overline{C} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \left\| \underline{F}_T \right\|_2^2
\end{aligned}
$$

Moreover, note that

$$
\begin{aligned}
E \left[\left\| \underline{Y}_T \right\|_2^2 \right] & \leq \left(E \left\| \underline{Y}_T \right\|_2^6 \right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
& \leq \overline{C}^{\frac{1}{3}} = C < \infty \quad \text{(by Lemma C-5)}
\end{aligned}
$$

and

$$
\begin{aligned}
E \left[\left\| \underline{F}_T \right\|_2^2 \right] & \leq \left(E \left\| \underline{F}_T \right\|_2^6 \right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
& \leq \overline{C}^{\frac{1}{3}} = C < \infty \quad \text{(by Lemma C-5)}
\end{aligned}
$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{C/\epsilon}$, and Markov's inequality then implies that, for all $T > p - 1$,

$$
\Pr \left\{ \left\| \underline{Y}_T \right\|_2 \geq C_\epsilon \right\} = \Pr \left\{ \left\| \underline{Y}_T \right\|_2^2 \geq C_\epsilon^2 \right\} \leq \frac{E \left[\left\| \underline{Y}_T \right\|_2^2 \right]}{C_\epsilon^2} = \frac{\epsilon E \left[\left\| \underline{Y}_T \right\|_2^2 \right]}{C} \leq \epsilon
$$

from which it follows that

$$
\left\| \underline{Y}_T \right\|_2 = O_p(1).
$$

In a similar way, we can also show that

$$
\left\| \underline{F}_T \right\|_2 = O_p(1).
$$

Application of the result given in Lemma D-18 then allows us to deduce that

$$\left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| \leq \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \left\| \underline{Y}_T \right\|_2^2 = o_p(1)$$

and

$$\begin{aligned}
& \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right| \\
\leq \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\underline{F}_T' Q Q' \underline{F}_T} \\
\leq \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max} \left(Q Q' \right)} \left\| \underline{F}_T \right\|_2 \\
= \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right)^{\frac{1}{2}} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma' \Gamma}{N_1} \right)^{\frac{1}{2}} \right\}} \left\| \underline{F}_T \right\|_2 \\
= \; & \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \sqrt{\lambda_{\max} \left\{ \left(\frac{\Gamma' \Gamma}{N_1} \right) \right\}} \left\| \underline{F}_T \right\|_2 \\
& \left(\text{since } \widehat{V} \widehat{V}' = I_{Kp} \text{ and } \Xi \Xi' = I_{Kp} \right) \\
\leq \; & \sqrt{\overline{C}} \sqrt{a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{B}_2 - Q^{-1} B_2 \right) a} \left\| \underline{F}_T \right\|_2 \quad \text{(by Assumption 3-6)} \\
= \; & o_p(1)
\end{aligned}$$

In addition, we can apply the CS inequality to get

$$\begin{aligned}
& \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
\leq \; & \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \sqrt{\left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)} \\
\leq \; & \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
= \; & o_p(1) \quad \text{(by Lemma D-18 and part (j) of Lemma D-15 in Appendix D)}
\end{aligned}$$

and

$$
\begin{aligned}
& \left|a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
\leq \;& \sqrt{a' B_2' Q^{-1\prime} Q^{-1} B_2 a} \sqrt{\left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)} \\
= \;& \sqrt{a' B_2' Q^{-1\prime} Q^{-1} B_2 a} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq \;& \sqrt{\left[\lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \right]^{-1} \lambda_{\max} \left(B_2' B_2 \right)} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq \;& \sqrt{C^*} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \quad \text{(for some positive constant } C^* \text{ as shown in} \\
& \text{expression (148) in Appendix D. See the proof of part (d) of Lemma D-17)} \\
= \;& o_p(1) \quad \text{(by part (j) of Lemma D-15)}
\end{aligned}
$$

Putting everything together and applying Slutsky's theorem, we then obtain

$$
\begin{aligned}
& \left| a' \widehat{Y}_{T+h} - a' \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) \right| \\
\leq \;& \left| a' \left(\widehat{\beta}_0 - \beta_0 \right) \right| + \left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
& + \left| a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right| \\
= \;& o_p(1).
\end{aligned}
$$

Since the above argument holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that

$$
\widehat{Y}_{T+h} - \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) = o_p(1).
$$

as required. $\square$

2 Appendix B: Supporting Lemmas Used in the Proof of Theorem 1

In this appendix, we first state and prove a number of lemmas which are used in the proof of Theorem 1.

Lemma B-1 (Weyl's inequality): Let A, B be real, symmetric $T \times T$ matrices and let the eigenvalues $\lambda_{(i)}(A)$, $\lambda_{(i)}(B)$, and $\lambda_{(i)}(A+B)$ be arranged in decreasing (or, more generally, nonincreasing) order, so that

$$\begin{aligned}
\lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(T)}(A), \\
\lambda_{(1)}(B) &\geq \lambda_{(2)}(B) \geq \cdots \geq \lambda_{(T)}(B), \\
\lambda_{(1)}(A+B) &\geq \lambda_{(2)}(A+B) \geq \cdots \geq \lambda_{(T)}(A+B).
\end{aligned}$$

Then, for each $j = 1, 2, ..., T$, we have

$$\lambda_{(j)}(A) + \lambda_{(T)}(B) \leq \lambda_{(j)}(A+B) \leq \lambda_{(j)}(A) + \lambda_{(1)}(B).$$

Proof of Lemma B-1: This inequality is well-known, and its proof can be found in many linear algebra textbooks. See, for example, Theorem 4.3.1 and its proof on pages 181-182 of Horn and Johnson (1985). Hence, we shall not provide an explicit proof here. □

Lemma B-2: Suppose that $\|\gamma\|_2^2 \to \infty$ as $N \to \infty$, and suppose that, given N,

$$\{\zeta_{1,t,N}\} \equiv i.i.d.N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right) \text{ for } t = 1, ..., T.$$

Let $\zeta_{1,N} = \begin{pmatrix} \zeta_{1,1,N} & \zeta_{1,2,N} & \cdots & \zeta_{1,T,N} \end{pmatrix}'$ and $\underset{T\times T}{A} = T^{-1}\|\gamma\|_2^2 \zeta_{1,N}\zeta_{1,N}'$. Then, as $N, T \to \infty$ such that $T/N \to 0$, we have

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

where $\lambda_{(1)}(A)$ denotes the largest eigenvalue of the matrix A.

Proof of Lemma B-2:

Note that, since $A = \|\gamma\|_2^2 \zeta_{1,N}\zeta_{1,N}'/T$, we can write its dual a_D as

$$\underset{1\times 1}{a_D} = \frac{1}{T}\|\gamma\|_2^2 \zeta_{1,N}'\zeta_{1,N}$$

Next, write

$$\frac{1}{T}\zeta_{1,N}'\zeta_{1,N} = \frac{1}{T}\sum_{t=1}^{T}\zeta_{1,t,N}^2 = \left(1 + \frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{t=1}^{T}\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^2$$

where, by assumption,

$$\{\zeta_{1,t,N}\} \equiv i.i.d.N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right) \text{ for each } N.$$

This implies that

$$\begin{aligned} \left\{\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1/2} \zeta_{1,t,N}\right\} &\equiv i.i.d.N(0,1) \text{ and} \\ \left\{\mathcal{X}_{t,N}^*\right\} &\equiv i.i.d.\chi_1^2 \end{aligned}$$

where

$$\mathcal{X}_{t,N}^* = \left[\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1/2} \zeta_{1,t,N}\right]^2 = \left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1} \zeta_{1,t,N}^2$$

and where χ_1^2 denotes a chi-square random variable with one degree of freedom. Hence, by direct calculation, we get

$$\begin{aligned} & E\left(\frac{1}{T}\zeta_{1\cdot,N}'\zeta_{1\cdot,N} - \left[1 + \frac{1}{\|\gamma\|_2^2}\right]\right)^2 \\ = \;& E\left[\left(1 + \frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^2 - 1\right)\right]^2 \\ = \;& \left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left\{\left[\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^2 - 1\right]\left[\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,s,N}^2 - 1\right]\right\} \\ = \;& \left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \frac{1}{T^2}\sum_{t=1}^{T} E\left\{\left[\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^2 - 1\right]^2\right\} \\ = \;& \frac{2}{T}\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \quad \left(\text{since } E\left[\chi_1^2\right] = 1 \text{ and } Var\left(\chi_1^2\right) = 2\right) \\ = \;& O\left(\frac{1}{T}\right) \end{aligned}$$

Applying Markov's inequality, we then obtain

$$\frac{1}{T}\zeta_{1\cdot,N}'\zeta_{1\cdot,N} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

Hence, as $N, T \to \infty$

$$\begin{aligned}
\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} &= \frac{a_D}{\|\gamma\|_2^2} \\
&= \left(\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\|\gamma\|_2^2 \zeta'_{1\cdot,N}\zeta_{1\cdot,N} \\
&= \frac{1}{T}\zeta'_{1\cdot,N}\zeta_{1\cdot,N} \\
&= 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

where the first equality above follows from the fact that $\lambda_{(1)}(A) = \lambda_{\max}(A) = \lambda_{\max}(a_D) = a_D$ given that a_D is a scalar. This proves Lemma B-2. □

Lemma B-3: Let $X_1, X_2, ..., X_N$ be N independent T dimensional sub-Gaussian random vectors with zero mean vector and identity covariance matrix and the sub-Gaussian norms bounded by a constant C_0. Then, for every $\tau \geq 0$, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\},$$

one has

$$\begin{aligned}
\overline{w} - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N}\sum_{i=1}^N w_i X_i X_i'\right) \\
&\leq \lambda_{(1)}\left(\frac{1}{N}\sum_{i=1}^N w_i X_i X_i'\right) \\
&= \overline{w} + \max\left\{\delta, \delta^2\right\}
\end{aligned}$$

where

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}}$$

for constants $C, c > 0$, depending on C_0. Here, $|w_i|$ is bounded for all i and

$$\overline{w} = \frac{1}{N}\sum_{i=1}^N w_i.$$

Remark: Lemma B-3 is Lemma A.1 given in Appendix A of Wang and Fan (2017), and so we state this result here without proof. As discussed there, this lemma is an extension of the classical Davidson-Szarek bound. See Davidson and Szarek (2001) and Vershynin (2010) for additional discussion.

Lemma B-4: Suppose that

$$\left\{\zeta_{i,t}\right\} \equiv i.i.d.N\left(0,1\right) \text{ for } i=2,...,N;\; t=1,...,T$$

Let $\zeta_i = \begin{pmatrix} \zeta_{i,1} & \zeta_{i,2} & \cdots & \zeta_{i,T} \end{pmatrix}'$. Also, let

$$\underset{T\times T}{B} = \frac{1}{T}\sum_{i=2}^{N}\zeta_i\zeta_i'$$

and let

$$\lambda_{(1)}\left(B\right) \geq \lambda_{(2)}\left(B\right) \geq \cdot\cdot\cdot \geq \lambda_{(T)}\left(B\right)$$

denote the eigenvalues of B. Then, for $k=1,...,T$;

$$\frac{T}{N-1}\lambda_{(k)}\left(B\right) = 1 + O_p\left(\sqrt{\frac{T}{N}}\right) = 1 + o_p\left(1\right),$$

as $N, T \to \infty$ such that $T/N \to 0$.

Proof of Lemma B-4:

Applying Lemma B-3 above for the case where $\tau = \sqrt{T}$ and where $w_i = 1$ for all i, we see that, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\} = 1 - 2\exp\left\{-cT\right\},$$

the following inequality holds for any $k \in \{1,...,T\}$

$$\begin{aligned} 1 - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &\leq \lambda_{(k)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &\leq \lambda_{(1)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &= 1 + \max\left\{\delta, \delta^2\right\}. \end{aligned}$$

Since in this case

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = \left(1+C\right)\sqrt{\frac{T}{N}},$$

the above inequality relationship simplifies to

$$1-(1+C)\sqrt{\frac{T}{N}} \leq \lambda_{(k)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \leq 1+(1+C)\sqrt{\frac{T}{N}}$$

or

$$1-(1+C)\sqrt{\frac{T}{N}} \leq \frac{T}{N-1}\lambda_{(k)}\left(\frac{1}{T}\sum_{j=2}^{N}\zeta_{j\cdot}\zeta_{j\cdot}'\right) = \frac{T}{N-1}\lambda_{(k)}\left(B\right) \leq 1+(1+C)\sqrt{\frac{T}{N}}$$

This shows that, as $N, T \to \infty$ such that $T/N \to 0$,

$$\frac{T}{N-1}\lambda_{(k)}\left(B\right) = 1+O_p\left(\sqrt{\frac{T}{N}}\right) = 1+o_p\left(1\right)$$

for $k = 1, ..., T$. $\square$

Lemma B-5: Suppose that $\{\mathbf{W}_{2,t}\} \equiv i.i.d.N\left(0, I_{N-1}\right)$. Now, let

$$\underset{(N-1)\times T}{\mathbf{W}_2'} = \begin{pmatrix} \underset{(N-1)\times 1}{\mathbf{W}_{2,1}} & \underset{(N-1)\times 1}{\mathbf{W}_{2,2}} & \cdots & \underset{(N-1)\times 1}{\mathbf{W}_{2,T}} \end{pmatrix}$$

and let

$$\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$$

be the $N-1$ eigenvalues of

$$\widehat{\Sigma}_{\mathbf{W_2}} = \frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{W}_{2,t}\mathbf{W}_{2,t}'.$$

Then, the following results hold as $N, T \to \infty$ such that $T/N \to 0$.

(a)

$$\widetilde{\lambda}_{(j)} = 0 \text{ for } j = T+2, ..., N$$

(b)

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} = 1+O_p\left(\sqrt{\frac{T}{N}}\right) = 1+o_p\left(1\right).$$

Proof of Lemma B-5:

To show part (a), note that, by assumption, for N, T sufficiently large, we have $N - 1 > T$, so that $\widehat{\Sigma}_{\mathbf{W_2}} = \mathbf{W}_2'\mathbf{W}_2/T$ is a $(N-1) \times (N-1)$ matrix with rank less than or equal to T, from which it follows trivially that

$$\widetilde{\lambda}_{(j)} = 0 \text{ for } j = T+2, ..., N.$$

Next, to show part (b), first write

$$\underset{T \times (N-1)}{\mathbf{W}_2} = \begin{pmatrix} \underline{W}_{2,1} & \underline{W}_{2,2} & \cdots & \underline{W}_{2,N-1} \end{pmatrix}$$

so that $\underline{W}_{2,i}$ denotes the i^{th} column of $\mathbf{W}_2$ for $i = 1, ..., N-1$. Note that, by Sylvester's determinantal identity, the non-zero eigenvalues of $\widehat{\Sigma}_{\mathbf{W_2}} = \mathbf{W}_2'\mathbf{W}_2/T$ (i.e., $\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(T+1)}$) are the same as those of the dual matrix

$$\underset{T \times T}{\widehat{\Sigma}_{\mathbf{W_2},D}} = \frac{\mathbf{W}_2\mathbf{W}_2'}{T} = \frac{1}{T}\sum_{i=1}^{N-1} \underline{W}_{2,i}\underline{W}_{2,i}'$$

Now, under our assumptions, $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\{0,1\}$ for $t = 1, ..., T$ and $i = 1, ..., N-1$ where $\mathbf{W}_{2,t,i}$ denotes the $(t,i)^{th}$ element of $\mathbf{W}_2$. Applying Lemma B-3 above with $\tau = \sqrt{T}$, we see that, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\} = 1 - 2\exp\left\{-cT\right\},$$

the following inequality holds for any $j \in \{2, ..., T+1\}$

$$\begin{aligned} 1 - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1} \underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &\leq \lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1} \underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &\leq \lambda_{(1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1} \underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &= 1 + \max\left\{\delta, \delta^2\right\} \end{aligned}$$

where

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = (1+C)\sqrt{\frac{T}{N}}$$

Moreover, by our definition,

$$\widetilde{\lambda}_{(j)} = \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right),$$

so that, by multiplying and dividing by T, we see that

$$\begin{aligned} 1 - (1 + C) \sqrt{\frac{T}{N}} &\leq \frac{T}{N-1} \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}_{2,i}' \right) = \frac{T}{N-1} \widetilde{\lambda}_{(j)} \\ &\leq 1 + (1 + C) \sqrt{\frac{T}{N}} \end{aligned}$$

Furthermore, since the above inequality relationship above holds for any $j \in \{2, ..., T+1\}$, it must be that

$$1 - (1 + C) \sqrt{\frac{T}{N}} \leq \frac{T}{N-1} \max_{2 \leq j \leq T+1} \widetilde{\lambda}_{(j)} \leq 1 + (1 + C) \sqrt{\frac{T}{N}}$$

It follows that, as $N, T \to \infty$ such that $T/N \to 0$,

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \widetilde{\lambda}_{(j)} = 1 + O_p \left(\sqrt{\frac{T}{N}} \right) = 1 + o_p(1). \; \square$$

Lemma B-6: Let X be a $N \times T$ random matrix, and let X_{it} be the $(i,t)^{th}$ element of X. Suppose that

$$\{X_{it}\} \equiv i.i.d.\,(0,1)$$

and suppose that $E\left[X_{it}^4\right] < \infty$. Moreover, let

$$B = \frac{1}{N} X'X.$$

Then, as $N, T \to \infty$ such that $T/N \to c \in [0, 1)$,

$$\begin{aligned} \lambda_{\min}(B) &\overset{a.s.}{\to} \left(1 - \sqrt{c}\right)^2, \\ \lambda_{\max}(B) &\overset{a.s.}{\to} \left(1 + \sqrt{c}\right)^2. \end{aligned}$$

Remark: Lemma B-6 is a special case of Lemma 1 given in Shen, Shen, Zhu, and Marron (2016) and is a slightly extended version of Theorem 2 of Bai and Yin (1993). Hence, we state this result here without proof.

Lemma B-7: Suppose that $\{\mathbf{W}_{2,t}\} \equiv i.i.d.N\left(0, I_{N-1}\right)$. Let

$$\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$$

be the $N-1$ eigenvalues of

$$\widehat{\Sigma}_{\mathbf{W_2}} = \frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{W}_{2,t}\mathbf{W}_{2,t}'.$$

where $\underset{T\times(N-1)}{\mathbf{W}_2} = \begin{pmatrix} \underset{(N-1)\times 1}{\mathbf{W}_{2,1}} & \underset{(N-1)\times 1}{\mathbf{W}_{2,2}} & \cdots & \underset{(N-1)\times 1}{\mathbf{W}_{2,T}} \end{pmatrix}'$. Then, as $N, T \to \infty$ such that $T/N \to 0$,

$$\frac{T}{N-1}\widetilde{\lambda}_{(j)} \overset{a.s.}{\to} 1 \text{ for any } j \in \{2, ..., T+1\}.$$

In particular,

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} \overset{a.s.}{\to} 1$$

and

$$\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)} - 1\right| \overset{a.s.}{\to} 0.$$

Proof of Lemma B-7:

To proceed, first define the dual matrix of $\widehat{\Sigma}_{\mathbf{W_2}}$ given by

$$\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W_2},D}} = \frac{\mathbf{W}_2\mathbf{W}_2'}{T} = \frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'$$

where $\underline{W}_{2,i}$ denotes the i^{th} column of $\mathbf{W}_2$ for $i = 1, ..., N-1$. Now, since $T/\left(N-1\right) \to 0$ and since $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\left\{0,1\right\}$ for $t = 1, ..., T$ and $i = 1, ..., N-1$; it follows from applying Lemma B-6 that

$$\begin{aligned}
\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} &= \frac{T}{N-1}\max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\widehat{\Sigma}_{\mathbf{W_2},D}\right) \\
&= \frac{T}{N-1}\max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\
&= \max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \overset{a.s.}{\to} 1 \text{ as } N, T \to \infty \qquad (30)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{T}{N-1}\min_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} &= \frac{T}{N-1}\min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\widehat{\Sigma}_{\mathbf{W_2},D}\right) \\
&= \frac{T}{N-1}\min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\
&= \min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right)\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty. \qquad (31)
\end{aligned}
$$

Expressions (30) and (31) then imply that, for any $j\in\{2,...,T+1\}$,

$$\frac{T}{N-1}\widetilde{\lambda}_{(j)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty,$$

so that

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} = \frac{T}{N-1}\widetilde{\lambda}_{(2)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty.$$

In addition, note that, for any $j\in\{2,...,T+1\}$,

$$\frac{T}{N}\widetilde{\lambda}_{(j)} = \frac{N-1}{N}\frac{T}{N-1}\widetilde{\lambda}_{(j)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty$$

from which it further follows that

$$\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right| \leq \left|\frac{T}{N}\widetilde{\lambda}_{(1)}-1\right| + \left|\frac{T}{N}\widetilde{\lambda}_{(T+1)}-1\right|\overset{a.s.}{\to} 0. \ \square$$

Lemma B-8: Consider the simple factor model

$$\underset{N\times 1}{Z_t} = \underset{N\times 1}{\gamma}\underset{1\times 1}{f_t} + \underset{N\times 1}{u_t},\ t=1,...,T;$$

where we assume that $\{u_t\}\equiv i.i.d.N(0,I_N)$, $\{f_t\}\equiv i.i.d.N(0,1)$, and u_s and f_t are independent for all t,s. Let $\Sigma_Z = E[Z_tZ_t']$; then, the eigenvalues of Σ_Z are given by

$$\lambda_{(1)} = \|\gamma\|_2^2+1 \text{ and } \lambda_{(j)}=1 \text{ for } j=2,...,N.$$

Moreover, let $\pi_{(1)}$ $(N\times 1)$ be the eigenvector assocated with the top eigenvalue $\lambda_{(1)}$; then,

$$\pi_{(1)} = \frac{\gamma}{\|\gamma\|_2}.$$

Proof of Lemma B-8: To show part (a), note first that

$$\begin{aligned}\Sigma_Z &= E\left[Z_t Z_t'\right] \\ &= E\left[(\gamma f_t + u_t)\left(\gamma' f_t + u_t'\right)\right] \\ &= \gamma\gamma' + I_N\end{aligned}$$

Consider the determinantal equation

$$\begin{aligned}0 &= \det\left\{\lambda I_N - \left(\gamma\gamma' + I_N\right)\right\} \\ &= \det\left\{(\lambda - 1) I_N - \gamma\gamma'\right\} \\ &= \det\left\{\kappa I_N - \gamma\gamma'\right\} \quad (\text{where } \kappa = \lambda - 1) \\ &= \kappa^N \det\left\{I_N - \kappa^{-1}\gamma\gamma'\right\} \\ &= \kappa^N\left(1 - \kappa^{-1}\gamma'\gamma\right) \;\; (\text{by Sylvester's determinantal theorem}) \\ &= \kappa^{N-1}\left(\kappa - \gamma'\gamma\right)\end{aligned}$$

so the roots of this equation are

$$\kappa_{(1)} = \gamma'\gamma = \|\gamma\|_2^2\,,\ \kappa_{(2)} = 0, ..., \kappa_{(N)} = 0$$

and, thus,

$$\lambda_{(1)} = \gamma'\gamma + 1 = \|\gamma\|_2^2 + 1, \lambda_{(2)} = 1, ..., \lambda_{(N)} = 1.$$

Next, note that

$$\begin{aligned}\left(\gamma\gamma' + I_N\right)\gamma &= \|\gamma\|_2^2\,\gamma + \gamma \\ &= \left(\|\gamma\|_2^2 + 1\right)\gamma\end{aligned}$$

so that γ is an (unnormalized) eigenvector of the matrix $\gamma\gamma' + I_N$ associated with the eigenvalue $\lambda_{(1)} = \|\gamma\|_2^2 + 1$. It follows that we can take

$$\pi_{(1)} = \gamma / \|\gamma\|_2$$

to be the (normalized) eigenvector of $\Sigma_Z = E\left[Z_t Z_t'\right] = \gamma\gamma' + I_N$ associated with the eigenvalue

$$\lambda_{(1)} = \|\gamma\|_2^2 + 1.\ \square$$

Lemma B-9: Let $A \in M_n$ be a Hermetian matrix, let r be an integer with $1 \leq r \leq n$, and let A_r denote any $r \times r$ principal submatrix of A (obtained by deleting $n-r$ rows and the corresponding columns of A). Let the eigenvalues of A and A_r be ordered as follows

$$\begin{aligned} \lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(n)}(A), \\ \lambda_{(1)}(A_r) &\geq \lambda_{(2)}(A_r) \geq \cdots \geq \lambda_{(r)}(A_r). \end{aligned}$$

Then, for each integer k such that $1 \leq k \leq r$, we have

$$\lambda_{(k)}(A) \geq \lambda_{(k)}(A_r) \geq \lambda_{(n-[r-k])}(A)$$

so that for $r = n-1$, we have

$$\lambda_{(1)}(A) \geq \lambda_{(1)}(A_{n-1}) \geq \lambda_{(2)}(A) \geq \lambda_{(2)}(A_{n-1}) \geq \cdots \geq \lambda_{(n-1)}(A) \geq \lambda_{(n-1)}(A_{n-1}) \geq \lambda_{(n)}(A)$$

Proof of Lemma B-9: This result is essentially Theorem 4.3.15 in Horn and Johnson (1985), except that we use different notations here. A proof of this lemma can be obtained by a slight adaptation of the proof given in Horn and Johnson (1985) for Theorem 4.3.15 using our notations here.

Lemma B-10: Let

$$\underset{N \times 1}{W_t} = \sum_{j=1}^{N} \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1} \eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, ..., N$; where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, ..., N$; and where $\mathbf{e}_{j,N}$ is an $N \times 1$ elementary vector whose j^{th} component is 1 and all remaining components are 0. Suppose that $\{\eta_t\} \equiv i.i.d.N(0, I_N)$, $\{f_t\} \equiv i.i.d.N(0,1)$, and f_t and η_s are independent for all t, s. In addition, suppose that the following assumptions hold.

(i) As $N \to \infty$

$$\|\gamma\|_2 \to \infty.$$

(ii) As $N, T \to \infty$

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty$$

for some κ such that $0 < \kappa < 1$.

Moreover, let $\widehat{\lambda}_{(1)}$ denote the largest eigenvalue of the sample covariance matrix

$$\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T}\sum_{t=1}^{T} W_t W_t',$$

where $\underset{N\times T}{\mathbf{W}} = (W_1, ..., W_T)$. Then, as $N, T \to \infty$ such that $T/N \to 0$; the largest sample eigenvalue $\widehat{\lambda}_{(1)}$ satisfy

$$\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} = c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \text{ for } 0 < \kappa < 1.$$

Proof of Lemma B-10:

Following Shen, Shen, Zhu, and Marron (2016), we shall study the sample eigenvalue properties via the dual matrix

$$\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W},D}} = \frac{1}{T}\mathbf{W}'\mathbf{W}$$

which shares the same nonzero eigenvalues with the sample covariance matrix

$$\underset{N\times N}{\widehat{\Sigma}_{\mathbf{W}}} = \frac{1}{T}\mathbf{W}\mathbf{W}'.$$

Define $\zeta_{j\cdot} = \begin{pmatrix} \zeta_{j,1} & \zeta_{j,2} & \cdots & \zeta_{j,T} \end{pmatrix}'$. Since $W_t = \sum_{j=1}^{N} \sqrt{\ell_j}\zeta_{j,t}\mathbf{e}_{j,N}$, we can write

$$\frac{1}{T}W_t' W_s = \sum_{k=1}^{N}\sum_{\ell=1}^{N} \ell_k^{1/2}\ell_\ell^{1/2}\zeta_{k,t}\zeta_{\ell,s}e_{k,N}^T e_{\ell,N} = \sum_{k=1}^{N} \ell_k \zeta_{k,t}\zeta_{k,s}$$

where

$$\begin{aligned}
\ell_1 &= \|\gamma\|_2^2,\ \ell_2 = \cdots = \ell_N = 1 \\
\zeta_{1,t} &= f_t + \frac{1}{\|\gamma\|_2}\eta_{1t},\ \zeta_{2,t} = \eta_{2t}, ..., \zeta_{N,t} = \eta_{Nt}.
\end{aligned}$$

so that

$$
\begin{aligned}
&\widehat{\Sigma}_{\mathbf{W},D} \\
= \;& \frac{1}{T} \underset{T\times N}{\mathbf{W}'} \underset{N\times T}{\mathbf{W}} = \frac{1}{T} \begin{pmatrix} W_1' \\ W_2' \\ \vdots \\ W_T' \end{pmatrix} \begin{pmatrix} W_1 & W_2 & \cdots & W_T \end{pmatrix} \\
= \;& \frac{1}{T} \begin{pmatrix} W_1'W_1 & W_1'W_2 & \cdots & W_1'W_T \\ W_2'W_1 & W_2'W_2 & \cdots & W_2'W_T \\ \vdots & \vdots & & \vdots \\ W_T'W_1 & W_T'W_2 & \cdots & W_T'W_T \end{pmatrix} = \frac{1}{T} \sum_{k=1}^{N} \ell_k \begin{pmatrix} \zeta_{k,1}^2 & \zeta_{k,1}\zeta_{k,2} & \cdots & \zeta_{k.1k,T} \\ \zeta_{k,2}\zeta_{k,1} & \zeta_{k,2}^2 & \cdots & \zeta_{k,2}\zeta_{k,T} \\ \vdots & \vdots & & \vdots \\ \zeta_{k,T}\zeta_{k,1} & \zeta_{k,T}\zeta_{k,2} & \cdots & \zeta_{k,T}^2 \end{pmatrix} \\
= \;& \frac{1}{T} \sum_{k=1}^{N} \ell_k \begin{pmatrix} \zeta_{k,1} \\ \zeta_{k,2} \\ \vdots \\ \zeta_{k,T} \end{pmatrix} \begin{pmatrix} \zeta_{k,1} & \zeta_{k,2} & \cdots & \zeta_{k,T} \end{pmatrix} = \frac{1}{T} \sum_{k=1}^{N} \ell_k \zeta_{k\cdot} \zeta_{k\cdot}'
\end{aligned}
$$

which can be decomposed into sum of two matrices as follows

$$
\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W},D}} = A + B
$$

where

$$
\underset{T\times T}{A} = \frac{1}{T} \ell_1 \underset{T\times 1}{\zeta_{1\cdot}} \underset{1\times T}{\zeta_{1\cdot}'} = \frac{1}{T} \|\gamma\|_2^2 \zeta_{1\cdot}\zeta_{1\cdot}' \text{ and } B = \frac{1}{T} \sum_{k=2}^{N} \zeta_{k\cdot}\zeta_{k\cdot}'.
$$

Next, we apply Weyl's inequality (given in Lemma B-1 above) to obtain

$$
\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} \leq \frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^2} = \frac{\lambda_{(1)}(A+B)}{\|\gamma\|_2^2} \leq \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2}
$$

Moreover, as $N, T \to \infty$, $\|\gamma\|_2^2 \to \infty$ under Assumption (i); whereas Assumption (ii) states that

$$
\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty
$$

from which it follows that

$$\begin{aligned}\frac{N-1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} &= \frac{N}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}+O\left(\frac{1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\right) \\ &= c+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)+O\left(\frac{1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\right) \\ &= c+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right) \qquad (32)\end{aligned}$$

In addition, recall that the result of Lemma B-4 shows that, as $N,T\to\infty$,

$$\frac{T\lambda_{(1)}(B)}{(N-1)}=1+O_p\left(\sqrt{\frac{T}{N}}\right) \text{ and } \frac{T\lambda_{(T)}(B)}{N-1}=1+O_p\left(\sqrt{\frac{T}{N}}\right)$$

Hence, applying Lemma B-4 and Assumpton (ii); we obtain, as $N,T\to\infty$

$$\begin{aligned}\frac{1}{\left\|\gamma\right\|_2^{2\kappa}}\frac{\lambda_{(1)}(B)}{\left\|\gamma\right\|_2^2} &= \frac{(N-1)}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{T\lambda_{(1)}(B)}{(N-1)} \\ &= \left[c+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)\right]\left(1+O_p\left(\sqrt{\frac{T}{N}}\right)\right) \\ &= c+O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right) \\ \frac{1}{\left\|\gamma\right\|_2^{2\kappa}}\frac{\lambda_{(T)}(B)}{\lambda_{(1)}\left\|\gamma\right\|_2^2} &= \frac{(N-1)}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{T\lambda_{(T)}(B)}{(N-1)} \\ &= \left[c+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)\right]\left(1+O_p\left(\sqrt{\frac{T}{N}}\right)\right) \\ &= c+O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)\end{aligned}$$

which, together with the inequality relationship

$$\frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2}+\frac{\lambda_{(T)}(B)}{\left\|\gamma\right\|_2^2}\le\frac{\widehat{\lambda}_{(1)}}{\left\|\gamma\right\|_2^2}\le\frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2}+\frac{\lambda_{(1)}(B)}{\left\|\gamma\right\|_2^2}$$

and the fact that, by Lemma B-2,

$$\frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2}=1+\frac{1}{\left\|\gamma\right\|_2^2}+O_p\left(\frac{1}{\sqrt{T}}\right)$$

imply that

$$\begin{aligned}
\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2}+\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} &= \frac{1}{\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right)+O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}\sqrt{T}}\right)+c \\
&\quad +O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2}+\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(B)}{\lambda_{(1)}\|\gamma\|_2^{2\kappa}} &= \frac{1}{\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right)+O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}\sqrt{T}}\right)+c \\
&\quad +O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)
\end{aligned}$$

so that

$$\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}}=\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^2}=c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right). \ \square$$

3 Appendix C: Supporting Lemmas Used in the Proofs of Theorems 2 and 3

Lemma C-1: Let a and θ be real numbers such that $a > 0$ and $\theta \geq 1$. Also, let G be a finite non-negative integer. Then,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

Proof of Lemma C-1: By the integral test,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty \text{ for finite non-negative integer } G$$

if

$$\int_1^{\infty} x^G \exp\left\{-ax^\theta\right\} dx < \infty \text{ for finite non-negative integer } G$$

In addition, note that since, by assumption, $a > 0$ and $\theta \geq 1$, we have

$$\int_1^{\infty} x^G \exp\left\{-ax^\theta\right\} dx \leq \int_1^{\infty} x^G \exp\left\{-ax\right\} dx$$

We will first consider the case where $G = 0$. In this case, note that

$$\int_1^\infty x^0 \exp\{-ax\}\,dx = \int_1^\infty \exp\{-ax\}\,dx$$

Let $u = -ax$, so that $-\frac{du}{a} = dx$; and we have

$$\begin{aligned}
\int_1^\infty \exp\{-ax\}\,dx &= -\frac{1}{a}\int_{-a}^{-\infty} \exp\{u\}\,du \\
&= \frac{1}{a}\int_{-\infty}^{-a} \exp\{u\}\,du \\
&= \frac{\exp\{-a\}}{a} \\
&< \infty \text{ for any } a > 0. \qquad (33)
\end{aligned}$$

Next, consider the case where G is an integer such that $G \geq 1$. Here, we will show that

$$\int_1^\infty x^G \exp\{-ax\}\,dx = \left[\frac{1}{a} + \sum_{k=1}^{G} \frac{1}{a}\left(\prod_{j=0}^{k-1} \frac{G-j}{a}\right)\right] \exp\{-a\} < \infty$$

using mathematical induction. To proceed, first consider the case where $G = 1$. Let

$$\begin{aligned}
u &= x,\; du = dx \\
dv &= \exp\{-ax\}\,dx,\; v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x \exp\{-ax\}\,dx &= -\frac{x}{a}\exp\{-ax\}\Big|_1^\infty + \int_1^\infty \frac{1}{a}\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} - \frac{1}{a^2}\exp\{-ax\}\Big|_1^\infty \\
&= \frac{1}{a}\exp\{-a\} + \frac{1}{a^2}\exp\{-a\} \\
&= \left(\frac{1}{a} + \frac{1}{a^2}\right)\exp\{-a\} \\
&= \left\{\frac{1}{a} + \sum_{k=1}^{1} \frac{1}{a}\left(\prod_{j=0}^{k-1} \frac{1-j}{a}\right)\right\} \exp\{-a\} < \infty
\end{aligned}$$

Next, for $G = 2$, let

$$\begin{aligned} u &= x^2,\ du = 2x dx \\ dv &= \exp\{-ax\}\, dx,\ v = -\frac{1}{a}\exp\{-ax\}\,; \end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned} \int_1^{\infty} x^2 \exp\{-ax\}\, dx &= -\frac{x^2}{a}\exp\{-ax\}\Big|_1^{\infty} + \frac{2}{a}\int_1^{\infty} x\exp\{-ax\}\, dx \\ &= \frac{1}{a}\exp\{-a\} + \frac{2}{a}\left(\frac{1}{a} + \frac{1}{a^2}\right)\exp\{-a\} \\ &= \frac{1}{a}\exp\{-a\} + 2\left(\frac{1}{a^2} + \frac{1}{a^3}\right)\exp\{-a\} \\ &= \left(\frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3}\right)\exp\{-a\} \\ &= \left[\frac{1}{a} + \sum_{k=1}^{2}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{2-j}{a}\right)\right]\exp\{-a\} \\ &< \infty \end{aligned}$$

Now, suppose that, for some $G \geq 2$,

$$\int_1^{\infty} x^{G-1}\exp\{-ax\}\, dx = \left[\frac{1}{a} + \sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\}\,;$$

then, let

$$\begin{aligned} u &= x^G,\ du = Gx^{G-1}dx \\ dv &= \exp\{-ax\}\, dx,\ v = -\frac{1}{a}\exp\{-ax\}\,; \end{aligned}$$

and, using integration-by-parts, we have

$$
\begin{aligned}
\int_1^\infty x^G \exp\{-ax\}\, dx &= -\frac{x^G}{a}\exp\{-ax\}\Big|_1^\infty + \frac{G}{a}\int_1^\infty x^{G-1}\exp\{-ax\}\, dx \\
&= \frac{1}{a}\exp\{-a\} + \frac{G}{a}\left[\frac{1}{a} + \sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\} \\
&= \frac{1}{a}\exp\{-a\} + \left[\frac{G}{a^2} + \sum_{k=1}^{G-1}\frac{1}{a}\frac{G}{a}\left(\prod_{j=0}^{k-1}\frac{G-(j+1)}{a}\right)\right]\exp\{-a\} \\
&= \left\{\frac{1}{a} + \frac{G}{a^2} + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right) + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\right. \\
&\quad \left. + \cdots + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\times\cdots\times\left(\frac{1}{a}\right)\right\}\exp\{-a\} \\
&= \left\{\frac{1}{a} + \sum_{k=1}^{G}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-j}{a}\right)\right\}\exp\{-a\} \\
&< \infty. \qquad (34)
\end{aligned}
$$

In view of expressions (33) and (34), it then follows by the integral test for series convergence that

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

for any finite non-negative integer G and for any constants a and θ such that $a > 0$ and $\theta \geq 1$. □

Lemma C-2: Let $\{V_t\}$ be a sequence of random variables (or random vectors) defined on some probability space $(\Omega, \mathcal{F}, P)$, and let

$$X_t = g\left(V_t, V_{t-1}, ..., V_{t-\varkappa}\right)$$

be a measurable function for some finite positive integer $\varkappa$. In addition, defne $\mathcal{G}_{-\infty}^t = \sigma\left(...., X_{t-1}, X_t\right)$, $\mathcal{G}_{t+m}^\infty = \sigma\left(X_{t+m}, X_{t+m+1}, ...\right)$, $\mathcal{F}_{-\infty}^t = \sigma\left(...., V_{t-1}, V_t\right)$, and $\mathcal{F}_{t+m-\varkappa}^\infty = \sigma\left(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...\right)$. Under this setting, the following results hold.

(a) Let

$$
\begin{aligned}
\beta_{V,m-\varkappa} &= \sup_t \beta\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^t\right) - P\left(B\right)\right| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\right\}\right], \\
\beta_{X,m} &= \sup_t \beta\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P\left(H\right)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right].
\end{aligned}
$$

If $\{V_t\}$ is β-mixing with

$$\beta_{V,m-\varkappa} \leq \overline{C}_1 \exp\{-C_2(m-\varkappa)\}$$

for all $m \geq \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also β-mixing with β-mixing coefficient satisfying

$$\beta_{X,m} \leq C_1 \exp\{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp\{C_2 \varkappa\}$.

(b) Let

$$\begin{aligned}
\alpha_{V,m-\varkappa} &= \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t \sup_{G\in\mathcal{F}_{-\infty}^t, H\in\mathcal{F}_{t+m-\varkappa}^\infty} |P(G\cap H) - P(G)P(H)|, \\
\alpha_{X,m} &= \sup_t \alpha\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t \sup_{G\in\mathcal{G}_{-\infty}^t, H\in\mathcal{G}_{t+m}^\infty} |P(G\cap H) - P(G)P(H)|
\end{aligned}$$

If $\{V_t\}$ is α-mixing with

$$\alpha_{V,m-\varkappa} \leq \overline{C}_1 \exp\{-C_2(m-\varkappa)\}$$

for all $m \geq \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also α-mixing with α-mixing coefficient satisfying

$$\alpha_{X,m} \leq C_1 \exp\{-C_2 m\} \text{ for all } m \geq \varkappa,$$

where C_1 is a positive constant such that $C_1 \geq \overline{C}_1 \exp\{C_2 \varkappa\}$.

Proof of Lemma C-2:

To show part (a), note first that it is well known that

$$\begin{aligned}
\beta_{X,m} &= \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P(H)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right] \\
&= \sup_t \left\{\frac{1}{2}\sup \sum_{i=1}^I \sum_{j=1}^J |P(G_i \cap H_j) - P(G_i)P(H_j)|\right\}
\end{aligned}$$

where the second supremum on the last line above is taken over all pairs of finite partitions $\{G_1, ..., G_I\}$ and $\{H_1, ..., H_J\}$ of Ω such that $G_i \in \mathcal{G}_{-\infty}^t$ for $i = 1, ..., I$ and $H_j \in \mathcal{G}_{t+m}^\infty$ for

$j = 1, ..., J$. See, for example, Borovkova, Burton, and Dehling (2001). Similarly,

$$\begin{aligned}
\beta_{V,m-\varkappa} &= \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^t\right) - P\left(B\right)\right| : B \in \mathcal{F}_{t+m-\varkappa}^{\infty}\right\}\right] \\
&= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P\left(A_i \cap B_j\right) - P\left(A_i\right)P\left(B_j\right)\right|\right\}
\end{aligned}$$

where, similar to the definition of $\beta_{X,m}$, the second supremum on the last line above is taken over all pairs of finite partitions $\{A_1, ..., A_L\}$ and $\{B_1, ..., B_M\}$ of Ω such that $A_i \in \mathcal{F}_{-\infty}^t$ for $i = 1, ..., I$ and $B_j \in \mathcal{F}_{t+m-\varkappa}^{\infty}$ for $j = 1, ..., M$. Moreover, since X_t is measurable on any σ-field on which $V_t, V_{t-1}, ..., V_{t-\varkappa}$ are measurable, we also have

$$\mathcal{G}_{-\infty}^t = \sigma\left(...., X_{t-1}, X_t\right) \subseteq \sigma\left(...., V_{t-1}, V_t\right) = \mathcal{F}_{-\infty}^t$$

and

$$\mathcal{G}_{t+m}^{\infty} = \sigma\left(X_{t+m}, X_{t+m+1}, ...\right) \subseteq \sigma\left(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...\right) = \mathcal{F}_{t+m-\varkappa}^{\infty}.$$

It, thus, follows that, for all $m \geq \varkappa$,

$$\begin{aligned}
\beta_{X,m} &= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{I}\sum_{j=1}^{J}\left|P\left(G_i \cap H_j\right) - P\left(G_i\right)P\left(H_j\right)\right|\right\} \\
&\leq \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P\left(A_i \cap B_j\right) - P\left(A_i\right)P\left(B_j\right)\right|\right\} \\
&= \beta_{V,m-\varkappa} \\
&\leq \overline{C}_1 \exp\left\{-C_2\left(m - \varkappa\right)\right\} \\
&= \overline{C}_1 \exp\left\{C_2\varkappa\right\}\exp\left\{-C_2 m\right\} \\
&\leq C_1 \exp\left\{-C_2 m\right\}
\end{aligned}$$

for some positive constant $C_1 \geq \overline{C}_1 \exp\{C_2\varkappa\}$ which exists given that $\varkappa$ is fixed. Moreover, we have

$$\beta_{X,m} \leq C_1 \exp\left\{-C_2 m\right\} \to 0 \text{ as } m \to \infty,$$

which establishes the required result for part (a).

Part (b) can be shown in a manner similar to part (a), so to avoid redundancy, we do not include an explicit proof here. □

Remark: Note that part (b) of Lemma C-2 is similar to a result given in Theorem 14.1 of Davidson

(1994) but adapted to suit our situation and our notatons here. Indeed, parts (a) and (b) of this lemma are both well-known results in the probability literature. We have chosen to state these results explicitly here only so that we can more easily refer to them in the proofs of some of our other results.

Lemma C-3: Let $\{X_t\}$ be a sequence of random variables that is α-mixing. Let $p > 1$ and $r \geq p/(p-1)$, and let $q = \max\{p, r\}$. Suppose that, for all t,

$$\|X_t\|_q = (E|X_t|^q)^{\frac{1}{q}} < \infty$$

Then,

$$|Cov(X_t, X_{t+m})| \leq 2\left(2^{1-1/p} + 1\right)\alpha_m^{1-1/p-1/r}\|X_t\|_p\|X_{t+m}\|_r$$

where

$$\alpha_m = \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^{\infty}\right) = \sup_{G\in\mathcal{F}_{-\infty}^t, H\in\mathcal{F}_{t+m}^{\infty}} |P(G\cap H) - P(G)P(H)|.$$

Remark: This is Corollary 14.3 of Davidson (1994). For a proof, see pages 212-213 of Davidson (1994).

Lemma C-4: Suppose that Assumption 3-3 hold. Let $\tau_1 = \lfloor T_0^{\alpha_1}\rfloor$, where $1 > \alpha_1 > 0$ and $T_0 = T - p + 1$. Then,

(a)

$$\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}]| = O\left(\frac{1}{\tau_1}\right)$$

(b)

$$\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} |E(u_{ih}u_{iv}u_{iw})| = O\left(\frac{1}{\tau_1^2}\right)$$

(c)

$$\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} |E[u_{ig}u_{ih}u_{iv}u_{iw}]| = O\left(\frac{1}{\tau_1^2}\right)$$

Proof of Lemma C-4:

To show part (a), first write

$$\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p \\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}\right]\right| = \frac{1}{\tau_1^2}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[u_{ig}^2\right] + \frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}\right]\right| \tag{35}$$

Consider now the first term on the right-hand side of expression (35). Note that, trivially, by Assumption 3-3(b), there exists a positive constant C such that

$$\frac{1}{\tau_1^2}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[u_{ig}^2\right] \leq \frac{C}{\tau_1} = O\left(\frac{1}{\tau_1}\right) \tag{36}$$

For the second term on the right-hand side of expression (35), note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right) \leq a_1 \exp\left\{-a_2 m\right\}.$$

for every i. Since $\alpha_{i,m} \leq \beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\left\{-a_2 m\right\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 6$ and $r = 5/4$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}\right]\right| \\
\leq\ &\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p \\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}
\end{aligned}$$

Next, by application of Liapunov's inequality, we have that there exists some positive constant $\overline{C}$ such that

$$\begin{aligned}
\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}} \\
&\leq \left(\sup_t E\left|u_{it}\right|^6\right)^{\frac{1}{3}} \\
&= \overline{C}^{\frac{1}{3}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho = h - g$, so that $h = g + \varrho$. Using these notations and the boundedness of $\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}\right]\right| \\
& \frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\; & \frac{\overline{C}^{\frac{1}{3}}}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{\frac{1}{30}} \\
\leq\; & \frac{C^*}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)\overline{C}^{\frac{1}{3}}a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
\leq\; & \frac{C^*}{\tau_1^2}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\; & \frac{C^*}{\tau_1}\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\; & O\left(\frac{1}{\tau_1}\right) \quad \text{(given Lemma C-1)}
\end{aligned}
\tag{37}
$$

It follows from expressions (35), (36), and (37) that

$$
\begin{aligned}
\frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}\right]\right| &= \frac{1}{\tau_1^2}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[u_{ig}^2\right] + \frac{1}{\tau_1^2}\sum_{\substack{g,h=(r-1)\tau+p\\ g<h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}\right]\right| \\
&= O\left(\frac{1}{\tau_1}\right) + O\left(\frac{1}{\tau_1}\right) \\
&= O\left(\frac{1}{\tau_1}\right).
\end{aligned}
$$

To show part (b), first write

$$\begin{aligned}\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| &= \frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3+\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\ &\quad +\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \qquad (38)\end{aligned}$$

For the first term on the right-hand side of expression (38) above, note that, trivially, we can apply Assumption 3-3(b) to obtain

$$\frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3\leq\frac{C}{\tau_1^2}=O\left(\frac{1}{\tau_1^2}\right). \qquad (39)$$

Next, for the second term on the right-hand side of expression (38) above, we can apply Lemma C-3 with $p=6$ and $r=5/4$ to obtain

$$\begin{aligned}&\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\ \leq\ &\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)^{\theta}\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\end{aligned}$$

Next, by application of Hölder's inequality, we have

$$\begin{aligned}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \\ &= \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{iw}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}} \\ &\leq \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\ &\quad \text{(by Liapunov's inequality)} \\ &= \overline{C}^{\frac{1}{2}}<\infty\ \text{ (by Assumption 3-3(b))}\end{aligned}$$

Moreover, let $\varrho_1=v-h$ and $\varrho_2=w-v$, so that $v=h+\varrho_1$ and $w=v+\ \varrho_2=h+\varrho_1+\ \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ & \frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\ & \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\leq v\leq w\\ v-h>w-v,v-h>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)\overline{C}^{\frac{1}{2}}a_1^{\frac{1}{30}}\leq C^*<\infty\right) \\
\leq\ & \frac{C^*}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1-1}\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
\leq\ & \frac{C^*}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & \frac{C^*}{\tau_1^2}\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & O\left(\frac{1}{\tau_1^2}\right)\quad\text{(given Lemma C-1)} \qquad (40)
\end{aligned}
$$

Similarly, for the third term on the right-hand side of expression (38), we can apply Lemma

C-3 with $p = 6$ and $r = 5/4$ to obtain

$$
\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
\leq\ & \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w \\ w-v\geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}}
\end{aligned}
$$

Next, by applying Hölder's inequality, we have

$$
\begin{aligned}
\left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} &\leq \left(\left(E\left|u_{ih}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}} \left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\
&= \left(E\left|u_{ih}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E\left|u_{iv}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\
&\leq \left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}} \left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= \overline{C}^{\frac{1}{2}} < \infty \ \text{ (by Assumption 3-3(b))}
\end{aligned}
$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$, so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left(u_{ih} u_{iv} u_{iw}\right) \right| \\
\leq\ & \frac{1}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \left(E\left|u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \\
\leq\ & \frac{\overline{C}^{\frac{1}{2}}}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{\tau_1^3} \sum_{\substack{h,v,w=(r-1)\tau+p \\ h \leq v \leq w \\ w-v \geq v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)\overline{C}^{\frac{1}{2}} a_1^{\frac{1}{30}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^*}{\tau_1^3} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_2=1}^{\infty} \left(\varrho_2+1\right) \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^*}{\tau_1^2} \left[\sum_{\varrho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{30}\varrho_2\right\} + \sum_{\varrho_2=1}^{\infty} \exp\left\{-\frac{a_2}{30}\varrho_2\right\}\right] \\
=\ & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma C-1)} \qquad (41)
\end{aligned}
$$

It follows from expressions (38), (39), (40), and (41) that

$$\begin{aligned}
\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\le v\le w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| &= \frac{1}{\tau_1^3}\sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left|u_{ih}\right|^3+\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\le v\le w\\ v-h>w-v,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\\
&\quad+\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p\\ h\le v\le w\\ w-v\ge v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\\
&= O\left(\frac{1}{\tau_1^2}\right)+O\left(\frac{1}{\tau_1^2}\right)+O\left(\frac{1}{\tau_1^2}\right)\\
&= O\left(\frac{1}{\tau_1^2}\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
&\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
= &\frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}^3\right]\right| + \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
&+\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
= &\frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}^3\right]\right| + \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)+E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&+\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)+E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq &\frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}^3\right]\right| + \frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&+\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&+\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g\leq h\leq v\leq w\\ w-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \qquad (42)
\end{aligned}$$

For the first term on the right-hand side of expression (42) above, note that, trivially, by Jensen's inequality and Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{ig}u_{ih}^3\right]\right| &\leq \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} E\left[\left|u_{ig}u_{ih}^3\right|\right] \\
&\leq \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left|u_{ig}\right|^2}\sqrt{E\left|u_{ih}\right|^6} \\
&\leq \frac{1}{\tau_1^4}\sum_{\substack{g,h=(r-1)\tau+p\\ g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left(E\left|u_{ih}\right|^6\right)^{\frac{1}{6}}\sqrt{E\left|u_{ih}\right|^6} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{\overline{C}^{\frac{2}{3}}\tau_1^2}{\tau_1^4} \quad \text{(by Assumption 3-3(b))} \\
&= O\left(\frac{1}{\tau_1^2}\right) \qquad (43)
\end{aligned}$$

Next, for the second term on the right-hand side of expression (42), we can apply Lemma C-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g<h<v<w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq\ &\frac{1}{\tau_1^4}\sum_{\substack{g,h,v,w=(r-1)\tau+p\\ g<h<v<w\\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}}\right. \\
&\qquad\qquad \left.\times\left(E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}}\right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$
\begin{aligned}
E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}} &\leq \left[E\left(u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right)^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}+\left|E\left[u_{ig}u_{ih}\right]\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}+E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}}E\left|u_{ig}u_{ih}\right|^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\left(E\left|u_{ih}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\right]^{\frac{7}{9}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\left(E\left|u_{ih}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\right]^{\frac{4}{3}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{4}{3}}\left[E\left|u_{iv}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left(\overline{C}\right)^{\frac{2}{9}}\left(\overline{C}\right)^{\frac{2}{9}}\left(\overline{C}\right)^{\frac{2}{9}} \quad \text{(by Assumption 3-3(b))} \\
&= 2^{\frac{4}{3}}\overline{C}^{\frac{2}{3}}
\end{aligned}
$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}\right|^{\frac{4}{3}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g<h<v<w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{ u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\} u_{iv}u_{iw}\right]\right| \\
\leq \ & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g<h<v<w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \qquad\qquad \left. \times \left(E\left|\left\{ u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\} u_{iv}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \right\} \\
\leq \ & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g<h<v<w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{4}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{12}} \left(2^{\frac{4}{3}}\overline{C}^{\frac{2}{3}}\right)^{\frac{3}{4}} \left(\overline{C}\right)^{\frac{1}{6}} \\
\leq \ & \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g<h<v<w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{4}}+1\right)\overline{C}^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty\right) \\
\leq \ & \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \ \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \ \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2-1} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
\leq \ & \frac{C^*}{\tau_1^2} \sum_{\rho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
= \ & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma C-1)} \qquad (44)
\end{aligned}
$$

Similarly, for the third term on the right-hand side of expression (42) above, we can apply

Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$\begin{aligned}
&\frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v \le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{ u_{ig} u_{ih} - E\left(u_{ig} u_{ih} \right) \right\} u_{iv} u_{iw} \right] \right| \\
\le\ & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g \le h \le v \le w \\ w-v \le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \Big\{ 2\left(2^{1-\frac{1}{2}} + 1 \right) \left[a_1 \exp\left\{ -a_2 \left(v - h \right) \right\} \right]^{1-\frac{1}{2}-\frac{1}{3}} \\
& \qquad \times \left(E\left| \left\{ u_{ig} u_{ih} - E\left(u_{ig} u_{ih} \right) \right\} \right|^2 \right)^{\frac{1}{2}} \left(E\left| u_{iv} u_{iw} \right|^3 \right)^{\frac{1}{3}} \Big\}
\end{aligned}$$

Next, applications of Hölder's inequality yield

$$\begin{aligned}
E\left| u_{iv} u_{iw} \right|^3 &\le \left(E\left| u_{iv} \right|^6 \right)^{\frac{1}{2}} \left(E\left| u_{iw} \right|^6 \right)^{\frac{1}{2}} \\
&\le \left(\overline{C} \right)^{\frac{1}{2}} \left(\overline{C} \right)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\
&= \overline{C} < \infty
\end{aligned}$$

and

$$\begin{aligned}
E\left| \left\{ u_{ig} u_{ih} - E\left(u_{ig} u_{ih} \right) \right\} \right|^2 &\le 2\left(E\left| u_{ig} u_{ih} \right|^2 + E\left| u_{ig} u_{ih} \right|^2 \right) \\
&\quad \text{(by Loève's } c_r \text{ inequality and Jensen's inequality)} \\
&= 4E\left| u_{ig} u_{ih} \right|^2 \\
&\le 4\left[\left(E\left| u_{ig} \right|^4 \right)^{\frac{1}{4}} \left(E\left| u_{ih} \right|^4 \right)^{\frac{1}{4}} \right]^2 \\
&\le 4\left[\left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^6 \right)^{\frac{1}{6}} \right]^2 \quad \text{(by Liapunov's inequality)} \\
&\le 4\left(\sup_{i,t} E\left| u_{it} \right|^6 \right)^{\frac{2}{3}} \\
&\le 4\left(\overline{C} \right)^{\frac{2}{3}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = v - h$ and $\varrho_2 = w - v$ so that $v = h + \varrho_1$ and $w = v + \varrho_2 = h + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left| u_{iv} u_{iw} \right|^3$ and $E\left| \left\{ u_{ig} u_{ih} - E\left(u_{ig} u_{ih} \right) \right\} \right|^2$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\le h\le v\le w \\ w-v\le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\left\{ u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right) \right\} u_{iv}u_{iw} \right] \right| \\
\le & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\le h\le v\le w \\ w-v\le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2\left(2^{1-\frac{1}{2}}+1\right) \left[a_1 \exp\left\{-a_2\left(v-h\right)\right\} \right]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{ig}u_{ih} - E\left(u_{ig}u_{ih}\right)\right\}\right|^2 \right)^{\frac{1}{2}} \left(E\left|u_{iv}u_{iw}\right|^3 \right)^{\frac{1}{3}} \right\} \\
\le & \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\le h\le v\le w \\ w-v\le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{2}}+1\right) \left[a_1 \exp\left\{-a_2\left(v-h\right)\right\} \right]^{\frac{1}{6}} \left(4\overline{C}^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(\overline{C}\right)^{\frac{1}{3}} \\
\le & \frac{C^*}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\le h\le v\le w \\ w-v\le v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)\overline{C}^{\frac{2}{3}} a_1^{\frac{1}{6}} \le C^* < \infty \right) \\
\le & \frac{C^*}{\tau_1^4} \sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{h=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & \frac{C^*}{\tau_1^2} \sum_{\rho_1=1}^{\infty} \left(\varrho_1+1\right) \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & O\left(\frac{1}{\tau_1^2}\right) \quad \text{(given Lemma C-1)} \qquad (45)
\end{aligned}
$$

Finally, consider the fourth term on the right-hand side of expression (42) above. For this term, we apply the result given in part (a) to obtain

$$
\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right| \left|E\left(u_{iv}u_{iw}\right)\right| \\
\leq & \left(\frac{1}{\tau_1^2} \sum_{\substack{g,h=(r-1)\tau+p \\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right|\right) \left(\frac{1}{\tau_1^2} \sum_{\substack{v,w=(r-1)\tau+p \\ v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
= & O\left(\frac{1}{\tau_1^2}\right). \qquad (46)
\end{aligned}
$$

It follows from expressions (42)-(46) that

$$
\begin{aligned}
& \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq & \frac{1}{\tau_1^4} \sum_{\substack{g,h=(r-1)\tau+p \\ g\leq h}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{ig}u_{ih}^3\right]\right| + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v>v-h,\ w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-v\leq v-h,\ v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[\left\{u_{ig}u_{ih}-E\left(u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& + \frac{1}{\tau_1^4} \sum_{\substack{g,h,v,w=(r-1)\tau+p \\ g\leq h\leq v\leq w \\ w-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left(u_{ig}u_{ih}\right)\right| \left|E\left(u_{iv}u_{iw}\right)\right| \\
= & O\left(\frac{1}{\tau_1^2}\right). \ \square
\end{aligned}
$$

Lemma C-5: Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, there exists a positve constant $\overline{C}$ such that

$$E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t$$

and, thus,

$$E\left\|\underline{Y}_t\right\|_2^6 \leq \overline{C} < \infty \text{ and } E\left\|\underline{F}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t,$$

where

$$\underset{dp\times 1}{\underline{Y}_t} = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \text{ and } \underset{Kp\times 1}{\underline{F}_t} = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}.$$

Proof of Lemma C-5:

To proceed, note that, given Assumption 3-1, we can write the vector moving-average (VMA) representation of the companion form of the FAVAR model as

$$\begin{aligned} \underline{W}_t &= \left(I_{(d+K)p} - A\right)^{-1} \alpha + \sum_{j=0}^{\infty} A^j E_{t-j} \\ &= \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K} J_{d+K} \alpha + \sum_{j=0}^{\infty} A^j J'_{d+K} J_{d+K} E_{t-j} \\ &= \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K} \mu + \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} \underline{W}_t &= \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \ E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \\ \underset{(d+K)\times(d+K)p}{J_{d+K}} &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}. \end{aligned}$$

By the triangle inequality,

$$\|\underline{W}_t\|_2 \leq \left\| \left(I_{(d+K)p} - A\right)^{-1} J'_{d+K} \mu \right\|_2 + \left\| \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\|_2$$

Moreover, using the inequality $\left|\sum_{i=1}^{m} a_i\right|^r \leq m^{r-1} \sum_{i=1}^{m} |a_i|^r$ for $r \geq 1$, we get

$$\|\underline{W}_t\|_2^6 \leq 2^5 \left(\left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 + \left\| \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j} \right\|_2^6 \right)$$

so that

$$E \|\underline{W}_t\|_2^6 \leq 32 \left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 + 32E \left\| \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j} \right\|_2^6 \tag{48}$$

Focusing first on the first term on the right-hand side of the inequality (48), we note that

$$\begin{aligned}
\left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 &= \left(\mu' J_{d+K} \left(I_{(d+K)p} - A\right)^{-1\prime} \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right)^3 \\
&= \left(\mu' J_{d+K} \left[\left(I_{(d+K)p} - A\right) \left(I_{(d+K)p} - A\right)' \right]^{-1} J_{d+K}' \mu \right)^3 \\
&\leq \left(\frac{1}{\lambda_{\min}\left\{ \left(I_{(d+K)p} - A\right) \left(I_{(d+K)p} - A\right)' \right\}} \right)^3 \left(\mu' J_{d+K} J_{d+K}' \mu \right)^3 \\
&= \left(\frac{1}{\lambda_{\min}\left\{ \left(I_{(d+K)p} - A\right) \left(I_{(d+K)p} - A\right)' \right\}} \right)^3 \left(\mu' \mu \right)^3
\end{aligned}$$

Now, by Assumption 3-7, there exists a constant $\underline{C} > 0$ such that

$$\begin{aligned}
\lambda_{\min}\left\{ \left(I_{(d+K)p} - A\right) \left(I_{(d+K)p} - A\right)' \right\} &= \lambda_{\min}\left\{ \left(I_{(d+K)p} - A\right)' \left(I_{(d+K)p} - A\right) \right\} \\
&= \sigma_{\min}^2 \left(I_{(d+K)p} - A\right) \\
&\geq \underline{C} \lambda_{\min}^2 \left(I_{(d+K)p} - A\right) \\
&\geq \underline{C} \left[1 - \phi_{\max}\right]^2 \\
&> 0
\end{aligned}$$

where $\phi_{\max} = \max\left\{ \left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right| \right\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. It follows by Assumption 3-5 that, there exists a positive constant $\overline{C}_1$ such that

$$\begin{aligned}
\left\|\left(I_{(d+K)p}-A\right)^{-1}J'_{d+K}\mu\right\|_2^6 &\leq \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p}-A\right)\left(I_{(d+K)p}-A\right)'\right\}}\right)^3\left(\mu'\mu\right)^3 \\
&\leq \frac{\|\mu\|_2^6}{\underline{C}^3\left[1-\phi_{\max}\right]^6}\leq\overline{C}_1<\infty.
\end{aligned}$$

To show the boundedness of the second term on the right-hand side of the inequality (48), let $e_{g,(d+K)p}$ be a $(d+K)\,p\times 1$ elementary vector whose g^{th} component is 1 and all other components are 0 for $g\in\{1,2,...,(d+K)\,p\}$, and note that

$$\begin{aligned}
\left\|\sum_{j=0}^{\infty}A^jJ'_{d+K}\varepsilon_{t-j}\right\|_2^2 &= \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}e'_{g,(d+K)p}A^jJ'_{d+K}\varepsilon_{t-j}\right)^2 \\
&= \sum_{g=1}^{(d+K)p}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}e'_{g,(d+K)p}A^jJ'_{d+K}\varepsilon_{t-j}\varepsilon'_{t-k}J_{d+K}\left(A'\right)^k e_{g,(d+K)p}
\end{aligned}$$

from which we obtain, by applying the inequality $\left|\sum_{i=1}^m a_i\right|^r\leq m^{r-1}\sum_{i=1}^m|a_i|^r$ for $r\geq 1$

$$\begin{aligned}
&\left\|\sum_{j=0}^{\infty}A^jJ'_{d+K}\varepsilon_{t-j}\right\|_2^6 \\
=\; &\left[\sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}e'_{g,(d+K)p}A^jJ'_{d+K}\varepsilon_{t-j}\right)^2\right]^3 \\
\leq\; &\left[(d+K)\,p\right]^2\sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}e'_{g,(d+K)p}A^jJ'_{d+K}\varepsilon_{t-j}\right)^6 \\
=\; &\left[(d+K)\,p\right]^2\sum_{g=1}^{(d+K)p}\left\{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}e'_{g,(d+K)p}A^jJ'_{d+K}\varepsilon_{t-j}\varepsilon'_{t-k}J_{d+K}\left(A'\right)^k e_{g,(d+K)p}\right. \\
&\left.\times e'_{g,(d+K)p}A^iJ'_d\varepsilon_{t-i}\varepsilon'_{t-\ell}J_{d+K}\left(A'\right)^\ell e_{g,(d+K)p}e'_{g,(d+K)p}A^rJ'_{d+K}\varepsilon_{t-r}\varepsilon'_{t-s}J_d\left(A'\right)^s e_{g,(d+K)p}\right\}
\end{aligned}$$

Hence,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right\|_{2}^{6} \\
\leq\; & \left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{6} \\
& +\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p}\binom{6}{3}\left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{3}\right)^{2} \\
& +\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p}\binom{6}{2}\binom{4}{2}\left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{2}\right)^{3} \\
& +\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p}\binom{6}{4} \sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{4} \sum_{k=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{k} J_{d+K}^{\prime} \varepsilon_{t-k}\right|^{2} \\
=\; & \left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{6} \\
& +20\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{3}\right)^{2} \\
& +90\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p}\left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{2}\right)^{3} \\
& +15\left[(d+K)\, p\right]^{2} \sum_{g=1}^{(d+K) p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{j} J_{d+K}^{\prime} \varepsilon_{t-j}\right|^{4} \sum_{k=0}^{\infty} E\left|e_{g,(d+K) p}^{\prime} A^{k} J_{d+K}^{\prime} \varepsilon_{t-k}\right|^{2}
\end{aligned}
$$

Next, applying the Cauchy-Schwarz inequality, we further obtain

$$\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}\right\|_2^6 \\
\leq\ & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad\qquad \left.\times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k J_{d+K}' J_{d+K} \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\} \\
=\ & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad\qquad \left.\times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\}
\end{aligned}$$

In addition, observe that, for every $g \in \{1, 2, ..., (d+K)p\}$

$$\begin{aligned}
& e'_{g,(d+K)p} A^j \left(A^j\right)' e_{g,(d+K)p} \\
\leq \; & \lambda_{\max}\left\{A^j \left(A^j\right)'\right\} \\
= \; & \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \\
= \; & \sigma^2_{\max}\left(A^j\right) \\
\leq \; & C \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \text{ (by Assumption 3-7)} \\
= \; & C \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \\
= \; & C\phi^{2j}_{\max}
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. Now, in light of Assumption 3-2(b), we can let $C \geq 1$ be a constant such that $E\|\varepsilon_{t-j}\|_2^6 \leq C < \infty$, so that, by Liapunov's inequality,

$$\begin{aligned}
E\|\varepsilon_{t-j}\|_2^2 & \leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{3}} \leq C^{\frac{1}{3}}, \; E\|\varepsilon_{t-j}\|_2^3 \leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}, \\
E\|\varepsilon_{t-j}\|_2^4 & \leq \left(E\|\varepsilon_{t-j}\|_2^6\right)^{\frac{2}{3}} \leq C^{\frac{2}{3}},
\end{aligned}$$

and, thus,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}^{\prime} \varepsilon_{t-j}\right\|_2^6 \\
\leq\ & [(d+K)\, p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j\left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\,[(d+K)\, p]^2 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j\left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\,[(d+K)\, p]^2 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j\left(A^j\right)^{\prime} e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\,[(d+K)\, p]^2 \sum_{g=1}^{(d+K)p}\left\{\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j\left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad \left.\times \sum_{k=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^k\left(A^k\right)^{\prime} e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\} \\
\leq\ & C\,[(d+K)\, p]^2\left\{\sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \phi_{\max}^{6j}+20 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{3j}\right)^2+90 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{2j}\right)^3\right. \\
& \qquad \left.+15 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{4j}\right)\left(\sum_{k=0}^{\infty} \phi_{\max}^{2k}\right)\right\} \\
\leq\ & C\,[(d+K)\, p]^3 \\
& \times\left\{\frac{1}{1-\phi_{\max}^6}+20\left(\frac{1}{1-\phi_{\max}^3}\right)^2+90\left(\frac{1}{1-\phi_{\max}^2}\right)^3+15\left(\frac{1}{1-\phi_{\max}^4}\right)\left(\frac{1}{1-\phi_{\max}^2}\right)\right\} \\
\leq\ & \overline{C}_2<\infty
\end{aligned}
$$

for some constant such that

$$
\begin{aligned}
& \overline{C}_2 \\
\geq\ & C\,[(d+K)\, p]^3 \\
& \times\left\{\frac{1}{1-\phi_{\max}^6}+20\left(\frac{1}{1-\phi_{\max}^3}\right)^2+90\left(\frac{1}{1-\phi_{\max}^2}\right)^3+15\left(\frac{1}{1-\phi_{\max}^4}\right)\left(\frac{1}{1-\phi_{\max}^2}\right)\right\}.
\end{aligned}
$$

Putting everything together, we see that

$$\begin{aligned} E\left\|\underline{W}_t\right\|_2^6 &\leq 32\left\|\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\right\|_2^6+32E\left\|\sum_{j=0}^{\infty}A^jJ_{d+K}'\varepsilon_{t-j}\right\|_2^6 \\ &\leq 32\left(\overline{C}_1+\overline{C}_2\right) \\ &\leq \overline{C}<\infty \end{aligned}$$

for a constant $\overline{C}$ such that $0<32\left(\overline{C}_1+\overline{C}_2\right)\leq\overline{C}<\infty$.

In addition, define$\mathcal{P}_{(d+K)p}$ to be the $(d{+}K)\,p\times(d{+}K)\,p$ permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t=\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t}\\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}; \tag{49}$$

and let $S_d'=\begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0}\end{pmatrix}$ and $S_K'=\begin{pmatrix}\underset{Kp\times dp}{0} & I_{Kp}\end{pmatrix}$. Note that

$$\begin{aligned} S_d'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0}\end{pmatrix}\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t}\\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}=\underline{Y}_t, \\ S_K'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix}\underset{Kp\times dp}{0} & I_{Kp}\end{pmatrix}\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t}\\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}=\underline{F}_t. \end{aligned}$$

so that

$$\begin{aligned} \left\|\underline{Y}_t\right\|_2 &\leq \left\|S_d'\right\|_2\left\|\mathcal{P}_{(d+K)p}\right\|_2\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_dS_d'\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}_{(d+K)p}'\mathcal{P}_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_d'S_d\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(I_{dp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \left\|\underline{W}_t\right\|_2 \end{aligned}$$

and

$$\begin{aligned}
\|\underline{F}_t\|_2 &\leq \left\|S_K'\right\|_2 \left\|\mathcal{P}_{(d+K)p}\right\|_2 \|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}\left(S_K S_K'\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}_{(d+K)p}'\mathcal{P}_{(d+K)p}\right)}\,\|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}\left(S_K' S_K\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\,\|\underline{W}_t\|_2 \\
&= \sqrt{\lambda_{\max}\left(I_{Kp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\,\|\underline{W}_t\|_2 \\
&= \|\underline{W}_t\|_2
\end{aligned}$$

It further follows that

$$E\,\|\underline{Y}_t\|_2^6 \leq E\,\|\underline{W}_t\|_2^6 \leq \overline{C} < \infty \text{ and } E\,\|\underline{F}_t\|_2^6 \leq E\,\|\underline{W}_t\|_2^6 \leq \overline{C} < \infty. \ \square$$

Lemma C-6: Suppose that Assumptions 3-1, 3-2(a)-(b), 3-3(a)-(c), 3-5, 3-7, and 3-10(b) hold. Then, the following statements are true as $N_1, T \to \infty$

(a)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right| \xrightarrow{p} 0.$$

(b)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2 \xrightarrow{p} 0$$

(c)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right| \xrightarrow{p} 0.$$

(d)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2 \xrightarrow{p} 0$$

(e)

$$\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \xrightarrow{p} 0$$

Proof of Lemma C-6.

To show part (a), first write

$$
\begin{aligned}
& P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right|\geq \epsilon\right\} \\
= \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\geq \epsilon^6\right\} \\
\leq \; & P\left\{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\geq \epsilon^6\right\} \\
& \text{(by Jensen's inequality)} \\
\leq \; & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6\geq \epsilon^6\right\} \\
\leq \; & \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
& \frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6 \\
\leq\ & \frac{1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right]^6 \\
& +\frac{20}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\left|\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right|\right]^3 E\left[\left|\gamma_i'\underline{F}_s\varepsilon_{\ell,s+1}\right|\right]^3 \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right]^4 E\left[\gamma_i'\underline{F}_s\varepsilon_{\ell,s+1}\right]^2 \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{r=(r-1)\tau+p\\ r\neq t,\ r\neq s}}^{(r-1)\tau+\tau_1+p-1}\Big\{E\left[\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right]^2 E\left[\gamma_i'\underline{F}_s\varepsilon_{\ell,s+1}\right]^2 \\
& \qquad\qquad \times E\left[\gamma_i'\underline{F}_s\varepsilon_{\ell,r+1}\right]^2\Big\} \\
\leq\ & \frac{1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\left(\gamma_i'\underline{F}_t\right)^6\right] E\left[\varepsilon_{\ell,t+1}^6\right] \\
& +\frac{20}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\frac{1}{64}E\left[\gamma_i'\underline{F}_t\underline{F}_t'\gamma_i+\varepsilon_{\ell,t+1}^2\right]^3 E\left[\gamma_i'\underline{F}_s\underline{F}_s'\gamma_i+\varepsilon_{\ell,s+1}^2\right]^3 \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i'\underline{F}_t\underline{F}_t'\gamma_i\right]^2 E\left[\varepsilon_{\ell,t+1}^4\right] E\left[\gamma_i'\underline{F}_s\underline{F}_s'\gamma_i\right] E\left[\varepsilon_{\ell,s+1}^2\right] \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\left\{\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i'\underline{F}_t\underline{F}_t'\gamma_i\right] E\left[\varepsilon_{\ell,t+1}^2\right] E\left[\gamma_i'\underline{F}_s\underline{F}_s'\gamma_i\right] E\left[\varepsilon_{\ell,s+1}^2\right]\right. \\
& \qquad\qquad \left.\times\sum_{\substack{r=(r-1)\tau+p\\ r\neq t,\ r\neq s}}^{(r-1)\tau+\tau_1+p-1} E\left[\gamma_i'\underline{F}_r\underline{F}_r'\gamma_i\right] E\left[\varepsilon_{\ell,r+1}^2\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\|\gamma_i\right\|_2^6 E\left[\left\|\underline{F}_t\right\|_2^6\right]E\left[\varepsilon_{\ell,t+1}^6\right] \\
& +\frac{(20\cdot 16)}{64q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\left\{\left(E\left[\left(\gamma_i'\underline{F}_t\right)^6\right]+E\left[\varepsilon_{\ell,t+1}^6\right]\right)\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times\left(E\left[\left(\gamma_i'\underline{F}_s\right)^6\right]+E\left[\varepsilon_{\ell,s+1}^6\right]\right)\right\} \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\left\{\left\|\gamma_i\right\|_2^4 E\left[\left\|\underline{F}_t\right\|_2^4\right]E\left[\varepsilon_{\ell,t+1}^4\right]\right. \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\times\left\|\gamma_i\right\|_2^2 E\left[\left\|\underline{F}_s\right\|_2^2\right]E\left[\varepsilon_{\ell,s+1}^2\right]\right\} \\
& +\frac{15}{q\tau_1^6}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}\left\{\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\|\gamma_i\right\|_2^2 E\left[\left\|\underline{F}_t\right\|_2^2\right]E\left[\varepsilon_{\ell,t+1}^2\right]\right. \\
& \quad\left.\times\sum_{\substack{s=(r-1)\tau+p\\ s\neq t}}^{(r-1)\tau+\tau_1+p-1}\left\|\gamma_i\right\|_2^2 E\left[\left\|\underline{F}_s\right\|_2^2\right]E\left[\varepsilon_{\ell,s+1}^2\right]\sum_{\substack{r=(r-1)\tau+p\\ r\neq t,\ r\neq s}}^{(r-1)\tau+\tau_1+p-1}\left\|\gamma_i\right\|_2^2 E\left[\left\|\underline{F}_r\right\|_2^2\right]E\left[\varepsilon_{\ell,r+1}^2\right]\right\} \\
\leq\ & C\left(\frac{N_1}{\tau_1^5}+5\frac{N_1}{\tau_1^4}+15\frac{N_1}{\tau_1^4}+15\frac{N_1}{\tau_1^3}\right) \\
& \text{(applying Assumptions 3-2(b), Assumption 3-5, and Lemma C-5)} \\
=\ & O\left(\frac{N_1}{\tau_1^3}\right).
\end{aligned}
$$

It follows that

$$
P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right|\geq\epsilon\right\}=O\left(\frac{N_1}{\tau_1^3}\right)=o(1).
$$

To show part (b), note that, for any $\epsilon > 0$

$$\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2 \geq \epsilon\right\} \\
= \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2\right|^3 \geq \epsilon^3\right\} \\
\leq \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6 \geq \epsilon^3\right\} \\
& \text{(by Jensen't inequality)} \\
\leq \; & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6 \geq \epsilon^3\right\} \\
\leq \; & \frac{1}{\epsilon^3}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^6
\end{aligned}$$

The rest of the proof for part (b) then follows in a manner similar to the argument given for part (a) above.

To show part (c), first note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right| \geq \epsilon\right\} \\
= \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^6\right\} \\
\leq \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^6\right\} \\
& \text{(by convexity or Jensen's inequality)} \\
\leq \; & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^6\right\} \\
\leq \; & \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \qquad (50)
\end{aligned}$$

Now, there exists a constant $C_1 > 1$ such that

$$
\begin{aligned}
&\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \\
\leq\ & \frac{C_1}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\left\{\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right|\right. \\
&\qquad\qquad\left. \times\sum_{\ell=1}^{d}\left|E\left[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}\right]\right|\right\}
\end{aligned}
$$

Next, note that, by repeated application of Hölder's inequality, we have by Lemma C-5 that there exists a positive constant $\overline{C}$ such that

$$
\begin{aligned}
&\sum_{\ell=1}^{d}\left|E\left[y_{\ell,t+1}y_{\ell,s+1}y_{\ell,g+1}y_{\ell,h+1}y_{\ell,v+1}y_{\ell,w+1}\right]\right| \\
\leq\ & \sum_{\ell=1}^{d}\left(E\left[y_{\ell,t+1}^2y_{\ell,s+1}^2y_{\ell,g+1}^2\right]\right)^{\frac{1}{2}}\left(E\left[y_{\ell,h+1}^2y_{\ell,v+1}^2y_{\ell,w+1}^2\right]\right)^{\frac{1}{2}} \\
\leq\ & \sum_{\ell=1}^{d}\left(\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{3}}\left(E\left[\left|y_{\ell,s+1}y_{\ell,g+1}\right|^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{3}}\left(E\left[\left|y_{\ell,v+1}y_{\ell,w+1}\right|^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
\leq\ & \sum_{\ell=1}^{d}\left[\left(\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,s+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,g+1}^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}}\right. \\
&\quad\left.\times\left(\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,v+1}^6\right]\right\}^{\frac{1}{3}}\left\{E\left[y_{\ell,w+1}^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}}\right] \\
\leq\ & \sum_{\ell=1}^{d}\left\{E\left[y_{\ell,t+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,s+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,g+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,h+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,v+1}^6\right]\right\}^{\frac{1}{6}}\left\{E\left[y_{\ell,w+1}^6\right]\right\}^{\frac{1}{6}} \\
\leq\ & d\max_{1\leq\ell\leq d}\sup_t E\left[y_{\ell,t}^6\right] \\
\leq\ & \overline{C} < \infty \\
&\Big(\text{since, given that } y_{\ell,t} = \mathbf{e}_{\ell,dp}'\underline{Y}_t;\ E\left[y_{\ell,t}^6\right] \leq E\left\|\underline{Y}_t\right\|_2^6 \leq \overline{C} \text{ by Lemma C-5} \\
&\text{where } \overline{C} \text{ is a constant not depending on } \ell \text{ or } t\ \Big)
\end{aligned}
$$

Hence, we can write

$$
\begin{aligned}
& \frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \\
\leq \;& \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq \;& \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p\\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
&\quad+\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
&\quad+\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{ih}-E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&\quad+\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
&\quad+\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}u_{ih}u_{iv}u_{iw}\right]\right| \\
&\quad+\frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&= \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5+\mathcal{T}_6,\ \ (say)\,. \qquad (51)
\end{aligned}
$$

Consider first $\mathcal{T}_1$. Note that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}E\left[\left|u_{it}u_{is}u_{ig}^4\right|\right] \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(E\left[\left|u_{it}u_{is}\right|^3\right]\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \quad \text{(by Hölder's inequality)} \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(\left[E\left\{\left|u_{it}\right|^6\right\}\right]^{\frac{1}{2}}\left[E\left\{\left|u_{is}\right|^6\right\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\quad \text{(by further application of Hölder's inequality)} \\
&= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{1}{6}} \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\overline{C}^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\
&\leq C_1\overline{C}^{\frac{3}{2}}\frac{N_1}{\tau_1^5} \\
&= O\left(\frac{N_1}{\tau_1^5}\right). \qquad (52)
\end{aligned}$$

Next, consider $\mathcal{T}_2$. For this term, note first that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1\exp\{-a_2 m\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1\exp\{-a_2 m\} \text{ for every } i.$$

Hence, we apply Lemma C-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned}
&\mathcal{T}_2 \\
= \;& \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1} \left|E\left[u_{it}u_{is}u_{ig}u_{ih}u_{iv}u_{iw}\right]\right| \\
\leq \;& \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1} \Big\{2\left(2^{1-\frac{4}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \\
&\qquad\qquad\qquad\qquad \times\left(E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}}\Big\}
\end{aligned}$$

Next, by Liapunov's inequality and Assumption 3-3(b), we obtain

$$\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \leq \left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}$$

Making use of this bound and by repeated application of Hölder's inequality, we have

$$\begin{aligned}
&E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}} \\
\leq\;& \left[E\left|u_{it}u_{is}u_{ig}\right|^{\frac{25}{12}}\right]^{\frac{3}{5}}\left[E\left|u_{ih}u_{iv}\right|^{\frac{25}{8}}\right]^{\frac{2}{5}} \\
\leq\;& \left[\left(E\left|u_{it}u_{is}\right|^{\frac{150}{47}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{ih}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq\;& \left[\left(\sqrt{E\left|u_{it}\right|^{\frac{300}{47}}}\sqrt{E\left|u_{is}\right|^{\frac{300}{47}}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{ih}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
=\;& \left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{ih}\right|^{6}\right)^{\frac{15}{72}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}}\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}} \\
=\;& \left[\left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{ih}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}} \\
\leq\;& \left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{ih}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \\
\leq\;& \left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}} \quad \text{(by Assumption 3-3(b))} \\
=\;& \overline{C}^{\frac{25}{28}}
\end{aligned}$$

Moreover, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}$

as shown above, we can further write

$$
\begin{aligned}
& \mathcal{T}_2 \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{4}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\right. \\
& \left.\times\left(E\left|u_{it}u_{is}u_{ig}u_{ih}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}}\right\} \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{\frac{1}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}}\overline{C}^{\frac{25}{28}}\overline{C}^{\frac{1}{7}} \\
\leq\ & \frac{C_1\overline{C}^{\frac{57}{28}}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}2\left(2^{\frac{1}{5}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ w-v\geq\max\{v-h,h-g\},w-v>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{1}{5}}+1\right)C_1\overline{C}^{\frac{57}{28}}a_1^{\frac{1}{30}}\leq C^*<\infty\right) \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\rho_3=1}^{\infty}\sum_{\rho_1=0}^{\rho_3}\sum_{\rho_2=0}^{\rho_3}\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\rho_3=1}^{\infty}\left(\rho_3+1\right)^2\exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
=\ & C^*\frac{N_1}{\tau_1^3}\left[\sum_{\rho_3=1}^{\infty}\rho_{3\backslash}^2\exp\left\{-\frac{a_2}{30}\rho_3\right\}+2\sum_{\rho_3=1}^{\infty}\rho_{3\backslash}\exp\left\{-\frac{a_2}{30}\rho_3\right\}+\sum_{\rho_3=1}^{\infty}\exp\left\{-\frac{a_2}{30}\rho_3\right\}\right] \\
=\ & O\left(\frac{N_1}{\tau_1^3}\right)\quad\text{(by Lemma C-1)}. \qquad (53)
\end{aligned}
$$

Now, consider $\mathcal{T}_3$. Here, we can apply Lemma C-3 with $p = 3/2$ and $r = 7/2$ to obtain

$$\begin{aligned}
\mathcal{T}_3 &= \frac{C_1 \overline{C}}{q\tau_1^6} \sum_{r=1}^{q} \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left| E\left[\{ u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih} \right) \} u_{iv}u_{iw} \right] \right| \\
&\leq \frac{C_1 \overline{C}}{q\tau_1^6} \sum_{r=1}^{q} \sum_{i \in H^c} \sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t \leq s \leq g \leq h \leq v \leq w \\ v-h \geq \max\{w-v, h-g\}, v-h>0}}^{(r-1)\tau+\tau_1+p-1} \left\{ 2\left(2^{1-\frac{2}{3}} + 1 \right) \left[a_1 \exp\left\{ -a_2 \left(v-h \right) \right\} \right]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\qquad \left. \times \left(E\left| \{ u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih} \right) \} \right|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(E\left| u_{iv}u_{iw} \right|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\}
\end{aligned}$$

Next, note that applications of Hölder's inequality yield

$$\begin{aligned}
E\left| u_{iv}u_{iw} \right|^{\frac{7}{2}} &\leq \left(E\left| u_{iv} \right|^7 \right)^{\frac{1}{2}} \left(E\left| u_{iw} \right|^7 \right)^{\frac{1}{2}} \\
&\leq \left(\overline{C} \right)^{\frac{1}{2}} \left(\overline{C} \right)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\
&= \overline{C} < \infty
\end{aligned}$$

and

$$\begin{aligned}
E\left| \{ u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih} \right) \} \right|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}} \left(E\left| u_{it}u_{is}u_{ig}u_{ih} \right|^{\frac{3}{2}} + E\left| u_{it}u_{is}u_{ig}u_{ih} \right|^{\frac{3}{2}} \right) \\
&\qquad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq 2^{\frac{3}{2}} E\left| u_{it}u_{is}u_{ig}u_{ih} \right|^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left(E\left| u_{it}u_{is} \right|^3 \right)^{\frac{1}{2}} \left(E\left| u_{ig}u_{ih} \right|^3 \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}} \left(\left(E\left| u_{it} \right|^6 \right)^{\frac{1}{2}} \left(E\left| u_{is} \right|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left(\left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{2}} \left(E\left| u_{ih} \right|^6 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}} \left[\left(E\left| u_{it} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{is} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ig} \right|^6 \right)^{\frac{1}{6}} \left(E\left| u_{ih} \right|^6 \right)^{\frac{1}{6}} \right]^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}} \left[\left(E\left| u_{it} \right|^7 \right)^{\frac{1}{7}} \left(E\left| u_{is} \right|^7 \right)^{\frac{1}{7}} \left(E\left| u_{ig} \right|^7 \right)^{\frac{1}{7}} \left(E\left| u_{ih} \right|^7 \right)^{\frac{1}{7}} \right]^{\frac{3}{2}} \\
&\qquad \text{(by Liapunov's inequality)} \\
&\leq 2^{\frac{3}{2}} \left[\left(\sup_{i,t} E\left| u_{it} \right|^7 \right)^{\frac{4}{7}} \right]^{\frac{3}{2}} \\
&= 2^{\frac{3}{2}} \overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Again, let $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}$ and $E\left|\left\{u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}}$ as shown above, we can further write

$$
\begin{aligned}
\mathcal{T}_3 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}u_{iv}u_{iw}\right]\right| \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{2}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{1-\frac{2}{3}-\frac{2}{7}}\right. \\
&\qquad\left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}u_{ih} - E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}}\right\} \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-h\right)\right\}\right]^{\frac{1}{21}}\left(2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}}\right)^{\frac{2}{3}}\left(\overline{C}\right)^{\frac{2}{7}} \\
&\leq \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&\quad\left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{3}}+1\right)C_1\overline{C}^{\frac{13}{7}}a_1^{\frac{1}{21}}\leq C^*<\infty\right) \\
&\leq \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\sum_{\varrho_3=0}^{\varrho_2}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&\leq C^*\frac{N_1}{\tau_1^3}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&= C^*\frac{N_1}{\tau_1^3}\left[\sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}\right] \\
&= O\left(\frac{N_1}{\tau_1^3}\right)\quad\left(\text{by Lemma C-1}\right).
\end{aligned}
\tag{54}
$$

Turning our attention to the term $\mathcal{T}_4$, note that, from the upper bounds given in the proofs of parts (a) and (c) of Lemma C-4, it is clear that there exists a positive constant C such that

$$\frac{1}{\tau_1^4}\sum_{\substack{t,s,g,h=(r-1)\tau+p\\ t\leq s\leq g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\leq\frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^2}\sum_{\substack{v,w=(r-1)\tau+p\\ v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{iv}u_{iw}\right)\right|\leq\frac{C}{\tau_1}$$

from which it follows that

$$\begin{aligned}
\mathcal{T}_4 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ v-h\geq\max\{w-v,h-g\},v-h>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{1}{\tau_1^4}\sum_{\substack{t,s,g,h=(r-1)\tau+p\\ t\leq s\leq g\leq h}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}u_{ih}\right)\right|\right)\left(\frac{1}{\tau_1^2}\sum_{\substack{v,w=(r-1)\tau+p\\ v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{C}{\tau_1^2}\right)\left(\frac{C}{\tau_1}\right) \\
&= C_1\overline{C}C^2\frac{N_1}{\tau_1^3} \\
&= O\left(\frac{N_1}{\tau_1^3}\right). \qquad (55)
\end{aligned}$$

Consider now $\mathcal{T}_5$. In this case, we apply Lemma C-3 with $p=2$ and $r=9/4$ to obtain

$$\begin{aligned}
\mathcal{T}_5 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left[\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}u_{ih}u_{iv}u_{iw}\right]\right| \\
&\leq \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p\\ t\leq s\leq g\leq h\leq v\leq w\\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left\{2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}}\right. \\
&\qquad\qquad\left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}}\left(E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}}\right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$
\begin{aligned}
& E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}} \\
\leq \; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[E\left|u_{iv}u_{iw}\right|^{\frac{63}{19}}\right]^{\frac{19}{28}} \\
\leq \; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\right]^{\frac{19}{28}} \\
= \; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \\
= \; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\right]^{\frac{9}{4}} \\
\leq \; & \left[E\left|u_{ih}\right|^{7}\right]^{\frac{9}{28}}\left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{9}{4}} \quad \text{(by Liapunov's inequality)} \\
\leq \; & \left(\sup_{i,t} E\left|u_{it}\right|^{7}\right)^{\frac{27}{28}} \\
\leq \; & \overline{C}^{\frac{27}{28}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^{2} \leq \; & 2\left(E\left|u_{it}u_{is}u_{ig}\right|^{2}+E\left|u_{it}u_{is}u_{ig}\right|^{2}\right) \\
& \text{(by Loève's } c_r \text{ inequality)} \\
\leq \; & 4E\left|u_{it}u_{is}u_{ig}\right|^{2} \\
\leq \; & 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(E\left|u_{is}u_{ig}\right|^{3}\right)^{\frac{2}{3}} \\
\leq \; & 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(\sqrt{E\left|u_{is}\right|^{6}}\sqrt{E\left|u_{ig}\right|^{6}}\right)^{\frac{2}{3}} \\
= \; & 4\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\right]^{2}\left[\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\right]^{2} \\
\leq \; & 4\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{2}\left[\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\right]^{2} \\
& \text{(by Liapunov's inequality)} \\
\leq \; & 4\left[\left(\sup_{i,t} E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{6} \\
\leq \; & 4\overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}
$$

Define again $\rho_1 = h - g$, $\rho_2 = v - h$, and $\rho_3 = w - v$, so that $h = g + \rho_1$, $v = h + \rho_2 = g + \rho_1 + \rho_2$,

$w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}$ and $E\left|\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2$ as shown above, we can further write

$$
\begin{aligned}
& \mathcal{T}_5 \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} \left\{2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)^{\theta}\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}}\right. \\
& \left. \times\left(E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}}\left(E\left|u_{ih}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}}\right\} \\
\leq\ & \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} 2\left(2^{\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(h-g\right)\right\}\right]^{\frac{1}{18}}\left(4\overline{C}^{\frac{6}{7}}\right)^{\frac{1}{2}}\left(\overline{C}^{\frac{27}{28}}\right)^{\frac{4}{9}} \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1} \exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)C_1\overline{C}^{\frac{13}{7}}a_1^{\frac{1}{18}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{g=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1}\sum_{\varrho_3=0}^{\varrho_1}\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq\ & C^*\frac{N_1}{\tau_1^3}\sum_{\varrho_1=1}^{\infty}\left(\varrho_1+1\right)^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
=\ & C^*\frac{N_1}{\tau_1^3}\left[\sum_{\varrho_1=1}^{\infty}\varrho_1^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+2\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{18}\varrho_1\right\}\right] \\
=\ & O\left(\frac{N_1}{\tau_1^3}\right) \quad \text{(by Lemma C-1)}
\end{aligned} \tag{56}
$$

Finally, consider $\mathcal{T}_6$. Note that, from the upper bounds given in the proofs of part (b) of Lemma C-4, it is clear that there exists a positive constant C such that

$$\frac{1}{\tau_1^3}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \leq \frac{C}{\tau_1^2}$$

and

$$\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\leq\frac{C}{\tau_1^2}$$

from which it follows that

$$\begin{aligned}
\mathcal{T}_6 &= \frac{C_1\overline{C}}{q\tau_1^6}\sum_{r=1}^{q}\sum_{i\in H^c}\sum_{\substack{t,s,g,h,v,w=(r-1)\tau+p \\ t\leq s\leq g\leq h\leq v\leq w \\ h-g\geq\max\{w-v,v-h\},h-g>0}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{1}{\tau_1^3}\sum_{\substack{t,s,g=(r-1)\tau+p \\ t\leq s\leq g}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\right)\left(\frac{1}{\tau_1^3}\sum_{\substack{h,v,w=(r-1)\tau+p \\ h\leq v\leq w}}^{(r-1)\tau+\tau_1+p-1}\left|E\left(u_{ih}u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1\overline{C}}{q}\sum_{r=1}^{q}\sum_{i\in H^c}\left(\frac{C}{\tau_1^2}\right)\left(\frac{C}{\tau_1^2}\right) \\
&= C_1C\overline{C}^2\frac{N_1}{\tau_1^4} \\
&= O\left(\frac{N_1}{\tau_1^4}\right). \qquad (57)
\end{aligned}$$

It follows from expressions (50)-(57) that, for any $\epsilon>0$,

$$\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right|\geq\epsilon\right\} \\
\leq\ & \frac{1}{\epsilon^6}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c}E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^6 \\
\leq\ & \frac{1}{\epsilon^4}\left(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5+\mathcal{T}_6\right) \\
=\ & O\left(\frac{N_1}{\tau_1^5}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^3}\right)+O\left(\frac{N_1}{\tau_1^4}\right) \\
=\ & O\left(\frac{N_1}{\tau_1^3}\right) \\
=\ & o\left(1\right)\quad\left(\text{by Assumption 3-10(b) which stipulates that }\frac{N_1}{\tau_1^3}\sim\frac{N_1}{T^{3\alpha_1}}\to 0\right)
\end{aligned}$$

which proves the required result.

Turning our attention to part (d), note that, for any $\epsilon > 0$,

$$\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2 \geq \epsilon\right\} \\
= \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2\right|^3 \geq \epsilon^3\right\} \\
\leq \; & P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^3\right\} \\
& \text{(by Jensen's inequality)} \\
\leq \; & P\left\{\sum_{\ell=1}^{d}\sum_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 \geq \epsilon^3\right\} \\
\leq \; & \frac{1}{\epsilon^3}\frac{1}{q}\sum_{r=1}^{q}\sum_{\ell=1}^{d}\sum_{i\in H^c} E\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^6 .
\end{aligned}$$

The rest of the proof for part (d) then follows in a manner similar to the argument given for part (c) above.

For part (e), note that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}
& \max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)\right| \\
\leq \; & \max_{1\leq\ell\leq d}\max_{i\in H^c}\sqrt{\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2}\sqrt{\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2} \\
\leq \; & \left\{\sqrt{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2}\right. \\
& \left.\times\sqrt{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right)^2}\right\} \\
= \; & o_p(1),
\end{aligned}$$

where the convergence in probability to zero in the last line above follows from applying the results in parts (b) and (c) of this lemma. □

Lemma C-7: Suppose that Assumptions 3-1 and 3-7 hold. Then, the following statements are true.

(a) There exists a positive constant $C^{\dagger}$ such that

$$\|A_{YY}\|_2 \leq C^{\dagger}\phi_{\max}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ with $0 < \phi_{\max} < 1$.

(b) There exists a positive constant $C^{\dagger}$ such that

$$\|A_{YF}\|_2 \leq C^{\dagger}\phi_{\max}$$

where $\phi_{\max}$ is as defined in part (a).

Proof of Lemma C-7:

To proceed, recall first that the FAVAR model, i.e.,

$$\begin{aligned} Y_t &= \mu_Y + A_{YY}\underline{Y}_{t-1} + A_{YF}\underline{F}_{t-1} + \varepsilon_t^Y \\ F_t &= \mu_F + A_{FY}\underline{Y}_{t-1} + A_{FF}\underline{F}_{t-1} + \varepsilon_t^F, \end{aligned}$$

can be written in the companion form

$$\underline{W}_t = \alpha + A\underline{W}_{t-1} + E_t$$

where $\underline{W}_t = \begin{pmatrix} W_t' & W_{t-1}' & \cdots & W_{t-p+2}' & W_{t-p+1}' \end{pmatrix}'$ with $W_t = \begin{pmatrix} Y_t' & F_t' \end{pmatrix}'$ and where

$$\alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & 0 & 0 \\ 0 & I_{d+K} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}, \text{ and } E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

with $\mu = \begin{pmatrix} \mu_Y' & \mu_F' \end{pmatrix}'$, $\varepsilon_t = \begin{pmatrix} \varepsilon_t^{Y\prime} & \varepsilon_t^{F\prime} \end{pmatrix}'$, and

$$A_\ell = \begin{pmatrix} A_{YY,\ell} & A_{YF,\ell} \\ A_{FY,\ell} & A_{FF,\ell} \end{pmatrix} \text{ for } \ell = 1, ..., p.$$

Let $\mathcal{P}_{(d+K)p}$ be the $(d+K)\,p \times (d+K)\,p$ permutation matrix defined by expression (49) in the proof of Lemma C-5; and it is easy to see that $\overline{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p}$ has the partitioned form

$$\overline{A} = \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} = \begin{pmatrix} \underset{d\times dp}{\overline{A}_{11}} & \underset{d\times Kp}{\overline{A}_{12}} \\ \underset{d(p-1)\times dp}{\overline{A}_{21}} & \underset{d(p-1)\times Kp}{\overline{A}_{22}} \\ \underset{K\times dp}{\overline{A}_{31}} & \underset{K\times Kp}{\overline{A}_{32}} \\ \underset{K(p-1)\times dp}{\overline{A}_{41}} & \underset{K(p-1)\times Kp}{\overline{A}_{42}} \end{pmatrix}$$

where $\overline{A}_{11} = A_{YY}$ and $\overline{A}_{12} = A_{YF}$, i.e., the first d rows of the matrix $\overline{A}$ as given by the submatrix $\begin{bmatrix} A_{YY} & A_{YF} \end{bmatrix}$.

Now, to show part (a), let $\overline{\upsilon} \in \mathbb{R}^{dp}$ such that $\|\overline{\upsilon}\|_2 = 1$ and such that

$$\|A_{YY}\|_2 = \overline{\upsilon}' A'_{YY} A_{YY} \overline{\upsilon} = \max_{\|\upsilon\|_2 = 1} \upsilon' A'_{YY} A_{YY} \upsilon = \overline{\upsilon}' \overline{A}'_{11} \overline{A}_{11} \overline{\upsilon}$$

and let $S_d = \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}'$. It follows that

$$\begin{aligned}
\|A_{YY}\|_2 &= \sqrt{\overline{\upsilon}' A'_{YY} A_{YY} \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' \overline{A}'_{11} \overline{A}_{11} \overline{\upsilon}} \\
&\leq \sqrt{\overline{\upsilon}' \overline{A}'_{11} \overline{A}_{11} \overline{\upsilon} + \overline{\upsilon}' \overline{A}'_{21} \overline{A}_{21} \overline{\upsilon} + \overline{\upsilon}' \overline{A}'_{31} \overline{A}_{31} \overline{\upsilon} + \overline{\upsilon}' \overline{A}'_{41} \overline{A}_{41} \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S'_d \overline{A}' \overline{A} S_d \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S'_d \mathcal{P}_{(d+K)p} A' \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} A \mathcal{P}'_{(d+K)p} S_d \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' S'_d \mathcal{P}_{(d+K)p} A' A \mathcal{P}'_{(d+K)p} S_d \overline{\upsilon}} \;\; \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&\leq \sqrt{\max_{\|\upsilon\|_2 = 1} \upsilon' A' A \upsilon} \;\; \left(\text{noting that } \left\|\mathcal{P}'_{(d+K)p} S_d \overline{\upsilon}\right\|_2 = \sqrt{\overline{\upsilon}' S'_d \mathcal{P}_{(d+K)p} \mathcal{P}'_{(d+K)p} S_d \overline{\upsilon}} = 1\right) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^{\dagger} \phi_{\max} \;\; (\text{by Assumption 3-7})
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. Note further that $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1.

To show part (b), let $\widetilde{v} \in \mathbb{R}^{Kp}$ such that $\|\widetilde{v}\|_2 = 1$ and such that

$$\|A_{YF}\|_2 = \widetilde{v}' A_{YF}' A_{YF} \widetilde{v} = \max_{\|v\|_2=1} v' A_{YF}' A_{YF} v = \widetilde{v}' \overline{A}_{12}' \overline{A}_{12} \widetilde{v}$$

and let

$$\underset{(d+K)p \times Kp}{S_K} = \begin{pmatrix} 0 \\ I_{Kp} \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\|A_{YF}\|_2 &= \sqrt{\widetilde{v}' A_{YF}' A_{YF} \widetilde{v}} \\
&= \sqrt{\widetilde{v}' \overline{A}_{12}' \overline{A}_{12} \widetilde{v}} \\
&\leq \sqrt{\widetilde{v}' \overline{A}_{12}' \overline{A}_{12} \widetilde{v} + \widetilde{v}' \overline{A}_{22}' \overline{A}_{22} \widetilde{v} + \widetilde{v}' \overline{A}_{32}' \overline{A}_{32} \widetilde{v} + \widetilde{v}' \overline{A}_{42}' \overline{A}_{42} \widetilde{v}} \\
&= \sqrt{\widetilde{v}' S_K' \overline{A}' \overline{A} S_K \widetilde{v}} \\
&= \sqrt{\widetilde{v}' S_K' \mathcal{P}_{(d+K)p} A' \mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p} A \mathcal{P}_{(d+K)p}' S_K \widetilde{v}} \\
&= \sqrt{\widetilde{v}' S_K' \mathcal{P}_{(d+K)p} A' A \mathcal{P}_{(d+K)p}' S_K \widetilde{v}} \ \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&\leq \sqrt{\max_{\|v\|_2=1} v' A' A v} \ \left(\text{noting that } \left\|\mathcal{P}_{(d+K)p}' S_K \widetilde{v}\right\|_2 = \sqrt{\widetilde{v}' S_K' \mathcal{P}_{(d+K)p} \mathcal{P}_{(d+K)p}' S_K \widetilde{v}} = 1\right) \\
&= \|A\|_2 \\
&= \sigma_{\max}(A) \\
&\leq C^{\dagger} \phi_{\max} \ \text{(by Assumption 3-7)}
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$. As noted in the proof for part (a), $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. $\square$

Lemma C-8: Consider the linear process

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}$$

Suppose the process satisfies the following assumptions

(i) Let $\{\varepsilon_t\}$ is an independent sequence of random vectors with $E[\varepsilon_t] = 0$ for all t. For some $\delta > 0$, suppose that there exists a positive constant K such that

$$E\|\varepsilon_t\|_2^{1+\delta} \leq K < \infty \text{ for all } t.$$

(ii) Suppose that ε_t has p.d.f. g_{ε_t} such that, for some positive constant $M < \infty$,

$$\sup_t \int \left| g_{\varepsilon_t} (v - u) - g_{\varepsilon_t} (v) \right| d\varepsilon \leq M |u|$$

whenever $|u| \leq \overline{\kappa}$ for some constant $\overline{\kappa} > 0$.

(iii) Suppose that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 < \infty$$

and

$$\det \left\{ \sum_{j=0}^{\infty} \Psi_j z^j \right\} \neq 0 \text{ for all } z \text{ with } |z| \leq 1$$

Under these conditions, suppose further that

$$\sum_{j=0}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_j\|_2 \right)^{\frac{\delta}{1+\delta}} < \infty;$$

then, for some positive constant $\overline{K}$,

$$\beta_\xi (m) \leq \overline{K} \sum_{j=m}^{\infty} \left(\sum_{k=j}^{\infty} \|\Psi_k\|_2 \right)^{\frac{\delta}{1+\delta}}$$

where

$$\beta_\xi (m) = \sup_t E \left[\sup \left\{ \left| P \left(B | \mathcal{F}_{\xi,-\infty}^t \right) - P (B) \right| : B \in \mathcal{F}_{\xi,t+m}^{\infty} \right\} \right].$$

with $\mathcal{F}_{\xi,-\infty}^t = \sigma \left(...., \xi_{t-2}, \xi_{t-1}, \xi_t \right)$ and $\mathcal{F}_{\xi,t+m}^{\infty} = \sigma \left(\xi_{t+m}, \xi_{t+m+1}, \xi_{t+m+2}, ... \right)$.

Remark: This is Theorem 2.1 of Pham and Tran (1985) restated here in our notation. For a proof, see Pham and Tran (1985).

Lemma C-9: Let A be an $n \times n$ square matrix with (ordered) singular values given by

$$\sigma_{(1)} (A) \geq \sigma_{(2)} (A) \geq \cdots \geq \sigma_{(n)} (A) \geq 0.$$

Suppose that A is diagonalizable, i.e.,

$$A = S \Lambda S^{-1}$$

where Λ is diagonal matrix whose diagonal elements are the eigenvalues of A. Let the modulus of these eigenvalues be ordered as follows:

$$\left|\lambda_{(1)}(A)\right| \geq \left|\lambda_{(2)}(A)\right| \geq \cdots \geq \left|\lambda_{(n)}(A)\right|.$$

Then, for $k \in \{1, ..., n\}$ and for any positive integer j, we have

$$\chi(S)^{-1}\left|\lambda_{(k)}\left(A^j\right)\right| \leq \sigma_{(k)}\left(A^j\right) \leq \chi(S)\left|\lambda_{(k)}\left(A^j\right)\right|$$

where

$$\chi(S) = \sigma_{(1)}(S)\,\sigma_{(1)}\left(S^{-1}\right).$$

Proof of Lemma C-9: Observe first that we can assume, without loss of generality, that the decomposition

$$A = S\Lambda S^{-1} = S \cdot diag\left(\lambda_1, \lambda_2, ..., \lambda_n\right) \cdot S^{-1}$$

is such that

$$\lambda_i = \lambda_{(i)}(A) \text{ for } i = 1, ..., n$$

with

$$\left|\lambda_{(1)}(A)\right| \geq \left|\lambda_{(2)}(A)\right| \geq \cdots \geq \left|\lambda_{(n)}(A)\right|.$$

This is because suppose we have the alternative representation where

$$A = \widetilde{S}\widetilde{\Lambda}\widetilde{S}^{-1} = \widetilde{S} \cdot diag\left(\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_n\right) \cdot \widetilde{S}^{-1}$$

and where $\widetilde{\lambda}_i \neq \lambda_{(i)}(A)$ for at least some of the $i's$. Then, we can always define a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}'\widetilde{\Lambda}\mathcal{P} = \Lambda$$

so that, given that $\mathcal{P}$ is an orthogonal matrix, we have

$$A = \widetilde{S}\widetilde{\Lambda}\widetilde{S}^{-1} = \widetilde{S}\mathcal{P}\mathcal{P}'\widetilde{\Lambda}\mathcal{P}\mathcal{P}'\widetilde{S}^{-1} = S\Lambda S^{-1}$$

where $S = \widetilde{S}\mathcal{P}$ and, thus, $S^{-1} = \left(\widetilde{S}\mathcal{P}\right)^{-1} = \mathcal{P}'\widetilde{S}^{-1}$.

Next, note that, for any positive integer j,

$$A^j = S\Lambda S^{-1} \times S\Lambda S^{-1} \times \cdots \times S\Lambda S^{-1} = S\Lambda^j S^{-1}$$

where

$$\Lambda^j = diag\left(\lambda_1^j, \lambda_2^j, ..., \lambda_n^j\right) = diag\left(\lambda_{(1)}^j (A), \lambda_{(2)}^j (A), ..., \lambda_{(n)}^j (A)\right).$$

Moreover, since $\lambda_{(k)}\left(A^j\right) = \lambda_{(k)}^j (A)$ for any $k \in \{1, ..., m\}$, we also have

$$\Lambda^j = diag\left(\lambda_1^j, \lambda_2^j, ..., \lambda_n^j\right) = diag\left(\lambda_{(1)}\left(A^j\right), \lambda_{(2)}\left(A^j\right), ..., \lambda_{(n)}\left(A^j\right)\right).$$

In addition, let $\overline{\lambda_{(k)}(A^j)}$ denote the complex conjugate of $\lambda_{(k)}\left(A^j\right)$ for $k \in \{1, ..., m\}$, and note that, by definition,

$$\sigma_{(k)}\left(\Lambda^j\right) = \sqrt{\overline{\lambda_{(k)}(A^j)}\lambda_{(k)}(A^j)} = \left|\lambda_{(k)}\left(A^j\right)\right|$$

Since $\left|\lambda_{(k)}\left(A^j\right)\right| = \left|\lambda_{(k)}^j (A)\right| = \left|\lambda_{(k)} (A)\right|^j$, the ordering

$$\left|\lambda_{(1)} (A)\right| \geq \left|\lambda_{(2)} (A)\right| \geq \cdots \geq \left|\lambda_{(n)} (A)\right|$$

implies that

$$\left|\lambda_{(1)}\left(A^j\right)\right| \geq \left|\lambda_{(2)}\left(A^j\right)\right| \geq \cdots \geq \left|\lambda_{(n)}\left(A^j\right)\right|$$

and, thus,

$$\sigma_{(1)}\left(\Lambda^j\right) \geq \sigma_{(2)}\left(\Lambda^j\right) \geq \cdots \geq \sigma_{(n)}\left(\Lambda^j\right)$$

for any positive integer j.

Now, apply the inequality

$$\sigma_{(i+\ell-1)}(BC) \leq \sigma_{(i)}(B)\,\sigma_{(\ell)}(C)$$

for $i, \ell \in \{1, ..., n\}$ and $i + \ell \leq n + 1$; we have

$$\begin{aligned}
\sigma_{(k)}\left(A^j\right) &= \sigma_{(k)}\left(S\Lambda^j S^{-1}\right) \\
&\leq \sigma_{(k)}\left(S\Lambda^j\right)\sigma_{(1)}\left(S^{-1}\right) \\
&\leq \sigma_{(k)}\left(\Lambda^j\right)\sigma_{(1)}(S)\,\sigma_{(1)}\left(S^{-1}\right) \\
&= \sigma_{(1)}(S)\,\sigma_{(1)}\left(S^{-1}\right)\left|\lambda_{(k)}\left(A^j\right)\right| \\
&= \chi(S)\left|\lambda_{(k)}\left(A^j\right)\right| \text{ for any } k \in \{1, ..., n\}
\end{aligned}$$

Moreover, for any $k \in \{1, ..., n\}$,

$$\begin{aligned}
\left|\lambda_{(k)}\left(A^j\right)\right| &= \sigma_{(k)}\left(\Lambda^j\right) \\
&= \sigma_{(k)}\left(S^{-1} S \Lambda^j S^{-1} S\right) \\
&= \sigma_{(k)}\left(S^{-1} A^j S\right) \\
&\leq \sigma_{(1)}\left(S^{-1}\right) \sigma_{(k)}\left(A^j\right) \sigma_{(1)}\left(S\right)
\end{aligned}$$

or

$$\frac{\left|\lambda_{(k)}\left(A^j\right)\right|}{\chi\left(S\right)} = \frac{\left|\lambda_{(k)}\left(A^j\right)\right|}{\sigma_{(1)}\left(S\right) \sigma_{(1)}\left(S^{-1}\right)} \leq \sigma_{(k)}\left(A^j\right)$$

Putting these two inequalities together, we have, for any $k \in \{1, ..., n\}$ and for all positive integer j,

$$\chi\left(S\right)^{-1} \left|\lambda_{(k)}\left(A^j\right)\right| \leq \sigma_{(k)}\left(A^j\right) \leq \chi\left(S\right) \left|\lambda_{(k)}\left(A^j\right)\right|. \ \square$$

Remark: Note that the case where $j = 1$ in Lemma C-9 has previously been obtained in Theorem 1 of Ruhe (1975). Hence, Lemma C-9 can be viewed as providing an extension to the first part of that theorem.

Lemma C-10: Let ρ be such that $|\rho| < 1$. Then,

$$\sum_{j=0}^{\infty} \left(j+1\right) \rho^j = \frac{1}{\left(1-\rho\right)^2} < \infty$$

Proof of Lemma C-10: Define

$$S_n\left(\rho\right) = 1 + \rho + \rho^2 + \cdots + \rho^n = \frac{1-\rho^{n+1}}{1-\rho}$$

Note that

$$\begin{aligned}
S_n'(\rho) &= 1+2\rho+3\rho^2+\cdots+n\rho^{n-1} \\
&= -\frac{(n+1)\rho^n}{1-\rho}+\frac{1-\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1}-(n+1)\rho^n(1-\rho)}{(1-\rho)^2} \\
&= \frac{1-\rho^{n+1}-(n+1)\rho^n+(n+1)\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-(n+1)\rho^n+n\rho^{n+1}}{(1-\rho)^2} \\
&= \frac{1-\rho^n-n\rho^n(1-\rho)}{(1-\rho)^2}
\end{aligned}$$

It follows that

$$S_n'(\rho)=\sum_{j=0}^{n-1}(j+1)\rho^j=\frac{1-\rho^n-n\rho^n(1-\rho)}{(1-\rho)^2}\to\frac{1}{(1-\rho)^2} \text{ as } n\to\infty. \ \square$$

Lemma C-11: Let $W_t=(Y_t',F_t')'$ be generated by the factor-augmented VAR process

$$W_{t+1}=\mu+A_1W_t+\cdots+A_pW_{t-p+1}+\varepsilon_{t+1}$$

described in section 3 of the main paper. Under Assumptions 3-1, 3-2(a)-(c), and 3-7; $\{W_t\}$ is a β-mixing process with β-mixing coefficient $\beta_W(m)$ such that

$$\beta_W(m)\le C_1\exp\{-C_2m\}$$

for some positive constants C_1 and C_2. Here,

$$\beta_W(m)=\sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{A}_{-\infty}^t\right)-P(B)\right|:B\in\mathcal{A}_{t+m}^\infty\right\}\right]$$

with $\mathcal{A}_{-\infty}^t=\sigma(...,W_{t-2},W_{t-1},W_t)$ and $\mathcal{A}_{t+m}^\infty=\sigma(W_{t+m},W_{t+m+1},W_{t+m+2},....)$.

Proof of Lemma C-11:

To prove this lemma, we shall verify the conditions of Lemma C-8 given above for the vector moving-average representation of W_t, i.e.,

$$W_t = J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} J_{d+K} A^j J_{d+K}' \varepsilon_{t-j} = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j},$$

where

$$\begin{aligned}
\mu_* &= J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu,\ \Psi_j = J_{d+K} A^j J_{d+K}', \\
\underset{(d+K)\times(d+K)p}{J_{d+K}} &= \left[\begin{array}{ccccc} I_{d+K} & 0 & \cdots & 0 & 0 \end{array}\right], \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}
\end{aligned}$$

To proceed, set

$$\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \tag{58}$$

and note first that, setting $\delta = 5$ in Lemma C-8, and we see that Assumptions (i) and (ii) of Lemma C-8 are the same as the conditions specified in Assumption 3-2 (a)-(c). Next, note that, since in this case $\Psi_j = J_{d+K} A^j J_{d+K}'$, we have

$$\begin{aligned}
\left\|\Psi_j\right\|_2 &\leq \left\|J_{d+K}\right\|_2 \left\|A^j\right\|_2 \left\|J_{d+K}'\right\|_2 \\
&\leq \sqrt{\lambda_{\max}\left(J_{d+K}' J_{d+K}\right)} \left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right) \sqrt{\lambda_{\max}\left(J_{d+K} J_{d+K}'\right)} \\
&= \lambda_{\max}\left(J_{d+K} J_{d+K}'\right) \left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right) \\
&= \sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}} \\
&= \sigma_{\max}\left(A^j\right) \\
&\leq C\left[\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}\right] \quad \text{(by Assumption 3-7)} \\
&= C\left[\max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}\right]^j \\
&= C\phi_{\max}^j
\end{aligned}$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. It follows that

$$\sum_{j=0}^{\infty} \|\Psi_j\|_2 \leq C \sum_{j=0}^{\infty} \phi_{\max}^j = \frac{C}{1-\phi_{\max}} < \infty.$$

Moreover, by Assumption 3-1,

$$\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \neq 0 \text{ for all } z \text{ such that } |z| \leq 1$$

and, by definition,

$$\sum_{j=0}^{\infty} \Psi_j z^j = \Psi(z) = \left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)^{-1} \text{ for all } z \text{ such that } |z| \leq 1$$

so that

$$\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right) = I_{(d+K)p} \text{ for all } z \text{ such that } |z| \leq 1$$

In addition, since

$$\begin{aligned} & \det\{\Psi(z)\} \det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \\ = \ & \det\left\{\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)\right\} \\ = \ & \det\left\{I_{(d+K)p}\right\} \\ = \ & 1, \end{aligned}$$

it follows that

$$\begin{aligned} \det\left\{\sum_{j=0}^{\infty} \Psi_j z^j\right\} &= \det\{\Psi(z)\} \\ &= \frac{1}{\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\}} \\ &\neq 0 \text{ for all } z \text{ such that } |z| \leq 1. \end{aligned}$$

Finally, note that, setting $\delta = 5$,

$$
\begin{aligned}
\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{\delta}{1+\delta}} &= \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
&\leq \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&= C^{\frac{5}{6}}\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&\leq C^{\frac{5}{6}}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^{k} \\
&\quad \left(\text{by the inequality } \left|\sum_{i=1}^{m}a_i\right|^{r} \leq c_r\sum_{i=1}^{m}|a_i|^{r} \text{ where } c_r = 1 \text{ for } r \leq 1\right) \\
&= C^{\frac{5}{6}}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^{j} \\
&= C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \quad \text{(by Lemma C-10)} \\
&< \infty \quad \left(\text{since } 0 < \phi_{\max}^{\frac{5}{6}} < 1 \text{ given that } 0 < \phi_{\max} < 1\right).
\end{aligned}
$$

Hence, all conditions of Lemma C-8 are fulfilled. Applying Lemma C-8, we then obtain that there exists a constant $\overline{C}$ such that

$$\begin{aligned}
\beta_{\xi}(m) &\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
&\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&= \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&\leq \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^{k} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^{j} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \\
&= \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2}\exp\left\{-\left[\frac{5}{6}\left|\ln\phi_{\max}\right|\right]m\right\} \quad (\text{since } 0<\phi_{\max}<1) \\
&\leq C_1\exp\{-C_2 m\} \to 0 \text{ as } m\to\infty.
\end{aligned}$$

for some positive constants C_1 and C_2 such that

$$C_1 \geq \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \text{ and } C_2 \leq \frac{5}{6}\left|\ln\phi_{\max}\right|$$

It follows that the process $\{\xi_t\}$ (as defined in expression (58)) is β mixing with beta coefficient $\beta_{\xi}(m)$ satisfying

$$\beta_{\xi}(m) \leq C_1\exp\{-C_2 m\}.$$

Since

$$W_t = \mu_* + \sum_{j=0}^{\infty}\Psi_j\varepsilon_{t-j} = \mu_* + \xi_t$$

and since μ_* is a nonrandom parameter, we can then apply part (a) of Lemma C-2 to deduce that $\{W_t\}$ is a β mixing process with β coefficient $\beta_W(m)$ satisfying the inequality

$$\beta_W(m) \leq C_1\exp\{-C_2 m\}. \ \square$$

Lemma C-12: Let $\underline{Y}_t = \begin{pmatrix} Y_t' & Y_{t-1}' & \cdots & Y_{t-p+2}' & Y_{t-p+1}' \end{pmatrix}'$ and $\underline{F}_t = \begin{pmatrix} F_t' & F_{t-1}' & \cdots & F_{t-p+2}' & F_{t-p+1}' \end{pmatrix}'$. Under Assumptions 3-1, 3-2(a)-(c), 3-5, 3-7, and 3-10(b); the following statements are true as $N, T \to \infty$

(a)
$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t' \right] \right) \alpha_{YY,\ell} \right| \xrightarrow{p} 0$$

(b)
$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right) \alpha_{YF,\ell} \right| \xrightarrow{p} 0$$

(c)
$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t - E\left[\underline{F}_t \right] \right) \mu_{Y,\ell} \right| \xrightarrow{p} 0$$

(d)
$$\max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^{q} \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left\{ \left(\underline{F}_t - E\left[\underline{F}_t \right] \right) \mu_{Y,\ell} + \left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t' \right] \right) \alpha_{YY,\ell} \right.\right.$$
$$\left.\left. + \left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right) \alpha_{YF,\ell} \right\} \right)^2$$
$$\xrightarrow{p} 0$$

(e) There exists a positive constant $\overline{C}$ such that

$$\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left(\frac{\pi_{i,\ell,T}}{q\tau_1^2} \right) \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^{q} \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left\{ E\left[\underline{F}_t \right] \mu_{Y,\ell} + E\left[\underline{F}_t \underline{Y}_t' \right] \alpha_{YY,\ell} + E\left[\underline{F}_t \underline{F}_t' \right] \alpha_{YF,\ell} \right\} \right)^2 \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \frac{1}{q} \sum_{r=1}^{q} \left(\frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' E\left[\underline{F}_t y_{\ell,t+1} \right] \right)^2 \\
\leq & \overline{C} < \infty
\end{aligned}$$

(f)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2=O_p\left(1\right).$$

(g)

$$\begin{aligned}&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left\{\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right.\right.\right.\\&\left.+\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)\\&\left.\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right\}\right|\\&\xrightarrow{p}0\end{aligned}$$

(h)

$$\begin{aligned}&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.\\&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right|\\&\xrightarrow{p}0\end{aligned}$$

(i)

$$\begin{aligned}&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right.\\&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right|\\&\xrightarrow{p}0\end{aligned}$$

Proof of Lemma C-12:

To show part (a), note that, for any $\epsilon>0$,

$$
\begin{aligned}
& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right|\ge\epsilon\right\} \\
= \;& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\ge\epsilon^2\right\} \\
\le \;& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2\ge\epsilon^2\right\} \\
& \text{(by Jensen's inequality)} \\
= \;& P\left\{\max_{i\in H^c}\max_{1\le\ell\le d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)^2\ge\epsilon^2\right\} \\
\le \;& P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)^2\ge\epsilon^2\right\} \\
\le \;& P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]'\right.\right. \\
& \qquad\qquad\left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right]\right)\ge\epsilon^2\right\} \\
= \;& P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\right. \\
& \qquad\qquad\left.\times\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\alpha_{YY,\ell}\ge\epsilon^2\right\} \\
\le \;& \frac{\max_{i\in H^c}\|\gamma_i\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right. \\
& \qquad\qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\} \\
& \text{(by Markov's inequality)} \\
\le \;& \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right. \\
& \qquad\qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\} \qquad (59) \\
& \text{(by Assumption 3-5)}
\end{aligned}
$$

Next, write

$$
\begin{aligned}
&\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\alpha_{YY,\ell}'\right.\right.\\
&\qquad\left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right\}\right)\\
=&\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right)\\
&+\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left\{\alpha_{YY,\ell}'\right.\right.\\
&\qquad\left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]\alpha_{YY,\ell}\right\}\right)\\
\leq&\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right)\\
&+\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|\alpha_{YY,\ell}'\right.\right.\\
&\qquad\left.\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]\alpha_{YY,\ell}\right|\right) \quad (60)
\end{aligned}
$$

Let $e_{\ell,d}$ be a $d\times 1$ elementary vector whose ℓ^{th} component is 1 and all other components are 0, and

note that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}' E\left[\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]\alpha_{YY,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e_{\ell,d}' A_{YY} E\left[\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right] A_{YY}' e_{\ell,d}\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e_{\ell,d}' A_{YY} E\left[\underline{Y}_t\underline{F}_t'\underline{F}_t\underline{Y}_t'\right] A_{YY}' e_{\ell,d}\right. \\
& \qquad\qquad \left. - \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e_{\ell,d}' A_{YY} E\left[\underline{Y}_t\underline{F}_t'\right] E\left[\underline{F}_t\underline{Y}_t'\right] A_{YY}' e_{\ell,d}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\|\underline{F}_t\|_2^2\left(e_{\ell,d}' A_{YY}\underline{Y}_t\right)^2\right] \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left(e_{\ell,d}' A_{YY}\underline{Y}_t\underline{Y}_t' A_{YY}' e_{\ell,d}\right)^2} \text{ (by CS inequality)} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{Y}_t\|_2^4\right]}\sqrt{\left(e_{\ell,d}' A_{YY} A_{YY}' e_{\ell,d}\right)^2} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{Y}_t\|_2^4\right]}\|A_{YY}\|_2^2\sqrt{\left(e_{\ell,d}' e_{\ell,d}\right)^2} \\
\leq & \frac{d\left(C^{\dagger}\right)^2}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{Y}_t\|_2^4\right]}\phi_{\max}^2 \\
& \text{(by part (a) of Lemma C-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector)} \\
\leq & \frac{\overline{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \qquad (61)
\end{aligned}
$$

for some positive constant $\overline{C} \geq d\left(C^{\dagger}\right)^2\sqrt{E\left[\|\underline{F}_t\|_2^4\right]}\sqrt{E\left[\|\underline{Y}_t\|_2^4\right]}\phi_{\max}^2$, which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1.

To analyze the second term on the right-hand side of expression (60), note first that by Lemma C-11, $\left\{(Y_t', F_t')'\right\}$ is β-mixing with β mixing coefficient satisfying

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\} \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{W,m} \leq \beta_W(m)$, it follows that $W_t = (Y_t', F_t')'$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{W,m} \leq C_1 \exp\{-C_2 m\}$$

Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = \underline{F}_t \underline{Y}_t' A_{YY}' e_{\ell,d}$ is also α-mixing with α mixing coefficient satisfying

$$\begin{aligned} \alpha_{X,m} &\leq C_1 \exp\{-C_2 (m - p + 1)\} \\ &\leq C_1^* \exp\{-C_2 m\} \end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2 (p-1)\}$. Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned} & \left|\alpha_{YY,\ell}' E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] \alpha_{YY,\ell}\right| \\ = & \left|e_{\ell,d}' A_{YY} E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] A_{YY}' e_{\ell,d}\right| \\ = & \left|\sum_{h=1}^{Kp} e_{\ell,d}' A_{YY} E\left[\left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' e_{h,Kp} e_{h,Kp}' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right)\right] A_{YY}' e_{\ell,d}\right| \\ \leq & \sum_{h=1}^{Kp} \left\{ 2\left(2^{\frac{2}{3}} + 1\right) \alpha_{X,m}^{\frac{1}{3}} \left(E\left| e_{\ell,d}' A_{YY} \left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right]\right)' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\ & \left. \times \left(E\left| e_{h,Kp}' \left(\underline{F}_{t+m} \underline{Y}_{t+m}' - E\left[\underline{F}_{t+m} \underline{Y}_{t+m}'\right]\right) A_{YY}' e_{\ell,d} \right|^3 \right)^{1/3} \right\} \end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $\{X_t\}$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp\left\{-\frac{C_2 m}{3}\right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\ &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \\ &\leq (C_1^*)^{\frac{1}{3}} \left[1 - \exp\left\{-\frac{C_2}{3}\right\}\right]^{-1} \\ &\leq C_3 \end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\{-(C_2 m/3)\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$\begin{aligned}
& E\left|e'_{\ell,d}A_{YY}\left(\underline{F}_t\underline{Y}'_t - E\left[\underline{F}_t\underline{Y}'_t\right]\right)' e_{h,Kp}\right|^3 \\
\leq\; & 2^2\left\{E\left|e'_{\ell,d}A_{YY}\underline{Y}_t\underline{F}'_t e_{h,Kp}\right|^3 + \left|E\left[e'_{\ell,d}A_{YY}\underline{Y}_t\underline{F}'_t e_{h,Kp}\right]\right|^3\right\} \text{(by Loève's } c_r \text{ inequality)} \\
\leq\; & 2^2\left\{E\left|e'_{\ell,d}A_{YY}\underline{Y}_t\underline{F}'_t e_{h,Kp}\right|^3 + \left(E\left[\left|e'_{\ell,d}A_{YY}\underline{Y}_t\underline{F}'_t e_{h,Kp}\right|\right]\right)^3\right\} \text{ (by Jensen's inequality)} \\
\leq\; & 2^2\left\{E\left|\frac{e'_{\ell,d}A_{YY}\underline{Y}_t\underline{Y}'_t A'_{YY}e_{\ell,d}}{2} + \frac{e'_{h,Kp}\underline{F}_t\underline{F}'_t e_{h,Kp}}{2}\right|^3 + \left(E\left[\left|e'_{\ell,d}A_{YY}\underline{Y}_t\underline{F}'_t e_{h,Kp}\right|\right]\right)^3\right\} \\
\leq\; & 4\frac{2^2}{8}\left[E\left|e'_{\ell,d}A_{YY}\underline{Y}_t\underline{Y}'_t A'_{YY}e_{\ell,d}\right|^3 + E\left|e'_{h,Kp}\underline{F}_t\underline{F}'_t e_{h,Kp}\right|^3\right] \\
& +4\left(\sqrt{E\left[e'_{\ell,d}A_{YY}\underline{Y}_t\underline{Y}'_t A'_{YY}e_{\ell,d}\right]}\sqrt{E\left[e_{h,Kp}\underline{F}_t\underline{F}'_t e_{h,Kp}\right]}\right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
\leq\; & 2\left|e'_{\ell,d}A_{YY}A'_{YY}e_{\ell,d}\right|^3 E\left\|\underline{Y}_t\right\|_2^6 + 2E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(e'_{\ell,d}A_{YY}A'_{YY}e_{\ell,d}\right)^{\frac{3}{2}}\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
\leq\; & 2\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + 2E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
=\; & 2\left(C^{\dagger}\right)^6\phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + 2E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(C^{\dagger}\right)^3\phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}} \\
& \left(\text{since } \left\|e_{\ell,d}\right\|_2 = 1 \text{ for every } \ell \in \{1,...,d\} \text{ given that } e_{\ell,d}\text{'s are elementary vectors}\right) \\
\leq\; & C_4
\end{aligned}$$

for some positive constant $C_4 \geq 2\left(C^{\dagger}\right)^6\phi_{\max}^6 E\left\|\underline{Y}_t\right\|_2^6 + 2E\left\|\underline{F}_t\right\|_2^6 + 4\left(E\left\|\underline{Y}_t\right\|_2^2\right)^{\frac{3}{2}}\left(C^{\dagger}\right)^3\phi_{\max}^3\left(E\left\|\underline{F}_t\right\|_2^2\right)^{\frac{3}{2}}$ which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1. In a similar way, we can also show that there exists a positive constant C_5 such that

$$\begin{aligned}
& E\left|e'_{h,Kp}\left(\underline{F}_{t+m}\underline{Y}'_{t+m} - E\left[\underline{F}_{t+m}\underline{Y}'_{t+m}\right]\right)A'_{YY}e_{\ell,d}\right|^3 \\
\leq\; & 2\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6 E\left\|\underline{Y}_{t+m}\right\|_2^6 + 2E\left\|\underline{F}_{t+m}\right\|_2^6 \\
& +4\left(E\left\|\underline{Y}_{t+m}\right\|_2^2\right)^{\frac{3}{2}}\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3\left(E\left\|\underline{F}_{t+m}\right\|_2^2\right)^{\frac{3}{2}} \\
\leq\; & C_5 < \infty
\end{aligned}$$

Hence,

$$
\begin{aligned}
& \frac{2}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|e_{\ell,d}'A_{YY} \\
& \qquad \times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]A_{YY}'e_{\ell,d}\Big| \\
\leq \;& \frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\sum_{h=1}^{Kp}\alpha_{W,m}^{\frac{1}{3}}\left(E\left|e_{\ell,d}'A_{YY}\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \\
& \qquad \times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)A_{YY}'e_{\ell,d}\right|^3\right)^{1/3} \\
\leq \;& \frac{4dKp\left(2^{\frac{2}{3}}+1\right)C_4^{\frac{1}{3}}C_5^{\frac{1}{3}}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}\left(C_1^*\right)^{\frac{1}{3}}\exp\left\{-\frac{C_2m}{3}\right\} \\
\leq \;& \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\}\;\;\left(\text{where } C^*\geq 4dKp\left(2^{\frac{2}{3}}+1\right)\left(C_1^*\right)^{\frac{1}{3}}C_4^{\frac{1}{3}}C_5^{\frac{1}{3}}\right) \\
\leq \;& \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
= \;& O\left(\frac{1}{\tau_1}\right) \qquad (62)
\end{aligned}
$$

It then follows from expressions (59), (60), (61), and (62) that

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right|\geq\epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}\right. \\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s'-E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]A_{YY}'e_{\ell,d}\right) \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}e_{\ell,d}'A_{YY}E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\right]A_{YY}'e_{\ell,d}\right) \\
& +\frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|e_{\ell,d}'A_{YY}\right. \\
& \qquad\left.\times E\left[\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{Y}_{t+m}'-E\left[\underline{F}_{t+m}\underline{Y}_{t+m}'\right]\right)\right]A_{YY}'e_{\ell,d}\right| \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\ & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\ & O\left(\frac{1}{\tau_1}\right)=o(1).
\end{aligned}
$$

Next, to show part (b), note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right|\ge\epsilon\right\} \\
= \; & P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2\ge\epsilon^2\right\} \\
\le \; & P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2\ge\epsilon^2\right\} \\
& \text{(by Jensen's inequality)} \\
= \; & P\left\{\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)^2\ge\epsilon^2\right\} \\
\le \; & P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)^2\ge\epsilon^2\right\} \\
\le \; & P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]'\right.\right. \\
& \qquad\qquad \left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right]\right)\ge\epsilon^2\right\} \\
= \; & P\left\{\max_{i\in H^c}\|\gamma_i\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\alpha_{YF,\ell}\ge\epsilon^2\right\} \\
\le \; & \frac{\max_{i\in H^c}\|\gamma_i\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& \text{(by Markov's inequality)} \\
\le \; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad\qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \qquad (63) \\
& \text{(by Assumption 3-5)}
\end{aligned}
$$

Note first that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'\right. \\
& \qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\alpha_{YF,\ell}'\right. \\
& \qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\right) \\
\leq & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\sum_{\ell=1}^{d}\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|\alpha_{YF,\ell}'\right. \\
& \qquad \left.\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\right| \qquad (64)
\end{aligned}
$$

Consider the first term on the majorant side of expression (64), whose order of magnitude we can analyze as follows

$$\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}' E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\right] A_{YF}' e_{\ell,d}\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{e_{\ell,d}' A_{YF} E\left[\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right] A_{YF}' e_{\ell,d} - e_{\ell,d}' A_{YF} E\left[\underline{F}_t\underline{F}_t'\right] E\left[\underline{F}_t\underline{F}_t'\right] A_{YF}' e_{\ell,d}\right\}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\left\|\underline{F}_t\right\|_2^2\left(e_{\ell,d}' A_{YF}\underline{F}_t\right)^2\right] \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left(e_{\ell,d}' A_{YF}\underline{F}_t\underline{F}_t' A_{YF}' e_{\ell,d}\right)^2} \text{ (by CS inequality)} \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{\left(e_{\ell,d}' A_{YF} A_{YF}' e_{\ell,d}\right)^2} \\
\leq & \frac{1}{q\tau_1^2}\sum_{\ell=1}^{d}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\sqrt{E\left[\left\|\underline{F}_t\right\|_2^4\right]}\left\|A_{YF}\right\|_2^2\sqrt{\left(e_{\ell,d}' e_{\ell,d}\right)^2} \\
\leq & \frac{\left(C^{\dagger}\right)^2}{q\tau_1^2}\sum_{\ell=1}^{d}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} E\left[\left\|\underline{F}_t\right\|_2^4\right]\phi_{\max}^2 \\
& \text{(by part (b) of Lemma C-7 and by the fact that } e_{\ell,d} \text{ is an elementary vector)} \\
\leq & \frac{\overline{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \qquad (65)
\end{aligned}$$

for some positive constant $\overline{C} \geq d\left(C^{\dagger}\right)^2 E\left[\left\|\underline{F}_t\right\|_2^4\right]\phi_{\max}^2$, which exists in light of Lemma C-5 and the fact that $0 < \phi_{\max} < 1$ given Assumption 3-1.

To analyze the second term on the right-hand side of expression (64), note first that by Lemma C-11, $\{F_t\}$ is β-mixing with β mixing coefficient satisfying

$$\beta_F(m) \leq C_1 \exp\{-C_2 m\} \text{ for some positive constants } C_1 \text{ and } C_2.$$

Since $\alpha_{F,m} \leq \beta_F(m)$, it follows that F_t is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp\{-C_2 m\}$$

Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = \underline{F}_t \underline{F}_t' A_{YF}' e_{\ell,d}$ is also α-mixing with α mixing coefficient satisfying

$$\begin{aligned}
\alpha_{X,m} &\leq C_1 \exp\{-C_2 (m - p + 1)\} \\
&\leq C_1^* \exp\{-C_2 m\}
\end{aligned}$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2 (p-1)\}$. Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned}
& \left| \alpha_{YF,\ell}' E\left[\left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right)' \left(\underline{F}_{t+m} \underline{F}_{t+m}' - E\left[\underline{F}_{t+m} \underline{F}_{t+m}' \right] \right) \right] \alpha_{YF,\ell} \right| \\
= & \left| e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right)' \left(\underline{F}_{t+m} \underline{F}_{t+m}' - E\left[\underline{F}_{t+m} \underline{F}_{t+m}' \right] \right) \right] A_{YF}' e_{\ell,d} \right| \\
= & \left| \sum_{h=1}^{Kp} e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right)' e_{h,Kp} e_{h,Kp}' \left(\underline{F}_{t+m} \underline{F}_{t+m}' - E\left[\underline{F}_{t+m} \underline{F}_{t+m}' \right] \right) \right] A_{YF}' e_{\ell,d} \right| \\
\leq & \sum_{h=1}^{Kp} \left\{ 2 \left(2^{\frac{2}{3}} + 1 \right) \alpha_{X,m}^{\frac{1}{3}} \left(E\left| e_{\ell,d}' A_{YF} \left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t' \right] \right)' e_{h,Kp} \right|^3 \right)^{\frac{1}{3}} \right. \\
& \quad \left. \times \left(E\left| e_{h,Kp}' \left(\underline{F}_{t+m} \underline{F}_{t+m}' - E\left[\underline{F}_{t+m} \underline{F}_{t+m}' \right] \right) A_{YF}' e_{\ell,d} \right|^3 \right)^{1/3} \right\}
\end{aligned}$$

where $\alpha_{X,m}$ denotes the alpha mixing coefficient for the process $\{X_t\}$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{3}} \leq (C_1^*)^{\frac{1}{3}} \exp\left\{ -\frac{C_2 m}{3} \right\} \text{ for all } m \text{ sufficiently large,}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned}
\sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{3}} &\leq (C_1^*)^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{ -\frac{C_2 m}{3} \right\} \\
&\leq (C_1^*)^{\frac{1}{3}} \sum_{m=0}^{\infty} \exp\left\{ -\frac{C_2 m}{3} \right\} \\
&\leq (C_1^*)^{\frac{1}{3}} \left[1 - \exp\left\{ -\frac{C_2}{3} \right\} \right]^{-1} \\
&\leq C_3
\end{aligned}$$

Next, note that

$$
\begin{aligned}
& E\left|e_{\ell,d}^{\prime}A_{YF}\left(\underline{F}_t\underline{F}_t^{\prime}-E\left[\underline{F}_t\underline{F}_t^{\prime}\right]\right)^{\prime}e_{h,Kp}\right|^3 \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|^3+\left|E\left[e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right]\right|^3\right\}\text{(by Loève's } c_r \text{ inequality)} \\
\leq\ & 2^2\left\{E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|^3+\left(E\left[\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|\right]\right)^3\right\}\ \text{(by Jensen's inequality)} \\
\leq\ & 2^2\left\{E\left|\frac{e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}}{2}+\frac{e_{h,Kp}^{\prime}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}}{2}\right|^3+\left(E\left[\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right|\right]\right)^3\right\} \\
\leq\ & 4\frac{2^2}{8}\left[E\left|e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}\right|^3+E\left|e_{h,Kp}^{\prime}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}^{\prime}\right|^3\right] \\
& +4\left(\sqrt{E\left[e_{\ell,d}^{\prime}A_{YF}\underline{F}_t\underline{F}_t^{\prime}A_{YF}^{\prime}e_{\ell,d}\right]}\sqrt{E\left[e_{h,Kp}\underline{F}_t\underline{F}_t^{\prime}e_{h,Kp}\right]}\right)^3 \\
& \text{(by Loève's } c_r \text{ inequality and by the CS inequality)} \\
\leq\ & 2\left|e_{\ell,d}^{\prime}A_{YF}A_{YF}^{\prime}e_{\ell,d}\right|^3E\left\|\underline{F}_t\right\|_2^6+2E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(e_{\ell,d}^{\prime}A_{YF}A_{YF}^{\prime}e_{\ell,d}\right)^{\frac{3}{2}} \\
\leq\ & 2\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+2E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
=\ & 2\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+2E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
& \left(\text{since } \left\|e_{\ell,d}\right\|_2=1 \text{ for every } \ell\in\{1,...,d\} \text{ given that } e_{\ell,d}\text{'s are elementary vectors}\right) \\
\leq\ & C_6
\end{aligned}
$$

for some positive constant $C_6\geq 2\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_t\right\|_2^6+2E\left\|\underline{F}_t\right\|_2^6+4\left(E\left\|\underline{F}_t\right\|_2^2\right)^3\left(C^{\dagger}\right)^3\phi_{\max}^3$ which exists in light of Lemma C-5 and the fact that $0<\phi_{\max}<1$ given Assumption 3-1. In a similar way, we can also show that there exists a positive constant C_7 such that

$$
\begin{aligned}
E\left|e_{h,Kp}^{\prime}\left(\underline{F}_{t+m}\underline{F}_{t+m}^{\prime}-E\left[\underline{F}_{t+m}\underline{F}_{t+m}^{\prime}\right]\right)A_{YY}^{\prime}e_{\ell,d}\right|^3 &\leq 2\left\|e_{\ell,d}\right\|_2^6\left(C^{\dagger}\right)^6\phi_{\max}^6E\left\|\underline{F}_{t+m}\right\|_2^6+2E\left\|\underline{F}_{t+m}\right\|_2^6 \\
&\quad +4\left(E\left\|\underline{F}_{t+m}\right\|_2^2\right)^3\left\|e_{\ell,d}\right\|_2^3\left(C^{\dagger}\right)^3\phi_{\max}^3 \\
&\leq C_7<\infty
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \frac{2}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|e_{\ell,d}'A_{YF} \\
& \qquad \times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]A_{YF}'e_{\ell,d}\Big| \\
\leq \;& \frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\sum_{\ell=1}^{d}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\sum_{h=1}^{Kp}\left\{\alpha_{F,m}^{\frac{1}{3}}\left(E\left|e_{\ell,d}'A_{YF}\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}}\right. \\
& \qquad \left.\times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)A_{YF}'e_{\ell,d}\right|^3\right)^{1/3}\right\} \\
\leq \;& \frac{4dKp\left(2^{\frac{2}{3}}+1\right)C_6^{\frac{1}{3}}C_7^{\frac{1}{3}}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}\left(C_1^*\right)^{\frac{1}{3}}\exp\left\{-\frac{C_2m}{3}\right\} \\
\leq \;& \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\}\quad\left(\text{where } C^*\geq 4dKp\left(2^{\frac{2}{3}}+1\right)\left(C_1^*\right)^{\frac{1}{3}}C_6^{\frac{1}{3}}C_7^{\frac{1}{3}}\right) \\
\leq \;& \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
= \;& O\left(\frac{1}{\tau_1}\right) \qquad (66)
\end{aligned}
$$

It then follows from expressions (63), (64), (65), and (66) that

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right|\geq\epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s'-E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
\leq\ & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\right]\alpha_{YF,\ell}\right) \\
& +\frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\Big|\alpha_{YF,\ell}' \\
& \qquad\qquad\times E\left[\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_{t+m}\underline{F}_{t+m}'-E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\right)\right]\alpha_{YF,\ell}\Big| \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
=\ & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2 m}{3}\right\} \\
=\ & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\ & O\left(\frac{1}{\tau_1}\right)=o\left(1\right).
\end{aligned}
$$

Now, to show part (c), note that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right| \geq \epsilon\right\} \\
= \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \geq \epsilon^2\right\} \text{ (by Jensen's inequality)} \\
= \;& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{i\in H^c}\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\left(\gamma_i'\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right)^2 \geq \epsilon^2\right\} \\
\leq \;& P\left\{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]'\right.\right. \\
& \qquad\qquad \left.\left.\times\left[\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right]\right) \geq \epsilon^2\right\} \\
= \;& P\left\{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2\sum_{\ell=1}^{d}\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\mu_{Y,\ell} \geq \epsilon^2\right\} \\
\leq \;& \frac{\max_{i\in H^c}\left\|\gamma_i\right\|_2^2}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \\
& \text{(by Markov's inequality)} \\
\leq \;& \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \qquad (67) \\
& \text{(by Assumption 3-5)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \\
= & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\sum_{\ell=1}^{d}\left(\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right]\right) \\
\leq & \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\frac{2}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \quad (68)
\end{aligned}
$$

Consider the first term on the majorant side of expression (68), whose order of magnitude we can analyze as follows

$$
\begin{aligned}
& \sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\right]\right) \\
= & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\left(\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2\left\{E\left[\underline{F}_t'\underline{F}_t\right] - E\left[\underline{F}_t\right]'E\left[\underline{F}_t\right]\right\}\right) \\
\leq & \sum_{\ell=1}^{d}\frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2 E\left[\left\|\underline{F}_t\right\|_2^2\right] \\
= & \frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\left\|\underline{F}_t\right\|_2^2\right]\sum_{\ell=1}^{d}\left(\mu_{Y,\ell}^2\right) \\
\leq & \frac{1}{q\tau_1^2}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left[\left\|\underline{F}_t\right\|_2^2\right]\left\|\mu_Y\right\|_2^2 \\
\leq & \frac{\overline{C}}{\tau_1} = O\left(\frac{1}{\tau_1}\right). \quad (69)
\end{aligned}
$$

for some positive constant $\overline{C} \geq \|\mu_Y\|_2^2 E\left[\|\underline{F}_t\|_2^2\right]$, which exists in light of Assumption 3-5 and Lemma C-5.

To analyze the second term on the right-hand side of expression (68), note first that by the same argument as given for part (b) above, we can apply Lemma C-11 to deduce that $\{F_t\}$ is β-mixing and, thus, also α-mixing with α mixing coefficient satisfying

$$\alpha_{F,m} \leq C_1 \exp\{-C_2 m\}$$

Hence, we can apply Lemma C-3 with $p = 3$ and $r = 3$ to obtain

$$\begin{aligned}
& \left|E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right]\right| \sum_{\ell=1}^{d} \mu_{Y,\ell}^2 \\
= & \left|\sum_{h=1}^{Kp} e_{\ell,d}' A_{YF} E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right] A_{YF}' e_{\ell,d}\right| \sum_{\ell=1}^{d} \mu_{Y,\ell}^2 \\
\leq & \sum_{h=1}^{Kp} 2\left(2^{\frac{2}{3}} + 1\right) \alpha_{F,m}^{\frac{1}{3}} \left(E\left|\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)' e_{h,Kp}\right|^3\right)^{\frac{1}{3}} \left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m} - E\left[\underline{F}_{t+m}\right]\right)\right|^3\right)^{1/3} \sum_{\ell=1}^{d} \mu_{Y,\ell}^2
\end{aligned}$$

Moreover, there exists a positive constant C_3 such that

$$\sum_{m=1}^{\infty} \alpha_{F,m}^{\frac{1}{3}} \leq C_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{3}\right\} \leq C_3$$

where again the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\{-(C_2 m/3)\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/3)\} < 1$ for $C_2 > 0$. Next, note that

$$\begin{aligned}
& E\left|\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)' e_{h,Kp}\right|^3 \\
\leq\ & 2^2\left\{E\left|\underline{F}_t' e_{h,Kp}\right|^3 + \left|E\left[\underline{F}_t' e_{h,Kp}\right]\right|^3\right\} \text{(by Loève's } c_r \text{ inequality)} \\
\leq\ & 2^2\left\{E\left|\underline{F}_t' e_{h,Kp}\right|^3 + \left(E\left[\left|\underline{F}_t' e_{h,Kp}\right|\right]\right)^3\right\} \text{ (by Jensen's inequality)} \\
\leq\ & 2^2\left\{E\left[\left(\underline{F}_t'\underline{F}_t\right)^{\frac{3}{2}}\left(e_{h,Kp}' e_{h,Kp}\right)^{\frac{3}{2}}\right] + \left(\sqrt{E\left[\underline{F}_t'\underline{F}_t\right]}\sqrt{e_{h,Kp}' e_{h,Kp}}\right)^3\right\} \text{ (by CS inequality)} \\
\leq\ & 4\left\{E\left[\left\|\underline{F}_t\right\|_2^3\right] + \left(E\left[\left\|\underline{F}_t\right\|_2^2\right]\right)^{\frac{3}{2}}\right\} \\
\leq\ & C_8
\end{aligned}$$

for some positive constant $C_8 \geq 4\left\{E\left[\left\|\underline{F}_t\right\|_2^3\right] + \left(E\left[\left\|\underline{F}_t\right\|_2^2\right]\right)^{\frac{3}{2}}\right\}$ which exists in light of the result given in Lemma C-5. In a similar way, we can also show that there exists a positive constant C_9

such that

$$\begin{aligned} E\left|e_\ell'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right|^3 &\leq 4\left\{E\left[\left\|\underline{F}_{t+m}\right\|_2^3\right]+\left(E\left[\left\|\underline{F}_{t+m}\right\|_2^2\right]\right)^{\frac{3}{2}}\right\} \\ &\leq C_9<\infty \end{aligned}$$

Finally, by Assumption 3-5, there exists a positive constant C_{10} such that $\max_{1\leq\ell\leq d}\mu_{Y,\ell}^2\leq\|\mu_Y\|_2^2\leq C_{10}<\infty$. Hence,

$$\begin{aligned} &\frac{2}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \\ \leq\;& \sum_{h=1}^{Kp}\frac{4\left(2^{\frac{2}{3}}+1\right)}{\tau_1^2}\|\mu_Y\|_2^2 \\ &\times\frac{1}{q}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left\{\alpha_{F,m}^{\frac{1}{3}}\left(E\left|\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'e_{h,Kp}\right|^3\right)^{\frac{1}{3}}\right. \\ &\qquad\qquad\left.\times\left(E\left|e_{h,Kp}'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right|^3\right)^{1/3}\right\} \\ \leq\;& \frac{4Kp\left(2^{\frac{2}{3}}+1\right)C_8^{\frac{1}{3}}C_9^{\frac{1}{3}}C_{10}}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{\infty}C_1^{\frac{1}{3}}\exp\left\{-\frac{C_2m}{3}\right\} \\ \leq\;& \frac{C^*}{\tau_1}\left(\frac{\tau_1-1}{\tau_1}\right)\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\}\quad\left(\text{where } C^*\geq 4Kp\left(2^{\frac{2}{3}}+1\right)C_1^{\frac{1}{3}}C_8^{\frac{1}{3}}C_9^{\frac{1}{3}}C_{10}\right) \\ \leq\;& \frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\ =\;& O\left(\frac{1}{\tau_1}\right) \end{aligned} \tag{70}$$

It then follows from expressions (67), (68), (69), and (70) that

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right|\geq\epsilon\right\} \\
\leq\; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s-E\left[\underline{F}_s\right]\right)\right]\right) \\
\leq\; & \frac{C}{\epsilon^2}\sum_{\ell=1}^{d}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\right]\right) \\
& +\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{2}{\tau_1^2}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-2}\sum_{m=1}^{(r-1)\tau+\tau_1+p-t-1}\left|E\left[\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)'\left(\underline{F}_{t+m}-E\left[\underline{F}_{t+m}\right]\right)\right]\right|\sum_{\ell=1}^{d}\mu_{Y,\ell}^2 \\
\leq\; & \frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{\overline{C}}{\tau_1}+\frac{C}{\epsilon^2}\frac{1}{q}\sum_{r=1}^{q}\frac{C^*}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\; & \frac{C\overline{C}}{\epsilon^2}\frac{1}{\tau_1}+\frac{CC^*}{\epsilon^2}\frac{1}{\tau_1}\sum_{m=1}^{\infty}\exp\left\{-\frac{C_2m}{3}\right\} \\
=\; & O\left(\frac{1}{\tau_1}\right)+O\left(\frac{1}{\tau_1}\right) \\
=\; & O\left(\frac{1}{\tau_1}\right)=o\left(1\right).
\end{aligned}
$$

Turning our attention to part (d), note that, by apply Loève's c_r inequality, we obtain

$$
\begin{aligned}
& \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
\leq\; & 3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \\
& +3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2 \\
& +3\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2
\end{aligned}
$$

It follows from the arguments given in the proofs of parts (a)-(c) above that, for any $\epsilon > 0$,

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right)^2 \geq \epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q\tau_1^2} \\
& \times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YY,\ell}'E\left[\left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\underline{F}_s\underline{Y}_s' - E\left[\underline{F}_s\underline{Y}_s'\right]\right)\right]\alpha_{YY,\ell}\right) \\
=\ & o\left(1\right),
\end{aligned}
$$

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right)^2 \geq \epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q\tau_1^2} \\
& \times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\alpha_{YF,\ell}'E\left[\left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)'\left(\underline{F}_s\underline{F}_s' - E\left[\underline{F}_s\underline{F}_s'\right]\right)\right]\alpha_{YF,\ell}\right) \\
=\ & o\left(1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& P\left\{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}\right)^2 \geq \epsilon\right\} \\
\leq\ & \frac{C}{\epsilon^2}\frac{1}{q\tau_1^2} \\
& \times\sum_{\ell=1}^{d}\left(\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\mu_{Y,\ell}^2E\left[\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)'\left(\underline{F}_s - E\left[\underline{F}_s\right]\right)\right]\right) \\
=\ & o\left(1\right),
\end{aligned}
$$

from which we deduce via the Slutsky's theorem that

$$
\begin{aligned}
& \max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t - E\left[\underline{F}_t\right]\right)\mu_{Y,\ell} + \left(\underline{F}_t\underline{Y}_t' - E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
& \qquad\qquad \left.\left. + \left(\underline{F}_t\underline{F}_t' - E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
=\ & o_p\left(1\right)
\end{aligned}
$$

as required.

For part (e), note that

$$
\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left(\frac{\pi_{i,\ell,T}}{q\tau_1^2}\right) \\
= \quad & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
= \quad & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]A_{YY}'e_{\ell,d}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]A_{YF}'e_{\ell,d}\right\}\right)^2 \\
\leq \quad & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{E\left[\left|\gamma_i'\underline{F}_t\right|\right]\left|\mu_{Y,\ell}\right|+E\left[\left|\gamma_i'\underline{F}_t\underline{Y}_t'A_{YY}'e_{\ell,d}\right|\right]+E\left[\left|\gamma_i'\underline{F}_t\underline{F}_t'A_{YF}'e_{\ell,d}\right|\right]\right\}\right)^2 \\
& \text{(by triangle and Jensen's inequalities)} \\
\leq \quad & \max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\sqrt{\|\gamma_i\|_2^2}\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq \ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right. \\
& +\sqrt{\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\gamma_i}\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'A_{YY}E\left[\underline{Y}_t\underline{Y}_t'\right]A_{YY}'e_{\ell,d}} \\
& \left.\left.\quad +\sqrt{\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\gamma_i}\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'A_{YF}E\left[\underline{F}_t\underline{F}_t'\right]A_{YF}'e_{\ell,d}}\right\}\right)^2 \\
\leq \quad & \left(\max_{i\in H^c}\|\gamma_i\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq \ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right. \\
& \left.\left.\quad +\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'A_{YY}A_{YY}'e_{\ell,d}}+E\left\|\underline{F}_t\right\|_2^2\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'A_{YF}A_{YF}'e_{\ell,d}}\right\}\right)^2 \\
\leq \quad & \left(\max_{i\in H^c}\|\gamma_i\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq \ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right. \\
& \left.\left.\quad +\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}C^{\dagger}\phi_{\max}\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'e_{\ell,d}}+E\left\|\underline{F}_t\right\|_2^2C^{\dagger}\phi_{\max}\sqrt{\max_{1\leq \ell\leq d}e_{\ell,d}'e_{\ell,d}}\right\}\right)^2 \\
& \text{(by Lemma C-7)} \\
= \quad & \left(\max_{i\in H^c}\|\gamma_i\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\sqrt{E\left\|\underline{F}_t\right\|_2^2}\max_{1\leq \ell\leq d}\left|\mu_{Y,\ell}\right|\right.\right. \\
& \left.\left.\qquad +\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{Y}_t\right\|_2^2}C^{\dagger}\phi_{\max}+E\left\|\underline{F}_t\right\|_2^2C^{\dagger}\phi_{\max}\right\}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\max_{i\in H^c}\|\gamma_i\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\sqrt{E\|\underline{F}_t\|_2^2}\max_{1\leq\ell\leq d}|\mu_{Y,\ell}|\right.\\
&\qquad\left.+\sqrt{E\|\underline{F}_t\|_2^2}\sqrt{E\|\underline{Y}_t\|_2^2}C^{\dagger}\phi_{\max}+E\|\underline{F}_t\|_2^2C^{\dagger}\phi_{\max}\right)^2\\
&\leq \left(\max_{i\in H^c}\|\gamma_i\|_2^2\right)\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left(\sqrt{E\|\underline{F}_t\|_2^2}\|\mu_Y\|_2^2\right.\\
&\qquad\left.+\sqrt{E\|\underline{F}_t\|_2^2}\sqrt{E\|\underline{Y}_t\|_2^2}C^{\dagger}\phi_{\max}+E\|\underline{F}_t\|_2^2C^{\dagger}\phi_{\max}\right)^2\\
&\leq \overline{C}<\infty
\end{aligned}
$$

for some positive constant $\overline{C}$ such that

$$
\overline{C}\geq\left(\sup_{i\in H^c}\|\gamma_i\|_2^2\right)E\|\underline{F}_t\|_2^2\left(\|\mu_Y\|_2^2+\sqrt{E\|\underline{Y}_t\|_2^2}C^{\dagger}\phi_{\max}+\sqrt{E\|\underline{F}_t\|_2^2}C^{\dagger}\phi_{\max}\right)^2
$$

where such a constant exists in light of Assumption 3-5, Lemma C-5, and the fact that $0<\phi_{\max}<1$ given Assumption 3-1.

To show part (f), note that

$$
\begin{aligned}
&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2 \\
\leq\;&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
&\qquad\qquad\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\} \\
&\qquad\left.+\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
\leq\;&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{2}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right. \\
&\qquad\left.\left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2 \\
&+\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{2}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
&\text{(by Loève's } c_r \text{ inequality)} \\
=\;&o_p(1)+O(1)\quad\text{(applying the results given in parts (d) and (e) of this lemma)} \\
=\;&O_p(1).
\end{aligned}
$$

To show part (g), we apply the Cauchy-Schwarz inequality as well as parts (d) and (e) of this lemma to obtain

$$\begin{aligned}
&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left\{\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\right.\\
&\qquad\qquad\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)\\
&\qquad\left.\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right\}\right|\\
\leq\;&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\\
&\qquad\qquad\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)\\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)\right|\\
\leq\;&\left[\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\gamma_i'\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.+\gamma_i'\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\right)^2\right]^{1/2}\\
&\times\left[\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2\right]^{1/2}\\
=\;&o_p(1)\,O(1)\\
=\;&o_p(1)\,.
\end{aligned}$$

For part (h), we apply the Cauchy-Schwarz inequality as well as part (d) of Lemma C-6 and part (f) of this lemma to obtain

$$
\begin{aligned}
& \max_{1\leq \ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
\leq\ & \max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
& \qquad\qquad \left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
\leq\ & \sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2} \\
& \times\sqrt{\max_{1\leq \ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2} \\
=\ & O_p(1)\,o_p(1) \\
=\ & o_p(1)
\end{aligned}
$$

Finally, for part (i), we apply the Cauchy-Schwarz inequality as well as part (b) of Lemma C-6 and part (f) of this lemma to obtain

$$\begin{aligned}
&\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right| \\
\leq\;&\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left|\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)\right. \\
&\qquad\left.\times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\right| \\
\leq\;&\sqrt{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right)^2} \\
&\times\sqrt{\max_{1\leq\ell\leq d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2} \\
=\;&O_p(1)\,o_p(1) \\
=\;&o_p(1).\;\square
\end{aligned}$$

Lemma C-13: Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-5, 3-7, and 3-9 hold and suppose that $N_1, N_2, T \to \infty$ such that $N_1/\tau_1^3 = N_1/\left\lfloor T_0^{\alpha_1}\right\rfloor^3 \to 0$. Then, the following statements are true.

(a)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\xrightarrow{p}0$$

(b)

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{\pi_{i,\ell,T}}\right|\xrightarrow{p}0$$

Proof of Lemma C-13:

To show part (a), note first that by applying parts (a) and (c) of Lemma C-6, parts (a)-(c) of

Lemma C-12, and the Slutsky theorem; we obtain

$$
\begin{aligned}
& \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{\overline{S}_{i,\ell,T} - \mu_{i,\ell,T}}{q\tau_1} \right| \\
= & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t \left[\mu_{Y,\ell} + \underline{Y}_t' \alpha_{YY,\ell} + \underline{F}_t' \alpha_{YF,\ell} \right] \right. \\
& + \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t \varepsilon_{\ell,t+1} + \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \\
& \left. - \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left\{ \gamma_i' E\left[\underline{F}_t\right] \mu_{Y,\ell} + \gamma_i' E\left[\underline{F}_t \underline{Y}_t'\right] \alpha_{YY,\ell} + \gamma_i' E\left[\underline{F}_t \underline{F}_t'\right] \alpha_{YF,\ell} \right\} \right| \\
\leq & \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t - E\left[\underline{F}_t\right] \right) \mu_{Y,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t \underline{Y}_t' - E\left[\underline{F}_t \underline{Y}_t'\right] \right) \alpha_{YY,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left(\underline{F}_t \underline{F}_t' - E\left[\underline{F}_t \underline{F}_t'\right] \right) \alpha_{YF,\ell} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \underline{F}_t \varepsilon_{\ell,t+1} \right| \\
& + \max_{1 \leq \ell \leq d} \max_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1} u_{it} \right| \\
= & \; o_p(1)
\end{aligned}
$$

Moreover, by Assumption 3-9, there exists a positive constant $\underline{c}$ such that for all N and T sufficiently large

$$
\begin{aligned}
& \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{\mu_{i,\ell,T}}{q\tau_1} \right| \\
= & \min_{1 \leq \ell \leq d} \min_{i \in H^c} \left| \frac{1}{q} \sum_{r=1}^{q} \frac{1}{\tau_1} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \gamma_i' \left\{ E\left[\underline{F}_t\right] \mu_{Y,\ell} + E\left[\underline{F}_t \underline{Y}_t'\right] \alpha_{YY,\ell} + E\left[\underline{F}_t \underline{F}_t'\right] \alpha_{YF,\ell} \right\} \right| \\
\geq & \; \underline{c} > 0
\end{aligned}
$$

It follows that

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{\mu_{i,\ell,T}}\right|\leq\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{S}_{i,\ell,T}-\mu_{i,\ell,T}}{q\tau_1}\right|\Big/\min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{\mu_{i,\ell,T}}{q\tau_1}\right|=o_p\left(1\right).$$

Now, for part (b), note that, applying parts (d), (g), (h), and (i) of Lemma C-12, parts (b), (d), and (e) of Lemma C-6, and the Slutsky theorem; we have

$$
\begin{aligned}
& \max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{q\tau_1^2}\right| \\
= \; & \max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\Bigg(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right. \\
& \qquad\qquad \left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\Bigg)^2 \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\Bigg|\frac{2}{q}\sum_{r=1}^{q}\Bigg\{\Bigg(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{\left(\underline{F}_t-E\left[\underline{F}_t\right]\right)\mu_{Y,\ell}+\left(\underline{F}_t\underline{Y}_t'-E\left[\underline{F}_t\underline{Y}_t'\right]\right)\alpha_{YY,\ell}\right. \\
& \qquad\qquad \left.+\left(\underline{F}_t\underline{F}_t'-E\left[\underline{F}_t\underline{F}_t'\right]\right)\alpha_{YF,\ell}\right\}\Bigg) \\
& \qquad \times\Bigg(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\Bigg)\Bigg\}\Bigg| \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)^2 \\
& +\max_{1\le\ell\le d}\max_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)^2 \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\right| \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\Bigg|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right) \\
& \qquad \times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right)\Bigg| \\
& +2\max_{1\le\ell\le d}\max_{i\in H^c}\Bigg|\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\left[\mu_{Y,\ell}+\underline{Y}_t'\alpha_{YY,\ell}+\underline{F}_t'\alpha_{YF,\ell}\right]\right) \\
& \qquad \times\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\underline{F}_t\varepsilon_{\ell,t+1}\right)\Bigg| \\
= \; & o_p(1)
\end{aligned}
$$

Moreover, note that, for all N and T sufficiently large,

$$\begin{aligned}
& \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{\pi_{i,\ell,T}}{q\tau_1^2} \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\left\{\gamma_i'E\left[\underline{F}_t\right]\mu_{Y,\ell}+\gamma_i'E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+\gamma_i'E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\frac{1}{q}\sum_{r=1}^{q}\left(\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
\geq \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\left(\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right)^2 \\
& \text{(by Jensen's inequality)} \\
= \quad & \min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right|^2 \\
= \quad & \left(\min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right|\right)^2 \\
\geq \quad & \underline{c}^2>0 \quad \text{(by Assumption 3-9)}.
\end{aligned}$$

It follows that

$$\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{\pi_{i,\ell,T}}\right|\leq\max_{1\leq\ell\leq d}\max_{i\in H^c}\left|\frac{\overline{V}_{i,\ell,T}-\pi_{i,\ell,T}}{q\tau_1^2}\right|/\min_{1\leq\ell\leq d}\min_{i\in H^c}\left(\frac{\pi_{i,\ell,T}}{q\tau_1^2}\right)=o_p(1).\ \square$$

Lemma C-14: Let $a,b\in\mathbb{R}$ such that $a\geq 0$ and $b\geq 0$. Then,

$$\left|\sqrt{a}-\sqrt{b}\right|\leq\sqrt{|a-b|}$$

Proof of Lemma C-14: Note that

$$\begin{aligned}\left(\sqrt{a}-\sqrt{b}\right)^2 &= a-2\sqrt{a}\sqrt{b}+b \\ &= \sqrt{a}\left(\sqrt{a}-\sqrt{b}\right)+\sqrt{b}\left(\sqrt{b}-\sqrt{a}\right) \\ &\leq \sqrt{a}\left|\sqrt{a}-\sqrt{b}\right|+\sqrt{b}\left|\sqrt{b}-\sqrt{a}\right| \\ &= \left(\sqrt{a}+\sqrt{b}\right)\left|\sqrt{a}-\sqrt{b}\right| \\ &= \left|\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)\right| \\ &= |a-b|\end{aligned}$$

Taking principal square root on both sides, we obtain

$$\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{|a-b|}. \ \Box$$

Lemma C-15:

$$P\left\{\bigcap_{i=1}^{m} A_i\right\} \geq \sum_{i=1}^{m} P\left(A_i\right)-(m-1)$$

Proof of Lemma C-15:

$$\begin{aligned}P\left\{\bigcap_{i=1}^{m} A_i\right\} &= 1-P\left\{\left(\bigcap_{i=1}^{m} A_i\right)^c\right\} \\ &= 1-P\left\{\bigcup_{i=1}^{m} A_i^c\right\} \quad \text{(by DeMorgan's Law)} \\ &\geq 1-\sum_{i=1}^{m} P\left(A_i^c\right) \\ &= 1-\sum_{i=1}^{m}\left[1-P\left(A_i\right)\right] \\ &= \sum_{i=1}^{m} P\left(A_i\right)-m+1 \\ &= \sum_{i=1}^{m} P\left(A_i\right)-(m-1). \ \Box\end{aligned}$$

Lemma C-16:

(a) For $t>0$,

$$\overline{\Phi}(t)=1-\Phi(t) \leq \frac{\phi(t)}{t},$$

where $\phi(t)$ and $\Phi(t)$ denote, respectively, the pdf and the cdf of a standard normal random variable.

(b) Let $N = N_1 + N_2$. Specify φ such that $\varphi \to 0$ as $N_1, N_2 \to \infty$ and such that, for some constant $a > 0$,

$$\varphi \geq \frac{1}{N^a}$$

for all N_1, N_2 sufficiently large. Then, for all N_1, N_2 sufficiently large such that

$$1 - \frac{\varphi}{2N} \geq \Phi(2)$$

we have

$$\Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \leq \sqrt{2(1+a)}\sqrt{\ln N}.$$

Proof of Lemma C-16:

(a)

$$\begin{aligned} 1 - \Phi(t) &= \int_t^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &= \int_t^\infty \frac{1}{z} \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \\ &\leq \frac{1}{t} \int_t^\infty \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz \end{aligned}$$

Let

$$u = -\frac{z^2}{2} \text{ and } du = -z dz$$

so that

$$\begin{aligned} \int_t^\infty \frac{z}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz &= -\int_{-\frac{t^2}{2}}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\ &= \int_{-\infty}^{-\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \exp\{u\} du \\ &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{t^2}{2}\right\} \\ &= \phi(t) \end{aligned}$$

It follows that

$$\overline{\Phi}(t) = 1 - \Phi(t) \leq \frac{\phi(t)}{t}.$$

(b) Let $t > 0$ and let

$$\Phi(t) = \Pr(Z \leq t) = 1 - \frac{\varphi}{2N}$$

Note that

$$\Phi^{-1}(\Phi(t)) = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) = t$$

and, by the result given in part (a) above,

$$1 - \Phi(t) = 1 - \left(1 - \frac{\varphi}{2N}\right) = \frac{\varphi}{2N} \leq \frac{\phi(t)}{t}$$

The latter inequality implies that

$$t \leq \phi(t)\,\frac{2N}{\varphi}$$

so that

$$\begin{aligned}
\ln t &\leq \ln\phi(t) + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\pi + \ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&= -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 - \frac{1}{2}\ln\pi + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \frac{1}{2}\ln 2 + \ln\left(\frac{N}{\varphi}\right) \\
&< -\frac{1}{2}t^2 + \ln 2 + \ln\left(\frac{N}{\varphi}\right)
\end{aligned}$$

or

$$\begin{aligned}
t^2 &\leq 2(\ln 2 - \ln t) + 2\ln\left(\frac{N}{\varphi}\right) \\
&= 2\ln\left(\frac{2}{t}\right) + 2\ln\left(\frac{N}{\varphi}\right) \\
&\leq 2\ln\left(\frac{N}{\varphi}\right) \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2
\end{aligned}$$

so that

$$t \leq \sqrt{2}\sqrt{\ln\left(\frac{N}{\varphi}\right)} \text{ for any } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2$$

Hence, for N_1, N_2 sufficiently large so that

$$1 - \frac{\varphi}{2N} \geq \Phi(2) \text{ or, equivalently, } t = \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \geq 2,$$

we have

$$\begin{aligned}
\Phi^{-1}\left(1-\frac{\varphi}{2N}\right) &= t \\
&\leq \sqrt{2}\sqrt{\ln\left(\frac{N}{\varphi}\right)} \\
&= \sqrt{2}\sqrt{\ln N - \ln\varphi} \\
&= \sqrt{2}\sqrt{\ln N}\sqrt{1-\frac{\ln\varphi}{\ln N}} \\
&\leq \sqrt{2}\sqrt{\ln N}\sqrt{1-\frac{\ln N^{-a}}{\ln N}} \\
&= \sqrt{2\left(1+a\right)}\sqrt{\ln N}. \ \square
\end{aligned}$$

Lemma C-17: Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c) 3-4, 3-5, 3-7, and 3-8 hold. Let $\Phi\left(\cdot\right)$ denote the cumulative distribution function of the standard normal random variable. Then, there exists a positive constant A such that

$$P\left(\left|S_{i,\ell,T}\right| \geq z\right) \leq 2\left[1-\Phi\left(z\right)\right]\left\{1+A\left(1+z\right)^{3}T^{-\left(1-\alpha_{1}\right)\frac{1}{2}}\right\} \tag{71}$$

for

$$i \in H = \left\{k \in \left\{1, ..., N\right\} : \gamma_{k} = 0\right\},$$

for $\ell \in \left\{1, ..., d\right\}$, for T sufficiently large, and for all z such that

$$0 \leq z \leq c_{0}\min\left\{T^{\left(1-\alpha_{1}\right)\frac{1}{6}}, T^{\frac{\alpha_{2}}{2}}\right\}$$

with c_0 being a positive constant.

Proof of Lemma C-17:

Note first that, for any i such that

$$i \in H = \left\{k \in \left\{1, ..., N\right\} : \gamma_{k} = 0\right\},$$

the formula for $S_{i,\ell,T}$ reduces to

$$S_{i,\ell,T} = \left(\sum_{r=1}^{q}\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_{1}+p-1} y_{\ell,t+1}u_{it}\right]^{2}\right)^{-\frac{1}{2}}\sum_{r=1}^{q}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_{1}+p-1} y_{\ell,t+1}u_{it}.$$

Hence, to verify the conditions of Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we set $X_{it} = u_{it}y_{\ell,t+1}$, and note that

$$\begin{aligned} E\left[X_{it}\right] &= E\left[u_{it}y_{\ell,t+1}\right] \\ &= E_Y\left[E\left[u_{it}\right]y_{\ell,t+1}\right] \quad \text{(by the law of iterated expectations} \\ &\quad \text{given the independence of } u_{it} \text{ and } y_{\ell,t+1} \text{ in light of Assumption 3-4)} \\ &= 0 \quad \text{(by Assumption 3-3(a))} \end{aligned}$$

so that the first part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is fulfilled. Moreover, in light of Assumptions 3-3(b) and Lemma C-5, we see that there exists some positive constant c_1 such that, for $\ell \in \{1..., d\}$,

$$\begin{aligned} E\left[\left|X_{it}\right|^{\frac{31}{10}}\right] &= E\left[\left|u_{it}y_{\ell,t+1}\right|^{\frac{31}{10}}\right] \\ &\leq \left(E\left|u_{it}\right|^{\frac{186}{29}}\right)^{\frac{29}{60}}\left(E\left|y_{\ell,t+1}\right|^{6}\right)^{\frac{31}{60}} \quad \text{(by Hölder's inequality)} \\ &\leq \left[\left(E\left|u_{it}\right|^{\frac{186}{29}}\right)^{\frac{29}{186}}\right]^{\frac{31}{10}}\left[E\left(\sum_{k=1}^{d}\sum_{j=0}^{p-1}y_{k,t+1-j}^{2}\right)^{3}\right]^{\frac{31}{60}} \\ &\leq \left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{31}{10}}\left[\left(E\left\|\underline{Y}_{t+1}\right\|_2^6\right)^{\frac{1}{6}}\right]^{\frac{31}{10}} \quad \text{(by Liapunov's inequality)} \\ &\leq c_1^{\frac{31}{10}} \end{aligned}$$

Hence, the second part of condition (4.1) of Chen, Shao, Wu, and Xu (2016) is also fulfilled with $r = \frac{31}{10} > 2$. Moreover, note that, by Assumption 3-8, for all $r \geq 1$, $\tau_1 \geq 1$, and $\tau_2 = \tau - \tau_1 \geq 1$

$$\begin{aligned} E\left\{\left[\frac{1}{\sqrt{\tau_1}}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}X_{it}\right]^2\right\} &= E\left\{\left[\frac{1}{\sqrt{\tau_1}}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right]^2\right\} \\ &= \frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\sum_{s=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}E\left\{y_{\ell,t+1}u_{it}u_{is}y_{\ell,s+1}\right\} \\ &\geq \underline{c} \end{aligned}$$

or

$$E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}X_{it}\right]^2\right\} = E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}y_{\ell,t+1}u_{it}\right]^2\right\} \geq \underline{c}\tau_1,$$

so that condition (4.2) of Chen, Shao, Wu, and Xu (2016) is also satisfied. Finally, by Lemma C-11, Assumption 3-3(c), and Assumption 3-4; $\left\{(y_{\ell,t+1}, u_{it})'\right\}$ is β mixing with β mixing coefficient satisfying

$$\beta(m) \leq a_1 \exp\{-a_2 m\}$$

for some constants $a_1 > 0$ and $a_2 > 0$. It follows by part (a) of Lemma C-2 that $\{X_{it}\}$ (with $X_{it} = u_{it}y_{\ell,t+1}$) satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for all $i \in H$. Hence, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016) for the case where $\delta = 1$[1], we obtain the Cramér-type moderate deviation result

$$\frac{P\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi(z)} = 1 + O(1)(1+z)^3 T^{-(1-\alpha_1)\frac{1}{2}} \tag{72}$$

for all $0 \leq z \leq c_0 \min\left\{T^{(1-\alpha_1)\frac{1}{6}}, T^{\frac{\alpha_2}{2}}\right\}$ and for $|O(1)| \leq A$ with A being an absolute constant.

In addition, note that

$$\begin{aligned}
\frac{P\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right\}}{\Phi(-z)} &= \frac{P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{\Phi(-z)} \\
&= \frac{P\left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}}{1-\Phi(z)}
\end{aligned}$$

Since

$$-\overline{S}_{i,\ell,T} = \sum_{r=1}^{q} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left(-u_{it}y_{\ell,t+1}\right),$$

for this case, we can take $X_{it} = -u_{it}y_{\ell,t+1}$, and note that, trivially, by calculations similar to those

[1] Note that Theorem 4.1 of Chen, Shao, Wu, and Xu (2016) requires that $0 < \delta \leq 1$ and $\delta < r-2$. These conditions are satisfied here given that we choose $\delta = 1$ and $r = 31/10$.

given above, we have

$$\begin{aligned}
E\left[X_{it}\right] &= E\left[-u_{it}y_{\ell,t+1}\right] = 0, \\
E\left[\left|X_{it}\right|^{\frac{31}{10}}\right] &= E\left[\left|-u_{it}y_{\ell,t+1}\right|^{\frac{31}{10}}\right] = E\left[\left|u_{it}y_{\ell,t+1}\right|^{\frac{31}{10}}\right] \leq c_1^{\frac{31}{10}}, \text{ and} \\
E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} X_{it}\right]^2\right\} &= E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} \left(-u_{it}y_{\ell,t+1}\right)\right]^2\right\} \\
&= E\left\{\left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} u_{it}y_{\ell,t+1}\right]^2\right\} \geq \underline{c}\tau_1.
\end{aligned}$$

Moreover, it is easily seen that $\{X_{it}\}$ (with $X_{it} = -u_{it}y_{\ell,t+1}$) also satisfies the β mixing condition (2.1) stipulated in Chen, Shao, Wu, and Xu (2016) for every i. Thus, by applying Theorem 4.1 of Chen, Shao, Wu, and Xu (2016), we also obtain the Cramér-type moderate deviation result

$$\frac{P\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right\}}{\Phi\left(-z\right)} = 1 + O\left(1\right)\left(1+z\right)^3 T^{-\left(1-\alpha_1\right)\frac{1}{2}} \tag{73}$$

for all $0 \leq z \leq c_0 \min\left\{T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}}\right\}$ and for $\left|O\left(1\right)\right| \leq A$ with A being an absolute constant.

Next, note that

$$
\begin{aligned}
& \left| \frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \right| \\
= & \left| \frac{P\left(\left|\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \right| \\
= & \left| \frac{P\left(\left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} \cup \left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\}\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \right| \\
= & \left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right) + P\left(-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - 1 \right| \\
& \left(\text{since } \left\{\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} \cap \left\{-\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right\} = \varnothing\right) \\
= & \left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{2\left[1-\Phi\left(z\right)\right]} - \frac{1}{2} + \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right)}{2\Phi\left(-z\right)} - \frac{1}{2} \right| \\
= & \left| \frac{1}{2}\left[\frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{\left[1-\Phi\left(z\right)\right]} - 1\right] + \frac{1}{2}\left[\frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right)}{\Phi\left(-z\right)} - 1\right] \right| \\
\leq & \frac{1}{2}\left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{\left[1-\Phi\left(z\right)\right]} - 1 \right| + \frac{1}{2}\left| \frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right)}{\Phi\left(-z\right)} - 1 \right|
\end{aligned}
$$

so that in light of expressions (72) and (73), we have

$$\begin{aligned}
\left|\frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]}-1\right| &= \left|\frac{P\left(\left|\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]}-1\right| \\
&\leq \frac{1}{2}\left|\frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \geq z\right)}{\left[1-\Phi\left(z\right)\right]}-1\right| + \frac{1}{2}\left|\frac{P\left(\overline{S}_{i,\ell,T}/\sqrt{\overline{V}_{i,\ell,T}} \leq -z\right)}{\Phi\left(-z\right)}-1\right| \\
&\leq \frac{A}{2}\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}} + \frac{A}{2}\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}} \\
&= A\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}}
\end{aligned}$$

It then follows that

$$-A\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}} \leq \frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]}-1 \leq A\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}} \tag{74}$$

Focusing on the right-hand part of the inequality in (74), we have

$$\frac{P\left(\left|S_{i,\ell,T}\right| \geq z\right)}{2\left[1-\Phi\left(z\right)\right]}-1 \leq A\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}}$$

Simple rearrangement of the inequality above then leads to the desired result

$$P\left(\left|S_{i,\ell,T}\right| \geq z\right) \leq 2\left[1-\Phi\left(z\right)\right]\left\{1+A\left(1+z\right)^{3}T^{-(1-\alpha_1)\frac{1}{2}}\right\},$$

which holds for

$$i \in H = \left\{k \in \{1,....,N\} : \gamma_k = 0\right\},$$

for $\ell \in \{1,...,d\}$, for all T sufficiently large, and for all z such that

$$0 \leq z \leq c_0 \min\left\{T^{\frac{1-\alpha_1}{6}}, T^{\frac{\alpha_2}{2}}\right\}. \ \square$$

4 Appendix D: Supporting Lemmas Used in the Proofs of Theorems 4 and 5

Derivation of the h-step Ahead Forecasting Equation Given in Expression (23) of the Main Paper:

Consider the FAVAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}, \tag{75}$$

where $W_t = (Y_t', F_t')'$. Suppose that equation (75) satisfies Assumptions 3-1 and 3-2 of the main paper. Then, similar to a VAR process, we can rewrite this model in the companion form

$$\underline{W}_t = \alpha + A\underline{W}_{t-1} + E_t$$

where

$$\begin{aligned}
\underline{W}_t &= \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \ W_t = \begin{pmatrix} Y_t \\ F_t \end{pmatrix}, \ E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \text{ and} \\
A &= \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.
\end{aligned} \tag{76}$$

Successive substitution for the lagged $\underline{W}_t$'s gives

$$\underline{W}_{t+h} = \sum_{j=0}^{h-1} A^j \alpha + A^h \underline{W}_t + \sum_{j=0}^{h-1} A^j E_{t+h-j}$$

Let

$$\underset{d\times(d+K)p}{J_d} = \begin{bmatrix} I_d & 0 & \cdots & 0 \end{bmatrix} \text{ and } \underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}$$

and note that

$$J_d \underline{W}_{t+h} = Y_{t+h}, \ \ J_{d+K} E_{t+h-j} = \varepsilon_{t+h-j},$$

and

$$J'_{d+K}J_{d+K}E_{t+h-j} = \begin{pmatrix} I_{d+K} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{aligned} Y_{t+h} &= J_d\underline{W}_{t+h} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} J_{d+K} E_{t+h-j} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \end{aligned} \tag{77}$$

Furthermore, let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}, \text{ where } \underline{Y}_t = \begin{pmatrix} Y_t \\ \vdots \\ Y_{t-p+1} \end{pmatrix} \text{ and } \underline{F}_t = \begin{pmatrix} F_t \\ \vdots \\ F_{t-p+1} \end{pmatrix}. \tag{78}$$

and note that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix, so that $\mathcal{P}'_{(d+K)p}\mathcal{P}_{(d+K)p} = I_{(d+K)p} = \mathcal{P}_{(d+K)p}\mathcal{P}'_{(d+K)p}$. Next, for $g = 1, ..., p$, let $e_{g,p}$ be a $p \times 1$ elementary vector whose g^{th} component is 1 and all other components are 0; and define

$$\underset{(d+K)p\times d}{S_{d,g}} = \begin{pmatrix} e_{g,p}\otimes I_d \\ \underset{Kp\times d}{0} \end{pmatrix}, \quad \underset{(d+K)p\times K}{S_{K,g}} = \begin{pmatrix} \underset{dp\times K}{0} \\ e_{g,p}\otimes I_K \end{pmatrix},$$

$$\begin{aligned}
\underset{(d+K)p\times dp}{S_d} &= \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} \\
&= \begin{pmatrix} e_{1,p}\otimes I_d & e_{2,p}\otimes I_d & \cdots & e_{p,p}\otimes I_d \\ \underset{Kp\times d}{0} & \underset{Kp\times d}{0} & \cdots & \underset{Kp\times d}{0} \end{pmatrix} \\
&= \begin{pmatrix} I_p\otimes I_d \\ \underset{Kp\times dp}{0} \end{pmatrix} = \begin{pmatrix} I_{dp} \\ \underset{Kp\times dp}{0} \end{pmatrix} \\
\underset{(d+K)p\times Kp}{S_K} &= \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} \\
&= \begin{pmatrix} \underset{dp\times K}{0} & \underset{dp\times K}{0} & \cdots & \underset{dp\times K}{0} \\ e_{1,p}\otimes I_K & e_{2,p}\otimes I_K & \cdots & e_{p,p}\otimes I_K \end{pmatrix} \\
&= \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_p\otimes I_K \end{pmatrix} = \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_{Kp} \end{pmatrix}
\end{aligned}$$

It follows that

$$\underset{(d+K)p\times(d+K)p}{S} = \begin{pmatrix} \underset{(d+K)p\times dp}{S_d} & \underset{(d+K)p\times Kp}{S_K} \end{pmatrix} = \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \\ \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix} = I_{(d+K)p} \tag{79}$$

In addition, using these notations, it is easy to see that

$$S'_{d,g}\mathcal{P}_{(d+K)p}\underline{W}_t = Y_{t-g+1} \text{ for } g = 1, ..., p \tag{80}$$

and, similarly,

$$S'_{K,g}\mathcal{P}_{(d+K)p}\underline{W}_t = F_{t-g+1} \text{ for } g = 1, ..., p. \tag{81}$$

Hence, making use of expressions (77) and (79) and the fact that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix, we can write

$$
\begin{aligned}
Y_{t+h} &= J_d \underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} S S' \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} \left(S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g} \right) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}
\end{aligned}
$$

so that, in light of expressions (80) and (81), we further deduce that

$$
\begin{aligned}
Y_{t+h} &= J_d \underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} \left(S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g} \right) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} S'_{d,g} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} S'_{K,g} \mathcal{P}_{(d+K)p} \underline{W}_t \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} Y_{t-g+1} + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} F_{t-g+1} \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \beta_0 + \sum_{g=1}^{p} B'_{1,g} Y_{t-g+1} + \sum_{g=1}^{p} B'_{2,g} F_{t-g+1} + \eta_{t+h}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= \sum_{j=0}^{h-1} J_d A^j \alpha, \ \eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}, \\
B'_{1,g} &= J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} \text{ and } B'_{2,g} = J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} \text{ for } g = 1, ..., p.
\end{aligned} \tag{82}
$$

Next, define $B'_1 = \begin{pmatrix} B'_{1,1} & B'_{1,2} & \cdots & B'_{1,p} \end{pmatrix}$ and $B'_2 = \begin{pmatrix} B'_{2,1} & B'_{2,2} & \cdots & B'_{2,p} \end{pmatrix}$, and note that, by expression (82) above,

$$\begin{aligned} B_1' &= J_d A^h \mathcal{P}_{(d+K)p}' \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} = J_d A^h \mathcal{P}_{(d+K)p}' S_d \\ B_2' &= J_d A^h \mathcal{P}_{(d+K)p}' \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} = J_d A^h \mathcal{P}_{(d+K)p}' S_K. \end{aligned}$$

Finally, let $\underline{Y}_t$ and $\underline{F}_t$ be as defined in expression (78), and we can write the h-step ahead forecast equation more succinctly as

$$\begin{aligned} Y_{t+h} &= \beta_0 + \sum_{g=1}^{p} B_{1,g}' Y_{t-g+1} + \sum_{g=1}^{p} B_{2,g}' F_{t-g+1} + \eta_{t+h} \\ &= \beta_0 + B_1' \underline{Y}_t + B_2' \underline{F}_t + \eta_{t+h}. \ \square \end{aligned}$$

Lemma D-1: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7 hold. Then, the following statements are true.

(a) There exists a positive constant $\underline{c}$ such that

$$\lambda_{\min} \left\{ \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' \right\} \geq \underline{c} > 0,$$

where A is the coefficient matrix of the companion form given in expression (76) and where

$$\underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}. \tag{83}$$

(b) The matrix

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t \underline{Y}_t'\right] & E\left[\underline{Y}_t \underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t \underline{Y}_t'\right] & E\left[\underline{F}_t \underline{F}_t'\right] \end{pmatrix}$$

is non-singular for all $T > h + p - 1$.

Proof of Lemma D-1:

For part (a), we prove by contradiction. To proceed, let

$$J_{d+K,r} = e_{r,p}' \otimes I_{d+K} \text{ for } r \in \{1, ..., p\}$$

where $e_{r,p}$ is a $p \times 1$ elementary vector whose r^{th} component is equal to 1 and all other components are equal to 0. Note that, under this definition, $J_{d+K,1} = J_{d+K}$, where J_{d+K} is as defined previously in expression (83). Suppose that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)'$$

is singular; then, there exists $b \in \mathbb{R}^{(d+K)p} \backslash \{0\}$ such that

$$\sum_{j=0}^{\infty} b' A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)' b = 0$$

This, in turn, implies that $J_{d+K,1} \left(A^j\right)' b = 0$ for all j. Now, partition

$$b = \begin{pmatrix} \underset{(d+K)\times 1}{b_1} \\ \underset{(d+K)\times 1}{b_2} \\ \vdots \\ \underset{(d+K)\times 1}{b_p} \end{pmatrix}$$

Note that, for $j = 0$, let $L_0 = I_{d+K}$, and it is easily seen that

$$\begin{aligned} 0 &= J_{d+K,1} \left(A^0\right)' b \\ &= J_{d+K,1} b \\ &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\ &= b_1 \ \ (= L_0 b_1) \end{aligned}$$

Now, for $j = 1$, define $\overline{A} = \left[\begin{array}{ccccc} A_1 & A_2 & \cdots & A_{p-1} & A_p \end{array} \right]$, and note that

$$
\begin{aligned}
0 &= J_{d+K,1} A' b \\
&= \left[\begin{array}{ccccc} I_{d+K} & 0 & \cdots & 0 & 0 \end{array} \right] \begin{pmatrix} A_1' & I_{d+K} & 0 & \cdots & 0 \\ A_2' & 0 & I_{d+K} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{p-1}' & \vdots & 0 & \ddots & I_{d+K} \\ A_p' & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\
&= J_{d+K,1} \left[\begin{array}{ccccc} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' \end{array} \right] b \\
&= \left[J_{d+K,1} \overline{A}' J_{d+K,1} + J_{d+K,2} \right] b \\
&= \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] b \\
&= L_1 b_1 + L_0 b_2
\end{aligned}
$$

where $L_1 = J_{d+K,1} \overline{A}' = A_1'$. Since previously we have shown that $b_1 = 0$, it follows that

$$b_2 = L_1 b_1 + L_0 b_2 = 0.$$

Moreover, for $j = 2$, using the fact that $J_{d+K,r} J_{d+K,r}' = I_{d+K}$ and $J_{d+K,r} J_{d+K,s}' = 0$ for $r \neq s$, we obtain

$$
\begin{aligned}
0 &= J_{d+K,1} \left(A' \right)^2 b \\
&= J_{d+K,1} \left[\begin{array}{cccccc} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' & J_{d+K,p}' \end{array} \right]^2 b \\
&= \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \left[\begin{array}{cccccc} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' & J_{d+K,p}' \end{array} \right] b \\
&= \left(\left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \overline{A}' J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3} \right) b \\
&= \left(L_2 J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3} \right) b \\
&= L_2 b_1 + L_1 b_2 + L_0 b_3
\end{aligned}
$$

where

$$L_2 = \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \overline{A}'$$

Given that $b_1 = 0$ and $b_2 = 0$, as we have previously shown, it then follows that

$$b_3 = L_2 b_1 + L_1 b_2 + L_0 b_3 = 0 \ \text{(since } L_0 = I_{d+K}\text{)}$$

We will show by mathematical induction that, in fact, $b_r = 0$ for every $r \in \{1, ..., p\}$. To proceed, suppose that $b_1 = b_2 = \cdots = b_j = 0$ and $0 = J_{d+K,1} (A')^j b$. By straightforward calculations, one can show (in a manner similar to the case where $j = 0$, 1, or 2 given earlier) that $J_{d+K,1} (A')^j b$ has the representation

$$J_{d+K,1} \left(A'\right)^j b = L_j b_1 + L_{j-1} b_2 + \cdots L_1 b_j + L_0 b_{j+1}$$

for coefficients $L_j, L_{j-1}, ..., L_1$, and L_0 where $L_0 = I_{d+K}$. It follows from the induction hypotheses that

$$\begin{aligned} b_{j+1} &= L_j b_1 + L_{j-1} b_2 + \cdots L_1 b_j + L_0 b_{j+1} \\ &= J_{d+K,1} \left(A'\right)^j b \\ &= 0. \end{aligned}$$

Hence, by mathematical induction, we conclude that $b_r = 0$ for every $r \in \{1, ..., p\}$, but this implies that

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} = \underset{(d+K)p \times 1}{0}$$

which contradicts our initial assumption that $b \neq 0$. It then follows that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)'$$

is positive definite and, thus, also non-singular, so that there exists a positive constant C_* such that

$$\lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)' \right\} \geq C_* > 0$$

Moreover, in light of Assumption 3-2(d), this further implies that

$$\begin{aligned}
& \lambda_{\min}\left\{\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
= \; & \lambda_{\min}\left\{\sum_{j=0}^{\infty}A^j J_{d+K,1}'\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K,1}\left(A^j\right)'\right\} \;\; (\text{since } J_{d+K,1} = J_{d+K}) \\
\geq \; & \lambda_{\min}\left\{\sum_{j=0}^{\infty}A^j J_{d+K,1}' J_{d+K,1}\left(A^j\right)'\right\}\lambda_{\min}\left\{\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq \; & \lambda_{\min}\left\{\sum_{j=0}^{\infty}A^j J_{d+K,1}' J_{d+K,1}\left(A^j\right)'\right\}\inf_t \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq \; & C_*\underline{C} \\
\geq \; & \underline{c} > 0 \quad \left(\text{by choosing } \underline{c} \leq C_*\underline{C}\right).
\end{aligned}$$

where the second inequality above follows from the fact that

$$\begin{aligned}
\lambda_{\min}\left\{\sum_{t=p}^{T-h}\frac{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]}{T_h}\right\} & \geq \sum_{t=p}^{T-h}\lambda_{\min}\left\{\frac{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]}{T_h}\right\} \\
& = \frac{1}{T_h}\sum_{t=p}^{T-h}\lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
& \geq \inf_t \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\}.
\end{aligned}$$

Now, to show part (b), note first that extpression (47) in the proof of Lemma C-5 gives a vector moving-average representation for $\underline{W}_t$ of the form

$$\underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty}A^j J_{d+K}'\varepsilon_{t-j},$$

where $J_{d+K} = J_{d+K,1} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}$. Now, let

$$\underset{(d+K)p\times dp}{S_d} = \begin{pmatrix} I_{dp} \\ \underset{Kp\times dp}{0} \end{pmatrix} \text{ and } \underset{(d+K)p\times Kp}{S_K} = \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_{Kp} \end{pmatrix},$$

and let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\underline{Y}_t &= S_d' \mathcal{P}_{(d+K)p} \underline{W}_t \\
&= S_d' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu + \sum_{j=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' \varepsilon_{t-j}
\end{aligned}$$

and

$$\begin{aligned}
\underline{F}_t &= S_K' \mathcal{P}_{(d+K)p} \underline{W}_t \\
&= S_K' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu + \sum_{j=0}^{\infty} S_K' \mathcal{P}_{(d+K)p} A^j J_{d+K}' \varepsilon_{t-j}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& E\left[\underline{Y}_t \underline{Y}_t' \right] \\
= \; & E\Bigg\{ \left(S_d' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu + \sum_{j=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' \varepsilon_{t-j} \right) \\
& \times \left(\mu' J_{d+K} \left(I_{(d+K)p} - A' \right)^{-1} \mathcal{P}_{(d+K)p}' S_d + \sum_{k=0}^{\infty} \varepsilon_{t-k}' J_{d+K} \left(A^j \right)' \mathcal{P}_{(d+K)p}' S_d \right) \Bigg\} \\
= \; & S_d' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu \mu' J_{d+K} \left(I_{(d+K)p} - A' \right)^{-1} \mathcal{P}_{(d+K)p}' S_d \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-k}' \right] J_{d+K} \left(A^j \right)' \mathcal{P}_{(d+K)p}' S_d \\
= \; & S_d' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu \mu' J_{d+K} \left(I_{(d+K)p} - A' \right)^{-1} \mathcal{P}_{(d+K)p}' S_d \\
& + \sum_{j=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}' \right] J_{d+K} \left(A^j \right)' \mathcal{P}_{(d+K)p}' S_d,
\end{aligned}$$

$$
\begin{aligned}
& E\left[\underline{F}_t \underline{F}_t'\right] \\
= \; & E\left\{\left(S_K' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu+\sum_{j=0}^{\infty} S_K' \mathcal{P}_{(d+K)p} A^j J_{d+K}' \varepsilon_{t-j}\right)\right. \\
& \left.\times\left(\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K+\sum_{k=0}^{\infty} \varepsilon_{t-k}' J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K\right)\right\} \\
= \; & S_K' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K \\
& +\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S_K' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-k}'\right] J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K \\
= \; & S_K' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K \\
& +\sum_{j=0}^{\infty} S_K' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K,
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\underline{Y}_t \underline{F}_t'\right] \\
= \; & E\left\{\left(S_d' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu+\sum_{j=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' \varepsilon_{t-j}\right)\right. \\
& \left.\times\left(\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K+\sum_{k=0}^{\infty} \varepsilon_{t-k}' J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K\right)\right\} \\
= \; & S_d' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K \\
& +\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-k}'\right] J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K \\
= \; & S_d' \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \mathcal{P}_{(d+K)p}' S_K \\
& +\sum_{j=0}^{\infty} S_d' \mathcal{P}_{(d+K)p} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \mathcal{P}_{(d+K)p}' S_K,
\end{aligned}
$$

In addition, since

$$
\begin{aligned}
E\left[\underline{W}_t\right] &= \left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \text{ and} \\
E\left[\underline{W}_t \underline{W}_t'\right] &= \left(I_{(d+K)p}-A\right)^{-1} J_{d+K}' \mu \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \\
&\quad +\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'
\end{aligned}
$$

and since

$$\left[\begin{array}{cc} S_d & S_K \end{array}\right] = \left(\begin{array}{cc} I_{dp} & \underset{dp\times Kp}{0} \\ \underset{Kp\times dp}{0} & I_{Kp} \end{array}\right) = I_{(d+K)p}$$

it is easy to see that

$$\begin{aligned}
& \left(\begin{array}{cc} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{array}\right) \\
= & \left(\begin{array}{c} S_d' \\ S_K' \end{array}\right)\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
& +\left(\begin{array}{c} S_d' \\ S_K' \end{array}\right)\sum_{j=0}^{\infty}\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
= & \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}' \\
& +\sum_{j=0}^{\infty}\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}' \\
= & \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'
\end{aligned}$$

and

$$\begin{aligned}
& \left(\begin{array}{cc} E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \end{array}\right) \\
= & \left(\begin{array}{cc} \mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_d & \mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \end{array}\right) \\
= & \mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
= & \mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}' \\
= & E\left[\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'
\end{aligned}$$

Making use of these expressions, we can then write

$$\begin{aligned}
\left(\begin{array}{ccc} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{array}\right) & = \left(\begin{array}{cc} 1 & E\left[\underline{W}_t'\right]\mathcal{P}_{(d+K)p}' \\ \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\right] & \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}' \end{array}\right) \\
& = \left(\begin{array}{cc} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{array}\right)\left(\begin{array}{cc} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p}' \end{array}\right).
\end{aligned}$$

Next, note that

$$
\begin{aligned}
& \det\begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{pmatrix} \\
= \quad & \det(1)\det\left\{E\left[\underline{W}_t\underline{W}_t'\right] - E\left[\underline{W}_t\right]E\left[\underline{W}_t'\right]\right\} \\
= \quad & \det\left\{E\left[\underline{W}_t\underline{W}_t'\right] - E\left[\underline{W}_t\right]E\left[\underline{W}_t'\right]\right\} \\
= \quad & \det\Big\{\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1} \\
& + \sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \\
& - \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1}\Big\} \\
= \quad & \det\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\}
\end{aligned}
$$

Now, by Assumption 3-2(d) and by the same argument as that used to prove part (a) above, we see that there exists a constant $\underline{c}$ such that

$$
\begin{aligned}
& \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
\geq \quad & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' J_{d+K}\left(A^j\right)'\right\} \inf_j \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq \quad & \underline{c} > 0
\end{aligned}
$$

for all t, which, in turn, implies that in this case

$$
\begin{aligned}
\det\begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{pmatrix} \quad = \quad & \det\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
\geq \quad & \underline{c}^{(d+K)p} > 0
\end{aligned}
$$

for all t. Furthermore, since the matrix

$$
\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix}
$$

is nonsingular, it follows that the matrix

$$
\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h}\begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix}\frac{1}{T_h}\sum_{t=p}^{T-h}\begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p}' \end{pmatrix}
\end{aligned}
$$

will be nonsingular and, thus, positive definite as required. □

Lemma D-2: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-5, and 3-7 hold. Then, the following statements are true.

(a)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ \vdots \\ W_{t-p+1} \end{pmatrix} \text{ and } W_t = \begin{bmatrix} Y_t \\ F_t \end{bmatrix}.$$

(b)

$$
\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right) \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right), \text{ and} \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{F}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}
$$

where $\underline{Y}_t$ and $\underline{F}_t$ are as defined in expression (78).

(c)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(d)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t = S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right),$$

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t = S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(e)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right), \text{ where } \eta_{t+h} = \sum_{j=0}^{h-1}J_dA^jJ_{d+K}'\varepsilon_{t+h-j}$$

with $\underset{d\times(d+K)p}{J_d} = \begin{bmatrix} I_d & 0 & \cdots & 0 \end{bmatrix}$ and $\underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}$.

(f)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right),$$

where η_{t+h} is as defined in part (e) above.

(g)

$$\frac{\mathfrak{H}'\iota_{T_h}}{T_h} = \frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1).$$

(h)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right] = O_p\left(\frac{1}{\sqrt{T}}\right),$$

where η_{t+h} is as defined in part (e) above.

Proof of Lemma D-2:

To show part (a), we note that for $a, b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we can write

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\right]^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] \\
& + \frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\}
\end{aligned}
$$

Note first that

$$
\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] &= \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\underline{W}_t\underline{W}_t'b\right)^2 - \frac{1}{T_h^2}\sum_{t=p}^{T-h}\left(E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2 \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'a\right)\left(b'\underline{W}_t\underline{W}_t'b\right)\right] \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left(a'\underline{W}_t\underline{W}_t'a\right)^2}\sqrt{E\left(b'\underline{W}_t\underline{W}_t'b\right)^2} \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left\|\underline{W}_t\right\|_2^4 \\
&\leq \frac{C}{T_h} = O\left(\frac{1}{T}\right)
\end{aligned}
$$

where the fourth inequality above follows from applying Liapunov's inequality and the result given in Lemma C-5.

Next, note that by Lemma C-11, $\{W_t\}$ is β-mixing with β mixing coefficient satisfying $\beta_W(m) \leq C_1 \exp\{-C_2 m\}$. Since $\alpha_{W,m} \leq \beta_W(m)$, it follows that W_t is α-mixing as well, with α mixing coefficient satisfying $\alpha_{W,m} \leq C_1 \exp\{-C_2 m\}$. Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = a'\underline{W}_t\underline{W}_t'b$ is also α-mixing with α mixing coefficient satisfying

$$
\begin{aligned}
\alpha_{X,m} &\leq C_1 \exp\{-C_2(m-p+1)\} \\
&\leq C_1^* \exp\{-C_2 m\}
\end{aligned}
$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2(p-1)\}$. Hence, we can apply Lemma C-3 with $p = 2$

and $r = 3$ to obtain

$$\begin{aligned} & \left| E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b \right] \right) \left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b \right] \right) \right\} \right| \\ \leq \; & 2\left(2^{\frac{1}{2}} + 1 \right) \alpha_{X,m}^{\frac{1}{6}} \sqrt{E\left(a'\underline{W}_t\underline{W}_t'b \right)^2} \left(E\left| a'\underline{W}_{t+m}\underline{W}_{t+m}'b \right|^3 \right)^{1/3} \end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $X_t = a'\underline{W}_t\underline{W}_t'b$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{6}} \leq (C_1^*)^{\frac{1}{6}} \exp\left\{ -\frac{C_2 m}{6} \right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{6}} & \leq (C_1^*)^{\frac{1}{6}} \sum_{m=1}^{\infty} \exp\left\{ -\frac{C_2 m}{6} \right\} \\ & \leq (C_1^*)^{\frac{1}{6}} \sum_{m=0}^{\infty} \exp\left\{ -\frac{C_2 m}{6} \right\} \\ & \leq (C_1^*)^{\frac{1}{6}} \left[1 - \exp\left\{ -\frac{C_2}{6} \right\} \right]^{-1} \\ & \leq C_3 \end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\{-(C_2 m/6)\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/6)\} < 1$ for $C_2 > 0$. Hence,

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right) \left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
\leq\ & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right) \left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
\leq\ & \frac{4}{T_h^2} \left(2^{\frac{1}{2}} + 1\right) \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \alpha_{X,m}^{\frac{1}{6}} \sqrt{E\left(a'\underline{W}_t\underline{W}_t'b\right)^2} \left(E\left|a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right|^3\right)^{1/3} \\
\leq\ & 4\left(\sqrt{2}+1\right) \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{ \alpha_{X,m}^{\frac{1}{6}} \left[E\left(a'\underline{W}_t\right)^4\right]^{1/4} \left[E\left(b'\underline{W}_t\right)^4\right]^{1/4} \left[E\left(a'\underline{W}_{t+m}\right)^6\right]^{\frac{1}{6}} \right. \\
& \qquad \left. \times \left[E\left(b'\underline{W}_{t+m}\right)^6\right]^{\frac{1}{6}} \right\} \\
\leq\ & 4\left(\sqrt{2}+1\right) \left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}} \left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{6}} \\
\leq\ & 4\left(\sqrt{2}+1\right) \left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}} \left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} C_3 \\
\leq\ & \frac{\overline{C}}{T_h} = O\left(\frac{1}{T}\right) \quad \left(\text{where } \overline{C} \geq 4\left(\sqrt{2}+1\right) \left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}} \left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} C_3\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& E\left[\frac{1}{T_h} \sum_{t=p}^{T-h} \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\right]^2 \\
\leq\ & \frac{1}{T_h^2} \sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] \\
& + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right) \left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
=\ & O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h} \sum_{t=p}^{T-h} a'\underline{W}_t\underline{W}_t'b - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[a'\underline{W}_t\underline{W}_t'b\right] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t \underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{W}_t \underline{W}_t'\right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

To show part (b), note first that

$$\begin{aligned}
S_d' \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp \times Kp}{0} \end{pmatrix} \begin{pmatrix} \underset{dp \times 1}{\underline{Y}_t} \\ \underset{Kp \times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S_K' \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} \underset{Kp \times dp}{0} & I_{Kp} \end{pmatrix} \begin{pmatrix} \underset{dp \times 1}{\underline{Y}_t} \\ \underset{Kp \times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

By the result given in part (a) above, it follows from applying Slutsky's theorem that

$$\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{Y}_t \underline{Y}_t'\right] \\
=\;& S_d' \mathcal{P}_{(d+K)p} \left(\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t \underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{W}_t \underline{W}_t'\right]\right) \mathcal{P}_{(d+K)p} S_d \\
=\;& O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{F}_t'\right] \\
=\;& S_K' \mathcal{P}_{(d+K)p} \left(\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t \underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{W}_t \underline{W}_t'\right]\right) \mathcal{P}_{(d+K)p} S_K \\
=\;& O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

and

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{F}_t'\right] \\
= \;& S_d'\mathcal{P}_{(d+K)p}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right]\right)\mathcal{P}_{(d+K)p}S_K \\
= \;& O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

To show part (c), let $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, and write

$$
\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}a'\underline{W}_t &= \frac{1}{T_h}\sum_{t=p}^{T-h}\left\{a'\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu + \sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}\right\} \\
&= a'\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu + \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}\right]^2 &= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{s-k}'\right]J_{d+K}\left(A^k\right)'a \\
&= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'a \\
&\quad + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^{m+j}\right)'a \\
&= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'a \\
&\quad + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'a
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' a \\
\leq \; & \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} E \left\|\varepsilon_{t-j}\right\|_2^2 a' A^j J_{d+K}' J_{d+K} \left(A^j\right)' a \\
\leq \; & \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \left(E \left\|\varepsilon_{t-j}\right\|_2^6\right)^{\frac{1}{3}} a' A^j \left(A^j\right)' a \\
& \left(\text{by Liapunov's inequality and } \lambda_{\max}\left(J_{d+K}' J_{d+K}\right) = 1\right) \\
\leq \; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} a' A^j \left(A^j\right)' a \\
& \left(\text{where } \overline{C} \geq 1 \text{ is a constant such that } E \left\|\varepsilon_{t-j}\right\|_2^6 \leq \overline{C} < \infty \text{ by Assumption 3-2(b)}\right) \\
\leq \; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \lambda_{\max}\left\{A^j \left(A^j\right)'\right\} a'a \\
= \; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \\
& \left(\text{since } \lambda_{\max}\left\{A^j \left(A^j\right)'\right\} = \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \text{ and } a'a = 1\right) \\
= \; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \sigma_{\max}^2\left(A^j\right) \\
\leq \; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} C \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \quad \text{(by Assumption 3-7)} \\
= \; & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \\
= \; & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} \phi_{\max}^{2j}
\end{aligned}
$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ'_{d+K}E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right]J_{d+K}\left(A^j\right)'a &\leq \overline{C}^{\frac{1}{3}}C\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}\phi_{\max}^{2j} \\
&= \overline{C}^{\frac{1}{3}}C\frac{T-h-p+1}{T_h^2}\frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}}C\frac{1}{T_h}\frac{1}{1-\phi_{\max}^2} \\
&\quad \text{(since } T_h = T-h-p+1\text{)} \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty}a'A^jJ'_{d+K}E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'a\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\left|\sum_{j=0}^{\infty}a'A^jJ'_{d+K}E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'a\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\left\{\sqrt{\sum_{j=0}^{\infty}a'A^jJ'_{d+K}E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right]E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right]J_{d+K}\left(A^j\right)'a}\right. \\
&\left.\times\sqrt{\sum_{j=0}^{\infty}\sum_{m_1=1}^{T-h-t}a'A^{m_1}A^jJ'_{d+K}J_{d+K}\left(A^j\right)'\sum_{m_2=1}^{T-h-t}\left(A^{m_2}\right)'a}\right\}
\end{aligned}$$

Observe that

$$
\begin{aligned}
& \sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' a \\
\leq & \sum_{j=0}^{\infty} \lambda_{\max}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) a' A^j J_{d+K}' J_{d+K} \left(A^j\right)' a \\
\leq & \sum_{j=0}^{\infty} \lambda_{\max}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) C\phi_{\max}^{2j} \\
= & \; C\sum_{j=0}^{\infty} \lambda_{\max}^{2}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) \phi_{\max}^{2j} \\
\leq & \; C\sum_{j=0}^{\infty} \left(tr\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\}\right)^2 \phi_{\max}^{2j} \\
= & \; C\sum_{j=0}^{\infty} \left(E\left\|\varepsilon_{t-j}\right\|_2^2\right)^2 \phi_{\max}^{2j} \\
\leq & \; C\sum_{j=0}^{\infty} \left(E\left\|\varepsilon_{t-j}\right\|_2^6\right)^{\frac{2}{3}} \phi_{\max}^{2j} \quad \text{(by Liapunov's inequality)} \\
\leq & \; \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J_{d+K}' J_{d+K} \left(A^j\right)' \sum_{m_2=1}^{T-h-t} \left(A^{m_2}\right)' a \\
\leq\; & \sum_{j=0}^{\infty}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} a' A^{m_1} A^j \left(A^j\right)' \left(A^{m_2}\right)' a \\
\leq\; & C\sum_{j=0}^{\infty}\phi_{\max}^{2j}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} \left|a' A^{m_1} \left(A^{m_2}\right)' a\right| \\
\leq\; & C\sum_{j=0}^{\infty}\phi_{\max}^{2j}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} \sqrt{a' A^{m_1} \left(A^{m_1}\right)' a}\sqrt{a' A^{m_2} \left(A^{m_2}\right)' a} \\
\leq\; & C\sum_{j=0}^{\infty}\phi_{\max}^{2j}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} \sqrt{C\phi_{\max}^{2m_1}}\sqrt{C\phi_{\max}^{2m_2}} \\
\leq\; & C^2\sum_{j=0}^{\infty}\phi_{\max}^{2j}\sum_{m_1=1}^{T-h-t}\phi_{\max}^{m_1}\sum_{m_2=1}^{T-h-t}\phi_{\max}^{m_2} \\
\leq\; & C^2\frac{1}{1-\phi_{\max}^2}\left(\frac{1}{1-\phi_{\max}}\right)^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)' a\right| \\
\leq\; & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\left\{\sqrt{\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' a}\right. \\
& \qquad\left.\times\sqrt{\sum_{j=0}^{\infty}\sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J_{d+K}' J_{d+K}\left(A^j\right)'\sum_{m_2=1}^{T-h-t}\left(A^{m_2}\right)' a}\right\} \\
\leq\; & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sqrt{\overline{C}^{\frac{2}{3}} C\frac{1}{1-\phi_{\max}^2}}\sqrt{C^2\frac{1}{1-\phi_{\max}^2}\left(\frac{1}{1-\phi_{\max}}\right)^2} \\
\leq\; & 2\overline{C}^{\frac{1}{3}} C^{\frac{3}{2}}\frac{1}{T_h}\left(\frac{1}{1-\phi_{\max}^2}\right)\left(\frac{1}{1-\phi_{\max}}\right) \\
=\; & O\left(\frac{1}{T}\right)
\end{aligned}
$$

Putting these results together, we obtain

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j}\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' a \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' a \\
= & O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, upon applying Markov's inequality, we get

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

from which we further deduce, upon applying Slutsky's theorem, that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t &= a'\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j} \\
&= a'\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Since the above result holds for all $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

To show part (d), note again that

$$\begin{aligned}
S_d'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S_K'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

By the result given in part (c) above, it follows by Slutsky's theorem that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t &= S_d'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t \\
&= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + S_d'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}A^jJ_{d+K}'\varepsilon_{t-j} \\
&= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t &= S_K'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t \\
&= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + S_K'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}A^jJ_{d+K}'\varepsilon_{t-j} \\
&= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Turning our attention to part (e), let $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; and, by direct calculation, we obtain

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}a'\underline{W}_t\eta_{t+h}'b\right]^2 \\
=\ & \frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[\left(a'\underline{W}_t\right)^2\left(\eta_{t+h}'b\right)^2\right] + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left\{\left(a'\underline{W}_t\eta_{t+h}'b\right)\left(a'\underline{W}_{t+m}\eta_{t+m+h}'b\right)\right\}
\end{aligned}$$

Let $\sigma_{\max}\left(A^j\right)$ denotes the max singular value of A^j and let $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$, and note first that

$$
\begin{aligned}
E\left(b'\eta_{t+h}\right)^4 &= E\left(\sum_{j=0}^{h-1} b' J_d A^j J_{d+K}' \varepsilon_{t+h-j}\right)^4 \\
&\leq h^3 \sum_{j=0}^{h-1} E\left[\left(b' J_d A^j J_{d+K}' \varepsilon_{t+h-j}\right)^4\right] \quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq h^3 \sum_{j=0}^{h-1} E\left[\left(b' J_d A^j J_{d+K}' J_{d+K} \left(A'\right)^j J_d' b\right)^2 \left(\varepsilon_{t+h-j}' \varepsilon_{t+h-j}\right)^2\right] \\
&= h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j J_{d+K}' J_{d+K} \left(A^j\right)' J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j \left(A^j\right)' J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4\left(A^j\right) \left(b' J_d J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&= h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4\left(A^j\right) E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\leq h^3 \sum_{j=0}^{h-1} \overline{C}\left[\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}\right]^4 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \quad \text{(by Assumption 3-7)} \\
&= h^3 \sum_{j=0}^{h-1} \overline{C} \phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\leq C^{\frac{2}{3}} \overline{C} h^3 \sum_{j=0}^{h-1} \phi_{\max}^{4j} \\
&\leq C^* \qquad (84)
\end{aligned}
$$

where the next to last inequality follows from the fact that $E\left\|\varepsilon_{t+h-j}\right\|_2^4 \leq \left(\sup_t E\left\|\varepsilon_t\right\|^6\right)^{\frac{2}{3}} \leq C^{\frac{2}{3}}$ by Liapunov's inequality and by application of Assumption 3-2(b) and where the last inequality follows from the fact that h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. Applying the Cauchy-Schwarz inequality and the existence of moment result given in Lemma C-5, it then follows that

$$
\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\right)^2\left(\eta'_{t+h}b\right)^2\right] &\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left(a'\underline{W}_t\underline{W}'_t a\right)^2}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left\|\underline{W}_t\right\|_2^4}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
&\leq \frac{C}{T_h} = \frac{C}{T-h-p+1} = O\left(\frac{1}{T}\right)
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
& E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\} \\
= \; & E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\} \\
= \; & E\left\{a'\underline{W}_t\underline{W}'_{t+m}a\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+h-j}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+m+h-k}\right\} \\
= \; & E\left\{a'\underline{W}_t\underline{W}'_{t+m}a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+h-j}\varepsilon'_{t+m+h-k}J_{d+K}\left(A^j\right)'J'_db\right\},
\end{aligned}
$$

so that, for $m \geq h$, we have

$$
\begin{aligned}
& E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\} \\
= \; & E\left\{a'\underline{W}_t\underline{W}'_{t+m}a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+h-j}\varepsilon'_{t+m+h-k}J_{d+K}\left(A^j\right)'J'_db\right\} \\
= \; & E\left\{a'\underline{W}_t\underline{W}'_{t+m}a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+h-j}E\left[\varepsilon'_{t+m+h-k}|\mathcal{F}_{-\infty}^{t+m}\right]J_{d+K}\left(A^j\right)'J'_db\right\} \\
= \; & E\left\{a'\underline{W}_t\underline{W}'_{t+m}a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}\varepsilon_{t+h-j}E\left[\varepsilon'_{t+m+h-k}\right]J_{d+K}\left(A^j\right)'J'_db\right\} \\
= \; & 0
\end{aligned}
$$

Hence, defining $\sum_{m=1}^{0} E\left|\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right| = 0$, we have

$$
\begin{aligned}
& \left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\}\right| \\
= & \left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\}\right| \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} E\left|\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right| \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(a'\underline{W}_t\underline{W}'_{t+m}a\right)^2}\sqrt{E\left(b'\eta_{t+h}\eta'_{t+m+h}b\right)^2} \\
= & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(a'\underline{W}_t\underline{W}'_t a a'\underline{W}_{t+m}\underline{W}'_{t+m}a\right)}\sqrt{E\left\{\left(b'\eta_{t+h}\right)^2\left(b'\eta_{t+m+h}\right)^2\right\}} \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(\left\|\underline{W}_t\right\|_2^2\left\|\underline{W}_{t+m}\right\|_2^2\right)}\sqrt{E\left\{\left(b'\eta_{t+h}\right)^2\left(b'\eta_{t+m+h}\right)^2\right\}} \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \left(E\left\|\underline{W}_t\right\|_2^4\right)^{\frac{1}{4}}\left(E\left\|\underline{W}_{t+m}\right\|_2^4\right)^{\frac{1}{4}}\left(E\left(b'\eta_{t+h}\right)^4\right)^{\frac{1}{4}}\left(E\left(b'\eta_{t+m+h}\right)^4\right)^{\frac{1}{4}} \\
\leq & \frac{2\left(T-h-p\right)\left(h-1\right)}{T_h^2}\overline{C} \quad \text{(applying Lemma C-5 and expression (84) above)} \\
< & \frac{2\left(h-1\right)\overline{C}}{T_h} \quad \text{(since } T_h = T-h-p+1\text{)} \\
= & O\left(\frac{1}{T}\right)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t\eta'_{t+h}b\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\right)^2\left(\eta'_{t+h}b\right)^2\right] + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\} \\
= & O\left(\frac{1}{T}\right)
\end{aligned}
$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t\eta'_{t+h}b = O_p\left(\frac{1}{\sqrt{T}}\right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Now, for part (f), note that

$$\begin{aligned}
S'_d\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S'_K\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

Hence, it follows by applying the result given in part (e) above and the Slutsky's theorem that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{Y}_t\eta'_{t+h} &= S'_d\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right) \text{ and} \\
\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t\eta'_{t+h} &= S'_K\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

To show part (g), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$ and write

$$\begin{aligned}
E\left(\frac{b'\mathfrak{H}'\iota_{T_h}}{\sqrt{T_h}}\right)^2 &= E\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h}\right)^2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{s+h-k}\right]J_{d+K}\left(A^k\right)'J'_db \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
&\quad + \frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^{m+j}\right)'J'_db \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
&\quad + \frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'J'_db
\end{aligned}$$

Now,

$$
\begin{aligned}
& \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j} \varepsilon_{t+h-j}'\right] J_{d+K} \left(A^j\right)' J_d' b \\
\leq\; & \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} E\left\|\varepsilon_{t+h-j}\right\|_2^2 b' J_d A^j J_{d+K}' J_{d+K} \left(A^j\right)' J_d' b \\
\leq\; & \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^6\right)^{\frac{1}{3}} b' J_d A^j \left(A^j\right)' J_d' b \\
& \left(\text{by Liapunov's inequality and the fact that } \lambda_{\max}\left(J_{d+K}' J_{d+K}\right) = 1\right) \\
\leq\; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} b' J_d A^j \left(A^j\right)' J_d' b \\
& \left(\text{where } \overline{C} \geq 1 \text{ is a constant such that } E\left\|\varepsilon_{t-j}\right\|_2^6 \leq \overline{C} < \infty \text{ by Assumption 3-2(b)}\right) \\
\leq\; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \lambda_{\max}\left\{A^j \left(A^j\right)'\right\} b' J_d J_d' b \\
=\; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \\
& \left(\text{since } \lambda_{\max}\left\{A^j \left(A^j\right)'\right\} = \lambda_{\max}\left\{\left(A^j\right)' A^j\right\},\ \lambda_{\max}\left(J_d J_d'\right) = 1, \text{ and } b'b = 1\right) \\
=\; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \sigma_{\max}^2\left(A^j\right) \\
\leq\; & \overline{C}^{\frac{1}{3}} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} C \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \quad \text{(by Assumption 3-7)} \\
=\; & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \\
=\; & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{h-1} \phi_{\max}^{2j}
\end{aligned}
$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' J_d' b &\leq \overline{C}^{\frac{1}{3}} C \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1}\phi_{\max}^{2j} \\
&\leq \overline{C}^{\frac{1}{3}} C \frac{T-h-p+1}{T_h}\frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}} C \frac{1}{1-\phi_{\max}^2} \\
&\quad \text{(since } T_h = T-h-p+1\text{)} \\
&= O(1)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left|\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b'J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' J_d' b\right| \\
\leq\ & \frac{2}{T_h}\sum_{t=p}^{T-h-1}\left|\sum_{j=0}^{\max\{0,h-2\}} b'J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' J_d' b\right| \\
\leq\ & \frac{2}{T_h}\sum_{t=p}^{T-h-1}\left\{\sqrt{\sum_{j=0}^{\max\{0,h-2\}} b'J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' J_d' b}\right. \\
&\left.\times\sqrt{\sum_{j=0}^{\max\{0,h-2\}}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} b'J_d A^{m_1} A^j J_{d+K}' J_{d+K}\left(A^j\right)'\left(A^{m_2}\right)' J_d' b}\right\}
\end{aligned}$$

Similar to the argument given previously, we have

$$\begin{aligned}
& \sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] J_{d+K}\left(A^j\right)' J'_d b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}\left(E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]\right) b' J_d A^j J'_{d+K} J_{d+K}\left(A^j\right)' J'_d b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}\left(E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]\right) C\phi_{\max}^{2j} \\
= & \; C \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}^2\left(E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]\right) \phi_{\max}^{2j} \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(tr\left\{E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]\right\}\right)^2 \phi_{\max}^{2j} \\
= & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^2\right)^2 \phi_{\max}^{2j} \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^6\right)^{\frac{2}{3}} \phi_{\max}^{2j} \\
\leq & \; \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2}
\end{aligned}$$

and

$$
\begin{aligned}
& \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} \left(A^j\right)' \left(A^{m_2}\right)' J'_d b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j \left(A^j\right)' \left(A^{m_2}\right)' J'_d b \\
\leq & \ C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \left| b' J_d A^{m_1} \left(A^{m_2}\right)' J'_d b \right| \\
\leq & \ C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{b' J_d A^{m_1} \left(A^{m_1}\right)' J'_d b} \sqrt{b' J_d A^{m_2} \left(A^{m_2}\right)' J'_d b} \\
\leq & \ C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{C \phi_{\max}^{2m_1}} \sqrt{C \phi_{\max}^{2m_2}} \\
\leq & \ C^2 \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \phi_{\max}^{m_1} \sum_{m_2=1}^{T-h-t} \phi_{\max}^{m_2} \\
\leq & \ C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left| \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{h-2} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] J_{d+K} \left(A^j\right)' \sum_{m=1}^{T-h-t} \left(A^m\right)' J'_d b \right| \\
\leq & \ \frac{2}{T_h} \sum_{t=p}^{T-h-1} \left\{ \sqrt{ \sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] J_{d+K} \left(A^j\right)' J'_d b } \right. \\
& \left. \times \sqrt{ \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} \left(A^j\right)' \left(A^{m_2}\right)' J'_d b } \right\} \\
\leq & \ \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sqrt{ \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2} } \sqrt{ C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2 } \\
= & \ 2 \overline{C}^{\frac{1}{3}} C^{\frac{3}{2}} \frac{T-h-p+1}{T_h} \left(\frac{1}{1-\phi_{\max}^2} \right) \left(\frac{1}{1-\phi_{\max}} \right) \\
= & \ O(1)
\end{aligned}
$$

Putting these results together, we obtain

$$
\begin{aligned}
& E\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h}\right)^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
& +\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'J'_db \\
= & O(1)
\end{aligned}
$$

so that, upon applying Markov's inequality, we get

$$
\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h} = O_p(1).
$$

Since the above result holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$
\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h}\eta_{t+h} = O_p(1)
$$

and that

$$
\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h} = \frac{1}{\sqrt{T_h}}\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h}\eta_{t+h}\right) = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1).
$$

Lastly, to show part (h), let $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$; and write

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)^2\right] \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left\{\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\right. \\
& \qquad \left.\times\left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\}
\end{aligned}
$$

Making use of the Cauchy-Schwarz inequality, we then have

$$
\begin{aligned}
& \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2\right] \\
= \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\eta_{t+h}\eta_{t+h}'b\right)^2 - \frac{1}{T_h^2}\sum_{t=p}^{T-h}\left(E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2 \\
\leq \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\eta_{t+h}\eta_{t+h}'b\right)^2 \\
\leq \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} \sqrt{E\left(a'\eta_{t+h}\right)^4}\sqrt{E\left(b'\eta_{t+h}\right)^4}
\end{aligned}
$$

In the proof of part (e) of this lemma, we have shown that, given Assumptions 3-2(b) and 3-7, there exists positive constants C and $\overline{C}$ such that

$$
E\left(b'\eta_{t+h}\right)^4 \leq h^3 \sum_{j=0}^{h-1} C\phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \leq \overline{C} < \infty .
$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}\left(A\right)\right|, \left|\lambda_{\min}\left(A\right)\right|\right\}$ and where h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. In a similar manner, we can also show that

$$
E\left(a'\eta_{t+h}\right)^4 \leq h^3 \sum_{j=0}^{h-1} C\phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \leq \overline{C} < \infty .
$$

It follows that

$$
\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2\right] &\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} \sqrt{E\left(a'\eta_{t+h}\right)^4}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} \overline{C} \\
&= \overline{C}\frac{T-h-p+1}{T_h^2} \\
&= \frac{\overline{C}}{T_h} \quad (\text{since } T_h = T-h-p+1) \\
&= O\left(\frac{1}{T}\right) \qquad (85)
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
& a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right] \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1} a'J_dA^jJ'_{d+K}\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-k} - E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-k}\right]\right)J_{d+K}\left(A^k\right)'J'_db \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)\left\{vec\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-k}\right) - vec\left(E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-k}\right]\right)\right\} \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)\left\{\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left[\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right] \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1} a'J_dA^\ell J'_{d+K}\left(\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-r} - E\left[\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-r}\right]\right)J_{d+K}\left(A^r\right)'J'_db \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left(b'J_dA^rJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)\left\{vec\left(\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-r}\right) - vec\left(E\left[\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-r}\right]\right)\right\} \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left(b'J_dA^rJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)\left\{\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left[\varepsilon_{t+m+h-k}\otimes\varepsilon_{t+m+h-r}\right]\right\}
\end{aligned}
$$

Moreover, note that, for $m \geq h$

$$
\begin{aligned}
& E\left\{\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\} \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\Big\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right) \\
& \times E\left(\left[\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)\right]\right. \\
& \qquad \left.\times\left[\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)\right]'\right) \\
& \times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\Big\} \\
= & \ 0
\end{aligned}
$$

Note further that, when $h = 1$, we will always have $m \geq h$, given that by definition m is an integer ≥ 1. This implies we need to distinguish between the case where $h = 1$ from the case where $h \geq 2$.

Consider first the case where $h = 1$. In this case, we have, for all $m \geq 1$

$$
\begin{aligned}
& E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} \\
= & \left(b'J_dA^0J_{d+K}' \otimes a'J_dA^0J_{d+K}'\right) \\
& \times E\left(\left[(\varepsilon_{t+1} \otimes \varepsilon_{t+1}) - E\left(\varepsilon_{t+1} \otimes \varepsilon_{t+1}\right)\right]\left[(\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1}) - E\left(\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1}\right)\right]'\right) \\
& \times \left(J_{d+K}\left(A^0\right)' J_d'b \otimes J_{d+K}\left(A^0\right)' J_d'a\right) \\
= & \; 0
\end{aligned}
$$

so that, in this case, we have

$$
\begin{aligned}
& E\left[\frac{1}{T_1}\sum_{t=p}^{T-1}\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\right]^2 \\
= & \frac{1}{T_1^2}\sum_{t=p}^{T-1}E\left[\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)^2\right] \\
& + \frac{2}{T_1^2}\sum_{t=p}^{T-1}\sum_{m=1}^{T-1-t}E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\right. \\
& \qquad \left. \times \left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} \\
= & \frac{1}{T_1^2}\sum_{t=p}^{T-1}E\left[\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)^2\right] \\
= & O\left(\frac{1}{T}\right) \quad \text{(as shown previously in expression (85))} \qquad (86)
\end{aligned}
$$

Consider next the case where $h \geq 2$. In this case,

$$
E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} = 0
$$

for all $m \geq h$ as previously shown; however, for $1 \leq m \leq h-1$, we have

$$
\begin{aligned}
& \left|E\left\{\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\}\right| \\
= & \left|\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)\right.\right. \\
& \times E\left(\left[\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)\right]\right. \\
& \times\left.\left[\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)\right]'\right) \\
& \left.\left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\}\right| \\
= & \left|\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+m+h-r}\right)\right.\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\} \\
& -\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)'\right. \\
& \qquad\qquad \left.\left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\}\right| \\
\leq & \sum_{j=0}^{h-1}\left|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right. \\
& \quad \left.\times\left(J_{d+K}\left(A^j\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right| \\
& +\sum_{j=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+h-k}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^k\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right| \\
& +\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\\ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^kJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right| \\
& +\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\\ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\varepsilon'_{t+h-k}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^k\right)'J'_da\right)\right|
\end{aligned}
$$

$$+\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left|\left(b'J_dA^jJ_{d+K}'\otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)'\right.$$
$$\left.\times\left(J_{d+K}\left(A^\ell\right)'J_d'b\otimes J_{d+K}\left(A^\ell\right)'J_d'a\right)\right|.$$

Analyzing each term on the majorant side of the function above, we have

$$\begin{aligned}
&\sum_{j=0}^{h-1}\left|\left(b'J_dA^jJ_{d+K}'\otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\otimes\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right)\right.\\
&\qquad\left.\times\left(J_{d+K}\left(A^j\right)'J_d'b\otimes J_{d+K}\left(A^j\right)'J_d'a\right)\right|\\
\le\ &\overline{C}\sum_{j=0}^{h-1}\left(b'J_dA^jJ_{d+K}'J_{d+K}\left(A^j\right)'J_d'b\right)\left(a'J_dA^jJ_{d+K}'J_{d+K}\left(A^j\right)'J_d'a\right)\\
=\ &\overline{C}\sum_{j=0}^{h-1}\left[b'J_dA^j\left(A^j\right)'J_d'b\right]\left[a'J_dA^j\left(A^j\right)'J_d'a\right]\quad\left(\text{since }\lambda_{\max}\left(J_{d+K}'J_{d+K}\right)=1\right)\\
\le\ &\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]^2\left[b'J_dJ_d'b\right]\left[a'J_dJ_d'a\right]\\
=\ &\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]^2\quad\left(\text{since }J_dJ_d'=I_d\text{ and }a'a=b'b=1\right)\\
=\ &\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]^2\\
=\ &\overline{C}\sum_{j=0}^{h-1}\sigma_{\max}^4\left(A^j\right)\\
\le\ &\overline{C}\sum_{j=0}^{h-1}C^*\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^4,\left|\lambda_{\min}\left(A^j\right)\right|^4\right\}\ \text{(by Assumption 3-7)}\\
=\ &\overline{C}\sum_{j=0}^{h-1}C^*\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{4j},\left|\lambda_{\min}\left(A\right)\right|^{4j}\right\}\\
=\ &\overline{C}\sum_{j=0}^{h-1}C^*\phi_{\max}^{4j}\ \left(\text{where }0<\phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}<1\right)\\
\le\ &\overline{C}hC^*\quad\left(\text{since }0<\phi_{\max}<1\text{ and }\phi_{\max}^0=1\right)\\
\le\ &C\quad\left(\text{for }\overline{C}hC^*\le C<\infty\right),
\end{aligned}$$

$$
\begin{aligned}
& \sum_{j=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left|\left(b'J_dA^kJ_{d+K}'\otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-k}\varepsilon_{t+h-k}'\otimes\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right)\right. \\
& \qquad \left.\times\left(J_{d+K}\left(A^k\right)'J_d'b\otimes J_{d+K}\left(A^j\right)'J_d'a\right)\right| \\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(b'J_dA^kJ_{d+K}'J_{d+K}\left(A^k\right)'J_d'b\right)\left(a'J_dA^jJ_{d+K}'J_{d+K}\left(A^j\right)'J_d'a\right) \\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[b'J_dA^k\left(A^k\right)'J_d'b\right]\left[a'J_dA^j\left(A^j\right)'J_d'a\right]\quad\left(\text{since }\lambda_{\max}\left(J_{d+K}'J_{d+K}\right)=1\right) \\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\left[b'J_dJ_d'b\right]\left[a'J_dJ_d'a\right] \\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\quad\left(\text{since }J_dJ_d'=I_d\text{ and }a'a=b'b=1\right) \\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]\left[\lambda_{\max}\left\{\left(A^k\right)'A^k\right\}\right] \\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\sigma_{\max}^2\left(A^j\right)\sigma_{\max}^2\left(A^k\right) \\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2,\left|\lambda_{\min}\left(A^j\right)\right|^2\right\}\max\left\{\left|\lambda_{\max}\left(A^k\right)\right|^2,\left|\lambda_{\min}\left(A^k\right)\right|^2\right\} \\
& \text{(by Assumption 3-7)} \\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2j},\left|\lambda_{\min}\left(A\right)\right|^{2j}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\} \\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\phi_{\max}^{2j}\phi_{\max}^{2k}\ \left(\text{where }\phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right) \\
=\ & \overline{C}h^2\left(C^*\right)^2\ \ (\text{since }0<\phi_{\max}<1\text{ given Assumption 3-1}) \\
\leq\ & C\quad\left(\text{for }\overline{C}h^2\left(C^*\right)^2\leq C<\infty\right),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right| \\
\leq \; & \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left\{ \left[\left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+h-k} \right) \right. \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^k \right)' J'_d b \otimes J_{d+K} \left(A^k \right)' J'_d a \right) \right]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \qquad \left. \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right]^{1/2} \right\} \\
\leq \; & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left(b' J_d A^k J'_{d+K} J_{d+K} \left(A^k \right)' J'_d b \right) \left(a' J_d A^k J'_{d+K} J_{d+K} \left(A^k \right)' J'_d a \right) } \\
& \qquad \times \sqrt{ \left(b' J_d A^\ell J'_{d+K} J_{d+K} \left(A^\ell \right)' J'_d b \right) \left(a' J_d A^\ell J'_{d+K} J_{d+K} \left(A^\ell \right)' J'_d a \right) } \\
= \; & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left[b' J_d A^k \left(A^k \right)' J'_d b \right] \left[a' J_d A^k \left(A^k \right)' J'_d a \right] } \sqrt{ \left[b' J_d A^\ell \left(A^\ell \right)' J'_d b \right] \left[a' J_d A^\ell \left(A^\ell \right)' J'_d a \right] } \\
& \left(\text{since } \lambda_{\max} \left(J'_{d+K} J_{d+K} \right) = 1 \right) \\
\leq \; & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \left[b' J_d J'_d b \right] \left[a' J_d J'_d a \right] \\
= \; & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \quad \left(\text{since } J_d J'_d = I_d \text{ and } a'a = b'b = 1 \right) \\
= \; & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ \left(A^k \right)' A^k \right\} \right] \left[\lambda_{\max} \left\{ \left(A^\ell \right)' A^\ell \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1} \sigma_{\max}^{2}\left(A^{k}\right)\sigma_{\max}^{2}\left(A^{\ell}\right) \\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1} \left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A^{k}\right)\right|^{2},\left|\lambda_{\min}\left(A^{k}\right)\right|^{2}\right\}\max\left\{\left|\lambda_{\max}\left(A^{\ell}\right)\right|^{2},\left|\lambda_{\min}\left(A^{\ell}\right)\right|^{2}\right\} \\
&\quad \text{(by Assumption 3-7)} \\
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1} \left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2\ell},\left|\lambda_{\min}\left(A\right)\right|^{2\ell}\right\} \\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1} \left(C^{*}\right)^{2}\phi_{\max}^{2k}\phi_{\max}^{2\ell} \ \ \left(\text{since } \phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right) \\
&= \overline{C}h^{2}\left(C^{*}\right)^{2} \ \ \left(\text{since } 0<\phi_{\max}<1 \text{ and } \phi_{\max}^{0}=1\right) \\
&\leq C \quad \left(\text{for } \overline{C}h^{2}\left(C^{*}\right)^{2}\leq C<\infty\right),
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\varepsilon'_{t+h-k}\right)\right.\\
&\qquad\left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^k\right)'J'_da\right)\right|\\
\leq\ &\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left\{\left[\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+h-k}\otimes\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-\ell}\right)\right.\right.\\
&\qquad\left.\times\left(J_{d+K}\left(A^k\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right]^{1/2}\\
&\quad\times\left[\left(b'J_dA^\ell J'_{d+K}\otimes a'J_dA^kJ'_{d+K}\right)E\left(\varepsilon_{t+m+h-\ell}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+h-k}\varepsilon'_{t+h-k}\right)\right.\\
&\qquad\left.\left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^k\right)'J'_da\right)\right]^{1/2}\right\}\\
\leq\ &\overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\sqrt{\left(b'J_dA^kJ'_{d+K}J_{d+K}\left(A^k\right)'J'_db\right)\left(a'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_da\right)}\\
&\qquad\times\sqrt{\left(b'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_db\right)\left(a'J_dA^kJ'_{d+K}J_{d+K}\left(A^k\right)'J'_da\right)}\\
=\ &\overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\sqrt{\left[b'J_dA^k\left(A^k\right)'J'_db\right]\left[a'J_dA^\ell\left(A^\ell\right)'J'_da\right]}\sqrt{\left[b'J_dA^\ell\left(A^\ell\right)'J'_db\right]\left[a'J_dA^k\left(A^k\right)'J'_da\right]}\\
&\left(\text{since }\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)=1\right)\\
\leq\ &\overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\left[b'J_dJ'_db\right]\left[a'J_dJ'_da\right]\\
=\ &\overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\quad\left(\text{since }J_dJ'_d=I_d\text{ and }a'a=b'b=1\right)\\
=\ &\overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left[\lambda_{\max}\left\{\left(A^k\right)'A^k\right\}\right]\left[\lambda_{\max}\left\{\left(A^\ell\right)'A^\ell\right\}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\sigma_{\max}^{2}\left(A^{k}\right)\sigma_{\max}^{2}\left(A^{\ell}\right)\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A^{k}\right)\right|^{2},\left|\lambda_{\min}\left(A^{k}\right)\right|^{2}\right\}\max\left\{\left|\lambda_{\max}\left(A^{\ell}\right)\right|^{2},\left|\lambda_{\min}\left(A^{\ell}\right)\right|^{2}\right\}\\
&\quad \text{(by Assumption 3-7)}\\
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2\ell},\left|\lambda_{\min}\left(A\right)\right|^{2\ell}\right\}\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\phi_{\max}^{2k}\phi_{\max}^{2\ell}\ \ \left(\text{since } \phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right)\\
&= \overline{C}h^{2}\left(C^{*}\right)^{2}\ \ \left(\text{since } 0<\phi_{\max}<1 \text{ and } \phi_{\max}^{0}=1\right)\\
&\leq C\ \ \ \left(\text{for } \overline{C}h^{2}\left(C^{*}\right)^{2}\leq C<\infty\right),
\end{aligned}
$$

and

$$\begin{aligned}
& \sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\Big|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)' \\
& \qquad\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\Big| \\
\leq\ & \sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\Big\{\Big[\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)' \\
& \qquad\times\left(J_{d+K}\left(A^j\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\Big]^{1/2} \\
& \quad\times\Big[\left(b'J_dA^\ell J'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)' \\
& \qquad\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\Big]^{1/2}\Big\} \\
\leq\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sqrt{\left(b'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_db\right)\left(a'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_da\right)} \\
& \qquad\times\sqrt{\left(b'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_db\right)\left(a'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_da\right)} \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sqrt{\left[b'J_dA^j\left(A^j\right)'J'_db\right]\left[a'J_dA^j\left(A^j\right)'J'_da\right]}\sqrt{\left[b'J_dA^\ell\left(A^\ell\right)'J'_db\right]\left[a'J_dA^\ell\left(A^\ell\right)'J'_da\right]} \\
& \left(\text{since }\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)=1\right) \\
\leq\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\left[b'J_dJ'_db\right]\left[a'J_dJ'_da\right] \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\quad\left(\text{since }J_dJ'_d=I_d\text{ and }a'a=b'b=1\right) \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]\left[\lambda_{\max}\left\{\left(A^\ell\right)'A^\ell\right\}\right] \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sigma^2_{\max}\left(A^j\right)\sigma^2_{\max}\left(A^\ell\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A^{j}\right)\right|^{2},\left|\lambda_{\min}\left(A^{j}\right)\right|^{2}\right\}\max\left\{\left|\lambda_{\max}\left(A^{\ell}\right)\right|^{2},\left|\lambda_{\min}\left(A^{\ell}\right)\right|^{2}\right\} \\
&\quad \text{(by Assumption 3-7)} \\
&= \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2j},\left|\lambda_{\min}\left(A\right)\right|^{2j}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2\ell},\left|\lambda_{\min}\left(A\right)\right|^{2\ell}\right\} \\
&\leq \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left(C^{*}\right)^{2}\phi_{\max}^{2j}\phi_{\max}^{2\ell}\ \ \left(\text{since } \phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right) \\
&= \overline{C}h^{2}\left(C^{*}\right)^{2}\ \ \left(\text{since } 0<\phi_{\max}<1 \text{ and } \phi_{\max}^{0}=1\right) \\
&\leq C\ \ \left(\text{for } \overline{C}h^{2}\left(C^{*}\right)^{2}\leq C<\infty\right),
\end{aligned}
$$

where upper bounds given above have made use of the fact that for all t and s

$$
\begin{aligned}
& E\left[\varepsilon_{t}\varepsilon_{t}'\otimes\varepsilon_{s}\varepsilon_{s}'\right] \\
=\ & E\left[\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)'\right] \\
\leq\ & tr\left\{E\left[\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)'\right]\right\}\cdot I_{(d+K)^{2}} \\
& \text{(where the inequality holds in positive semi-definite sense)} \\
=\ & E\left[tr\left\{\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)'\right\}\right]\cdot I_{(d+K)^{2}} \\
=\ & E\left[tr\left\{\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)'\left(\varepsilon_{t}\otimes\varepsilon_{s}\right)\right\}\right]\cdot I_{(d+K)^{2}} \\
=\ & E\left[\varepsilon_{t}'\varepsilon_{t}\varepsilon_{s}'\varepsilon_{s}\right]\cdot I_{(d+K)^{2}} \\
=\ & E\left[\left\|\varepsilon_{t}\right\|_{2}^{2}\left\|\varepsilon_{s}\right\|_{2}^{2}\right]\cdot I_{(d+K)^{2}} \\
\leq\ & \sup_{t} E\left[\left\|\varepsilon_{t}\right\|_{2}^{4}\right]\cdot I_{(d+K)^{2}} \\
\leq\ & \overline{C}\cdot I_{d^{2}}\ \ \text{(by Assumption 3-2(b))}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(\varepsilon_t \otimes \varepsilon_t\right) E\left(\varepsilon_t \otimes \varepsilon_t\right)' &\leq tr\left\{E\left(\varepsilon_t \otimes \varepsilon_t\right) E\left(\varepsilon_t \otimes \varepsilon_t\right)'\right\} \cdot I_{(d+K)^2} \\
&\quad \text{(where the inequality holds in positive semi-definite sense)} \\
&= E\left(\varepsilon_t \otimes \varepsilon_t\right)' E\left(\varepsilon_t \otimes \varepsilon_t\right) \cdot I_{(d+K)^2} \\
&= \sum_{g=1}^{d} \sum_{\ell=1}^{d} \left(E\left[\varepsilon_{gt}\varepsilon_{\ell t}\right]\right)^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^{d} \sum_{\ell=1}^{d} \left(E\left|\varepsilon_{gt}\varepsilon_{\ell t}\right|\right)^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^{d} \sum_{\ell=1}^{d} E\left[\varepsilon_{gt}^2\right] E\left[\varepsilon_{\ell t}^2\right] \cdot I_{(d+K)^2} \\
&= E\left[\sum_{g=1}^{d} \varepsilon_{gt}^2\right] E\left[\sum_{\ell=1}^{d} \varepsilon_{\ell t}^2\right] \cdot I_{(d+K)^2} \\
&= \left(E\left\|\varepsilon_t\right\|_2^2\right)^2 \cdot I_{(d+K)^2} \\
&\leq \overline{C} \cdot I_{(d+K)^2} \text{ (by Assumption 3-2(b))}
\end{aligned}
$$

for some positive constant $\overline{C}$. It follows from these calculations that, for $1 \leq m \leq h-1$ where

$h \geq 2$, we have

$$
\begin{aligned}
& \left| E\left\{ \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b \right] \right) \left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b \right] \right) \right\} \right| \\
\leq \; & \sum_{j=0}^{h-1} \left| \left(b'J_dA^jJ'_{d+K} \otimes a'J_dA^jJ'_{d+K} \right) E\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-j} \otimes \varepsilon_{t+h-j}\varepsilon'_{t+h-j} \right) \right. \\
& \qquad \left. \times \left(J_{d+K}\left(A^j\right)'J'_db \otimes J_{d+K}\left(A^j\right)'J'_da \right) \right| \\
& + \sum_{j=0}^{h-1} \sum_{\substack{k=0 \\ k \neq j}}^{h-1} \left| \left(b'J_dA^kJ'_{d+K} \otimes a'J_dA^jJ'_{d+K} \right) E\left(\varepsilon_{t+h-k}\varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-j}\varepsilon'_{t+h-j} \right) \right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^k\right)'J'_db \otimes J_{d+K}\left(A^j\right)'J'_da \right) \right| \\
& + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b'J_dA^kJ'_{d+K} \otimes a'J_dA^kJ'_{d+K} \right) E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell} \right) \right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J'_db \otimes J_{d+K}\left(A^\ell\right)'J'_da \right) \right| \\
& + \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b'J_dA^kJ'_{d+K} \otimes a'J_dA^\ell J'_{d+K} \right) E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell}\varepsilon'_{t+h-k} \right) \right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J'_db \otimes J_{d+K}\left(A^k\right)'J'_da \right) \right| \\
& + \sum_{j=0}^{h-1} \sum_{\ell=0}^{h-1} \left| \left(b'J_dA^jJ'_{d+K} \otimes a'J_dA^jJ'_{d+K} \right) E\left(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j} \right) E\left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \right)' \right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J'_db \otimes J_{d+K}\left(A^\ell\right)'J'_da \right) \right| \\
\leq \; & 5C
\end{aligned}
$$

so that, when $h \geq 2$,

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right) \left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\} \right| \\
= & \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} E\left\{ \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right) \right.\right. \\
& \qquad \left.\left. \times \left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\} \right| \\
\leq & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \left| E\left\{ \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right) \right.\right. \\
& \qquad \left.\left. \times \left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\} \right| \\
\leq & \frac{2}{T_h} \frac{T-h-p}{T_h} (h-1) 5C \\
< & \frac{10(h-1)C}{T_h} \quad (\text{ since } T_h = T-h-p+1) \\
= & O\left(\frac{1}{T}\right)
\end{aligned}$$

Putting everything together for the case where $h \geq 2$, we see that

$$\begin{aligned}
& E\left[\frac{1}{T_h} \sum_{t=p}^{T-h} \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right) \right]^2 \\
= & \frac{1}{T_h^2} \sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)^2 \right] \\
& + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right) \left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\} \\
= & O\left(\frac{1}{T}\right) + O\left(\frac{1}{T}\right) \\
= & O\left(\frac{1}{T}\right) \qquad (87)
\end{aligned}$$

In light of the results given in expressions (86) and (87), we can apply Markov's inequality to show that regardless of whether $h = 1$ or $h \geq 2$

$$\frac{1}{T_h} \sum_{t=p}^{T-h} a'\eta_{t+h}\eta'_{t+h}b - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[a'\eta_{t+h}\eta'_{t+h}b\right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that for all (fixed) positive integer h

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right] = O_p\left(\frac{1}{\sqrt{T}}\right). \ \square$$

Lemma D-3: Suppose that A is an $N \times N$ symmetric matrix which we can partition as

$$A = \begin{pmatrix} \underset{r\times r}{A_{11}} & \underset{r\times(N-r)}{A_{12}} \\ \underset{(N-r)\times r}{A_{21}} & \underset{(N-r)\times(N-r)}{A_{22}} \end{pmatrix}$$

Then,

$$\left\|A_{21}\right\|_2 \leq \left\|A\right\|_2.$$

Proof of Lemma D-3: Define

$$\underset{N\times r}{B_1} = \begin{pmatrix} I_r \\ 0 \end{pmatrix}.$$

Let $\overline{v} \in \mathbb{R}^r$ be such that $\left\|\overline{v}\right\|_2 = 1$ and

$$\overline{v}'A_{21}'A_{21}\overline{v} = \max_{\|v\|_2=1} v'A_{21}'A_{21}v$$

It follows that

$$\begin{aligned} \left\|A_{21}\right\|_2 &= \sqrt{\overline{v}'A_{21}'A_{21}\overline{v}} \\ &\leq \sqrt{\overline{v}'A_{11}'A_{11}\overline{v} + \overline{v}'A_{21}'A_{21}\overline{v}} \\ &= \sqrt{\overline{v}'B_1'A'AB_1\overline{v}} \\ &\leq \sqrt{\max_{\|v\|_2=1} v'A'Av} \ \left(\text{noting that } \left\|B_1\overline{v}\right\|_2 = \sqrt{\overline{v}'B_1'B_1\overline{v}} = \sqrt{\overline{v}'\overline{v}} = 1\right) \\ &= \left\|A\right\|_2. \ \square \end{aligned}$$

Remark: This is a well-known linear algebraic result. A similar result has also been given in the beginning of section 6 of Johnstone and Lu (2009).

Lemma D-4: Let

$$M_{FF} = \frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{F}_t\underline{F}_t'\right] \tag{88}$$

where $T_0 = T - p + 1$. Then, under Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7; there exists a positive constant $\underline{C}$ such that

$$\lambda_{\min}\left\{M_{FF}\right\} \geq \underline{C} > 0$$

for all $T > p - 1$.

Proof of Lemma D-4:

To proceed, note that we can write

$$\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} = \mathcal{P}_{(d+K)p}\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'$$

from which it follows that

$$\begin{aligned}
\lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix}\right\} &= \lambda_{\min}\left\{\mathcal{P}_{(d+K)p}\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'\right\} \\
&\geq \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right]\right\}\lambda_{\min}\left\{\mathcal{P}_{(d+K)p}\mathcal{P}_{(d+K)p}'\right\} \\
&= \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right]\right\}\lambda_{\min}\left\{I_{(d+K)p}\right\} \\
&\quad \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&= \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right]\right\}
\end{aligned}$$

Next, note that

$$\begin{aligned}
\frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{W}_t\underline{W}_t'\right] &= \frac{1}{T_0}\sum_{t=p}^{T}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1} \\
&\quad + \frac{1}{T_0}\sum_{t=p}^{T}\sum_{j=0}^{\infty}A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)' \\
&= \left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1} \\
&\quad + \sum_{j=0}^{\infty}A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T}E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'
\end{aligned}$$

so that there exists a positive constant $\underline{C}$ such that

$$\begin{aligned}
& \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\} \\
\geq\ & \lambda_{\min}\left\{\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\right\} \\
& +\lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
& \text{(by Weyl's Theorem (see Theorem 4.3.1 of Horn and Johnson, 1985))} \\
\geq\ & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
\geq\ & \underline{c} > 0 \text{ for all } T > p-1 \ \text{ (by the result given in part (a) of Lemma D-1)}
\end{aligned}$$

It then follows that

$$\begin{aligned}
& \lambda_{\min}\left\{M_{FF}\right\} \\
=\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{F}_t\underline{F}_t'\right]\right\} \\
\geq\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right]\end{pmatrix}\right\} \\
& \text{(by the Poincaré separation theorem (see Corollary 4.3.16 of Horn and Johnson, 1985))} \\
\geq\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\} \\
\geq\ & \underline{C} > 0 \text{ for all } T > p-1,
\end{aligned}$$

as required. □

Lemma D-5: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumption 3-3 hold. Then,

(a)

$$\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| = O(1)$$

(b)

$$\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|=O\left(1\right)$$

(c)

$$\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right|=O\left(1\right)$$

Proof of Lemma D-5:

To show part (a), first write

$$\frac{1}{T_h}\sum_{\substack{v,w=p\\ v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right|=\frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^2\right]+\frac{1}{T_h}\sum_{\substack{v,w=p\\ v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| \tag{89}$$

Consider now the first term on the right-hand side of expression (89). Note that, trivially, by Assumption 3-3(b),

$$\frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^2\right]\leq C=O\left(1\right) \tag{90}$$

For the second term on the right-hand side of expression (89), note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right)\leq a_1\exp\left\{-a_2 m\right\}.$$

for every i. Since $\alpha_{i,m}\leq\beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m}\leq a_1\exp\left\{-a_2 m\right\}\text{ for every }i.$$

Hence, in this case, we can apply Lemma C-3 with $p=6$ and $r=5/4$ to obtain

$$\begin{aligned}&\frac{1}{T_h}\sum_{\substack{v,w=p\\ v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right|\\ \leq\ &\frac{1}{T_h}\sum_{\substack{v,w=p\\ v<w}}^{T-h}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\end{aligned}$$

Application of Liapunov's inequality then gives us

$$\begin{aligned}
\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
&\leq \left(\sup_{t} E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}} \\
&= C^{\frac{1}{3}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho = w - v$, so that $w = v + \varrho$. Using these notations and the boundedness of $\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
&\frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| \\
&\frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\ & \frac{C^{\frac{1}{3}}}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^{*}}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho\right\} \\
&\left(\text{for some constant } C^{*} \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{3}}a_1^{\frac{1}{30}} \leq C^{*} < \infty\right) \\
\leq\ & \frac{C^{*}}{T_h}\sum_{v=p}^{T-h}\sum_{\varrho=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\ & C^{*}\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\ & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (91)
\end{aligned}$$

It follows from expressions (89), (90), and (91) that

$$\begin{aligned}
\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| &= \frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^{2}\right] + \frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h}\left|E\left[u_{ig}u_{ih}\right]\right| \\
&= O\left(1\right) + O\left(1\right) \\
&= O\left(1\right).
\end{aligned}$$

To show part (b), first write

$$\frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| = \frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3+\frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g\\s-t>g-s,s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| + \frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g\\g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \tag{92}$$

For the first term on the right-hand side of expression (92) above, note that, trivially, we can apply Assumption 3-3(b) to obtain

$$\frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3\leq C=O\left(1\right). \tag{93}$$

Next, note that, for the second term on the right-hand side of expression (92) above, we can apply Lemma C-3 with $p=6$ and $r=5/4$ to obtain

$$\begin{aligned}&\frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g\\s-t>g-s,s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\\ \leq\ &\frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g\\s-t>g-s,s-t>0}}^{T-h}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(s-t\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\end{aligned}$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{ig}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}}\\ &= \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{ig}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\\ &\leq \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\\ &\quad\text{(by Liapunov's inequality)}\\ &= C^{\frac{1}{2}}<\infty\ \text{(by Assumption 3-3(b))}\end{aligned}$$

Moreover, let $\varrho_1=s-t$ and $\varrho_2=g-s$, so that $s=t+\varrho_1$ and $g=s+\ \varrho_2=t+\varrho_1+\ \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\ & \frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(s-t\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\ & \frac{C^{\frac{1}{2}}}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(s-t\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^{*}}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \left(\text{for some constant } C^{*} \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{2}}a_1^{\frac{1}{30}}\leq C^{*}<\infty\right) \\
\leq\ & \frac{C^{*}}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1-1}\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
\leq\ & \frac{C^{*}}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & C^{*}\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\ & O\left(1\right)\quad\text{(given Lemma C-1)}
\end{aligned}
\tag{94}
$$

Similarly, for the third term on the right-hand side of expression (92), we can apply Lemma C-3 with $p=6$ and $r=5/4$, we have

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\ & \frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}
\end{aligned}
$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}
\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} &\leq \left(\left(E\left|u_{it}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&= \left(E\left|u_{it}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&\leq \left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= C^{\frac{1}{2}} < \infty \ \text{ (by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = s - t$ and $\varrho_2 = g - s$, so that $s = t + \varrho_1$ and $g = s + \varrho_2 = t + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\ & \frac{1}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
\leq\ & \frac{C^{\frac{1}{2}}}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(g-s\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^*}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h}\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{2}}a_1^{\frac{1}{30}}\leq C^*<\infty\right) \\
\leq\ & \frac{C^*}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^*}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & C^*\left[\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{30}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho_2\right\}\right] \\
=\ & O\left(1\right) \quad \text{(given Lemma C-1)}
\end{aligned} \tag{95}$$

It follows from expressions (38), (39), (40), and (41) that

$$\begin{aligned}
\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| &= \frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3+\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
&\quad +\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
&= O\left(1\right)+O\left(1\right)+O\left(1\right) \\
&= O\left(1\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
&\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right| \\
&= \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right| \\
&= \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)+E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&\quad +\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)+E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&\quad +\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-s>0}}^{T-h}\left|E\left(u_{it}u_{is}\right)\right|\left|E\left(u_{ig}u_{iv}\right)\right| \qquad (96)
\end{aligned}$$

For the first term on the right-hand side of expression (96) above, note that, by Jensen's inequality, the Cauchy-Schwarz inequality, and Assumption 3-3(b); we have

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right| &\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}E\left[\left|u_{it}u_{is}^3\right|\right] \\
&\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\sqrt{E\left|u_{it}\right|^2}\sqrt{E\left|u_{is}\right|^6} \\
&\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\sqrt{E\left|u_{is}\right|^6} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{C^{\frac{2}{3}}T_h^2}{T_h^2}\ \text{(by Assumption 3-3(b))} \\
&= O(1) \qquad (97)
\end{aligned}$$

Next, for the second term on the right-hand side of expression (96), we can apply Lemma C-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
\leq\ &\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left\{2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(v-g\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}}\right. \\
&\qquad\left.\times\left(E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality,

$$\begin{aligned}
E\left|\{u_{it}u_{is}-E(u_{it}u_{is})\}u_{ig}\right|^{\frac{4}{3}} &\leq \left[E\left(u_{it}u_{is}-E(u_{it}u_{is})\right)^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{it}u_{is}\right|^{\frac{12}{7}}+\left|E\left[u_{it}u_{is}\right]\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{it}u_{is}\right|^{\frac{12}{7}}+E\left|u_{it}u_{is}\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}}E\left|u_{it}u_{is}\right|^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\left(E\left|u_{is}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\right]^{\frac{4}{3}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{4}{3}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}(C)^{\frac{2}{9}}(C)^{\frac{2}{9}}(C)^{\frac{2}{9}} \quad \text{(by Assumption 3-3(b))} \\
&= 2^{\frac{4}{3}}C^{\frac{2}{3}}
\end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s + \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|\{u_{it}u_{is}-E(u_{it}u_{is})\}u_{ig}\right|^{\frac{4}{3}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right) \right\} u_{ig}u_{iv} \right] \right| \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \left\{ 2\left(2^{1-\frac{3}{4}}+1\right) \left[a_1 \exp\left\{ -a_2 \left(v-g\right) \right\} \right]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \qquad \left. \times \left(E\left| \left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right) \right\} u_{ig} \right|^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(E\left|u_{iv}\right|^6 \right)^{\frac{1}{6}} \right\} \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g>g-s,\ v-g>0}}^{T-h} 2\left(2^{\frac{1}{4}}+1\right) \left[a_1 \exp\left\{ -a_2 \left(v-g\right) \right\} \right]^{\frac{1}{12}} \left(2^{\frac{4}{3}} C^{\frac{2}{3}} \right)^{\frac{3}{4}} \left(C\right)^{\frac{1}{6}} \\
\leq & \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \exp\left\{ -\frac{a_2}{12}\varrho_2 \right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{4}}+1\right) C^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty \right) \\
\leq & \frac{C^*}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{\varrho_2=1}^{\infty} \sum_{\varrho_1=0}^{\varrho_2-1} \exp\left\{ -\frac{a_2}{12}\varrho_2 \right\} \\
= & C^* \sum_{\rho_2=1}^{\infty} \varrho_2 \exp\left\{ -\frac{a_2}{12}\varrho_2 \right\} \\
= & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (98)
\end{aligned}
$$

Similarly, for the third term on the right-hand side of expression (96) above, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right) \right\} u_{ig}u_{iv} \right] \right| \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}}+1\right) \left[a_1 \exp\left\{ -a_2 \left(g-s\right) \right\} \right]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \qquad \left. \times \left(E\left| \left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right) \right\} \right|^{2} \right)^{\frac{1}{2}} \left(E\left|u_{ig}u_{iv}\right|^3 \right)^{\frac{1}{3}} \right\}
\end{aligned}
$$

Next, applications of Hölder's inequality yield

$$\begin{aligned} E\left|u_{ig}u_{iv}\right|^{3} &\leq \left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{2}} \\ &\leq (C)^{\frac{1}{2}}(C)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\ &= C < \infty \end{aligned}$$

and

$$\begin{aligned} E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\}\right|^{2} &\leq 2\left(E\left|u_{it}u_{is}\right|^{2} + \left|E\left[u_{it}u_{is}\right]\right|^{2}\right) \quad \text{(by Loève's } c_r \text{ inequality)} \\ &\leq 2\left(E\left|u_{it}u_{is}\right|^{2} + E\left|u_{it}u_{is}\right|^{2}\right) \quad \text{(by Jensen's inequality)} \\ &= 4E\left|u_{it}u_{is}\right|^{2} \\ &\leq 4\left[\left(E\left|u_{it}\right|^{4}\right)^{\frac{1}{4}}\left(E\left|u_{is}\right|^{4}\right)^{\frac{1}{4}}\right]^{2} \\ &\leq 4\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\right]^{2} \quad \text{(by Liapunov's inequality)} \\ &\leq 4\left(\sup_{t} E\left|u_{it}\right|^{6}\right)^{\frac{2}{3}} \\ &\leq 4\,(C)^{\frac{2}{3}} < \infty \quad \text{(by Assumption 3-3(b))} \end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s + \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|u_{ig}u_{iv}\right|^{3}$ and $E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\}\right|^{2}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\} u_{ig}u_{iv}\right]\right| \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{ig}u_{iv}\right|^3\right)^{\frac{1}{3}} \right\} \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} 2\left(2^{\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{\frac{1}{6}} \left(4C^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(C\right)^{\frac{1}{3}} \\
\leq & \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right) C^{\frac{2}{3}} a_1^{\frac{1}{6}} \leq C^* < \infty\right) \\
\leq & \frac{C^*}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & C^* \sum_{\rho_1=1}^{\infty} \left(\varrho_1+1\right) \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (99)
\end{aligned}
$$

Finally, consider the fourth term on the right-hand side of expression (96) above. For this term, we apply the result given in part (a) to obtain

$$
\begin{aligned}
\frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-s>0}}^{T-h} \left|E\left(u_{it}u_{is}\right)\right| \left|E\left(u_{ig}u_{iv}\right)\right| & \leq \left(\frac{1}{T_h} \sum_{\substack{t,s=p \\ t\leq s}}^{T-h} \left|E\left(u_{it}u_{is}\right)\right| \right) \left(\frac{1}{T_h} \sum_{\substack{g,v=p \\ g\leq v}}^{T-h} \left|E\left(u_{ig}u_{iv}\right)\right| \right) \\
& = O\left(1\right). \qquad (100)
\end{aligned}
$$

It follows from expressions (96)-(100) that

$$\begin{aligned}
&\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right| \\
\leq\ & \frac{1}{T_h^2}\sum_{\substack{t,s=p\\t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v\\v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v\\v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v\\v-s>0}}^{T-h}\left|E\left(u_{it}u_{is}\right)\right|\left|E\left(u_{ig}u_{iv}\right)\right| \\
=\ & O(1).\ \square
\end{aligned}$$

Lemma D-6: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, as $N_1, N_2, T \to \infty$,

$$\max_{i\in H}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T}\right).$$

Proof of Lemma D-6:

To proceed, we first show the boundedness of the quantity

$$\frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_2T_h^3}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^6$$

Note first that there exist a constant $C_1 > 1$ such that

$$\begin{aligned}
&\frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_2T_h^3}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^6 \\
\leq\ & \frac{C_1}{N_1N_2T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\t\leq s\leq g\leq \ell\leq v\leq w}}^{T-h}\left\{\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right|\right. \\
&\left.\times\left|E\left[\left(\gamma_k'\underline{F}_t\right)\left(\gamma_k'\underline{F}_s\right)\left(\gamma_k'\underline{F}_g\right)\left(\gamma_k'\underline{F}_\ell\right)\left(\gamma_k'\underline{F}_v\right)\left(\gamma_k'\underline{F}_w\right)\right]\right|\right\}
\end{aligned}$$

Next, note that, by repeated application of Hölder's inequality, we have by Assumption 3-5 and

Lemma C-5 that there exists a positive constant C such that

$$\begin{aligned}
& \left|E\left[\left(\gamma_k'\underline{F}_t\right)\left(\gamma_k'\underline{F}_s\right)\left(\gamma_k'\underline{F}_g\right)\left(\gamma_k'\underline{F}_\ell\right)\left(\gamma_k'\underline{F}_v\right)\left(\gamma_k'\underline{F}_w\right)\right]\right| \\
\leq\; & E\left[\left|\gamma_k'\underline{F}_t\right|\left|\gamma_k'\underline{F}_s\right|\left|\gamma_k'\underline{F}_g\right|\left|\gamma_k'\underline{F}_\ell\right|\left|\gamma_k'\underline{F}_v\right|\left|\gamma_k'\underline{F}_w\right|\right] \\
\leq\; & \|\gamma_k\|_2^6\, E\left[\|\underline{F}_t\|_2\,\|\underline{F}_s\|_2\,\left\|\underline{F}_g\right\|_2\,\|\underline{F}_\ell\|_2\,\|\underline{F}_v\|_2\,\|\underline{F}_w\|_2\right] \\
\leq\; & \|\gamma_k\|_2^6\left(E\left[\|\underline{F}_t\|_2^2\,\|\underline{F}_s\|_2^2\,\left\|\underline{F}_g\right\|_2^2\right]\right)^{\frac{1}{2}}\left(E\left[\|\underline{F}_\ell\|_2^2\,\|\underline{F}_v\|_2^2\,\|\underline{F}_w\|_2^2\right]\right)^{\frac{1}{2}} \\
\leq\; & \|\gamma_k\|_2^6\left(\left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{3}}\left(E\left[\|\underline{F}_s\|_2^3\,\left\|\underline{F}_g\right\|_2^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
& \times\left(\left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{3}}\left(E\left[\|\underline{F}_v\|_2^3\,\|\underline{F}_w\|_2^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
\leq\; & \|\gamma_k\|_2^6\left(\left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{3}}\left\{E\left[\|\underline{F}_s\|_2^6\right]\right\}^{\frac{1}{3}}\left\{E\left[\left\|\underline{F}_g\right\|_2^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
& \times\left(\left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{3}}\left\{E\left[\|\underline{F}_v\|_2^6\right]\right\}^{\frac{1}{3}}\left\{E\left[\|\underline{F}_w\|_2^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
\leq\; & \|\gamma_k\|_2^6\left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{6}}\left\{E\left[\|\underline{F}_s\|_2^6\right]\right\}^{\frac{1}{6}}\left\{E\left[\left\|\underline{F}_g\right\|_2^6\right]\right\}^{\frac{1}{6}} \\
& \times\left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{6}}\left\{E\left[\|\underline{F}_v\|_2^6\right]\right\}^{\frac{1}{6}}\left\{E\left[\|\underline{F}_w\|_2^6\right]\right\}^{\frac{1}{6}} \\
\leq\; & \|\gamma_k\|_2^6\sup_t E\left[\|\underline{F}_t\|_2^6\right] \\
\leq\; & C<\infty
\end{aligned}$$

Hence, we can write

$$
\begin{aligned}
& \frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_2 T_h^3} E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^6 \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ w-v\geq \max\{v-\ell,\ell-g\},w-v>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ w-v\geq \max\{v-\ell,\ell-g\},w-v>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right|
\end{aligned}
$$

$$+\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}u_{i\ell}u_{iv}u_{iw}\right]\right|$$

$$+\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right|$$

$$= \mathcal{TT}_1+\mathcal{TT}_2+\mathcal{TT}_3+\mathcal{TT}_4+\mathcal{TT}_5+\mathcal{TT}_6,\ \ (say).$$

Consider first $\mathcal{TT}_1$. Note that

$$\begin{aligned}
\mathcal{TT}_1 &= \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h} E\left[\left|u_{it}u_{is}u_{ig}^4\right|\right] \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left(E\left[\left|u_{it}u_{is}\right|^3\right]\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \text{ (by Hölder's inequality)} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left(\left[E\left\{\left|u_{it}\right|^6\right\}\right]^{\frac{1}{2}}\left[E\left\{\left|u_{is}\right|^6\right\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\quad \text{(by further application of Hölder's inequality)} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left\{\left|u_{is}\right|^6\right\}\right)^{\frac{1}{6}}\left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left(\sup_t E\left\{\left|u_{it}\right|^7\right\}\right)^{\frac{6}{7}} \\
&\quad \text{(using Liapunov's inequality)} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\overline{C}^{\frac{6}{7}} \text{ (by Assumption 3-3(b))} \\
&\leq C_1 C\overline{C}^{\frac{6}{7}}\frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \\
&= C_1 C\overline{C}^{\frac{6}{7}} = O(1) \qquad (101)
\end{aligned}$$

Next, consider $\mathcal{TT}_2$. For this term, note first that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned}
&\mathcal{TT}_2 \\
= \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left| E\left[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}\right]\right| \\
\leq \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{1-\frac{4}{5}} + 1\right) \left[a_1 \exp\{-a_2 (w-v)\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\
& \qquad \left. \times \left(E\left|u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \right\} \\
= \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{5}} + 1\right) \left[a_1 \exp\{-a_2 (w-v)\}\right]^{\frac{1}{30}} \right. \\
& \qquad \left. \times \left(E\left|u_{it} u_{is} u_{ig} u_{i\ell} u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$
\begin{aligned}
& E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}} \\
\leq\ & \left[E\left|u_{it}u_{is}u_{ig}\right|^{\frac{25}{12}}\right]^{\frac{3}{5}}\left[E\left|u_{i\ell}u_{iv}\right|^{\frac{25}{8}}\right]^{\frac{2}{5}} \\
\leq\ & \left[\left(E\left|u_{it}u_{is}\right|^{\frac{150}{47}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{i\ell}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq\ & \left[\left(\sqrt{E\left|u_{it}\right|^{\frac{300}{47}}}\sqrt{E\left|u_{is}\right|^{\frac{300}{47}}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{i\ell}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq\ & \left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{15}{72}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}}\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}} \\
=\ & \left[\left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}} \\
\leq\ & \left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{i\ell}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \\
& \text{(by Liapunov's inequality)} \\
\leq\ & \left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}} \quad \text{(by Assumption 3-3(b))} \\
=\ & \overline{C}^{\frac{25}{28}}
\end{aligned}
$$

By Liapunov's inequality and Assumption 3-3(b), we also obtain

$$\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \leq \left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}.$$

Moreover, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$.. Using these notations and the boundedness of $E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}}$

as shown above, we can further write

$$
\begin{aligned}
& \mathcal{TT}_2 \\
\leq \; & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \right. \\
& \left. \times \left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \right\} \\
\leq \; & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \left(\overline{C}^{\frac{25}{28}}\right)^{\frac{4}{5}} \overline{C}^{\frac{1}{7}} \\
\leq \; & \frac{C_1 C \overline{C}^{\frac{6}{7}}}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq \; & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
& \left(\text{for some constant } C_1^* \text{ such that } 2\left(2^{\frac{1}{5}}+1\right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{30}} \leq C_1^* < \infty\right) \\
\leq \; & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} \sum_{\rho_1=0}^{\rho_3} \sum_{\rho_2=0}^{\rho_3} \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
\leq \; & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} \left(\rho_3+1\right)^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
= \; & C_1^* \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \left[\sum_{\rho_3=1}^{\infty} \rho_{3\backslash}^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} + 2\sum_{\rho_3=1}^{\infty} \rho_{3\backslash} \exp\left\{-\frac{a_2}{30}\rho_3\right\} + \sum_{\rho_3=1}^{\infty} \exp\left\{-\frac{a_2}{30}\rho_3\right\}\right] \\
\leq \; & C_1^* \overline{\overline{C}}_1 \qquad (102)
\end{aligned}
$$

for some positive constant

$$
\overline{\overline{C}}_1 \geq \sum_{\rho_3=1}^{\infty} \rho_{3\backslash}^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} + 2\sum_{\rho_3=1}^{\infty} \rho_{3\backslash} \exp\left\{-\frac{a_2}{30}\rho_3\right\} + \sum_{\rho_3=1}^{\infty} \exp\left\{-\frac{a_2}{30}\rho_3\right\}.
$$

which exists in light of Lemma C-1.

Now, consider $\mathcal{TT}_3$. Here, we apply Lemma C-3 with $p = 3/2$ and $r = 7/2$ to obtain

$$\begin{aligned}
\mathcal{TT}_3 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left| E\left[\left\{ u_{it} u_{is} u_{ig} u_{i\ell} - E\left(u_{it} u_{is} u_{ig} u_{i\ell} \right) \right\} u_{iv} u_{iw} \right] \right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left\{ 2\left(2^{1-\frac{2}{3}} + 1 \right) \left[a_1 \exp\left\{ -a_2 \left(v - \ell \right) \right\} \right]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\qquad \left. \times \left(E\left| \left\{ u_{it} u_{is} u_{ig} u_{i\ell} - E\left(u_{it} u_{is} u_{ig} u_{i\ell} \right) \right\} \right|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(E\left| u_{iv} u_{iw} \right|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{3}} + 1 \right) \left[a_1 \exp\left\{ -a_2 \left(v - \ell \right) \right\} \right]^{\frac{1}{21}} \right. \\
&\qquad \left. \times \left(E\left| \left\{ u_{it} u_{is} u_{ig} u_{i\ell} - E\left(u_{it} u_{is} u_{ig} u_{i\ell} \right) \right\} \right|^{\frac{3}{2}} \right)^{\frac{2}{3}} \left(E\left| u_{iv} u_{iw} \right|^{\frac{7}{2}} \right)^{\frac{2}{7}} \right\}
\end{aligned}$$

Next, observe that by applying of Hölder's inequality, we have

$$\begin{aligned}
E\left| u_{iv} u_{iw} \right|^{\frac{7}{2}} &\leq \left(E\left| u_{iv} \right|^7 \right)^{\frac{1}{2}} \left(E\left| u_{iw} \right|^7 \right)^{\frac{1}{2}} \\
&\leq \left(\overline{C} \right)^{\frac{1}{2}} \left(\overline{C} \right)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\
&= \overline{C} < \infty,
\end{aligned}$$

and

$$
\begin{aligned}
E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}}\left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}+\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}\right]\right|^{\frac{3}{2}}\right) \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq 2^{\frac{1}{2}}\left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}+E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}\right) \\
&\quad \text{(by Jensen's inequality)} \\
&\leq 2^{\frac{3}{2}}E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}}\left(E\left|u_{it}u_{is}\right|^{3}\right)^{\frac{1}{2}}\left(E\left|u_{ig}u_{i\ell}\right|^{3}\right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}}\left(\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{i\ell}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{3}{2}} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq 2^{\frac{3}{2}}\left[\left(\sup_{t} E\left|u_{it}\right|^{7}\right)^{\frac{4}{7}}\right]^{\frac{3}{2}} \\
&= 2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}
$$

Again, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}$ as shown above, we can further write

$$
\begin{aligned}
&\mathcal{TT}_3 \\
= \;& \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq \;& \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left\{2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{\frac{1}{21}}\right. \\
& \left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}}\right\} \\
\leq \;& \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h} 2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{\frac{1}{21}}\left(2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}}\right)^{\frac{2}{3}}\left(\overline{C}\right)^{\frac{2}{7}} \\
\leq \;& \frac{C_2^*}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
& \left(\text{for some constant } C_2^* \text{ such that } 4\left(2^{\frac{1}{3}}+1\right)C_1 C\overline{C}^{\frac{6}{7}}a_1^{\frac{1}{21}}\leq C_2^*<\infty\right) \\
\leq \;& \frac{C_2^*}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{g=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\sum_{\varrho_3=0}^{\varrho_2}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
= \;& C_2^*\frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
= \;& C_2^*\left[\sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}\right] \\
\leq \;& C_2^*\overline{\overline{C}}_2 \qquad (103)
\end{aligned}
$$

for some positive constant

$$\overline{\overline{C}}_2\geq\sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}$$

which exists in light of Lemma C-1.

Turning our attention to the term $\mathcal{TT}_4$, note that, from the upper bounds given in the proofs of parts (a) and (c) of Lemma D-5, it is clear that there exists a positive constant C^{**} such that, for all i and for all T sufficiently large,

$$\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left(u_{iv}u_{iw}\right)\right| \leq C_1^{**}$$

and

$$\frac{1}{T_h^2}\sum_{\substack{t,s,g,\ell=p \\ t\leq s\leq g\leq \ell}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right| \leq C_1^{**}$$

from which it follows that

$$\begin{aligned}
\mathcal{TT}_4 &= \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1 C}{N_1 N_2}\sum_{k\in H^c}\sum_{i\in H}\left(\frac{1}{T_h^2}\sum_{\substack{t,s,g,\ell=p \\ t\leq s\leq g\leq \ell}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\right)\left(\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1 C}{N_1 N_2}\sum_{k\in H^c}\sum_{i\in H}\left(C_1^{**}\right)^2 \\
&= C_1 C\left(C_1^{**}\right)^2\frac{N_1 N_2}{N_1 N_2} \\
&= C_1 C\left(C_1^{**}\right)^2 \qquad (104)
\end{aligned}$$

Consider now $\mathcal{TT}_5$. In this case, we apply Lemma C-3 with $p = 2$ and $r = 9/4$ to obtain

$$\begin{aligned}
\mathcal{TT}_5 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left| E\left[\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\} u_{i\ell}u_{iv}u_{iw}\right]\right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}} + 1\right)\left[a_1 \exp\left\{-a_2\left(\ell - g\right)\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}} \right\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left\{ 2\left(2^{\frac{1}{2}} + 1\right)\left[a_1 \exp\left\{-a_2\left(\ell - g\right)\right\}\right]^{\frac{1}{18}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
& E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[E\left|u_{iv}u_{iw}\right|^{\frac{63}{19}}\right]^{\frac{19}{28}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\right]^{\frac{19}{28}} \\
=\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \\
=\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\right]^{\frac{9}{4}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^7\right)^{\frac{1}{7}} \left(E\left|u_{iw}\right|^7\right)^{\frac{1}{7}}\right]^{\frac{9}{4}} \quad \text{(by Liapunov's inequality)} \\
\leq\ & \left(\sup_t E\left|u_{it}\right|^7\right)^{\frac{27}{28}} \\
\leq\ & \overline{C}^{\frac{27}{28}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

and

$$
\begin{aligned}
E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^{2} &\leq 2\left(E\left|u_{it}u_{is}u_{ig}\right|^{2}+\left|E\left[u_{it}u_{is}u_{ig}\right]\right|^{2}\right) \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq 2\left(E\left|u_{it}u_{is}u_{ig}\right|^{2}+E\left|u_{it}u_{is}u_{ig}\right|^{2}\right) \\
&\quad \text{(by Jensen's inequality)} \\
&\leq 4E\left|u_{it}u_{is}u_{ig}\right|^{2} \\
&\leq 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(E\left|u_{is}u_{ig}\right|^{3}\right)^{\frac{2}{3}} \\
&\leq 4\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}}\left(\sqrt{E\left|u_{is}\right|^{6}}\sqrt{E\left|u_{ig}\right|^{6}}\right)^{\frac{2}{3}} \\
&= 4\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\right]^{2}\left[\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\right]^{2} \\
&\leq 4\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{2}\left[\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\right]^{2} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq 4\left[\left(\sup_{t} E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\right]^{6} \\
&\leq 4\overline{C}^{\frac{6}{7}} \text{ (by Assumption 3-3(b))}
\end{aligned}
$$

Define again $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2$ as shown above, we can further write

$$
\begin{aligned}
& \mathcal{TT}_5 \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq\max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left\{2\left(2^{\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(\ell-g\right)\right\}\right]^{\frac{1}{18}}\right. \\
& \left.\times\left(E\left|\left\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}}\left(E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}}\right\} \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq\max\{w-v,v-\ell\},\ell-g>0}}^{T-h}2\left(2^{\frac{1}{2}}+1\right)\left[a_1\exp\left\{-a_2\left(\ell-g\right)\right\}\right]^{\frac{1}{18}}\left(4\overline{C}^{\frac{6}{7}}\right)^{\frac{1}{2}}\left(\overline{C}^{\frac{27}{28}}\right)^{\frac{4}{9}} \\
\leq\ & \frac{C_3^*}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq\max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
& \left(\text{for some constant } C_3^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)C_1 C\overline{C}^{\frac{6}{7}}a_1^{\frac{1}{18}}\leq C_3^*<\infty\right) \\
\leq\ & \frac{C_3^*}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{g=p}^{T-h}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1}\sum_{\varrho_3=0}^{\varrho_1}\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq\ & \frac{C_3^* N_1 N_2 T_h^3}{N_1 N_2 T_h^3}\sum_{\varrho_1=1}^{\infty}\left(\varrho_1+1\right)^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq\ & C_3^*\left[\sum_{\varrho_1=1}^{\infty}\varrho_1^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+2\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{18}\varrho_1\right\}\right] \\
\leq\ & C_3^*\overline{\overline{C}}_3 \qquad (105)
\end{aligned}
$$

for some positive constant

$$\overline{\overline{C}}_3\geq\sum_{\varrho_1=1}^{\infty}\varrho_1^2\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+2\sum_{\varrho_1=1}^{\infty}\varrho_1\exp\left\{-\frac{a_2}{18}\varrho_1\right\}+\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{18}\varrho_1\right\}$$

which exists in light of Lemma C-1.

Finally, consider $\mathcal{TT}_6$. Note that, from the upper bounds given in the proofs of part (b) of Lemma D-5, it is clear that there exists a positive constant C_2^{**} such that, for all i and for all T

sufficiently large,

$$\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\leq C_2^{**}$$

and

$$\frac{1}{T_h}\sum_{\substack{\ell,v,w=p\\ \ell\leq v\leq w}}^{T-h}\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right|\leq C_2^{**}$$

from which it follows that

$$\begin{aligned}
\mathcal{TT}_6 &= \frac{C_1C}{N_1N_2T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq\max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1C}{N_1N_2T_h}\sum_{k\in H^c}\sum_{i\in H}\left(\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\right)\left(\frac{1}{T_h}\sum_{\substack{\ell,v,w=p\\ \ell\leq v\leq w}}^{T-h}\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1C}{N_1N_2T_h}\sum_{k\in H^c}\sum_{i\in H}\left(C^{**}\right)^2 \\
&= C_1C\left(C_2^{**}\right)^2\frac{N_1N_2}{N_1N_2T_h} \\
&= \frac{C_1C\left(C_2^{**}\right)^2}{T_h}=O\left(\frac{1}{T}\right). \qquad (106)
\end{aligned}$$

It follows from expressions (101)-(106) that, for all N_1, N_2, and T sufficiently large,

$$\begin{aligned}
&\frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_1^3T_h^3}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_tu_{i,t}\right)^6 \\
&\leq \mathcal{TT}_1+\mathcal{TT}_2+\mathcal{TT}_3+\mathcal{TT}_4+\mathcal{TT}_5+\mathcal{TT}_6 \\
&\leq C_1C\overline{C}^{\frac{6}{7}}+C_1^*\overline{\overline{C}}_1+C_2^*\overline{\overline{C}}_2+C_1C\left(C_1^{**}\right)^2+C_3^*\overline{\overline{C}}_3+\frac{C_1C\left(C_2^{**}\right)^2}{T_h} \\
&\leq \widetilde{C}
\end{aligned}$$

for some positive constant $\widetilde{C}$ such that

$$\widetilde{C}\geq C_1C\overline{C}^{\frac{6}{7}}+C_1^*\overline{\overline{C}}_1+C_2^*\overline{\overline{C}}_2+C_1C\left(C_1^{**}\right)^2+C_3^*\overline{\overline{C}}_3+\frac{C_1C\left(C_2^{**}\right)^2}{T_h}.$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \left(\widetilde{C}/\epsilon\right)^{\frac{1}{3}}$, and note that

$$
\begin{aligned}
& \Pr\left\{\frac{N_1 T_h}{N_2^{\frac{1}{3}}} \max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 \geq C_\epsilon\right\} \\
= & \Pr\left\{\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^2 \geq C_\epsilon\right\} \\
= & \Pr\left\{\max_{i\in H} \left[\frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^2\right]^3 \geq C_\epsilon^3\right\} \\
\leq & \Pr\left\{\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \geq C_\epsilon^3\right\} \text{ (by Jensen's inequality)} \\
\leq & \Pr\left\{\frac{1}{N_1} \sum_{k\in H^c} \sum_{i\in H} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \geq C_\epsilon^3\right\} \\
\leq & \frac{\epsilon}{\widetilde{C}} \frac{1}{N_1} \sum_{k\in H^c} \sum_{i\in H} \frac{1}{N_2 T_h^3} E\left(\sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \\
\leq & \frac{\epsilon}{\widetilde{C}} \widetilde{C} \\
= & \epsilon
\end{aligned}
$$

This shows that

$$
\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T_h}\right) = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T}\right). \ \square
$$

Before stating the next lemma, we first introduce some more notations. Let $\mathbb{S}_{i,T}^+$ denote either the statistic $\sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}|$ or the statistic $\max_{1\leq \ell \leq d} |S_{i,\ell,T}|$, and define

$$\widehat{H^c} = \left\{ i \in \{1, ..., N\} : \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\}, \tag{107}$$

$$\widehat{H} = \left\{ i \in \{1, ..., N\} : \mathbb{S}_{i,T}^{+} < \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\}, \tag{108}$$

$$\widehat{N}_1 = \#\left(\widehat{H^c}\right), \text{ i.e., the cardinality of the set } \widehat{H^c}, \tag{109}$$

$$\Gamma\left(\widehat{H^c}\right) = \begin{pmatrix} \gamma_1\left(\widehat{H^c}\right)' \\ \gamma_2\left(\widehat{H^c}\right)' \\ \vdots \\ \gamma_N\left(\widehat{H^c}\right)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\left\{1 \in \widehat{H^c}\right\}\gamma_1' \\ \mathbb{I}\left\{2 \in \widehat{H^c}\right\}\gamma_2' \\ \vdots \\ \mathbb{I}\left\{N \in \widehat{H^c}\right\}\gamma_N' \end{pmatrix}, \text{ and}$$

$$U\left(\widehat{H^c}\right) = \begin{pmatrix} u_{1\cdot}\left(\widehat{H^c}\right)' \\ u_{2\cdot}\left(\widehat{H^c}\right)' \\ \vdots \\ u_{N\cdot}\left(\widehat{H^c}\right)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\left\{1 \in \widehat{H^c}\right\}u_{1\cdot}' \\ \mathbb{I}\left\{2 \in \widehat{H^c}\right\}u_{2\cdot}' \\ \vdots \\ \mathbb{I}\left\{N \in \widehat{H^c}\right\}u_{N\cdot}' \end{pmatrix}, \tag{110}$$

where $u_{i\cdot} = (u_{i,p}, u_{i,p+1}, ..., u_{i,T-h})'$ for $i = 1, ..., N$.

Lemma D-7: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11 hold. Then, as $N_1, N_2, T \to \infty$, the following statements are true.

(a)

$$\sum_{i\in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} = O_p(\varphi)$$

(b)

$$\sum_{i\in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}\right).$$

(c)

$$\frac{1}{N_1} \sum_{i\in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 = O_p\left(\frac{1}{T}\right)$$

Proof of Lemma D-7:

To show part (a), let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}|$ or the statistic $\max_{1\leq \ell \leq d} |S_{i,\ell,T}|$. Following arguments similar to that given in the proof of part (a) of Theorem 2, we see that there exists a constant $C > 2d$ such that

$$\begin{aligned}\sum_{i\in H} E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] &= \sum_{i\in H} \Pr\left(i \in \widehat{H^c}\right) \\ &= \sum_{i\in H} \Pr\left\{\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\} \\ &\leq C\frac{N_2\varphi}{N} \\ &\leq C\varphi\end{aligned}$$

for all N_1, N_2, and T sufficiently large. Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}\Pr\left\{\frac{1}{\varphi}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\} \geq C_\epsilon\right\} &\leq \frac{1}{C_\epsilon\varphi}\sum_{i\in H} E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad \text{(by Markov's inequality)} \\ &\leq \frac{\epsilon}{C\varphi}C\varphi \\ &= \epsilon\end{aligned}$$

which shows that

$$\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\} = O_p(\varphi)$$

Next, to show part (b), we combine the result given in part (a) of this lemma with the result of Lemma D-6 to obtain

$$\begin{aligned}&\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 \\ \leq\ &\max_{i\in H}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2\left[\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad \text{(by Hölder's inequality)} \\ =\ &O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T}\right)O_p(\varphi) \\ =\ &O_p\left(\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}\right).\end{aligned}$$

Finally, to show part (c), note first that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 \leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{T_h}\right)^2$$

Moreover, write

$$
\begin{aligned}
0 &\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c} E\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c} E\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2 \\
&= \frac{1}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c} E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2 \\
&= \frac{1}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h} E\left\{\gamma_k'\underline{F}_s u_{i,s}u_{i,t}\underline{F}_t'\gamma_k\right\} \\
&= \frac{1}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k' E_F\left[\underline{F}_t E\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E_F\left[\gamma_k'\underline{F}_t E\left(u_{i,t}u_{i,t+m}\right)\underline{F}_{t+m}'\gamma_k\right] \\
&= \frac{1}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k' E_F\left[\underline{F}_t E\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left(u_{i,t}u_{i,t+m}\right) E_F\left[\gamma_k'\underline{F}_t\underline{F}_{t+m}'\gamma_k\right] \\
&\leq \frac{1}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k' E_F\left[\underline{F}_t E\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad + \frac{2}{N_1^2 T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} \left|E\left(u_{i,t}u_{i,t+m}\right)\right|\left|\gamma_k' E_F\left[\underline{F}_t\underline{F}_{t+m}'\right]\gamma_k\right|
\end{aligned}
$$

Note that by Assumption 3-3(c), $\{u_{i,t}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, applying Lemma C-3 with $p = 3$ and $r = 3$ as well as Assumptions 3-3(b) and 3-5 and

Lemma C-5; we get

$$
\begin{aligned}
& \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} E\left[\left(\sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^2\right] \\
\leq\ & \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma_k' E_F\left[\underline{F}_t E\left(u_{i,t}^2\right) \underline{F}_t'\right] \gamma_k \\
& + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left|E\left(u_{i,t} u_{i,t+m}\right)\right| \left|\gamma_k' E_F\left[\underline{F}_t \underline{F}_{t+m}'\right] \gamma_k\right| \\
\leq\ & \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} \gamma_k' E_F\left[\underline{F}_t E\left(u_{i,t}^2\right) \underline{F}_t'\right] \gamma_k \\
& + \frac{2}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left|E\left(u_{i,t} u_{i,t+m}\right)\right| E\left|\gamma_k' \underline{F}_t \underline{F}_{t+m}' \gamma_k\right| \\
\leq\ & \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} E\left(u_{i,t}^2\right) \|\gamma_k\|_2^2 E\left[\|\underline{F}_t\|_2^2\right] \\
& + \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} 2\left(2^{\frac{2}{3}}+1\right) 2 \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{\alpha_m^{\frac{1}{3}} \left(E\left|u_{i,t}\right|^3\right)^{\frac{1}{3}} \left(E\left|u_{i,t+m}\right|^3\right)^{\frac{1}{3}}\right. \\
& \qquad\qquad \left. \times \sqrt{\gamma_k' E\left[\underline{F}_t \underline{F}_t'\right] \gamma_k} \sqrt{\gamma_k' E\left[\underline{F}_{t+m} \underline{F}_{t+m}'\right] \gamma_k}\right\} \\
\leq\ & \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h} E\left(u_{i,t}^2\right) \|\gamma_k\|_2^2 E\left[\|\underline{F}_t\|_2^2\right] \\
& + \frac{1}{N_1^2 T_h^2} \sum_{i \in H^c} \sum_{k \in H^c} 4\left(2^{\frac{2}{3}}+1\right) \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{\alpha_m^{\frac{1}{3}} \left(E\left|u_{i,t}\right|^3\right)^{\frac{1}{3}} \left(E\left|u_{i,t+m}\right|^3\right)^{\frac{1}{3}}\right. \\
& \qquad\qquad \left. \times \|\gamma_k\|_2^2 \sqrt{E\|\underline{F}_t\|_2^2} \sqrt{E\left\|\underline{F}_{t+m}\right\|_2^2}\right\} \\
\leq\ & \frac{C_1}{T_h} + C_2 \frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} a_1^{\frac{1}{3}} \exp\left\{-\frac{a_2}{3} m\right\} \\
\leq\ & \frac{C_1}{T_h} + C_2 a_1^{\frac{1}{3}} \frac{1}{T_h} \sum_{m=1}^{\infty} \exp\left\{-\frac{a_2}{3} m\right\} \\
\leq\ & \frac{\overline{C}}{T_h}
\end{aligned}
$$

for some positive constant

$$
\overline{C} \geq C_1 + C_2 a_1^{\frac{1}{3}} \sum_{m=1}^{\infty} \exp\left\{-\frac{a_2}{3} m\right\}
$$

which exists in light of Lemma C-1. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and note that

$$\begin{aligned}
& \Pr\left\{\frac{T_h}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 \geq C_\epsilon\right\} \\
\leq\ & \Pr\left\{\frac{T_h}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{T_h}\right)^2 \geq C_\epsilon\right\} \\
\leq\ & \frac{T_h}{C_\epsilon}\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left[\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2\right] \\
\leq\ & \frac{\epsilon}{\overline{C}}T_h\frac{\overline{C}}{T_h} \\
=\ & \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 = O_p\left(\frac{1}{T_h}\right) = O_p\left(\frac{1}{T}\right). \ \square$$

Lemma D-8: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)
$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p(1).$$

(b)
$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) = o_p(1).$$

Proof of Lemma D-8:

To show part (a), note first that

$$\begin{aligned}
\frac{1}{N_1}\sum_{i\in H^c} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] &\leq \frac{1}{N_1}\sum_{i\in H^c} E\left[\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c} E\left[\frac{1}{T_h}\sum_{t=p}^{T-h} u_{i,t}^2\right] \\
&\leq \frac{1}{N_1}\sum_{i\in H^c} \frac{1}{T_h}\sum_{t=p}^{T-h} \sup_{i,t} E\left[u_{i,t}^2\right] \\
&\leq \frac{1}{N_1}\sum_{i\in H^c} \frac{1}{T_h}\sum_{t=p}^{T-h} C \\
&= C
\end{aligned}$$

for some positive constant $C \geq \sup_{i,t} E\left[u_{i,t}^2\right]$ which exists in light of Assumption 3-3(b). Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}
&\Pr\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\geq C_\epsilon\right\} \\
\leq\ & \frac{1}{C_\epsilon}\frac{1}{N_1}\sum_{i\in H^c} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \quad \text{(by Markov's inequality)} \\
\leq\ & \frac{\epsilon}{C}C \\
=\ & \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p(1).$$

Next, to show part (b), note that

$$
\begin{aligned}
& \frac{1}{N_1}\sum_{i\in H}E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \\
\leq \ & \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{i\in\widehat{H^c}\right\}\right)^{\frac{5}{7}}\left(E\left[\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)^{\frac{7}{2}}\right]\right)^{\frac{2}{7}} \text{ (by Hölder's inequality)} \\
= \ & \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{i\in\widehat{H^c}\right\}\right)^{\frac{5}{7}}\left(E\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2\right)^{\frac{7}{2}}\right]\right)^{\frac{2}{7}} \\
\leq \ & \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\sup_{i,t}E\left[\left|u_{i,t}\right|^7\right]\right)^{\frac{2}{7}} \\
\leq \ & C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}}
\end{aligned}
$$

for some positive constant $C_1 \geq \sup_{i,t} E\left[\left|u_{i,t}\right|^7\right]$ which exists in light of Assumption 3-3(b). Now, let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the statistic $\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$; and, following arguments similar to that given in the proof of part (a) of Theorem 2, we see that, for any $i\in H$, there exists a constant $C_2 > 2d$ such that

$$
\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\leq C_2\frac{\varphi}{N}
$$

for all N_1, N_2, and T sufficiently large, from which it follows that

$$
\begin{aligned}
\frac{1}{N_1}\sum_{i\in H}E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] & \leq C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}} \\
& \leq C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}C_2^{\frac{5}{7}}\left(\frac{\varphi}{N}\right)^{\frac{5}{7}} \\
& = C_1^{\frac{2}{7}}C_2^{\frac{5}{7}}\frac{N_2\varphi^{\frac{5}{7}}}{N_1 N^{\frac{5}{7}}} \\
& \leq C_3\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}
\end{aligned}
$$

for all N_1, N_2, and T sufficiently large and for some positive constant $C_3 \geq C_1^{\frac{2}{7}}C_2^{\frac{5}{7}}$. Hence, for any

$\epsilon > 0$, set $C_\epsilon = C_3/\epsilon$, and note that

$$\begin{aligned}
& \Pr\left\{ \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{i \cdot}}{T_h} \right) \geq C_\epsilon \right\} \\
\leq & \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{1}{C_\epsilon} \frac{1}{N_1} \sum_{i \in H} E\left[\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{i \cdot}}{T_h} \right) \right] \quad \text{(by Markov's inequality)} \\
\leq & \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{\epsilon}{C_3} C_3 \frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \\
= & \ \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{i \cdot}}{T_h} \right) = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1). \ \square$$

Lemma D-9: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)

$$\mathcal{T}_1 = \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 = O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) = o_p(1).$$

(b)

$$\mathcal{T}_2 = \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(c)

$$\mathcal{T}_3 = \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(d)

$$\mathcal{T}_4 = \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{4}{7}}\varphi^{\frac{10}{7}}}{N_1^2} \right) = o_p(1).$$

Proof of Lemma D-9:

To show part (a), note that

$$
\begin{aligned}
0 &\leq \mathcal{T}_1 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 \\
&\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left(\frac{u_{i \cdot}' u_{k \cdot}}{T_h} \right)^2 \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} u_{i,t} u_{k,t} u_{i,s} u_{k,s} \\
&= \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} u_{i,t}^2 u_{k,t}^2 \\
&\quad + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}
\end{aligned}
$$

From the non-negativity of $\mathcal{T}_1$, we get

$$
\begin{aligned}
E\left|\mathcal{T}_1\right| &= E\left[\mathcal{T}_1\right] \\
&\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E\left[u_{i,t}^2 u_{k,t}^2\right] \\
&\quad + \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left[u_{i,t} u_{k,t} u_{i,t+m} u_{k,t+m}\right]
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} E\left[u_{i,t}^2 u_{k,t}^2\right] &\leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{1}{T_h^2} \sum_{t=p}^{T-h} \sqrt{E\left[u_{i,t}^4\right]} \sqrt{E\left[u_{k,t}^4\right]} \\
&\leq \left(\sup_{i,t} E\left[u_{i,t}^4\right] \right) \frac{1}{T_h} \\
&\leq \frac{C_1}{T_h}
\end{aligned}
$$

for some positive constant $C_1 \geq \sup_{i,t} E\left[u_{i,t}^4\right]$ which exists in light of Assumption 3-3(b). Moreover,

$$\begin{aligned}
& \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left[u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}\right] \\
= & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left[\left(u_{i,t}u_{k,t} - E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m} - E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right] \\
& + \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left[u_{i,t}u_{k,t}\right]E\left[u_{i,t+m}u_{k,t+m}\right] \\
\leq & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} \left|E\left[\left(u_{i,t}u_{k,t} - E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m} - E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right]\right| \\
& + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right|
\end{aligned}$$

Consider the first term on the right-hand side above. Note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right) \leq a_1 \exp\left\{-a_2 m\right\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\left\{-a_2 m\right\} \text{ for every } i.$$

Hence, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$
\begin{aligned}
& \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left[\left(u_{i,t} u_{k,t} - E\left[u_{i,t} u_{k,t} \right] \right) \left(u_{i,t+m} u_{k,t+m} - E\left[u_{i,t+m} u_{k,t+m} \right] \right) \right] \right| \\
\leq \; & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} 2 \left(\sqrt{2} + 1 \right) \alpha_m^{\frac{1}{6}} \sqrt{E\left[u_{i,t}^2 u_{k,t}^2 \right]} \left(E\left| u_{i,t+m} u_{k,t+m} \right|^3 \right)^{\frac{1}{3}} \\
\leq \; & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{4\left(\sqrt{2} + 1 \right)}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} a_1^{\frac{1}{6}} \exp\left\{ -\frac{a_2}{6} m \right\} \sqrt{E\left[u_{i,t}^2 u_{k,t}^2 \right]} \left(E\left| u_{i,t+m} u_{k,t+m} \right|^3 \right)^{\frac{1}{3}} \\
\leq \; & \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \frac{4 a_1^{\frac{1}{6}} \left(\sqrt{2} + 1 \right) \left(E\left[u_{i,t}^4 \right] \right)^{\frac{1}{4}} \left(E\left[u_{k,t}^4 \right] \right)^{\frac{1}{4}} \left(E\left[u_{i,t+m}^6 \right] E\left[u_{k,t+m}^6 \right] \right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
\leq \; & \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \frac{4 a_1^{\frac{1}{6}} \left(\sqrt{2} + 1 \right) \left(E\left[u_{i,t}^6 \right] \right)^{\frac{1}{6}} \left(E\left[u_{k,t}^6 \right] \right)^{\frac{1}{6}} \left(E\left[u_{i,t+m}^6 \right] E\left[u_{k,t+m}^6 \right] \right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
\leq \; & \frac{4\overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} \left(\sup_{i,t} E\left[u_{i,t}^6 \right] \right)^{\frac{2}{3}}}{T_h} \\
& \left(\text{for some positive constant } \overline{C} \text{ such that } \overline{C} \; \geq \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \right) \\
\leq \; & \frac{4\overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} C^{\frac{2}{3}}}{T_h} \\
& \left(\text{by Assumption 3-3(b), there exists positive constant } C \text{ such that } \sup_{i,t} E\left| u_{i,t} \right|^6 \leq C < \infty \right) \\
\leq \; & \frac{C_2}{T_h} \quad \left(\text{setting } C_2 \geq 4\overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} C^{\frac{2}{3}} \right) \\
= \; & O\left(\frac{1}{T} \right)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\sqrt{E\left[u_{i,t+m}^2\right]}\sqrt{E\left[u_{k,t+m}^2\right]} \\
\leq\ & \left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right| \\
\leq\ & \frac{2}{N_1}\left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\sup_t\left(\frac{1}{N_1}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\right) \\
\leq\ & \frac{C_3}{N_1}.
\end{aligned}
$$

for some positive constant C_3 such that

$$
2\left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\sup_t\left(\frac{1}{N_1}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\right)\leq C_3<\infty
$$

which exists in light of Assumptions 3-3(b) and 3-3(d). It follows from these results that

$$
\begin{aligned}
& E\left|\mathcal{T}_1\right| \\
=\ & E\left[\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right] \\
\leq\ & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right]+\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}\right] \\
\leq\ & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right] \\
& +\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left[\left(u_{i,t}u_{k,t}-E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m}-E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right]\right| \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right| \\
\leq\ & \frac{C_1}{T_h}+\frac{C_2}{T_h}+\frac{C_3}{N_1} \\
\leq\ & \frac{\overline{C}}{\min\left\{N_1,T_h\right\}}
\end{aligned}
$$

for some positive constant $\overline{C} \geq C_1 + C_2 + C_3$. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and applying Markov's inequality, we obtain

$$\begin{aligned}
& \Pr\left(\min\left\{N_1, T_h\right\}\left|\mathcal{T}_1\right| \geq C_\epsilon\right) \\
= & \Pr\left(\min\left\{N_1, T_h\right\}\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right| \geq C_\epsilon\right) \\
\leq & \frac{\min\left\{N_1, T_h\right\}}{C_\epsilon} E\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right\} \\
\leq & \min\left\{N_1, T_h\right\}\frac{\epsilon}{\overline{C}}\frac{\overline{C}}{\min\left\{N_1, T_h\right\}} \\
= & \epsilon
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= O_p\left(\frac{1}{\min\left\{N_1, T_h\right\}}\right) = O_p\left(\frac{1}{\min\left\{N_1, T\right\}}\right) = O_p\left(\max\left\{\frac{1}{N_1}, \frac{1}{T}\right\}\right).
\end{aligned}$$

Next, to show part (b), we apply parts (a) and (b) of Lemma D-8 to obtain

$$\begin{aligned}
\mathcal{T}_2 &= \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right) \text{ (by CS inequality)} \\
&= \left[\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right]\left[\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right)\right] \\
&= O_p(1)\, O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) \\
&= O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) = o_p(1)
\end{aligned}$$

Part (c) can be shown in the same way as part (b) above. Hence, to avoid redundancy, we do not give an explicit proof here.

Finally, to show part (d), we apply part (b) of Lemma D-8 to obtain

$$\begin{aligned}
\mathcal{T}_4 &= \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2}\sum_{i\in H}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right) \text{ (by CS inequality)} \\
&= \left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right]^2 \\
&= O_p\left(\frac{N^{\frac{4}{7}}\varphi^{\frac{10}{7}}}{N_1^2}\right) = o_p(1)\,. \ \square
\end{aligned}$$

Lemma D-10: Let

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} \tag{111}$$

where $T_0 = T - p + 1$, where $\widehat{H^c}$ and $\widehat{N}_1$ are as defined, respectively, in expressions (107) and (109) above, and where

$$\underset{T_0\times N}{Z}\left(\widehat{H^c}\right) = \left[\begin{array}{cccc} Z_{1\cdot}\mathbb{I}\left\{1\in\widehat{H^c}\right\} & Z_{2\cdot}\mathbb{I}\left\{2\in\widehat{H^c}\right\} & \cdots & Z_{N\cdot}\mathbb{I}\left\{N\in\widehat{H^c}\right\}\end{array}\right] \tag{112}$$

with $Z_{i\cdot} = (Z_{i,p}, Z_{i,p+1}, ..., Z_{i,T})'$ for $i = 1, ..., N$. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10, and 3-11* hold.

Under the assumed conditions,

$$\left\|\widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_2 = o_p(1) \text{ as } N_1,\ N_2,\ T\to\infty,$$

where

$$M_{FF} = \frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{F}_t\underline{F}_t'\right].$$

Proof of Lemma D-10:

To proceed, note that we can write

$$Z\left(\widehat{H^c}\right) = \underline{F}\Gamma\left(\widehat{H^c}\right)' + U\left(\widehat{H^c}\right),$$

so that

$$
\begin{aligned}
& \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
= \quad & \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
= \quad & \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
= \quad & \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \left\{ \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} + \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right. \\
& \qquad \left. + \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} + \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\
= \quad & -\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right) \frac{\Gamma M_{FF} \Gamma'}{N_1} + \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \left\{ \frac{1}{N_1} \Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}' \underline{F}}{T_0} - M_{FF}\right] \Gamma\left(\widehat{H^c}\right) \right. \\
& + \frac{1}{N_1}\left(\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)' - \Gamma M_{FF} \Gamma'\right) + \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \\
& \left. + \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} + \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\}
\end{aligned} \tag{113}
$$

where M_{FF} is as defined in (88), where $\Gamma\left(\widehat{H^c}\right)$ and $U\left(\widehat{H^c}\right)$ are as defined in (110), and where $Z\left(\widehat{H^c}\right)$ is as defined in expression (112).

Consider first the term $-\left[\left(\widehat{N}_1 - N_1\right)/\widehat{N}_1\right] \left(\Gamma M_{FF} \Gamma'/N_1\right)$. Note that, for some positive constant $\overline{C}$ such that

$$
\begin{aligned}
\|M_{FF}\|_F &= \left\| \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \right\|_F \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} \left\| E\left[\underline{F}_t \underline{F}_t'\right] \right\|_F \\
&\quad \text{(by the homogeneity of matrix norm and the triangle inequality)} \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} E\left\| \underline{F}_t \underline{F}_t' \right\|_F \text{ (by the Jensen's inequality)} \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\left[\sqrt{tr\left\{ \underline{F}_t \underline{F}_t' \underline{F}_t \underline{F}_t' \right\}} \right] \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\left[\|\underline{F}_t\|_2^2 \right] \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} \left(E\left[\|\underline{F}_t\|_2^6 \right] \right)^{\frac{1}{3}} \text{ (by Liapunov's inequality)} \\
&\leq \overline{C}^{\frac{1}{3}} \text{ (by Lemma C-5)} \qquad (114) \\
&< \infty
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F &= \sqrt{tr\left\{ \frac{\Gamma M_{FF} \Gamma'}{N_1} \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right) tr\left\{ \frac{\Gamma M_{FF}^2 \Gamma'}{N_1} \right\}} \\
&= \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right) tr\left\{ \frac{M_{FF} \Gamma'\Gamma M_{FF}}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max}^2\left(\frac{\Gamma'\Gamma}{N_1} \right) tr\left\{ M_{FF}^2 \right\}} \\
&= \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right) \|M_{FF}\|_F \\
&\leq C^* \overline{C}^{\frac{1}{3}} < \infty \text{ for all } N_1,\, N_2 \text{ sufficiently large,}
\end{aligned}
$$

since, by Assumption 3-6, there exists some positive constant C^* such that $\lambda_{\max}(\Gamma'\Gamma/N_1) \leq C^* < \infty$ for all N_1, N_2 sufficiently large. Moreover, applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left|-\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right)\right| = \left|\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right| = \left|\frac{\widehat{N}_1 - N_1}{N_1}\right| \left|\frac{1}{\left(\widehat{N}_1 - N_1\right)/N_1 + 1}\right| \xrightarrow{p} 0$$

so that by a further application of the Slutsky's theorem, we can deduce that

$$\left\|-\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right)\frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F = \left|\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right| \left\|\frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \xrightarrow{p} 0. \tag{115}$$

Consider now the other terms on the right-hand side of expression (113). To proceed, we first note that, by applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left|\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1}\right| = \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \xrightarrow{p} 1.$$

Next, note that

$$\begin{aligned}
&\left\|\frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F^2 \\
=\ & \sum_{i=1}^{N}\sum_{k=1}^{N}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\mathbb{I}\left\{k \in \widehat{H^c}\right\}\gamma_i' M_{FF}\gamma_k - \gamma_i' M_{FF}\gamma_k\right)^2 \\
=\ & \sum_{i \in H^c}\sum_{k \in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\mathbb{I}\left\{k \in \widehat{H^c}\right\}\gamma_i' M_{FF}\gamma_k - \gamma_i' M_{FF}\gamma_k\right)^2
\end{aligned}$$

where $H^c = \{k \in \{1, ..., N\} : \gamma_k \neq 0\}$, where $\widehat{H^c} = \left\{i \in \{1, ..., N\} : \mathbb{S}_{i,T}^+ \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\}$, and where $\mathbb{S}_{i,T}^+$ denotes either the statistic $\sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the statistic $\max_{1 \leq \ell \leq d}\left|S_{i,\ell,T}\right|$. Note that

$$\sum_{i \in H^c}\sum_{k \in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\mathbb{I}\left\{k \in \widehat{H^c}\right\}\gamma_i' M_{FF}\gamma_k - \gamma_i' M_{FF}\gamma_k\right)^2 = 0 \text{ if } \mathbb{I}\left\{i \in \widehat{H^c}\right\} = 1 \text{ for every } i \in H^c,$$

so that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{ \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \geq \epsilon \right\} &\subseteq \left\{ i \notin \widehat{H^c} \text{ for at least one } i \in H^c \right\} \\
&= \bigcup_{i \in H^c} \left\{ i \notin \widehat{H^c} \right\} \\
&= \bigcup_{i \in H^c} \left\{ \mathbb{S}^+_{i,T} < \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right\} \\
&= \left\{ \bigcap_{i \in H^c} \left\{ \mathbb{S}^+_{i,T} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right\} \right\}^c
\end{aligned}$$

Hence, applying either part (a) or part (b) of Theorem 3 depending on whether $\mathbb{S}^+_{i,T} = \sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or $\mathbb{S}^+_{i,T} = \max_{1 \leq \ell \leq d} \left|S_{i,\ell,T}\right|$, we obtain

$$\begin{aligned}
&\Pr\left(\left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \geq \epsilon \right) \\
\leq\ & 1 - \Pr\left(\bigcap_{i \in H^c} \left\{ \mathbb{S}^+_{i,T} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right\} \right) \\
=\ & 1 - \Pr\left(\min_{i \in H^c} \mathbb{S}^+_{i,T} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right) \\
\to\ & 1 - 1 = 0 \text{ as } N_1, N_2, T \to \infty,
\end{aligned}$$

so that

$$\left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F = o_p(1) \tag{116}$$

Now, consider the term $\Gamma\left(\widehat{H^c}\right)\left[\frac{F'F}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'/N_1$. For this term, note first that, by sub-multiplicativity of matrix norms, we have that

$$\begin{aligned}
\left\| \frac{\Gamma\left(\widehat{H^c}\right)\left[\frac{F'F}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'}{N_1} \right\|_F &\leq \left\| \frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_F \left\| \frac{F'F}{T_0} - M_{FF} \right\|_F \left\| \frac{\Gamma\left(\widehat{H^c}\right)'}{\sqrt{N_1}} \right\|_F \\
&= \left\| \frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_F^2 \left\| \frac{F'F}{T_0} - M_{FF} \right\|_F
\end{aligned}$$

Note that

$$
\begin{aligned}
\left\| \frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_F^2 &= tr\left\{ \frac{\Gamma\left(\widehat{H^c}\right)' \Gamma\left(\widehat{H^c}\right)}{N_1} \right\} \\
&= tr\left\{ \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \gamma_i \gamma_i' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} tr\left\{ \gamma_i \gamma_i' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \|\gamma_i\|_2^2 \, \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&\leq \sup_i \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&= \sup_{i \in \in H^c} \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&\quad (\text{since } \gamma_i = 0 \text{ for all } \gamma_i \in H) \\
&\leq C_1 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\}
\end{aligned}
$$

for some positive constant $C_1 \geq \sup_i \|\gamma_i\|_2^2 = \sup_{i \in \in H^c} \|\gamma_i\|_2^2$ which exists in light of Assumption 3-5. Moreover, write

$$
\frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} = \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \tag{117}
$$

For the first term on the right-hand side of expression (117) above, we can apply part (a) of Lemma D-7 to obtain

$$
\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} = O_p\left(\frac{\varphi}{N_1} \right) = o_p(1).
$$

With regard to the second term on the right-hand side of expression (117), note that

$$
\frac{1}{N_1} \sum_{i \in H^c} E\left[\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right] \leq 1
$$

since, by definition, N_1 is the cardinality of the set$\{i \in \{1, ..., N\} : i \in H^c\}$. Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$ for any positive constant $C \geq 1$, and note that

$$\begin{aligned} \Pr\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\geq C_\epsilon\right\} &\leq \frac{1}{C_\epsilon}\frac{1}{N_1}\sum_{i\in H^c}E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right] \text{ (by Markov's inequality)} \\ &\leq \frac{\epsilon}{C}C \\ &= \epsilon \end{aligned}$$

which shows that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\} = O_p(1).$$

It follows that

$$\begin{aligned} \left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F^2 &\leq C_1\frac{1}{N_1}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\} \\ &= \frac{C_1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}+\frac{C_1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\} \\ &= O_p\left(\frac{\varphi}{N_1}\right)+O_p(1) \\ &= O_p(1). \end{aligned}$$

In addition, applying the result of part (b) of Lemma D-2, we have that

$$\left\|\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right\|_F = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1)$$

from which we further deduce that

$$\begin{aligned} \left\|\frac{\Gamma\left(\widehat{H^c}\right)\left[\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'}{N_1}\right\|_F &\leq \left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F^2\left\|\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right\|_F \\ &= O_p(1)\,O_p\left(\frac{1}{\sqrt{T}}\right) \\ &= O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned} \tag{118}$$

Turning our attention to the term $U\left(\widehat{H^c}\right)'\underline{F}\Gamma\left(\widehat{H^c}\right)'/(N_1T_0)$, we first write

$$
\begin{aligned}
&\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F^2 \\
= \;& \sum_{i=1}^{N}\sum_{k=1}^{N}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\frac{u_{i\cdot}'\underline{F}\gamma_k}{N_1 T_0}\right)^2 \\
= \;& \frac{1}{N_1^2 T_0^2}\sum_{i=1}^{N}\sum_{k=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\gamma_k'\underline{F}'u_{i\cdot}u_{i\cdot}'\underline{F}\gamma_k \\
= \;& \frac{1}{N_1^2 T_0^2}\sum_{i=1}^{N}\sum_{k\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\gamma_k'\underline{F}'u_{i\cdot}u_{i\cdot}'\underline{F}\gamma_k \\
= \;& \frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\frac{1}{N_1 T_0^2}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
\leq \;& \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
= \;& \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 + \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
= \;& \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 + \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2
\end{aligned}
$$

Applying parts (b) and (c) of Lemma D-7, we obtain

$$
\begin{aligned}
&\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F^2 \\
\leq \;& \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 + \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 \\
= \;& O_p\left(\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}\right) + O_p\left(\frac{1}{T}\right) \\
= \;& O_p\left(\max\left\{\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}, \frac{1}{T}\right\}\right) \\
= \;& o_p(1) \quad \text{(by Assumption 3-11*)}
\end{aligned}
$$

so that

$$\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F = O_p\left(\max\left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\} \right) = o_p(1). \tag{119}$$

Since

$$\left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F = \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F$$

it follows immediately also that

$$\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F = O_p\left(\max\left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\} \right) = o_p(1). \tag{120}$$

Finally, consider the term $\left\| U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right) / N_1 T_0 \right\|_F^2$, where

$$U\left(\widehat{H^c}\right) = \left[\begin{array}{cccc} u_{1\cdot}\mathbb{I}\left\{1 \in \widehat{H^c}\right\} & u_{2\cdot}\mathbb{I}\left\{2 \in \widehat{H^c}\right\} & \cdots & u_{N\cdot}\mathbb{I}\left\{N \in \widehat{H^c}\right\} \end{array} \right].$$

Given that

$$\begin{aligned} & U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right) \\ = & \begin{pmatrix} u_{1\cdot}' u_{1\cdot}\mathbb{I}\left\{1 \in \widehat{H^c}\right\} & \cdots & u_{1\cdot}' u_{N\cdot}\mathbb{I}\left\{1 \in \widehat{H^c}\right\}\mathbb{I}\left\{N \in \widehat{H^c}\right\} \\ u_{1\cdot}' u_{2\cdot}\mathbb{I}\left\{1 \in \widehat{H^c}\right\}\mathbb{I}\left\{2 \in \widehat{H^c}\right\} & \cdots & u_{2\cdot}' u_{N\cdot}\mathbb{I}\left\{2 \in \widehat{H^c}\right\}\mathbb{I}\left\{N \in \widehat{H^c}\right\} \\ \vdots & & \vdots \\ u_{1\cdot}' u_{N\cdot}\mathbb{I}\left\{1 \in \widehat{H^c}\right\}\mathbb{I}\left\{N \in \widehat{H^c}\right\} & \cdots & u_{N\cdot}' u_{N\cdot}\mathbb{I}\left\{N \in \widehat{H^c}\right\} \end{pmatrix}, \end{aligned}$$

we can write

$$
\begin{aligned}
\left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F^2 &= \sum_{i=1}^{N}\sum_{k=1}^{N}\left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \frac{u_{i\cdot}' u_{k\cdot}}{N_1 T_0} \right)^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^{N}\sum_{k=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(u_{i\cdot}' u_{k\cdot} \right)^2 \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k=1}^{N} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \ \ (say),
\end{aligned}
$$

where the order of magnitude in probability of the terms $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$ are given in parts (a)-(d) of Lemma D-9. It, thus, follows by applying parts (a)-(d) of Lemma D-9 with $h = 0$ that

$$
\begin{aligned}
&\left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^{N}\sum_{k=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(u_{i\cdot}' u_{k\cdot} \right)^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \\
&= O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) + O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p\left(\frac{N^{\frac{4}{7}} \varphi^{\frac{10}{7}}}{N_1^2} \right) \\
&= O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T}, \frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right\} \right) \\
&= o_p(1).
\end{aligned}
$$

from which we further deduce that

$$\left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F = O_p\left(\max\left\{\frac{1}{\sqrt{N_1}}, \frac{1}{\sqrt{T}}, \frac{N^{\frac{1}{7}}\varphi^{\frac{5}{14}}}{\sqrt{N_1}}\right\}\right) = o_p(1) \text{ as } N_1, N_2, T \to \infty. \quad (121)$$

Expressions (115)-(121) together imply that

$$\begin{aligned}
& \left\| \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F \\
= & \left\| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F \\
\leq & \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1}\Gamma\left(\widehat{H^c}\right)\left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)' \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F \\
= & \; o_p(1) \text{ as } N_1, N_2, T \to \infty.
\end{aligned}$$

Since $\|A\|_2 \leq \|A\|_F$, we also have

$$
\begin{aligned}
& \|E\|_2 \\
= \; & \left\| \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
= \; & \left\| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
\leq \; & \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1} \Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF} \right] \Gamma\left(\widehat{H^c}\right)' \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_2 \\
\leq \; & \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right|^{-1} \left\| \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1} \Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF} \right] \Gamma\left(\widehat{H^c}\right)' \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F \\
= \; & o_p(1) \text{ as } N_1, N_2, T \to \infty. \; \square
\end{aligned}
$$

Lemma D-11: Let

$$\underset{N \times N}{A} = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

where

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \text{ with } T_0 = T - p + 1.$$

Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, 3-6 and 3-7 hold; and let G be an $N \times N$ orthogonal matrix whose columns are the eigenvectors of A. Under the assumed conditions, the following statements are true.

(a) $Rank\,(A) = Kp$ for all N_1, N_2 sufficiently large, and, hence, 0 is an eigenvlaue of A with algebraic multiplicity equaling $N - Kp$.

(b) Partition G as follows:

$$\underset{N\times N}{G} = \left[\begin{array}{cc} \underset{N\times Kp}{G_1} & \underset{N\times(N-Kp)}{G_2} \end{array}\right]$$

Without loss of generality, suppose that the columns of G_1 are eigenvectors associated with the non-zero eigenvalues of A, whereas G_2 contains the eigenvectors associated with the zero eigenvalue. Then, the matrix $G'AG$ can be partitioned as follows:

$$G'AG = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{\Lambda_2} \end{pmatrix} = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{0} \end{pmatrix}. \tag{122}$$

where Λ_1 is a diagonal matrix whose diagonal elements are the non-zero eigenvalues of A and where $\Lambda_2 = 0$.

(c) Define the separation measure

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \min_{X\neq 0} \frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F};$$

then, there exists a positive constant $\underline{c}$ such that

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \text{sep}\,(\Lambda_1, 0) = \min_{X\neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \geq \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \geq \underline{c} > 0.$$

Proof of Lemma D-11: To show part (a), note first that, by the result of Lemma D-4 above, there exists a positive constant $\underline{C}$ such that

$$\lambda_{\min}\left\{M_{FF}\right\} \geq \underline{C} > 0$$

for all $T > p-1$; and, by Assumption 3-6, we have,

$$\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \frac{1}{\overline{C}} \text{ for } N_1, N_2 \text{ sufficiently large.}$$

for some constant $\overline{C}$ such that $0 < \overline{C} < \infty$. Combining these two inequalities, we see that

$$\begin{aligned}
\lambda_{\min}\left\{\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} &\geq \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\lambda_{\min}\left\{M_{FF}\right\} \\
&\geq \frac{\underline{C}}{\overline{C}} > 0 \text{ for all } N_1,\ N_2, \text{ and } T \text{ sufficiently large.}
\end{aligned}$$

This implies that the $Kp \times Kp$ matrix

$$\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}$$

is a positive definite (and, therefore, also non-singular) for N_1, N_2, and T sufficiently large. Moreover, observe that

$$\begin{aligned}
&\det\left\{\lambda I_N - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\} \\
= \ & \lambda^N \det\left\{I_N - \lambda^{-1}\frac{\Gamma M_{FF}\Gamma'}{N_1}\right\} \\
= \ & \lambda^N \det\left\{I_{Kp} - \lambda^{-1}\ \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} \text{ (by Sylvester's determinantal theorem)} \\
= \ & \lambda^{N-Kp} \det\left\{\lambda I_{Kp} - \ \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} \qquad (123)
\end{aligned}$$

Hence, the non-zero eigenvalues of the matrix $\Gamma M_{FF}\Gamma'/N_1$ correspond exactly to the eigenvalues of the positive definite matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$, from which we further deduce that the matrix

$$A = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

must be of rank Kp for N_1, N_2, T sufficiently large. Since A is an $N \times N$ matrix with $N = N_1 + N_2$, it follows immediately that 0 is an eigenvalue of A with algebraic multiplicity equaling $N - Kp$ for N_1, N_2, T sufficiently large.

To show part (b), let $\Lambda_1 = diag\left(\lambda_{1,1}, ..., \lambda_{1,Kp}\right)$, whose diagonal elements $\lambda_{1,i} > 0$, for $i \in \{1, ..., Kp\}$, denote the non-zero eigenvalues of A (which must all be positive given that they correspond to the eigenvalues of the positive definite matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$ as shown in the proof of part (a)). Moreover, let

$$\Lambda_2 = \underset{(N-Kp)\times(N-Kp)}{0}$$

whose diagonal elements are the $N-Kp$ zero eigenvalues of A.Since A is a symmetric matrix, the representation given in expression (122) follows immediately from the usual spectral decomposition.

Finally, to show part (c), note that for any $Kp \times (N-Kp)$ matrix $X \neq 0$, we have

$$\begin{aligned}
\|\Lambda_1 X - X\Lambda_2\|_F &= \|\Lambda_1 X\|_F \quad (\text{since } \Lambda_2 = 0) \\
&= \sqrt{tr\left\{X'\Lambda_1'\Lambda_1 X\right\}} \\
&\geq \lambda_{\min}(\Lambda_1)\sqrt{tr\left\{X'X\right\}} \\
&= \lambda_{\min}(\Lambda_1)\|X\|_F
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname{sep}(\Lambda_1, \Lambda_2) &= \min_{X\neq 0}\frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F} \\
&= \min_{X\neq 0}\frac{\|\Lambda_1 X\|_F}{\|X\|_F} \quad (\text{since } \Lambda_2 = 0 \text{ in this case}) \\
&\geq \frac{\lambda_{\min}(\Lambda_1)\|X\|_F}{\|X\|_F} \\
&= \lambda_{\min}(\Lambda_1)
\end{aligned}$$

Furthermore, in light of expression (123), the diagonal elements of Λ_1, being the non-zero eigenvalues of A, must all be the solutions of the determinantal equation

$$\det\left\{\lambda I_{Kp} - \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} = 0$$

so that, as noted in the proof of part (a) above, they are also the eigenvalues of the dual matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$. It follows from the proof of part (a) that there exists a positive constant $\underline{c}$ such that for all N_1, N_2, and T sufficiently large.

$$\begin{aligned}
\operatorname{sep}(\Lambda_1, \Lambda_2) &= \operatorname{sep}(\Lambda_1, 0) \\
&\geq \lambda_{\min}(\Lambda_1) \\
&= \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \\
&\geq \underline{c} > 0. \ \square
\end{aligned}$$

Lemma D-12: Suppose that A and E are both $n \times n$ symmetric matrices and that

$$G = \left[\underset{n \times r}{G_1} \quad \underset{n \times (n-r)}{G_2} \right]$$

is an orthogonal matrix such that

$$\text{ran}\,(G_1) = \{ y \in \mathbb{R}^n : y = G_1 x \text{ for some } x \in \mathbb{R}^r \}$$

is an invariant subspace for A, i.e., for any $\widetilde{q} \in \text{ran}(G_1)$ and let $q^* = A\widetilde{q}$; then $q^* \in \text{ran}(G_1)$. Partition the matrices $G'AG$ and $G'EG$ as follows:

$$G'AG = \begin{pmatrix} \underset{r \times r}{\Lambda_1} & \underset{r \times (n-r)}{0} \\ \underset{(N-r) \times r}{0} & \underset{(n-r) \times (n-r)}{\Lambda_2} \end{pmatrix} \text{ and } G'EG = \begin{pmatrix} \underset{r \times r}{E_{11}} & \underset{r \times (n-r)}{E'_{21}} \\ \underset{(n-r) \times r}{E_{21}} & \underset{(n-r) \times (n-r)}{E_{22}} \end{pmatrix}.$$

If

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F} > 0 \tag{124}$$

and if

$$\begin{aligned} \|E\|_2 &\leq \frac{\text{sep}\,(\Lambda_1, \Lambda_2)}{5} \\ &= \frac{1}{5} \min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F}, \end{aligned} \tag{125}$$

then, there exists a matrix $R \in \mathbb{R}^{(n-r) \times r}$ satisfying

$$\begin{aligned} \|R\|_2 &\leq \frac{4}{\text{sep}\,(\Lambda_1, \Lambda_2)} \|E_{21}\|_2 \\ &= 4 \left(\min_{X \neq 0} \frac{\|\Lambda_1 X - X \Lambda_2\|_F}{\|X\|_F} \right)^{-1} \|E_{21}\|_2 \end{aligned}$$

such that the columns of

$$\widehat{G}_1 = (G_1 + G_2 R) \left(I_r + R'R \right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$.

Remark: Lemma D-12 is a well-known result in linear algebra restated here in our notations. It is given in Golub and van Loan (1996) as Theorem 8.1.10. As noted in Golub and van Loan (1996), this result is also a slight adaptation of Theorem 4.11 in Stewart (1973), which could be consulted for proof details.

Lemma D-13: Let $\mathcal{X}$ be an invariant subspace of A, and let the columns of X form a basis for $\mathcal{X}$. Then, there is a unique matrix L such that

$$AX = XL.$$

The matrix L is the representation of A on $\mathcal{X}$ with respect to the basis X. In particular, (v, λ) is an eigenpair of L if and only if (Xv, λ) is an eigenpair of A.

Proof of Lemma D-13: This is Theorem 3.9 of Stewart and Sun (1990). For a proof of this theorem, see Stewart and Sun (1990).

A straightforward application of Lemma D-12 (or Theorem 8.1.10 of Golub and van Loan, 1996) to our setting here leads to the following lemma.

Lemma D-14: Let $\widehat{\Sigma}\left(\widehat{H^c}\right)$ be the post-variable-selection sample covariance matrix as defined in expression (111) in Lemma D-10. Decompose $\widehat{\Sigma}\left(\widehat{H^c}\right)$ as follows:

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = A + E,$$

where

$$A = \frac{\Gamma M_{FF} \Gamma'}{N_1} \tag{126}$$

and where

$$\begin{aligned} E &= \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\ &= \left(\frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{\widehat{N}_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right) + \frac{1}{\widehat{N}_1} \Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF} \right] \Gamma\left(\widehat{H^c}\right)' \\ &\quad + \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{\widehat{N}_1 T_0} + \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} + \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0}, \end{aligned} \tag{127}$$

with $T_0 = T - p + 1$ and

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right].$$

Suppose that Assumptions 3-1, 3-2, 3-3, 3-4 3-5, 3-6, 3-7, 3-8, 3-10, and 3-11* hold, and define

$$\underset{N\times N}{G} = \begin{bmatrix} \underset{N\times Kp}{G_1} & \underset{N\times(N-Kp)}{G_2} \end{bmatrix}$$

to be an orthogonal matrix whose columns are the eigenvectors of the matrix A. Without loss of generality, suppose that the columns of G_1 are the eigenvectors associated with the non-zero eigenvalues of A, whereas G_2 contains the eigenvectors associated with the zero eigenvalue which has an algebraic multiplicity of $N - Kp$ in this case[2]. Partition the matrices $G'AG$ and $G'EG$ as follows:

$$G'AG = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{\Lambda_2} \end{pmatrix} = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{0} \end{pmatrix} \text{ and}$$

$$G'EG = \begin{pmatrix} \underset{Kp\times Kp}{E_{11}} & \underset{Kp\times(N-Kp)}{E'_{21}} \\ \underset{(N-Kp)\times Kp}{E_{21}} & \underset{(N-Kp)\times(N-Kp)}{E_{22}} \end{pmatrix},$$

where Λ_1 is a diagonal matrix whose diagonal elements are the Kp largest eigevalues of the matrix A.[3]

Under the assumed conditions, the following statements are true.

(a) There exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $\widehat{\Sigma}\left(\widehat{H^c}\right) = A + E$. Moreover,

$$\|R\|_2 = o_p(1) \text{ as } N_1, N_2, \text{ and} T \to \infty$$

(b) $\left\|\widehat{G}_1 - G_1\right\|_2 = o_p(1)$ as N_1, N_2, and $T \to \infty$

(c) The exists a unique symmetric matrix L such that

$$(A + E)\widehat{G}_1 = \widehat{G}_1 L.$$

Moreover, let

$$\widehat{\Lambda} = diag\left(\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right) \tag{128}$$

[2] That 0 is an eigenvalue of the matrix

$$A = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

with algebraic multiplicity equaling $N - Kp$ has already been shown previously in Lemma D-11.

[3] We have also previously shown in Lemma D-11 that $G'AG$ can be partitioned in the manner given here.

denote a diagonal matrix whose diagonal elements are the eigenvalues of the matrix L, and let

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} \tag{129}$$

be a $Kp \times Kp$ matrix whose ℓ^{th} column (i.e., $\widehat{v}_\ell$) is an eigenvector of L associated with the eigenvalue $\widehat{\lambda}_\ell$ for $\ell = 1, ..., Kp$. Then, $\widehat{V}$ is an orthogonal matrix and $\left(\widehat{G}_1 \widehat{v}_\ell, \widehat{\lambda}_\ell\right)$ is an eigenpair for the matrix $A + E$ for $\ell = 1, ..., Kp$.

(d) The columns of the matrix

$$\widehat{G}_1 \widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1 \widehat{v}_1 & \widehat{G}_1 \widehat{v}_2 & \cdots & \widehat{G}_1 \widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the post-variable-selection sample covariance matrix

$$A + E = \widehat{\Sigma}\left(\widehat{H^c}\right).$$

Proof of Lemm D-14:

To show part (a), we first verify that the conditions (124) and (125) of Lemma D-12 are satisfied here. To proceed, let $\text{ran}(G_1)$ denote the range space of G_1, i.e.,

$$\text{ran}\left(G_1\right) = \left\{g \in \mathbb{R}^N : g = G_1 b \text{ for some } b \in \mathbb{R}^{Kp}\right\}$$

and, by definition, Λ_1 is a $Kp \times Kp$ diagonal matrix whose diagonal elements are the non-zero eigenvalues of the matrix $A = \Gamma M_{FF} \Gamma' / N_1$. Now, for any $\widetilde{g} \in \text{ran}(G_1)$, note that

$$\begin{aligned} g^* &= A\widetilde{g} \\ &= \left(\frac{\Gamma M_{FF} \Gamma'}{N_1}\right) G_1 b \\ &= G_1 \Lambda_1 b \\ &= G_1 b^* \text{ where } b^* = \Lambda_1 b. \end{aligned}$$

from which it follows that $g^* \in \text{ran}(G_1)$, so that $\text{ran}(G_1)$ is an invariant subspace of A. Next, by applying the result of Lemma D-11, we have

$$\begin{aligned}
\operatorname{sep}(\Lambda_1, \Lambda_2) &= \operatorname{sep}(\Lambda_1, 0) \\
&= \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \\
&\geq \lambda_{\min}(\Lambda_1) \\
&= \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \\
&\geq \underline{c} > 0 \text{ for } N_1 \text{ and } N_2 \text{ sufficiently large,}
\end{aligned}$$

so that condition (124) of Lemma D-12 is fulfilled. Next, note that, from the result of Lemma D-10, we have

$$\|E\|_2 = \left\|\widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_2 = o_p(1) \quad \text{as } N_1,\ N_2, \text{ and } T \to 0;$$

from which it follows that

$$\|E\|_2 \leq \frac{\operatorname{sep}(\Lambda_1, 0)}{5} \text{ w.p.a.1 as } N_1,\ N_2, \text{ and } T \to 0.$$

so that condition (125) of Lemma D-12 is also satisfied here w.p.a.1. Hence, application of Lemma D-12 allows us to conclude that there exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$. In addition,

$$\begin{aligned}
\|R\|_2 &\leq \frac{4}{\operatorname{sep}(\Lambda_1, 0)}\|E\|_2 \\
&\leq 4\left[\lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right)\right]^{-1}\|E\|_2 \\
&\leq \frac{4}{\underline{c}}\|E\|_2 \quad \text{(for some } \underline{c} > 0 \text{ by Assumption 3-6 and Lemma D-4)} \\
&= o_p(1),
\end{aligned}$$

which shows result (a).

To show that $\left\|\widehat{G}_1 - G_1\right\|_2 = o_p(1)$, we first show that an explicit representation for G_1 can be given as

$$G_1 = \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$$

where Ξ is an orthogonal matrix whose columns are eigenvectors of the matrix

$$\underset{Kp\times Kp}{M_{FF}^{*}} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}$$

To see that this representation satisfies the various properties we require of G_1, note first that

$$G_1'G_1 = \Xi' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \frac{\Gamma'\Gamma}{N_1} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \ = I_{Kp};$$

hence, G_1 so represented does have orthonormal columns. Moreover, note that

$$\begin{aligned}
\frac{\Gamma M_{FF}\Gamma'}{N_1}G_1 &= \frac{\Gamma}{\sqrt{N_1}}M_{FF}\frac{\Gamma'\Gamma}{\sqrt{N_1}}\left(\Gamma'\Gamma\right)^{-1/2}\Xi \\
&= \frac{\Gamma}{\sqrt{N_1}}M_{FF}\frac{\Gamma'\Gamma}{N_1}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi \\
&= \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}M_{FF}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'\Gamma}{N_1}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi \\
&= \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}M_{FF}^{*}\Xi \\
&= \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi\Lambda_1 \\
&= G_1\Lambda_1 \qquad (130)
\end{aligned}$$

where Λ_1 is a $Kp \times Kp$ diagonal matrix whose diagonal elements are the eigenvalues of the matrix M_{FF}^{*}, which also happen to be the non-zero eigenvalues of the matrix $A = \Gamma M_{FF}\Gamma'/N_1$. Premultiplying the above equation by G_1', we obtain

$$G_1'\frac{\Gamma M_{FF}\Gamma'}{N_1}G_1 = G_1'G_1\Lambda_1 = \Lambda_1.$$

Since equation (130) shows that the columns of $\Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$ are indeed the eigenvectors of the matrix $A = \Gamma M_{FF}\Gamma'/N_1$, by the argument given previously in the proof of part (a) above, we can then deduce that $\mathrm{ran}(G_1)$, the range space of G_1 with $G_1 = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$, is an invariant subspace of A. It follows that setting

$$G_1 = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$$

fulfills all the required properties of G_1 specified in Lemma D-12 above.

Next, write

$$
\begin{aligned}
\widehat{G}_1 - G_1 &= (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2} - G_1 \\
&= G_1\left[\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right] + G_2 R\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi\left[\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right] + G_2 R\left(I_{Kp} + R'R\right)^{-1/2}
\end{aligned}
$$

Applying the submultiplicative property of matrix norms and the triangle inequality, we obtain

$$
\begin{aligned}
& \left\|\widehat{G}_1 - G_1\right\|_2 \\
\leq & \left\|\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\|_2 \|\Xi\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right\|_2 \\
& + \|G_2\|_2 \|R\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \\
= & \left\|\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right\|_2 + \|R\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2
\end{aligned}
$$

where the last equality follows from the fact that

$$
\begin{aligned}
\|\Xi\|_2 &= \sqrt{\lambda_{\max}\left(\Xi'\Xi\right)} = \sqrt{\lambda_{\max}\left(I_{Kp}\right)} = 1, \\
\|G_2\|_2 &= \sqrt{\lambda_{\max}\left(G_2'G_2\right)} = \sqrt{\lambda_{\max}\left(I_{N-Kp}\right)} = 1, \text{ and} \\
\left\|\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\|_2 &= \sqrt{\lambda_{\max}\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'\Gamma}{N_1}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\}} = \sqrt{\lambda_{\max}\left\{I_{Kp}\right\}} = 1.
\end{aligned}
$$

Now, if (λ, ρ) is an eigen-pair of $R'R$ so that

$$
R'R\rho = \lambda\rho \text{ with } \lambda \geq 0 \text{ given that } R'R \text{ is positive semidefinite;}
$$

then,

$$
\begin{aligned}
\left(I_{Kp} + R'R\right)\rho &= (1+\lambda)\rho, \\
\left(I_{Kp} + R'R\right)^{-1/2}\rho &= \frac{1}{\sqrt{1+\lambda}}\rho, \text{ and} \\
\left[I_{Kp} - \left(I_{Kp} + R'R\right)^{-1/2}\right]\rho &= \left(I_{Kp} - \frac{1}{\sqrt{1+\lambda}}I_{Kp}\right)\rho \\
&= \frac{\sqrt{1+\lambda} - 1}{\sqrt{1+\lambda}}\rho
\end{aligned}
$$

since

$$\frac{1}{\sqrt{1+\lambda}} \text{ is an eigenvalue of } \left(I_{Kp}+R'R\right)^{-1/2} \text{ associated with the eigenvector } \rho$$

and

$$\frac{\sqrt{1+\lambda}-1}{\sqrt{1+\lambda}} \text{ is an eigenvalue of } I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2} \text{ associated with the eigenvector } \rho$$

Moreover, let

$$g(\lambda)=\frac{\sqrt{1+\lambda}-1}{\sqrt{1+\lambda}}$$

and note that

$$\begin{aligned}
g'(\lambda) &= \frac{1}{2}\frac{1}{1+\lambda}-\frac{1}{2}\frac{\sqrt{1+\lambda}-1}{(1+\lambda)^{3/2}} \\
&= \frac{1}{2}\frac{\sqrt{1+\lambda}-\sqrt{1+\lambda}+1}{(1+\lambda)^{3/2}} \\
&= \frac{1}{2(1+\lambda)^{3/2}}>0
\end{aligned}$$

so that, in particular, $g(\lambda)$ is an increasing function of λ for $\lambda \geq 0$. It follows that

$$\begin{aligned}
& \left\|\widehat{G}_1-G_1\right\|_2 \\
\leq\ & \left\|\left(I_{Kp}+R'R\right)^{-1/2}-I_{Kp}\right\|_2+\|R\|_2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \\
=\ & \left\|I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2+\|R\|_2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \\
=\ & \sqrt{\lambda_{\max}\left(\left[I_{Kp}-(I_{Kp}+R'R)^{-1/2}\right]'\left[I_{Kp}-(I_{Kp}+R'R)^{-1/2}\right]\right)} \\
& +\|R\|_2\sqrt{\lambda_{\max}\left((I_{Kp}+R'R)^{-1/2\prime}(I_{Kp}+R'R)^{-1/2}\right)} \\
=\ & \lambda_{\max}\left[I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2}\right]+\|R\|_2\lambda_{\max}\left[\left(I_{Kp}+R'R\right)^{-1/2}\right] \\
& \left(\text{since } I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2} \text{ and } \left(I_{Kp}+R'R\right)^{-1/2} \text{ are both symmetric and positive semidefinite}\right) \\
\leq\ & \frac{\sqrt{1+\lambda_{\max}(R'R)}-1}{\sqrt{1+\lambda_{\min}(R'R)}}+\frac{\|R\|_2}{\sqrt{1+\lambda_{\min}(R'R)}} \\
\leq\ & \sqrt{1+\|R\|_2^2}-1+\|R\|_2 \quad \left(\text{since } \lambda_{\min}\left(R'R\right)\geq 0 \text{ given that } R'R \text{ is positive semi-definite}\right) \\
=\ & o_p(1) \text{ as } N_1,\ N_2, \text{ and } T\to\infty \text{ (since } \|R\|_2=o_p(1)).
\end{aligned}$$

This shows result (b).

To show part (c), note that, by the result given in part (a) above, the columns of $\widehat{G}_1 = (G_1 + G_2 R)(I_r + R'R)^{-1/2}$ form an orthonormal basis for a subspace that is invariant for $A + E$. It then follows immediately from Lemma D-13 that there exists a unique matrix L such that

$$\begin{aligned}
(A+E)\widehat{G}_1 &= (A+E)(G_1 + G_2 R)\left(I_r + R'R\right)^{-1/2} \\
&= (G_1 + G_2 R)\left(I_r + R'R\right)^{-1/2} L \\
&= \widehat{G}_1 L.
\end{aligned}$$

Note further that

$$\begin{aligned}
\widehat{G}_1'\widehat{G}_1 &= \left(I_{Kp} + R'R\right)^{-1/2}\left(G_1' + R'G_2'\right)(G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \left(I_{Kp} + R'R\right)^{-1/2}\left(G_1'G_1 + R'G_2'G_1 + G_1'G_2 R + R'G_2'G_2 R\right)\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \left(I_{Kp} + R'R\right)^{-1/2}\left(I_{Kp} + R'R\right)\left(I_{Kp} + R'R\right)^{-1/2} \\
&\quad \left(\text{since by assumption } G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \text{ is an orthogonal matrix}\right) \\
&= I_{Kp}
\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}
\widehat{G}_1'(A+E)\widehat{G}_1 &= \widehat{G}_1'\left(\frac{\Gamma M_{FF}\Gamma'}{N_1} + E\right)\widehat{G}_1 = \widehat{G}_1'\widehat{G}_1 L \\
&= L
\end{aligned}$$

so that L must be symmetric since, in our situation here,

$$A + E = \frac{\Gamma M_{FF}\Gamma'}{N_1} + \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} = \widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0}$$

is a symmetric matrix. Now, let $\widehat{\Lambda} = diag\left(\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right)$ and

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix}$$

be as defined in expressions (128) and (129). The fact that L is symmetric implies that $\widehat{V}$ is an orthogonal matrix. In addition, further application of Lemma C-13 shows that $\left(\widehat{G}_1\widehat{v}_g, \widehat{\lambda}_g\right)$ is an eigenpair for the matrix $A + E$ for $g = 1, ..., Kp$.

Finally, to show part (d), let $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, and note that, by assumption,

$$G'AG = \begin{pmatrix} G_1'AG_1 & G_1'AG_2 \\ G_2'AG_1 & G_2'AG_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \Lambda$$

where $\Lambda_1 = diag\left(\lambda_{1,1.}, ..., \lambda_{1,Kp}\right)$ contains the Kp largest eigenvalues of A. Without loss of generality, we can further assume that $\lambda_{1,1.}, ..., \lambda_{1,Kp}$ are ordered, so that $\lambda_{1,j} = \lambda_{(j)}(A)$, i.e., $\lambda_{1,j}$ is the j^{th} largest eigenvalue of A.[4] Given that, $G'G = GG' = I_N$, we have

$$\begin{pmatrix} AG_1 & AG_2 \end{pmatrix} = AG = G\Lambda = \begin{pmatrix} G_1\Lambda_1 & 0 \end{pmatrix}$$

from which it follows that

$$AG_1G_1'\widehat{G}_1\widehat{v}_\ell = G_1\Lambda_1G_1'\widehat{G}_1\widehat{v}_\ell, \text{ for } \ell \in \{1, ..., Kp\}. \tag{131}$$

Now, the result of part (c) above shows $\left(\widehat{G}_1\widehat{v}_\ell, \widehat{\lambda}_\ell\right)$ to be an eigenpair of the matrix $A + E$ for $\ell \in \{1, ..., Kp\}$, so that

$$(A + E)\,\widehat{G}_1\widehat{v}_\ell = \widehat{\lambda}_\ell\widehat{G}_1\widehat{v}_\ell \text{ for } \ell \in \{1, ..., Kp\} \tag{132}$$

where $\widehat{G}_1 = (G_1 + G_2R)\left(I_{Kp} + R'R\right)^{-1/2}$ as given in the result for part (a). Multiplying both sides of expression (132) by $\widehat{v}_\ell'\widehat{G}_1'G_1G_1'$, we get

$$\begin{aligned} \widehat{\lambda}_\ell\widehat{v}_\ell'\widehat{G}_1'G_1G_1'\widehat{G}_1\widehat{v}_\ell &= \widehat{v}_\ell'\widehat{G}_1'G_1G_1'(A + E)\,\widehat{G}_1\widehat{v}_\ell \\ &= \widehat{v}_\ell'\widehat{G}_1'G_1G_1'A\widehat{G}_1\widehat{v}_\ell + \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell \end{aligned} \tag{133}$$

Since $A = \Gamma M_{FF}\Gamma'/N_1$ is symmetric, it further follows by expression (131) that

$$\widehat{v}_\ell'\widehat{G}_1'G_1G_1'A = \widehat{v}_\ell'\widehat{G}_1'G_1G_1'A' = \widehat{v}_\ell'\widehat{G}_1'G_1\Lambda_1G_1' \tag{134}$$

[4] If this is not the case; then, we can always define a permutation matrix $\mathcal{P}$ such that

$$\Lambda^* = \mathcal{P}'\Lambda\mathcal{P}$$

results in a diagonal matrix whose diagonal elements are repermutated in such a way, so that the required ordering of the eigenvalues is satisfied. Moreover, since $\mathcal{P}$ is an orthogonal matrix, it further follows that

$$A = G\mathcal{P}\mathcal{P}'\Lambda\mathcal{P}\mathcal{P}'G' = G\mathcal{P}\Lambda^*\mathcal{P}'G'.$$

Now, define $\widetilde{G} = G\mathcal{P}$, and note that $\widetilde{G}$ is an orthogonal matrix whose columns are just the columns of G repermutated. Hence, we can simply proceed with our analysis using $\widetilde{G}$ in lieu of G, and the associated eigenvalues will be in the order which we have assumed.

Moreover, note that

$$
\begin{aligned}
0 &\leq \left(\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell\right)^2 \\
&\leq \left(\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell\right)\left(\widehat{v}_\ell' \widehat{G}_1' E' E \widehat{G}_1 \widehat{v}_\ell\right) \text{ (by CS inequality)} \\
&= \left(\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell\right)\left(\widehat{v}_\ell' \widehat{G}_1' E' E \widehat{G}_1 \widehat{v}_\ell\right) \text{ (since } G_1' G_1 = I_{Kp}) \\
&= \left[\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1/2} \left(G_1' + R' G_2'\right) G_1 G_1' \left(G_1 + G_2 R\right)\left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell\right]\left(\widehat{v}_\ell' \widehat{G}_1' E' E \widehat{G}_1 \widehat{v}_\ell\right) \\
&= \left[\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell\right]\left(\widehat{v}_\ell' \widehat{G}_1' E' E \widehat{G}_1 \widehat{v}_\ell\right) \\
&\leq \left[\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell\right] \lambda_{\max}\left(E'E\right)
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
-\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}\, \|E\|_2 &= -\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell} \sqrt{\lambda_{\max}\left(E'E\right)} \\
&\leq -\sqrt{\left(\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell\right)^2} \\
&\leq -\left|\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell\right| \\
&\leq \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell \qquad (135)
\end{aligned}
$$

where the last inequality follows from the fact that

$$
\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell > -\left|\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell\right| \text{ if } \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell > 0
$$

whereas

$$
\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell = -\left|\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell\right| \text{ if } \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell \leq 0
$$

Combining expressions (133), (134), and (135), we see that

$$
\begin{aligned}
\widehat{\lambda}_\ell \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell &= \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' A \widehat{G}_1 \widehat{v}_\ell + \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' E \widehat{G}_1 \widehat{v}_\ell \\
&\geq \widehat{v}_\ell' \widehat{G}_1' G_1 \Lambda_1 G_1' \widehat{G}_1 \widehat{v}_\ell - \sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}\, \|E\|_2 \qquad (136)
\end{aligned}
$$

for $\ell \in \{1, ..., Kp\}$. In addition, note that

$$\begin{aligned}
\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell &= \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \left(G_1 + G_2 R\right) \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \widehat{G}_1' G_1 \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1/2} \left(G_1' + R'G_2'\right) G_1 \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell \\
&> 0
\end{aligned}$$

Hence, dividing both sides of expression (136) by $\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell$, we obtain

$$\begin{aligned}
\widehat{\lambda}_\ell &\geq \frac{\widehat{v}_\ell' \widehat{G}_1' G_1 \Lambda_1 G_1' \widehat{G}_1 \widehat{v}_\ell}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} - \frac{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell} \, \|E\|_2}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} \\
&= \widetilde{v}_\ell' \Lambda_1 \widetilde{v}_\ell - \frac{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell} \, \|E\|_2}{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell} \\
&= \widetilde{v}_\ell' \Lambda_1 \widetilde{v}_\ell - \frac{\|E\|_2}{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{\|E\|_2}{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}}
\end{aligned}$$

where

$$\widetilde{v}_\ell = \frac{G_1' \widehat{G}_1 \widehat{v}_\ell}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} \text{ so that } \|\widetilde{v}_\ell\|_2^2 = \sum_{\ell=1}^{Kp} \widetilde{v}_{\ell,j}^2 = 1.$$

Note also that

$$
\begin{aligned}
\widehat{v}_\ell' \left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell &\geq \lambda_{\min}\left\{\left(I_{Kp}+R'R\right)^{-1}\right\}\widehat{v}_\ell'\widehat{v}_\ell \\
&= \lambda_{\min}\left\{\left(I_{Kp}+R'R\right)^{-1}\right\} \quad \left(\text{since } \|\widehat{v}_\ell\|_2^2=1\right) \\
&= \frac{1}{\lambda_{\max}\left(I_{Kp}+R'R\right)} \\
&\geq \frac{1}{1+\lambda_{\max}\left(R'R\right)} \\
&= \frac{1}{1+\|R\|_2^2} \\
&\geq \left[1+\frac{16\left\|E_{21}\right\|_2^2}{\left(\operatorname{sep}\left(\Lambda_1,\Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-12)} \\
&\geq \left[1+\frac{16\left\|E\right\|_2^2}{\left(\operatorname{sep}\left(\Lambda_1,\Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-3)} \\
&\geq \left[1+\frac{16\left(\operatorname{sep}\left(\Lambda_1,\Lambda_2\right)\right)^2/25}{\left(\operatorname{sep}\left(\Lambda_1,\Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-12)} \\
&= \frac{25}{41}
\end{aligned}
$$

Making use of this lower bound, we obtain

$$
\begin{aligned}
\widehat{\lambda}_\ell &\geq \sum_{j=1}^{Kp}\widetilde{v}_{\ell,j}^2\lambda_{1,j}-\frac{\|E\|_2}{\sqrt{\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp}\widetilde{v}_{\ell,j}^2\lambda_{1,j}-\frac{25}{41}\|E\|_2 .
\end{aligned}
$$

Next, recall the notations we have introduced previously on the ordering of the eigenvalues of the matrices $A+E$ and A, i.e.,

$$
\begin{aligned}
\lambda_{(1)}(A+E) &\geq \cdots \geq \lambda_{(Kp)}(A+E) \geq \lambda_{(Kp+1)}(A+E) \geq \cdots \geq \lambda_{(N)}(A+E), \\
\lambda_{(1)}(A) &\geq \cdots \geq \lambda_{(Kp)}(A) \geq \lambda_{(Kp+1)}(A) \geq \cdots \geq \lambda_{(N)}(A)
\end{aligned}
$$

Since $A=\Gamma M_{FF}\Gamma'/\!\!/N_1$ and since part (a) of Lemma D-11 shows that $Rank\,(A)=Kp$ for all N_1, N_2, and T sufficiently large; it follows that

$$
\lambda_{(Kp+1)}(A)=\cdots=\lambda_{(N)}(A)=0. \tag{137}
$$

In addition, by Corollary 8.1.6 of Golub and van Loan (1996), we have the inequality.

$$\lambda_{(Kp+1)}(A+E) \leq \lambda_{(Kp+1)}(A) + \|E\|_2 . \tag{138}$$

Making use of expressions (137) and (138); we see that, for any $\ell \in \{1, ..., Kp\}$,

$$\begin{aligned}
\widehat{\lambda}_\ell - \lambda_{(Kp+1)}(A+E) &\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{25}{41}\|E\|_2 - \left\{\lambda_{(Kp+1)}(A) + \|E\|_2\right\} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{66}{41}\|E\|_2 \quad \left(\text{since } \lambda_{(Kp+1)}(A) = 0 \text{ here}\right) \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41}\|E\|_2 \\
&\quad \left(\text{since } \lambda_{1,j} = \lambda_{(j)}(A) \text{ as discussed previously}\right) \\
&\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41}\frac{\operatorname{sep}(\Lambda_1, \Lambda_2)}{5} \qquad \text{(by Lemma D-12)} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{205}\operatorname{sep}(\Lambda_1, 0) \quad (\text{since } \Lambda_2 = 0 \text{ here}) \\
&\geq \lambda_{\min}(\Lambda_1) - -\frac{66}{205}\operatorname{sep}(\Lambda_1, 0) \quad \left(\text{since } \Lambda_1 = diag\left(\lambda_{(1)}(A), ..., \lambda_{(Kp)}(A)\right)\right) \\
&= \frac{139}{205}\operatorname{sep}(\Lambda_1, 0) \\
&\quad (\text{since } \operatorname{sep}(\Lambda_1, 0) = \lambda_{\min}(\Lambda_1) \text{ by Theorem 3.1 of Stewart and Sun (1990)}) \\
&\geq \frac{139}{205}\underline{c} > 0 \quad (\text{by part (c) of Lemma D-11}).
\end{aligned}$$

This shows that the set $\left\{\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right\}$ contains the Kp largest eigenvalues of the matrix $A+E$. It further follows from the result given in part (c) that the columns of the matrix

$$\widehat{G}_1 \widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1 \widehat{v}_1 & \widehat{G}_1 \widehat{v}_2 & \cdots & \widehat{G}_1 \widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the matrix $A+E$. □

Lemma D-15: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{\widehat{N}_1 - N_1}{N_1} \xrightarrow{p} 0$$

(b)

$$\left\| \frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0$$

(c) Let

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

where G_1, G_2, and R are as defined in Lemma D-14 above. Also, let $\widehat{V}$ be the $Kp \times Kp$ orthogonal matrix given in expression (129) of Lemma D-14. Then, there exists some positive constant $\overline{C}$ such that

$$\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \leq \overline{C} < \infty$$

for N_1, N_2, and T sufficiently large. In addition,

$$\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0$$

where

$$Q = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \Xi \widehat{V},$$

with Ξ being the $Kp \times Kp$ orthogonal matrix whose columns are the eigenvectors of the matrix

$$M_{FF}^* = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} \frac{1}{T-p+1} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}.$$

(d) For all fixed index t

$$\left\| \frac{G_1' U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2 = o_p(1).$$

(e) For all fixed index t

$$\left\| \frac{U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2^2 = O_p(1).$$

(f) For all fixed index t,

$$\left\| \frac{G_2' U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

(g) Let

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

where G_1, G_2, and R are as defined in Lemma D-14 above. Also, let $\widehat{V}$ be the $Kp \times Kp$ orthogonal matrix given in expression (129) of Lemma D-14. Then, for all fixed index t,

$$\left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1,\ N_2, \text{ and } T \to \infty.$$

(h)

$$\left\| \frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 = \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 = o_p(1) \text{ as } N_1, N_2, T \to \infty.$$

where Q is as defined in part (c) above.

(i)

$$\|\underline{F}_t\|_2 = O_p(1) \text{ for all } t.$$

(j)

$$\left\| \widehat{\underline{F}}_T - Q'\underline{F}_T \right\|_2 = o_p(1) \text{ as } N_1,\ N_2, \text{ and } T \to \infty$$

where $\widehat{\underline{F}}_T$ denotes the principal component estimator of the factor vector $\underline{F}_T$ obtained after the variables have been pre-screened based on the decision rule

$$i \in \begin{cases} \widehat{H}^c & \text{if } \mathbb{S}_{i,T}^{+} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \\ \widehat{H} & \text{if } \mathbb{S}_{i,T}^{+} \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \end{cases},$$

as described in section 3. Here, $\mathbb{S}_{i,T}^{+}$ may be either the statistic $\sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}|$or the statistic $\max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$.

Proof of Lemma D-15:

To show part (a), note first that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{\left|\frac{\widehat{N}_1 - N_1}{N_1}\right| \geq \epsilon\right\} &= \left\{\left|\frac{1}{N_1}\sum_{i=1}^{N}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \epsilon\right\} \\
&= \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} + \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \epsilon\right\} \\
&\subseteq \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| + \left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \epsilon\right\} \\
&\subseteq \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\} \\
&= \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\}
\end{aligned}$$

By Markov's inequality, we have

$$\begin{aligned}
& \Pr\left(\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right) \\
= \;& \Pr\left(\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right|^2 \geq \frac{\epsilon^2}{4}\right) \\
\leq \;& \frac{4}{\epsilon^2}E\left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right|^2\right\} \\
= \;& \frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c} E\left[\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\left(\mathbb{I}\left\{k \in \widehat{H^c}\right\} - 1\right)\right] \\
= \;& \frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\mathbb{I}\left\{k \in \widehat{H^c}\right\}\right] - E\left[\mathbb{I}\left\{k \in \widehat{H^c}\right\}\right] - E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] + 1\right) \\
= \;& \frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{\Pr\left(\left\{i \in \widehat{H^c}\right\} \cap \left\{k \in \widehat{H^c}\right\}\right) - \Pr\left(k \in \widehat{H^c}\right)\right\} \\
& + \frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{1 - \Pr\left(i \in \widehat{H^c}\right)\right\} \\
\leq \;& \frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{\Pr\left(k \in \widehat{H^c}\right) - \Pr\left(k \in \widehat{H^c}\right)\right\} + \frac{4}{\epsilon^2}\frac{1}{N_1}\sum_{i\in H^c}\left\{1 - \Pr\left(i \in \widehat{H^c}\right)\right\} \\
\leq \;& \frac{4}{\epsilon^2}\frac{1}{N_1}\sum_{i\in H^c}\left\{1 - \min_{i\in H^c}\Pr\left(i \in \widehat{H^c}\right)\right\} \to 0 \text{ as } N_1, N_2, \text{ and } T \to \infty.
\end{aligned}$$

where the last line above follows from the fact that, for $i \in H^c$ and for either the case where

$\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we can apply the results of Theorem 3 to obtain

$$\begin{aligned}
\min_{i\in H^c} \Pr\left(i \in \widehat{H^c}\right) &\geq \Pr\left(\bigcap_{i\in H^c}\left\{\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right) \\
&= P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
&\to 1
\end{aligned}$$

Also, making use of Markov's inequality, we obtain, for either the case where $\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$,

$$\begin{aligned}
& \Pr\left(\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right) \\
= \;& \Pr\left(\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\} \geq \frac{\epsilon}{2}\right) \\
\leq \;& \frac{2}{\epsilon}E\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right] \\
= \;& \frac{2}{\epsilon}\frac{1}{N_1}\sum_{i\in H}\Pr\left(i\in\widehat{H^c}\right) \\
= \;& \frac{2}{\epsilon}\frac{1}{N_1}\sum_{i\in H}\Pr\left(\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq \;& \frac{2}{\epsilon}\frac{dN_2\varphi}{NN_1}\left[1+o\left(1\right)\right] \\
& \text{(following an argument similar to that given in the proof of Theorem 2)} \\
\to \;& 0 \; \left(\text{since } \frac{\varphi}{N_1} \to 0 \text{ and } \frac{N_2}{N} = O\left(1\right)\right).
\end{aligned}$$

Combining these results, we have that

$$\begin{aligned}
& \Pr\left(\left|\frac{\widehat{N}_1 - N_1}{N_1}\right| \geq \epsilon\right) \\
\leq\ & \Pr\left(\left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\}\right) \\
\leq\ & \Pr\left(\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right) + \Pr\left(\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right) \\
& \text{(by the union bound)} \\
\to\ & 0
\end{aligned}$$

For part (b), note that

$$\begin{aligned}
\left\|\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right\|_F^2 &= \frac{1}{N_1} tr\left\{\left(\Gamma\left(\widehat{H^c}\right) - \Gamma\right)'\left(\Gamma\left(\widehat{H^c}\right) - \Gamma\right)\right\} \\
&= \frac{1}{N_1}\sum_{i=1}^{N} tr\left\{\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)'\right\} \\
&= \frac{1}{N_1}\sum_{i=1}^{N}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)'\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right) \\
&= \frac{1}{N_1}\sum_{i=1}^{N}\gamma_i'\gamma_i\left[1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i\left[1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad \text{(since } \gamma_i = 0 \text{ for } i \in H\text{)}
\end{aligned}$$

Applying Markov's inequality, we have, for any $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_F^2 \geq \epsilon\right) &\leq \frac{1}{\epsilon}E\left\{\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i\left[1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\right\} \\
&= \frac{1}{\epsilon}\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i\left[1-\Pr\left(i\in\widehat{H^c}\right)\right] \\
&\leq \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i \\
&\leq \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\left(\sup_{i\in H^c}\|\gamma_i\|_2\right)^2 \\
&\leq \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\overline{C}^2 \quad \text{(by Assumption 3-5)} \\
&\to 0 \quad \left(\text{since } \min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\to 1 \text{ by Theorem 3}\right)
\end{aligned}$$

from which we further deduce that

$$\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_2 \leq \left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_F \xrightarrow{p} 0.$$

Turning our attention to part (c), note that since, by definition,

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

where $G_1'G_1 = I_{Kp}$, $G_2'G_2 = I_{N-Kp}$, and $G_1'G_2 = 0$; it follows that

$$\begin{aligned}
\widehat{G}_1'\widehat{G}_1 &= \left(I_{Kp}+R'R\right)^{-1/2}\left(G_1'+R'G_2'\right)(G_1+G_2R)\left(I_{Kp}+R'R\right)^{-1/2} \\
&= \left(I_{Kp}+R'R\right)^{-1/2}\left(I_{Kp}+R'R\right)\left(I_{Kp}+R'R\right)^{-1/2} \\
&= I_{Kp}
\end{aligned}$$

Hence, by Assumption 3-6,

$$\begin{aligned}
\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 &\leq \left\| \widehat{V}'\widehat{G}_1' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&= \sqrt{\lambda_{\max}\left(\widehat{G}_1\widehat{V}\widehat{V}'\widehat{G}_1'\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}\left(\widehat{V}'\widehat{G}_1'\widehat{G}_1\widehat{V}\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}\left(I_{Kp}\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \quad \left(\text{since } \widehat{V} \text{ is an orthogonal matrix}\right) \\
&= \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \leq \overline{C} < \infty \text{ for } N_1, N_2 \text{ sufficiently large}
\end{aligned}$$

Now, to show the second result in part (c), note that, since

$$Q = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \Xi\widehat{V} \text{ and } G_1 = \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi\ ,$$

we can write

$$\begin{aligned}
\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' &= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'\Gamma}{N_1} \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \frac{\widehat{V}'G_1'\Gamma}{\sqrt{N_1}} \\
&= \widehat{V}'\left(\widehat{G}_1 - G_1\right)'\frac{\Gamma}{\sqrt{N_1}}
\end{aligned}$$

from which it follows that

$$
\begin{aligned}
\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 &\leq \left\| \widehat{V}' \right\|_2 \left\| \left(\widehat{G}_1 - G_1 \right)' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&= \sqrt{\lambda_{\max}\left(\widehat{V}\widehat{V}' \right)} \sqrt{\lambda_{\max}\left\{ \left(\widehat{G}_1 - G_1 \right)\left(\widehat{G}_1 - G_1 \right)' \right\}} \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right)} \\
&= \sqrt{\lambda_{\max}\left(I_{Kp} \right)} \sqrt{\lambda_{\max}\left\{ \left(\widehat{G}_1 - G_1 \right)' \left(\widehat{G}_1 - G_1 \right) \right\}} \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right)} \\
&\quad \left(\text{since } \widehat{V} \text{ is an orthogonal matrix and since } \lambda_{\max}\left(AA' \right) = \lambda_{\max}\left(A'A \right) \right) \\
&\leq \sqrt{\overline{C}} \left\| \widehat{G}_1 - G_1 \right\|_2 \quad \text{(by Assumption 3-6)} \\
&= o_p(1) \quad \text{as } N_1,\ N_2, \text{ and } T \to \infty \ \text{(by part (b) of Lemma D-14)}.
\end{aligned}
$$

Next, to show part (d), we first write

$$
\begin{aligned}
\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c} \right)}{\sqrt{N_1}} \right\|_2^2 &= \sum_{k=1}^{Kp} \left(\sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} + \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 + 2 \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \qquad (139)
\end{aligned}
$$

where $g_{1,ik}$ denotes the $(i,k)^{th}$ element of G_1. Now, consider the first term on the right-hand side of expression (139). Write

$$
\begin{aligned}
& \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1+1\right)\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1+1\right)g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}g_{1,jk}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
= & \mathcal{E}_{1,1,t}+\mathcal{E}_{1,2,t}+\mathcal{E}_{1,3,t}+\mathcal{E}_{1,4,t}
\end{aligned}
$$

Focusing first on the term $\mathcal{E}_{1,1,t}$, we have

$$
\begin{aligned}
& \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right)^2 \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\left|\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right|\right)^2 \\
\leq & 2\sum_{k=1}^{Kp}\left(\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2\right)\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-2\mathbb{I}\left\{i\in\widehat{H^c}\right\}+1\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right)
\end{aligned}
$$

Now, for either the case where $\mathbb{S}_{i,T}^+=\sum_{\ell=1}^d\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^+=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have

$$
\begin{aligned}
0 &\leq E\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c}\left[1-\Pr\left(i\in\widehat{H^c}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c}\left[1-P\left(\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\right] \\
&\leq 1-P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
&\quad \text{(given that } N_1=\#\left\{H^c\right\}\text{, where } \#\left\{H^c\right\} \text{ denotes the cardinality of the set } H^c\text{)} \\
&\to 0\ \left(\text{since, by Theorem 3, } P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\to 1\right).
\end{aligned}
$$

Moreover, by part (b) of Assumption 3-3, we have

$$
E\left[\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right]=\sum_{i\in H^c}g_{1,ik}^2E\left[u_{i,t}^2\right]\leq C\sum_{i=1}^{N}g_{1,ik}^2\leq C
$$

It follows by Markov's inequality that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)=o_p(1)\ \text{ and }\ \sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2=O_p(1)
$$

from which we deduce that

$$
\begin{aligned}
\mathcal{E}_{1,1,t} &= \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right)^2 \\
&\leq 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
&= o_p(1)
\end{aligned}
$$

Consider next the term $\mathcal{E}_{1,2,t}$. To proceed, let $U_{t,N}(H^c)$ denote an $N\times 1$ vector whose i^{th} component $U_{i,t,N}(H^c)$ is given by

$$
U_{i,t,N}(H^c)=\begin{cases} u_{i,t} & \text{if } i\in H^c \\ 0 & \text{if } i\in H \end{cases}.
$$

and we can write

$$
\begin{aligned}
\mathcal{E}_{1,2,t} &= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= 2 \left\| \frac{G_1' U_{t,N}(H^c)}{\sqrt{N_1}} \right\|_2^2 \\
&\leq 2tr \left\{ \frac{G_1' U_{t,N}(H^c) U_{t,N}(H^c)' G_1}{N_1} \right\} \\
&= 2tr \left\{ \Xi' \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \Xi \right\} \\
&= 2tr \left\{ \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}(H^c) U_{t,N}(H^c)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \right\} \\
&= 2tr \left\{ \frac{\Gamma_*' U_{t,N}(H^c) U_{t,N}(H^c)' \Gamma_*}{N_1^2} \right\} \quad \left(\text{where } \Gamma_* = \Gamma \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \right) \\
&= \frac{2}{N_1^2} U_{t,N}(H^c)' \Gamma_* \Gamma_*' U_{t,N}(H^c) \\
&= \frac{2}{N_1^2} \sum_{i \in H^c} \sum_{j \in H^c} \gamma_{*,i}' \gamma_{*,j} u_{i,t} u_{j,t}
\end{aligned}
$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma\left(\Gamma'\Gamma/N_1\right)^{-1/2}$. Hence,

$$\begin{aligned}
0 &\leq E\left[\mathcal{E}_{1,2,t}\right] \\
&= \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_{*,i}\gamma_{*,j}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j E\left[u_{i,t}u_{j,t}\right] \\
&\leq \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\left|\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j\right|\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\quad \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \left.\sup_{i\in H^c}\left\|\gamma_i\right\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\right) \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1} \to 0 \text{ as } N_1 \to \infty. \Big(\text{since, under Assumption 3-3(d), there exists a positive constant } \overline{C} \\
&\qquad\qquad \left.\text{such that } \sup_t \frac{1}{N_1}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}$$

It follows by Markov's inequality that

$$\mathcal{E}_{1,s,t} = o_p\left(1\right).$$

Now, for $\mathcal{E}_{1,3,t}$, write

$$
\begin{aligned}
& |\mathcal{E}_{1,3,t}| \\
= \quad & \left| \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
= \quad & \left| \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{j \in H^c} g_{1,jk} u_{j,t} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
\leq \quad & \frac{2}{N_1} \sum_{k=1}^{Kp} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \left| \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \right| \\
\leq \quad & \frac{2}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right)^2} \sqrt{\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2} \left| \sum_{j \in H^c} g_{1,jk} u_{j,t} \right| \\
\leq \quad & \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right)^2} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
& + \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right)^2} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
& \left(\text{by the inequality } |XY| \leq \frac{1}{2} X^2 + \frac{1}{2} Y^2 \right) \\
= \quad & \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right)} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
& + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right)} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \;& \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sum_{j\in H^c}\sum_{\ell\in H^c} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= \;& \frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sum_{k=1}^{Kp} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= \;& \frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\frac{e_{j,N}'\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\sum_{k=1}^{Kp}\Xi e_{k,Kp}e_{k,Kp}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma' e_{\ell,N}}{\sqrt{N_1}}E\left[u_{j,t}u_{\ell,t}\right] \\
& \left(\text{since } G_1 = \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi\right) \\
= \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Xi\Xi'\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \quad \left(\text{where } \Gamma_* = \Gamma\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right) \\
= \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
& (\text{since } \Xi \text{ is an orthogonal matrix}) \\
= \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} \gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right]
\end{aligned}
$$

where we take

$$
\gamma_{*,j} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j,
$$

Applying the triangle and Cauchy-Schwarz inequalities, we further obtain

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \; & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right] \\
= \; & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell E\left[u_{j,t}u_{\ell,t}\right] \\
\leq \; & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell\right|\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \; & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sqrt{\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_j}\sqrt{\gamma_\ell'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_\ell}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \; & \frac{\overline{c}}{\underline{C}}\frac{1}{\sqrt{N_1}}\frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
& \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
& \left.\sup_{i\in H^c}\|\gamma_i\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\right) \\
\leq \; & \frac{\overline{c}}{\underline{C}}\frac{\overline{C}}{\sqrt{N_1}} \to 0 \text{ as } N_1 \to \infty. \text{(since, under Assumption 3-3(d) that there exists a} \\
& \left.\text{positive constant } \overline{C} \text{ such that } \sup_t \frac{1}{N_1}\sum_{j\in H}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}
$$

from which we further deduce, upon applying Markov's inequality, that

$$
\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2 = o_p(1).
$$

Moreover, since we have previously shown that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right) = o_p(1) \text{ and } \sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1),
$$

it follows from these calculations that

$$
\begin{aligned}
\left|\mathcal{E}_{1,3,t}\right| &\leq \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right)} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
&\quad + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right)} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t} \right)^2 \\
&= o_p(1).
\end{aligned}
$$

In a similar way, we can also show that

$$
\left|\mathcal{E}_{1,4,t}\right| = o_p(1).
$$

Finally, application of the Slutsky's theorem then allows us to deduce that

$$
\begin{aligned}
\frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} &= \mathcal{E}_{1,1,t} + \mathcal{E}_{1,2,t} + \mathcal{E}_{1,3,t} + \mathcal{E}_{1,4,t} \\
&= o_p(1) + o_p(1) + o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}
$$

Next, consider the second term on the right-hand side of expression (139). In this case, write

$$
\begin{aligned}
&\frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right)^2 \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right| \right)^2 \\
&\leq 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right] \left[\sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right]
\end{aligned}
$$

Note that, for either the case where $\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1 \leq \ell \leq d} \left|S_{i,\ell,T}\right|$, we have, by applying an argument similar to that given in the proof of Theorem 2,

$$
\begin{aligned}
0 &\leq E\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right] \\
&= \frac{1}{N_1}\sum_{i\in H}\Pr\left(i\in\widehat{H^c}\right) \\
&= \frac{1}{N_1}\sum_{i\in H}P\left(\mathbb{S}^+_{i,T}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
&\leq \frac{dN_2\varphi}{NN_1}\left\{1+2^2AT^{-(1-\alpha_1)\frac{1}{2}}+2^2A\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)^3T^{-(1-\alpha_1)\frac{1}{2}}\right\} \\
&= \frac{dN_2\varphi}{N_1\left(N_1+N_2\right)}\left[1+o\left(1\right)\right] \\
&\rightarrow 0 \text{ as } N_1,N_2,T\rightarrow\infty
\end{aligned}
$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$
E\left[\sum_{i\in H}g^2_{1,ik}u^2_{i,t}\right]=\sum_{i\in H}g^2_{1,ik}E\left[u^2_{i,t}\right]\leq C\sum_{i=1}^{N}g^2_{1,ik}\leq C.
$$

It follows by Markov's inequality that

$$
\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}=o_p\left(1\right)\text{ and }\sum_{i\in H}g^2_{1,ik}u^2_{i,t}=O_p\left(1\right)
$$

from which we deduce that

$$
\begin{aligned}
&\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
\leq\ & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\left[\sum_{i\in H}g^2_{1,ik}u^2_{i,t}\right] \\
=\ & o_p\left(1\right).
\end{aligned}
$$

Combining these results and using the inequality $\sqrt{a_1+a_2}\leq\sqrt{a_1}+\sqrt{a_2}$ for $a_1,a_2\geq0$, we further obtain, for all t,

$$\begin{aligned}
\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 &\leq \sqrt{\frac{2}{N_1} \sum_{k=1}^{K} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
&\quad + \sqrt{\frac{2}{N_1} \sum_{k=1}^{K} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
&= o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

For part (e), write

$$\begin{aligned}
\left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &= \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&\leq \frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2
\end{aligned}$$

Note that, by Assumption 3-3(b),

$$E\left[\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2\right] = \frac{1}{N_1} \sum_{i \in H^c} E\left[u_{i,t}^2\right] \leq C \ \ (\text{since } N_1 = \#\left\{H^c\right\})$$

so that, by applying Markov's inequality, we obtain

$$\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i \in H} \left\{i \notin \widehat{H^c}\right\} \subseteq \left\{\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 < \epsilon\right\}$$

so that by DeMorgan's law

$$\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2\geq\epsilon\right\}\subseteq\left\{\bigcap_{i\in H}\left\{i\notin\widehat{H^c}\right\}\right\}^c=\bigcup_{i\in H}\left\{i\in\widehat{H^c}\right\}$$

Hence, for either the case where $\mathbb{S}_{i,T}^{+}=\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+}=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have, by applying an argument similar to that given in the proof of Theorem 2,

$$\begin{aligned}
&\Pr\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2\geq\epsilon\right\}\\
\leq\;&\Pr\left\{\bigcup_{i\in H}\left\{i\in\widehat{H^c}\right\}\right\}\\
\leq\;&\sum_{i\in H}\Pr\left\{i\in\widehat{H^c}\right\}\\
=\;&\sum_{i\in H}P\left(\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\\
\leq\;&\frac{dN_2\varphi}{N}\left\{1+2^2AT^{-(1-\alpha_1)\frac{1}{2}}+2^2A\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)^3T^{-(1-\alpha_1)\frac{1}{2}}\right\}\\
=\;&\frac{dN_2\varphi}{N_1+N_2}\left[1+o\left(1\right)\right]\\
\to\;&0\text{ as }N_1,N_2,T\to\infty
\end{aligned}$$

Hence,

$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2=o_p\left(1\right)$$

from which it further follows that

$$\begin{aligned}
\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2&\leq\frac{1}{N_1}\sum_{i\in H^c}u_{i,t}^2+\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2\\
&=O_p\left(1\right)+o_p\left(1\right)\\
&=O_p\left(1\right).
\end{aligned}$$

Turning our attention to part (f), note first that since $G=\left[\begin{array}{cc}G_1 & G_2\end{array}\right]$ is an orthogonal matrix, we have $I_N = GG' = G_1 G_1' + G_2 G_2'$ or $G_2 G_2' = I_N - G_1 G_1'$. Hence, we can write

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &= \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} - \frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&\leq \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} + \frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Applying the results from parts (d) and (e) above, we then obtain

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &\leq \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

so that

$$\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

Now, to show part (g), first write

$$\begin{aligned}
\frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} &= \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}
\end{aligned}$$

Note that

$$
\begin{aligned}
&\left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
= \quad & \left\| \frac{\widehat{V}'\left(I_{Kp} + R'R\right)^{-1/2}\left[G_1' U_{t,N}\left(\widehat{H^c}\right) + R'G_2' U_{t,N}\left(\widehat{H^c}\right)\right]}{\sqrt{N_1}} \right\|_2 \\
\leq \quad & \left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
& + \left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
= \quad & \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
& \left(\text{since } \widehat{V}'\widehat{V} = I_{Kp} \text{ so that } \left\|\widehat{V}\right\|_2 = 1\right)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
= \;& \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
\leq \;& \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \left\| \left(I_{Kp} + R'R\right)^{-1/2} \right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \left\| \left(I_{Kp} + R'R\right)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
\leq \;& \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \frac{1}{\sqrt{1 + \lambda_{\min}\left(R'R\right)}} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \frac{\|R\|_2}{\sqrt{1 + \lambda_{\min}\left(R'R\right)}} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
\leq \;& \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
= \;& o_p(1)
\end{aligned}
$$

where the last line follows from the fact that

$$
\|R\|_2 \xrightarrow{p} 0, \; \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \xrightarrow{p} 1, \; \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \text{ and } \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1)
$$

as shown in part (a) in Lemma D-14 and in parts (a), (d), and (f) of this lemma.

Turning our attention to part (h), we write

$$
\begin{aligned}
& \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
= \;& Q' + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right) + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}}\right) \\
= \;& Q' + \left(\left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right) \\
& + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right) \\
= \;& Q' + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right) + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \\
& + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right) + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right)
\end{aligned}
$$

so that, by the triangle inequality

$$
\begin{aligned}
& \left\| \frac{\Gamma\left(\widehat{H^c}\right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 \\
= & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
\leq & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right] \widehat{V}' \widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 + \left\| \widehat{V}' \widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 \\
\leq & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \widehat{V}' \widehat{G}_1' \right\|_2 \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \widehat{V}' \widehat{G}_1' \right\|_2 \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \\
= & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2
\end{aligned}
$$

where the last equality follows from the fact that

$$
\left\| \widehat{V}' \widehat{G}_1' \right\|_2 = \left\| \widehat{G}_1 \widehat{V} \right\|_2 = \sqrt{\lambda_{\max}\left(\widehat{V}' \widehat{G}_1' \widehat{G}_1 \widehat{V} \right)} = \sqrt{\lambda_{\max}\left(I_{Kp} \right)} = 1.
$$

Now, by parts (a), (b), and (c) of this lemma, we have that

$$
\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \xrightarrow{p} 0, \quad \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0, \text{ and}
$$

and

$$
\left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max}\left(\frac{\Gamma' \Gamma}{N_1} \right)} \leq \overline{C} < \infty \text{ for all } N_1, N_2 \text{ sufficiently large.}
$$

It follows that

$$
\begin{aligned}
\left\| \frac{\Gamma\left(\widehat{H^c}\right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 &= \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
&\leq \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \\
&= o_p(1).
\end{aligned}
$$

To show part (i), let $\overline{C}$ be the positive constant given in Lemma C-5 such that

$$E \left\| \underline{F}_t \right\|_2^6 \leq \overline{C} < \infty \text{ for all } t;$$

and, for any $\epsilon > 0$, we let $C_\epsilon = \overline{C}^{\frac{1}{6}} / \sqrt{\epsilon}$. Applying Markov's inequality, we see that

$$
\begin{aligned}
\Pr\left(\left\| \underline{F}_t \right\|_2 \geq C_\epsilon\right) &\leq \Pr\left(\left\| \underline{F}_t \right\|_2^2 \geq C_\epsilon^2\right) \\
&\leq \frac{1}{C_\epsilon^2} E \left\| \underline{F}_t \right\|_2^2 \\
&\leq \frac{1}{C_\epsilon^2} \left(E \left\| \underline{F}_t \right\|_2^6\right)^{\frac{1}{3}} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{\epsilon}{\overline{C}^{\frac{1}{3}}} \overline{C}^{\frac{1}{3}} \\
&\leq \epsilon
\end{aligned}
$$

from which it follows that $\left\| \underline{F}_t \right\|_2 = O_p(1)$ for all t.

Lastly, to show part (j), note that, similar to the derivation given in the proof of Theorem 4, except that we replace the fixed index t with the sample size T, we can write

$$\begin{aligned}
\widehat{\underline{F}}_T - Q'\underline{F}_T &= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right) \underline{F}_T + \frac{\widehat{V}'\widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right) \underline{F}_T + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \underline{F}_T \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right] \widehat{V}'\widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \underline{F}_T + \frac{\widehat{V}'\widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Next, note that, by following the same derivation as that given for the proof of part (g), we can show that

$$\begin{aligned}
&\left\| \frac{\widehat{V}'\widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
\leq &\left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \|R\|_2 \left\| \frac{G_2' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\}
\end{aligned}$$

Moreover, by argument similar to that given for parts (d) and (f) of this lemma, we can show that, as N_1, N_2, and $T \to \infty$;

$$\left\| \frac{G_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0 \tag{140}$$

and

$$\left\| \frac{G_2' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1). \tag{141}$$

It follows from applying expressions (140) and (141), part (a) of this lemma, and part (a) of Lemma D-14 that

$$\left\| \frac{\widehat{V}'\widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1, N_2, \text{ and } T \to \infty. \tag{142}$$

In addition, note that by applying Lemma C-5 and the Markov's inequality in a way similar to the argument given for the proof of part (i) above, we can show that

$$\|\underline{F}_T\|_2 = O_p(1). \tag{143}$$

Making use of the results given in expressions (142) and (143) and applying the triangle inequality as well as parts (a)-(c) of this lemma, expression (143), and the Slutsky's theorem; we then obtain, as N_1, N_2, and $T \to \infty$;

$$
\begin{aligned}
& \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq \; & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \| \underline{F}_T \|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \| \underline{F}_T \|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \widehat{V}' \widehat{G}_1' \right\|_2 \left\| \frac{\Gamma \left(\widehat{H^c} \right) - \Gamma}{\sqrt{N_1}} \right\|_2 \| \underline{F}_T \|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{T,N} \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
= \; & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \| \underline{F}_T \|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \| \underline{F}_T \|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \frac{\Gamma \left(\widehat{H^c} \right) - \Gamma}{\sqrt{N_1}} \right\|_2 \| \underline{F}_T \|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{T,N} \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
& \left(\text{again since } \left\| \widehat{V}' \widehat{G}_1' \right\|_2 = \lambda_{\max} \left(\widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}_1' \right) = \lambda_{\max} \left(\widehat{V}' \widehat{G}_1' \widehat{G}_1 \widehat{V} \right) = \lambda_{\max} \left(I_{Kp} \right) = 1 \right) \\
= \; & o_p(1) O_p(1) + o_p(1) O_p(1) O_p(1) + O_p(1) o_p(1) O_p(1) + o_p(1) \\
= \; & o_p(1) . \; \square
\end{aligned}
$$

Lemma D-16: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true as $N_1, N_2, T \to \infty$.

(a)

$$
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N} \left(\widehat{H^c} \right)}{\sqrt{N_1}} \right\|_2 = o_p(1), \text{ where } T_h = T - h - p + 1.
$$

(b)

$$
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N} \left(\widehat{H^c} \right)}{\sqrt{N_1}} \right\|_2^2 = O_p(1) .
$$

(c)

$$
, \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N} \left(\widehat{H^c} \right)}{\sqrt{N_1}} \right\|_2 = O_p(1)
$$

(d)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 = O_p(1) \text{ and } \left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 = O_p(1)$$

(e)

$$\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}\right\|_2 = o_p(1)$$

(f)

$$\left\|\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 = o_p(1)$$

(g)

$$\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{F}_t - Q'\underline{F}_t\right)\left(\widehat{F}_t - Q'\underline{F}_t\right)'\right\|_2 = o_p(1).$$

Proof of Lemma D-16:

For part (a), first write

$$\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}} + \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
\leq \;& \frac{2}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 + \frac{2}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
= \;& \frac{2}{N_1T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
& + \frac{2}{N_1T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \qquad (144)
\end{aligned}$$

where $g_{1,ik}$ denotes the $(i,k)^{th}$ element of

$$G_1 = \frac{\Gamma_* \Xi}{\sqrt{N_1}} = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi$$

Now, where

$$\begin{aligned}
& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 + 1 \right) \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 + 1 \right) g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 \right) g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 \right) g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \underline{\mathcal{E}}_{1,1} + \underline{\mathcal{E}}_{1,2} + \underline{\mathcal{E}}_{1,3} + \underline{\mathcal{E}}_{1,4}
\end{aligned}$$

Focusing first on the term $\underline{\mathcal{E}}_{1,1}$, we have

$$\begin{aligned}
& \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
= & \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right)^2 \\
\leq & \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\left|\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right|\right)^2 \\
\leq & 2\sum_{k=1}^{Kp}\left(\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-2\mathbb{I}\left\{i\in\widehat{H^c}\right\}+1\right)\right]\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right)
\end{aligned}$$

Now, for either the case where $\mathbb{S}_{i,T}^{+}=\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+}=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have, by applying Theorem 3,

$$\begin{aligned}
0 \leq\ & E\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right] \\
=\ & \frac{1}{N_1}\sum_{i\in H^c}\left[1-\Pr\left(i\in\widehat{H^c}\right)\right] \\
=\ & \frac{1}{N_1}\sum_{i\in H^c}\left[1-P\left(\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\right] \\
\leq\ & 1-P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
& \text{(given that } N_1=\#\left\{H^c\right\}\text{, where } \#\left\{H^c\right\}\text{ denotes the cardinality of the set } H^c\text{)} \\
\to\ & 0\ \left(\text{since } P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\to 1\right).
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$
\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2\right] &= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 E\left[u_{i,t}^2\right] \\
&\leq C\frac{T-h-p+1}{T_h}\sum_{i=1}^{N} g_{1,ik}^2 \\
&\leq C \quad \left(\text{since } \sum_{i=1}^{N} g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1\right)
\end{aligned}
$$

It follows by Markov's inequality that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right) = o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1)
$$

from which we deduce that

$$
\begin{aligned}
\underline{\mathcal{E}}_{1,1} &= \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right) g_{1,ik}u_{i,t}\right)^2 \\
&\leq 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2\right) \\
&= o_p(1).
\end{aligned}
$$

Next, consider the term $\underline{\mathcal{E}}_{1,2}$. To proceed, write

$$
\begin{aligned}
& \left|\underline{\mathcal{E}}_{1,2}\right| \\
= & \left|\frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t}\right| \\
= & \left|\frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{j \in H^c} g_{1,jk} u_{j,t} \sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right) g_{1,ik} u_{i,t}\right| \\
\leq & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left|\sum_{j \in H^c} g_{1,jk} u_{j,t}\right| \left|\sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right) g_{1,ik} u_{i,t}\right| \\
\leq & \frac{2}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2} \left|\sum_{j \in H^c} g_{1,jk} u_{j,t}\right| \\
\leq & \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
& + \frac{1}{N_1} \sum_{k=1}^{Kp} \sqrt{\sum_{i \in H^c} \left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)^2} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t}\right)^2 \\
& \left(\text{by the inequality } |XY| \leq \frac{1}{2} X^2 + \frac{1}{2} Y^2\right) \\
= & \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right)} \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \\
& + \sqrt{\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right)} \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{1}{\sqrt{N_1}} \sum_{k=1}^{Kp} \left(\sum_{j \in H^c} g_{1,jk} u_{j,t}\right)^2
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sum_{j\in H^c}\sum_{\ell\in H^c} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sum_{k=1}^{Kp} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\frac{e_{j,N}'\Gamma_*}{\sqrt{N_1}}\sum_{k=1}^{Kp}\Xi e_{k,Kp}e_{k,Kp}'\Xi'\frac{\Gamma_*'e_{\ell,N}}{\sqrt{N_1}}E\left[u_{j,t}u_{\ell,t}\right] \\
& \left(\text{since } G_1 = \frac{\Gamma_*\Xi}{\sqrt{N_1}} \text{ with } \Gamma_* = \Gamma\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right) \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Xi\Xi'\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
& (\text{since } \Xi \text{ is an orthogonal matrix}) \\
= \quad & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} \gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right]
\end{aligned}
$$

where $\gamma_{*,j} = \left(\Gamma'\Gamma/N_1\right)^{-1/2}\gamma_j$. Applying the triangle and Cauchy-Schwarz inequalities, we further obtain

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right] \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell E\left[u_{j,t}u_{\ell,t}\right] \\
\leq \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell\right|\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sqrt{\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_j}\sqrt{\gamma_\ell'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_\ell}\left|E\left[u_{j,t}u_{\to,t}\right]\right| \\
\leq \; & \frac{\overline{c}}{\underline{C}}\frac{1}{\sqrt{N_1}T_h}\sum_{t=p}^{T-h}\frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
& \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
& \left.\sup_{i\in H^c}\|\gamma_i\|_2\leq\overline{c}<\infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\geq\underline{C}>0\right) \\
\leq \; & \frac{\overline{c}}{\underline{C}}\frac{\overline{C}}{\sqrt{N_1}}\;\to 0 \text{ as } N_1\to\infty. \text{(since, under Assumption 3-3(d), there exists a} \\
& \left.\qquad\text{positive constant } \overline{C} \text{ such that } \sup_t\frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right|\leq\overline{C}<\infty\right)
\end{aligned}
$$

from which we further deduce, upon applying Markov's inequality, that

$$
\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2 = o_p(1)
$$

Moreover, since we have previously shown that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)=o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2u_{i,t}^2=O_p(1),
$$

it follows from these calculations that

$$\begin{aligned}
\left|\underline{\mathcal{E}}_{1,2}\right| &\leq \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
&\quad+\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2 \\
&= o_p\left(1\right).
\end{aligned}$$

In a similar way, we can also show that

$$\left|\underline{\mathcal{E}}_{1,3}\right| = o_p\left(1\right).$$

Finally, let $U_{t,N}\left(H^c\right)$ denote an $N\times 1$ vector whose i^{th} component $U_{i,t,N}\left(H^c\right)$ is given by

$$U_{i,t,N}\left(H^c\right)=\begin{cases} u_{i,t} & \text{if } i\in H^c \\ 0 & \text{if } i\in H \end{cases}.$$

and we can write

$$\begin{aligned}
\underline{\mathcal{E}}_{1,4} &= \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&= \frac{2}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(H^c\right)}{\sqrt{N_1}}\right\|_2^2 \\
&= \frac{2}{T_h}\sum_{t=p}^{T-h} tr\left\{\frac{G_1' U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)' G_1}{N_1}\right\} \\
&= \frac{2}{T_h}\sum_{t=p}^{T-h} tr\left\{\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'}{\sqrt{N_1}}\frac{U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'}{N_1}\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi\right\} \\
&= \frac{2}{T_h}\sum_{t=p}^{T-h} tr\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'}{\sqrt{N_1}}\frac{U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'}{N_1}\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\} \\
&= \frac{2}{T_h}\sum_{t=p}^{T-h} tr\left\{\frac{\Gamma_*' U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'\Gamma_*}{N_1^2}\right\} \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h} U_{t,N}\left(H^c\right)'\Gamma_*\Gamma_*' U_{t,N}\left(H^c\right) \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\gamma_{*,i}'\gamma_{*,j}u_{i,t}u_{j,t}
\end{aligned}$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma\left(\Gamma'\Gamma/N_1\right)^{-1/2}$. Taking expectation, we then obtain

$$\begin{aligned}
0 &\leq E\left[\underline{\mathcal{E}}_{1,4}\right] \\
&= \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_{*,i}\gamma_{*,j}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j E\left[u_{i,t}u_{j,t}\right] \\
&\leq \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\left|\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j\right|\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{1}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\quad \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \left.\sup_{i\in H^c}\left\|\gamma_i\right\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\right) \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1}\frac{T-h-p+1}{T_h} = \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1} \to 0 \text{ as } N_1, T \to \infty. \\
&\quad \Bigg(\text{since, under Assumption 3-3(d), there exist a positive constant } \overline{C} \\
&\qquad \left.\text{such that } \sup_t \frac{1}{N_1}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}$$

It follows by Markov's inequality that

$$\underline{\mathcal{E}}_{1,4} = o_p(1).$$

Application of the Slutsky's theorem then allows us to deduce that

$$\begin{aligned}
\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} &= \underline{\mathcal{E}}_{1,1}+\underline{\mathcal{E}}_{1,2}+\underline{\mathcal{E}}_{1,3}+\underline{\mathcal{E}}_{1,4} \\
&= o_p(1)+o_p(1)+o_p(1)+o_p(1) \\
&= o_p(1).
\end{aligned}$$

Consider now the second term on the extreme right-hand side of expression (144)

$$
\begin{aligned}
& \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right)^2 \\
= \; & \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right| \right)^2 \\
\leq \; & 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right] \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right]
\end{aligned}
$$

Note that, for either the case where $\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right|$, we have

$$
\begin{aligned}
0 \leq \; & E\left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right] \\
= \; & \frac{1}{N_1} \sum_{i \in H} \Pr\left(i \in \widehat{H^c}\right) \\
= \; & \frac{1}{N_1} \sum_{i \in H} P\left(\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
\leq \; & \frac{N_2 \varphi}{N_1 N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\
= \; & \frac{N_2 \varphi}{N_1 N} \left[1 + o(1)\right] \\
& \text{(following an argument similar to that given in the proof of Theorem 2)} \\
\to \; & 0 \text{ as } N_1, N_2, T \to \infty
\end{aligned}
$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2\right] &= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 E\left[u_{i,t}^2\right] \\
&\leq C\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i=1}^{N} g_{1,ik}^2 \\
&\leq C\frac{T-h-p+1}{T_h} \\
&\quad \left(\text{given that } \sum_{i=1}^{N} g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1\right) \\
&\leq C < \infty
\end{aligned}$$

It follows by Markov's inequality that

$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\} = o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
&\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
\leq\ & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2\right] \\
=\ & o_p(1)
\end{aligned}$$

Combining these results, we further obtain

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&\quad +\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&= o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

To show part (b), write

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &= \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2 \\
&= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2 + \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2 \\
&\leq \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}u_{i,t}^2 + \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2
\end{aligned}$$

Next, note that, by making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E\left[\frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}u_{i,t}^2\right] &= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}E\left[u_{i,t}^2\right] \\
&\leq C\frac{T-h-p+1}{T_h}\ \text{(since } N_1=\#\{H\}, \\
&\quad \text{where } \#\{H\} \text{ denotes the cardinality of the set } H) \\
&\leq C\ \text{(since } T_h = T-h-p+1)
\end{aligned}$$

so that, by Markov's inequality,

$$\frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i\in H}\left\{i\notin\widehat{H^c}\right\} \subseteq \left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2 < \epsilon\right\}$$

so that, applying DeMorgan's law, we obtain

$$\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2 \geq \epsilon\right\} \subseteq \left\{\bigcap_{i\in H}\left\{i\notin\widehat{H^c}\right\}\right\}^c = \bigcup_{i\in H}\left\{i\in\widehat{H^c}\right\}$$

It follows that, for any $\epsilon > 0$ and for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have

$$\begin{aligned}
& \Pr\left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \geq \epsilon \right\} \\
\leq \quad & \Pr\left\{ \bigcup_{i \in H} \left\{ i \in \widehat{H^c} \right\} \right\} \\
\leq \quad & \sum_{i \in H} \Pr\left\{ i \in \widehat{H^c} \right\} \\
= \quad & \sum_{i \in H} P\left(\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
\leq \quad & \frac{N_2 \varphi}{N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\
= \quad & \frac{N_2 \varphi}{N} \left[1 + o(1) \right] \\
& \text{(following an argument similar to that given in the proof of Theorem 2)} \\
\to \quad & 0 \text{ as } N_1, N_2, T \to \infty
\end{aligned}$$

Hence,

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 = o_p(1)$$

from which it we further deduce that

$$\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &\leq \frac{1}{T_h N_1} \sum_{t=p}^{T-h} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{T_h} \sum_{t=p}^{T-h} u_{i,t}^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

Now, for part (c), note first that since $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ is an orthogonal matrix, we have

$I_N = GG' = G_1 G_1' + G_2 G_2'$ or $G_2 G_2' = I_N - G_1 G_1'$. Hence, we can write

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &= \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} - \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2
\end{aligned}$$

Applying the results from parts (a) and (b) of this lemma, we then obtain

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

Next, to show part (d), let $\overline{C}$ be the constant given in Lemma C-5 such that

$$E\left\|\underline{F}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t.$$

Now, for any $\epsilon > 0$, let $C_\epsilon^* = \overline{C}^{\frac{1}{3}}/\epsilon$; then, upon application of Markov's inequality, we have

$$\begin{aligned}
\Pr\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\right\|_2^2 \geq C_\epsilon^*\right) &\leq \frac{1}{C_\epsilon^*}\frac{1}{T_h}\sum_{t=p}^{T-h} E\left\|\underline{F}_t\right\|_2^2 \\
&\leq \frac{1}{C_\epsilon^*}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(E\left\|\underline{F}_t\right\|_2^6\right)^{\frac{1}{3}} \text{ (by Liapunov's inequality)} \\
&= \frac{\epsilon}{\overline{C}^{\frac{1}{3}}}\frac{1}{T_h}\sum_{t=p}^{T-h}\overline{C}^{\frac{1}{3}} \\
&= \epsilon\frac{T-h-p+1}{T_h} \\
&\leq \epsilon \text{ (since } T_h = T-h-p+1\text{)}
\end{aligned}$$

so that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 = O_p(1)$$

In addition, note that

$$\begin{aligned}
\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\underline{F}_t'\right\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^2\,\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^2\,\lambda_{\max}\left(\underline{F}_t'\underline{F}_t\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 \\
&= O_p(1)
\end{aligned}$$

Turning our attention to part (e), write

$$
\begin{aligned}
& \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1'\widehat{V}}{T_h\widehat{N}_1} \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'\mathbf{e}_{t,T}}{\sqrt{\widehat{N}_1}}\frac{\mathbf{e}_{t,T}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1-N_1+N_1\right)/N_1}}\frac{U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1-N_1+N_1\right)/N_1}} \\
= \; & \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\frac{1}{N_1T_h}\sum_{t=p}^{T-h}\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V} \\
= \; & \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\frac{1}{N_1T_h}\sum_{t=p}^{T-h}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}\left[G_1'+R'G_2'\right]U_{t,N}\left(\widehat{H^c}\right)\right. \\
& \qquad\left.\times U_{t,N}\left(\widehat{H^c}\right)'\left[G_1+G_2R\right]\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
= \; & \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}G_1'\right. \\
& \left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_1\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}G_1'\right. \\
& \quad\left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_2R\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}R'G_2'\right. \\
& \quad\left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_1\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}R'G_2'\right. \\
& \quad\left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_2R\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\}
\end{aligned}
$$

To analyze the four terms on the right-hand side of the expression above, note first that, by the homogeneity of matrix norm and the triangle inequality,

$$
\begin{aligned}
\left\| G_1' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1 \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left\{ \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right)^2 \right\}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}^2 \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \lambda_{\max} \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right) \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\left\| G_1' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2 \right\|_2 \\
\leq\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right\|_2 \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left\{ \left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1^2} \right) \right\}} \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \lambda_{\max}\left\{ \left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right) \right\}} \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \sqrt{\lambda_{\max}\left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right)} \\
\leq\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| G_2' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2 \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left\| \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1'\widehat{V}}{T_h\widehat{N}_1} \right\|_2 \\
\leq\ & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\widehat{V}\right\|_2^2 \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\widehat{V}\right\|_2^2 \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \\
&+\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\widehat{V}\right\|_2^2 \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
=\ & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \|R\|_2 \left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \\
&+\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&\left(\text{since } \widehat{V}'\widehat{V}=I_{Kp} \text{ so that } \left\|\widehat{V}\right\|_2=1\right) \\
\leq\ & 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1} \right\|_2 \\
\leq \;& 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
& +2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq \;& 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left(\frac{1}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\right)^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
& +2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left(\frac{\|R\|_2}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\right)^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
= \;& 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\frac{1}{1+\lambda_{\min}\left(R'R\right)} \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
& +2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\frac{\|R\|_2^2}{1+\lambda_{\min}\left(R'R\right)} \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq \;& 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right] \\
= \;& o_p(1) \quad \text{(applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
& \qquad\qquad \text{parts (a) and (c) of this lemma, and Slutsky's theorem)}
\end{aligned}
$$

To show part (f), first write

$$
\begin{aligned}
\left\| \frac{\underline{F}' U\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 &= \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \frac{\underline{F}_t U_{t,N}'\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\underline{F}_t U_{t,N}'\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left(\frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right) \underline{F}_t' \underline{F}_t U_{t,N}'\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{\lambda_{\max}\left(\frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}'\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{\frac{U_{t,N}'\left(\widehat{H^c}\right) \widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \left\| \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2^2}
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \\
= & \left(\left\| \frac{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}\left[G_1'+R'G_2'\right]U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
\leq & \left(\left\| \widehat{V} \right\|_2 \left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2 \left\| \frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \left\| \widehat{V} \right\|_2 \left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
= & \left(\left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2 \left\| \frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
& \left(\text{since } \widehat{V}'\widehat{V} = I_{Kp} \text{ so that } \left\| \widehat{V} \right\|_2 = 1\right) \\
\leq & \ 2 \left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2^2 \left\| \frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + 2 \left\| \left(I_{Kp}+R'R\right)^{-1/2} \right\|_2^2 \|R\|_2^2 \left\| \frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(N_1+\widehat{N}_1-N_1\right)/N_1}}\right\|_2^2 \\
= & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{2\left\|\left(I_K+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right. \\
& \left.+2\left\|\left(I_K+R'R\right)^{-1/2}\right\|_2^2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{\frac{2}{1+\lambda_{\min}\left(R'R\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right. \\
& \left.+\frac{2\|R\|_2^2}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{\frac{2}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2+2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
= & o_p(1) \quad \text{(applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
& \qquad \text{parts (a) and (c) of this lemma, and Slutsky's theorem)}
\end{aligned}
$$

It then follows from part (d) of this lemma and the Slutsky's theorem that

$$
\begin{aligned}
\left\|\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 & \leq \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\right\|_2^2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2^2} \\
& = O_p(1)\,o_p(1) \\
& = o_p(1)
\end{aligned}
$$

Lastly, to show part (g), first write

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\left\{\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right)\right. \\
& \qquad \left.\times\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right)\right\}' \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)\frac{U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\frac{U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
= & \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} - Q\right) \\
& + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}} \\
& + \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'\underline{F}}{T_h\sqrt{\widehat{N}_1}}\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} - Q\right) + \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}
\end{aligned}
$$

where $U_{t,N}\left(\widehat{H^c}\right) = U\left(\widehat{H^c}\right)'\mathbf{e}_{t,T} = \begin{pmatrix} \mathbb{I}\left\{1 \in \widehat{H^c}\right\}u_{1,t} & \mathbb{I}\left\{2 \in \widehat{H^c}\right\}u_{2,t} & \cdots & \mathbb{I}\left\{N \in \widehat{H^c}\right\}u_{N,t}\end{pmatrix}'$. Applying part (h) of Lemma D-15 and parts (d), (e), and (f) of this lemma and the Slutsky's theorem, we obtain

$$\begin{aligned}
&\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{F}_t-Q'\underline{F}_t\right)\left(\widehat{F}_t-Q'\underline{F}_t\right)'\right\|_2 \\
\leq\ & \left\|\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}}-Q\right)\right\|_2 \\
&+\left\|\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right)\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 \\
&+\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'\underline{F}}{T_h\sqrt{\widehat{N}_1}}\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}}-Q\right)\right\|_2+\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}\right\|_2 \\
=\ & \left\|\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}}-Q\right)\right\|_2 \\
&+2\left\|\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right)\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2+\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}\right\|_2 \\
\leq\ & \left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right\|_2^2\left\|\frac{1}{T_h}\sum_{t=p}^{T}\underline{F}_t\underline{F}_t'\right\|_2+2\left\|\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}-Q'\right\|_2\left\|\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 \\
&+\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}\right\|_2 \\
=\ & o_p(1)\,O_p(1)+o_p(1)\,o_p(1)+o_p(1) \\
=\ & o_p(1).\ \square
\end{aligned}$$

Lemma D-17: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{\widehat{\underline{F}}'\widehat{\underline{F}}}{T_h}-\frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q=o_p(1)\,,\text{ where } T_h=T-h-p+1.$$

(b)

$$\frac{\underline{\widehat{F}}'\underline{Y}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t \underline{Y}_t'\right] = o_p(1)$$

(c)

$$\frac{\underline{\widehat{F}}'\iota_{T_h}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] = o_p(1),$$

where $\iota_{T_h} = (1, 1, ..., 1)'$ is a $T_h \times 1$ vector.

(d)

$$\frac{\underline{\widehat{F}}'\left(\underline{\widehat{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1)$$

(e)

$$\frac{\underline{Y}'\left(\underline{\widehat{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1)$$

(f)

$$\frac{\iota_{T_h}'\left(\underline{\widehat{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1)$$

(g)

$$\frac{\underline{\widehat{F}}'\mathfrak{H}}{T_h} = \frac{1}{T_h}\sum_{t=p}^{T-h} \underline{\widehat{F}}_t \eta_{t+h}' = o_p(1)$$

Proof of Lemma D-17:

To show part (a), first write

$$\begin{aligned}
& \frac{\underline{\widehat{F}}'\underline{\widehat{F}}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q' E\left[\underline{F}_t \underline{F}_t'\right] Q \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} \underline{\widehat{F}}_t \underline{\widehat{F}}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} Q' E\left[\underline{F}_t \underline{F}_t'\right] Q \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)' - \frac{1}{T_h}\sum_{t=p}^{T-h} Q' E\left[\underline{F}_t \underline{F}_t'\right] Q \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' + Q'\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\underline{F}_t' Q + Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{F}_t'\right]\right) Q
\end{aligned}$$

Now, by part (g) of Lemma D-16, we have that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \xrightarrow{p} 0$$

Moreover, for any $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$

$$\begin{aligned}
& \left| a'Q'\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' b \right| \\
\leq \; & \sqrt{a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t'\right) Qa} \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' b} \\
\leq \; & \sqrt{a'Q'Qa\lambda_{\max}\left(\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t'\right)} \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' b} \\
= \; & \sqrt{a'Q'Qa\left\|\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t'\right\|_2} \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' b} \\
& \text{(since, for a symmetric psd matrix } A, \\
& \left.\|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} = \lambda_{\max}(A)\right)
\end{aligned}$$

Now, by Assumption 3-6, there exists a positive constant C such that

$$\begin{aligned}
a'Q'Qa &= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\Xi\widehat{V}a \\
&= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)\Xi\widehat{V}a \\
&\leq \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)a'\widehat{V}'\Xi'\Xi\widehat{V}a \\
&= \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right) \quad \Big(\text{since } \Xi'\Xi = I_{Kp},\ \widehat{V}'\widehat{V} = I_{Kp}, \text{ and } a'a = 1\Big) \\
&\leq C \text{ for all } N_1, N_2 \text{ sufficiently large.} \qquad (145)
\end{aligned}$$

while, applying the triangle inequality and part (d) of Lemma D-16 allow us to show that

$$\begin{aligned}
\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\underline{F}_t'\right\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\left[\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\right)\right]^2} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\left\|\underline{F}_t\right\|_2^4} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\right\|_2^2 \\
&= O_p(1)
\end{aligned}$$

Combining this result with part (g) of Lemma D-16 and the Slutsky's Theorem, we deduce that

$$\begin{aligned}
&\left|a'Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'b\right| \\
\leq\ & \sqrt{a'Q'Qa\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'b} \\
=\ & o_p(1)
\end{aligned}$$

Since this argument holds for all $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further obtain

$$Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' = o_p\left(1\right)$$

Now, given that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\underline{F}_t'Q = \left[Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right]',$$

a similar argument also shows that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\underline{F}_t'Q = o_p\left(1\right).$$

Making use of part (b) of Lemma D-2 and the Slutsky's theorem, we also see that

$$Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right]\right)Q \xrightarrow{p} 0$$

Putting these results together and apply Slutsky's theorem, we then obtain

$$\begin{aligned}
& \frac{\widehat{\underline{F}}'\widehat{\underline{F}}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' + Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\underline{F}_t'Q + Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right]\right)Q \\
= \; & o_p\left(1\right)
\end{aligned}$$

To show part (b), first write, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$,

$$
\begin{aligned}
& \frac{a'\underline{\widehat{F}}'\underline{Y}b}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{\widehat{F}}_t\underline{Y}_t'b - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\underline{Y}_t'b - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\underline{Y}_t'b + a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b
\end{aligned}
$$

Focusing first on the first term on last line above, we note that,

$$
\begin{aligned}
& \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \underline{Y}_t' b \right| \\
\leq \ & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \underline{Y}_t \underline{Y}_t' b} \\
= \ & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \\
& \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b + \frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
\leq \ & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
& \left(\text{since } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \right) \\
\leq \ & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=r+1}^{T-h} E \left\| \underline{Y}_t \right\|_2^2} \\
& \left(\text{since } b' E \left[\underline{Y}_t \underline{Y}_t' \right] b = E \left[\left(b' \underline{Y}_t \right)^2 \right] \leq E \left[b' b \underline{Y}_t' \underline{Y}_t \right] = E \left[\left\| \underline{Y}_t \right\|_2^2 \right] \right) \\
= \ & o_p(1)
\end{aligned}
$$

by part (b) of Lemma D-2 and parts (d) and (g) of Lemma D-16. In addition, note that, by making use of part (b) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
& \left| a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b\right| \\
\leq & \sqrt{a'Q'Qa}\sqrt{b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b} \\
\leq & \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)}\sqrt{b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b} \\
= & o_p(1).
\end{aligned}$$

Combining these results, we then get

$$\begin{aligned}
& \left|\frac{a'\underline{\widehat{F}}'\underline{Y}b}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h}a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b\right| \\
\leq & \left|\frac{1}{T_h}\sum_{t=p}^{T-h}a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\underline{Y}_t'b\right| + \left|a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b\right| \\
= & o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\underline{\widehat{F}}'\underline{Y}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{Y}_t'\right] = o_p(1).$$

To show part (c), first write, for any $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$,

$$\begin{aligned}
& \frac{a'\underline{\widehat{F}}'\iota_{T_h}}{T_h} - a'Q'S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}a'\underline{\widehat{F}}_t - a'Q'S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right) - a'Q'S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right) + a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)
\end{aligned}$$

Focusing first on the first term on last line above, we note that,

$$\begin{aligned}
\left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left|a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| \quad \text{(by triangle inequality)} \\
&\leq \sqrt{a'\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'a} \quad \text{(by Liapunov's inequality)} \\
&\leq \sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2} \\
&= o_p(1)
\end{aligned}$$

by part (g) of Lemma D-16 and Slutsky's theorem. In addition, note that, by making use of part (d) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
&\left|a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)\right| \\
\leq\ & \sqrt{a'Q'Qa}\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)'\right. \\
&\qquad\left.\times\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)\right]^{1/2} \\
\leq\ & \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)}\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)'\right. \\
&\qquad\left.\times\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)\right]^{1/2} \\
=\ & o_p(1).
\end{aligned}$$

Combining these results and applying Slutsky's theorem, we then get

$$\begin{aligned}
&\left|\frac{a'\widehat{\underline{F}}'\iota_{T_h}}{T_h} - a'Q'S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right| \\
\leq\ & \left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| + \left|a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu\right)\right| \\
=\ & o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$; we further deduce that

$$\frac{\underline{\widehat{F}}' \iota_{T_h}}{T_h} - Q' S_K' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu = o_p(1).$$

Turning our attention to part (d), note that for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{a' \underline{\widehat{F}}' \left(\underline{\widehat{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
= \; & \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \underline{\widehat{F}}_t \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b \right| \\
\leq \; & \sqrt{a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' \right) a} \sqrt{b' B_2' Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b} \\
\leq \; & \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right| + \frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E \left[\underline{F}_t \underline{F}_t' \right] Q a} \\
& \times \sqrt{b' B_2' Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b} \\
\leq \; & \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right|} \right. \\
& \left. \times \sqrt{b' B_2' Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b} \right\} \\
& + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} a' Q' E \left[\underline{F}_t \underline{F}_t' \right] Q a} \sqrt{b' B_2' Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b} \\
& \left(\text{using the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0 \right) \\
\leq \; & \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right|} \right. \\
& \left. \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \right\} \\
& + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\| \underline{F}_t \|_2^2 \right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right)' \left(\underline{\widehat{F}}_t' - \underline{F}_t' Q \right) \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
& \left(\text{since for a symmetric psd matrix } A, \; \|A\|_2 = \sqrt{\lambda_{\max} \left(A' A \right)} = \sqrt{\lambda_{\max} \left(A^2 \right)} = \sqrt{\left[\lambda_{\max} \left(A \right) \right]^2} \right. \\
& \left. = \lambda_{\max} \left(A \right) \text{ and since } a' Q' E \left[\underline{F}_t \underline{F}_t' \right] Q a = E \left[\left(a' Q' \underline{F}_t \right)^2 \right] \leq E \left[a' Q' Q a \underline{F}_t' \underline{F}_t \right] = a' Q' Q a E \left[\| \underline{F}_t \|_2^2 \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right|} \\
&\quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \Bigg\} \\
&\quad + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\| \underline{F}_t \|_2^2 \right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b}
\end{aligned}
$$

Now, by part (a) of this lemma and Slutsky's theorem, we have

$$
\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right| = o_p (1) \tag{146}
$$

Note also that

$$
\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\| \underline{F}_t \|_2^2 \right] &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E \left[\| \underline{F}_t \|_2^6 \right] \right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\overline{C} \right)^{\frac{1}{3}} \quad \text{(by Lemma C-5)} \\
&= \left(\overline{C} \right)^{\frac{1}{3}}. \qquad (147)
\end{aligned}
$$

In addition, note that, by Assumption 3-7, there exists a positive constant C such that

$$
\begin{aligned}
& \lambda_{\max}\left(B_2' B_2\right) \\
= \; & \lambda_{\max}\left(J_d A^h \mathcal{P}_{(d+K)p}' S_K S_K' \mathcal{P}_{(d+K)p} \left(A^h\right)' J_d'\right) \\
\leq \; & \lambda_{\max}\left(S_K S_K'\right) \lambda_{\max}\left(\mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p}\right) \lambda_{\max}\left\{A^h \left(A^h\right)'\right\} \lambda_{\max}\left(J_d J_d'\right) \\
= \; & \lambda_{\max}\left(S_K S_K'\right) \lambda_{\max}\left\{A^h \left(A^h\right)'\right\} \; \Big(\text{since } \mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p} = I_{(d+K)p} \text{ and } J_d J_d' = I_d \\
& \qquad\qquad\qquad \text{so } \lambda_{\max}\left(\mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p}\right) = \lambda_{\max}\left(J_d J_d'\right) = 1\Big) \\
= \; & \lambda_{\max}\left(S_K' S_K\right) \lambda_{\max}\left\{\left(A^h\right)' A^h\right\} \\
= \; & \lambda_{\max}\left\{\left(A^h\right)' A^h\right\} \; \left(\text{since } S_K' S_K = I_{Kp} \text{ so } \lambda_{\max}\left(S_K' S_K\right) = 1\right) \\
= \; & \sigma_{\max}^2\left(A^h\right) \\
\leq \; & C \max\left\{\left|\lambda_{\max}\left(A^h\right)\right|^2, \left|\lambda_{\min}\left(A^h\right)\right|^2\right\} \; (\text{by Assumption 3-7}) \\
= \; & C \max\left\{\left|\lambda_{\max}(A)\right|^{2h}, \left|\lambda_{\min}(A)\right|^{2h}\right\} \\
= \; & C \phi_{\max}^{2h} \\
< \; & C \text{ for integer } h \geq 1,
\end{aligned}
$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where the last equality follows from the fact that $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. The boundedness of $\lambda_{\max}\left(B_2' B_2\right)$ allows us to further deduce that

$$
\begin{aligned}
& b' B_2' Q'^{-1} Q^{-1} B_2 b \\
= \; & b' B_2' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} B_2 b \\
= \; & b' B_2' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1} B_2 b \\
\leq \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} b' B_2' B_2 b \\
\leq \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}\left(B_2' B_2\right) b' b \\
= \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}\left(B_2' B_2\right) \\
\leq \; & C^* < \infty
\end{aligned}
\tag{148}
$$

for some positive constant C^* in light of Assumption 3-6. It follows by applying expression (145) in the proof for part (a), expressions (146)-(148) here, as well as the result given in part (g) of Lemma D-16 and the Slutsky' theorem that

$$
\begin{aligned}
& \left| \frac{a'\underline{\widehat{F}}'\left(\underline{\widehat{F}} - \underline{F}Q\right)Q^{-1}B_2 b}{T_h} \right| \\
\leq & \left\{ \sqrt{\left| a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} Q' E\left[\underline{F}_t \underline{F}_t'\right] Q \right) a \right|} \right. \\
& \left. \times \sqrt{\left\| \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b} \right\} \\
& + \sqrt{a'Q'Qa\frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\left\|\underline{F}_t\right\|_2^2\right]} \sqrt{\left\| \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b} \\
= & \ o_p(1).
\end{aligned}
$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$
\frac{\underline{\widehat{F}}'\left(\underline{\widehat{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1).
$$

To show part (e), note that for any $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{a'\underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 b}{T_h} \right| \\
= & \left| \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{Y}_t\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b \right| \\
\leq & \sqrt{a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t'\right)a}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
\leq & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right| + \left|\frac{1}{T_h}\sum_{t=p}^{T-h}a'E\left[\underline{Y}_t\underline{Y}_t'\right]a\right|}\right. \\
& \left. \times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b}\right\} \\
\leq & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right|}\right. \\
& \left. \times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b}\right\} \\
& +\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}a'E\left[\underline{Y}_t\underline{Y}_t'\right]a}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
& \left(\text{using the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right|}\right.\\
&\quad\left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\right\}\\
&\quad+\sqrt{a'a\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\|\underline{Y}_t\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\\
&\quad\left(\text{since for a symmetric psd matrix } A,\ \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{\left[\lambda_{\max}(A)\right]^2}\right.\\
&\quad\left. = \lambda_{\max}(A) \text{ and since } a'E\left[\underline{Y}_t\underline{Y}_t'\right]a = E\left[\left(a'\underline{Y}_t\right)^2\right] \leq E\left[a'a\underline{Y}_t'\underline{Y}_t\right] = E\left[\|\underline{Y}_t\|_2^2\right]\right)\\
&= \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right|}\right.\\
&\quad\left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\right\}\\
&\quad+\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\|\underline{Y}_t\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\\
&\quad\left(\text{since } \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right) = \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right)
\end{aligned}$$

Now, by part (b) of Lemma D-2 and Slutsky's theorem, we have

$$\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right| = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1) \tag{149}$$

Note also that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\|\underline{Y}_t\|_2^2\right] &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left(E\left[\|\underline{Y}_t\|_2^6\right]\right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)}\\
&\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\overline{C}\right)^{\frac{1}{3}} \quad \text{(by Lemma C-5)}\\
&= \left(\overline{C}\right)^{\frac{1}{3}}. \qquad (150)
\end{aligned}$$

It follows by applying expressions (148), (149), and (150) as well as the result given in part (g) of

Lemma D-16 and the Slutsky' theorem that

$$
\begin{aligned}
& \left| \frac{a' \underline{Y}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
\leq \; & \left\{ \sqrt{ \left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) a \right| } \right. \\
& \left. \times \sqrt{ \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b } \right\} \\
& + \sqrt{ \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\left\| \underline{Y}_t \right\|_2^2 \right] } \sqrt{ \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b } \\
= \; & o_p (1) .
\end{aligned}
$$

Since the above argument holds for all $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$
\frac{\underline{Y}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p (1) .
$$

To show part (f), note that for any $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
= & \left| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b \right| \\
\leq & \frac{1}{T_h} \sum_{t=p}^{T-h} \left| \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b \right| \quad \text{(by triangle inequality)} \\
\leq & \sqrt{b' B'_2 Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) Q^{-1} B_2 b} \quad \text{(by Liapunov's inequality)} \\
\leq & \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \ \text{(since for a symmetric} \\
& \text{psd matrix } A,\ \left\| A \right\|_2 = \sqrt{\lambda_{\max} \left(A' A \right)} = \sqrt{\lambda_{\max} \left(A^2 \right)} = \sqrt{\left[\lambda_{\max} \left(A \right) \right]^2} = \lambda_{\max} \left(A \right) \Big) \\
= & \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \\
& \left(\text{since } \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right)' \left(\widehat{\underline{F}}'_t - \underline{F}'_t Q \right) = \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right)
\end{aligned}
$$

It follows by applying expression (148), the result given in part (g) of Lemma D-16, and the Slutsky' theorem that

$$
\begin{aligned}
\left| \frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| & \leq \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B'_2 Q'^{-1} Q^{-1} B_2 b} \\
& = o_p \left(1 \right).
\end{aligned}
$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\left\| b \right\|_2 = 1$, we further deduce that

$$
\frac{\iota'_{T_h} \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p \left(1 \right).
$$

For part (g), note that, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\left\| a \right\|_2 = 1$ and $\left\| b \right\|_2 = 1$, we have, by direct calculation,

$$\begin{aligned}
& \frac{a'\underline{\widehat{F}}'\mathfrak{H}b}{T_h} \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{\widehat{F}}_t\eta_{t+h}'b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\eta_{t+h}'b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}'b + \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t\eta_{t+h}'b
\end{aligned}$$

Focusing first on the first term on last line above, we note that

$$\begin{aligned}
& \left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}'b\right| \\
\leq \; & \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'a}\sqrt{b'\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}'b} \\
\leq \; & \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'a} \\
& \times\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right| + \left|\frac{1}{T_h}\sum_{t=p}^{T-h}b'E\left[\eta_{t+h}\eta_{t+h}'\right]b\right|} \\
\leq \; & \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'a}\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right|} \\
& +\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'a}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'E\left[\eta_{t+h}\eta_{t+h}'\right]b} \\
& \left(\text{by the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0\right)
\end{aligned}$$

Note that, by part (g) of Lemma D-16, we have

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' = o_p(1).$$

and, by part (h) of Lemma D-2,

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, note that

$$\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J_{d+K}' \varepsilon_{t+h-j}$$

and, using expression (84) given in the proof of part (e) of Lemma D-2 and Assumption 3-2(b), we see that there exists a positive constant C^* such that

$$\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h} b' E\left[\eta_{t+h}\eta_{t+h}'\right] b \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\left(b'\eta_{t+h}\right)^2\right] \\
\leq \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(E\left[\left(b'\eta_{t+h}\right)^4\right]\right)^{\frac{1}{2}} \\
\leq \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\,C^*\right)^{\frac{1}{2}} \\
& \text{(for some positive constant } C^* \text{ as shown in expression (84))} \\
\leq \;& \left(\,C^*\right)^{\frac{1}{2}} < \infty
\end{aligned}$$

Making use of these calculations and applying Slutsky's theorem, we deduce that

$$\begin{aligned}
& \left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}' b\right| \\
\leq \;& \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right|} \\
& + \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b' E\left[\eta_{t+h}\eta_{t+h}'\right] b} \\
= \;& o_p(1).
\end{aligned}$$

Next, note that, by part (f) of Lemma D-2 and Slutsky's theorem, we see that

$$\begin{aligned}\frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t\eta'_{t+h}b &= a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\eta'_{t+h}\right)b \\ &= O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1)\end{aligned}$$

Putting everything together and applying Slutsky's theorem once more, we then obtain

$$\begin{aligned}\frac{a'\underline{\widehat{F}}'\mathfrak{H}b}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{\widehat{F}}_t\eta'_{t+h}b \\ &= \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\eta'_{t+h}b + \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t\eta'_{t+h}b \\ &= o_p(1).\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\underline{\widehat{F}}'\mathfrak{H}}{T_h} = \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{\widehat{F}}_t\eta'_{t+h} = o_p(1). \ \square$$

Lemma D-18: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then,

$$\begin{pmatrix} \widehat{\beta}'_0 - \beta'_0 \\ \widehat{B}_1 - B_1 \\ \widehat{B}_2 - Q^{-1}B_2 \end{pmatrix} = o_p(1).$$

Here, $\widehat{\beta}_0$, $\widehat{B}_1$, and $\widehat{B}_2$ denote the OLS estimators of the coefficient parameters in the (feasible) h-step ahead forecast equation

$$\begin{aligned}Y_{t+h} &= \beta_0 + \sum_{g=1}^{p} B'_{1,g}Y_{t-g+1} + \sum_{g=1}^{p} B'_{2,g}\widehat{F}_{t-g+1} + \widehat{\eta}_{t+h} \\ &= \beta_0 + B'_1\underline{Y}_t + B'_2\underline{\widehat{F}}_t + \widehat{\eta}_{t+h},\end{aligned}$$

for $t = p, ..., T-h$, where the unobserved factor vector $\underline{F}_t$ is replaced by the estimate $\underline{\widehat{F}}_t$ and where $\widehat{\eta}_{t+h} = \eta_{t+h} - B'_2\left(\underline{\widehat{F}}_t - \underline{F}_t\right)$ with $\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K}\varepsilon_{t+h-j}$ as previously defined.

Proof of Lemma D-18: To proceed, we first stack the observations to obtain the representation

$$\underset{T_h \times d}{Y(h)} = \underset{T_h \times 1}{\iota_{T_h}} \underset{1 \times d}{\beta_0'} + \underset{T_h \times dp}{\underline{Y}} \underset{dp \times d}{B_1} + \underset{T_h \times Kp}{\widehat{\underline{F}}} \underset{Kp \times d}{B_2} + \underset{T_h \times d}{\widehat{\mathfrak{H}}} \tag{151}$$

where $T_h = T - h - p + 1$ and where

$$\underset{T_h \times d}{Y(h)} = \begin{pmatrix} Y_{h+p}' \\ \vdots \\ Y_T' \end{pmatrix}, \underset{T_h \times dp}{\underline{Y}} = \begin{pmatrix} \underline{Y}_p' \\ \vdots \\ \underline{Y}_{T-h}' \end{pmatrix}, \underset{T_h \times Kp}{\widehat{\underline{F}}} = \begin{pmatrix} \widehat{\underline{F}}_p' \\ \vdots \\ \widehat{\underline{F}}_{T-h}' \end{pmatrix}, \text{ and } \underset{T_h \times d}{\widehat{\mathfrak{H}}} = \begin{pmatrix} \widehat{\eta}_{h+p}' \\ \vdots \\ \widehat{\eta}_T' \end{pmatrix}.$$

It is easily seen from expression (151) that the OLS estimators of the coefficients β_0, B_1, and B_2 are given by

$$\begin{pmatrix} \widehat{\beta}_0' \\ \widehat{B}_1 \\ \widehat{B}_2 \end{pmatrix} = \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'Y(h) \\ \underline{Y}'Y(h) \\ \widehat{\underline{F}}'Y(h) \end{bmatrix}.$$

Now, rewrite expression (151) as

$$\begin{aligned} Y(h) &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}B_2 + \widehat{\mathfrak{H}} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}B_2 + \mathfrak{H} - \left(\widehat{\underline{F}} - \underline{F}\right)B_2 \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \underline{F}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \underline{F}QQ^{-1}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \left(\widehat{\underline{F}} + \underline{F}Q - \widehat{\underline{F}}\right)Q^{-1}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}Q^{-1}B_2 - \left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 + \mathfrak{H} \\ &= \begin{bmatrix} \iota_{T_h} & \underline{Y} & \widehat{\underline{F}} \end{bmatrix} \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} - \left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 + \mathfrak{H}, \end{aligned}$$

and it follows that

$$
\begin{aligned}
&\begin{pmatrix} \widehat{\beta}_0' - \beta_0' \\ \widehat{B}_1 - B_1 \\ \widehat{B}_2 - Q^{-1}B_2 \end{pmatrix} \\
= \;& \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \widehat{\underline{F}}' \end{bmatrix} Y(h) - \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
= \;& \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \widehat{\underline{F}}' \end{bmatrix} \begin{bmatrix} \iota_{T_h} & \underline{Y} & \widehat{\underline{F}} \end{bmatrix} \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} - \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
&- \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \widehat{\underline{F}}' \end{bmatrix} \left(\widehat{\underline{F}} - \underline{F}Q\right) Q^{-1}B_2 \\
&+ \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \widehat{\underline{F}}' \end{bmatrix} \mathfrak{H}. \\
= \;& - \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'\left(\widehat{\underline{F}} - \underline{F}Q\right) Q^{-1}B_2 \\ \underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right) Q^{-1}B_2 \\ \widehat{\underline{F}}'\left(\widehat{\underline{F}} - \underline{F}Q\right) Q^{-1}B_2 \end{bmatrix} \\
&+ \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'\mathfrak{H} \\ \underline{Y}'\mathfrak{H} \\ \widehat{\underline{F}}'\mathfrak{H} \end{bmatrix}
\end{aligned}
$$

Next, applying parts (b) and (d) of Lemma D-2 and parts (a), (b), (c), and (d) of Lemma D-17, we obtain

$$\begin{aligned}
&\begin{pmatrix} 1 & \iota_{T_h}'\underline{Y}/T_h & \iota_{T_h}'\underline{\widehat{F}}/T_h \\ \underline{Y}'\iota_{T_h}/T_h & \underline{Y}'\underline{Y}/T_h & \underline{Y}'\underline{\widehat{F}}/T_h \\ \underline{\widehat{F}}'\iota_{T_h}/T_h & \underline{\widehat{F}}'\underline{Y}/T_h & \underline{\widehat{F}}'\underline{\widehat{F}}/T_h \end{pmatrix} \\
&- \begin{pmatrix} 1 & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{F}_t'\right]Q \\ T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{F}_t'\right]Q \\ T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \end{pmatrix} \\
= \;& o_p(1).
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
&\begin{pmatrix} 1 & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{F}_t'\right]Q \\ T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{F}_t'\right]Q \\ T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \end{pmatrix} \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right]Q \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right]Q \\ Q'E\left[\underline{F}_t\right] & Q'E\left[\underline{F}_t\underline{Y}_t'\right] & Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \end{pmatrix} \\
= \;& \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q' \end{pmatrix} \frac{1}{T_h}\sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q \end{pmatrix}
\end{aligned}$$

which is non-singular and, therefore, also positive definite for all T sufficiently large in light of the result given in part (b) of Lemma D-1.

In addition, applying parts (f) and (g) of Lemma D-2 and parts (d), (e), (f), and (g) of Lemma D-17, we have

$$\begin{aligned}
\frac{\iota_{T_h}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{\underline{Y}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\iota_{T_h}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t'-\underline{F}_t'Q\right)Q^{-1}B_2=o_p(1),\\
\frac{\underline{Y}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\left(\widehat{\underline{F}}_t'-\underline{F}_t'Q\right)Q^{-1}B_2=o_p(1),\\
\frac{\widehat{\underline{F}}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{\underline{F}}_t\left(\widehat{\underline{F}}_t'-\underline{F}_t'Q\right)Q^{-1}B_2=o_p(1),\\
\frac{\widehat{\underline{F}}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{\underline{F}}_t\eta_{t+h}'=o_p(1)
\end{aligned}$$

Putting everything together and applying the Slutsky's theorem

$$\begin{aligned}
&\begin{pmatrix}\widehat{\beta}_0'-\beta_0'\\ \widehat{B}_1-B_1\\ \widehat{B}_2-Q^{-1}B_2\end{pmatrix}\\
=&-\begin{pmatrix}T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}}\\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}}\\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}}\end{pmatrix}^{-1}\begin{bmatrix}\iota_{T_h}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\\ \underline{Y}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\\ \widehat{\underline{F}}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\end{bmatrix}\\
&+\begin{pmatrix}T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}}\\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}}\\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}}\end{pmatrix}^{-1}\begin{bmatrix}\iota_{T_h}'\mathfrak{H}\\ \underline{Y}'\mathfrak{H}\\ \widehat{\underline{F}}'\mathfrak{H}\end{bmatrix}\\
=&-\begin{pmatrix}1 & T_h^{-1}\iota_{T_h}'\underline{Y} & T_h^{-1}\iota_{T_h}'\widehat{\underline{F}}\\ T_h^{-1}\underline{Y}'\iota_{T_h} & T_h^{-1}\underline{Y}'\underline{Y} & T_h^{-1}\underline{Y}'\widehat{\underline{F}}\\ T_h^{-1}\widehat{\underline{F}}'\iota_{T_h} & T_h^{-1}\widehat{\underline{F}}'\underline{Y} & T_h^{-1}\widehat{\underline{F}}'\widehat{\underline{F}}\end{pmatrix}^{-1}\begin{bmatrix}T_h^{-1}\iota_{T_h}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\\ T_h^{-1}\underline{Y}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\\ T_h^{-1}\widehat{\underline{F}}'\left(\widehat{\underline{F}}-\underline{F}Q\right)Q^{-1}B_2\end{bmatrix}\\
&+\begin{pmatrix}1 & T_h^{-1}\iota_{T_h}'\underline{Y} & T_h^{-1}\iota_{T_h}'\widehat{\underline{F}}\\ T_h^{-1}\underline{Y}'\iota_{T_h} & T_h^{-1}\underline{Y}'\underline{Y} & T_h^{-1}\underline{Y}'\widehat{\underline{F}}\\ T_h^{-1}\widehat{\underline{F}}'\iota_{T_h} & T_h^{-1}\widehat{\underline{F}}'\underline{Y} & T_h^{-1}\widehat{\underline{F}}'\widehat{\underline{F}}\end{pmatrix}^{-1}\begin{bmatrix}T_h^{-1}\iota_{T_h}'\mathfrak{H}\\ T_h^{-1}\underline{Y}'\mathfrak{H}\\ T_h^{-1}\widehat{\underline{F}}'\mathfrak{H}\end{bmatrix}\\
=&\ o_p(1).\ \square
\end{aligned}$$

References

[1] Anatolyev, S. and A. Mikusheva (2021): "Factor Models with Many Assets: Strong Factors, Weak Factors, and the Two-Pass Procedure," *Journal of Econometrics*, forthcoming.

[2] Andrews, D.W.K. (1984): "Non-strong Mixing Autoregressive Processes," *Journal of Applied Probability*, 21, 930-934.

[3] Bai, J. and S. Ng (2002): "Determining the Number of Factors in Approximate Factor Models," *Econometrica*, 70, 191-221.

[4] Bai, J. (2003): "Inferential Theory for Factor Models of Large Dimensions," *Econometrica*, 71, 135-171.

[5] Bai, J. and S. Ng (2008): "Forecasting Economic Time Series Using Targeted Predictors," *Journal of Econometrics*, 146, 304-317.

[6] Bai, J. and S. Ng (2021): "Approximate Factor Models with Weaker Loading," Working Paper, Columbia University.

[7] Bai, Z. D. and Y. Q. Yin (1993): "Limit of the Smallest Eigenvalue of a Large Dimensional Sample Covariance Matrix," *Annals of Probability*, 21, 1275-1294.

[8] Bair, E., T. Hastie, D. Paul, and R. Tibshirani (2006): "Prediction by Supervised Principal Components," *Journal of the American Statistical Association*, 101, 119-137.

[9] Belloni, A., D. Chen, V. Chernozhukov, and C. Hansen (2012): "Sparse Models and Methods for Optimal Instruments with an Application to Eminent Domain," *Econometrica*, 80, 2369-2429.

[10] Billingsley, P. (1995): *Probability and Measure*. New York: John Wiley & Sons.

[11] Borovkova, S., R. Burton, and H. Dehling (2001): "Limit Theorems for Functionals of Mixing Processes to U-Statistics and Dimension Estimation," *Transactions of the American Mathematical Society*, 353, 4261-4318.

[12] Bryzgalova, S. (2016): "Spurious Factors in Linear Asset Pricing Models," Working Paper, Stanford Graduate School of Business.

[13] Burnside, C. (2016): "Identification and Inference in Linear Stochastic Discount Factor Models with Excess Returns," *Journal of Financial Econometrics*, 14, 295-330.

[14] Chen, X., Q. Shao, W. B. Wu, and L. Xu (2016): "Self-normalized Cramér-type Moderate Deviations under Dependence," *Annals of Statistics*, 44, 1593-1617.

[15] Davidson. J. (1994): *Stochastic Limit Theory: An Introduction for Econometricians*. New York: Oxford University Press.

[16] Davidson, K. R. and S. J. Szarek (2001): "Local Operator Theory, Random Matrices and Banach Spaces." In *Handbook of the Geometry of Banach Spaces*, 1, 317-366. Amsterdam: North-Holland.

[17] Fan, J., Y. Liao, and M. Mincheva (2011): "High-dimensional Covariance Matrix Estimation in Approximate Factor Models," *Annals of Statistics*, 39, 3320-3356.

[18] Fan, J., Y. Liao, and M. Mincheva (2013): "Large Covariance Estimation by Thresholding Principal Orthogonal Complements," *Journal of the Royal Statistical Society, Series B*, 75, 603-680.

[19] Freyaldenhoven, S. (2021a): "Factor Models with Local Factors - Determining the Number of Relevant Factors," *Journal of Econometrics*, forthcoming.

[20] Freyaldenhoven, S. (2021b): "Identification through Sparsity in Factor Models: The ℓ_1-Rotation Criterion," Working Paper, Federal Reserve Bank of Philadelphia.

[21] Giglio, S., D. Xiu, and D. Zhang (2021): "Test Assets and Weak Factors," Working Paper, Yale School of Management and the Booth School of Business, University of Chicago.

[22] Golub, G. H. and C. F. van Loan (1996): *Matrix Computations*, 3rd Edition. Baltimore: The Johns Hopkins University Press.

[23] Goroketskii, V. V. (1977): "On the Strong Mixing Property for Linear Sequences," *Theory of Probability and Applications*, 22, 411-413.

[24] Gospodinov, N., R. Kan, and C. Robotti (2017): "Spurious Inference in Reduced-Rank Asset Pricing Models," *Econometrica*, 85, 1613-1628.

[25] Harding, M. C. (2008): "Explaining the Single Factor Bias of Arbitrage Pricing Models in Finite Samples," *Economics Letters*, 99, 85-88.

[26] Horn, R. and C. Johnson (1985): *Matrix Analysis*. Cambridge University Press.

[27] Jagannathan, R. and Z. Wang (1998): "An Asymptotic Theory for Estimating Beta-Pricing Models Using Cross-Sectional Regression," *Journal of Finance*, 53, 1285-1309.

[28] Johnstone, I. M. and A. Lu (2009): "On Consistency and Sparsity for Principal Components Analysis in High Dimensions," *Journal of the American Statistical Association*, 104, 682-697.

[29] Johnstone, I. M. and D. Paul (2018): "PCA in High Dimensions: An Orientation," *Proceedings of the IEEE*, 106, 1277-1292.

[30] Kan, R. and C. Zhang (1999): "Two-Pass Tests of Asset Pricing Models with Useless Factors," *Journal of Finance*, 54, 203-235.

[31] Kleibergen, F. (2009): "Tests of Risk Premia in Linear Factor Models," *Journal of Econometrics*, 149, 149-173.

[32] Lütkepohl, H. (2005): *New Introduction to Multiple Time Series Analysis*. New York: Springer.

[33] Nadler, B. (2008): "Finite Sample Approximation Results for Principal Component Analysis: A Matrix Perturbation Approach," *Annals of Statistics*, 36, 2791-2817.

[34] Onatski, A. (2012): "Asymptotics of the Principal Components Estimator of Large Factor Models with Weakly Influential Factors," *Journal of Econometrics*, 168, 244-258.

[35] Paul, D. (2007): "Asymptotics of Sample Eigenstructure for a Large Dimensional Spiked Covariance Model," *Statistica Sinica*, 17, 1617-1642.

[36] Pham, T. D. and L. T. Tran (1985): "Some Mixing Properties of Time Series Models," *Stochastic Processes and Their Applications*, 19, 297-303.

[37] Ruhe, A. (1975): "On the Closeness of Eigenvalues and Singular Values for Almost Normal Matrices," *Linear Algebra and Its Applications*, 11, 87-94.

[38] Shen, D., H. Shen, H. Zhu, J.S. Marron (2016): "The Statistics and Mathematics of High Dimension Low Sample Size Asymptotics," *Statistica Sinica*, 26, 1747-1770.

[39] Stewart, G.W. (1973): "Error and Perturbation Bounds for Subspaces Associated with Certain Eigenvalue Problems," *SIAM Review*, 15, 727-764.

[40] Stewart, G.W. and J. Sun (1990): *Matrix Perturbation Theory*. Boston: Academic Press.

[41] Stock, J. H. and M. W. Watson (2002a): "Forecasting Using Principal Components from a Large Number of Predictors," *Journal of the American Statistical Association*, 97, 1167-1179.

[42] Stock, J. H. and M. W. Watson (2002b): "Macroeconomic Forecasting Using Diffusion Indexes," *Journal of Business and Economic Statistics*, 20, 147-162.

[43] Vershynin, R. (2012): "Introduction to the Non-asymptotic Analysis of Random Matrices," In *Compressed Sensing, Theory and Applications,* 210-268. Cambridge University Press.

[44] Wang, W. and J. Fan (2017): "Asymptotics of Empirical Eigenstructure for High Dimensional Spiked Covariance," *Annals of Statistics*, 45, 1342-1374.