Supplemental Appendix: Robust Forecast Superiority Testing with an Application to Assessing Pools of Expert Forecasters *

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November 2021

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1 Appendix SA1: Forecast Superiority Tests under Convex Loss Functions

In this Supplemental Section we state the counterpart of Lemmas 1 and 2, and Theorems 1 and 2 for Convex Loss (CL) forecast superiority testing. Let $[a]_+ = a1\{a \geq 0\}$, define for $x \in \mathcal{X}^+$

$$\begin{aligned}\widehat{\sigma}_{j,n}^{2,C+}(x) &= \frac{1}{n}\sum_{t=1}^{n}(\widehat{\eta}_{j,t}(x) - \widehat{\eta}_{1,t}(x))^2 \\ &+2\frac{1}{n}\sum_{\tau=1}^{l_n}\sum_{t=\tau+1}^{n} w_\tau(\widehat{\eta}_{j,t}(x) - \widehat{\eta}_{1,t}(x))(\widehat{\eta}_{j,t-\tau}(x) - \widehat{\eta}_{1,t-\tau}(x)),\end{aligned}$$

where

$$\widehat{\eta}_{j,t}(x) = [(e_{j,t} - x)]_+ - \frac{1}{n}\sum_{t=1}^{n}[(e_{j,t} - x)]_+,$$

we have

Lemma S1: *Let Assumptions A1-A3 hold. Then, if as $n \to \infty$, $\frac{l_n}{n^\delta} \to c$, with $0 < c < \infty$ and $0 < \delta < \frac{1}{2}$, with δ defined as in Assumption A1:*

$$\sup_{x \in \mathcal{X}^+}\left|\widehat{\sigma}_{j,n}^{2,C+}(x) - \sigma_j^{2,C+}(x)\right| = o_p(1),$$

with $\sigma_j^{2C+}(x) = avar\left(\sqrt{n}C_{j,n}^+(x)\right)$.

Now let $\eta_{j,t}^*(x) = \left[e_{j,t}^* - x\right]_+ - \frac{1}{n}\sum_{t=1}^{n} 1\left[e_{j,t}^* - x\right]_+$ and define

$$\widehat{\sigma}_{j,n}^{2*C+}(x) = \frac{1}{b_n}\sum_{k=1}^{b_n}\left(\frac{1}{l_n^{1/2}}\sum_{i=1}^{l_n}\left(\eta_{j,(k-1)l_n+i}^*(x) - \eta_{1,(k-1)l_n+i}^*(x)\right)\right)^2,$$

We have

Lemma S2: *Let Assumptions A1-A3 hold. Then, if as $n \to \infty$, $\frac{l_n}{n^\delta} \to c$, with $0 < c < \infty$ and $0 < \delta < \frac{1}{2}$, with δ defined as in Assumption A1:*

$$\sup_{x \in \mathcal{X}^+}\left|\widehat{\sigma}_{j,n}^{*C+}(x) - \widehat{\sigma}_{j,n}^{C+}(x)\right| = o_p^*(1),$$

where $o_p^*(1)$ denotes convergence to zero according to the bootstrap law, P^*, conditional on the sample.

Finally, define

$$\begin{aligned}C_{j,n}(x) &= \int_{-\infty}^{x}\left(\widehat{F}_{1,n}(t) - \widehat{F}_{j,n}(t)\right)dt1(x<0) - \int_{x}^{\infty}\left(\widehat{F}_{j,n}(t) - \widehat{F}_{1,n}(t)\right)dt1(x \geq 0) \\ &= \frac{1}{n}\sum_{t=1}^{n}\left([(e_{1,t} - x)\, sgn(x)]_+ - [(e_{j,t} - x)\, sgn(x)]_+\right), \qquad (1.1)\end{aligned}$$

where $sgn(x) = 1$ if $x \geq 0$ and $sgn(x) = -1$ if $x < 0$, and

$$S_n^{C+} = \int_{x \in \mathcal{X}^+}\sum_{j=2}^{k}\left(\max\left\{0, \sqrt{n}\frac{C_{j,n}(x)}{\overline{\sigma}_{j,n}^{C+}(x)}\right\}\right)^2 dQ(x)$$

where $\overline{\sigma}^{2C}_{j,n}(x) = \widehat{\sigma}^{2C}_{j,n}(x) + \varepsilon$.
As shown in Proposition 2.3 in Jin, Corradi and Swanson (2017; JCS), the null hypothesis is CL forecast superiority and its negation write as

$$\begin{aligned} H_0^C &= H_0^{C-} \cap H_0^{C+} \\ &: \left(\int_{-\infty}^{x} (F_1(t) - F_j(t))dt \leq 0, \text{ for } j = 2, ..., k, \text{ and for all } x \in \mathcal{X}^- \right) \\ &\quad \cap \left(\int_{x}^{\infty} (F_j(t) - F_1(t))dt \leq 0, \text{ for } j = 2, ..., k, \text{ and for all } x \in \mathcal{X}^+ \right) \end{aligned}$$

versus

$$\begin{aligned} H_A^C &= H_A^{C-} \cup H_A^{C+} \\ &: \left(\int_{-\infty}^{x} (F_1(t) - F_j(t))dt > 0, \text{ for some } j = 2, ..., k, \text{ and for some } x \in \mathcal{X}^- \right) \\ &\quad \cup \left(\max_{j=2,...,k} \int_{x}^{\infty} (F_j(t) - F_1(t))dt > 0, \text{ for some } j = 2, ..., k, \text{ and for some } x \in \mathcal{X}^+ \right). \end{aligned}$$

We now find lower and upper bound for S_n^{C+} under H_0^{C+}.Let

$$v^{C+}(.) = (v_2^{C+}(.), ..., v_k^{C+}(.))',$$

be a $(k-1)-$dimensional zero mean Gaussian process with covariance kernel equal to acov$(\sqrt{n}C^+(x), \sqrt{n}C^+(x'))$. Also, let

$$h_{j,A,n}^{C+}(x) = \sigma_j^{C+}(x)^{-1} \sqrt{n} C_j^+(x)$$
$$h_{j,B}^{C+}(x) = \sigma_j^{2,C+}(x)^{-1} \left(\sigma_j^{C+}(x) + \varepsilon \right)^2$$

and $h_B^{C+}(x) = \left(h_{2,B}^{C+}(x), ..., h_{k,B}^{C+}(x) \right)$

$$S_n^{\dagger C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max \left\{ 0, \frac{v_j^{C+}(x) + h_{j,A,n}^{C+}(x)}{\sqrt{h_{j,B}^{C+}(x)}} \right\} \right)^2 \mathrm{d}Q(x),$$

and

$$S_\infty^{\dagger C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max \left\{ 0, \frac{v_j^{C+}(x) + h_{j,A,\infty}^{C+}(x)}{\sqrt{h_{j,B}^{C+}(x)}} \right\} \right)^2 \mathrm{d}Q(x),$$

We have the CL counterpart of Theorem 1.
Theorem S1: *Let Assumptions A1-A4 hold. Then, under* H_0^{C+}*, there exists* $\delta > 0$ *and* a^{C+} *such that*

$$\limsup_{n\to\infty} \sup_{P \in \mathcal{P}_0^{C+}} \left[P\left(S_n^{C+} > a^{C+} \right) - P\left(S_n^{\dagger C+} + \delta > a^{C+} \right) \right] \leq 0$$

and

$$\liminf_{n\to\infty} \inf_{P \in \mathcal{P}_0^{C+}} \left[P\left(S_n^{C+} > a^{C+} \right) - P\left(S_n^{\dagger C+} - \delta > a^{C+} \right) \right] \geq 0.$$

We now construct asymptotically valid bootstrap critical values. Let

$$C_{j,n}^{*+}(x) = \frac{1}{n}\sum_{t=1}^{n}\left(\left[e_{j,t}^{*} - x\right]_{+} - \frac{1}{n}\sum_{t=1}^{n}\left[e_{1,t}^{*} - x\right]_{+}\right)$$

and $v_n^{*C+}(x) = \left(v_{2,n}^{*C+}(x), ..., v_{k,n}^{*C+}(x)\right)$ with

$$v_{j,n}^{*C+}(x) = \frac{\sqrt{n}\left(C_{j,n}^{*+}(x) - C_{j,n}^{+}(x)\right)}{\widehat{\sigma}_{j,n}^{C+}(x)}.$$

Then, define:

$$\xi_{j,n}^{C+}(x) = \kappa_n^{-1} n^{1/2} \frac{C_{j,n}^{+}(x)}{\overline{\sigma}_{j,n}^{C+}(x)},$$

and

$$\phi_{j,n}^{G+}(x) = c_n 1\left\{\xi_{j,n}^{C+}(x) < -1\right\},$$

and $\phi_n^{C+} = \left(\phi_{2,n}^{C+}, ..., \phi_{k,n}^{C+}\right)$. Now, let

$$S_n^{*C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \max\left(\left\{0, \frac{v_{j,n}^{*C+}(x) - \phi_{j,n}^{C+}(x)}{\sqrt{h_{j,B}^{*C+}(x)}}\right\}\right)^2 \mathrm{d}Q(x), \tag{1.2}$$

where

$$h_{j,B}^{*C+}(x) = \overline{\sigma}_{j,n}^{2,*C+}(x)/\widehat{\sigma}_{j,n}^{2,C+}(x), \tag{1.3}$$

Then, construct:

$$S_n^{*C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \max\left(\left\{0, \frac{v_{j,n}^{*C+}(x) - \phi_{j,n}^{C+}(x)}{\sqrt{h_{B,jj}^{*C+}(x)}}\right\}\right)^2 \mathrm{d}Q(x).$$

and let $c_{0,n,1-\alpha}^{*C+}\left(\phi_n^{C+}, h_{B,n}^{C+}\right) = \lim_{B\to\infty} c_{n,B,1-\alpha+\eta}^{*C+}\left(\phi_n^{C+}, h_{B,n}^{*C+}\right) + \eta$. We have the CL counterpart of Theorem 2.

Theorem S2: *Let Assumptions A1-A4 hold, and let $l_n \to \infty$ and $l_n n^{\frac{1}{3}-\varepsilon} \to 0$ as $n \to \infty$. Under H_0^{C+},:*

(i) if as $n \to \infty$, $\kappa_n \to \infty$ and $c_n/\kappa_n \to 0$, then

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{C+}} P\left(S_n^{C+} \geq c_{0,n,1-\alpha}^{*C+}\left(\phi_n^{C+}, h_{B,n}^{*C+}\right)\right) \leq \alpha;$$

(ii) Let $\mathcal{B}^{C+}$ be defined analogously to $\mathcal{B}^{G+}$ Eq.(3.12) in main text. If as $n \to \infty$, $\kappa_n \to \infty$, $c_n \to \infty$, $\sqrt{n}/\kappa_n \to \infty$, and $Q\left(\mathcal{B}^{C+}\right) > 0$, then

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{C+}} P\left(S_n^{C+} \geq c_{0,n,1-\alpha}^{*C+}\left(\phi_n^{C+}, h_{B,n}^{*C+}\right)\right) = \alpha.$$

Finally, the CL counterparts Theorem 3 and Theorem 4 follow straightforwardly, and are not stated for brevity.

The proofs for Theorem 1 and 2 do not depend on whether we construct GL or CL forecast superiority tests. Hence, below we only provide the proofs for Lemma S1 and Lemma S2.

Proof of Lemma S1:
By noting that,

$$\begin{aligned}
& [e_{j,t} - s_j]_+ - [e_{j,t} - x]_+ \\
= \; & (x - s_j)1\{e_t \geq x\} + (x - s_j)\left(1\{e_t \geq x\} - 1\{e_t \geq s_j\}\right) \\
& + (e_t - x)\left(1\{e_t \geq s_j\} - 1\{e_t \geq x\}\right),
\end{aligned}$$

the statement follows by the same argument as that used the proof of Lemma 1.
Proof of Lemma S2:
By the same argument as the proof of Lemma 2, replacing $\widehat{u}^*_{j,t}$ with $\widehat{\eta}^*_{j,t}$.

2 Appendix SA2: Forecast Superiority Tests in the Presence of Recursive Estimation Error

2.1 The Statistic

This Supplement extends all Lemmas and Theorems in the paper to the case in which there is non vanishing, recursive estimation error.

Let $T = R + n$. At each point in time, $t > R$, update model parameter estimates prior to the construction of each new forecast, using all the available information.[1]

For $j = 1, ..., k$, use the first R observations to compute $\widehat{\theta}_{j,R}$, and construct the first prediction error:

$$\widehat{e}_{j,R+1} = X_{R+1} - \phi_j\left(Z_{j,R}, \widehat{\theta}_{j,R}\right),$$

where $Z_{j,R}$ contains lags of X as well as other regressors. Then, use the first $R+1$ observations to construct

$$\widehat{e}_{j,R+2} = X_{R+2} - \phi_j\left(Z_{j,R+1}, \widehat{\theta}_{j,R+1}\right).$$

Proceed in the same manner until a sequence of n prediction errors has been constructed, defined as:

$$\widehat{e}_{j,t+1} = X_{t+1} - \phi_j\left(Z_{j,t}, \widehat{\theta}_{j,t}\right), \tag{2.1}$$

for $t = R, ..., R+n-1$, where $\widehat{\theta}_{j,t}$ is the estimator computed using observations up to time t. In the sequel, assume that $\widehat{\theta}_{j,t}$ is a recursive $m-$estimator, so that:

$$\widehat{\theta}_{j,t} = \arg\min_{\theta_j \in \Theta_j} \frac{1}{t}\sum_{i=2}^{t} m_j(X_i, Z_{j,i-1}, \theta_j), \quad R \leq t \leq n-1,\ j = 1, ..., k, \tag{2.2}$$

and

$$\theta_j^{\dagger} = \arg\min_{\theta_j \in \Theta_j} E(m_j(X_i, Z_{j,i-1}, \theta_j)).$$

For $x \geq 0$, define:

$$\widetilde{G}^{+}_{j,n}(x) = \frac{1}{n}\sum_{t=R}^{T-1}\left(1\left\{\widehat{e}_{j,t+1} \leq x\right\} - 1\left\{\widehat{e}_{1,t+1} \leq x\right\}\right) = \left(\widetilde{F}_{j,n}(x) - \widetilde{F}_{1,n}(x)\right) \tag{2.3}$$

and

$$\begin{aligned}
\widetilde{C}^{+}_{j,n}(x) &= \int_x^{\infty}\left(\widetilde{F}_{j,n}(t) - \widetilde{F}_{1,n}(t)\right)dt \\
&= \frac{1}{n}\sum_{t=R}^{T-1}\left\{\left[(\widehat{e}_{1,t+1} - x)\right]_+ - \left[(\widehat{e}_{j,t+1} - x)\right]_+\right\}.
\end{aligned} \tag{2.4}$$

Define the following forecast superiority test statistics:

$$\widetilde{S}^{G+}_n = \int_{\mathcal{X}^+}\sum_{j=2}^{k}\left(\max\left\{0, \frac{\sqrt{n}\widetilde{G}^{+}_{j,n}(x)}{\widetilde{\sigma}^{G+}_{j,n}(x) + \varepsilon}\right\}\right)^2 \mathrm{d}Q(x)$$

[1] In the rolling estimation case, we use only the most recent R observations to re-estiamte the forecasdting model, for each $t > R$. The rolling case can be treated analogously, and it is omitted only for brevity.

and

$$\widetilde{S}_n^{C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{ 0, \frac{\sqrt{n}\widetilde{C}_{j,n}^{+}(x)}{\widetilde{\sigma}_{j,n}^{C+}(x) + \varepsilon} \right\} \right)^2 \mathrm{d}Q(x),$$

where $\widetilde{\sigma}_{j,n}^{G+}(x)$ and $\widetilde{\sigma}_{j,n}^{C+}(x)$ include terms accounting for the contribution of parameter estimation error to asymptotic covariance. Here, $\widetilde{\sigma}_{j,n}^{2,G+}(x)$ is defined as:

$$\begin{aligned} \widetilde{\sigma}_{j,n}^{2,G+}(x) &= \widehat{\sigma}_{j,n}^{2,G+}(x) + 2\widehat{\Pi}\widehat{f}_{1,n,h}^{2}(x)\widehat{A}_1\widehat{\Sigma}_{11}\widehat{A}_1' + 2\widehat{\Pi}\widehat{f}_{j,n,h}^{2}(x)\widehat{A}_j\widehat{\Sigma}_{jj}\widehat{A}_j' \\ & -4\widehat{\Pi}\widehat{f}_{1,n,h}(x)\widehat{A}_1\widehat{\Sigma}_{1j}\widehat{A}_j'\widehat{f}_{j,n,h}(x) + 2\widehat{\Pi}\widehat{f}_{1,n,h}(x)\widehat{A}_1\widehat{\Sigma}_{u1}(x) - 2\widehat{\Pi}\widehat{f}_{j,n,h}(x)\widehat{A}_j\widehat{\Sigma}_{uj}(x), \end{aligned}$$

where $\widehat{\sigma}_{j,n}^{2,G+}(x)$ is defined as in the text, but using only the last n observations, $\widehat{\Pi} = 1 - \frac{R}{n}\ln\left(1 + \frac{n}{R}\right)$,

$$\widehat{f}_{j,n,h}(x) = \frac{1}{nh}\sum_{t=R+1}^{n} K\left(\frac{\widehat{e}_{j,t} - x}{h}\right),$$

$$\widehat{A}_j = \frac{1}{n}\sum_{t=R+1}^{T} \nabla_{\theta_j}\phi_j\left(Z_{j,t+1}, \widehat{\theta}_{j,R}\right)' \left(\frac{1}{R}\sum_{t=1}^{R} \nabla_{\theta_j}^2 m_j(X_t, Z_{j,t-1}, \widehat{\theta}_{j,R})\right)^{-1},$$

$$\begin{aligned} \widehat{\Sigma}_{jj} &= \frac{1}{n}\sum_{t=R+1}^{T} \nabla_{\theta_j} m_j(X_t, Z_{t,i-1}, \widehat{\theta}_{j,R})\nabla_{\theta_j} m_j(X_t, Z_{t,i-1}, \widehat{\theta}_{j,R})' \\ & +2\frac{1}{n}\sum_{\tau=1}^{l_n}\sum_{t=R+\tau+1}^{T} w_\tau \nabla_{\theta_j} m_j(X_t, Z_{t,i-1}, \widehat{\theta}_{j,R})\nabla_{\theta_j} m_j(X_{t-\tau}, Z_{t-\tau,i-1}, \widehat{\theta}_{j,R})', \end{aligned}$$

and

$$\begin{aligned} \widehat{\Sigma}_{uj}(x) &= \frac{1}{n}\sum_{t=R+1}^{T} \nabla_{\theta_j} m_j(X_t, Z_{t,i-1}, \widehat{\theta}_{j,R}) \left(\left(1\left\{\widehat{e}_{j,t} \le x\right\} - \frac{1}{n}\sum_{t=1}^{n} 1\left\{\widehat{e}_{j,t} \le x\right\} \right) \right. \\ & \left. - \left(1\left\{\widehat{e}_{1,t} \le x\right\} - \frac{1}{n}\sum_{t=1}^{n} 1\left\{\widehat{e}_{1,t} \le x\right\} \right) \right) \\ & +2\frac{1}{n}\sum_{\tau=1}^{l_n}\sum_{t=R+\tau+1}^{T} w_\tau \nabla_{\theta_j} m_j(X_t, Z_{t,i-1}, \widehat{\theta}_{j,R}) \left(\left(1\left\{\widehat{e}_{j,t-\tau} \le x\right\} - \frac{1}{n}\sum_{t=1}^{n} 1\left\{\widehat{e}_{j,t-\tau} \le x\right\} \right) \right. \\ & \left. - \left(1\left\{\widehat{e}_{1,t-\tau} \le x\right\} - \frac{1}{n}\sum_{t=1}^{n} 1\left\{\widehat{e}_{1,t-\tau} \le x\right\} \right) \right), \end{aligned}$$

By noting that

$$
\begin{aligned}
& \widetilde{C}_{j,n}^{+}(x) \\
= \; & \frac{1}{n}\sum_{t=R}^{T-1}\Big([(e_{1,t+1}-x)]_{+}-[(e_{j,t+1}-x)]_{+}\Big) \\
& +\frac{1}{n}\sum_{t=R}^{T-1}\left((\widehat{e}_{1,t}-e_{1,t})\,1\left\{e_{1,t}\geq x\right\}-(\widehat{e}_{j,t}-e_{j,t})\,1\left\{e_{j,t}\geq x\right\}\right) \\
& +\frac{1}{n}\sum_{t=R}^{T-1}\left((e_{1,t}-x)\left(1\left\{\widehat{e}_{1,t}\geq x\right\}-1\left\{\widehat{e}_{1,t}\geq x\right\}\right)-(e_{j,t}-x)\left(1\left\{\widehat{e}_{j,t}\geq x\right\}-1\left\{e_{j,t}\geq x\right\}\right)\right) \qquad (2.5) \\
& +\frac{1}{n}\sum_{t=R}^{T-1}\left((\widehat{e}_{1,t}-e_{1,t})\left(1\left\{\widehat{e}_{1,t}\geq x\right\}-1\left\{e_{1,t}\geq x\right\}\right)-(\widehat{e}_{j,t}-e_{j,t})\left(1\left\{\widehat{e}_{j,t}\geq x\right\}-1\left\{e_{j,t}\geq x\right\}\right)\right)
\end{aligned}
$$

we see that $\widetilde{\sigma}_{j,n}^{2,C+}(x)$ is defined as:

$$
\begin{aligned}
& \widetilde{\sigma}_{j,n}^{2,G+}(x) \\
= \; & \widehat{\sigma}_{j,n}^{2,C+}(x)+2\widehat{\Pi}\widehat{f}_{1,n,h}^{2}(x)\widetilde{A}_{1}(x)\widehat{\Sigma}_{11}\widetilde{A}_{1}'(x)+2\widehat{\Pi}\widehat{f}_{j,n,h}^{2}(x)\widetilde{A}_{j}(x)\widehat{\Sigma}_{jj}\widetilde{A}_{j}'(x) \\
& -4\widehat{\Pi}\widehat{f}_{1,n,h}(x)\widetilde{A}_{1}(x)\widehat{\Sigma}_{1j}\widetilde{A}_{j}'(x)\widehat{f}_{j,n,h}(x)+2\widehat{\Pi}\widehat{f}_{1,n,h}(x)\widetilde{A}_{1}(x)\widehat{\Sigma}_{u1}(x)-2\widehat{\Pi}\widehat{f}_{j,n,h}(x)\widetilde{A}_{j}(x)\widehat{\Sigma}_{uj}(x) \\
& +2\widehat{\Pi}\widetilde{B}_{1}(x)\widehat{\Sigma}_{11}\widetilde{B}_{1}'(x)+2\widehat{\Pi}\widetilde{B}_{j}'(x)\widehat{\Sigma}_{jj}\widetilde{B}_{j}'(x)-4\widehat{\Pi}\widetilde{B}_{1}(x)\widehat{\Sigma}_{1j}\widetilde{B}_{j}'(x) \\
& +2\widehat{\Pi}\widetilde{B}_{1}(x)\widehat{\Sigma}_{u1}(x)-2\widehat{\Pi}\widetilde{B}_{j}(x)'\widehat{\Sigma}_{uj}(x) \\
& +2\widehat{\Pi}\widehat{f}_{1,n,h}(x)\widetilde{A}_{1}(x)\widehat{\Sigma}_{11}\widetilde{B}_{1}'(x)+2\widehat{\Pi}\widehat{f}_{j,n,h}(x)\widetilde{A}_{j}\widehat{\Sigma}_{jj}\widetilde{B}_{j}'(x)-2\widehat{\Pi}\widetilde{B}_{1}(x)\widehat{\Sigma}_{1j}\widetilde{A}_{j}'(x)\widehat{f}_{j,n,h}(x) \\
& -2\widehat{\Pi}\widetilde{B}_{j}(x)\widehat{\Sigma}_{1j}\widetilde{A}_{1}'(x)\widehat{f}_{1,n,h}(x),
\end{aligned}
$$

where $\widehat{\sigma}_{j,n}^{2,C+}(x)$ is defined as in the statement of Lemma 1, but computed using only the last n observations. Also,

$$
\widetilde{A}_{j}(x)=\frac{1}{n}\sum_{t=R+1}^{T}(\widehat{e}_{t+1,j}-x)\,\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\widehat{\theta}_{j,R}\right)'\left(\frac{1}{R}\sum_{t=1}^{R}\nabla_{\theta_j}^{2}m_j(X_t,Z_{j,t-1},\widehat{\theta}_{j,R})\right)^{-1}
$$

and

$$
\widetilde{B}_{j}(x)=\frac{1}{n}\sum_{t=R+1}^{T}1\left\{\widehat{e}_{t+1,j}>x\right\}\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\widehat{\theta}_{j,R}\right)'\left(\frac{1}{R}\sum_{t=1}^{R}\nabla_{\theta_j}^{2}m_j(X_t,Z_{j,t-1},\widehat{\theta}_{j,R})\right)^{-1}.
$$

In order to formalize the case of asymptotically non-vanishing parameter estimation error, we require the following assumptions.

Assumption A5: ϕ_j is twice continuously differentiable on the interior of Θ_j and the elements of $\nabla_{\theta_j}\phi_j(Z_{j,i-1},\theta_i)$ and $\nabla_{\theta_j}^{2}\phi_j(Z_{j,i-1},\theta_i)$ are $p-$dominated on Θ_i, for $j=1,...,k$, with $p>4$.

Assumption A6: For $j=1,...,k$: (i) $\theta_j^{\dagger}$ is uniquely identified (i.e. $E(m_j(X_t,Z_{j,t-1},\theta_j))>E(m_j(X_t,Z_{j,t-1},\theta_j^{\dagger}))$, for any $\theta_j\neq\theta_j^{\dagger}$); (ii) m_j is twice continuously differentiable on the interior of Θ_j; (iii) the elements of $\nabla_{\theta_j}m_j$ and $\nabla_{\theta_j}^{2}m_j$ are $p-$dominated on Θ_j, with $p>4$; and (iii) $E\left(-\nabla_{\theta_j}^{2}m_j(\theta_j)\right)$ is positive definite, uniformly on Θ_j.[2]

Assumption A7: $T = R + n$, and as $T \to \infty$, $n/R \to \pi$, with $0 \leq \pi < \infty$.

As explained earlier, it is crucial to have a consistent estimator of the variance of the moment conditions. Otherwise, bootstrap critical values are not scale invariant. Hence, we need to construct estimators which properly capture parameters estimation error, regardless the fact that we rely on bootstrap critical values. We have the following result.

Lemma 3: *Let Assumptions A1-A3, and A5-A7 hold. If $l_n \approx n^{\delta}$ $\delta < \frac{1}{2}$, as defined in Assumption A1, then:*

(i) $\sup_{x\in\mathcal{X}^+} \left|\widetilde{\sigma}^{2,G+}_{j,n}(x) - \omega^{2,G+}_{j}(x)\right| = o_p(1)$, *with* $\omega^{2G+}_{j}(x) = avar\left(\sqrt{n}\widetilde{G}^{+}_{j,n}(x)\right)$; *and*

(ii) $\sup_{x\in\mathcal{X}^+} \left|\widetilde{\sigma}^{2,C+}_{j,n}(x) - \omega^{2,C+}_{j}(x)\right| = o_p(1)$, *with* $\omega^{2C+}_{j}(x) = avar\left(\sqrt{n}\widetilde{C}^{+}_{j,n}(x)\right)$.

Lemma 3 mirrors Lemma 1 for the case of non-vanishing estimation error. In order to provide the analog of Theorem 1, we need define the counterparts of $S^{\dagger G+}_{n}$ and $S^{\dagger C+}_{n}$ which take into account of parameter estimation error, i.e.

$$S^{\ddagger G+}_{n} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{0, \frac{v^{G+}_{j}(x) + h^{G+}_{j,A,n}(x)}{\sqrt{h^{G+}_{j,B}(x)}}\right\}\right)^2 \mathrm{d}Q(x),$$

with $h^{G+}_{j,A,n}(x)$ and $h^{G+}_{j,B,n}(x)$ be defined as in Section 3.2 of the main paper,

$$S^{\ddagger C+}_{n} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{0, \frac{v^{C+}_{j}(x) + h^{C+}_{j,A,n}(x)}{\sqrt{h^{C+}_{j,2}(x)}}\right\}\right)^2 \mathrm{d}Q(x).$$

The following result holds.

Theorem 5: *Let Assumptions A1-A7 hold.*

(i) Under H^{G+}_0, tthere exists $\delta > 0$ and $a^{G+} > 0$, such that

$$\lim\sup_{n\to\infty} \sup_{P\in\mathcal{P}^{G+}_0} \left[P\left(\widetilde{S}^{G+}_{n} > a^{G+}\right) - P\left(S^{\ddagger G+}_{n} + \delta > a^{G+}\right)\right] \leq 0$$

and

$$\lim\inf_{n\to\infty} \inf_{P\in\mathcal{P}^{G+}_0} \left[P\left(\widetilde{S}^{G+}_{n} > a^{G+}\right) - P\left(S^{\ddagger G+}_{n} - \delta > a^{G+}\right)\right] \geq 0.$$

(ii) Under H^{C+}_0, there exist $\delta > 0$ such that:

$$\lim\sup_{n\to\infty} \sup_{P\in\mathcal{P}^{C+}_0} \left[P\left(\widetilde{S}^{C+}_{n} > a^{C+}\right) - P\left(S^{\ddagger C+}_{n} + \delta > a^{C+}\right)\right] \leq 0$$

and

$$\lim\inf_{n\to\infty} \inf_{P\in\mathcal{P}^{C+}_0} \left[P\left(\widetilde{S}^{C+}_{n} > a^{C+}_{h_{A,n}}\right) - P\left(S^{\ddagger C+}_{n} - \delta > a^{C+}_{h_{A,n}}\right)\right] \geq 0.$$

Theorem 5 provides upper and lower bounds for $P\left(\widetilde{S}^{G+}_{n} > a^{G+}\right)$ and $P\left(\widetilde{S}^{C+}_{n} > a^{C+}\right)$, uniformly, over the probabilities under the null H^{G+}_0 and H^{C+}_0, respectively.

[2] We say that $\nabla_{\theta_j} \ln f_j(y_t, Z^{t-1}, \theta_j)$ is $2r-$dominated on Θ_j if its $v - th$ element, $v = 1, ..., \varrho(j)$, is such that $\left|\nabla_{\theta_j} \ln f_j(y_t, Z^{t-1}, \theta_j)\right|_v \leq D_t$, and $E(|D_t|^{2r}) < \infty$. For more details on domination conditions, see Gallant and White (1988, pp. 33).

2.2 Bootstrap Estimators

When computing recursive $m-$estimators, it is important to note that earlier observations are used more frequently than temporally subsequent observations. On the other hand, in the standard block bootstrap, all blocks from the original sample have the same probability of being selected, regardless of the dates of the observations in the blocks. Thus, the bootstrap estimator, say $\widehat{\theta}^*_{j,t}$, which is constructed as a direct analog of $\widehat{\theta}_{j,t}$ in (2.2), is characterized by a location bias that can be either positive or negative, depending on the sample that we observe. In order to circumvent this problem, Corradi and Swanson (2007) suggest a re-centering of the bootstrap score which ensures that the new bootstrap estimator is asymptotically unbiased. Also, assume that $T = R + n = b_T l_T$, with $b_T = b_n \frac{T}{n}$ and $l_T = l_n \frac{T}{n}$, and define:

$$\widetilde{\theta}^*_{j,t} = \arg\min_{\theta_j \in \Theta_j} \frac{1}{t} \sum_{i=1}^{t} \left(m_j(X^*_i, Z^*_{j,i-1}, \theta_j) - \theta'_j \left(\frac{1}{T} \sum_{k=1}^{T-1} \nabla_{\theta_j} m_j(X_k, Z_{j,k-1}, \widehat{\theta}_{j,t}) \right) \right),$$

where $X^*_i, Z^*_{j,i-1}$ are resampled via the "standard" block bootstrap outlined in the previous section, but with block length l_T. Theorem 1 in Corradi and Swanson (2007) establish that $\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(\widetilde{\theta}^*_{j,t} - \widehat{\theta}_{j,t} \right)$ has the same limiting distribution as $\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(\widehat{\theta}_{j,t} - \theta^\dagger_j \right)$, conditional of the sample.

With a slight abuse of notation, let $u^*_{j,t}(x) = 1\{e^*_{j,t} \leq x\} - \frac{1}{T} \sum_{t=1}^{T} 1\{\widehat{e}_{j,t} \leq x\}$ and $\eta^*_{j,t}(x) = \left[e^*_{j,t} - x\right]_+ - \frac{1}{T} \sum_{t=1}^{n} \left[\widehat{e}_{j,t} - x\right]_+$, with $e^*_{j,t+1} = X^*_{t+1} - \phi_j \left(Z^*_{j,t}, \widehat{\theta}_{j,t} \right)$, and let $\widehat{u}^*_{j,t}(x) = 1\{\widehat{e}^*_{j,t} \leq x\} - \frac{1}{T} \sum_{t=1}^{T} 1\{\widehat{e}_{j,t} \leq x\}$ and $\widehat{\eta}^*_{j,t}(x) = \left[\widehat{e}^*_{j,t} - x\right]_+ - \frac{1}{T} \sum_{t=1}^{n} \left[\widehat{e}_{j,t} - x\right]_+$, with $\widehat{e}^*_{j,t+1} = X^*_{t+1} - \phi_j \left(Z^*_{j,t}, \widetilde{\theta}^*_{j,t} \right)$. Our first goal is to construct the bootstrap counterparts of $\widetilde{\sigma}^{2,G+}_{j,n}(x)$ and $\widetilde{\sigma}^{2,C+}_{j,n}(x)$, called $\widetilde{\sigma}^{*2,G+}_{j,n}(x)$ and $\widetilde{\sigma}^{*2,C+}_{j,n}(x)$. Define:

$$\begin{aligned}
& \widetilde{\sigma}^{*2,G+}_{j,n}(x) \\
= \; & \widehat{\text{avar}^*} \left(\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(\widehat{u}^*_{j,t}(x) - \widehat{u}^*_{1,t}(x) \right) \right) \\
= \; & \widehat{\text{avar}^*} \left(\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(u^*_{j,t}(x) - u^*_{1,t}(x) \right) \right) + \widehat{\text{avar}^*} \left(\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(\widehat{f}^*_{j,n,h}(x) \widehat{PEE^*}_{j,t} - \widehat{f}^*_{1,n,h}(x) \widehat{PEE^*}_{1,t} \right) \right) \\
& -2\widehat{\text{acov}^*} \left(\frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(u^*_{j,t}(x) - u^*_{1,t}(x) \right), \frac{1}{\sqrt{n}} \sum_{t=R}^{n-1} \left(\widehat{f}^*_{j,n,h}(x) \widehat{PEE^*}_{j,t} - \widehat{f}^*_{1,n,h}(x) \widehat{PEE^*}_{1,t} \right) \right),
\end{aligned}$$

and

$$
\begin{aligned}
& \widetilde{\sigma}_{j,n}^{*2,C+}(x) \\
= \ & \widehat{\text{avar}^*}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\widehat{\eta}_{j,t}^*(x)-\widehat{\eta}_{1,t}^*(x)\right)\right) \\
= \ & \widehat{\text{avar}^*}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\eta_{j,t}^*(x)-\eta_{1,t}^*(x)\right)\right) \\
& +\widehat{\text{avar}^*}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\left[\widehat{f}_{j,n,h}^*\widehat{PEE}_{j,t}^*-x\right]_+-\left[\widehat{f}_{1,n,h}^*\widehat{PEE}_{1,t}^*-x\right]_+\right)\right)+ \\
& -2\widehat{\text{acov}^*}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\eta_{j,t}^*(x)-\eta_{1,t}^*(x)\right),\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\left[\widehat{f}_{j,n,h}^*\widehat{PEE}_{j,t}^*-x\right]_+-\left[\widehat{f}_{1,n,h}^*\widehat{PEE}_{1,t}^*-x\right]_+\right)\right),
\end{aligned}
$$

where avar* and acov* denote asymptotic variances and covariances, with respect to the bootstrap probability measure, $\widehat{f}_{j,n,h}^*$ is an estimator of the density of e_j based on the resampled observations, and $\widehat{PEE^*}_{j,t}$ is an estimator of:

$$
\begin{aligned}
PEE_{j,t}^* \ = \ & \mathrm{E}^*\left(\nabla_{\theta_j}\phi_j\left(Z_{j,t}^*,\widetilde{\theta}_{j,t}^*\right)\right)\mathrm{E}^*\left(\nabla_\theta^2 m_j\left(X_i^*,Z_{j,i-1}^*,\widetilde{\theta}_{j,t}^*\right)\right) \\
& \frac{1}{t}\sum_{i=1}^{t}\left(\nabla_\theta m_j\left(X_i^*,Z_{j,i-1}^*,\widehat{\theta}_{j,t}\right)-\frac{1}{T}\sum_{i=1}^{T}\nabla_{\theta_j}m_j(X_k,Z_{j,k-1},\widehat{\theta}_{j,t}\right). \qquad (2.6)
\end{aligned}
$$

Closed form expressions for $\widehat{PEE^*}_{j,t}$, $\widehat{\text{avar}^*}$, and $\widehat{\text{acov}^*}$ are given in the proof of Lemma 4.

Lemma 4: *Let Assumptions A1-A3 and A5-A7 hold. Then, if* $l_n \approx n^\delta$ $\delta < \frac{1}{2}$, *and* β *the mixing coefficient in Assumption A1 is such that* $\beta > \frac{6\delta}{1-2\delta}$:

(i) $\sup_{x\in\mathcal{X}^+}\left|\widetilde{\sigma}_{j,n}^{*2,G+}(x)-\widetilde{\sigma}_{j,n}^{2,G+}(x)\right| = o_p^*(1)$ *and*

(ii) $\sup_{x\in\mathcal{X}^+}\left|\widetilde{\sigma}_{j,n}^{*2,C+}(x)-\widetilde{\sigma}_{j,n}^{2,C+}(x)\right| = o_p^*(1)$.

2.3 Bootstrap Critical Values

The bootstrap statistics in the non-vanishing recursive parameter estimation error case are:

$$
\widetilde{S}_n^{*G+} = \int_{\mathcal{X}^+}\sum_{j=2}^{k}\max\left(\left\{0,\frac{\widetilde{v}_{j,n}^{*G+}(x)-\widetilde{\phi}_{j,n}^{G+}(x)}{\sqrt{\widetilde{\overline{h}}_{B,j}^{*G+}(x)}}\right\}\right)^2 \mathrm{d}Q(x), \qquad (2.7)
$$

with

$$
\begin{aligned}
\widetilde{v}_n^{*G+}(x) \ = \ & \frac{1}{\widetilde{\sigma}_{j,n}^{*2,G+}(x)\sqrt{n}}\sum_{i=R+1}^{n}\left(\left(1\left\{\widehat{e}_{j,i}^*\le x\right\}-1\left\{\widehat{e}_{1,i}^*\le x\right\}\right)\right. \\
& \left.\frac{1}{T}\sum_{t=1}^{T}\left(1\left\{\widehat{e}_{j,t}\le x\right\}-1\left\{\widehat{e}_{1,t}\le x\right\}\right)\right)
\end{aligned}
$$

$$\widetilde{\overline{h}}_{B,j}^{*G+}(x) = \sqrt{\frac{\widetilde{\sigma}_{j,n}^{*2,G+}(x) + \varepsilon}{\widetilde{\sigma}_{j,n}^{*2,G+}(x)}}$$

$$\widetilde{\phi}_{j,n}^{G+}(x) = c_n 1\left\{\widetilde{\xi}_{j,n}^{G+}(x) < -1\right\}$$

with $\widetilde{\xi}_{j,n}^{G+}(x) = \frac{\widetilde{G}_{jn}^{+}(x)}{\sqrt{\widetilde{\sigma}_{j,n}^{*2,G+}(x)+\varepsilon}}$. Finally, also define

$$\widetilde{S}_n^{*C+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \max\left(\left\{0, \frac{\widetilde{v}_{j,n}^{*C+}(x) - \widetilde{\phi}_{j,n}^{C+}(x)}{\sqrt{\widetilde{\overline{h}}_{B,j}^{*C+}(x)}}\right\}\right)^2 \mathrm{d}Q(x),$$

where $\widetilde{v}_n^{*C+}(x), \widetilde{\xi}_{j,n}^{C+}(x)$, and $\widetilde{\phi}_{j,n}^{C+}(x)$ are defined analogously to $\widetilde{v}_n^{*G+}(x), \widetilde{\xi}_{j,n}^{G+}(x)$, and $\widetilde{\phi}_{j,n}^{G+}(x)$. It is immediate to see that estimation error contributes to the bootstrap statistics not only as a scaling factor, but also in determining which moment conditions are binding. This is why we need an estimator of the variance, even if inference is based on bootstrap critical values.

We now define the GMS bootstrap critical values for the case of non-vanishing recursive estimation error. Let $\widetilde{c}_{n,B,1-\alpha}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right)$ be the $(1-\alpha)$-th critical value of $\widetilde{S}_n^{*G+}$, based on B bootstrap replications, with $\widetilde{\phi}_n^{G+}$ and $\widetilde{\overline{h}}_{B,j}^{*G+}(x)$ defined above. The $(1-\alpha)$-th GMS bootstrap critical value, $\widetilde{c}_{0,n,1-\alpha}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right)$ is defined as:

$$\widetilde{c}_{0,n,1-\alpha}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right) = \lim_{B\to\infty} \widetilde{c}_{n,B,1-\alpha+\eta}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right) + \eta,$$

for arbitrarily small $\eta > 0$. Also, $\widetilde{c}_{n,B,1-\alpha+\eta}^{*C+}\left(\widetilde{\phi}_n^{C+}, \widetilde{\overline{h}}_{B,j}^{*C+}\right)$ and $\widetilde{c}_{0,n,1-\alpha}^{*C+}\left(\widetilde{\phi}_n^{C+}, \widetilde{\overline{h}}_{B,j}^{*C+}\right)$ are defined analogously. The following result then holds.

Theorem 6: Let Assumptions A1-A7 hold, and let $l_n \to \infty$ and $l_n n^{\frac{1}{3}-\varepsilon} \to 0$, as $n \to \infty$. Under H_0^{G+} :
(i) if as $n \to \infty$, $\kappa_n \to \infty$ and $c_n/\kappa_n \to 0$, then

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{G+}} P\left(\widetilde{S}_n^{G+} \geq \widetilde{c}_{n,B,1-\alpha+\eta}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right)\right) \leq \alpha;$$

and (ii) if as $n \to \infty$, $\kappa_n \to \infty$, $c_n \to \infty$, $\sqrt{n}/\kappa_n \to \infty$ and $Q\left(\mathcal{B}^{G+}\right) > 0$, $\mathcal{B}^{G+}$ as in Eq. (3.13), then

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{G+}} P\left(\widetilde{S}_n^{G+} \geq \widetilde{c}_{n,B,1-\alpha+\eta}^{*G+}\left(\widetilde{\phi}_n^{G+}, \widetilde{\overline{h}}_{B,j}^{*G+}\right)\right) = \alpha.$$

Also, under H_0^{C+},
(iii) if as $n \to \infty$, $\kappa_n \to \infty$ and $c_n/\kappa_n \to 0$, then

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{C+}} P\left(\widetilde{S}_n^{C+} \geq \widetilde{c}_{0,n,1-\alpha}^{*C+}\left(\widetilde{\phi}_n^{C+}, \widetilde{\overline{h}}_{B,j}^{*C+}\right)\right) \leq \alpha;$$

and (iv) if as $n \to \infty$, $\kappa_n \to \infty$, $c_n \to \infty$, $\sqrt{n}/\kappa_n \to \infty$ and $Q\left(\mathcal{B}^{C+}\right) > 0$, $\mathcal{B}^{C+}$ as in Eq. (3.14), then

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{C+}} P\left(\widetilde{S}_n^{C+} \geq \widetilde{c}_{0,n,1-\alpha}^{*C+}\left(\widetilde{\phi}_n^{C+}, \widetilde{\overline{h}}_{B,j}^{*C+}\right)\right) = \alpha.$$

Statements (i) and (iii) of Theorem 6 establish that inference based on GMS bootstrap critical values has uniform correct size, in the parameter estimation error case. Statements (ii) and (iv) of the theorem establish that inference based on the GMS bootstrap critical values is asymptotically non-conservative, whenever $Q\left(\mathcal{B}^{+}\right)>0$ or $Q\left(\mathcal{B}^{C+}\right)>0$.

2.4 Proofs

Proof of Lemma 3: (i) Letting $\overline{F}_j(x)=\frac{1}{n}\sum_{t=R}^{T-1}1\left\{\widehat{e}_{j,t+1}\leq x\right\}$, by an intermediate value expansion, in the case of a recursive estimation scheme, we have that

$$
\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{\widehat{e}_{j,t+1}\leq x\right\}-F_j(x)\right) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)+\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{\widehat{e}_{j,t+1}\leq x\right\}-1\left\{e_{j,t+1}\leq x\right\}\right) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)+\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right\}\right.\right. \\
& \left.\left.-F_j\left(x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right)\right)-\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)\right) \\
& +\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(F_j\left(x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right)-F_j(x)\right) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)+\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(F_j\left(x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right)-F_j(x)\right) \\
& +o_p(1) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)-f_j(x)\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)+o_p(1) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right) \qquad (2.8) \\
& -f_j(x)\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)'\frac{1}{t}\sum_{i=1}^{t}\left(\nabla^2_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)^{-1}\left(\nabla_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)+o_p(1) \\
= \; & \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)-f_j(x)\widehat{A}_j\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)+o_p(1)
\end{aligned}
$$

where the $o_p(1)$ term on the RHS of the third equality in (2.8) comes from the fact that

$$
\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right\}\right. \\
& \left.-F_j\left(x-\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\overline{\theta}_{j,t}\right)\left(\widehat{\theta}_{j,t}-\theta_j^{\dagger}\right)\right)\right)-\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)=o_p(1),
\end{aligned}
$$

because of stochastic equicontinuity.

Hence,

$$
\begin{aligned}
& \operatorname{var}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(\left(1\left\{\widehat{e}_{1,t+1}\leq x\right\}-F_1(x)\right)-\left(1\left\{\widehat{e}_{j,t+1}\leq x\right\}-F_j(x)\right)\right)\right) \\
= \ & \operatorname{var}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(\left(1\left\{e_{1,t+1}\leq x\right\}-F_1(x)\right)-\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)\right)\right) \\
& +f_1(x)^2\mathrm{E}\left(\nabla_{\theta_1}\phi_1\left(Z_{1,t+1},\theta_1^{\dagger}\right)\right)'\left(\mathrm{E}\left(\nabla^2_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right)^{-1} \\
& \operatorname{var}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right) \\
& \left(\mathrm{E}\left(\nabla^2_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right)^{-1}\mathrm{E}\left(\nabla_{\theta_1}\phi_1\left(Z_{1,t+1},\theta_1^{\dagger}\right)\right) \\
& +f_j(x)^2\mathrm{E}\left(\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\theta_j^{\dagger}\right)\right)'\left(\mathrm{E}\left(\nabla^2_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right)^{-1} \\
& \operatorname{var}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right) \\
& \left(\mathrm{E}\left(\nabla^2_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right)^{-1}\mathrm{E}\left(\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\theta_j^{\dagger}\right)\right) \\
& -2f_1(x)f_j(x)\mathrm{E}\left(\nabla_{\theta_1}\phi_1\left(Z_{1,t+1},\theta_1^{\dagger}\right)\right)'\left(\mathrm{E}\left(\nabla^2_{\theta_1}m_j(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right)^{-1} \\
& \operatorname{cov}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right) \\
& \left(\mathrm{E}\left(\nabla^2_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right)^{-1}\mathrm{E}\left(\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\theta_j^{\dagger}\right)\right) \\
& +2f_1(x)\mathrm{E}\left(\nabla_{\theta_1}\phi_1\left(Z_{1,t+1},\theta_1^{\dagger}\right)\right)'\left(\mathrm{E}\left(\nabla^2_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right)^{-1} \\
& \operatorname{cov}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(\left(1\left\{e_{1,t+1}\leq x\right\}-F_1(x)\right)-\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)\right)\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_1}m_1(X_i,Z_{1,i-1},\theta_1^{\dagger})\right)\right) \\
& -2f_j(x)^2\mathrm{E}\left(\nabla_{\theta_j}\phi_j\left(Z_{j,t+1},\theta_j^{\dagger}\right)\right)'\left(\mathrm{E}\left(\nabla^2_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right)^{-1} \\
& \operatorname{cov}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\left(\left(1\left\{e_{1,t+1}\leq x\right\}-F_1(x)\right)-\left(1\left\{e_{j,t+1}\leq x\right\}-F_j(x)\right)\right)\frac{1}{\sqrt{n}}\sum_{t=R}^{T-1}\frac{1}{t}\sum_{i=1}^{t}\left(\nabla_{\theta_j}m_j(X_i,Z_{j,i-1},\theta_j^{\dagger})\right)\right)
\end{aligned}
$$

(ii) Recalling (2.5) by a similar argument as in part (i).

Proof of Theorem 5:
Given Lemma 3, the statement follows by the same argument as in Theorem 1.

Proof of Lemma 4:
(i) Note that $\widehat{\text{avar}^*}\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\eta^*_{j,t}(x)-\eta^*_{1,t}(x)\right)\right)=\widehat{\sigma}^{2*G+}_{j,n}(x)$ as defined in Eq. (3.1), $\widehat{PEE^*}_{j,t}$ is defined

as $PEE^*_{j,t}$ with E^* replaced by an average, also

$$\begin{aligned}
&\widehat{\text{avar}}^*\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\widehat{PEE^*}_{j,t}-\widehat{PEE^*}_{1,t}\right)\right)\\
=\ & \frac{1}{b_n}\sum_{k=1}^{b_n}\left(\frac{1}{l_n^{1/2}}\sum_{i=1}^{l_n}\left(\widehat{PEE^*}_{j,(k-1)l_n+i}-\widehat{PEE^*}_{j,(k-1)l_n+i}\right)\right)^2
\end{aligned}$$

and by Theorem 1 in Corradi and Swanson (2007),

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\widehat{PEE^*}_{j,t}-\widehat{PEE^*}_{1,t}\right)\\
=\ & \frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\widehat{PEE}_{j,t}-\widehat{PEE}_{1,t}\right)+o_p(1)^*.
\end{aligned}$$

$$\begin{aligned}
&\widehat{\text{acov}}^*\left(\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(u^*_{j,t}(x)-u^*_{1,t}(x)\right),\frac{1}{\sqrt{n}}\sum_{t=R}^{n-1}\left(\widehat{PEE^*}_{j,t}-\widehat{PEE^*}_{1,t}\right)\right)\\
=\ & \frac{1}{b_n}\sum_{k=1}^{b_n}\left(\frac{1}{l_n^{1/2}}\sum_{i=1}^{l_n}\left(\widehat{PEE^*}_{j,(k-1)l_n+i}-\widehat{PEE^*}_{j,(k-1)l_n+i}\right)\right.\\
& \left.\frac{1}{l_n^{1/2}}\sum_{i=1}^{l_n}\left(u^*_{j,t}(x)-u^*_{1,t}(x)\right)\right)
\end{aligned}$$

and for $h\to 0$, $nh\to\infty$, $\widehat{f}^*_{j,n,h}(x)=\widehat{f}_{j,n,h}(x)+o_{p^*}(1)=f(x)+o_p(1)+o_{p^*}(1)$. The statement then follow by the same argument as in Lemma 2 and Lemma 3.
(ii) By a similar argument as in Part (i).

Proof of Theorem 6:
(i) By a similar argument as in the proof of Theorem 2 in Corradi and Swanson (2007),

$$\begin{aligned}
\widetilde{S}_n^{*G+} &= \int_{\mathcal{X}^+}\sum_{j=2}^{k}\max\left(\left\{0,\frac{\widetilde{v}^{*G+}_{j,n}(x)-\widetilde{\phi}^{G+}_{j,n}(x)}{\sqrt{\widetilde{\overline{h}}^{*G+}_{2,jj}(x)}}\right\}\right)^2\mathrm{d}Q(x)\\
&= \int_{\mathcal{X}^+}\sum_{j=2}^{k}\max\left(\left\{0,\frac{\widetilde{v}^{G+}_{j,n}(x)-\widetilde{\phi}^{G+}_{j,n}(x)}{\sqrt{\widetilde{\overline{h}}^{G+}_{2,jj}(x)}}\right\}\right)^2\mathrm{d}Q(x)+o_{p^*}(1)
\end{aligned}$$

The statement then follows from Lemma 4 and Theorem 2.
(ii) By a similar argument as in Part (i).

3 Appendix SA3: The JCS Test

In this section, we provide details on the JCS test used in the Monte Carlo and empirical sections of the paper, which deal with forecast superiority of judgmental forecasts. For the general case in which we compare model based prediction, please refer to Jin, Corradi, and Swanson (2017). For $j = 2, ..., k$, let $G_{j,n}(x)$ be defined as in Eq.(2.3) in Section 2 of the paper, and $C_{j,n}(x)$ be defined as in Eq.(1.1) in Appendix SA1. Then,

$$JCS_n^{G^+} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^+} \sqrt{n}G_{j,n}(x) \text{ and } JCS_n^{G^-} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^-} \sqrt{n}G_{j,n}(x)$$

and

$$JCS_n^{C^+} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^+} \sqrt{n}C_{j,n}(x) \text{ and } JCS_n^{C^-} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^-} \sqrt{n}C_{j,n}(x)$$

JCS uses the the stationary boostrap of Politis and Romano (1994) with smoothing parameter J_n. For $j = 2, ..., k$, let $G^*_{k,n}(x)$ and $C^*_{k,n}(x)$ be the bootstrap analogs of $G_{k,n}(x)$ and $G_{k,n}(x)$, then boostrap counterparts of $JCS_n^{G^+}$, $JCS_n^{G^-}$, $JCS_n^{C^+}$ and $JCS_n^{C^-}$ read as

$$JCS_n^{*G^+} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^+} \sqrt{n}\left(G^*_{j,n}(x) - G_{j,n}(x)\right) \text{ and } JCS_n^{*G^-} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^-} \sqrt{n}\left(G^*_{j,n}(x) - G_{j,n}(x)\right)$$

and

$$JCS_n^{*C^+} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^+} \sqrt{n}\left(C^*_{j,n}(x) - C_{j,n}(x)\right) \text{ and } JCS_n^{*G^-} = \max_{j=2,..,k} \sup_{x\in\mathcal{X}^-} \sqrt{n}\left(C^*_{j,n}(x) - C_{j,n}(x)\right)$$

JCS bootstrap critical values are then obtained via the empirical distribution of $JCS_n^{*G^+}$, $JCS_n^{*G^-}$, $JCS_n^{*C^+}$ and $JCS_n^{*C^-}$. They key difference with the new tests is that in JCS forecasts which are dominated by the benchmark do not contribute to the statistic but do contribute to its bootstrap counterpart. Hence, inference based on JCS tests is asymptotically non conservative only in the least favorable case in the null.

4 Appendix SA4: Additional Monte Carlo Experimental Results

In this section, experimental results are tabulated for the following DGPs:
DGP1: $e_{1t} \sim i.i.d.N(0, 1)$ and $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3$.
DGP2: $e_{1t} \sim i.i.d.N(0, 1)$ and $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3, 4, 5$.
DGP3: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.4^2)$, $k = 4, 5$
DGP4: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.6^2)$, $k = 4, 5$
DGP5: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.2^2)$, $k = 4, 5$.
DGP6: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3, 4, 5$ and $e_{kt} \sim i.i.d.N(0, 1.2^2)$, $k = 6, 7, 8, 9$.
DGP7: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 4, 5$.
DGP8: $e_{1t} \sim i.i.d.N(0, 1)$, $e_{kt} \sim i.i.d.N(0, 1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 0.6^2)$, $k = 4, 5$.
DGP9: $e_{1t} \sim i.i.d.N(0, 1)$ and $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3, 4, 5$.
DGP10: $e_{1t} \sim i.i.d.N(0, 1)$ and $e_{kt} \sim i.i.d.N(0, 0.6^2)$, $k = 2, 3, 4, 5$.

See Tables 1-2 for tabulated results, and Section 4 of the paper for complete details regarding the experiments that were run.

5 Appendix SA5: Additional Empirical Results

Tables 3-4 gather root mean square forecast errors associated with the models reported on in Tables 3 and 4 of the paper. These results are for nominal GDP. Results based on the same set of empirical experiments carried out using real GDP are reported in Tables 5-8. See Section 5 of the paper for a complete discussion.

6 References

Jin, S., V. Corradi and N.R. Swanson (2017). Robust Forecast Comparison. *Econometric Theory,* 33, 1306-1351.

Politis, D. N., J. P. Romano (1994a), The Stationary Bootstrap, *Journal of the American Statistical Association* 89, 1303-1313.

Table 1: Supplemental Monte Carlo Results: JCS_n^{G+}, JCS_n^{G-}, JCS_n^{C+}, and JCS_n^{C-} Tests*

DGP	n	$J_n = 0.20$	$J_n = 0.35$	$J_n = 0.50$	$J_n = 0.65$	$J_n = 0.20$	$J_n = 0.35$	$J_n = 0.50$	$J_n = 0.65$
		GL Forecast Superiority				CL Forecast Superiority			
		Empirical Size							
DGP1	300	0.113	0.100	0.101	0.112	0.113	0.107	0.120	0.115
	600	0.105	0.110	0.099	0.108	0.091	0.089	0.094	0.091
	900	0.102	0.095	0.093	0.096	0.094	0.092	0.082	0.094
DGP2	300	0.110	0.105	0.115	0.113	0.097	0.097	0.092	0.106
	600	0.077	0.073	0.082	0.079	0.090	0.092	0.089	0.081
	900	0.085	0.084	0.086	0.095	0.089	0.101	0.097	0.092
DGP3	300	0.070	0.065	0.065	0.060	0.030	0.030	0.035	0.035
	600	0.050	0.040	0.050	0.045	0.030	0.020	0.030	0.025
	900	0.070	0.070	0.080	0.075	0.020	0.020	0.020	0.020
DGP4	300	0.065	0.070	0.060	0.065	0.010	0.015	0.020	0.015
	600	0.040	0.040	0.040	0.055	0.015	0.015	0.015	0.020
	900	0.070	0.065	0.065	0.065	0.015	0.015	0.015	0.015
		Empirical Power							
DGP5	300	0.496	0.485	0.490	0.477	0.753	0.759	0.745	0.759
	600	0.775	0.771	0.773	0.773	0.991	0.986	0.989	0.981
	900	0.943	0.951	0.947	0.938	1.000	1.000	1.000	1.000
DGP6	300	0.483	0.480	0.476	0.474	0.758	0.741	0.745	0.736
	600	0.768	0.778	0.772	0.774	0.984	0.975	0.981	0.980
	900	0.954	0.949	0.954	0.947	1.000	1.000	1.000	1.000
DGP7	300	0.490	0.475	0.475	0.445	0.875	0.865	0.845	0.855
	600	0.835	0.820	0.820	0.800	0.995	0.995	0.995	0.995
	900	0.975	0.970	0.970	0.965	1.000	1.000	1.000	1.000
DGP8	300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP9	300	0.643	0.660	0.650	0.629	0.949	0.944	0.937	0.948
	600	0.913	0.885	0.890	0.896	1.000	1.000	1.000	1.000
	900	0.990	0.986	0.984	0.983	1.000	1.000	1.000	1.000
DGP10	300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

* Notes: Entries denote rejection frequencies of (JCS_n^{G+},JCS_n^{G-}) tests (i.e., GL forecast superiority) and (JCS_n^{C+},JCS_n^{C-}) tests (i.e., CL forecast superiority) under a variety of data generating processes denoted by DGP1-DGP10. In DGP1-DGP4, no alternative outperforms the benchmark model. In DGP5-DGP10, at least one alternative model outperfroms the benchmark model. Sample sizes include n=300, 600, and 900 observations, as indicated in the second column of entries in the table. Nominal test size is 10%, and tests are carried out using critical values constructed for values of J_n including 0.20, 0.35, 0.50, and 0.65. See Section 4 for complete details.

Table 2: Supplemental Monte Carlo Results: S_n^{G+}, S_n^{G-}, S_n^{C+}, and S_n^{C-} Tests*

DGP	n	$\eta = 0.0015$	$\eta = 0.002$	$\eta = 0.0025$	$\eta = 0.003$	$\eta = 0.0015$	$\eta = 0.002$	$\eta = 0.0025$	$\eta = 0.003$
		GL Forecast Superiority				CL Forecast Superiority			
		Empirical Size							
DGP1	300	0.078	0.078	0.076	0.076	0.095	0.094	0.094	0.091
	600	0.096	0.096	0.095	0.095	0.116	0.116	0.115	0.114
	900	0.120	0.119	0.118	0.117	0.096	0.096	0.095	0.095
DGP2	300	0.068	0.067	0.067	0.066	0.095	0.095	0.095	0.094
	600	0.097	0.096	0.095	0.095	0.096	0.096	0.095	0.094
	900	0.111	0.108	0.108	0.105	0.106	0.105	0.105	0.105
DGP3	300	0.021	0.021	0.021	0.021	0.038	0.038	0.038	0.038
	600	0.070	0.070	0.069	0.068	0.086	0.086	0.085	0.085
	900	0.071	0.070	0.069	0.069	0.083	0.083	0.083	0.082
DGP4	300	0.010	0.010	0.01	0.010	0.041	0.041	0.041	0.040
	600	0.064	0.064	0.064	0.063	0.077	0.077	0.076	0.076
	900	0.069	0.067	0.064	0.063	0.083	0.083	0.083	0.083
		Empirical Power							
DGP5	300	0.911	0.911	0.910	0.909	0.972	0.972	0.971	0.971
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP6	300	0.957	0.956	0.956	0.956	0.989	0.989	0.989	0.989
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP7	300	0.874	0.872	0.871	0.870	0.925	0.924	0.923	0.923
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP8	300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP9	300	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP10	300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

* Notes: Entries denote rejection frequencies of (S_n^{G+},S_n^{G-}) tests (i.e., GL forecast superiority) and (S_n^{C+},S_n^{C-}) tests (i.e., CL forecast superiority) under a variety of data generating processes denoted by DGP1-DGP10. In DGP1-DGP4, no alternative outperforms the benchmark model. In DGP5-DGP10, at least one alternative model outperfroms the benchmark model. Sample sizes include n=300, 600, and 900 observations, as indicated in the second column of entries in the table. Nominal test size is 10%, and tests are carried out using critical values constructed for values of η including 0.0015, 0.002, 0.0025, and 0.0030. See Section 4 for complete details.

Table 3: Supplemental Empirical Results (RMSFEs)– SPF Forecast Pooling Analysis of Quarterly Nominal GDP Using Mean Benchmark Model and Mean Expert Pool Predictions*

Group	*Model*	*Forecast Horizon*				
		$h = 0$	$h = 1$	$h = 2$	$h = 3$	$h = 4$
Group 1	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007237	0.012054	0.016277	0.020490	0.024470
	alternative 2	0.007264	0.012108	0.016358	0.020575	0.024611
	alternative 3	0.007272	0.012141	0.016328	0.020596	0.025276
Group 2	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.008794	0.012535	0.016267	0.021398	0.026000
	alternative 2	0.007476	0.012562	0.017698	0.021331	0.028178
	alternative 3	0.007602	0.012931	0.018113	0.022797	0.025831
Group 3	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007517	0.012197	0.015320	0.019503	0.023099
	alternative 2	0.007128	0.012595	0.016236	0.021133	0.024353
	alternative 3	0.007110	0.012292	0.016534	0.020799	0.024430
Group 4	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007811	0.011955	0.015533	0.019227	0.022871
	alternative 2	0.006971	0.012656	0.017039	0.021008	0.026196
	alternative 3	0.007306	0.012764	0.017251	0.021353	0.025116
Group 5	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007257	0.012007	0.015367	0.019122	0.023173
	alternative 2	0.007197	0.012401	0.016357	0.020830	0.024536
	alternative 3	0.007219	0.012215	0.016720	0.020553	0.024250
Group 6	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007237	0.012054	0.016277	0.020490	0.024470
	alternative 2	0.008794	0.012535	0.016267	0.021398	0.026000
	alternative 3	0.007517	0.012197	0.015320	0.019503	0.023099
	alternative 4	0.007811	0.011955	0.015533	0.019227	0.022871
	alternative 5	0.007257	0.012007	0.015367	0.019122	0.023173
Group 7	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007264	0.012108	0.016358	0.020575	0.024611
	alternative 2	0.007476	0.012562	0.017698	0.021331	0.028178
	alternative 3	0.007128	0.012595	0.016236	0.021133	0.024353
	alternative 4	0.006971	0.012656	0.017039	0.021008	0.026196
	alternative 5	0.007197	0.012401	0.016357	0.020830	0.024536
Group 8	benchmark	0.007245	0.012063	0.016287	0.020506	0.024494
	alternative 1	0.007272	0.012141	0.016328	0.020596	0.025276
	alternative 2	0.007602	0.012931	0.018113	0.022797	0.025831
	alternative 3	0.007110	0.012292	0.016534	0.020799	0.024430
	alternative 4	0.007306	0.012764	0.017251	0.021353	0.025116
	alternative 5	0.007219	0.012215	0.016720	0.020553	0.024250

* Notes: Entries are root mean square forecast errors (RMSFEs) of benchmark and alternative forecasting models for h =0,1,2,3,4. See Section 5 for complete details.

Table 4: Supplemental Empirical Results (RMSFEs) – SPF Forecast Pooling Analysis of Quarterly Nominal GDP Using Median Benchmark Model and Median Expert Pool Predictions*

Group	*Model*	*Forecast Horizon*				
		$h = 0$	$h = 1$	$h = 2$	$h = 3$	$h = 4$
Group 1	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007301	0.012033	0.016464	0.020595	0.024785
	alternative 2	0.007340	0.012049	0.016419	0.020566	0.024728
	alternative 3	0.007364	0.012067	0.016498	0.020632	0.025513
Group 2	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.008794	0.012535	0.016267	0.021398	0.026000
	alternative 2	0.007476	0.012562	0.017698	0.021331	0.028178
	alternative 3	0.007602	0.012931	0.018113	0.022797	0.025831
Group 3	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007825	0.012264	0.016016	0.019494	0.023117
	alternative 2	0.007291	0.012425	0.016649	0.021205	0.024090
	alternative 3	0.007372	0.012410	0.016961	0.021245	0.024526
Group 4	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007893	0.012048	0.015983	0.018890	0.022662
	alternative 2	0.007058	0.012606	0.017110	0.021198	0.026052
	alternative 3	0.007231	0.012771	0.017289	0.021477	0.025041
Group 5	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007459	0.011891	0.015555	0.019435	0.023215
	alternative 2	0.007197	0.012299	0.016555	0.020814	0.024760
	alternative 3	0.007400	0.012164	0.017024	0.020698	0.024653
Group 6	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007301	0.012033	0.016464	0.020595	0.024785
	alternative 2	0.008794	0.012535	0.016267	0.021398	0.026000
	alternative 3	0.007825	0.012264	0.016016	0.019494	0.023117
	alternative 4	0.007893	0.012048	0.015983	0.018890	0.022662
	alternative 5	0.007459	0.011891	0.015555	0.019435	0.023215
Group 7	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007340	0.012049	0.016419	0.020566	0.024728
	alternative 2	0.007476	0.012562	0.017698	0.021331	0.028178
	alternative 3	0.007291	0.012425	0.016649	0.021205	0.024090
	alternative 4	0.007058	0.012606	0.017110	0.021198	0.026052
	alternative 5	0.007197	0.012299	0.016555	0.020814	0.024760
Group 8	benchmark	0.007300	0.012036	0.016398	0.020542	0.024699
	alternative 1	0.007364	0.012067	0.016498	0.020632	0.025513
	alternative 2	0.007602	0.012931	0.018113	0.022797	0.025831
	alternative 3	0.007372	0.012410	0.016961	0.021245	0.024526
	alternative 4	0.007231	0.012771	0.017289	0.021477	0.025041
	alternative 5	0.007400	0.012164	0.017024	0.020698	0.024653

* Notes: See notes to Table 3.

Table 5: Supplemental Empirical Results (Test Statistics) – SPF Forecast Pooling Analysis of Quarterly Real GDP Using Mean Benchmark Model and Mean Expert Pool Predictions*

Group	*Statistic*	*Forecast Horizon*				
		$h=0$	$h=1$	$h=2$	$h=3$	$h=4$
Group 1	S_n^G	0.000824	0.002252	0.000001	0.002324	0.002887
	S_n^C	0.000002	0.002343	0.012414	0.020483	0.014605
	JCS_n^G	0.155230	0.077615	0.077615	0.155230	0.232845
	JCS_n^C	0.000054	0.001349	0.001398	0.002951	0.002554
Group 2	S_n^G	0.001799	0.002707	0.011380	0.005501	0.002788
	S_n^C	0.000152	0.009679	0.031745	0.028096	0.094510*
	JCS_n^G	0.232845	0.388075	1.008996*	0.155230	0.543305
	JCS_n^C	0.000560	0.001153	0.005452	0.006671	0.013698
Group 3	S_n^G	0.000988	0.000887	0.006405	0.008404	0.002418
	S_n^C	0.000902	0.003550	0.020512	0.051268	0.044420
	JCS_n^G	0.155230	0.077615	0.620920	0.543305	0.310460
	JCS_n^C	0.001551	0.002792	0.002252	0.011279	0.006633
Group 4	S_n^G	0.001005	0.000740	0.008447	0.006479	0.008698
	S_n^C	0.000206	0.007098	0.023468	0.038963	0.055278
	JCS_n^G	0.310460	0.077615	0.698535	0.388075	0.388075
	JCS_n^C	0.000912	0.003747	0.002675	0.010032	0.006216
Group 5	S_n^G	0.001896	0.003768	0.000848	0.004601	0.009403
	S_n^C	0.000287	0.004256	0.033755	0.029876	0.097775
	JCS_n^G	0.465690	0.155230	0.155230	0.543305*	0.155230
	JCS_n^C	0.000808	0.000114	0.004158*	0.006916	0.008606*
Group 6	S_n^G	0.000673	0.002223	0.011862	0.005185	0.003483
	S_n^C	0.000058	0.001989	0.037812	0.044141	0.113234
	JCS_n^G	0.232845	0.077615	0.388075	0.698535	0.543305
	JCS_n^C	0.000431	0.005559	0.003439	0.003138	0.013694
Group 7	S_n^G	0.003083	0.002330	0.002004	0.006939	0.005146
	S_n^C	0.001370	0.001372	0.016322	0.052538	0.062255
	JCS_n^G	0.388075	0.077615	0.232845	0.232845	0.232845
	JCS_n^C	0.001549	0.000073	0.002155	0.008447	0.004336
Group 8	S_n^G	0.001306	0.005194	0.003816	0.014861	0.021174
	S_n^C	0.000062	0.024702	0.065485	0.071887	0.131380
	JCS_n^G	0.155230	0.077615	0.232845	0.698535	0.543305
	JCS_n^C	0.000713	0.003734	0.004056	0.007718	0.008207

* Notes: Entries are S_n^G, S_n^C, JCS_n^G, and JCS_n^C test statistics reported for forecast horizons $h = 0, 1, 2, 3, 4$. More specifically, $S_n^G = S_n^{G+}$ if $p^{G+}_{B,n,S_n^{G+}} \leq p^{G-}_{B,n,S_n^{G-}}$; $otherwise$ $S_n^G = S_n^{G-}$. S_n^C, JCS_n^G, and JCS_n^C are defined analogously. Rejections of the null of no forecast superiority at a 10% level are denoted by a superscipt *. See Section 5 for complete details.

Table 6: Supplemental Empirical Results (Test Statistics) – SPF Forecast Pooling Analysis of Quarterly Real GDP Using Median Benchmark Model and Median Expert Pool Predictions*

Group	*Statistic*	*Forecast Horizon*				
		$h = 0$	$h = 1$	$h = 2$	$h = 3$	$h = 4$
Group 1	S_n^G	0.000000	0.000000	0.000000	0.005349	0.000000
	S_n^C	0.000054	0.001003	0.003760	0.022297	0.005915
	JCS_n^G	0.000000	0.077615	0.077615	0.155230	0.000000
	JCS_n^C	0.000186	0.000548	0.000630	0.002178	0.001092
Group 2	S_n^G	0.002210	0.003952	0.007185	0.003319	0.004003
	S_n^C	0.000192	0.010594	0.029682	0.015513	0.071462
	JCS_n^G	0.155230	0.155230	0.853766*	0.077615	0.543305
	JCS_n^C	0.000632	0.001430	0.005290	0.004591	0.011701
Group 3	S_n^G	0.001890	0.002990	0.008263	0.003040	0.010323
	S_n^C	0.001800	0.002079	0.043400	0.017020	0.026739
	JCS_n^G	0.310460	0.232845	0.543305*	0.232845	0.388075
	JCS_n^C	0.002035	0.000282	0.004896	0.010389	0.008321
Group 4	S_n^G	0.000853	0.002997	0.006742	0.003129	0.002204
	S_n^C	0.000241	0.007327	0.034950	0.028378	0.049325
	JCS_n^G	0.310460	0.155230	0.698535	0.388075	0.388075
	JCS_n^C	0.000854	0.003705	0.005250	0.011190	0.007344
Group 5	S_n^G	0.000740	0.000748	0.000147	0.005364	0.010838
	S_n^C	0.001083	0.004036	0.030223	0.031610	0.047832
	JCS_n^G	0.155230	0.077615	0.155230	0.232845	0.388075
	JCS_n^C	0.001163	0.002133	0.004524*	0.005652	0.008434*
Group 6	S_n^G	0.001219	0.002221	0.012655	0.003963	0.003489
	S_n^C	0.000206	0.002231	0.066063	0.034762	0.103800
	JCS_n^G	0.310460	0.077615	0.388075	0.310460	0.620920
	JCS_n^C	0.000496	0.005717	0.004404	0.011729	0.011775
Group 7	S_n^G	0.001638	0.004847	0.001687	0.002461	0.007289
	S_n^C	0.003125	0.001426	0.016467	0.037540	0.014719
	JCS_n^G	0.310460	0.155230	0.155230	0.077615	0.232845
	JCS_n^C	0.002033	0.000234	0.002944	0.005121	0.001894
Group 8	S_n^G	0.002070	0.005195	0.002250	0.007771	0.002820
	S_n^C	0.000044	0.022723	0.055627	0.041172	0.082716
	JCS_n^G	0.155230	0.077615	0.077615	0.388075	0.310460
	JCS_n^C	0.000293	0.003702	0.004050	0.005970	0.005798

* Notes: See notes to Table 6.

Table 7: Supplemental Empirical Results (RMSFEs) – SPF Forecast Pooling Analysis of Quarterly Real GDP Using Mean Benchmark Model and Mean Expert Pool Predictions*

Group	*Model*	*Forecast Horizon*				
		$h=0$	$h=1$	$h=2$	$h=3$	$h=4$
Group 1	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064388	0.096675	0.120861	0.140806	0.158194
	alternative 2	0.064363	0.096571	0.120695	0.140615	0.157738
	alternative 3	0.064371	0.096558	0.120535	0.140318	0.15863
Group 2	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064301	0.096925	0.120608	0.140357	0.156271
	alternative 2	0.064669	0.096909	0.12127	0.140601	0.157682
	alternative 3	0.064602	0.096323	0.120285	0.140601	0.156228
Group 3	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064426	0.09699	0.120392	0.140216	0.157832
	alternative 2	0.064184	0.096764	0.12081	0.140282	0.157844
	alternative 3	0.064511	0.096548	0.120742	0.140341	0.157491
Group 4	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064287	0.096933	0.120559	0.140288	0.157503
	alternative 2	0.064347	0.096876	0.120944	0.140352	0.158064
	alternative 3	0.06455	0.096397	0.120589	0.140437	0.156927
Group 5	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064395	0.096659	0.120372	0.14074	0.157251
	alternative 2	0.064199	0.09671	0.120781	0.140915	0.157701
	alternative 3	0.064597	0.096458	0.120587	0.140237	0.157023
Group 6	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064388	0.096675	0.120861	0.140806	0.158194
	alternative 2	0.064301	0.096925	0.120608	0.140357	0.156271
	alternative 3	0.064426	0.09699	0.120392	0.140216	0.157832
	alternative 4	0.064287	0.096933	0.120559	0.140288	0.157503
	alternative 5	0.064395	0.096659	0.120372	0.14074	0.157251
Group 7	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064363	0.096571	0.120695	0.140615	0.157738
	alternative 2	0.064669	0.096909	0.12127	0.140601	0.157682
	alternative 3	0.064184	0.096764	0.12081	0.140282	0.157844
	alternative 4	0.064347	0.096876	0.120944	0.140352	0.158064
	alternative 5	0.064199	0.09671	0.120781	0.140915	0.157701
Group 8	benchmark	0.064387	0.096676	0.120863	0.140802	0.158234
	alternative 1	0.064371	0.096558	0.120535	0.140318	0.15863
	alternative 2	0.064602	0.096323	0.120285	0.140601	0.156228
	alternative 3	0.064511	0.096548	0.120742	0.140341	0.157491
	alternative 4	0.06455	0.096397	0.120589	0.140437	0.156927
	alternative 5	0.064597	0.096458	0.120587	0.140237	0.157023

* Notes: Entries are root mean square forecast errors (RMSFEs) of benchmark and alternative forecasting models for h =0,1,2,3,4. See Section 5 for complete details.

Table 8: Supplemental Empirical Results (RMSFEs) – SPF Forecast Pooling Analysis of Quarterly Real GDP Using Median Benchmark Model and Median Expert Pool Predictions*

Group	*Model*	*Forecast Horizon*				
		$h=0$	$h=1$	$h=2$	$h=3$	$h=4$
Group 1	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064387	0.096706	0.120832	0.14062	0.157779
	alternative 2	0.064384	0.09664	0.120718	0.140338	0.157551
	alternative 3	0.064365	0.096643	0.12069	0.140163	0.158671
Group 2	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064301	0.096925	0.120608	0.140357	0.156271
	alternative 2	0.064669	0.096909	0.12127	0.140601	0.157682
	alternative 3	0.064602	0.096323	0.120285	0.140601	0.156228
Group 3	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064304	0.097125	0.119762	0.140274	0.156968
	alternative 2	0.06403	0.096705	0.120802	0.140387	0.157693
	alternative 3	0.064528	0.096639	0.120692	0.140597	0.157651
Group 4	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064307	0.096955	0.12033	0.14019	0.156868
	alternative 2	0.064356	0.09688	0.120832	0.140262	0.157751
	alternative 3	0.064542	0.096429	0.120483	0.14044	0.156674
Group 5	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064254	0.096676	0.120342	0.14018	0.156869
	alternative 2	0.06408	0.096744	0.120701	0.140605	0.157766
	alternative 3	0.064595	0.09655	0.120673	0.140136	0.157148
Group 6	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064387	0.096706	0.120832	0.14062	0.157779
	alternative 2	0.064301	0.096925	0.120608	0.140357	0.156271
	alternative 3	0.064304	0.097125	0.119762	0.140274	0.156968
	alternative 4	0.064307	0.096955	0.12033	0.14019	0.156868
	alternative 5	0.064254	0.096676	0.120342	0.14018	0.156869
Group 7	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064384	0.09664	0.120718	0.140338	0.157551
	alternative 2	0.064669	0.096909	0.12127	0.140601	0.157682
	alternative 3	0.06403	0.096705	0.120802	0.140387	0.157693
	alternative 4	0.064356	0.09688	0.120832	0.140262	0.157751
	alternative 5	0.06408	0.096744	0.120701	0.140605	0.157766
Group 8	benchmark	0.064398	0.096714	0.120815	0.1406	0.157726
	alternative 1	0.064365	0.096643	0.12069	0.140163	0.158671
	alternative 2	0.064602	0.096323	0.120285	0.140601	0.156228
	alternative 3	0.064528	0.096639	0.120692	0.140597	0.157651
	alternative 4	0.064542	0.096429	0.120483	0.14044	0.156674
	alternative 5	0.064595	0.09655	0.120673	0.140136	0.157148

* Notes: See notes to Table 7.