A Simulation Based Specification Test for Diffusion Processes*

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Abstract

This paper makes two contributions. First, we outline a simple simulation based framework for constructing conditional distributions for multi-factor and multi-dimensional diffusion processes, for the case where the functional form of the conditional density is unknown. The distributions can be used, for example, to form conditional confidence intervals for time period $t+\tau$, say, given information up to period t. Second, we use the simulation based approach to construct a test for the correct specification of a diffusion process. The suggested test is in the spirit of the conditional Kolmogorov test of Andrews (1997). However, in the present context the null conditional distribution is unknown and is replaced by its simulated counterpart. The limiting distribution of the test statistic is not nuisance parameter free. In light of this, asymptotically valid critical values are obtained via appropriate use of the block bootstrap. The suggested test has power against a larger class of alternatives than tests that are constructed using marginal distributions/densities, such as those in Aït-Sahalia (1996) and Corradi and Swanson (2005). The findings of a small Monte Carlo experiment underscore the good finite sample properties of the proposed test, and an empirical illustration underscores the ease with which the proposed simulation and testing methodology can be applied.

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1 Introduction

In this paper, we outline a simple simulation based framework for constructing conditional distributions for multi-factor and multi-dimensional diffusion processes, in the case where the functional form of the conditional density is unknown. The distributions can be used, for example, to form conditional confidence intervals for time period $t+\tau$, say, given information up to period t. We then use the simulation based approach to construct tests for the correct specification of a given diffusion model. What distinguishes our approaches from that followed by numerous other authors, is that we evaluate conditional distributions and confidence intervals, rather than focussing on marginal and/or joint distributions.

Precedents to our paper include Corradi and Swanson (CS: 2005), who construct various tests, including a Kolmogorov type test, based on comparison of the empirical cumulative distribution function and the cumulative distribution function (CDF) implied by the specification of the drift and the variance, under the null model (see also the related nonparametric test introduced by Aït-Sahalia (AS: 1996)). Both the AS and CS procedures determine whether the drift and variance components of a particular single factor continuous time model are correctly specified, although the CS test is based on a comparison of CDFs, while Aït-Sahalia's is based on the comparison of densities. Thus, his approach requires the use of a nonparametric density estimator (and hence the choice of the bandwidth parameter), and is characterized by a nonparametric rate, while the CS test has a parametric rate. In the case of multi-factor and multi-dimensional models characterized by stochastic volatility, say, the functional form of the invariant density of the return(s) is no longer guaranteed to be given in closed form, upon joint specification of the drift and variance terms, so that the Kolmogorov type test of CS is no longer applicable, for example. To get around this problem, CS compare the empirical *joint* distribution of the actual data and the empirical *joint* distribution of the (model) simulated data (see also Corradi and Swanson (2004a)). This paper can be seen as an extension of CS (2005), as *conditional* Kolmogorov type tests that are easy to implement, and that are based upon evaluation of *conditional* distributions and/or confidence intervals are discussed, and are implemented via use of our proposed simulation methodology. Furthermore, given that we develop conditional Kolmogorov (CK) type tests, this paper can also be viewed as providing an extension of the CK test developed in Andrews (1997).

In the literature on the evaluation of continuous time financial models, the main goal is to construct specification tests based on the transition density associated with a model. In fact, tests based on comparisons of marginal distributions have no power against *iid* alternatives with the same marginal, for example. The main difficulty that arises in this context stems from the fact that knowledge of the drift and variance terms of a diffusion process does not in turn imply knowledge of the transition density, in general. Indeed, if the functional form for the transition density were known, we could test the hypothesis of correct specification of a diffusion via the probability integral transform approach of Diebold, Gunther and Tay (1998), the cross spectrum approach of Hong (2001), Hong and Li (2004), Hong, Li and Zhao (2004), the test of Bai (2003) based on the joint use of a Kolmogorov test and a martingalization method, or via the normality transformation approach of Bontemps and Meddahi (2005) and Duan (2003). For the case in which the transition density is unknown, a test can be constructed by comparing the kernel (conditional) density estimator of the actual and simulated data, as in Altissimo and Mele (2002, 2005), and Thompson (2004). Alternatively, a closed form approximation of the transition density (and hence the likelihood function) is proposed by Aït-

Sahalia (1999, 2002), who also provides conditions under which the argmax of the approximated likelihood is asymptotically equivalent to the "true" maximum likelihood estimator. Whether the same approach can be used to obtain an approximation of the conditional distribution to be used in the implementation of a conditional Kolmogorov test, along the lines of Andrews (1997), is left to future research. As mentioned above, our approach in this paper is to focus on a simulation based methodology for diffusion processes specification testing.

In our framework, parameters are estimated via the simulated generalized method of moments (SGMM) approach of Duffie and Singleton (1993), assuming exact identification. Of note in this context is that $\sqrt{T}-$consistency does not apply to overidentified (S)GMM estimators of misspecified models, as shown by Hall and Inoue (2003). Additionally, and as is common with the type of test proposed here, limiting distributions are functionals of zero mean Gaussian processes with covariance kernels that reflect both the contribution of parameter estimation error (PEE) as well as the time series nature of the data. Thus, the limiting distributions are not nuisance parameters free and critical values cannot be tabulated.[1] In light of this fact, we provide valid asymptotic critical values via an extension of the empirical process version of the block bootstrap which properly captures the contribution of PEE, for the case where parameters are estimated via SGMM. Of note in this context is that when the simulation error is negligible (i.e. the simulated sample grows faster than the historical sample), and given exact identification, the results developed by Goncalves and White (2004) for QMLE estimators extend to SGMM estimators.

The potential usefulness of our proposed bootstrap based tests is examined via a simple Monte Carlo experiments using discrete samples of 400 and 800 observations, and based on the use of bootstrap critical values constructed using as few as 100 replications. Rejection rates under the null are quite close to nominal values, and rejection rates under the alternative are generally high.[2] Additionally, an empirical illustration is provided which underscores the ease with which one can apply our simulation and testing methodology.

The rest of the paper is organized as follows. In Section 2, we outline a setup which is appropriate for discussing our simulation and testing methodology, in the context of one-dimensional diffusion processes. In Section 3, we discuss a simple approach to simulation of conditional distributions and confidence intervals. Section 4 outlines our specification test, and Section 5 extends all results to the case of multi-factor and multi-dimensional models. Section 6 contains the results of our small Monte Carlo experiment, and discusses the findings from an empirical illustration based on the Eurodollar deposit rate. Concluding remarks are gathered in Section 7, and all proofs are collected in an appendix.

2 Setup

In this section we outline the set-up for the case of one-dimensional diffusion processes. All results, carry through to the more complicated cases of multi-dimensional and multi-factor stochastic volatility models, as outlined in Section 5.

[1] Note, though, that in the special case of testing for normality, Bontemps and Meddahi (2005) provide a GMM type test based on moment conditions that is robust to parameter estimation error.

[2] It is worth noting that the joint problem of simulating paths and simulating bootstrap replications make this Monte Carlo study rather computationally intensive, and we are not aware of other simulation studies which analyze the performance of bootstrap tests for diffusion processes other than Corradi and Swanson (2005).

Let $X(t)$, $t \geq 0$, be a one-dimensional diffusion process solution to the following stochastic differential equation:

$$dX(t) = b_0(X(t), \theta_0)dt + \sigma_0(X(t), \theta_0)dW(t), \tag{1}$$

where $\theta_0 \in \Theta$, $\Theta \subset \Re^p$, and Θ is a compact set. In general, assume that the following model is specified and estimated:

$$dX(t) = b(X(t), \theta^{\dagger})dt + \sigma(X(t), \theta^{\dagger})dW(t), \tag{2}$$

Thus, correct specification of the diffusion process corresponds to $b(\cdot,\cdot) = b_0(\cdot,\cdot)$ and $\sigma(\cdot,\cdot) = \sigma_0(\cdot,\cdot)$. Note that the drift and variance terms ($b(\cdot)$ and $\sigma^2(\cdot)$, respectively) uniquely determine the stationary density, say $f(x, \theta^{\dagger})$, associated with the invariant probability measure of the above diffusion process (see e.g. Karlin and Taylor (1981), pp. 241). In particular:

$$f(x, \theta^{\dagger}) = \frac{c(\theta^{\dagger})}{\sigma^2(x, \theta^{\dagger})} \exp\left(\int^x \frac{2b(v, \theta^{\dagger})}{\sigma^2(v, \theta^{\dagger})} dv\right), \tag{3}$$

where $c(\theta^{\dagger})$ is a constant ensuring that the density integrates to one. However, knowledge of the drift and variance terms does not ensure knowledge of a closed functional form for the transition density.

Now, suppose that we observe a discrete sample (skeleton) of T observations, say $(X_1, X_2, \ldots, X_T)'$, from the underlying diffusion. Furthermore, suppose that we use these sample data in conjunction with a simulated "path" of data in order to construct an estimator of θ, say $\widehat{\theta}_{T,N,h}$, where N denotes the simulation path length and h is the discretization interval.[3] Finally, note that we use the notation $X(t)$ to denote the continuous time process, and the notation X_t to denote the skeleton (i.e. the discrete sample). In light of this, let $X^{\theta}_{t,h}$ denote pathwise simulated data, constructed using some $\theta \in \Theta$.

Assume that $\widehat{\theta}_{T,N,h}$ is the simulated generalized method of moments (SGMM) estimator, which is defined as:

$$\begin{aligned} \widehat{\theta}_{T,N,h} &= \arg\min_{\theta\in\Theta} \left(\frac{1}{T}\sum_{t=1}^{T} g(X_t) - \frac{1}{N}\sum_{t=1}^{N} g(X^{\theta}_{t,h})\right)' W_T \left(\frac{1}{T}\sum_{t=1}^{T} g(X_t) - \frac{1}{N}\sum_{t=1}^{N} g(X^{\theta}_{t,h})\right) \\ &= \arg\min_{\theta\in\Theta} G_{T,N,h}(\theta)' W_T G_{T,N,h}(\theta), \end{aligned} \tag{4}$$

where g denotes a vector of p moment conditions, $\Theta \subset \Re^p$ (so that we have as many moment conditions as parameters), and $X^{\theta}_{t,h} = X^{\theta}_{[Kth/N]}$, with $N = Kh$. Finally, W_T is the inverse of a heteroskedasticity and autocorrelation (HAC) robust covariance matrix estimator. That is:

$$W_T^{-1} = \frac{1}{T}\sum_{\nu=-l_T}^{l_T} w_{\nu} \sum_{t=\nu+1+l_T}^{T-l_T} \left(g(X_t) - \frac{1}{T}\sum_{t=1}^{T} g(X_t)\right)\left(g(X_{t-\nu}) - \frac{1}{T}\sum_{t=1}^{T} g(X_t)\right)', \tag{5}$$

where $w_v = 1 - v/(l_T + 1)$. In order to construct simulated estimators, we require simulated sample paths. If we use a Milstein scheme (see e.g. Pardoux and Talay (1985)), then:

$$\begin{aligned} X^{\theta}_{kh} - X^{\theta}_{(k-1)h} &= b(X^{\theta}_{(k-1)h}, \theta)h + \sigma(X^{\theta}_{(k-1)h}, \theta)\epsilon_{kh} - \frac{1}{2}\sigma(X^{\theta}_{(k-1)h}, \theta)'\sigma(X^{\theta}_{(k-1)h}, \theta)h \\ &\quad + \frac{1}{2}\sigma(X^{\theta}_{(k-1)h}, \theta)'\sigma(X^{\theta}_{(k-1)h}, \theta)\epsilon^2_{kh}, \end{aligned} \tag{6}$$

[3] In the case in which the moment conditions can be written in closed form, we have that $\widehat{\theta}_{T,N,h} = \widehat{\theta}_T$, given that N is the sample length of the simulated path used in estimation of θ, and h is the discretization parameter used in the application of Euler and/or Milstein approximation schemes, for example (see below for further details).

where $\epsilon_{kh} \overset{iid}{\sim} N(0,h)$, $k=1,\ldots,N$, $Nh=S$, and $\sigma(\cdot,\cdot)'$ is the derivative of $\sigma(\cdot,\cdot)$, with respect to its first argument. Here, the pseudo true value, $\theta^{\dagger}$, is defined to be:

$$\theta^{\dagger} = \arg\min_{\theta\in\Theta} G_{\infty}(\theta)' W_0 G_{\infty}(\theta),$$

where $G_{\infty}(\theta)'W_0G_{\infty}(\theta)$ $=\text{plim}_{T\to\infty,h\to 0} G_{T,N,h}(\theta)'W_T G_{T,N,h}(\theta)$, and $\theta^{\dagger} = \theta_0$, if the model is correctly specified. Note that the reason why we limit our attention to the exactly identified case is that this ensures that $G_{\infty}(\theta^{\dagger})=0$, even when the model used to simulate the diffusion is misspecified, in the sense of differing from the underlying DGP (see e.g. Hall and Inoue (2003) for discussion of the asymptotic behavior of misspecified overidentified GMM estimators).[4]

Before discussing conditional simulation, we first state the assumptions under which $\sqrt{T}\left(\widehat{\theta}_{T,N,h}-\theta^{\dagger}\right)$ is asymptotically normal. In particular, we shall rely on the following assumption in the sequel.

Assumption A:

(i) $X(t), t\in\Re^{+}$, is a strictly stationary, geometric ergodic diffusion.

(ii) $b(\cdot,\theta^{\dagger})$ and $\sigma(\cdot,\theta^{\dagger})$, as defined in (2), are twice continuously differentiable. Also, $b(\cdot,\cdot), b(\cdot,\cdot)', \sigma(\cdot,\cdot)$, and $\sigma(\cdot,\cdot)'$ are Lipschitz, with Lipschitz constant independent of θ, where $b(\cdot,\cdot)'$ and $\sigma(\cdot,\cdot)'$ denote derivatives with respect to the first argument of the function.

(iii) For any fixed h and $\theta\in\Theta$, X_{kh}^{θ} is geometrically ergodic and strictly stationary.

(iv) $W_T \overset{a.s.}{\to} W_0 = \sum_0^{-1}$, where, $\sum_0 = \sum_{j=-\infty}^{\infty} E\left((g(X_1)-E(g(X_1)))(g(X_{1+j})-E(g(X_{1+j})))'\right)$.

(v) For $\theta\in\Theta$ and for all h, $||g(X_{t,h}^{\theta})||_{2+\delta} < C < \infty$, $g(X_{t,h}^{\theta})$ is Lipschitz, uniformly on Θ, $\theta \to E(g(X_{t,h}^{\theta}))$ is continuous, and $g(X_t), g(X_{t,h}^{\theta}), \nabla_{\theta}X_{t,h}^{\theta}$ are $2r-$dominated (the last two also on Θ) for $r>3/2$.[5]

(vi) Unique identifiability: $G_{\infty}(\theta^{\dagger})'W_0G_{\infty}(\theta^{\dagger}) < G_{\infty}(\theta)'W_0G_{\infty}(\theta)$, $\forall\theta\neq\theta^{\dagger}$.

(vii) $\widehat{\theta}_{T,N,h}$ and $\theta^{\dagger}$ are in the interior of Θ, $g(X_t^{\theta})$ is twice continuously differentiable in the interior of Θ; and $D^{\dagger} = E(\partial g(X_t^{\theta})/\partial\theta|_{\theta=\theta^{\dagger}})$ exists and is of full rank, p.

All results stated in the sequel rely on the following Lemma.

Lemma 1: Let Assumption A hold. Assume that $T,N\to\infty$. Then, if $h\to 0$, $T/N\to 0$, and $h^2T\to 0$, the following result holds:

$$\sqrt{T}\left(\widehat{\theta}_{T,N,h}-\theta^{\dagger}\right) \overset{d}{\to} N(0, \left(D^{\dagger\prime}W_0D^{\dagger}\right)^{-1}),$$

where W_0 and $D^{\dagger}$ are defined in Assumption A(iv) and A(vii). The above normality result for the SGMM estimator can be extended in a straightforward manner to the case of the EMM estimator of Gallant and Tauchen (1996) (see also Dridi (1999)).

3 Simulated Conditional Distributions

In this section, we outline how to construct in-sample $\tau-$step ahead simulated conditional distributions, when the functional form of the conditional distribution is not known in closed form. Conditional confidence

[4]Note that first order conditions imply that

$$\nabla_{\theta}G_{\infty}(\theta^{\dagger})'W^{\dagger}G_{\infty}(\theta^{\dagger})=0.$$

However, in the case for which the number of parameters and the number of moment conditions is the same, $\nabla_{\theta}G_{\infty}(\theta^{\dagger})'W^{\dagger}$ is invertible, and so the first order conditions also imply that $G_{\infty}(\theta^{\dagger})=0$.

[5]More precisely, we require $\sup_{\theta\in\Theta}|g(X_{t,h}^{\theta})|\leq D_t$, where $\sup_t E(D_t)^{2r}<\infty$.

interval construction follows immediately, and is discussed in Section 6. Let S be the number of simulated paths. Then, for $s = 1, ...S$, $t \geq 1$, and $k = 1, ..., \tau/h$, define:

$$\begin{aligned}
& X^{\widehat{\theta}_{T,N,h}}_{s,t+kh} - X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h} \\
= \; & b(X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h}, \widehat{\theta}_{T,N,h})h + \sigma(X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h}, \widehat{\theta}_{T,N,h})\epsilon_{s,t+kh} \\
& -\frac{1}{2}\sigma(X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h}, \widehat{\theta}_{T,N,h})'\sigma(X^{\widehat{\theta}_{T,N,h}}_{s+(k-1)h}, \widehat{\theta}_{T,N,h})h \\
& +\frac{1}{2}\sigma(X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h}, \widehat{\theta}_{T,N,h})'\sigma(X^{\widehat{\theta}_{T,N,h}}_{s,t+(k-1)h}, \widehat{\theta}_{T,N,h})\epsilon^2_{s,t+kh},
\end{aligned} \tag{7}$$

where $\epsilon_{s,t+kh} \overset{iid}{\sim} N(0, h)$. That is, we simulate S paths of length τ, with τ finite, all having the same starting value, X_t. This allows for the preservation of the starting value effect on the finite length simulation paths. It should be stressed, however, that the simulated diffusion is ergodic. Thus, the effect of the starting value approaches zero at an exponential rate, as $\tau \to \infty$.

Now, for any given starting value, X_t, the simulated randomness is assumed to be independent across simulations, so that $E(\epsilon_{s,t+kh}\epsilon_{j,t+kh}) = 0$, for all $s \neq j$. On the other hand, it is important to retain the same simulated randomness across different starting values (i.e. use the same set of random errors for each starting value), so that $E(\epsilon_{s,t+kh}\epsilon_{s,l+kh}) = h$, for any t, l.[6]

Given the above setup, note that we can equivalently define $X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}$ to be the final value for a given simulation path, s, when started at X_t. As an estimate for the distribution, at time $t+\tau$, conditional on X_t, define:

$$\widehat{F}(u|X_t, \widehat{\theta}_{T,N,h}) = \frac{1}{S}\sum_{s=1}^{S} 1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau} \leq u\right\}$$

As stated in Proposition 2 below, if the model is correctly specified, then $\frac{1}{S}\sum_{s=1}^{S} 1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau} \leq u\right\}$ provides a consistent estimate of the "true" conditional distribution.[7] Hereafter, let $F(u|X_t, \theta)$ denote the conditional distribution of $X_{t+\tau}$, given X_t.

The statement of Proposition 2 requires the following assumption.

Assumption B: (i) $F(u|X_t, \theta)$ is twice continuously differentiable in the interior of Θ. Also, $\nabla_\theta F(u|X_t, \theta)$ and $\nabla^2_\theta F(u|X_t, \theta)$ are jointly continuous in the interior of Θ, almost surely, and $2r$-dominated on Θ_i, $r > 2$. (ii) $X^{\theta}_{s,t+\tau}$ is continuously differentiable in the interior of Θ, for $s = 1, ..., S$; and $\nabla_\theta X^{\theta}_{s,t+\tau}$ is $2r-$dominated in Θ, uniformly in s for $r > 2$.

Proposition 2: Let Assumptions A and B hold. Assume that $T, N, S \to \infty$. Then, if $h \to 0$, $T/N \to 0$, and $h^2 T \to 0$, the following result holds for any X_t, $t \geq 1$, uniformly in u :

$$\widehat{F}(u|X_t, \widehat{\theta}_{T,N,h}) - F(u|X_t, \theta^\dagger) \overset{pr}{\to} 0, \tag{8}$$

In addition, if the model is correctly specified (i.e. if $b(\cdot, \cdot) = b_0(\cdot, \cdot)$ and $\sigma(\cdot, \cdot) = \sigma_0(\cdot, \cdot)$) then:

$$\widehat{F}(u|X_t, \widehat{\theta}_{T,N,h}) - F_0(u|X_t, \theta_0) \overset{pr}{\to} 0. \tag{9}$$

[6]The test discussed in the next section requires the construction of multiple paths of length τ, for a number of different starting values, and hence we have attempted to underscore the importance of keeping the random errors used in the construction of each set of paths for each starting value the same.

[7]A related approach for approximating distribution functions has been suggested by Thompson (2004), in the context of tests for the correct specification of term structure models.

In practice, once we have an estimate for the τ period ahead conditional distribution, we still do not know whether our estimate is based on the correct model or not. This suggests using the test for correct specification outlined in the next section, in order to assess the "relevance" of the distribution estimator.

4 Specification Testing

In the first sub-section, we outline the test. The second sub-section discusses bootstrapping procedures for obtaining asymptotically valid critical values.

4.1 The Test

The test outlined in this section can be viewed as a simulation-based extension of Andrews (1997) and Corradi and Swanson (2003,2004b), which has been adapted to the use of continuous time models. Of additional note is that in the Andrews and Corradi-Swanson papers, the functional form of conditional distribution under the null hypothesis is assumed to be known, while in the current context we replace the conditional distribution (or conditional mean) with a simulation based estimator. Furthermore, in the Andrews paper, a conditional-Kolmogorov test is developed under the assumption of *iid* data, so that the bootstrap used in that paper is not applicable in our context.

Recall that in the previous section we discussed the construction of simulation paths for a given starting value. In order to carry out a specification test, however, we now require the construction of a sequence of $T-\tau$ conditional distributions that are $\tau-$steps ahead, say. In this way, S paths of length τ are thus available for each starting value from $t = 1, ..., T-\tau$.

The hypotheses of interest are:

$H_0 : F(u|X_t, \theta^\dagger) = F_0(u|X_t, \theta_0)$, for all u, a.s.

$H_A : \Pr\left(F(u|X_t, \theta^\dagger) - F_0(u|X_t, \theta_0) \neq 0\right) > 0$, for some u, with non-zero Lebesgue measure,

Thus, the null hypothesis coincides with that of correct specification of the conditional distribution, and is implied by the correct specification of the drift and variance terms used in simulating the paths. The alternative is simply the negation of the null. In practice, we observe neither $F(u|X_t, \theta^\dagger)$ nor $F_0(u|X_t, \theta_0)$. However, we can simply construct a statistic replacing $F(u|X_t, \theta^\dagger)$ with $\frac{1}{S}\sum_{s=1}^{S} 1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}} \leq u\right\}$ and $F_0(u|X_t, \theta_0)$ with $\frac{1}{T}\sum_{t=1}^{T} 1\{X_{t+\tau} \leq u\}$. This is a sensible choice because of Proposition 2 above, and because $E\left(1\{X_{t+\tau} \leq u\}|X_t\right) = F_0(u|X_t, \theta_0)$. The natural test statistic in this context is thus:

$$V_T = \sup_{u \times v \in U \times V} |V_T(u, v)| \tag{10}$$

where

$$V_T(u, v) = \frac{1}{\sqrt{T-\tau}} \sum_{t=1}^{T-\tau} \left(\frac{1}{S} \sum_{s=1}^{S} 1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}} \leq u\right\} - 1\{X_{t+\tau} \leq u\} \right) 1\left\{X_t \leq v\right\},$$

where U and V are compact sets on the real line. The asymptotic behavior of V_T is described by the following theorem.

Theorem 3: Let Assumptions A and B hold. Assume that $T, N, S \rightarrow \infty$. Then, if $h \rightarrow 0$, $T/N \rightarrow 0$, $T/S \rightarrow 0$, $T^2/S \rightarrow \infty$, $Nh \rightarrow 0$, and $h^2 T \rightarrow 0$, the following result holds under H_0 :

$$V_T \xrightarrow{d} \sup_{u \times v \in U \times V} |Z(u,v)|,$$

where $Z(u,v)$ is a Gaussian process with covariance kernel $K(u,u',v,v')$ given by:

$$\begin{aligned} K(u,u',v,v') &= \sum_{j=-\infty}^{\infty} E\left(\left(F_0(u|X_1,\theta_0) - 1\{X_{1+\tau} \leq u\}\right) 1\left\{X_1 \leq v\right\}\right. \\ &\quad \left.\times \left(F_0(u'|X_{1+j},\theta_0) - 1\{X_{1+\tau+j} \leq u'\}\right) 1\left\{X_{1+j} \leq v'\right\}\right) \end{aligned}$$

$$\begin{aligned} &+ \sum_{j=-\infty}^{\infty} E\left(f_0(u|X_1,\theta_0) E_s\left(\nabla_{\theta_0} X^{\theta_0}_{s,1+\tau}\right)' 1\left\{X_1 \leq v\right\} D^{0\prime} W_0 D^0 \right. \\ &\left. \times\ f_0(u'|X_{1+j},\theta_0) E_s\left(\nabla_{\theta_0} X^{\theta_0}_{s,1+\tau+j}\right) 1\left\{X_{1+j} \leq v'\right\}\right) \end{aligned}$$

$$\begin{aligned} &-2 \sum_{j=-\infty}^{\infty} E\left(f_0(u|X_1,\theta_0) E_s\left(\nabla_{\theta_0} X^{\theta_0}_{s,1+\tau}\right)' \left(D^{0\prime} W_0 D^0\right)^{-1} D^{0\prime} W^0 \right. \\ &\left. \left(g(X_{1+j}) - E(g(X_1))\right)\left(F_0(u|X_{1+j},\theta_0) - 1\{X_{1+\tau+j} \leq u'\}\right) 1\left\{X_{1+j} \leq v'\right\}\right). \end{aligned}$$

Furthermore, under H_A, there exists some $\varepsilon > 0$ such that:

$$\lim_{P\rightarrow\infty} \Pr\left(\frac{1}{\sqrt{T}} V_T > \varepsilon\right) = 1.$$

In Theorem 3, $E(\cdot)$ denotes expectation with respect to the probability measure governing the sample, and $E_s(\cdot)$ denotes expectation with respect to the probability measure governing the simulated randomness, conditional on the sample. Also, recall that under the null hypothesis, $\theta^\dagger = \theta_0$. Notice that the limiting distribution is a zero mean Gaussian process, with a covariance kernel that reflects the contribution of parameter estimation error. Thus, the limiting distribution is not nuisance parameter free and hence critical values cannot be tabulated. In the next section we thus outline a bootstrap procedure for calculating asymptotically valid critical values for V_T.

4.2 Bootstrap Critical Values

Given that the limiting distributions of V_T is not nuisance parameter free, our approach is to construct bootstrap critical values for the test. In order to show the first order validity of the bootstrap, we shall obtain the limiting distribution of the bootstrapped statistic and show that it coincides with the limiting distribution of the actual statistic under H_0. Then, a test with correct asymptotic size and unit asymptotic power can be obtained by comparing the value of the original statistic with bootstrapped critical values.

If the data consisted of *iid* observations, we could consider proceeding along the lines of Andrews (1997), by drawing bootstrap samples of our simulated observations from the distribution under H_0, conditional on the observed values for the covariates. However, given that we have dependence, and that we observe neither

$F(u|X_t,\theta^{\dagger})$ nor $F_0(u|X_t,\theta_0)$, asymptotically valid bootstrap critical values for the test should be constructed as follows:

Step 1*:* At each replication, draw b blocks (with replacement) of length l, where $bl = T$. Thus, each block is equal to $X_{i+1},...,X_{i+l}$, for some $i = 0,...,T-l+1$, with probability $1/(T-l+1)$. More formally, let I_k, $k = 1,...,b$ be *iid* discrete uniform random variables on $[0,1,...,T-l+1]$. Then, the resampled series, X_t^* is such that $X_1^*,X_2^*,...,X_l^*,X_{l+1}^*,...,X_T^* = X_{I_1+1},X_{I_1+2},...,X_{I_1+l},X_{I_2},...,X_{I_b+l}$, and so a resampled series consists of b blocks that are discrete *iid* uniform random variables, conditional on the sample. Use these data to construct $\widehat{\theta}^*_{T,N,h}$. As $N/T \to \infty$, then GMM and simulated GMM are asymptotically equivalent. More precisely,

$$\widehat{\theta}^*_{T,N,h} = \arg\min_{\theta\in\Theta}\left(\frac{1}{T}\sum_{t=1}^{T}g(X_t^*) - \frac{1}{N}\sum_{t=1}^{N}g(X_{t,h}^{\theta})\right)' W_T\left(\frac{1}{T}\sum_{t=1}^{T}g(X_t^*) - \frac{1}{N}\sum_{t=1}^{N}g(X_{t,h}^{\theta})\right),$$

where W_T and $g(\cdot)$ are defined in (4).

Step 2*:* Using the same set of random errors used in the construction of the actual statistic, construct $X_{s,t+\tau,*}^{\widehat{\theta}^*_{T,N,h}}$, $s = 1,...,S$, and $t = 1,...,T-\tau$. Note that we do not resample the simulated series (as $S/T \to \infty$, simulation error is asymptotically negligible). Instead, simply simulate the series using bootstrap estimators and using bootstrapped values as starting values.

Step 3*:* Construct the following bootstrap statistic, which is the bootstrap counterpart of V_T:

$$V_T^* = \sup_{u\times v\in U\times V}\left|V_T^*(u,v)\right|, \tag{11}$$

where

$$\begin{aligned} V_T^*(u,v) &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau,*}^{\widehat{\theta}^*_{T,N,h}} \le u\right\} - 1\{X_{t+\tau}^* \le u\}\right)1\left\{X_t^* \le v\right\} \\ &\quad -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}} \le u\right\} - 1\{X_{t+\tau} \le u\}\right)1\left\{X_t \le v\right\} \end{aligned}$$

Step 4*:* Repeat *Steps 1-3* B times, and generate the empirical distribution of the B bootstrap statistics.

Theorem 4: Let Assumptions A and B hold. Assume that $T,N,S \to \infty$. Then, if $h \to 0$, $T/N \to 0$, $T/S \to 0$, $T^2/S \to \infty$, $Nh \to 0$, $h^2T \to 0$, $l \to \infty$, and $l^2/T \to 0$, the following result holds:

$$P\left[\omega : \sup_{x\in\Re}\left|P^*\left(V_T^*(\omega) \le x\right) - P\left((V_T - E(V_T)) \le x\right)\right| > \varepsilon\right] \to 0,$$

where P^* denotes the probability law of the resampled series, conditional on the sample.

The above results suggest proceeding in the following manner. For any bootstrap replication, compute the bootstrap statistic, V_T^*. Perform B bootstrap replications (B large) and compute the quantiles of the empirical distribution of the B bootstrap statistics. Reject H_0, if V_T is greater than the $(1-\alpha)th$-percentile. Otherwise, do not reject. Now, for all samples except a set with probability measure approaching zero, V_T has the same limiting distribution as the corresponding bootstrapped statistic, ensuring asymptotic size equal to α. Under the alternative, V_T diverges to (plus) infinity, while the corresponding bootstrap statistic has a well defined limiting distribution, ensuring unit asymptotic power.

5 Extension: Stochastic Volatility Models

In providing an extension to the multi-dimensional case of the above test for one-dimensional models, a first difficulty lies in the choice of the discrete approximation scheme. In particular, the diffusion process, $X(t)$, can be expressed as a function of the driving Brownian motion, $W(t)$, in the one-dimensional case. However, in the multi-dimensional case, where we have $X(t) \in R^p$, note that $X(t)$ cannot in general be expressed as a function of the p driving Brownian motions, but is instead a function of $(W_j(t), \int_0^t W_j(s)dW_i(s))$, $i, j = 1, ...p$ (see e.g. Pardoux and Talay (1985), pp. 30-32). For this reason, simple approximations like the Euler and Milstein schemes, which do not involve approximation of stochastic integrals, may not be adequate. One case in which the Milstein scheme does straigthforwardly generalize to the multidimensional case is when the diffusion matrix is commutative. For example, let $\Sigma(X) = \begin{pmatrix} \sigma_1(X) & . & . & . & \sigma_P(X) \end{pmatrix}$, where $\sigma_i(X)$ is a $p \times 1$ vector, for $i = 1, ..., p$. If for all $i, j = 1, ..., p$,

$$\begin{pmatrix} \frac{\partial\sigma_j(X)}{\partial X_1} & . & . & . & \frac{\partial\sigma_j(X)}{\partial X_p} \end{pmatrix} \sigma_i(X) = \begin{pmatrix} \frac{\partial\sigma_i(X)}{\partial X_1} & . & . & . & \frac{\partial\sigma_i(X)}{\partial X_p} \end{pmatrix} \sigma_j(X),$$

then $\Sigma(X)$ is commutative. It is immediate to see that almost all of the most frequently used stochastic volatility (SV) models violate the commutativity property. Of course, commutativity holds in the case of no leverage, but this rules out many important cases. For example, the non-leverage assumption is suitable for exchange rates, but not for stock returns. The intuitive reason for this is that both the variance of the observable asset and the variance of the (unobservable) variance process depend only on the volatility process. In this situation, more "sophisticated" approximation schemes are necessary.

We now focus our attention on two-factor stochastic volatility models. Extension to general multi-factor and multi-dimensional models follows directly. Consider the following model:

$$\begin{pmatrix} dX(t) \\ dV(t) \end{pmatrix} = \begin{pmatrix} b_1(X(t), \theta^\dagger) \\ b_2(V(t), \theta^\dagger) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(V(t), \theta^\dagger) \\ 0 \end{pmatrix} dW_1(t) + \begin{pmatrix} \sigma_{21}(V(t), \theta^\dagger) \\ \sigma_{22}(V(t), \theta^\dagger) \end{pmatrix} dW_2(t), \tag{12}$$

where $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions, and where $b_1(\cdot, \cdot)$, $b_2(\cdot, \cdot)$, $\sigma_{11}(\cdot, \cdot)$, $\sigma_{21}(\cdot, \cdot)$, $\sigma_{22}(\cdot, \cdot)$, and $\theta^\dagger$ are replaced with $b_{1,0}(\cdot, \cdot)$, $b_{2,0}(\cdot, \cdot)$, $\sigma_{11,0}(\cdot, \cdot)$, $\sigma_{21,0}(\cdot, \cdot)$, $\sigma_{22,0}(\cdot, \cdot)$, and θ_0, respectively, under correct specification. It is immediate to see that the diffusion in (12) violates the commutativity property. Also, note that most of the popular SV models, such as the square-root model of Heston (1993), the GARCH diffusion model of Nelson (1990), the lognormal model of Hull and White (1987), and in general the eigenfunction stochastic volatility models of Meddahi (2001) can be written as (12) above.

Now, let

$$b(\cdot, \cdot) = \begin{pmatrix} b_1(\cdot, \cdot) \\ b_2(\cdot, \cdot) \end{pmatrix}, \ \sigma = \begin{pmatrix} \sigma_{11}(\cdot, \cdot) & 0 \\ \sigma_{21}(\cdot, \cdot) & \sigma_{22}(\cdot, \cdot) \end{pmatrix}, \tag{13}$$

and define the following generalized Milstein scheme (see eq. (3.3), p.346 in Kloeden and Platen (1999)),

$$\begin{aligned} X^\theta_{(k+1)h} \quad = \quad & X^\theta_{kh} + \widetilde{b}_1(X^\theta_{kh}, \theta)h + \sigma_{11}(V^\theta_{kh}, \theta)\epsilon_{1,(k+1)h} + \sigma_{12}(V^\theta_{kh}, \theta)\epsilon_{2,(k+1)h} \\ & + \frac{1}{2}\sigma_{22}(V^\theta_{kh}, \theta)\frac{\partial\sigma_{12}(V^\theta_{kh}, \theta)}{\partial V}\epsilon^2_{2,(k+1)h} \\ & + \sigma_{22}(V^\theta_{kh}, \theta)\frac{\partial\sigma_{11}(V^\theta_{kh}, \theta)}{\partial V}\int_{kh}^{(k+1)h}\left(\int_{kh}^{s} dW_{2,\tau}\right) dW_{1,s} \end{aligned} \tag{14}$$

$$
\begin{aligned}
V^{\theta}_{(k+1)h} &= V^{\theta}_{kh} + \widetilde{b}_2(V^{\theta}_{kh},\theta)h + \sigma_{22}(V^{\theta}_{kh},\theta)\epsilon_{2,(k+1)h} \\
&\quad + \frac{1}{2}\sigma_{22}(V^{\theta}_{kh},\theta)\frac{\partial\sigma_{22}(V^{\theta}_{kh},\theta)}{\partial V}\epsilon^2_{2,(k+1)h}
\end{aligned} \tag{15}
$$

where $h^{-1/2}\epsilon_{i,kh} \sim N(0,1)$, $i=1,2$, $E(\epsilon_{1,kh}\epsilon_{2,mh})=0$ for all k and m, and

$$
\widetilde{b}(V,\theta) = \begin{pmatrix} \widetilde{b}_1(V,\theta) \\ \widetilde{b}_2(V,\theta) \end{pmatrix} = \begin{pmatrix} b_1(V,\theta) - \frac{1}{2}\sigma_{22}(V,\theta)\frac{\partial\sigma_{12}(V,\theta)}{\partial V} \\ b_2(V,\theta) - \frac{1}{2}\sigma_{22}(V,\theta)\frac{\partial\sigma_{22}(V,\theta)}{\partial V} \end{pmatrix}.
$$

The last terms on the RHS of (14) involves stochastic integrals and cannot be explicitly computed. However, they can be approximated, up to an error of order $o(h)$ by (see eq. (3.7), p. 347 in Kloeden and Platen (1999)):

$$
\begin{aligned}
\int_{kh}^{(k+1)h}\left(\int_{kh}^{s} dW_{2,\tau}\right)dW_{1,s} &\approx h\left(\frac{1}{2}\xi_1\xi_2 + \sqrt{\rho_p}\left(\mu_{2,p}\xi_1 - \mu_{1,p}\xi_2\right)\right) \\
&\quad + \frac{h}{2\pi}\sum_{r=1}^{p}\frac{1}{r}\left(\varsigma_{2,r}\left(\sqrt{2}\xi_1 + \eta_{1,r}\right) - \varsigma_{1,r}\left(\sqrt{2}\xi_2 + \eta_{2,r}\right)\right),
\end{aligned}
$$

where for $j=1,2$, $\xi_j, \mu_{j,p}, \varsigma_{j,r}, \eta_{j,r}$ are $iidN(0,1)$ with $\xi_j = h^{-1/2}\epsilon_{j,(k+1)h}$, and $\rho_p = \frac{1}{12} - \frac{1}{2\pi^2}\sum_{r=1}^{p}\frac{1}{r^2}$, and p is such that as $h \to 0$, $p \to \infty$.

In order to construct conditional distributions, given the observable state variables, we need to perform the following steps.

Step 1: Simulate a path of length N using the schemes in (14)(15) and estimate θ by SGMM, as in (4). Also, retrieve $V^{\widehat{\theta}_{T,N,h}}_{kh}$, for $k=1/h,...,N/h$, and hence obtain $V^{\widehat{\theta}_{T,N,h}}_{j,h}$, $j=1,...,N$ (i.e. we sample the simulated volatility at the same frequency as the data).

Step 2: Using the schemes in (14)(15), simulate $S \times N$ path of length τ, setting the initial value for the observable state variable to be X_t. As we do not observe data on volatility, use the values simulated in the previous step as the initial value for the volatility process (i.e. as initial values for unobservable state variable, use $V^{\widehat{\theta}_{T,N,h}}_{j,h}$, $j=1,...,N$). Also, keep the simulated randomness (i.e. $\epsilon_{1,kh}, \epsilon_{2,kh}$, $\int_{kh}^{(k+1)h}\left(\int_{kh}^{s} dW_{1,\tau}\right)dW_{2,s}$) constant across j (i.e. constant across the different starting values for the unobservable and observable state variable). Define $X^{\widehat{\theta}_{T,N,h}}_{j,s,t+\tau}$ to be the simulated $\tau-$step ahead value for the return series at replication s, and using initial values X_t and $V^{\widehat{\theta}_{T,N,h}}_{j,h}$.

Step 3: As an estimator of $F(u|X_t,\theta^{\dagger})$, construct $\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{j,s,t+\tau} \leq u\right\}$. Note that, by averaging over the initial value of the volatility process, we have integrated out its effect.[8]

Step 4: Construct the statistic of interest:

$$
SV_T = \sup_{u\times v \in U\times V}\left|SV_T(u,v)\right|,
$$

where

$$
SV_T(u,v) = \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{j,s,t+\tau} \leq u\right\} - 1\{X_{t+\tau} \leq u\}\right)1\{X_t \leq v\}, \tag{16}
$$

[8] In other words, $\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{j,s,t+\tau} \leq \overline{u}\right\}$ is an estimate of $F_i(u|X_t, V^{\theta^{\dagger}}_{j,h}, \theta^{\dagger})$.

Assumption A$'$: This assumption is the same as Assumption A, except that Assumption A(ii) is replaced by (ii$'$): $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ (as defined in (12) and (13)), and $\sigma_{lj}(V,\theta)\frac{\partial\sigma_{k\iota}(V,\theta)}{\partial V}$ are twice continuously differentiable, Lipschitz, with Lipschitz constant independent of θ, and grow at most at a linear rate, uniformly in Θ, for $l,j,k,\iota=1,2$.

All of the results outlined in Sections 3 and 4 generalize to the current setting. In particular, the following results hold.

Proposition 5: Let Assumptions A$'$ and B hold. Assume that $T,N,S\to\infty$. Then, if $h\to 0$, $T/N\to 0$, $T/S\to 0$, and $h^2T\to 0$, the following result holds for any X_t, $t\geq 1$:

$$\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-F(u|X_t,\theta^{\dagger})\xrightarrow{pr}0,\ \text{uniformly in } u$$

In addition, if the model is correctly specified, i.e. if $b(\cdot,\cdot)=b_0(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)=\sigma_0(\cdot,\cdot)$, then

$$\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq \overline{u}\right\}-F_0(u|X_t,\theta_0)\xrightarrow{pr}0,\ \text{uniformly in } u$$

Now, given the same hypotheses as stated in Section 4.1, it follows that the proceeding result holds.

Theorem 6: Let Assumptions A$'$ and B hold. Assume that $T,N,S\to\infty$. Then, if $h\to 0$, $T/N\to 0$, $T/S\to 0$, $T^2/S\to\infty$, $Nh\to 0$, and $h^2T\to 0$, the following result holds under H_0 :

$$SV_T\xrightarrow{d}\sup_{u\times v\in U\times V}|SZ(u,v)|,$$

where $SZ(v)$ is a Gaussian process with covariance kernel $SK(v,v')$ given by:

$$\begin{aligned}
SK(u,u',v,v') &= \sum_{k=-\infty}^{\infty}E\left(\left(F_0(u|X_1,\theta_0)-1\{X_{1+\tau}\leq u\}\right)1\left\{X_1\leq v\right\}\right.\\
&\quad\left.\left(F_0(u'|X_{1+k},\theta_0)-1\{X_{1+\tau+k}\leq u'\}\right)1\left\{X_{1+k}\leq v'\right\}\right)\\
&+\sum_{k=-\infty}^{\infty}E\left(f_0(u|X_1,\theta_0)E_{j,s}\left(\nabla_{\theta_0}X_{j,s,1+\tau}^{\theta_0}\right)'1\left\{X_1\leq v\right\}D^{0\prime}W_0D^0\right.\\
&\quad\left.\times f_0(u'|X_{1+k},\theta_0)E_{j,s}\left(\nabla_{\theta_0}X_{j,s,1+\tau+k}^{\theta_0}\right)1\left\{X_{1+k}\leq v'\right\}\right)\\
&-2\sum_{k=-\infty}^{\infty}E\left(f_0(u|X_1,\theta_0)E_{j,s}\left(\nabla_{\theta_0}X_{j,s,1+\tau}^{\theta_0}\right)'\left(D^{0\prime}W_0D^0\right)^{-1}D^{0\prime}W^0\right.\\
&\quad\left.\left(g(X_{1+k})-E(g(X_1))\right)\left(F_0(u|X_{1+k},\theta_0)-1\{X_{1+\tau+k}\leq u'\}\right)1\left\{X_{1+k}\leq v'\right\}\right).
\end{aligned}$$

Furthermore, under H_A, there exists some $\varepsilon>0$ such that:

$$\lim_{P\to\infty}\Pr\left(\frac{1}{\sqrt{T}}SV_T>\varepsilon\right)=1.$$

In Theorem 6, $E(\cdot)$ denotes expectation with respect to the probability measure governing the sample, and $E_{j,s}(\cdot)$ denotes expectation with respect to the joint probability measure governing the simulated randomness and the volatility process, $V_{h,j}^{\theta_0}$, conditional on the sample. Also, recall that under the null, $\theta_i^{\dagger}=\theta_0$.

Finally, in order to obtain asymptotically valid critical values, resample as discussed above for the one-factor model, and note that there is no need to resample $V_{h,j}^{\theta}$. In particular, form bootstrap statistics as follows:

$$SV_T^* = \sup_{u\times v\in U\times V} |SV_T^*(u,v)|, \tag{17}$$

where

$$\begin{aligned} SV_T^*(u,v) &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau,*}^{\widehat{\theta}^*_{i,T,N,h}}\leq u\right\}-1\{X_{t+\tau}^*\leq u\}\right)1\left\{X_t^*\leq v\right\} \\ &-\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{i,T,N,h}}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_t\leq v\right\}, \end{aligned}$$

where $X_{j,s,t+\tau,*}^{\widehat{\theta}^*_{i,T,N,h}}$ is the simulated value at simulation s, constructed using $\widehat{\theta}^*_{i,T,N,h}$ and using as initial value X_t^* and $V_{j,h}^{\widehat{\theta}^*_{i,T,N,h}}$.

The following result then holds.

Theorem 7: Let Assumptions A$'$ and B hold. Assume that $T,N,S\to\infty$. Then, if $h\to 0$, $T/N\to 0$, $T/S\to 0$, $T^2/S\to\infty$, $Nh\to 0$, $h^2T\to 0$, $l\to\infty$, and $l^2/T\to 0$, the following result holds:

$$P\left[\omega:\sup_{x\in\Re}\left|P^*\left(SV_T^*(\omega)\leq x\right)-P\left(\left(SV_T-E(SV_T)\right)\leq x\right)\right|>\varepsilon\right]\to 0.$$

As in the single-factor case, the above results suggest proceeding in the following manner. For any bootstrap replication, compute the bootstrap statistic, SV_T^*. Perform B bootstrap replications (B large) and compute the quantiles of the empirical distribution of the B bootstrap statistics. Reject H_0, if SV_T is greater than the $(1-\alpha)th$-percentile. Otherwise, do not reject. Now under the null, for all samples except a set with probability measure approaching zero, SV_T and SV_T^* have the same limiting distribution, ensuring asymptotic size equal to α. Under the alternative, SV_T diverges to (plus) infinity, while the corresponding bootstrap statistic has a well defined limiting distribution, ensuring unit asymptotic power.

6 Experimental and Empirical Results

In this section we briefly illustrate the above testing methodology via an illustration in which we are interested in specification testing from the perspective of conditional confidence intervals. In particular, we consider the special case in which $V_T=\sup_{v\in V}|V_T(v)|$,where

$$V_T(v)=\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}\leq\overline{u}\}\right)1\left\{X_t\leq v\right\}.$$

Additionally, define the following bootstrap statistic: $V_T^*=\sup_{v\in V}|V_T^*(v)|$, where

$$\begin{aligned} V_T^*(v) &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{s,t+\tau,*}^{\widehat{\theta}^*_{T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}^*\leq\overline{u}\}\right)1\left\{X_t^*\leq v\right\} \\ &-\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}\leq\overline{u}\}\right)1\left\{X_t\leq v\right\}. \end{aligned}$$

It is immediate to see that the above statistic is a simplified version of the distributional test discussed above, so that all of the theoretical results outlined in Section 4 and 5 hold. Although we shall focus in this simple illustration on single factor models, it also follows that the confidence interval version of the test can be applied to stochastic volatility models, in the sense that one need simply define: $SV_T = \sup_{v\in V}|SV_p(v)|$, where

$$SV_T(v) = \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}\leq\overline{u}\}\right)1\left\{X_t\leq v\right\};$$

and $SV_T^* = \sup_{v\in V}|SV_T^*(v)|$, where

$$\begin{aligned} SV_T^*(v) \quad = \quad & \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{j,s,t+\tau,*}^{\widehat{\theta}^*_{i,T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}^*\leq\overline{u}\}\right)1\left\{X_t^*\leq v\right\} \\ & -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{\underline{u}\leq X_{j,s,t+\tau}^{\widehat{\theta}_{i,T,N,h}}\leq\overline{u}\right\}-1\{\underline{u}\leq X_{t+\tau}\leq\overline{u}\}\right)1\left\{X_t\leq v\right\}. \end{aligned}$$

The particular single factor models that we use for this illustration are a common version of the square root process discussed in Cox, Ingersoll, and Ross (CIR: 1985), and a simple model in which logged data are generated according to an Ornstein-Uhlenbeck (OU) process. In particular the models are:

CIR: $dX(t) = \phi_1(\alpha_1 - X(t))\,dt + \sigma_1\sqrt{X(t)}dW_1(t)$, where $\phi_1 > 0$, $\sigma_1 > 0$ and $2\phi_1\alpha_1 \geq \sigma_1^2$,

OU: $d\ln X(t) = \phi_2(\alpha_2 - \ln X(t))\,dt + \sigma_2 dW_1(t)$, where $\phi_2 > 0$, and $\sigma_2 > 0$.

6.1 Monte Carlo Experiment

Data were generated using the above models. The parameters used in data generation were selected via examination of models estimated using the interest rate data discussed in the next subsection. Parameterizations considered included: *CIR:* $(\phi_1, \alpha_1, \sigma_1) = (0.30, 0.05, 0.10)$, $(0.30, 0.07, 0.10)$, $(0.50, 0.05, 0.10)$, $(0.50, 0.07, 0.10)$; and *OU*: $(\phi_2, \alpha_2, \sigma_2) = (0.10, -2.00, 0.30)$, $(0.10, -3.00, 0.30)$, $(0.20, -2.00, 0.30)$, $(0.20, -3.00, 0.30)$.

Data were generated using the Milstein scheme discussed above with h=$1/T$, for $T = \{400, 800\}$. All subsequent model estimation was done using both exactly identified GMM and SGMM, where the moments used in SGMM are $E[X_t]$, $Var[X_t]$ and $E[(X_t - E[X_t])(X_{t-1} - E[X_t])]$. Results were qualitatively the same using either approach, and hence only results for the GMM estimation case are discussed below. Additionally, and for the sake of brevity, we report results only for the following two parameterizations: *CIR:* $(\phi_1, \alpha_1, \sigma_1) = (0.30, 0.05, 0.10)$ and *OU*: $(\phi_2, \alpha_2, \sigma_2) = (0.10, -2.00, 0.30)$. Results for the other parameterizations are qualitatively similar, and are available upon request.

In our experiment, all the empirical bootstrap distributions were constructed using 100 bootstrap replications, and critical values were set equal to the 90^{th} percentile of the bootstrap distribution. For the bootstrap, block lengths of 5, 10, 20 and 50 were tried. Additionally, we set $S = \{10T, 20T, 30T\}$. Tests were carried out using $\tau-step$ ahead confidence intervals, for $\tau = \{1, 2, 4, 12\}$, and we set $(\underline{u}, \overline{u})$; $\overline{X} \pm 0.25\sigma_X$, $\overline{X} \pm 0.5\sigma_X$, $\overline{X} \pm \sigma_X$, and $\overline{X} \pm 2\sigma_X$, where $\overline{X}$ and σ_X are the mean and variance of an initial sample of data of length 1000 observations, generated using the model under investigation. Finally, the set V was set equal to $[X_{\min}, X_{\max}]$, and a grid of 100 equally spaced values for v across this range was used, where $X_{\min}$ and

$X_{\max}$ are again fixed using the sample of 1000 observations mentioned previously. All results are based on 500 Monte Carlo iterations.

Results are gathered in Table 1-4. Tables 1 and 2 report results for the empirical level of the test while Tables 3 and 4 report results for power experiments. In the level experiments, estimated models are the same as the models used for data generation, while in the power experiments, we estimated the CIR model (when data were generated according to the OU model), and the OU model (when data were generated according to the CIR model). Each table contains two panels, one for T =400 and one for T =800. Recalling that the nominal rejection rate is 10%, note that the test does not have particularly good empirical power (level) for sample size of 400, as empirical rejection rates range from around 13% to 21%, and power can be as low as 0.27. However, empirical power and level improve significantly as the sample size is increased to 800. In particular, for $T = 800$, nominal rejection rates range from approximately 10% to 13%. Interestingly, these results seem to be quite robust to the choice of bootstrap block length. Additionally, empirical power increases rather substantially, as empirical rejection frequencies range from around 0.70 to 0.92. Finally, results appear to be quite robust to the values of τ and S, as well as to the width of the confidence interval.

6.2 Empirical Illustration

In this subsection, the CIR and OU models discussed above are fit to the one-month Eurodollar deposit rate for the periods January 6, 1971 - April 8, 2005 (1,789 weekly observations) and January 3, 1990 - April 8, 2005 (798 observations).[9] Thereafter, specification tests for the two models are constructed.

Table 5 reports the estimation results along with the summary statistics. These include mean, standard deviation, skewness and kurtosis, as well as the parameter estimates and standard errors. As expected, the longer sample exhibits more volatility than the recent sample (see Table 1 and Figure 1). Additionally, parameter estimates for the CIR and OU models are plausible and in line with those reported in the literature.

Tables 6a-7b report specification test results for the CIR and the OU models, respectively. Intervals examined, block lengths considered, simulation samples size used, bootstrap replications, and values of τ considered are the same as discussed the previous subsection. For example, we again consider $\tau = \{1, 2, 4, 12\}$, corresponding to one week, two weeks, one month, and one quarter. The exception is that the interval endpoints are chosen using the actual interest rate data. In particular, $\overline{X}$, σ_X, $X_{\min}$, and $X_{\max}$ are constructed using 1789 observation dataset or the 798 observation dataset, depending upon which sample is being used in test statistic construction. Note, for example, that using this approach, $\overline{X}\pm 0.25\sigma_X$, $\overline{X}\pm 0.5\sigma_X$, $\overline{X}\pm\sigma_X$, and $\overline{X}\pm 2\sigma_X$ correspond to intervals with 17.4%, 46.3%, 72.4% and 94.8% coverage, respectively, for the period January 6, 1971 - April 8, 2005. In the tables, test statistic values (denoted by V_T) and 5%, 10%, and 20% level bootstrap critical values are given. Single, double, and triple starred entries denote rejection at the 20%, 10%, and 5% levels, respectively. Notice for both the specifications and both sample sizes, for smaller values of τ (1 and 2) we clearly reject the null of correct specification for all the confidence intervals $(\underline{u}, \overline{u})$. However for the wider confidence intervals, we sometimes fail to reject the null of correct specification

[9] Other interest rate datasets examined in the literature include the monthly federal funds rate (Ait-Sahalia (1999)), the weekly 3-month T-bill rate (Andersen, Benzoni and Lund (2004)), and the weekly US dollar swap rate (Dai and Singleton (2000)), to name but a few.

for 12-step (in some cases even 4-step) ahead confidence interval, when the block length is small. This is as expected, and indeed for bigger block lengths, the null hypothesis is again rejected. Figures 2 and 3 contain plots of $\tau - step$ ahead simulated densities and also contain plots of densities constructed using the historical data. Notice that the densities associated with $12 - step$ ahead simulations are quite far away from the historical distribution for both the specifications, as expected. Notice also that the simple models considered in this illustration poorly simulate tail behavior, and tend to yield densities that are too concentrated are mean.

7 Concluding Remarks

In this paper we outline a simple simulation based framework for constructing conditional distributions for multi-factor and multi-dimensional diffusion processes, for the case where the functional form of the conditional density is unknown. In a Monte Carlo experiment and an empirical illustration, we show how the distributions can be used, for example, to form conditional confidence intervals for time period $t+\tau$, say, given information up to period t. In addition, we use the simulation based framework to construct a test for the correct specification of a diffusion process, and establish the asymptotic validity of the block bootstrap for use in the construction of critical values.

This work represents a starting point in our investigation of the usefulness of simulation based methods for examining continuous time financial models. From a theoretical perspective, it is of interest to establish whether or not the simulation methodology discussed herein can be extended to contexts in which recursively constructed predictions are evaluated, and are used in model selection tests. From an empirical perspective, it should be of interest to compare the finite sample performance of the specification test proposed here with alternative tests available in the literature.

8 Appendix

Proof of Lemma 1: Immediate from the proof of Lemma A1 in Corradi and Swanson (2005).

Proof of Proposition 2: Given Assumption A, by Lemma 1, $\left(\widehat{\theta}_{T,N,h}-\theta^{\dagger}\right)=o_P(1)$. Given B(ii) and noting that the indicator function is a measurable function,

$$\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}=\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta^{\dagger}}\leq u\right\}+o_P(1).$$

Finally, for any given X_t, $X_{s,t+\tau}^{\theta^{\dagger}}$ is identically distributed and independent of $X_{j,t+\tau}^{\theta^{\dagger}}$, for all $j\neq s$. The statement in (8) then follows from the uniform law of large number for *iid* random variables. In fact, conditional on X_t, $X_{s,t+\tau}^{\theta^{\dagger}}$ is *iid*, as randomness is independent across simulations. Finally, the statement in (9) follows immediately, as in this case $F(u|X_t,\theta^{\dagger})=F_0(u|X_t,\theta_0)$.

Proof of Theorem 3: We begin by showing convergence in distribution under the null, pointwise in u and v. Recalling that $\theta^{\dagger}=\theta_0$ under H_0,

$$\begin{aligned}V_T(u,v) &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_t\leq v\right\}\\ &+\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}\right)1\left\{X_t\leq v\right\} \quad (18)\end{aligned}$$

Recalling that $F(u|X_t,\theta^{\dagger})=F_0(u|X_t,\theta_0)$, under H_0, the first term on the RHS can be written as:

$$\begin{aligned}&\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)\right)1\left\{X_t\leq v\right\}\\ &-\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(1\left\{X_{t+\tau}\leq u\right\}-F_0(u|X_t,\theta_0)\right)1\left\{X_t\leq v\right\} \quad (19)\end{aligned}$$

We now show that the first term on the RHS of (19) is $o_P(1)$, uniformly in u. Given that $1\left\{X_t\leq v\right\}$ is either 0 or 1, it suffices to show that $\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)\right)=o_P(1)$, uniformly in u. By Chebyshev's inequality, it thus suffices to show that

$$Var\left(\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)\right)\right)\rightarrow 0,\ \text{as } P,S\rightarrow\infty. \quad (20)$$

First note that,

$$\begin{aligned}&E\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)\right)\\ =\ &\frac{1}{S}\sum_{s=1}^{S}E_X\left(E_S\left(1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)\right)|X_t\right)=0,\end{aligned}$$

where E_X denotes expectation with respect to the probability law governing the sample, and E_S denotes expectation with respect to the probability law governing the simulated randomness, conditional on the sample. Hereafter, let $\phi_s(X_t)=1\left\{X_{s,t+\tau}^{\theta_0}\leq u\right\}-F_0(u|X_t,\theta_0)$, and note that,

$$E\left(\phi_s(X_t)\phi_j(X_l)\right)=0,\ \text{for all } s\neq j,$$

given that:

$$
\begin{aligned}
& E\left(1\left\{X^{\theta_0}_{s,t+\tau}\leq u\right\}1\left\{X^{\theta_0}_{j,l+\tau}\leq u\right\}\right)=E_X\left(E_S\left(1\left\{X^{\theta_0}_{s,t+\tau}\leq u\right\}1\left\{X^{\theta_0}_{j,l+\tau}\leq u\right\}\right)\right)\\
= \; & E_X\left(F_0(u|X_t,\theta_0)F_0(u|X_l,\theta_0)\right)
\end{aligned}
$$

Thus, (20) can be written as:

$$
\frac{1}{TS^2}\sum_{t=1}^{T-\tau}\sum_{l=1}^{T-\tau}\sum_{s=1}^{S}E\left(\phi_s(X_t)\phi_s(X_l)\right)=O\left(\frac{T}{S}\right)=o(1),
$$

for $T/S\rightarrow 0$. This establishes that the first term on the RHS of (19) is $o_P(1)$, pointwise in u. Uniformity in u follows from the stochastic equicontinuity of $1\left\{X^{\theta_0}_{s,t+\tau}\leq u\right\}-F_0(u|X_t,\theta_0)$.

Now, the second term on the RHS of (18), can be written as:

$$
\begin{aligned}
& \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\theta_0}_{s,t+\tau}\leq u-\left(X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}-X^{\theta_0}_{s,t+\tau}\right)\right\}-F_0\left(u-\left(X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}-X^{\theta_0}_{s,t+\tau}\right)|X_t,\theta_0\right)\right)\\
& \times 1\left\{X_t\leq v\right\}\\
& +\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\frac{1}{S}\sum_{s=1}^{S}\left(F_0\left(u-\left(X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}-X^{\theta_0}_{s,t+\tau}\right)|X_t,\theta_0\right)-F_0\left(u|X_t,\theta_0\right)\right)1\left\{X_t\leq v\right\}. \qquad (21)
\end{aligned}
$$

The first term of (21) is $o_P(1)$, uniformly in u and v, by an argument analogous to that used in the proof of Theorem 3 in Corradi and Swanson (2005). Given Assumption B(i)-(ii), it thus follows that:

$$
\begin{aligned}
& V_T(u,v)\\
= \; & -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(1\left\{X_{t+\tau}\leq u\right\}-F_0(u|X_t,\theta_0)\right)1\left\{X_t\leq v\right\}\\
& +\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\frac{1}{S}\sum_{s=1}^{S}\left(F_0\left(u-\left(X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}-X^{\theta_0}_{s,t+\tau}\right)|X_t,\theta_0\right)-F_0\left(u|X_t,\theta_0\right)\right)1\left\{X_t\leq v\right\}+o_P(1)\\
= \; & -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(1\left\{X_{t+\tau}\leq u\right\}-F_0(u|X_t,\theta_0)\right)1\left\{X_t\leq v\right\}\\
& +\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}\left(f_0\left(u-\left(X^{\overline{\theta}_{T,N,h}}_{s,t+\tau}-X^{\theta_0}_{s,t+\tau}\right)|X_t\right)\nabla_{\theta_i}X^{\theta}_{s,t+\tau}|_{\theta=\overline{\theta}_{T,N,h}}\right)'\right.\\
& \left.1\left\{X_t\leq v\right\}\sqrt{T-\tau}\left(\widehat{\theta}_{T,N,h}-\theta_0\right)\right)+o_P(1), \qquad (22)
\end{aligned}
$$

where the $o_P(1)$ term holds uniformly in u and v. Finally, recalling Lemma 1 and Assumption A1(v), the second term on the RHS of the last equality in (22) can be appropriately restated, so that:

$$
\begin{aligned}
V_T(u,v) \; = \; & -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(1\left\{X_{t+\tau}\leq u\right\}-F_0(u|X_t,\theta_0)\right)1\left\{X_t\leq v\right\}\\
& +\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}f_0\left(\overline{u}|X_t\right)E_S\left(\nabla_{\theta_i}X^{\theta_0}_{s,1+\tau+j}\right)'1\left\{X_t\leq v\right\}\sqrt{T-\tau}\left(\widehat{\theta}_{T,N,h}-\theta_0\right)\\
& +o_P(1),\text{ uniformly in } v,
\end{aligned}
$$

where $E_S\left(\nabla_{\theta_i} X^{\theta_0}_{s,1+\tau+j}\right)$ is a measurable function of X_t.
The covariance expression in the statement the follows by noting that:

$$\sqrt{T}\left(\widehat{\theta}_{T,N,h}-\theta_0\right)=\left(-\nabla_\theta G_{T,N,h}(\widehat{\theta}_{T,N,h})'W_T\nabla_\theta G_{T,N,h}(\overline{\theta}_{T,N,h})\right)^{-1}\nabla_\theta G_{T,N,h}(\widehat{\theta}_{T,N,h})'W_T\sqrt{T}G_{T,N,h}(\theta_0)$$

Proof of Theorem 4: Consider the bootstrap statistic:

$$V_T^* = \sup_{u\times v\in U\times V}\left|V_T^*(u,v)\right|,$$

where

$$\begin{aligned}
V_T^*(u,v) &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau,*}\leq u\right\}-1\{X^*_{t+\tau}\leq u\}\right)1\left\{X^*_t\leq v\right\}\\
&\quad -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_t\leq v\right\}\\
&\quad +\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}^*_{T,N,h}}_{s,t+\tau,*}\leq u\right\}-\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau,*}\leq u\right\}\right)1\left\{X^*_t\leq v\right\} \qquad (23)
\end{aligned}$$

The term in the first two lines in (23) has the same limiting distribution as:

$$\begin{aligned}
&\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\theta^\dagger}_{s,t+\tau}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)\right.\\
&\left.-E\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\theta^\dagger}_{s,t+\tau}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)\right)1\left\{X_t\leq v\right\},
\end{aligned}$$

where $\theta^\dagger=\theta_0$ if the null is true, conditional on the sample and for all samples except a subset of probability measure approaching zero. In fact,

$$\begin{aligned}
&\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}E^*\left(\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau,*}\leq u\right\}-1\{X^*_{t+\tau}\leq u\}\right)1\left\{X^*_t\leq v\right\}\right.\\
&\left.-\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X^{\widehat{\theta}_{T,N,h}}_{s,t+\tau}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_t\leq v\right\}\right)\\
=\;& O\left(l/T\right),\ \Pr-P.
\end{aligned}$$

Furthermore, and given Lemma 1, and using the same arguments as those used in Theorem 4 of Corradi and

Swanson (2003):

$$
\begin{aligned}
& Var^*\left(\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau,*}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-1\{X_{t+\tau}^{*}\leq u\}\right)1\left\{X_{t}^{*}\leq v\right\}\right.\right.\\
& \left.\left.-\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_{t}\leq v\right\}\right)\right)\\
= \;& \frac{1}{T-\tau}\sum_{t=l}^{T-\tau-l}\sum_{k=1}^{l}\left(\left(\left(\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_{t}\leq v\right\}-\mu(u,v)\right)\right.\right.\\
& \times\left.\left.\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau+k}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-1\{X_{t+\tau+k}\leq u\}\right)1\left\{X_{t+k}\leq v\right\}-\mu(u,v)\right)+O\left(l/T^{1/2}\right),\ \Pr-P\\
= \;& Var\left(\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta^{\dagger}}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_{t}\leq v\right\}\right)+O(l/T^{1/2}),\ \Pr-P,
\end{aligned}
$$

where $\mu(u,v)=E\left(\left(\frac{1}{S}\sum_{s=1}^{S}1\left\{X_{s,t+\tau}^{\theta^{\dagger}}\leq u\right\}-1\{X_{t+\tau}\leq u\}\right)1\left\{X_{t}\leq v\right\}\right)$. Thus, it remains to show that the term in the last line of (23) properly captures the contribution of parameter estimation error. Now, conditional on the resampling variability:

$$
\frac{1}{S}\sum_{s=1}^{S}\left(1\left\{X_{s,t+\tau,*}^{\widehat{\theta}^{*}_{R,N,h}}\leq\overline{u}\right\}-F\left(u|X_{t}^{*},\widehat{\theta}^{*}_{T,N,h}\right)\right)=O_{S}\left(S^{-1/2}\right),\ \Pr-P^{*}.
$$

Furthermore, conditional on sample and resampling variability:

$$
\frac{1}{S}\sum_{s=1}^{S}\left(1\left\{X_{s,t+\tau,*}^{\widehat{\theta}_{R,N,h}}\leq u\right\}-F\left(u|X_{t}^{*},\widehat{\theta}_{T,N,h}\right)\right)=O_{S}\left(S^{-1/2}\right),\ \Pr-P^{*},\ \Pr-P,
$$

where the subscript S denotes convergence in terms of the probability law governing the simulated randomness, which is independent of sampling and resampling (bootstrap) variability. Note that $F=F_0$ when the null is true. Thus, as $S/T\to\infty$, the last term on the RHS of (23) can be written as:

$$
\begin{aligned}
& -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T}\left(F\left(u|X_{t}^{*},\widehat{\theta}^{*}_{T,N,h}\right)-F\left(u|X_{t}^{*},\widehat{\theta}_{T,N,h}\right)\right)+o_{S}(1)\ \Pr-P,\ \Pr-P^{*}\\
= \;& -\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T}f(u|X_{t}^{*},\overline{\theta}^{*}_{T,N,h})\nabla_{\theta}'X_{t}^{*,\overline{\theta}^{*}_{T,N,h}}\left(\widehat{\theta}^{*}_{T,N,h}-\widehat{\theta}_{T,N,h}\right)+o_{S}(1)\ \Pr-P,\ \Pr-P^{*}\\
= \;& -E\left(f_{0}(u|X_{1},\theta_{0})E_{s}\left(\nabla_{\theta_{0}}X_{s,1+\tau}^{\theta_{0}}\right)\right)'\sqrt{T-\tau}\left(\widehat{\theta}^{*}_{T,N,h}-\widehat{\theta}_{T,N,h}\right)+o(1),\ \Pr-P,\ \Pr-P^{*},
\end{aligned}
$$

where $\overline{\theta}^{*}_{T,N,h}\in\left(\widehat{\theta}^{*}_{T,N,h},\widehat{\theta}_{R,N,h}\right)$. The statement in the theorem then follows from Lemma 1.

Proof of Proposition 5: Note that:

$$
\begin{aligned}
& \frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-F(u|X_{t},\theta^{\dagger})\\
= \;& \frac{1}{NS}\sum_{j=1}^{N}\left(\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\leq u\right\}-F(u|X_{t},V_{j,h}^{\widehat{\theta}_{T,N,h}},\theta^{\dagger})\right)\\
& +\frac{1}{N}\sum_{j=1}^{N}\left(F(u|X_{t},V_{j,h}^{\widehat{\theta}_{T,N,h}},\theta^{\dagger})-F(u|X_{t},\theta^{\dagger})\right)=I+II
\end{aligned}
$$

Now,

$$
\begin{aligned}
II &= \frac{1}{N}\sum_{j=1}^{N}\left(F(u|X_t,V_{j,h}^{\widehat{\theta}_{T,N,h}},\theta^\dagger)-F(u|X_t,V_{j,h}^{\theta^\dagger},\theta^\dagger)\right)+\frac{1}{N}\sum_{j=1}^{N}\left(F(u|X_t,V_{j,h}^{\theta^\dagger},\theta^\dagger)-F(u|X_t,\theta^\dagger)\right)\\
&= o_P(1)+o_P(1),\ \text{uniformly in } u,
\end{aligned}
$$

where the first $o_P(1)$ term follows from the fact that, given Assumption A', $\left(\widehat{\theta}_{T,N,h}-\theta^\dagger\right)=o_P(1)$; and the second $o_P(1)$ term follows from the uniform law of large numbers, given that $V_{j,h}^{\theta^\dagger}$ is a geometrically ergodic process. Also,

$$
\begin{aligned}
I &= \frac{1}{NS}\sum_{j=1}^{N}\left(\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\le u\right\}-F(u|X_t,V_{j,h}^{\widehat{\theta}_{T,N,h}},\theta^\dagger)\right)\\
&= \frac{1}{NS}\sum_{j=1}^{N}\left(\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\theta_i^\dagger}\le u\right\}-F_i(u|X_t,V_{j,h}^{\theta_i^\dagger},\theta^\dagger)\right)+o_P(1),
\end{aligned}
$$

where the $o_P(1)$ term holds uniformly in u. Furthermore, and the first term on the RHS above is $o_P(1)$, uniformly in u, by the uniform law of large numbers, given that:

$$
E\left(1\left\{X_{j,s,t+\tau}^{\theta^\dagger}\le u\right\}|X_t,V_{j,h}^{\theta^\dagger}\right)=F(u|X_t,V_{j,h}^{\theta^\dagger},\theta^\dagger).
$$

Proof of Theorem 6: Note that:

$$
\begin{aligned}
&\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\le u\right\}-1\{X_{t+\tau}\le u\}\right)1\{X_t\le v\}\\
=&\ \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\theta_0}\le u\right\}-1\{X_{t+\tau}\le u\}\right)1\{X_t\le v\}\\
&+\frac{1}{\sqrt{T-\tau}}\sum_{t=R}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}\left(1\left\{X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}}\le u\right\}-1\left\{X_{j,s,t+\tau}^{\theta_0}\le u\right\}\right)\right)1\{X_t\le v\}\\
=&\ I+II
\end{aligned}
$$

Recalling that as $T\to\infty$, $N/T\to\infty$ and $S/T\to\infty$, by a similar argument to that used in the proof of Theorem 3:

$$
II=\frac{1}{T-\tau}\sum_{t=1}^{T-\tau}f_0\left(\overline{u}|X_t\right)E_{j,s}\left(\nabla_{\theta_i}X_{s,1+\tau+j}^{\theta_0}\right)'1\{X_t\le v\}\sqrt{T-\tau}\left(\widehat{\theta}_{T,N,h}-\theta_0\right)+o_P(1),
$$

where $E_{j,s}(\cdot)$ denotes expectation with respect to the joint probability measure governing the simulated randomness and the volatility process, $V_{h,j}^{\theta_0}$, conditional on the sample. With regard to I, note that:

$$
\begin{aligned}
I &= \frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{NS}\sum_{j=1}^{N}\sum_{s=1}^{S}1\left\{X_{j,s,t+\tau}^{\theta_0}\leq u\right\}-F_0\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)\right)1\left\{X_t\leq v\right\} \\
&+\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{N}\sum_{j=1}^{N}\left(F_0\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F_0\left(u|X_t,\theta_0\right)\right)\right)1\left\{X_t\leq v\right\} \\
&-\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(1\{X_{t+\tau}\leq u\}-F_0\left(u|X_t,\theta_0\right)\right)1\left\{X_t\leq v\right\}. \qquad (24)
\end{aligned}
$$

We need to show that the first and second terms on the RHS of (24) are $o_P(1)$. The statement in the theorem will then follow by the same argument as that used in the proof of Theorem 3. With regard to the second term on the RHS of (24), note that:

$$
\begin{aligned}
& E\left(\left(\frac{1}{N}\sum_{j=1}^{N}\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F_1\left(u|X_t,\theta_0\right)\right)\right)1\left\{X_t\leq v\right\}\right) \\
= \; & E\left(\frac{1}{N}\sum_{j=1}^{N}E_j\left(\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F\left(u|X_t,\theta_0\right)\right)|X_t\right)1\left\{X_t\leq v\right\}\right)=0,
\end{aligned}
$$

where E_j denotes expectation with respect to the probability measure governing $V_{h,j}^{\theta_0}$, conditional on the sample. Also, note that $V_{h,j}^{\theta_0}$ and X_t are independent each other, as the simulated randomness does not depend on X_t. Thus, by noting that $1\left\{X_t\leq v\right\}$ is equal to either 1 or 0, it suffices to show that:

$$
\lim_{P,N\to\infty}Var\left(\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{N}\sum_{j=1}^{N}\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F\left(u|X_t,\theta_0\right)\right)\right)\right)=0.
$$

Now,

$$
\begin{aligned}
& Var\left(\frac{1}{\sqrt{T-\tau}}\sum_{t=1}^{T-\tau}\left(\frac{1}{N}\sum_{j=1}^{N}\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F_1\left(u|X_t,\theta_0\right)\right)\right)\right) \\
= \; & \frac{1}{TN^2}\sum_{t=1}^{T-\tau}\sum_{l=1}^{T-\tau}\sum_{j=1}^{N}\sum_{s=j}^{N}E\left(\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F\left(u|X_t,\theta_0\right)\right)\left(F\left(u|X_l,V_{h,s}^{\theta_0},\theta_0\right)-F\left(u|X_l,\theta_0\right)\right)\right) \\
& +\frac{1}{TN^2}\sum_{t=1}^{T-\tau}\sum_{l=1}^{T-\tau}\sum_{j=1}^{N}\sum_{s\neq j}^{N}E\left(\left(F\left(u|X_t,V_{h,j}^{\theta_0},\theta_0\right)-F_1\left(u|X_t,\theta_0\right)\right)\left(F\left(u|X_l,V_{h,s}^{\theta_0},\theta_0\right)-F\left(u|X_l,\theta_0\right)\right)\right). \qquad (25)
\end{aligned}
$$

The first term on the RHS of (25) is $O\left(\frac{T}{N}\right)=o(1)$, for $T/N\to 0$. Given that $V_{h,j}^{\theta_0}$ is a geometric mixing process, it can be shown via standard mixing inequality arguments that the second term on the RHS of (25) is $O\left(\frac{T}{N}\right)=o(1)$, for $T/N\to 0$.

Now, consider the first term on the RHS of (24). It can be shown that this term is $o_P(1)$ by a similar argument as that used in the proof of Theorem 3. Finally, the statement under the alternative follows by the same argument as that used in the proof of Theorem 3.

Proof of Theorem 7: Note that:

$$
\begin{aligned}
& SV_T^*(u,v) \\
= \; & \frac{1}{\sqrt{T-\tau}} \sum_{t=1}^{T-\tau} \left(\frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau,*}^{\widehat{\theta}_{T,N,h}} \leq u \right\} - 1\{X_{t+\tau}^* \leq u\} \right) 1\left\{X_t^* \leq v\right\} \\
& - \frac{1}{\sqrt{T-\tau}} \sum_{t=1}^{T-\tau} \left(\frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau}^{\widehat{\theta}_{T,N,h}} \leq u \right\} - 1\{X_{t+\tau} \leq u\} \right) 1\left\{X_t \leq v\right\} \\
& + \frac{1}{\sqrt{T-\tau}} \sum_{t=1}^{T-\tau} \left(\frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau,*}^{\widehat{\theta}^*_{T,N,h}} \leq u \right\} - \frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau,*}^{\widehat{\theta}_{T,N,h}} \leq u \right\} \right) \\
& \times 1\left\{X_t^* \leq v\right\}.
\end{aligned} \tag{26}
$$

By the same argument as that used in the proof of Theorem 4, the first two terms on the RHS of (26) have the the same limiting distribution as:

$$
\begin{aligned}
& \frac{1}{\sqrt{T-\tau}} \sum_{t=1}^{T-\tau} \left(\left(\frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau}^{\theta^\dagger} \leq u \right\} - 1\{X_{t+\tau} \leq u\} \right) \right. \\
& \left. - E\left(\frac{1}{NS} \sum_{j=1}^{N} \sum_{s=1}^{S} 1\left\{ X_{j,s,t+\tau}^{\theta^\dagger} \leq u \right\} - 1\{X_{t+\tau} \leq u\} \right) \right) 1\left\{X_t \leq v\right\},
\end{aligned}
$$

conditional on the sample. Furthermore, given Proposition 5, the last term on the RHS of (26) properly mimics the contribution of parameter estimation error, by the same argument as that used in the proof of Theorem 4.

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Table 1: Simulated Density Based Specification Test Rejection Frequencies - Empirical Level$^{(*)}$

Data Generated using the *CIR* Model

τ	$(\underline{u}, \overline{u})$	S,l											
		10T,5	20T,5	30T,5	10T,10	20T,10	30T,10	10T,20	20T,20	30T,20	10T,50	20T,50	30T,50
		Panel A: T = 400											
1	$\overline{X} \pm 0.25\sigma_X$	0.212	0.164	0.164	0.148	0.176	0.218	0.186	0.170	0.164	0.132	0.190	0.228
	$\overline{X} \pm 0.5\sigma_X$	0.192	0.156	0.134	0.144	0.152	0.156	0.128	0.172	0.182	0.198	0.156	0.186
	$\overline{X} \pm \sigma_X$	0.134	0.134	0.128	0.132	0.144	0.132	0.126	0.134	0.138	0.134	0.134	0.136
	$\overline{X} \pm 2\sigma_X$	0.132	0.134	0.128	0.126	0.134	0.128	0.128	0.128	0.136	0.128	0.136	0.126
2	$\overline{X} \pm 0.25\sigma_X$	0.214	0.166	0.144	0.136	0.214	0.136	0.216	0.176	0.176	0.136	0.174	0.142
	$\overline{X} \pm 0.5\sigma_X$	0.192	0.162	0.154	0.186	0.176	0.158	0.126	0.166	0.172	0.184	0.136	0.198
	$\overline{X} \pm \sigma_X$	0.128	0.126	0.126	0.132	0.132	0.132	0.126	0.132	0.128	0.128	0.136	0.132
	$\overline{X} \pm 2\sigma_X$	0.128	0.128	0.128	0.128	0.128	0.142	0.128	0.132	0.134	0.134	0.128	0.136
4	$\overline{X} \pm 0.25\sigma_X$	0.152	0.162	0.168	0.198	0.200	0.202	0.184	0.194	0.218	0.188	0.226	0.156
	$\overline{X} \pm 0.5\sigma_X$	0.134	0.144	0.122	0.158	0.186	0.156	0.170	0.144	0.178	0.172	0.162	0.122
	$\overline{X} \pm \sigma_X$	0.132	0.128	0.134	0.132	0.134	0.126	0.128	0.132	0.134	0.134	0.134	0.126
	$\overline{X} \pm 2\sigma_X$	0.132	0.126	0.134	0.134	0.132	0.134	0.132	0.132	0.132	0.138	0.132	0.136
12	$\overline{X} \pm 0.25\sigma_X$	0.178	0.182	0.150	0.166	0.166	0.142	0.222	0.158	0.206	0.138	0.186	0.192
	$\overline{X} \pm 0.5\sigma_X$	0.208	0.186	0.182	0.136	0.206	0.172	0.154	0.188	0.134	0.172	0.144	0.168
	$\overline{X} \pm \sigma_X$	0.128	0.126	0.134	0.138	0.134	0.136	0.126	0.136	0.132	0.128	0.132	0.128
	$\overline{X} \pm 2\sigma_X$	0.132	0.132	0.128	0.134	0.134	0.132	0.134	0.132	0.128	0.136	0.126	0.134
		Panel B: T = 800											
1	$\overline{X} \pm 0.25\sigma_X$	0.126	0.136	0.124	0.132	0.134	0.124	0.118	0.136	0.124	0.118	0.130	0.120
	$\overline{X} \pm 0.5\sigma_X$	0.136	0.126	0.124	0.126	0.134	0.122	0.118	0.126	0.124	0.134	0.118	0.122
	$\overline{X} \pm \sigma_X$	0.126	0.124	0.128	0.126	0.130	0.138	0.122	0.122	0.136	0.122	0.132	0.122
	$\overline{X} \pm 2\sigma_X$	0.116	0.132	0.134	0.118	0.132	0.136	0.132	0.134	0.134	0.122	0.132	0.122
2	$\overline{X} \pm 0.25\sigma_X$	0.118	0.134	0.122	0.118	0.136	0.128	0.124	0.118	0.128	0.134	0.120	0.128
	$\overline{X} \pm 0.5\sigma_X$	0.116	0.118	0.122	0.132	0.136	0.122	0.124	0.122	0.122	0.130	0.122	0.120
	$\overline{X} \pm \sigma_X$	0.132	0.124	0.126	0.128	0.136	0.118	0.118	0.134	0.128	0.136	0.130	0.126
	$\overline{X} \pm 2\sigma_X$	0.122	0.118	0.136	0.118	0.120	0.134	0.118	0.126	0.130	0.136	0.134	0.126
4	$\overline{X} \pm 0.25\sigma_X$	0.124	0.128	0.124	0.126	0.124	0.122	0.124	0.118	0.120	0.126	0.126	0.126
	$\overline{X} \pm 0.5\sigma_X$	0.132	0.132	0.128	0.120	0.128	0.118	0.124	0.132	0.124	0.134	0.122	0.118
	$\overline{X} \pm \sigma_X$	0.128	0.118	0.134	0.120	0.126	0.126	0.122	0.132	0.128	0.130	0.118	0.128
	$\overline{X} \pm 2\sigma_X$	0.136	0.132	0.122	0.118	0.122	0.134	0.126	0.136	0.126	0.136	0.126	0.120
12	$\overline{X} \pm 0.25\sigma_X$	0.134	0.128	0.128	0.134	0.134	0.128	0.132	0.124	0.134	0.128	0.128	0.126
	$\overline{X} \pm 0.5\sigma_X$	0.128	0.126	0.120	0.130	0.124	0.124	0.134	0.128	0.134	0.132	0.122	0.134
	$\overline{X} \pm \sigma_X$	0.132	0.122	0.124	0.132	0.134	0.124	0.134	0.134	0.128	0.132	0.124	0.122
	$\overline{X} \pm 2\sigma_X$	0.122	0.122	0.122	0.128	0.122	0.122	0.126	0.132	0.130	0.118	0.126	0.122

$^{(*)}$ Notes: Entries in the table are empirical rejection frequencies, for tests constructed using intervals given in the second column of the table, and for τ =1, 2, 4, 12 . (S, l) combinations used in test construction are given in the second row of the table, so that the simulation periods considered are $S = (10T, 20T, 30T)$ and the block lengths considered are $l = (5, 10, 20, 50)$, where T is the sample size, here set to 400 and 800 observations. Empirical bootstrap distributions are constructed using 100 bootstrap replications, and critical values are set equal to the 90^{th} percentile of the bootstrap distribution. Note additionally that $\overline{X}$ and σ_X are the mean and variance of an initial sample of data of length 1000 observations that is generated using the model under investigation. See Section 6.1 for further details. All results are based on 500 Monte Carlo simulations.

Table 2: Simulated Density Based Specification Test Rejection Frequencies - Empirical Level(*)

Data Generated using the OU Model

τ	$(\underline{u}, \overline{u})$	S,l											
		10T,5	20T,5	30T,5	10T,10	20T,10	30T,10	10T,20	20T,20	30T,20	10T,50	20T,50	30T,50
							Panel A: $T = 400$						
1	$\overline{X} \pm 0.25\sigma_X$	0.218	0.178	0.178	0.218	0.182	0.174	0.202	0.196	0.172	0.186	0.174	0.196
	$\overline{X} \pm 0.5\sigma_X$	0.206	0.186	0.190	0.208	0.198	0.178	0.214	0.190	0.176	0.218	0.212	0.188
	$\overline{X} \pm \sigma_X$	0.132	0.136	0.138	0.132	0.136	0.132	0.140	0.138	0.136	0.132	0.136	0.134
	$\overline{X} \pm 2\sigma_X$	0.136	0.136	0.136	0.144	0.132	0.136	0.132	0.132	0.136	0.138	0.132	0.134
2	$\overline{X} \pm 0.25\sigma_X$	0.186	0.188	0.192	0.188	0.214	0.218	0.192	0.184	0.202	0.184	0.202	0.192
	$\overline{X} \pm 0.5\sigma_X$	0.182	0.194	0.212	0.204	0.214	0.208	0.186	0.206	0.178	0.176	0.178	0.182
	$\overline{X} \pm \sigma_X$	0.134	0.134	0.134	0.130	0.134	0.142	0.132	0.132	0.136	0.140	0.134	0.140
	$\overline{X} \pm 2\sigma_X$	0.136	0.132	0.138	0.136	0.134	0.136	0.138	0.136	0.136	0.138	0.136	0.134
4	$\overline{X} \pm 0.25\sigma_X$	0.194	0.182	0.188	0.172	0.172	0.176	0.178	0.200	0.208	0.212	0.174	0.184
	$\overline{X} \pm 0.5\sigma_X$	0.214	0.176	0.188	0.178	0.178	0.176	0.196	0.196	0.172	0.178	0.178	0.192
	$\overline{X} \pm \sigma_X$	0.138	0.138	0.132	0.136	0.136	0.136	0.142	0.136	0.132	0.138	0.136	0.130
	$\overline{X} \pm 2\sigma_X$	0.142	0.138	0.132	0.138	0.132	0.140	0.136	0.136	0.134	0.136	0.140	0.140
12	$\overline{X} \pm 0.25\sigma_X$	0.210	0.202	0.198	0.184	0.206	0.212	0.188	0.212	0.180	0.202	0.216	0.176
	$\overline{X} \pm 0.5\sigma_X$	0.198	0.176	0.192	0.178	0.214	0.222	0.218	0.192	0.188	0.218	0.196	0.182
	$\overline{X} \pm \sigma_X$	0.134	0.134	0.136	0.138	0.136	0.138	0.142	0.134	0.132	0.138	0.138	0.136
	$\overline{X} \pm 2\sigma_X$	0.134	0.136	0.132	0.132	0.136	0.140	0.138	0.132	0.138	0.134	0.138	0.132
							Panel B: $T = 800$						
1	$\overline{X} \pm 0.25\sigma_X$	0.108	0.122	0.116	0.120	0.116	0.114	0.112	0.116	0.114	0.116	0.112	0.116
	$\overline{X} \pm 0.5\sigma_X$	0.110	0.114	0.118	0.116	0.112	0.118	0.112	0.114	0.118	0.112	0.112	0.118
	$\overline{X} \pm \sigma_X$	0.120	0.114	0.116	0.112	0.112	0.114	0.116	0.116	0.118	0.112	0.116	0.114
	$\overline{X} \pm 2\sigma_X$	0.120	0.112	0.112	0.112	0.120	0.120	0.112	0.116	0.120	0.118	0.112	0.118
2	$\overline{X} \pm 0.25\sigma_X$	0.110	0.116	0.114	0.118	0.112	0.122	0.122	0.112	0.116	0.114	0.114	0.120
	$\overline{X} \pm 0.5\sigma_X$	0.114	0.120	0.120	0.114	0.116	0.118	0.114	0.116	0.118	0.122	0.118	0.116
	$\overline{X} \pm \sigma_X$	0.116	0.112	0.122	0.114	0.116	0.116	0.120	0.116	0.114	0.114	0.120	0.118
	$\overline{X} \pm 2\sigma_X$	0.110	0.112	0.124	0.112	0.112	0.118	0.118	0.116	0.122	0.118	0.114	0.122
4	$\overline{X} \pm 0.25\sigma_X$	0.108	0.112	0.112	0.116	0.118	0.124	0.112	0.118	0.114	0.126	0.116	0.118
	$\overline{X} \pm 0.5\sigma_X$	0.112	0.116	0.118	0.114	0.118	0.112	0.120	0.116	0.112	0.112	0.116	0.112
	$\overline{X} \pm \sigma_X$	0.116	0.112	0.126	0.114	0.112	0.118	0.116	0.112	0.126	0.118	0.118	0.118
	$\overline{X} \pm 2\sigma_X$	0.116	0.114	0.116	0.118	0.118	0.114	0.124	0.114	0.116	0.114	0.112	0.116
12	$\overline{X} \pm 0.25\sigma_X$	0.114	0.112	0.114	0.122	0.120	0.116	0.116	0.118	0.122	0.118	0.116	0.120
	$\overline{X} \pm 0.5\sigma_X$	0.112	0.114	0.112	0.112	0.118	0.112	0.116	0.118	0.114	0.120	0.118	0.116
	$\overline{X} \pm \sigma_X$	0.112	0.108	0.112	0.116	0.112	0.114	0.116	0.114	0.118	0.118	0.120	0.116
	$\overline{X} \pm 2\sigma_X$	0.110	0.114	0.112	0.120	0.116	0.118	0.116	0.118	0.114	0.118	0.114	0.112

(*) Notes: See notes to Table 1.

Table 3: Simulated Density Based Specification Test Rejection Frequencies - Empirical Power(*)

Data Generated using the *CIR* Model

τ	$(\underline{u}, \overline{u})$	S,l											
		10T,5	20T,5	30T,5	10T,10	20T,10	30T,10	10T,20	20T,20	30T,20	10T,50	20T,50	30T,50
						Panel A: $T = 400$							
1	$\overline{X} \pm 0.25\sigma_X$	0.342	0.352	0.388	0.344	0.356	0.356	0.448	0.454	0.446	0.422	0.424	0.456
	$\overline{X} \pm 0.5\sigma_X$	0.382	0.354	0.374	0.378	0.342	0.348	0.464	0.466	0.448	0.432	0.458	0.452
	$\overline{X} \pm \sigma_X$	0.272	0.268	0.262	0.254	0.278	0.248	0.322	0.344	0.334	0.344	0.328	0.326
	$\overline{X} \pm 2\sigma_X$	0.288	0.232	0.248	0.244	0.252	0.238	0.328	0.338	0.314	0.336	0.322	0.356
2	$\overline{X} \pm 0.25\sigma_X$	0.356	0.358	0.398	0.348	0.346	0.368	0.442	0.456	0.438	0.468	0.436	0.426
	$\overline{X} \pm 0.5\sigma_X$	0.378	0.378	0.368	0.378	0.376	0.382	0.448	0.448	0.432	0.456	0.464	0.478
	$\overline{X} \pm \sigma_X$	0.288	0.262	0.278	0.236	0.278	0.256	0.344	0.332	0.358	0.358	0.358	0.352
	$\overline{X} \pm 2\sigma_X$	0.248	0.264	0.248	0.272	0.242	0.236	0.328	0.354	0.328	0.322	0.332	0.354
4	$\overline{X} \pm 0.25\sigma_X$	0.374	0.382	0.368	0.344	0.378	0.346	0.436	0.434	0.468	0.448	0.442	0.444
	$\overline{X} \pm 0.5\sigma_X$	0.368	0.344	0.378	0.348	0.372	0.378	0.442	0.458	0.428	0.458	0.426	0.448
	$\overline{X} \pm \sigma_X$	0.246	0.254	0.246	0.234	0.262	0.234	0.318	0.348	0.356	0.316	0.352	0.316
	$\overline{X} \pm 2\sigma_X$	0.256	0.236	0.264	0.264	0.236	0.276	0.358	0.338	0.358	0.322	0.332	0.336
12	$\overline{X} \pm 0.25\sigma_X$	0.368	0.368	0.342	0.374	0.348	0.398	0.432	0.458	0.468	0.438	0.452	0.454
	$\overline{X} \pm 0.5\sigma_X$	0.378	0.364	0.358	0.374	0.368	0.358	0.464	0.438	0.426	0.424	0.458	0.432
	$\overline{X} \pm \sigma_X$	0.244	0.252	0.238	0.258	0.248	0.238	0.328	0.316	0.348	0.348	0.368	0.356
	$\overline{X} \pm 2\sigma_X$	0.278	0.246	0.268	0.272	0.256	0.242	0.314	0.358	0.328	0.354	0.318	0.318
						Panel B: $T = 800$							
1	$\overline{X} \pm 0.25\sigma_X$	0.868	0.938	0.912	0.838	0.806	0.832	0.938	0.886	0.806	0.916	0.932	0.848
	$\overline{X} \pm 0.5\sigma_X$	0.824	0.802	0.878	0.854	0.926	0.934	0.822	0.844	0.952	0.802	0.928	0.898
	$\overline{X} \pm \sigma_X$	0.704	0.802	0.722	0.712	0.734	0.838	0.872	0.816	0.842	0.940	0.816	0.814
	$\overline{X} \pm 2\sigma_X$	0.734	0.774	0.718	0.754	0.766	0.752	0.910	0.868	0.906	0.912	0.934	0.814
2	$\overline{X} \pm 0.25\sigma_X$	0.858	0.846	0.818	0.928	0.908	0.918	0.898	0.864	0.852	0.862	0.864	0.928
	$\overline{X} \pm 0.5\sigma_X$	0.928	0.938	0.944	0.806	0.816	0.886	0.886	0.934	0.884	0.818	0.902	0.818
	$\overline{X} \pm \sigma_X$	0.714	0.798	0.754	0.838	0.766	0.776	0.920	0.936	0.932	0.946	0.918	0.882
	$\overline{X} \pm 2\sigma_X$	0.796	0.836	0.834	0.724	0.746	0.802	0.836	0.906	0.824	0.914	0.818	0.838
4	$\overline{X} \pm 0.25\sigma_X$	0.922	0.806	0.908	0.858	0.856	0.878	0.832	0.924	0.938	0.848	0.824	0.832
	$\overline{X} \pm 0.5\sigma_X$	0.858	0.942	0.918	0.856	0.888	0.864	0.818	0.896	0.878	0.916	0.906	0.858
	$\overline{X} \pm \sigma_X$	0.714	0.788	0.768	0.832	0.776	0.716	0.936	0.884	0.814	0.838	0.918	0.858
	$\overline{X} \pm 2\sigma_X$	0.738	0.726	0.786	0.738	0.822	0.728	0.866	0.918	0.806	0.882	0.928	0.944
12	$\overline{X} \pm 0.25\sigma_X$	0.878	0.852	0.928	0.916	0.924	0.838	0.884	0.926	0.836	0.904	0.85	0.934
	$\overline{X} \pm 0.5\sigma_X$	0.862	0.928	0.898	0.874	0.932	0.862	0.944	0.902	0.882	0.892	0.848	0.834
	$\overline{X} \pm \sigma_X$	0.756	0.786	0.798	0.724	0.722	0.818	0.938	0.816	0.840	0.816	0.814	0.834
	$\overline{X} \pm 2\sigma_X$	0.808	0.728	0.842	0.742	0.756	0.754	0.822	0.834	0.850	0.804	0.916	0.816

(*) Notes: See notes to Table 1.

Table 4: Simulated Density Based Specification Test Rejection Frequencies - Empirical Power(*)

Data Generated using the OU Model

τ	$(\underline{u}, \overline{u})$	S,l											
		10T,5	20T,5	30T,5	10T,10	20T,10	30T,10	10T,20	20T,20	30T,20	10T,50	20T,50	30T,50
						Panel A: $T = 400$							
1	$\overline{X} \pm 0.25\sigma_X$	0.414	0.414	0.424	0.388	0.418	0.432	0.462	0.488	0.466	0.466	0.462	0.484
	$\overline{X} \pm 0.5\sigma_X$	0.386	0.416	0.384	0.398	0.424	0.404	0.502	0.478	0.474	0.508	0.466	0.512
	$\overline{X} \pm \sigma_X$	0.312	0.344	0.348	0.324	0.342	0.318	0.418	0.428	0.406	0.408	0.418	0.428
	$\overline{X} \pm 2\sigma_X$	0.368	0.362	0.362	0.318	0.358	0.332	0.436	0.408	0.436	0.392	0.428	0.408
2	$\overline{X} \pm 0.25\sigma_X$	0.394	0.392	0.402	0.394	0.426	0.412	0.498	0.518	0.508	0.506	0.492	0.466
	$\overline{X} \pm 0.5\sigma_X$	0.418	0.428	0.392	0.428	0.418	0.404	0.498	0.518	0.508	0.502	0.496	0.464
	$\overline{X} \pm \sigma_X$	0.328	0.336	0.322	0.318	0.314	0.332	0.402	0.436	0.426	0.428	0.412	0.398
	$\overline{X} \pm 2\sigma_X$	0.358	0.346	0.352	0.312	0.352	0.352	0.43	0.404	0.434	0.418	0.426	0.412
4	$\overline{X} \pm 0.25\sigma_X$	0.388	0.426	0.398	0.394	0.428	0.406	0.494	0.508	0.504	0.464	0.474	0.512
	$\overline{X} \pm 0.5\sigma_X$	0.384	0.398	0.388	0.398	0.428	0.406	0.462	0.492	0.478	0.482	0.478	0.468
	$\overline{X} \pm \sigma_X$	0.314	0.314	0.354	0.338	0.338	0.316	0.398	0.436	0.438	0.406	0.408	0.414
	$\overline{X} \pm 2\sigma_X$	0.334	0.346	0.338	0.328	0.312	0.314	0.406	0.398	0.416	0.436	0.408	0.438
12	$\overline{X} \pm 0.25\sigma_X$	0.426	0.402	0.382	0.382	0.422	0.384	0.474	0.466	0.504	0.488	0.486	0.506
	$\overline{X} \pm 0.5\sigma_X$	0.418	0.418	0.384	0.412	0.382	0.428	0.474	0.472	0.468	0.482	0.508	0.468
	$\overline{X} \pm \sigma_X$	0.352	0.342	0.358	0.352	0.328	0.344	0.402	0.412	0.392	0.422	0.404	0.394
	$\overline{X} \pm 2\sigma_X$	0.328	0.314	0.334	0.358	0.358	0.326	0.418	0.436	0.438	0.396	0.432	0.426
						Panel B: $T = 800$							
1	$\overline{X} \pm 0.25\sigma_X$	0.812	0.854	0.858	0.834	0.928	0.804	0.838	0.914	0.828	0.828	0.902	0.906
	$\overline{X} \pm 0.5\sigma_X$	0.842	0.886	0.818	0.878	0.942	0.928	0.892	0.936	0.838	0.888	0.894	0.872
	$\overline{X} \pm \sigma_X$	0.726	0.786	0.756	0.802	0.806	0.792	0.858	0.932	0.918	0.888	0.878	0.902
	$\overline{X} \pm 2\sigma_X$	0.846	0.716	0.828	0.826	0.806	0.728	0.918	0.944	0.828	0.958	0.846	0.946
2	$\overline{X} \pm 0.25\sigma_X$	0.942	0.816	0.872	0.844	0.822	0.934	0.946	0.906	0.896	0.914	0.958	0.818
	$\overline{X} \pm 0.5\sigma_X$	0.872	0.932	0.898	0.864	0.828	0.862	0.802	0.828	0.858	0.878	0.948	0.958
	$\overline{X} \pm \sigma_X$	0.772	0.782	0.802	0.732	0.726	0.846	0.878	0.928	0.826	0.904	0.826	0.918
	$\overline{X} \pm 2\sigma_X$	0.846	0.762	0.722	0.822	0.718	0.812	0.858	0.854	0.818	0.802	0.804	0.838
4	$\overline{X} \pm 0.25\sigma_X$	0.878	0.834	0.858	0.864	0.886	0.884	0.936	0.894	0.918	0.882	0.918	0.846
	$\overline{X} \pm 0.5\sigma_X$	0.868	0.898	0.896	0.948	0.818	0.808	0.898	0.818	0.948	0.896	0.822	0.852
	$\overline{X} \pm \sigma_X$	0.752	0.702	0.702	0.716	0.718	0.726	0.886	0.844	0.882	0.882	0.946	0.948
	$\overline{X} \pm 2\sigma_X$	0.826	0.722	0.722	0.834	0.798	0.832	0.828	0.864	0.874	0.936	0.874	0.862
12	$\overline{X} \pm 0.25\sigma_X$	0.918	0.812	0.848	0.902	0.906	0.888	0.828	0.848	0.946	0.848	0.902	0.918
	$\overline{X} \pm 0.5\sigma_X$	0.828	0.928	0.924	0.802	0.918	0.932	0.946	0.838	0.942	0.802	0.932	0.908
	$\overline{X} \pm \sigma_X$	0.772	0.742	0.704	0.794	0.848	0.724	0.856	0.918	0.932	0.808	0.904	0.894
	$\overline{X} \pm 2\sigma_X$	0.776	0.804	0.712	0.732	0.764	0.772	0.884	0.944	0.916	0.804	0.808	0.898

(*) Notes: See notes to Table 1.

Table 5: Summary Statistics and Estimation Results(*)

	Jan-6-71 to Apr-8-05	Jan-3-90 to Apr-8-05
Panel A: Summary Statistics		
Mean	0.06827	0.04464
Std. Dev.	0.03477	0.01977
Skewness	0.94406	-0.15232
Kurtosis	4.43616	2.18807
Panel B: CIR Model Estimation Results		
ϕ_1	0.3864	0.5275
	(0.2196)	(0.1682)
α_1	0.0684	0.0527
	(0.0017)	(0.0020)
σ_1	0.1118	0.1115
	(0.0482)	(0.0141)
Panel C: OU Model Estimation Results		
ϕ_2	0.1481	0.2095
	(0.0928)	(0.1432)
α_2	-2.8329	-3.2444
	(0.0373)	(0.0429)
σ_2	0.3404	0.3079
	(0.1010)	(0.1175)

(*) Notes: This table reports summary statistics, estimated parameters, and parameter standrd errors (in brackets) using the 1-month Eurodollar deposit rate, for the periods January 6, 1971 - April 8, 2005 and January 3, 1990 - April 8, 2005. See Section 6 for a discussion of the CIR and OU models, and Section 6.1 for a discussion of estimation methodology.

Table 6a: Specification Test Results - CIR Model$^{(*)}$

Sample Period: 06-Jan-71 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	$S=T$				$S=2T$				$S=3T$			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
	Panel A: $l=5$												
1	$\overline{X}\pm0.25\sigma_X$	2.96***	2.24	2.07	1.71	2.84***	2.25	2.06	1.72	2.98***	2.25	2.07	1.72
	$\overline{X}\pm0.5\sigma_X$	4.98***	4.23	4.19	4.09	4.95***	4.43	4.26	4.11	4.95***	4.31	4.27	4.19
	$\overline{X}\pm\sigma_X$	3.03***	1.62	1.49	1.13	3.02***	1.60	1.45	1.12	3.03***	1.62	1.48	1.13
	$\overline{X}\pm2\sigma_X$	2.05***	1.11	1.08	0.81	2.02***	1.10	1.09	0.85	2.03***	1.13	1.09	0.83
2	$\overline{X}\pm0.25\sigma_X$	2.89***	1.97	1.91	1.66	2.95***	2.01	1.97	1.67	2.96***	2.00	1.96	1.63
	$\overline{X}\pm0.5\sigma_X$	4.76***	4.01	3.73	3.15	4.65***	4.02	3.74	3.17	4.64***	4.03	3.74	3.18
	$\overline{X}\pm\sigma_X$	3.02***	1.93	1.57	1.51	3.03***	1.97	1.61	1.49	3.03***	1.95	1.61	1.54
	$\overline{X}\pm2\sigma_X$	1.94***	1.38	1.36	1.28	1.92***	1.38	1.35	1.27	2.11***	1.39	1.35	1.27
4	$\overline{X}\pm0.25\sigma_X$	2.91**	2.94	2.79	2.51	2.98***	2.88	2.75	2.55	2.97***	2.89	2.78	2.55
	$\overline{X}\pm0.5\sigma_X$	4.92**	5.02	4.44	4.28	4.79**	5.04	4.42	4.36	4.85**	5.08	4.47	4.42
	$\overline{X}\pm\sigma_X$	3.03**	3.05	2.98	2.96	3.03	3.15	3.09	3.03	3.02*	3.17	3.09	2.98
	$\overline{X}\pm2\sigma_X$	1.94*	1.96	1.94	1.79	1.84*	2.03	1.98	1.82	1.96*	2.18	2.02	1.81
12	$\overline{X}\pm0.25\sigma_X$	2.77***	2.71	2.41	2.25	2.79***	2.74	2.36	2.23	2.84***	2.75	2.34	2.20
	$\overline{X}\pm0.5\sigma_X$	4.84***	4.53	4.44	4.19	4.99***	4.51	4.46	4.14	4.78***	4.45	4.43	4.04
	$\overline{X}\pm\sigma_X$	3.16	3.94	3.57	3.25	3.17	3.73	3.34	3.18	3.19*	3.81	3.38	3.08
	$\overline{X}\pm2\sigma_X$	1.72	1.84	1.79	1.74	1.72	1.91	1.87	1.80	1.68	1.92	1.87	1.81
	Panel B: $l=10$												
1	$\overline{X}\pm0.25\sigma_X$	2.96***	2.27	1.87	1.83	2.84***	2.12	1.96	1.79	2.98***	2.15	2.07	1.74
	$\overline{X}\pm0.5\sigma_X$	4.98***	3.32	2.94	2.25	4.95***	2.54	2.22	2.17	4.95***	2.29	2.25	1.98
	$\overline{X}\pm\sigma_X$	3.03***	1.91	1.79	1.59	3.02***	1.55	1.49	1.20	3.03***	1.45	1.44	1.36
	$\overline{X}\pm2\sigma_X$	2.05***	1.04	1.02	0.84	2.02***	0.93	0.84	0.82	2.03***	1.49	1.03	0.97
2	$\overline{X}\pm0.25\sigma_X$	2.89***	2.18	1.86	1.68	2.95***	1.36	1.34	1.18	2.96***	2.66	2.61	2.21
	$\overline{X}\pm0.5\sigma_X$	4.76***	3.71	3.46	2.93	4.65***	3.55	3.29	3.23	4.64***	3.18	2.85	2.49
	$\overline{X}\pm\sigma_X$	3.02***	2.08	2.04	1.97	3.03***	1.71	1.67	1.39	3.03***	1.97	1.94	1.77
	$\overline{X}\pm2\sigma_X$	1.94***	1.13	1.11	1.08	1.92***	1.18	1.10	0.92	2.11***	1.87	1.51	1.08
4	$\overline{X}\pm0.25\sigma_X$	2.91***	2.79	2.21	1.85	2.98***	2.13	2.04	1.87	2.97***	2.24	2.23	2.01
	$\overline{X}\pm0.5\sigma_X$	4.92***	4.51	4.46	4.22	4.79***	3.15	3.06	2.73	4.85***	2.76	2.71	2.65
	$\overline{X}\pm\sigma_X$	3.03***	2.25	2.15	2.08	3.03***	2.61	2.28	2.13	3.02***	2.59	2.56	2.11
	$\overline{X}\pm2\sigma_X$	1.94***	1.25	1.13	1.11	1.84***	1.28	1.24	1.13	1.96***	1.81	1.54	1.47
12	$\overline{X}\pm0.25\sigma_X$	2.77***	2.73	2.69	2.55	2.79**	2.88	2.48	2.35	2.84***	2.8	2.38	2.21
	$\overline{X}\pm0.5\sigma_X$	4.84*	5.22	4.92	4.55	4.99***	4.54	4.37	4.31	4.78***	4.54	4.27	4.12
	$\overline{X}\pm\sigma_X$	3.16	3.95	3.84	3.78	3.17	4.17	4.12	3.78	3.19	3.68	3.64	3.53
	$\overline{X}\pm2\sigma_X$	1.72	2.92	2.64	2.54	1.72	2.92	2.64	2.54	1.68	2.32	2.12	1.94
	Panel C: $l=20$												
1	$\overline{X}\pm0.25\sigma_X$	2.96***	1.64	1.30	1.18	2.84***	2.28	2.01	1.97	2.98***	2.03	1.81	1.11
	$\overline{X}\pm0.5\sigma_X$	4.98***	2.35	2.16	2.09	4.95***	2.21	2.02	1.65	4.95***	1.98	1.89	1.60
	$\overline{X}\pm\sigma_X$	3.03***	1.91	1.69	1.56	3.02***	1.86	1.62	1.46	3.03***	1.96	1.94	1.35
	$\overline{X}\pm2\sigma_X$	2.05***	0.82	0.78	0.55	2.02***	0.81	0.77	0.71	2.03***	1.29	1.01	0.93
2	$\overline{X}\pm0.25\sigma_X$	2.89***	1.49	1.15	0.86	2.95***	2.63	2.21	1.26	2.96***	1.87	1.53	1.06
	$\overline{X}\pm0.5\sigma_X$	4.76***	3.07	2.47	2.25	4.65***	2.74	2.44	2.06	4.64***	2.79	2.77	2.23
	$\overline{X}\pm\sigma_X$	3.02***	2.16	2.03	1.68	3.03***	2.21	1.93	1.75	3.03***	2.12	1.95	1.51
	$\overline{X}\pm2\sigma_X$	1.94***	1.05	0.75	0.74	1.92***	0.88	0.82	0.61	2.11***	1.32	1.08	0.83
4	$\overline{X}\pm0.25\sigma_X$	2.91***	2.72	2.36	1.96	2.98***	2.38	2.17	2.11	2.97***	1.68	1.58	1.44
	$\overline{X}\pm0.5\sigma_X$	4.92***	3.37	3.26	2.37	4.79***	2.84	2.68	2.55	4.85***	4.09	3.77	3.17
	$\overline{X}\pm\sigma_X$	3.03***	2.14	2.11	1.69	3.03***	2.38	2.31	1.89	3.02***	1.98	1.95	1.75
	$\overline{X}\pm2\sigma_X$	1.94***	1.01	0.97	0.82	1.84***	1.05	0.93	0.84	1.96***	1.29	1.23	0.82
12	$\overline{X}\pm0.25\sigma_X$	2.77***	2.37	2.36	2.11	2.79***	2.76	2.23	2.21	2.84**	2.86	2.59	2.13
	$\overline{X}\pm0.5\sigma_X$	4.84***	4.39	4.11	3.62	4.99***	4.19	4.13	3.51	4.78***	4.26	3.87	3.63
	$\overline{X}\pm\sigma_X$	3.16***	3.07	2.97	2.81	3.17**	3.17	2.77	2.72	3.19*	3.49	3.21	2.62
	$\overline{X}\pm2\sigma_X$	1.72***	1.61	1.59	1.54	1.72***	1.43	1.39	1.31	1.68***	1.48	1.39	1.33

Table 6a (continued): Specification Test Results - CIR Model$^{(*)}$

Sample Period: 06-Jan-71 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	$S=T$				$S=2T$				$S=3T$			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
						Panel D: $l=50$							
1	$\overline{X}\pm0.25\sigma_X$	2.96**	3.07	2.40	1.71	2.84***	2.03	2.02	1.68	2.98**	3.07	2.40	1.71
	$\overline{X}\pm0.5\sigma_X$	4.98***	3.02	2.39	2.31	4.95***	3.47	2.67	2.40	4.95***	2.83	2.60	2.30
	$\overline{X}\pm\sigma_X$	3.03***	2.11	2.08	1.83	3.02***	2.39	2.37	2.27	3.03**	4.19	2.60	2.16
	$\overline{X}\pm2\sigma_X$	2.05***	1.69	1.64	1.22	2.02***	1.19	1.15	1.06	2.03***	1.66	1.16	0.94
2	$\overline{X}\pm0.25\sigma_X$	2.89*	3.96	3.19	2.24	2.95***	2.55	2.44	1.86	2.96*	3.96	3.19	2.24
	$\overline{X}\pm0.5\sigma_X$	4.76***	3.22	2.87	2.44	4.65***	3.58	3.39	2.69	4.64***	2.98	2.48	2.18
	$\overline{X}\pm\sigma_X$	3.02***	2.02	1.97	1.73	3.03***	2.52	2.51	2.37	3.03**	3.73	2.43	2.27
	$\overline{X}\pm2\sigma_X$	1.94***	1.63	1.53	1.15	1.92***	1.23	1.22	1.04	2.11***	1.66	1.28	1.03
4	$\overline{X}\pm0.25\sigma_X$	2.91*	4.54	3.01	2.28	2.98**	3.07	2.65	2.16	2.97*	4.54	3.01	2.28
	$\overline{X}\pm0.5\sigma_X$	4.92***	3.31	2.97	2.87	4.79***	3.57	3.23	3.05	4.85***	3.02	3.01	2.55
	$\overline{X}\pm\sigma_X$	3.03***	2.26	2.11	1.78	3.03***	2.78	2.77	2.37	3.02**	3.56	2.67	2.27
	$\overline{X}\pm2\sigma_X$	1.94***	1.23	1.19	1.04	1.84***	1.37	1.18	0.98	1.96***	1.58	1.44	1.13
12	$\overline{X}\pm0.25\sigma_X$	2.77*	4.04	3.30	2.20	2.79*	3.38	3.05	2.58	2.84*	4.04	3.30	2.20
	$\overline{X}\pm0.5\sigma_X$	4.84***	3.91	3.59	3.00	4.99***	4.16	4.06	3.99	4.78***	4.14	3.80	3.41
	$\overline{X}\pm\sigma_X$	3.16***	2.23	2.18	1.97	3.17***	3.07	2.92	2.31	3.19***	3.12	2.84	2.66
	$\overline{X}\pm2\sigma_X$	1.72***	1.34	1.33	1.08	1.72***	1.23	1.15	0.84	1.68***	1.12	1.12	0.93

$^{(*)}$ Notes: Entries in the table are test statistics and 5constructed using intervals given in the second column of the table, and for τ =1, 2, 4, 12 . Single, double, and triple starred entries denote rejection at the 20The simulation periods considered is denoted by S, and T denotes the number of observations in the sample. Block lengths considered are $l = (5, 10, 20, 50)$. Empirical bootstrap distributions are constructed using 100 bootstrap replications, Note additionally that $\overline{X}$ and σ_X are the mean and variance of the sample of historical observations. See Section 6.2 for further details.

Table 6b: Specification Test Results - CIR Model$^{(*)}$

Sample Period: 03-Jan-90 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	$S=T$				$S=2T$				$S=3T$			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
Panel A: $l=5$													
1	$\overline{X}\pm\sigma_X$	3.79***	1.58	1.44	1.27	3.83***	1.40	1.35	1.27	3.83***	1.88	1.58	1.36
2	$\overline{X}\pm\sigma_X$	3.80***	2.14	1.97	1.77	3.79***	2.00	1.93	1.78	3.86***	2.24	2.10	2.00
4	$\overline{X}\pm\sigma_X$	3.73***	3.72	3.50	3.32	3.79***	3.58	3.3	3.12	3.76***	3.59	3.58	3.28
12	$\overline{X}\pm\sigma_X$	3.48	3.84	3.79	3.71	3.39	3.74	3.61	3.46	3.45	3.81	3.76	3.66
Panel B: $l=10$													
1	$\overline{X}\pm\sigma_X$	3.79***	2.18	2.15	1.34	3.83***	1.85	1.72	1.61	3.83***	1.53	1.48	1.27
2	$\overline{X}\pm\sigma_X$	3.8***	2.21	1.95	1.43	3.79***	2.00	1.87	1.67	3.86***	1.59	1.54	1.30
4	$\overline{X}\pm\sigma_X$	3.73***	2.38	2.19	1.99	3.79***	2.69	2.67	2.29	3.76***	2.11	1.98	1.84
12	$\overline{X}\pm\sigma_X$	3.48	4.52	4.22	3.62	3.39	3.78	3.77	3.71	3.45	4.12	4.09	3.73
Panel C: $l=20$													
1	$\overline{X}\pm\sigma_X$	3.79***	1.72	1.70	1.65	3.83***	1.54	1.47	1.26	3.83***	2.20	1.68	1.56
2	$\overline{X}\pm\sigma_X$	3.80***	2.01	1.94	1.80	3.79***	2.02	1.63	1.51	3.86***	2.40	1.88	1.67
4	$\overline{X}\pm\sigma_X$	3.73***	2.16	1.94	1.73	3.79***	2.02	2.01	1.61	3.76***	2.42	1.75	1.59
12	$\overline{X}\pm\sigma_X$	3.48***	3.06	2.85	2.74	3.39*	3.48	3.45	3.36	3.45***	3.21	3.08	2.73
Panel D: $l=50$													
1	$\overline{X}\pm\sigma_X$	3.79***	2.54	2.49	1.80	3.83***	2.15	2.12	1.91	3.83***	2.53	2.27	1.97
2	$\overline{X}\pm\sigma_X$	3.80***	2.69	2.49	1.91	3.79***	1.85	1.76	1.74	3.86***	2.68	2.14	1.89
4	$\overline{X}\pm\sigma_X$	3.73***	2.64	2.58	1.91	3.79***	2.20	2.07	1.80	3.76***	2.74	2.05	1.83
12	$\overline{X}\pm\sigma_X$	3.48***	3.30	3.07	2.46	3.39***	3.11	2.98	2.81	3.45***	3.31	3.03	2.63

$^{(*)}$ Notes: See notes to Table 6a.

Table 7a: Specification Test Results - OU Model$^{(*)}$

Sample Period: 06-Jan-71 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	$S=T$				$S=2T$				$S=3T$			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
Panel A: $l=5$													
1	$\overline{X}\pm0.25\sigma_X$	2.84***	1.75	1.74	1.25	2.88***	1.35	1.27	1.15	2.84***	1.47	1.43	1.24
	$\overline{X}\pm0.5\sigma_X$	4.98**	5.82	4.18	3.92	5.07**	5.47	5.06	4.66	4.99***	4.95	4.85	3.78
	$\overline{X}\pm\sigma_X$	3.14***	2.82	2.81	2.45	3.03**	3.14	2.83	2.59	3.03***	2.85	2.79	2.33
	$\overline{X}\pm2\sigma_X$	2.05***	0.71	0.66	0.48	2.02***	0.61	0.59	0.52	2.00***	0.88	0.85	0.71
2	$\overline{X}\pm0.25\sigma_X$	2.82***	1.91	1.83	1.71	2.86***	2.19	2.18	2.12	3.01***	1.98	1.85	1.72
	$\overline{X}\pm0.5\sigma_X$	4.82*	6.01	5.64	4.61	4.76*	4.97	4.93	4.15	4.77*	5.61	5.09	4.51
	$\overline{X}\pm\sigma_X$	3.11*	3.13	3.12	2.97	3.14**	3.21	3.09	2.95	3.13**	3.19	3.02	2.91
	$\overline{X}\pm2\sigma_X$	2.00***	0.89	0.86	0.69	1.98***	1.21	1.16	1.09	2.01***	1.21	1.16	1.04
4	$\overline{X}\pm0.25\sigma_X$	2.77**	2.77	2.56	2.46	2.67***	2.64	2.54	2.28	2.77**	2.78	2.69	2.65
	$\overline{X}\pm0.5\sigma_X$	4.78	5.90	5.88	5.43	4.98	5.80	5.74	5.39	4.68	5.97	5.44	5.38
	$\overline{X}\pm\sigma_X$	3.09	3.60	3.52	3.47	2.99	3.81	3.68	3.52	3.01	3.61	3.44	3.12
	$\overline{X}\pm2\sigma_X$	1.98***	1.52	1.46	1.31	2.02***	1.82	1.68	1.56	2.02***	1.42	1.35	1.31
12	$\overline{X}\pm0.25\sigma_X$	3.16**	3.37	3.15	2.71	3.39***	3.17	2.88	2.71	3.39***	3.11	2.75	2.56
	$\overline{X}\pm0.5\sigma_X$	4.69	6.02	5.56	5.49	4.93	5.54	5.36	5.25	4.92	5.98	5.85	5.77
	$\overline{X}\pm\sigma_X$	3.09	4.33	4.25	4.01	3.09	4.45	4.29	4.07	3.06	4.23	4.01	3.86
	$\overline{X}\pm2\sigma_X$	1.87***	1.55	1.39	1.33	1.72***	1.66	1.47	1.35	1.87***	1.47	1.42	1.41
Panel B: $l=10$													
1	$\overline{X}\pm0.25\sigma_X$	2.84***	2.56	2.19	1.93	2.88***	2.41	2.02	1.43	2.84***	2.34	2.33	2.09
	$\overline{X}\pm0.5\sigma_X$	4.98***	3.34	2.66	2.35	5.07***	3.69	3.25	3.14	4.99***	4.45	4.45	3.45
	$\overline{X}\pm\sigma_X$	3.14***	1.75	1.66	1.65	3.03***	1.91	1.74	1.34	3.03***	2.12	1.93	1.74
	$\overline{X}\pm2\sigma_X$	2.05***	0.78	0.74	0.69	2.02***	0.74	0.63	0.45	2.00***	0.72	0.71	0.69
2	$\overline{X}\pm0.25\sigma_X$	2.82***	1.75	1.31	1.29	2.86***	2.39	2.36	1.61	3.01***	2.02	1.69	1.42
	$\overline{X}\pm0.5\sigma_X$	4.82***	4.81	4.17	3.99	4.76***	4.31	4.14	3.92	4.77*	5.95	4.79	4.58
	$\overline{X}\pm\sigma_X$	3.11***	2.98	2.88	2.48	3.14***	2.97	2.71	2.57	3.13**	3.24	3.05	2.66
	$\overline{X}\pm2\sigma_X$	2.00***	0.68	0.67	0.64	1.98***	1.07	0.98	0.79	2.01***	0.96	0.89	0.76
4	$\overline{X}\pm0.25\sigma_X$	2.77***	2.45	2.14	1.84	2.67**	2.69	2.38	2.25	2.77***	2.54	2.45	1.91
	$\overline{X}\pm0.5\sigma_X$	4.78***	4.64	4.51	4.18	4.98***	4.63	4.59	4.19	4.68**	4.73	4.51	4.21
	$\overline{X}\pm\sigma_X$	3.09*	3.45	3.39	3.03	2.99**	3.14	2.77	2.71	3.01**	3.21	2.98	2.89
	$\overline{X}\pm2\sigma_X$	1.98***	1.52	1.41	1.24	2.02***	1.28	1.19	0.76	2.02***	1.41	1.38	0.98
12	$\overline{X}\pm0.25\sigma_X$	3.16**	3.17	3.08	2.89	3.39*	3.78	3.41	3.12	3.39**	3.49	3.34	3.02
	$\overline{X}\pm0.5\sigma_X$	4.69	5.68	5.51	5.02	4.93	5.87	5.76	5.52	4.92	5.57	5.46	5.18
	$\overline{X}\pm\sigma_X$	3.09	4.14	4.09	3.99	3.09	4.98	4.61	4.28	3.06	4.62	4.45	4.03
	$\overline{X}\pm2\sigma_X$	1.87***	1.53	1.52	1.33	1.72*	1.95	1.81	1.67	1.87***	1.78	1.66	1.49
Panel C: $l=20$													
1	$\overline{X}\pm0.25\sigma_X$	2.84***	2.42	2.29	2.06	2.88***	2.48	2.20	1.70	2.84**	3.24	2.35	2.08
	$\overline{X}\pm0.5\sigma_X$	4.98***	3.07	2.70	2.34	5.07***	3.40	3.02	2.57	4.99***	2.38	2.35	2.32
	$\overline{X}\pm\sigma_X$	3.14***	2.27	1.98	1.68	3.03***	1.97	1.75	1.68	3.03***	2.17	2.12	1.78
	$\overline{X}\pm2\sigma_X$	2.05***	1.32	1.04	0.79	2.02***	0.95	0.77	0.66	2.00***	0.69	0.67	0.53
2	$\overline{X}\pm0.25\sigma_X$	2.82***	1.90	1.71	1.51	2.86***	2.16	2.12	1.62	3.01***	2.60	2.43	2.15
	$\overline{X}\pm0.5\sigma_X$	4.82***	4.29	3.64	3.25	4.76***	4.49	3.76	3.21	4.77***	3.91	3.56	3.33
	$\overline{X}\pm\sigma_X$	3.11***	2.79	2.24	1.78	3.14***	2.34	2.06	1.75	3.13***	2.48	2.47	2.26
	$\overline{X}\pm2\sigma_X$	2.00***	1.41	0.81	0.57	1.98***	0.86	0.78	0.55	2.01***	0.79	0.74	0.52
4	$\overline{X}\pm0.25\sigma_X$	2.77***	2.16	2.07	1.89	2.67***	1.82	1.69	1.48	2.77***	2.61	2.01	1.92
	$\overline{X}\pm0.5\sigma_X$	4.78*	5.10	4.85	4.40	4.98**	5.25	4.59	4.30	4.68***	4.25	4.20	3.95
	$\overline{X}\pm\sigma_r$	3.09**	3.34	2.78	2.62	2.99*	3.45	3.17	2.59	3.01*	3.54	3.41	2.09
	$\overline{X}\pm2\sigma_r$	1.98***	1.31	1.07	0.85	2.02***	0.83	0.74	0.71	2.02***	0.76	0.74	0.73
12	$\overline{X}\pm0.25\sigma_r$	3.16*	3.85	3.44	2.43	3.39***	3.31	3.08	2.52	3.39**	3.43	2.95	2.40
	$\overline{X}\pm0.5\sigma_r$	4.69***	4.30	4.20	4.14	4.93**	5.44	4.88	4.72	4.92**	5.13	4.46	3.99
	$\overline{X}\pm\sigma_r$	3.09**	3.36	3.01	2.96	3.09**	3.16	2.85	2.24	3.06**	3.14	2.93	2.36
	$\overline{X}\pm2\sigma_r$	1.87***	1.68	1.55	1.35	1.72***	1.33	1.22	1.16	1.87***	1.33	1.32	0.97

Table 7.a (continued): Specification Test Results - *OU* Model(*)

Sample Period: 06-Jan-71 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	S=T				S=2T				S=3T			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
						Panel D: l=50							
1	$\overline{X}\pm0.25\sigma_r$	2.84***	1.75	1.59	1.24	2.88***	2.00	1.92	1.46	2.84***	2.36	2.33	1.67
	$\overline{X}\pm0.5\sigma_r$	4.98***	2.99	2.97	2.63	5.07***	3.05	2.72	2.18	4.99***	2.99	2.72	2.06
	$\overline{X}\pm\sigma_r$	3.14**	3.23	2.98	2.59	3.03***	2.40	2.16	1.97	3.03***	2.80	2.69	2.00
	$\overline{X}\pm2\sigma_r$	2.05***	1.54	1.15	0.95	2.02***	2.00	1.96	1.45	2.00***	1.75	1.45	0.98
2	$\overline{X}\pm0.25\sigma_r$	2.82**	2.82	2.15	1.75	2.86***	1.92	1.89	1.53	3.01***	2.27	2.26	1.99
	$\overline{X}\pm0.5\sigma_r$	4.82***	2.80	2.70	2.60	4.76***	3.10	3.01	2.61	4.77***	3.11	2.80	2.75
	$\overline{X}\pm\sigma_r$	3.11***	2.20	2.16	1.97	3.14***	2.47	2.22	1.99	3.13***	3.01	2.92	2.54
	$\overline{X}\pm2\sigma_r$	2.00***	1.41	1.38	0.72	1.98***	1.90	1.80	1.21	2.01***	1.73	1.49	1.01
4	$\overline{X}\pm0.25\sigma_r$	2.77**	3.42	2.14	1.51	2.67***	2.13	1.96	1.32	2.77**	2.84	2.58	2.01
	$\overline{X}\pm0.5\sigma_r$	4.78***	3.62	3.17	2.82	4.98***	3.72	3.19	2.96	4.68***	4.17	3.43	2.97
	$\overline{X}\pm\sigma_r$	3.09***	2.78	2.63	2.20	2.99***	2.24	1.97	1.95	3.01***	2.99	2.95	2.38
	$\overline{X}\pm2\sigma_r$	1.98***	1.40	1.20	0.65	2.02***	1.51	1.47	1.21	2.02***	1.71	1.22	1.05
12	$\overline{X}\pm0.25\sigma_r$	3.16**	3.51	2.66	2.39	3.39***	2.63	2.44	2.03	3.39**	3.75	2.50	1.99
	$\overline{X}\pm0.5\sigma_r$	4.69**	4.85	4.15	3.36	4.93*	5.60	5.09	3.97	4.92**	5.69	4.65	3.29
	$\overline{X}\pm\sigma_r$	3.09*	3.53	3.30	2.86	3.09*	3.52	3.17	2.97	3.06*	3.93	3.82	2.89
	$\overline{X}\pm2\sigma_r$	1.87***	1.23	1.11	0.94	1.72***	1.37	1.25	1.15	1.87***	1.46	1.41	1.25

(*) Notes: See notes to Table 6a.

Table 7b: Specification Test Results - OU Model$^{(*)}$

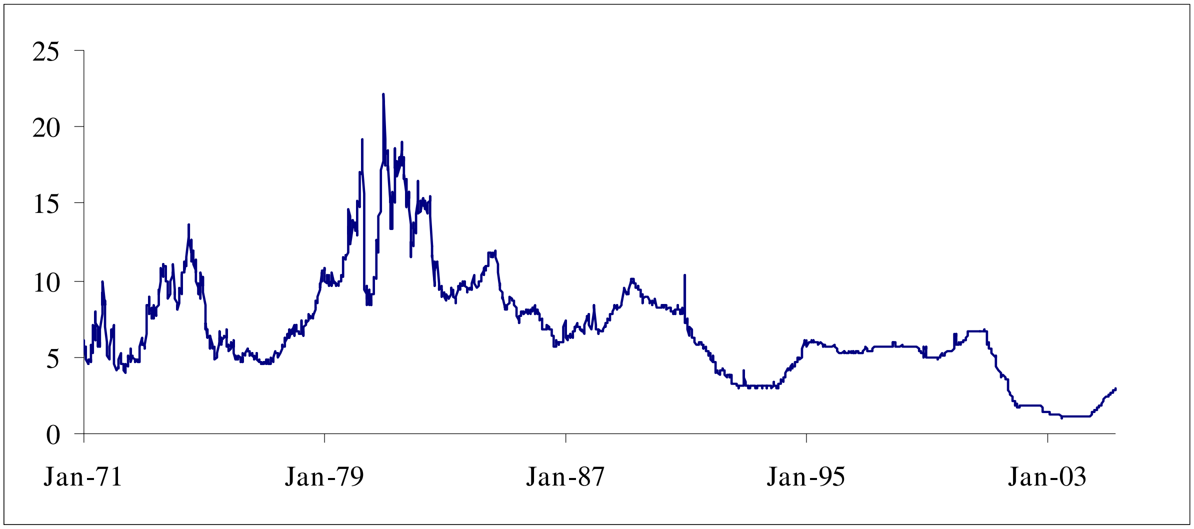
Sample Period: 03-Jan-90 to 08-Apr-05

τ	$(\underline{u},\overline{u})$	$S=T$				$S=2T$				$S=3T$			
		V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV	V_T	5% CV	10% CV	20% CV
Panel A: $l=5$													
1	$\overline{X}\pm\sigma_r$	3.83***	3.50	3.36	2.76	3.86***	3.28	3.23	2.93	3.84***	2.71	2.50	2.22
2	$\overline{X}\pm\sigma_r$	3.77***	3.55	3.19	2.94	3.79***	2.87	2.84	2.80	3.81***	3.05	2.96	2.64
4	$\overline{X}\pm\sigma_r$	3.67**	3.70	3.51	3.48	3.70**	3.80	3.66	3.59	3.73*	3.83	3.78	3.67
12	$\overline{X}\pm\sigma_r$	3.30	4.57	4.21	4.04	3.25	4.18	4.15	4.06	3.27	4.31	4.25	4.12
Panel B: $l=10$													
1	$\overline{X}\pm\sigma_r$	3.83***	3.17	2.90	2.61	3.86***	3.38	3.29	3.03	3.84***	3.67	3.04	2.86
2	$\overline{X}\pm\sigma_r$	3.77***	3.42	3.33	2.83	3.79***	2.49	2.45	2.23	3.81***	3.27	3.22	2.83
4	$\overline{X}\pm\sigma_r$	3.67***	3.10	2.73	2.55	3.70***	3.42	3.28	2.61	3.73***	3.44	3.08	2.69
12	$\overline{X}\pm\sigma_r$	3.30	4.44	4.41	4.23	3.25	4.78	4.53	4.30	3.27	4.28	4.20	4.11
Panel C: $l=20$													
1	$\overline{X}\pm\sigma_r$	3.83***	2.97	2.92	2.37	3.86***	2.09	1.88	1.73	3.84***	2.97	2.48	2.36
2	$\overline{X}\pm\sigma_r$	3.77***	2.86	2.78	2.54	3.79***	3.18	3.10	2.89	3.81***	2.47	2.47	2.31
4	$\overline{X}\pm\sigma_r$	3.67***	3.05	3.05	2.83	3.70***	2.82	2.60	2.36	3.73***	3.31	2.89	2.67
12	$\overline{X}\pm\sigma_r$	3.30**	3.76	3.13	3.06	3.25***	3.01	2.97	2.87	3.27**	3.74	3.21	2.75
Panel D: $l=50$													
1	$\overline{X}\pm\sigma_r$	3.83***	2.27	2.19	1.82	3.86***	2.16	1.99	1.87	3.84***	2.69	2.23	1.83
2	$\overline{X}\pm\sigma_r$	3.77***	2.56	2.39	2.27	3.79***	2.26	2.17	1.94	3.81***	3.17	2.68	2.18
4	$\overline{X}\pm\sigma_r$	3.67***	3.02	2.83	2.47	3.70***	2.92	2.90	2.44	3.73***	3.15	2.89	2.73
12	$\overline{X}\pm\sigma_r$	3.30***	2.74	2.33	2.17	3.25***	2.62	2.46	1.99	3.27***	3.16	2.64	2.22

$^{(*)}$ **See notes to Table 6a.**

Figure 1: 1-month Eorodollar Deposit Rate Observed Series

Panel 1: 1-month Eorodollar Deposit Rate, 06-Jan-71 to 08-Apr-05



Panel 2: 1-month Eorodollar Deposit Rate, 03-Jan-90 to 08-Apr-05

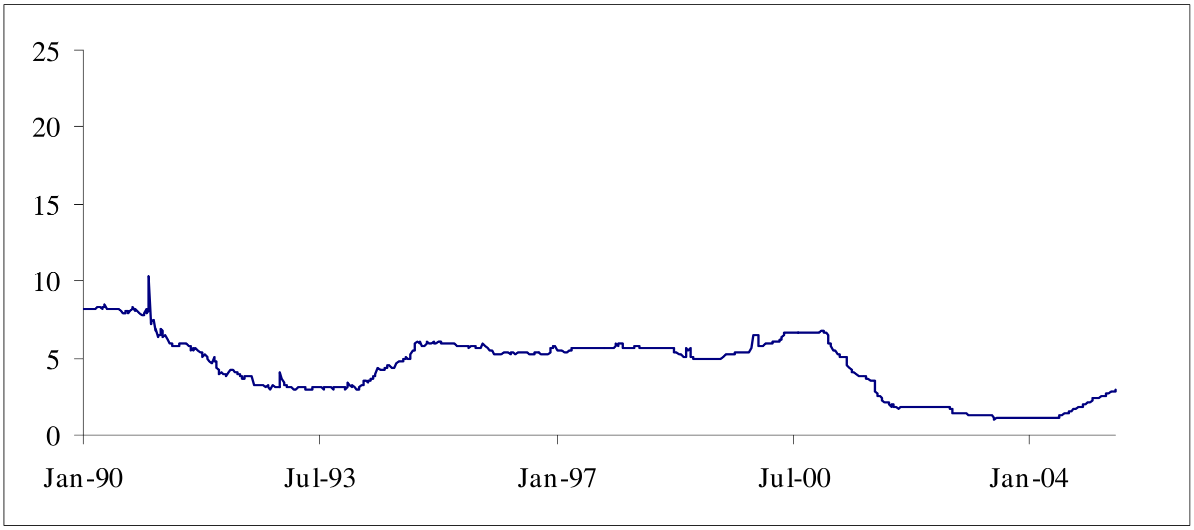
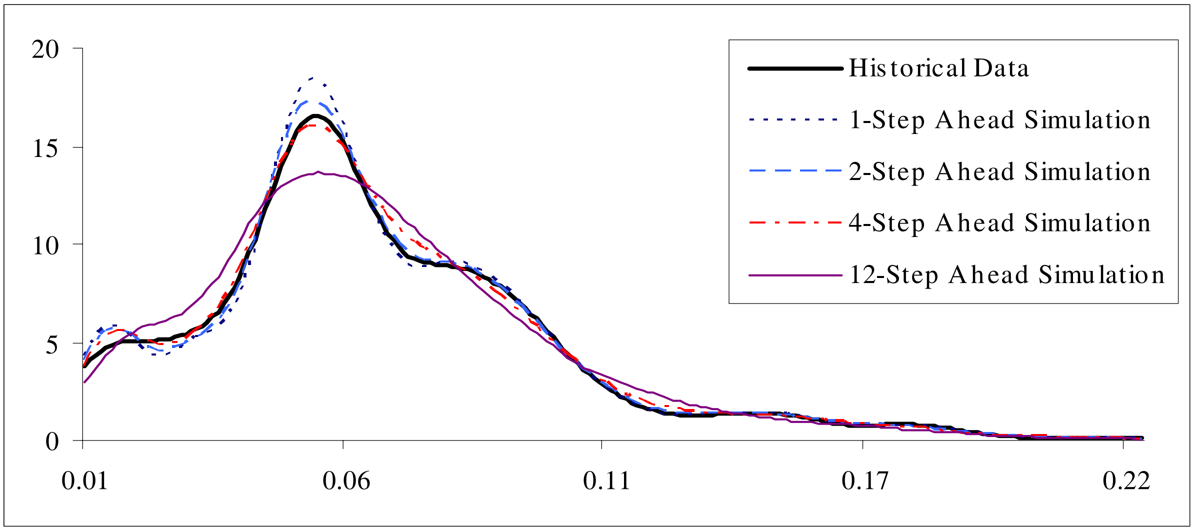


Figure 2: Historical and Simulated Empirical Densities
06-Jan-71 to 08-Apr-05

Panel 1: CIR Process



Panel 2: OU Process

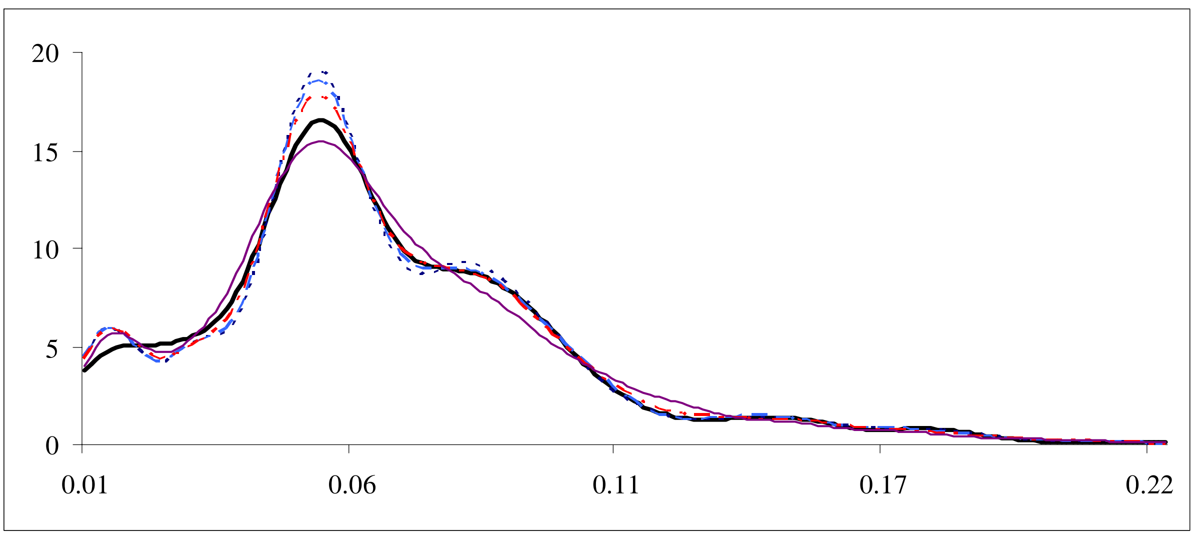
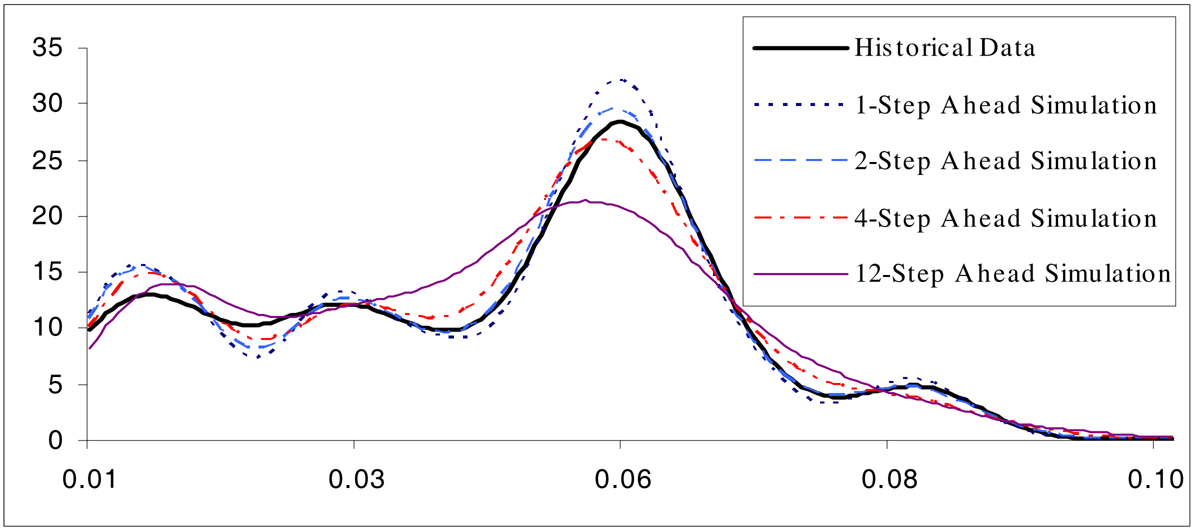


Figure 3: Historical and Simulated Empirical Densities
03-Jan-90 to 08-Apr-05

Panel 1: CIR Process



Panel 2: OU Process