Testing for Jumps and Jump Intensity Path Dependence*

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Abstract

In this paper, we develop a "jump test" for the null hypothesis that the probability of a jump is zero, building on earlier work by Aït-Sahalia (2002). The test is based on realized third moments, and uses observations over an increasing time span. The test offers an alternative to standard finite time span tests, and is designed to detect jumps in the data generating process rather than detecting realized jumps over a fixed time span. More specifically, we make two contributions. First, we introduce our largely model free jump test for the null hypothesis of zero jump intensity. Second, under the maintained assumption of strictly positive jump intensity, we introduce a "self-excitement test" for the null of constant jump intensity against the alternative of path dependent intensity. The latter test has power against autocorrelation in the jump component, and is a direct test for Hawkes diffusions (see, e.g. Aït-Sahalia, Cacho-Diaz and Laeven (2015)). The limiting distributions of the proposed statistics are analyzed via use of a double asymptotic scheme, wherein the time span goes to infinity and the discrete interval approaches zero; and the distributions of the tests are normal and half normal, respectively. The results from a Monte Carlo study indicate that the tests have reasonable finite sample properties. An empirical illustration is provided.

Keywords: diffusion model, jump intensity, jump size density, tricity.
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1 Introduction

Jump diffusions are widely used in the financial econometrics literature when analyzing returns or exchange rates, as discussed in Duffie, Pan and Singleton (2000), Singleton (2001), Anderson, Benzoni and Lund (2002), Jiang and Knight (2002), Chacko and Viceira (2003) and Eraker, Johannes and Polson (2003), among others. In this context, various estimation techniques have been developed, and the common practice is to jointly estimate the parameters of both the continuous time and the jump components of models. Thus, parameters characterizing the drift, variance, jump intensity, and jump size probability density are jointly estimated. However, an obvious non-standard feature of this class of models is that the parameters characterizing the jump size density are not identified when the jump intensity is identically zero. This is an issue both when the intensity parameter is constant, as in standard stochastic volatility models with jumps (see, e.g. Andersen, Benzoni and Lund (2002)) as well as when the intensity follows a diffusion process, as in the important case of the Hawkes diffusion models analyzed by Aït-Sahalia, Cacho-Diaz and Laeven (2015). If one estimates a jump diffusion model that contains a jump intensity parameter and if the population jump intensity happens to be zero, then a subset of the parameters in the model is not identified, which in turn precludes consistent estimation of other parameters (see Andrews and Cheng (2012)).

The above estimation problem serves to underscore the importance of pretesting for jumps. The first paper addressing the issue of discrimination between diffusion processes and jump processes was Aït-Sahalia (2002). He derived a set of necessary and sufficient conditions, based on the properties of the transition density, which have to be satisfied by any diffusion sampled at discrete times. Hence, he provided a criterion for checking whether there are jumps in the data generating process. Since then, there have been a large variety of tests for the null of no jumps versus the alternative of jumps. Tests include those based on the comparison of two realized volatility measures, one which is robust, and the other which is not robust to the presence of jumps (see, e.g. Barndorff-Nielsen, Shephard and Winkel (2006) and Podolskji and Vetter (2009a)), tests based on a thresholding approach (see, e.g. Corsi, Pirino, and Renò (2010), Lee and Mykland (2008), and Lee, Loretan and Ploberger (2013)), and tests based on power variation, as discussed in Aït-Sahalia and Jacod (2009). Such tests are consistent against realized jumps. One feature of these tests is that they are based on observations drawn on a given finite time span, and they can thus only detect whether jumps occurred during this given time span. While this is hardly a weakness of the existing tests, there are clearly situations for which interest lies in testing for the existence of jumps in the data generating process, or within a class of models. For example, this is the case if one is interested in using (transformations of) jump diffusion processes in a variety of valuation problems, such as option pricing and default modelling (see, e.g. Duffie, Pan and Singleton (2000)).

In this paper we make two contributions to the literature on jumps. First, we develop a "jump test" for the null hypothesis that the probability of a jump is zero, building on earlier work of Aït-Sahalia (2002). Second, under the maintained assumption of strictly positive jump intensity, we introduce a "self-excitement test" for the null of constant jump intensity against the alternative of path dependent intensity. This test has power against autocorrelation in the jump component, and is a direct test for Hawkes diffusions (see Aït-Sahalia, Cacho-Diaz and Laeven 2015), in which jump intensity is modeled as a mean-reverting diffusion process. When the proposed tests are implemented prior to model specification, standard estimation of jump diffusions can be subsequently carried out, avoiding the identification problems discussed above. Recently, Boswijk, Laeven and Yang (2017) and Dungey, Erdemlioglu, Matei and Yang (2017) have suggested tests for self-excitation and mutual excitation in realized jumps. Our tests instead detect jump self-excitation in the data generating process.

Our jump test is based on realized third moments, or so-called tricity. Various realized tricity-type statistics over a finite time span have already been examined in the literature in order to: detect realized jumps, as in Jacod (2012); study the contribution of realized skewness when predicting the cross-section of equity returns, as in Amaya, Christoffersen, Jacobs and Vasquez (2015); and to test for the endogeneity of sampling times, as in Li, Mykland, Renault, Zhang and Zheng (2014). What distinguishes our tricity-type test from these is that it is analyzed using both in-fill and long-span asymptotics. The use of long-span asymptotics ensures that the suggested statistic has power against jump intensity rather than against realized jumps. Importantly, our test is also robust to the presence of leverage. The limiting behavior of the proposed statistic is readily analyzed via use of a double asymptotic scheme wherein the time span goes to infinity and the discrete interval approaches zero. Under the null hypothesis of zero intensity, the statistic has a normal limiting distribution. Under the alternative, it is necessary to distinguish between jumps with zero or non-zero third moment. In the latter case, the proposed test has a well defined Pitman drift and has power against $\sqrt{T}-$local alternatives, where T is the time span, in days. In the former case, the sample third moment approaches zero, but the probability order of the statistic is larger than that which obtains under the null, since the jump component does not contribute to the mean, while it does contribute to the variance. As the order of magnitude of the variance depends on whether the null hypothesis is true or not, we introduce a threshold estimator for the variance, which is consistent under the null of zero intensity, and bounded in probability under the alternative. Thus, inference can be performed via use of a simple t-statistic.

We suggest two versions of our self-excitement test, one is based on the autocorrelation function of returns, the other on the autocorrelation of squared returns. The advantage of the latter over the former is that it does not require non-zero mean jump size.

In principle, one might consider testing for the null of zero intensity using a score, Wald or likelihood ratio test, based on discrete observations (see, e.g. Andrews (2001)). This approach requires treating jump size density parameters as nuisance parameters unidentified under the null, and requires correct specification of both the continuous and the jump components of the diffusion. Misspecification of one or both components will invalidate the test. Additionally, the likelihood function of a jump diffusion is not generally known in closed form, and therefore estimation (which is needed for test statistic construction) is usually based on either simulated GMM (see Duffie and Singleton (1993) and Anderson, Benzoni and Lund (2002)), indirect inference (see Gourieroux and Monfort (1993) and Gallant and Tauchen (1996)), or nonparametric simulated maximum likelihood (see Fermanian and Salanié (2004) and Corradi and Swanson (2011)). However, it goes without saying that one cannot simulate a diffusion with a negative intensity parameter. This, in turn, precludes the existence of a quadratic approximation around the null parameters of the criterion function to be maximized (minimized). Given that the existence of such quadratic approximations is a necessary condition for estimation and inference about parameters on the boundary (see Andrews (1999, 2001), Beg, Silvapulle and Silvapulle (2001), and Chapter 4 in Silvapulle and Sen (2005)), we cannot rely on simulation-based estimators when testing using standard score, Wald or likelihood ratio tests.

The finite sample behavior of the tests is studied in a series of Monte Carlo experiments. Since the tests are not robust to microstructure noise, one needs to choose a frequency for which the noise is not too binding. For this reason, in our Monte Carlo exercise, we set the discretization interval $\Delta = 1/78$ and $\Delta = 1/156$, corresponding to moderate frequencies. We also study test sensitivity to the presence of non-zero microstructure noise. The empirical size of the jump test is sensitive to the smallest values of T and Δ^{-1}, but performance is markedly better as the magnitude of these parameters is increased. Moreover, the power is quite good across all parameterizations, even in the case of jumps with zero third moment. We then assess and compare the finite sample properties of the two self-excitation tests. Overall, we find that the test based on returns behaves better than that based on squared returns, in the sense of suffering from less size distortion. As expected, both tests have good power as the level of path dependence increases.

We also discuss a small empirical illustration in which 11 stocks are examined. In the illustration, strong evidence of jumps is found, regardless of T, and moderate evidence of self-excitement is found.

The rest of the paper is organized as follows. Section 2 describes the set-up. Section 3 and Section 4 discuss the jump intensity and self-excitement tests, and derive their asymptotic properties, respectively. Section 5 reports the findings of a Monte Carlo study designed to examine the finite sample properties of the tests, Section 6 contains the results of an empirical illustration, and concluding remarks are gathered in Section 7. All proofs are collected in an Appendix.

2 Set-Up

We consider stochastic volatility jump diffusions, with either constant or path dependent intensity. For $t \in \mathbb{R}^+$, consider

$$\mathrm{d}\ln X_t = \mu \mathrm{d}t + V_t^{1/2}\sqrt{1-\rho^2}\mathrm{d}W_{1,t} + V_t^{1/2}\rho \mathrm{d}W_{2,t} + Z_t \mathrm{d}N_t, \tag{1}$$

and

$$\mathrm{d}V_t = \mu(V_t, \theta)\mathrm{d}t + g\left(V_t, \theta\right)\mathrm{d}W_{2,t}, \tag{2}$$

with $-1 \leq \rho \leq 1$. Here, $W_{1,t}$ and $W_{2,t}$ are independent standard Brownian motions. From (1) and (2), it is immediate to see that the specification of the volatility process is rather general, as the drift and variance terms in (2) need only ensure the existence of a strong solution, $V_t > 0$. For example, V_t can be generated by a square root process, which is the case considered in the Monte Carlo study.

In the sequel, we assume that the jump process, N_t, is a finite activity process. Namely, we focus on the case of a small number of large jumps. More precisely,

$$\Pr\left(N_{t+\Delta} - N_t = 1 | \mathcal{F}_t\right) = \lambda_t \Delta + o\left(\Delta\right), \tag{3}$$

$$\Pr\left(N_{t+\Delta} - N_t = 0 | \mathcal{F}_t\right) = 1 - \lambda_t \Delta + o\left(\Delta\right), \tag{4}$$

and

$$\Pr\left(N_{t+\Delta} - N_t > 1 | \mathcal{F}_t\right) = o\left(\Delta\right), \tag{5}$$

where $\mathcal{F}_t = \sigma\left(N_s,\ 0 \leq s \leq t\right)$. Additionally, in the sequel $1_{\Delta_{N_{(k+1)\Delta}}} = 1$ if a jump occurs between time $k\Delta$ and $(k+1)\Delta$, and the associated jump size, Z_k, is identically and independently distributed with density $f(z;\gamma)$.

We consider two general cases. The first is that of Poisson jumps, in which $\lambda_t = \lambda$, for all t. The second is that of a Hawkes diffusion, in which the intensity is an increasing function of past jumps (see Bowsher (2007) and Aït-Sahalia, Cacho-Diaz and Laeven (2015)). In this case,

$$\lambda_t = \lambda_\infty + \beta \int_0^t \exp\left(-a\left(t-s\right)\right)\mathrm{d}N_s,$$

with $\lambda_\infty \geq 0$, $\beta \geq 0$, $a > 0$, and $a > \beta$ (in order to ensure intensity mean reversion). Thus,

$$\mathrm{d}\lambda_t = a\left(\lambda_\infty - \lambda_t\right)\mathrm{d}t + \beta \mathrm{d}N_s \tag{6}$$

and

$$\mathrm{E}\left(\lambda_t\right) = \frac{a\lambda_\infty}{a-\beta} = \lambda.$$

If $\lambda_\infty = 0$, then $\mathrm{E}(\lambda_t) = 0$; and since λ_t can never be negative, this in turn implies that $\lambda_t = 0$ a.s., for all t (i.e., $N_t = 0$ a.s., for all t). But, if $N_t = 0$ a.s., for all t, then β cannot be identified, and consequently a is not identified. Furthermore, if $N_t = 0$ a.s., for all t, then γ cannot be identified.

In summary, if $\lambda_\infty = 0$, then β, α,and γ are not identified. By contrast, if $\lambda_\infty > 0$, then γ and β are identified. However, if $\lambda_\infty > 0$ but $\beta = 0$, then a is not identified.

These observations highlight the importance of being very clear as to which of the two assumptions, $\lambda_\infty = 0$ or $\lambda_\infty > 0$, is made for statistical inference in the foregoing Hawkes diffusion model. In practice, thus, we are concerned with the following jump test hypotheses: $H_0 : \lambda = 0$ versus $H_A : \lambda > 0$, where hereafter λ denotes expected intensity (i.e. $\lambda = \mathrm{E}(\lambda_t)$).[1] This is a nonstandard inference problem because, under H_0, some parameters are not identified and a parameter lies on the boundary of the null parameter space. Additionally, depending upon the outcome of tests of the above hypotheses, we are also interested in the following self-excitement test hypotheses: $H_0 : \beta = 0$ versus $H_A : \beta > 0$.

Another important class of jump diffusions is the affine jump diffusion, in which intensity is an affine function of a state variable (see, e.g. Duffie, Pan and Singleton (2000) and Singleton (2001)). For example, in our set-up one can define intensity to be an affine function of the volatility process,

$$\lambda_t = \lambda_0 + \lambda_1 V_t, \ \ \lambda_0, \lambda_1 \geq 0, \tag{7}$$

so that the probability of a jump is positively correlated with the volatility level. The main similarity between (6) and (7) is that both models generate clusters of jumps, inducing periods of positive correlation. The main difference between (6) and (7) is that there is no jump feedback in the latter model. Our self-excitation tests have power against positive correlation in (squared) returns.

If we are willing to parametrically specify the continuous and the jump components of the model, and most importantly if the transition density is known in closed form, then it is easy to construct a consistent test for jumps, based only on a long time span of discrete observations. In particular one can easily test $H_0 : \lambda = 0$ versus $H_A : \lambda > 0$. This fact can be illustrated by considering a score test. Suppose that the skeleton of the process, $\ln X_t$ in (1) is observed. Namely, $\ln X_1, \ln X_2, ..., \ln X_T$, is observed. Now, using the notation in (1)-(5), let $\delta = (\theta, \mu, \rho, \lambda, \gamma) = (\vartheta, \gamma)$. It follows immediately that, provided the transition density is known in closed form, the likelihood can be written as

$$l_T(\vartheta, \gamma) = \frac{1}{T} \sum_{t=1}^{T-1} l_t(\vartheta, \gamma) = \frac{1}{T} \sum_{t=1}^{T-1} \ln f_{t+1|t} \left(Y_{t+1} | Y_t, \vartheta, \gamma \right).$$

[1]Note that testing for $\lambda = 0$ (> 0) implies and is implied by $\lambda_\infty = 0$ (> 0). Also, if $\beta = 0$, $\lambda = \lambda_\infty$.

The score statistic for testing H_0 is thus[2]

$$K_T(\gamma) = \max\left\{0, \left(R\widehat{\mathcal{I}}_T(\gamma)^{-1}\widehat{V}_T(\gamma)\widehat{\mathcal{I}}_T(\gamma)^{-1}R'\right)^{-1/2} U_T(\gamma)\right\},$$

where R is a $1 \times p$ matrix, with p denoting the dimension of ϑ, and where

$$\begin{aligned} U_T(\gamma) &= \sqrt{T}\left(R\widehat{\mathcal{I}}_T(\gamma)^{-1}\nabla_\vartheta l_T\left(\widehat{\vartheta}_T,\gamma\right)\right), \\ \widehat{\mathcal{I}}_T(\gamma) &= \frac{1}{T}\sum_{t=1}^{T}\nabla_{\vartheta\vartheta}l_t\left(\widehat{\vartheta}_T,\gamma\right), \\ \widehat{\vartheta}_T &= \arg\max_{\vartheta} l_T(\vartheta,\gamma) \text{ s.t. } R\vartheta = \lambda_\infty = 0, \end{aligned} \tag{8}$$

and

$$\widehat{V}_T(\gamma) = \frac{1}{T}\sum_{j=-\tau_T}^{\tau_T}\sum_{t=\tau_T}^{T-\tau_T}\omega_j\nabla_\vartheta l_t\left(\widehat{\vartheta}_T,\gamma\right)\nabla_\vartheta l_{t+j}\left(\widehat{\vartheta}_T,\gamma\right)',\ \omega_j = 1 - \frac{j}{1+\tau_T}. \tag{9}$$

Now, given mild regularity assumptions controlling the smoothness of the likelihood, under the null of $\lambda = 0$,

$$\sup_{\gamma\in\Gamma} K_T(\gamma) \xrightarrow{d} \sup_{\gamma\in\Gamma}\max\left\{0, \left(R\mathcal{I}(\gamma)^{-1}V(\gamma)\mathcal{I}(\gamma)^{-1}R'\right)^{-1/2} G(\gamma)\right\},$$

where $\sup_{\gamma\in\Gamma}\left|\widehat{\mathcal{I}}_T(\gamma) - \mathcal{I}(\gamma)\right| = o_p(1)$, $\sup_{\gamma\in\Gamma}\left|\widehat{V}_T(\gamma) - V(\gamma)\right| = o_p(1)$, $\mathcal{I}$ is the Hessian and V the variance of the score, and $G(\cdot)$ is a Gaussian process with covariance kernel given by

$$C(\gamma_1,\gamma_2) = \begin{pmatrix} R\mathcal{I}(\gamma_1)^{-1}V(\gamma_1,\gamma_1)\mathcal{I}(\gamma_1)^{-1}R' & R\mathcal{I}(\gamma_1)^{-1}V(\gamma_1,\gamma_2)\mathcal{I}(\gamma_2)^{-1}R' \\ R\mathcal{I}(\gamma_2)^{-1}V(\gamma_1,\gamma_2)\mathcal{I}(\gamma_1)^{-1}R' & R\mathcal{I}(\gamma_2)^{-1}V(\gamma_2,\gamma_2)\mathcal{I}(\gamma_2)^{-1}R' \end{pmatrix},$$

where $V(\gamma_1,\gamma_2) = \mathrm{p}\lim_{\mathrm{T}\to\infty}\widehat{V}_T(\gamma_1,\gamma_2)$.

Note also that $\sup_{\gamma\in\Gamma} K_T(\gamma)$ diverges to infinity under the alternative. This test has power against $\sqrt{T}-$local alternatives. Additionally, the limiting behavior of the test depends on the quadratic approximation of the likelihood around $\lambda = 0$ (see Andrews (2001)). Hence, if the likelihood is known in closed form, and if both the continuous and the jump components of the model, including the density of the jumps size, are correctly specified, then inference can be easily carried out using this score test, or using analogous Wald or likelihood ratio tests. However, it is well known that for most empirically relevant models the likelihood is not known in closed form. In such cases,

[2] If λ is not scalar (for example, consider allowing for different up and down jump intensities, as in Chacko and Viceira (2003)), then the score statistic can be written as:

$$\begin{aligned} K_T(\gamma) &= U_T(\gamma)'\left(R\widehat{\mathcal{I}}_T(\gamma)^{-1}\widehat{V}_T(\gamma)\widehat{\mathcal{I}}_T(\gamma)^{-1}R'\right)^{-1}U_T(\gamma) \\ &\quad - \inf_{\lambda\geq 0}(U_T(\gamma)-\lambda)'\left(R\widehat{\mathcal{I}}_T(\gamma)^{-1}\widehat{V}_T(\gamma)\widehat{\mathcal{I}}_T(\gamma)^{-1}R'\right)^{-1}(U_T(\gamma)-\lambda) \end{aligned}$$

as discussed in the introduction, one often relies on simulation based estimation techniques such as simulated GMM, indirect inference, or nonparametric simulated maximum likelihood. However, as one cannot simulate observations with negative intensity, a quadratic approximation of the criterion function cannot be constructed, and these sorts of tests are not applicable. It is for this reason that we instead focus on simple moment based jump and self-excitement tests in the sequel.

3 Test of $\lambda = 0$ (Jump Test)

As mentioned in the introduction, tests based on high frequency observations over a finite time span are model free, but have power only against realized jumps, and thus cannot be consistent against the alternative $\lambda > 0$. On the other hand, tests based on discrete observations over a long time span are consistent against $\lambda > 0$, but require correct specification of both the continuous and jump components, as well as knowledge of the transition density. In order to have tests that are consistent against $\lambda > 0$, but are still to a large extent model free, we use functions of sample moments and rely on double in-fill and long-time span asymptotic approximations.

In the sequel, assume the existence of a sample of n^+ observations over an increasing time span T^+ and a shrinking discrete interval Δ, so that $n^+ = \frac{T^+}{\Delta}$, with $T^+ \to \infty$ and $\Delta \to 0$. Define $n = \frac{T}{\Delta} = n^+ - \frac{T^+ - T}{\Delta}$, with $T^+ > T$, and $T^+/T \to \infty$. We first test for zero jump intensity ($\lambda = 0$). The hypotheses of interest are

$$H_0^\lambda : \lambda = 0$$

versus

$$H_A^\lambda : \lambda > 0,$$

where

$$H_A^\lambda = H_A^{\lambda(1)} \cup H_A^{\lambda(2)} : \left(\lambda > 0 \text{ and } \mathrm{E}\left(Z_k^3\right) \neq 0\right) \cup \left(\lambda > 0 \text{ and } \mathrm{E}\left(Z_k^3\right) = 0\right).$$

Notice that the alternative hypothesis is the union of two different alternatives, depending on whether $\mathrm{E}\big(Z_k^3\big) \neq 0$ or $\mathrm{E}\big(Z_k^3\big) = 0$. Let

$$\begin{aligned}\widehat{\mu}_{3,T,\Delta} &= \frac{1}{T}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3 \\ &\quad - \frac{1}{T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\},\end{aligned} \tag{10}$$

with $T^+/T \to \infty$, $\tau(\Delta) \to 0$, and $\tau(\Delta)/\Delta^{1/2} \to \infty$, and define the statistic[3]

$$S_{T,\Delta} = \frac{T^{1/2}}{\Delta} \widehat{\mu}_{3,T,\Delta}. \tag{11}$$

The logic underlying the suggested statistic is the following. As outlined in the proof of Theorem 1,

$$\begin{aligned}
&\frac{1}{T}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3 \\
&= \mathrm{E}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right)^3 + \frac{3}{2}\Delta\rho^3 \mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta}, \theta\right)\right) + o_p\left(\frac{\Delta}{T^{1/2}}\right),
\end{aligned}$$

so that $\left|\frac{1}{\sqrt{T}\Delta}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3\right|$ may diverge to infinity in probability either because of the presence of jumps or because of the presence of leverage, or both. Hence, we need to correct for skewness due to the presence of leverage. This is the role played by the second term on the RHS of (10). In fact, regardless of the presence of jumps, provided that $\tau(\Delta) \to 0$ at an appropriate rate,

$$\begin{aligned}
&\frac{1}{T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau(\Delta)\right\} \\
&= \frac{3}{2}\Delta\rho^3 \mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta}, \theta\right)\right) + o_p\left(\frac{\Delta}{T^{+1/2}}\right).
\end{aligned}$$

The requirement of $T^+/T \to \infty$ ensures that the contribution of leverage estimation error is negligible.

Furthermore, as shown in a number of lemmata in the Appendix, the thresholding effect is asymptotically negligible under the null, and thus for $T = T^+$ the statistic is degenerate under H_0^{λ}. On the other hand, if $T^+/T \to 0$, the limiting behavior is driven by the threshold term, and the statistic lacks power against jumps.

With regard to the $\tau(\Delta)$ term in (10), note that thresholding is a well established technique for disentangling the jump component from the continuous component in various estimation and testing frameworks (see Mancini (2009) and Mancini and Renò (2011)). In these papers the threshold sequence, $\tau(\Delta)$, is selected so that $\tau(\Delta) \to 0$ and $\frac{\tau(\Delta)}{\sqrt{\Delta \log(1/\Delta)}} \to 0$, which is dictated by the law of the iterated logarithm of the Brownian component of the model. In the theorems below we require mildly stronger conditions on $\tau(\Delta)$ because the time span is growing.

Before establishing the asymptotic properties of $S_{T,\Delta}$, it should be pointed out that a central

[3] The first term on the RHS of $\widehat{\mu}_{3,T,\Delta}$ can also be expressed as $\frac{1}{T}\sum_{k=n^+-n+1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3$. However, it is necessary to use a longer sample size for the second term.

limit theorem for realized third moments has been proved in Li, Mykland, Renault, Zhang and Zheng (2014). Their Theorem 2 establishes asymptotic mixed normality for tricity in the case of unequal random times and finite time spans. Namely, they show that $\frac{1}{\Delta}\sum_{t_k\leq 1}\left(\ln X_{t_{k+1}}-\ln X_{t_k}\right)^3$ has a mixed normal limiting distribution under the null of exogenous sampling time, and diverges otherwise. Here, we have assumed equal spacing. We could allow for irregular, possibly random spacing, provided the latter is independent of the return process. In this case, we simply need to express our rate conditions in terms of the largest or smallest time interval, e.g. $\sqrt{T}\Delta \to 0$, should be replaced by $\sqrt{T}\max_{k\leq n}(t_k-t_{k-1}) \to 0$, and $T\Delta \to \infty$ by $T\min_{k\leq n}(t_k-t_{k-1}) \to \infty$. Heuristically, in this case we can condition on the observed sampling intervals, as they are independent of the return process. On the other hand, if the sampling interval is random and endogenous, then tricity does not converge to zero, even in the absence of leverage (see Li, Mykland, Renault, Zhang and Zheng (2014)). Here, the "endogeneity" bias is of smaller order than the jump contribution, so that power is not affected, but size is. Indeed, such randomness leads to severely oversized jump tests, resulting in spurious rejections.

In the sequel, we need the following assumption.

Assumption A: **(i)** $\ln X_t$ and V_t are generated as in (1) and (2), with $\mu(v,\theta)$ and $g(v,\theta)$ twice continuously differentiable, satisfying local Lipschitz and growth conditions, for all $\theta\in\Theta$, **(ii)** V_t is geometrically ergodic, **(iii)** $\mathrm{E}\left(V_t^{\frac{m}{2}}\right)<\infty$ and $\mathrm{E}\left(g(V_t)^2\right)<\infty$, for even integer $m>6$, (**iv**) N_t satisfies (3)-(5), and λ_t is either constant, or satisfies (6), with $\lambda_\infty\geq 0$, $\beta\geq 0$, $a>0$, and $a>\beta$, and (**v**) the jump size, Z_k, is independently and identically distributed, with density $f(z;\gamma)$, and $\mathrm{E}\left(|Z_k|^\kappa\right)<\infty$, for $\kappa\geq 6$.

Assumption A requires constant drift and one-factor stochastic volatility. However, as discussed in Remark 4 below, the assumption can be readily modified to allow for multi-factor stochastic volatility processes. On the other hand, as discussed in Remark 5 below, the constancy of the drift is important for the self-excitation test.

Theorem 1: *Let Assumption* **A** *hold. Also, assume that as* $T\to\infty$, $\Delta\to 0$, $T\Delta\to\infty$, $\sqrt{T}\Delta\to 0$, and $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)}\to 0$, *for even* $m>6$, *and* $T^+/T\to\infty$. *Then,*

(i) Under H_0^λ :

$$S_{T,\Delta}\xrightarrow{d}N\left(0,\omega_0\right),$$

where

$$\omega_0=\left(15\left(1-\rho^2\right)^3+15\rho^6+45\left(1-\rho^2\right)^2\rho^2+45\left(1-\rho^2\right)\rho^4\right)\mathrm{E}\left(V_{k\Delta}^3\right).$$

(ii) Under $H_A^{\lambda(1)}$, *there exists an* $\varepsilon>0$, *such that:*

$$\lim_{T\to\infty,\Delta\to 0}\Pr\left(\frac{\Delta}{\sqrt{T}}\left|S_{T,\Delta}\right|>\varepsilon\right)=1.$$

(iii) Under $H_A^{\lambda(2)}$, there exists an $\varepsilon > 0$, such that:

$$\lim_{T\to\infty,\Delta\to 0} \Pr\left(\Delta\left|S_{T,\Delta}\right| > \varepsilon\right) = 1.$$

It follows immediately that $S_{T,\Delta}$ converges to a normal random variable under the null hypothesis, diverges at rate $\frac{\sqrt{T}}{\Delta}$ under the alternative of jumps with non-zero third moment, and diverges at the slower rate of $\frac{1}{\Delta}$ under the alternative of jumps that are symmetric around zero. As shown in the Appendix (see equation (25)), as $\Delta \to 0$ and $T \to \infty$, we have that $\widehat{\mu}_{3,T,\Delta} = \lambda\mathrm{E}\left(Z^3\right) + 2\lambda^2\Delta\mathrm{E}\left(Z\right)\mathrm{E}\left(Z^2\right) + o_p\left(\Delta\right)$. Now, if $\mathrm{E}\left(Z_k^3\right) \neq 0$, (i.e., under $H_A^{\lambda(1)}$), the test has a well defined Pitman drift against $\sqrt{T}-$alternatives. On the other hand, if jumps are symmetric around zero (i.e., $\mathrm{E}\left(Z_k^3\right) = \mathrm{E}\left(Z_k\right) = 0$), then $\lambda\mathrm{E}\left(Z_k^3\right)$ is not identified, and under $H_A^{\lambda(2)}$ the Pitman drift is zero. Indeed, $\widehat{\mu}_{3,T,\Delta} \xrightarrow{p} 0$ regardless of whether $\lambda = 0$ or $\lambda > 0$ in this case. Although it is not possible to distinguish between H_0^{λ} and $H_A^{\lambda(2)}$ on the basis of the different locations of the limiting distribution (i.e., the Pitman drift), it is possible to distinguish between them on the basis of different scales of the limiting distribution of $\frac{T^{1/2}}{\Delta}\widehat{\mu}_{3,T,\Delta}$. This is because the order of magnitude of the variance of $\frac{T^{1/2}}{\Delta}\widehat{\mu}_{3,T,\Delta}$ is larger when $\lambda > 0$ and $\mathrm{E}\left(Z_k^3\right) = \mathrm{E}\left(Z_k\right) = 0$ than when $\lambda = 0$. Broadly speaking, under H_0^{λ}, $S_{T,\Delta} \xrightarrow{d} N\left(0,\omega_0\right)$, while under $H_A^{\lambda(2)}$, $\Delta S_{T,\Delta} \xrightarrow{d} N\left(0,\omega_1\right)$, with $\omega_1 \neq \omega_0$. This is what allows one to distinguish between H_0^{λ} and $H_A^{\lambda(2)}$.

The test has power not only against constant and self-exciting intensity, but also against affine jump diffusions where the intensity is an affine function of volatility, for example.

As the variance of the statistic is of larger order under the alternative of positive jump intensity, we cannot construct a variance estimator which is consistent under all hypotheses. Thus, our aim is to construct an estimator for the variance of $S_{T,\Delta}$ which is consistent under the null and bounded in probability under the (union of) alternatives. This is done by using a threshold variance estimator, which filters out the contribution of the jump component. In particular, define:

$$\begin{aligned}
&\widehat{\sigma}^2_{\lambda,T,\Delta} \\
&= \frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^6 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\}. \quad (12)
\end{aligned}$$

It follows that the t-statistic version of the jump test is,

$$t_{\lambda,T,\Delta} = \frac{S_{T,\Delta}}{\widehat{\sigma}_{\lambda,T,\Delta}}.$$

The following corollary summarizes the limiting behavior of $t_{\lambda,T,\Delta}$.

Corollary 2: *Let Assumption* ***A*** *hold. Also, assume that as $T \to \infty$, $\Delta \to 0$, $T\Delta \to \infty$, $\sqrt{T}\Delta \to 0$, $T^+/T \to \infty$, $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)} \to 0$, for even $m > 6$, and $\tau^7\left(\Delta\right)\Delta^{-2} \to 0$. Then,* *(i) Under H_0^{λ} :*

$$t_{\lambda,T,\Delta} \xrightarrow{d} N(0,1).$$

(ii) Under $H_A^{\lambda(1)}$, there exists an $\varepsilon > 0$, such that:

$$\lim_{T\to\infty,\Delta\to 0} \Pr\left(\frac{\Delta}{\sqrt{T}}\left|t_{\lambda,T,\Delta}\right| > \varepsilon\right) = 1.$$

(iii) Under $H_A^{\lambda(2)}$ there exists an $\varepsilon > 0$, such that:

$$\lim_{T\to\infty,\Delta\to 0} \Pr\left(\Delta\left|t_{\lambda,T,\Delta}\right| > \varepsilon\right) = 1.$$

Note that the variance estimator in (12) can be constructed using the entire time span of T^+ observations, and the statement in Corollary 2 still holds, provided that we replace $\sqrt{T}\Delta \to 0$ and $\sqrt{T}\tau^2(\Delta) \to 0$ with $\sqrt{T^+}\Delta \to 0$ and $\sqrt{T^+}\tau^2(\Delta) \to 0$. In general, the price of having a statistic which allows for possible leverage effects is that we need to use a longer time span for estimating the leverage contribution. A possible alternative approach would be to pretest for $H_0^{\rho}: \rho = 0$ vs. $H_A^{\rho}: \rho \neq 0$. Here, under H_0^{ρ},

$$\frac{\Delta}{\sqrt{T^+}}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau(\Delta)\right\}$$

is asymptotically normal, while it diverges to minus infinity under H_A^{ρ}. This means that if we do not reject the null of no leverage, we do not need to recenter the statistic in (10) and we can use all T^+ observations for tricity estimation.

Remark 1: Consider selection of the threshold sequence. From Corollary 2, it follows that $\tau(\Delta)$ should approach zero faster than $T^{-1/4}$ and faster than $\Delta^{2/7}$, but slower than $\Delta^{\frac{1}{2}-\frac{3}{m}}$, where m (even) denotes the number of finite moments of $V_t^{1/2}$, with $m > 6$ by Assumption A(iii). For example, if V_t follows a square root process, so that all finite moments exist, we can set $\tau(\Delta) = c\Delta^{\eta}$, with $\frac{2}{7} < \eta < \frac{1}{2}$ and $\Delta = T^{-\delta}$, where $\frac{7}{8} < \delta < 1$. These conditions ensure that $T\Delta \to \infty$, $\sqrt{T}\Delta \to 0$, and $\sqrt{T}\tau^2(\Delta) \to 0$. Additionally, in order to implement the statistic, we choose the constant c in a data driven manner. A natural solution is to use $\widehat{\sigma}_{\mu_Z}$ as defined in (15) below.

Remark 2: Since the suggested statistic is not robust to the presence of microstructure noise, the optimal discrete interval, Δ, is the highest frequency at which microstructure noise doesn't bind. Visual inspection of the signature plots of Andersen, Bollerslev and Diebold (2000) provides a useful tool for the choice of interval. It should also be noted that the statistic is constructed over an increasing time span; and hence it is not straightforward to ascertain whether simple pre-averaging will yield a statistic that is robust to microstructure noise (as in the case of the realized pre-average power variation discussed in Podolskji and Vetter (2009b)). According to our simulation results, empirical size is quite robust to the presence of microstructure noise. More specifically, the test is only slightly oversized under various noise scenarios (see Table 1, Case 3).

Remark 3: In this paper, we only derive tests for the null of zero jump intensity in asset returns. However, the same approach can be used for testing equivalent hypotheses for volatility. Such tests would require estimators of the spot volatility, say $V^2_{k\Delta}$, which can be constructed using a finer grid of observations than that used in the above tests, such as when there are M observations over each interval of order Δ. The order of magnitude of the error due to the estimation of the spot volatility is derived in Bandi and Renò (2012), under various settings.

Remark 4: We can allow for two-factor or general multifactor stochastic volatility. Consider the following two-factor stochastic volatility model,

$$\mathrm{d}\ln X_t = \mu \mathrm{d}t + V_{1,t}^{1/2}\sqrt{1-\rho_1^2}\mathrm{d}W_{1,t} + V_{1,t}^{1/2}\rho_1 \mathrm{d}W_{2,t} + V_{2,t}^{1/2}\sqrt{1-\rho_2^2}\mathrm{d}W_{3,t} + V_{2,t}^{1/2}\rho_2 \mathrm{d}W_{4,t} + Z_t \mathrm{d}N_t,$$

where $W_{1,t}, W_{2,t}, W_{3,t}$,and $W_{4,t}$ are independent Brownian motions. Also,

$$\mathrm{d}V_{1,t} = \mu_1(V_{1,t}, \theta_1)\mathrm{d}t + g_1\left(V_{1,t}, \theta_1\right)\mathrm{d}W_{2,t}$$

and

$$\mathrm{d}V_{2,t} = \mu_2(V_{2,t}, \theta_2)\mathrm{d}t + g_2\left(V_{2,t}, \theta_2\right)\mathrm{d}W_{4,t}.$$

For this three dimensional diffusion, the multivariate Milstein approximation in Eqs. (23)-(24) in the Appendix is no longer valid, and the leverage contribution to our statistic is different, as it depends on both ρ_1 and ρ_2. Nevertheless, the same arguments used in the proof of Theorem 1 can be extended to this two-factor volatility case, at the sole cost of further notational complication. In particular, the asymptotic normality result in Theorem 1 holds, but the statement of the theorem will be different, as the variance will depend on both ρ_1 and ρ_2. Importantly, the statements in Corollary 2 also hold, as the variance estimator is consistent, regardless the number of factors.

Remark 5: Assumption A(i) requires constant drift. However, in our Monte Carlo experiments, we consider the case in which $\mu_t = \mu - V_t/2$, where V_t is the variance. This drift specification is quite popular in empirical work. For our jump intensity test based on tricity, this results in an extra term, $\frac{1}{\sqrt{T}\Delta}\sum_{k=1}^{n}\left(V_{(k-1)\Delta}\Delta - \frac{1}{n}\sum_{k=1}^{n} V_{(k-1)\Delta}\Delta\right)^3$, as well as additional related cross product terms. This additional generality does not affect the limiting distribution in Theorem 1, however. This is confirmed in our Monte Carlo findings, which suggest that the size of the jump tests is not affected by replacing the constant drift with μ_t (see Table 1, Case 2).

4 Testing for Self-Exciting Jumps

If the null hypothesis of zero jump intensity is rejected, one can proceed to test the null of no self-excitation or path dependence. If the drift in (1) and the intensity are constant, and if the jump size is independently distributed, then $\left(\ln X_{(k+1)\Delta} - \ln X_{k\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)$ is a martingale difference process and so it is uncorrelated over time. If λ_t is instead generated as in (6), then it follows from Aït-Sahalia, Cacho-Diaz and Laeven (2015), that

$$\begin{aligned} &\mathrm{E}\left(\left(\ln X_{(k+\tau)\Delta} - \ln X_{(k+\tau-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)\left(X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)\right) \\ &= \frac{\beta\lambda_{\infty}(2a-\beta)}{2(a-\beta)} \exp\left(-(a-\beta)\tau\right)\left(\mathrm{E}(Z)\right)^2 \Delta^2 + o\left(\Delta^2\right). \end{aligned} \tag{13}$$

In this case, it is natural to test the null hypothesis of $\beta = 0$, against the alternative that $\beta > 0$, in order to test for path dependence. However, it is immediate to see that in order to identify β, we require not only $\lambda_{\infty} > 0$, but also $\mathrm{E}(Z_k) \neq 0$. In fact, failure to reject the null may be simply due to the fact that $\mathrm{E}(Z_k) = 0$. Hence, before testing for jump self-excitation, it remains to pretest for the null of $\mathrm{E}(Z_k) = 0$ versus $\mathrm{E}(Z_k) \neq 0$.

4.1 Test of $(Z_k) = 0$ (Zero Mean Jump Test)

We test the null of zero mean jumps, against its negation. The hypotheses of interest are:

$$H_0^{\mu_Z} : \mathrm{E}(Z_k) = 0$$

and

$$H_A^{\mu_Z} : \mathrm{E}(Z_k) \neq 0.$$

Let

$$\begin{aligned} \widehat{\mu}_{T,\Delta}^{Z} &= \frac{1}{T}\sum_{k=0}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right) \\ &\quad - \frac{1}{T^{+}}\sum_{k=1}^{n^{+}}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right) 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau(\Delta)\right\} \end{aligned} \tag{14}$$

and

$$\widehat{\sigma}_{\mu_Z}^2 = \frac{1}{T}\sum_{k=0}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^2; \tag{15}$$

and define

$$t_{\mu_Z,T,\Delta} = \sqrt{T}\frac{\widehat{\mu}_{T,\Delta}^{Z}}{\widehat{\sigma}_{\mu_Z}}. \tag{16}$$

Since $\mathrm{E}\left(\frac{1}{T}\sum_{k=0}^{n}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}\right)\right)=\mu+\lambda\mathrm{E}\left(Z_k\right)+O\left(\Delta\right)$, from the first term on the RHS of (14), we cannot disentangle the contribution to the mean of the continuous and jump components. However, due to the thresholding, the second term on the RHS of (14) provides a $\sqrt{T^+}-$consistent estimator of the drift, μ; and as $T^+/T\to\infty$, estimation error is negligible. As in the case of Theorem 1, if $T=T^+$, the statistic is degenerate under the null. Note also that the first term on the RHS of (14) is not recentered using $\frac{\ln X_{n\Delta}-\ln X_{\Delta}}{n}$. This is because otherwise we would recenter both the continuous as well as the jump components, and the statistic would not have power against non-zero jump mean.

Theorem 3: *Let Assumption **A** hold. Also, assume that as* $T\to\infty$, $\Delta\to 0$, $T\Delta\to\infty$, $T^+/T\to\infty$, $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)}\to 0$, *for even* $m>6$, *and* $\sqrt{T}\tau^2\left(\Delta\right)\to 0$.[4] *Then,*
(i) Under $H_0^{\mu_Z}$:

$$t_{\mu_Z,T,\Delta}\xrightarrow{d}N\left(0,1\right).$$

(ii) Under $H_A^{\mu_Z}$, *there exists an* $\varepsilon>0$, *such that:*

$$\lim_{T\to\infty,\Delta\to 0}\Pr\left(\left|\frac{t_{\mu_Z,T,\Delta}}{\sqrt{T}}\right|>\varepsilon\right)=1,$$

where $\hat{\mu}^Z_{T,\Delta},\hat{\sigma}_{\mu_Z}$, and $t_{\mu_Z,T,\Delta}$ are defined as in (14), (15), and (16), respectively.
Note that $\widehat{\sigma}^2_{\mu_Z}$ in (15) can be constructed using the entire time span, T^+, provided that we replace $\sqrt{T}\Delta\to 0$ and $\sqrt{T}\tau^2\left(\Delta\right)\to 0$ with $\sqrt{T^+}\Delta\to 0$ and $\sqrt{T^+}\tau^2\left(\Delta\right)\to 0$, respectively.

4.2 Test of $\beta=0$ (self-excitement Test)

Note that for this test, we can use the entire time span, T^+, as leverage plays no role in autocorrelation calculations. Our objective is to test the following hypotheses:

$$H_0^{\beta}:\beta=0$$

and

$$H_A^{\beta}:\beta>0,$$

under the maintained assumption that $\mathrm{E}\left(Z_k\right)\neq 0$. Define the statistic:

$$S^{\beta}_{T^+,\Delta}=\max\left\{0,t_{\beta,T^+,\Delta}\right\},$$

where

$$t_{\beta,T^+,\Delta}=\frac{\sqrt{\frac{T^+}{\Delta}}\widehat{\beta}_{T^+,\Delta}}{\widehat{\sigma}_{\beta,T^+,\Delta}},\tag{17}$$

[4]Note that $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)}\to 0$, for even $m>6$, and $\sqrt{T}\tau^2\left(\Delta\right)\to 0$ imply $\sqrt{T}\Delta\to 0$.

with

$$\widehat{\beta}_{T^+,\Delta} = \frac{1}{T^+}\sum_{k=2}^{n^+-1}\left(\ln X_{(k+1)\Delta} - \ln X_{k\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right) \tag{18}$$

and

$$\widehat{\sigma}^2_{\beta,T^+,\Delta} = \frac{1}{T^+\Delta}\sum_{k=2}^{n^+-1}\left(\ln X_{(k+1)\Delta} - \ln X_{k\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^2\left(\ln X_{(k+1)\Delta} - \ln X_{k\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^2. \tag{19}$$

From (13), and recalling that $a > 0$, $\beta \geq 0$, and $a > \beta$, it follows immediately that the autocorrelation can never be negative. This is why the test is one-sided. Additionally, recall that $T\Delta \to \infty$ implies that $T^+\Delta \to \infty$. The following result thus holds.

Theorem 4: *Let Assumption* ***A*** *hold. Also, assume that* $E(Z) \neq 0$, $\lambda_\infty > 0$, *and as* $n \to \infty$, $T \to \infty$, $\Delta \to 0$ *and* $T\Delta \to \infty$. *Then,*
(i) Under H_0 :

$$S^{\beta}_{T^+,\Delta} \xrightarrow{d} \max\{0, \mathcal{Z}\},$$

where $\mathcal{Z}$ *is a standard normal random variable.*
(ii) Under H_A, *there exists an* $\varepsilon > 0$ *such that:*

$$\lim_{T^+\to\infty,\Delta\to 0} \Pr\left(\frac{1}{\sqrt{T^+\Delta}} S^{\beta}_{T^+,\Delta} > \varepsilon\right) = 1.$$

It follows that $S^{\beta}_{T^+,\Delta}$ converges to an half-normal random variable under the null, and diverges at rate $\sqrt{T^+\Delta}$ under the alternative.

An obvious limitation of the test outlined above is the requirement of asymmetric jumps. This can be overcome by considering a test based on the autocorrelation of squared returns.[5] Consider

$$\widetilde{S}^{\beta}_{T^+,\Delta} = \max\left\{0, \widetilde{t}_{\beta,T^+,\Delta}\right\}, \tag{20}$$

where

$$\widetilde{t}_{\beta,T^+,\Delta} = \frac{\sqrt{\frac{T^+}{\Delta}}\widetilde{\beta}_{T^+,\Delta}}{\widetilde{\sigma}_{\beta,T^+,\Delta}}, \tag{21}$$

[5] We thank an anonymous referee for pointing out that by looking at squared returns we no longer require pretesting for (non)-zero jump mean.

with

$$\widetilde{\beta}_{T^+,\Delta} = \frac{1}{T^+\Delta}\sum_{k=2}^{n^+-1}\left(\left(\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\right)^2-\frac{1}{n^+}\sum_{k=2}^{n^+-1}\left(\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\right)^2\right)$$
$$\times\left(\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}\right)^2-\frac{1}{n^+}\sum_{k=2}^{n^+-1}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}\right)^2\right)$$

$$\widetilde{\sigma}^2_{\beta,T^+,\Delta}$$
$$=\frac{1}{T^+\Delta^2}\sum_{k=2}^{n^+-1}\left(\left(\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\right)^2-\frac{1}{n^+}\sum_{k=2}^{n^+-1}\left(\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\right)^2\right)^2$$
$$\times\left(\left(\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\right)^2-\frac{1}{n^+}\sum_{k=2}^{n^+-1}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}\right)^2\right)^2.$$

Heuristically, both $S^{\beta}_{T^+,\Delta}$ and $\widetilde{S}^{\beta}_{T^+,\Delta}$ have the same limiting distribution, under the null. Under the alternative,

$$\mathrm{p}\lim{}_{T,\Delta^{-1}\to\infty}\left(\frac{\widehat{\beta}_{T^+,\Delta}}{\Delta}-\frac{\widetilde{\beta}_{T^+,\Delta}}{\Delta}\right)=\frac{\beta\lambda\left(2a-\beta\right)}{2\left(a-\beta\right)}\exp\left(-\left(a-\beta\right)\right)\left(\mathrm{E}\left(Z_k\right)^2-\mathrm{E}\left(Z_k^2\right)^2\right).$$

In our Monte Carlo experiments, we compare the two alternative tests in various scenarios. The test based on squared returns tends to be rather oversized in presence of exponentially distributed jumps with large mean. One reason may be that in this case the sample mean does not accurately mimic the “true” jump mean, thus leading to spurious rejections. Interestingly, in our empirical illustration we find substantial evidence of jump excitation using the squared return test and very little self excitation using the return test. This is explained by the fact that most stocks display jumps with mean close to zero, for the time spans that we consider. We conclude this section with four remarks.

Remark 6: We have considered test statistics which are only a function of the first autocovariance term. It follows immediately that one can construct tests based on an increasing number of autocovariance terms, with the number chosen adaptively (see, e.g. Escanciano and Lobato (2009)).

Remark 7: If the nulls of zero intensity, zero jump mean and no self-excitation are all rejected, then one can proceed to estimate the full Hawkes diffusion using GMM, as in Aït-Sahalia, Cacho-Diaz and Laeven (2015).

Remark 8: It is immediate to see that the test based on returns is not robust to the presence of a mean-reverting drift. This is because the autocorrelation in the drift component cannot be disentangled from the autocorrelation in the jump component. On the other hand, in the case of the self-excitation tests based on squared returns, the autocorrelation in the drift component is of smaller order than the autocorrelation in the jump component. Nevertheless, our simulation results show that the empirical size of this test varies substantially across different values of T and Δ. For example, when T is high relative to Δ^{-1}, empirical size is close to power.[6]

Remark 9: In this section, we consider self-exciting intensity. However, from an empirical point of view, an interesting case is that of financial contagion, where the contagion is due to "common" jumps. In this case, the jump intensity is an increasing function not only of its own past jumps but also of past jumps in other assets. In order to test for (no) cross-excitation, it suffices to construct a statistic based on cross correlations instead of autocorrelations (see Theorem 4 in Aït-Sahalia, Cacho-Diaz and Laeven (2015)). For example, let:

$$\widehat{\beta}^{(I,II)}_{T^+,\Delta} = \frac{1}{T^+}\sum_{k=2}^{n^+-1}\left(\ln X^{(I)}_{(k+1)\Delta} - \ln X^{(I)}_{k\Delta} - \frac{\ln X^{(I)}_{n^+\Delta} - \ln X^{(I)}_{\Delta}}{n^+}\right)\left(\ln X^{(II)}_{k\Delta} - \ln X^{(II)}_{(k-1)\Delta} - \frac{\ln X^{(II)}_{n^+\Delta} - \ln X^{(II)}_{\Delta}}{n^+}\right),$$

and note that if the jump intensity in asset II does not depend on past jumps in asset I, then $\widehat{\beta}^{(I,II)}_{T,\Delta} \xrightarrow{p} 0$. On the other hand, if the intensity in asset II increases when there is a jump in asset I, then $\left|\widehat{\beta}^{(I,II)}_{T^+,\Delta}\right|$ has a strictly positive probability limit.

5 Monte Carlo Findings

In this section we present the findings of a series of Monte Carlo experiments designed to evaluate the finite sample properties of: (i) the "jump test" for the null of zero jump intensity, based on $t_{\lambda,T,\Delta} = \frac{S_{T,\Delta}}{\widehat{\sigma}_{\lambda,T,\Delta}}$, where $S_{T,\Delta} = \frac{T^{1/2}}{\Delta}\widehat{\mu}_{3,T,\Delta}$, $\widehat{\mu}_{3,T,\Delta}$ is defined in (10), and $\widehat{\sigma}^2_{\lambda,T,\Delta}$ is defined in (12); (ii) the "zero mean jump test" for the null of zero mean jumps, based on $t_{\mu_Z,T,\Delta} = \sqrt{T}\frac{\widehat{\mu}^Z_{T,\Delta}}{\widehat{\sigma}_{\mu_Z}}$, where $\widehat{\mu}^Z_{T,\Delta}$ is defined in (14), and $\widehat{\sigma}^2_{\mu_Z}$ is defined in (15); and (iii) the "self-excitement test" for the null of no jump-path dependence, based on $S^{\beta}_{T^+,\Delta} = \max\left\{0, t_{\beta,T^+,\Delta}\right\}$, where $t_{\beta,T^+,\Delta} = \frac{\sqrt{\frac{T^+}{\Delta}}\widehat{\beta}_{T^+,\Delta}}{\widehat{\sigma}_{\beta,T^+,\Delta}}$, $\widehat{\beta}_{T^+,\Delta}$ is defined in (18), and $\widehat{\sigma}^2_{\beta,T^+,\Delta}$ is defined in (19).

Data used in our experiments are generated according to the following data generating processe (DGP):

$$d\ln X_t = \mu dt + \sqrt{V_t}dW_{1,t} + Z_t dN_t,$$

where volatility is modeled as a square-root process:

$$dV_t = \kappa_v(\theta_v - V_t)dt + \zeta\sqrt{V_t}dW_{2,t},$$

[6] The presence of a constant drift is also assumed in Ait-Sahalia, Cacho-Chavez and Laeven (2015).

with $\mathrm{E}(W_{1,t}W_{2,t}) = \rho$. We set $\mu = 0.1$, $\rho = \{0, -0.25, -0.50, -0.75\}$, $\kappa_v = 5$, $\theta_v = 0.16$, and $\zeta = \{0.25, 0.50\}$. Additionally, N_t satisfies the conditions in (3)-(5). The jump size, Z_k, is identically and independently distributed with density $f(z;\gamma)$. We consider three jump densities: $f(z;\gamma) = N(0.0, \sigma)$, $f(z;\gamma) = N(0.5, \sigma)$, and $f(z;\gamma) = \varsigma e^{-\varsigma z}$. For the cases where Z_k is a normal random variable, $\sigma = \{0.1, 0.2, 0.3, 0.4, 0.7\}$; and for the case where Z_t is characterized by the exponential density, $\varsigma = \{2, 2.5, 5, 10, 20\}$.

The jump intensity evolves according to:

$$\lambda_t = \lambda_\infty + \beta \int_0^t \exp\left(-a\left(t-s\right)\right) \mathrm{d}N_s, \tag{22}$$

where $\lambda_\infty = \{0.3, 0.5, 0.7, 0.9\}$ and $(a, \beta) = \{(0,0), (3,2),\ (5,4),\ (7,5)\}$. Note that the case where $(a, \beta) = (0,0)$ is consistent with both the case of no jumps (i.e., $\lambda_\infty = 0$) and with the case of constant jump intensity (i.e., $\lambda_t = \lambda_\infty > 0$, for all t). In the constant jump intensity case, we consider Poisson jumps, with parameter λ_∞.

The various above parameterizations were found to adequately describe the manner in which finite sample test performance evolves by model specification; and, importantly, to fully characterize the regions of the parameter space where the tests "break down", in the sense of deteriorating finite sample test performance. However, all of these parameterizations satisfy the assumptions used in our asymptotic analysis of the tests. In order to examine test performance in settings not allowable under our assumptions, we also considered two "misspecification" cases. In the first case, we impose mean reversion in the above pricing equation. Namely, we set

$$\mathrm{d}\ln X_t = \left(\mu - V_t/2\right) \mathrm{d}t + \sqrt{V_t} \mathrm{d}W_{1,t} + Z_t \mathrm{d}N_t.$$

In the second case, we introduce market frictions by assuming that the log price process is given by:

$$Y_{t+l/M} = X_{t+j/M} + \epsilon_{t+j/M},\ t = 0, ..., T \text{ and } j = 1, ..., M,$$

with $\epsilon_{t+j/M} \sim N(0.0, \omega^2)$, where $\omega = \{0.005, 0.007, 0.014\}$.

We simulate observations using a Milstein discretization scheme, with discrete interval $h = 1/312$, and consider two intra-daily sampling frequencies: $\Delta = 1/78$ and $\Delta = 1/156$. In an empirical context, these values are consistent with 5-minute and 2.5-minute sampling frequencies, assuming a 6.5 hour trading day (see e.g., Aït-Sahalia and Jacod (2009)). Loosely speaking, we view our values of Δ as associated with noise which is either not binding or moderately binding. Recall also that for the $t_{\lambda,T,\Delta}$ and $t_{\mu_Z,T,\Delta}$ tests, a key assumption is that $T\Delta \to \infty$ and $\sqrt{T}\Delta \to 0$, leading to a restriction that $1/\Delta < T < 1/\Delta^2$. In particular, when $\Delta = 1/78$ we set $T = \{60, 70, 80, 90, 100, 110, 120, 130\}$ and when $\Delta = 1/156$ we set $T = \{160, 180, 200, 220, 240, 260, 280, 300\}$. Notice that all values of T satisfy the condition, with the exception of $T = 60$ and $T = 70$. These sample sizes are included in order to provide some evidence on the performance of the tests when the condition is broken. In all experiments, we perform 1000 Monte Carlo replications.

When implementing the jump test, $t_{\lambda,T,\Delta}$, in order to disentangle the contribution of jumps from that of leverage, we also need to select the thresholding sequence $\tau(\Delta)$ and the "longer" time span T^+. We set $T^+ = 10T$, to satisfy the requirement that $T^+/T \to \infty$, which ensures that the contribution of leverage estimation error is negligible. Clearly, finite sample performance of our test may hinge to some degree on the ration of T to T^+. We leave this issue to future research. We then set $\tau(\Delta) = c\Delta^\eta$, with $\frac{2}{7} < \eta < \frac{1}{2}$, and given that for most choices of T, $\Delta = T^{-\delta}$, $\frac{7}{8} < \delta < 1$, we have $\tau(\Delta) = cT^{-\delta\eta}$, and for $\delta\eta \in (\frac{1}{4}, \frac{1}{2})$, the rate conditions $\frac{\Delta^{\frac{1}{2}}}{\tau(\Delta)} \to 0$, $\sqrt{\Delta}\tau(\Delta)^{-1}$ and $\sqrt{T}\tau^2(\Delta) \to 0$ are satisfied.[7] Finally, we set $c = \widehat{\sigma}_{\mu_Z}$, where $\widehat{\sigma}^2_{\mu_Z}$ is defined in (15). The choice of $\delta\eta$ is also important in finite sample applications. Consider the case where there are no jumps, so that $\widehat{\sigma}^2_{\lambda,T,\Delta}$, which is used in the construction of $t_{\lambda,T,\Delta}$, is a consistent variance estimator. Recalling (10)-(12), it is immediate to see that $\tau(\Delta)$ plays a role both in the numerator and the denominator of $t_{\lambda,T,\Delta}$. In our experimental setup, small thresholds (e.g. $\delta\eta = 0.4$) result in a too small variance estimator, leading to an oversized test. Not surprisingly, the empirical power is not affected by the choice of the threshold parameter. Below, we only report results for the case where $\delta\eta = 0.251$.

A small subset of our experimental findings are reported in Tables 1-7. These findings are chosen to be representative of the patterns observed in all of the Monte Carlo experiments that were run. The complete set of Monte Carlo results are posted at http://econweb.rutgers.edu/nswanson/comp.htm. In the tables, rejection frequencies based on tests implemented at a 10% nominal level are reported.

Tables 1 and 2 contain results for the jump intensity test. Empirical rejection frequencies under $H_0^\lambda : \lambda_\infty = 0$ are given in Table 1. Consider the results reported under Case 1, with $\rho = 0$. When $\Delta = 1/78$, rejection frequencies are very near nominal levels only for $T = 70$, 80 and 90. Indeed, when T is increased to 130, the empirical size deteriorates substantively and is over 20%. However, rejection frequencies are close to the nominal 10% level for $T = 260, 280$, and 300 when $\Delta^{-1} = 1/156$. This finding is quite interesting, and it suggests that the range of permissible T and Δ permutations for which the size properties of our jump test are adequate is wide. Of course, for extremely large values of T, performance should be again expected to deteriorate, since we require that $T < 1/\Delta^2$. Turning to the case of leverage, the test becomes oversized more rapidly as ρ is increased from 0 to -0.25, and then to -0.5 and -0.75, when T increases, for fixed Δ. This is not surprising. Still, it is clear that the test performs adequately, as long as T and Δ are carefully monitored, regardless of the presence of leverage. Moreover, the above findings hold under mild forms of misspecification, such as when there is mean reversion in the pricing equation (see Case 2 of Table 1), and when there is microstructure noise (see Case 3 of Table 1).[8]

[7] Note that the condition $\sqrt{T}\tau(\Delta) \to 0$ is required only for Theorem 3.

[8] Our jump test, under the alternative, is similarly robust to these forms of misspecification. However, as discussed above, our self-excitation test is not robust to the presence of mean-reverting drift. Please refer to the full set of experiments posted at http://econweb.rutgers.edu/nswanson/comp.htm for further information.

Empirical rejection frequencies under $H_A^{\lambda(1)}: \lambda_\infty > 0$ and $\mathrm{E}(Z_k) \neq 0$ are given as entries in Table 1 for which $\varsigma \neq 0$. Recall, that in this case all jump densities have non-zero third moment, and the test has well defined Pitman drift against $\sqrt{T}-$ alternatives. Not surprisingly, rejection frequencies are thus near unit, regardless of jump density specification. The more challenging alternative is $H_A^{\lambda(2)}: \lambda_\infty > 0$ and $\mathrm{E}(Z_k) = 0$. In this case, the test has zero Pitman drift, and that the ability of the test to distinguish between H_0^λ and $H_A^{\lambda(2)}$ derives solely from the different order of magnitude of the variance under the two hypotheses. Results for DGPs generated under $H_A^{\lambda(2)}$ are also gathered in Table 2, for $\sigma = 0.2$ and 0.4, with $\lambda_\infty = 0.3$ and 0.7. As might be expected, the power increases as σ and λ_∞ increase. However, the value of σ plays a much bigger role than that of λ_∞. This is not surprising, given that what drives the power is the order of magnitude of the variance.

Table 3 summarizes experimental findings for the zero mean jump test, based on $t_{\mu_Z,T,\Delta} = \sqrt{T}\frac{\widehat{\mu}^Z_{T,\Delta}}{\widehat{\sigma}_{\mu_Z}}$. This test can be thought of as a pre-test, prior to testing for self-excitement, and after testing for jumps, say. The reason for this is that in order to identify β when testing $H_0^\beta: \beta = 0$ vs. $H_A^\beta: \beta > 0$, we require not only that $\lambda_\infty > 0$ but also that $E(Z_k) \neq 0$. Inspection of the rejection frequencies reported in the table indicates that the test is well sized, for all values of T, when $\lambda_\infty = 0.3$ and $\sigma = 0.2$. Similar results obtain for other values of λ_∞ and σ, and are hence not reported. Overall, the power is quite good, except for the cases in which T is small, λ_∞ is small, $\Delta = 1/78$, and there is no self-excitation.

Tables 4 (empirical size) and 5 (empirical power) summarize experimental findings for our self-excitement test, based on $S^\beta_{T^+,\Delta} = \max\left\{0, \frac{\sqrt{\frac{T^+}{\Delta}}\widehat{\beta}_{T^+,\Delta}}{\widehat{\sigma}_{\beta,T^+,\Delta}}\right\}$. Although dozens of $(\lambda_\infty,\sigma,\Delta,T)$ permutations are reported in Table 4, even cursory examination of the table indicates that the test is very well sized, with rejection frequencies very close to the nominal 10% level, in all cases. It remains to examine the performance of the self-excitement test, under $H_A^\beta: \beta > 0$. These findings are reported in Table 5, where empirical power is summarized for various values of λ_∞,(a,β), Δ^{-1}, and T.[9] For the case where Z_k is a normal random variable, we report rejection frequencies for only one value of σ (i.e., $\sigma = 0.2$). However, it should be noted that empirical power generally declines as σ increases from 0.1 to 0.7, for fixed values of the other parameters. One possible explanation for this finding is that "noisiness" is induced when estimating the first autocovariance term, when jumps are extremely large. Empirical power also declines when the value of a is increased, with $(a-\beta)$ fixed (compare rejection frequencies for $(a,\beta) = (3,2)$ with those for $(a,\beta) = (5,4)$). The reason for this follows from (13)-(22), where it is immediate to see that the smaller is a and the smaller is $(a-\beta)$, the higher is the degree of self-excitation. This means that our lowest degree of self-excitation is associated with the case where $(a,\beta) = (7,5)$. Empirical power is correspondingly the lowest in this case, bottoming out at around 0.30. When the degree of self-excitation is strongest (i.e.,

[9] As the test is based on the first autocovariance term, leverage plays no role in the test statistic. Not surprisingly, then, including leverage (or not) has no qualitatively noteworthy impact on our findings, and only results for the non-zero leverage case are reported.

$(a, \beta) = (3, 2)$) and there are "enough" jumps (i.e., $\lambda_\infty = 0.7$), rejection frequencies are (roughly) in the range 0.70 to 0.80. Finally, note that all of the above results are based on experiments with $\rho = -0.25$. Results are similar for the other values of ρ considered in our experiments.

Tables 6 (empirical size) and 7 (empirical power) summarize experimental findings for our alternative self-excitement test, based on $\widetilde{S}^{\beta}_{T^+,\Delta} = \max\left\{0, \sqrt{\frac{T^+}{\Delta}} \frac{\widetilde{\beta}_{T^+,\Delta}}{\widetilde{\sigma}_{\beta,T^+,\Delta}}\right\}$. This test does not require pre-testing $H_0 : \mathrm{E}(Z_k) = 0$, and so is of some interest. Due to problems with the size of this test, as discussed in Remark 8 above, this test is not recommended for use at this time. In order to point out the problems with the test, we focus on somewhat different parameterizations than those presented in Tables 4 and 5. However, please note that the performance of the $S^{\beta}_{T^+,\Delta}$ test remains unchanged under these alternative parameterizations. In particular, cursory inspection of the tables indicates that while empirical power remains satisfactory (see Table 7), the test is (severely) oversized for many parameter permutations. In particular, when T is high, relative to Δ^{-1}, size is severely distorted.

In summary, all of our tests perform adequately, with the possible exception of our alternative $\widetilde{S}^{\beta}_{T^+,\Delta}$ self-excitement test; and perform as expected, given the asymptotic theory describing their large sample behavior. Moreover, the finite sample performance of the tests is found to be adequate, with the important obvious caveat that jump process characteristics and sampling frequencies affect the ability of the tests to perform adequately.

6 Empirical Illustration

For our empirical illustration, we examine intraday observations on 11 stocks, including: 3M Company (3M); Apple Inc. (AAPL); Amazon.com, Inc. (AMZN); Bank of America Corporation (BAC); Costco Wholesale Corporation (COST); General Electric Company (GE); The Goldman Sachs Group, Inc. (GS); International Business Machines Corporation (IBM); Intel Corporation (INTC); Johnson & Johnson (JNJ); and JPMorgan Chase & Co. (JPM). Our sample period is from January 2, 2004 to December 31, 2013, and data are collected from the TAQ database. In our empirical implementation of the tests outlined in this paper, we considered time spans (in days) of 1 quarter, 2 quarters, and 1 year. Results from self-excitation testing based on our entire sample period did not differ from results based on these time spans. Thus, the number of days in our time span varies from approximately 60 to 250. Two intra-daily data frequencies were utilized, including 5-minute and 10-minute data, in order to examine the robustness of our findings to the possible presence of microstructure noise.

Results for the jump intensity test are gathered in Table 8. As is evident upon cursory inspection of the results in this table, we find in favor of the presence of jumps in the data generating process, regardless of T and Δ.[10] Results based on the application of our two alternative self-excitement

[10] Results are only tabulated for seleted values of T. However, our findings are consistent across all quarterly,

tests are not tabulated, for the sake of brevity, and are available upon request. However, it is worth noting that application of the $\widetilde{S}^{\beta}_{T^+,\Delta}$ self-excitement test indicates that self-excitement characterizes many jump processes for the stocks considered. On the other hand, application of the $S^{\beta}_{T^+,\Delta}$ self-excitement test indicates that self-excitement characterizes very few jump processes for the stocks considered. This finding is likely explained by the fact that most of the stocks that we examine display near zero mean jumps for the values of T considered. Because of these findings, we conjecture that self-excitement is quite prevalent in the data generating processes that characterize the stock prices that we study, although the size issues noted earlier suggest that this conclusion should be made with caution until further empirical research is undertaken.

7 Concluding Remarks

If the intensity parameter in a jump diffusion model is identically zero, then parameters characterizing the jump size density cannot be identified. In general, this lack of identification precludes consistent estimation of identified parameters. In the extant literature, there are a large variety of tests for the null of no jumps versus the alternative of jumps, including tests based on the comparison of two realized volatility measures, one which is robust, and the other which is not robust to the presence of jumps (see, e.g. Barndorff-Nielsen, Shephard and Winkel (2006) and Podolskji and Vetter (2009a)), tests based on a thresholding approach (see, e.g. Corsi, Pirino, and Renò (2010), Lee and Mykland (2008), and Lee, Loretan and Ploberger (2013)), and tests based on power variation (see e.g. Aït-Sahalia and Jacod (2009)). One feature of these tests is that they are based on observations drawn on a given finite time span, and thus they can only detect realized jumps. This paper introduces a test which is instead able to detect jumps in the data generating process. Our test is based on realized tricity and make use of high frequency observations measured over a long time-span. Importantly, the test is robust to the presence of leverage. It has a normal limiting distribution, and so inference is straightforward. A so-called "self-excitement" test is also introduced, which is designed to have power against path dependent intensity, thus providing a direct test for the Hawkes diffusion model of Aït-Sahalia, Cacho-Diaz and Laeven (2015). The finite sample behavior of the suggested statistics is studied via Monte Carlo experimentation, and is found to be adequate under a variety of realistic data generating processes. Finally, strong evidence of jumps,

semi-annual, and annual periods studied.

and moderate evidence of self-excitement is found in an empirical illustration.

8 Appendix

In the sequel with C we denote a positive constant that may change from line to line.

Lemma 1: Let Assumptions **A(i)-(iii)** hold. Also, as $T \to \infty$, $T\Delta^2 \to 0$, $\tau(\Delta) \to 0$ and $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)} \to 0$, with $m > 6$ and even. Then if $\lambda = 0$,

$$P\left(\max_{k\le n}\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| > \varepsilon\tau(\Delta)\right) \to 0.$$

Proof of Lemma 1: Recalling that $T\Delta^2 \to 0$,

$$\begin{aligned}
&P\left(\max_{k\le n}\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| > \varepsilon\tau(\Delta)\right)\\
&\le \sum_{k=1}^{n} P\left(\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| > \varepsilon\tau(\Delta)\right)\\
&= \sum_{k=1}^{n} P\left(\left|\mu\Delta + V_{k\Delta}^{1/2}\sqrt{1-\rho^2}\left(W_{1,(k+1)\Delta} - W_{1,k\Delta}\right) + V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta} - W_{2,k\Delta}\right)\right| > \varepsilon\tau(\Delta)\right)\\
&\le \frac{T}{\Delta}\frac{1}{\varepsilon^m\tau(\Delta)^m}\mathrm{E}\left(\left|\mu\Delta + V_{k\Delta}^{1/2}\sqrt{1-\rho^2}\left(W_{1,(k+1)\Delta} - W_{1,k\Delta}\right) + V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta} - W_{2,k\Delta}\right)\right|^m\right)\\
&\le C\mathrm{E}\left(V_{k\Delta}^{m/2}\right)\Delta^{\frac{m}{2}-3}\tau(\Delta)^{-m} \to 0,
\end{aligned}$$

for $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)} \to 0$ and $m > 6$, as $\mathrm{E}\left(V_{k\Delta}^{m/2}\right)$ is finite, by A(iii).

Lemma 2: Let Assumptions **A(iv)-(v)** hold. Then, for all $l \ge 2$ even,

$$\mathrm{E}\left(\frac{1}{T}\sum_{k=1}^{n}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}} - \Delta\mathrm{E}(Z_k)\lambda\right)^l 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right| \le \tau(\Delta)\right\}\right) \le C\tau(\Delta)^{l+1}.$$

Proof of Lemma 2: For l even, and for $c_{l-k,k} > 0$, whenever l and k are even,

$$\begin{aligned}
&\mathrm{E}\left(\frac{1}{T}\sum_{k=1}^{n}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}} - \Delta\mathrm{E}(Z_k)\lambda\right)^l 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right| \le \tau(\Delta)\right\}\right)\\
&= \frac{1}{\Delta}\int_{-\tau(\Delta)}^{\tau(\Delta)}\left((z - \Delta\mathrm{E}(Z)\lambda)\right)^l \mathrm{E}\left(1_{\Delta_{N_{(k+1)\Delta}}}\right) f_Z(z)\mathrm{d}z\\
&= \lambda\int_{-\tau(\Delta)}^{\tau(\Delta)}\left(z - \Delta\mathrm{E}(Z_k)\lambda\right)^l f_Z(z)\mathrm{d}z\\
&= \lambda\sum_{k=0}^{l} c_{l-k,k}\int_{-\tau(\Delta)}^{\tau(\Delta)} Z^{l-k}\left(\Delta\mathrm{E}(Z_k)\lambda\right)^k f_Z(z)\mathrm{d}z\,(1+o(\Delta))
\end{aligned}$$

$$
\begin{aligned}
&= \lambda c_{l,0} \int_{-\tau(\Delta)}^{\tau(\Delta)} Z^l f_Z(z) \mathrm{d}z \left(1 + o(\Delta)\right) \\
&\leq C\tau\left(\Delta\right)^{l+1}.
\end{aligned}
$$

Lemma 3: Let Assumptions **A(i)-(v)** hold. If as $T, \Delta^{-1} \to \infty$, $T^+/T \to \infty$, $\frac{\Delta^{\frac{1}{2}-\frac{3}{m}}}{\tau(\Delta)} \to 0$, with $m > 6$ even and $\sqrt{T}\tau^2\left(\Delta\right) \to 0$, then,

$$
\sqrt{T}\left(\frac{1}{T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right) 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\} - \mu\right) = o_p(1).
$$

Proof of Lemma 3: Note that $\sqrt{T}\tau\left(\Delta\right) \to 0$ implies that $T\Delta^2 \to 0$, and so by Lemma 1,

$$
\begin{aligned}
&\frac{\sqrt{T}}{T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right) 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\} \\
&= \frac{\sqrt{T}}{T^+}\sum_{k=1}^{n^+}\left(\mu\Delta + V_{k\Delta}^{1/2}\sqrt{1-\rho^2}\left(W_{1,(k+1)\Delta} - W_{1,k\Delta}\right) + V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta} - W_{2,k\Delta}\right)\right)\left(1 - 1_{\Delta_{N_{(k+1)\Delta}}}\right) \\
&+ \frac{\sqrt{T}}{T^+}\sum_{k=1}^{n^+}\left(\mu\Delta + V_{k\Delta}^{1/2}\sqrt{1-\rho^2}\left(W_{1,(k+1)\Delta} - W_{1,k\Delta}\right) + V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta} - W_{2,k\Delta}\right) + Z_k\right) \\
&\times 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right| \leq \tau\left(\Delta\right)\right\} + o_p(1) \\
&= I_{T,\Delta} + II_{T,\Delta}.
\end{aligned}
$$

We need to show that: (i) $I_{T,\Delta} = \sqrt{T}\mu + o_p(1)$; and (ii) $II_{T,\Delta} = o_p(1)$.

For $T^+/T \to \infty$,

$$
\frac{\sqrt{T}}{T^+}\sum_{k=1}^{n^+}\left(V_{k\Delta}^{1/2}\sqrt{1-\rho^2}\left(W_{1,(k+1)\Delta} - W_{1,k\Delta}\right) + V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta} - W_{2,k\Delta}\right)\right)\left(1 - 1_{\Delta_{N_{(k+1)\Delta}}}\right) = o_p(1)
$$

and

$$
\frac{\sqrt{T}}{T^+}\sum_{k=1}^{n^+} 1_{\Delta_{N_{(k+1)\Delta}}} = o_p(1)
$$

as $\operatorname{var}\left(1_{\Delta_{N_{(k+1)\Delta}}}\right) = O\left(\Delta\right)$ and $T/T^+ \to 0$. Thus, (i) is established. With regard to (ii), as

$$
\begin{aligned}
\mathrm{E}\left(1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right| \leq \tau\left(\Delta\right)\right\}\right) &= \mathrm{E}\left(1\left\{\left|Z_k\right| \leq \tau\left(\Delta\right)\right\}\right)\mathrm{E}\left(1_{\Delta_{N_{(k+1)\Delta}}}\right) \\
&\leq O\left(\tau\left(\Delta\right)\Delta\right)
\end{aligned}
$$

it follows that

$$\frac{\sqrt{T}}{T^{+}}\sum_{k=1}^{n^{+}}\left(\mu\Delta+V_{k\Delta}^{1/2}\sqrt{1-\rho^{2}}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\right)$$
$$\times 1\left\{\left|Z_{k}1_{\Delta_{N_{(k+1)\Delta}}}\right|\leq\tau\left(\Delta\right)\right\}=o_{p}(1).$$

Finally,

$$\begin{aligned}
&\frac{\sqrt{T}}{T^{+}}\sum_{k=1}^{n^{+}}\mathrm{E}\left(|Z_{k}|\right)1\left\{\left|Z_{k}1_{\Delta_{N_{(k+1)\Delta}}}\right|\leq\tau\left(\Delta\right)\right\}\\
&=\frac{\sqrt{T}}{\Delta}\int_{-\tau(\Delta)}^{\tau(\Delta)}|z|f_{Z}(z)\mathrm{d}z\mathrm{E}\left(1_{\Delta_{N_{(k+1)\Delta}}}\right)\\
&\leq C\sqrt{T}\tau^{2}\left(\Delta\right)=o(1).
\end{aligned}$$

Then (ii) follows by a straightforward application of the Markov inequality.

Proof of Theorem 1:

Part (i): From the multivariate Milstein formula (see Kloeden and Platen (1999), Section 10.3), we have that:

$$\begin{aligned}
&\ln X_{(k+1)\Delta}-\ln X_{k\Delta}\\
&=\Big(\mu\Delta+V_{k\Delta}^{1/2}\sqrt{1-\rho^{2}}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+V_{k\Delta}^{1/2}\rho\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\\
&+\frac{1}{4}\rho V_{(k-1)\Delta}^{-1/2}g\left(V_{(k-1)\Delta},\theta\right)\left(\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)^{2}-\Delta\right)\\
&+\frac{1}{4}\sqrt{1-\rho^{2}}V_{(k-1)\Delta}^{-1/2}g\left(V_{k\Delta},\theta\right)\int_{k\Delta}^{(k+1)\Delta}\int_{k\Delta}^{s_{2}}\mathrm{d}W_{2,s_{1}}\mathrm{d}W_{1,s_{2}}\Big)\left(1+o_{p}(1)\right)
\end{aligned}\tag{23}$$

Also,

$$\frac{\ln X_{T}}{n}=\left(\mu\Delta+\sqrt{1-\rho^{2}}\frac{\Delta}{T}\int_{0}^{T}V_{s}^{1/2}\mathrm{d}W_{1,s}+\rho\frac{\Delta}{T}\int_{0}^{T}V_{s}^{1/2}\mathrm{d}W_{2,s}\right)\left(1+o_{p}(1)\right)\tag{24}$$

and

$$\frac{\ln X_{\Delta}}{n}=\left(\mu\frac{\Delta^{2}}{T}+\sqrt{1-\rho^{2}}\frac{\Delta}{T}\int_{0}^{\Delta}V_{s}^{1/2}\mathrm{d}W_{1,s}+\rho\frac{\Delta}{T}\int_{0}^{\Delta}V_{s}^{1/2}\mathrm{d}W_{2,s}\right)\left(1+o_{p}(1)\right).$$

Thus, $\frac{\ln X_{\Delta}}{n}=O_{p}\left(\frac{\Delta^{3/2}}{T}\right)$ and $\frac{\ln X_{\Delta}}{n^{+}}=O_{p}\left(\frac{\Delta^{3/2}}{T^{+}}\right)$. These terms can thus be ignored given that they are $o_{p}\left(\Delta\right)$. Now,

$$\begin{aligned}
&\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_{T}}{n}\right)^{3}\\
&=\Big(\sqrt{1-\rho^{2}}V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)
\end{aligned}$$

$$+\frac{1}{4}\rho V_{k\Delta}^{-1/2} g\left(V_{k\Delta},\theta\right)\left(\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)^2-\Delta\right)+$$
$$\left.+\frac{1}{4}\sqrt{1-\rho^2} V_{(k-1)\Delta}^{-1/2} g\left(V_{k\Delta},\theta\right)\int_{k\Delta}^{(k+1)\Delta}\int_{k\Delta}^{s_2} \mathrm{d}W_{2,s_1}\mathrm{d}W_{1,s_2}\right)^3 (1+o_p(1)).$$

Straightforward but tedious algebra shows that

$$\begin{aligned}
&\mathrm{E}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_T}{n}\right)^3\\
&=\frac{3}{2}\rho^3 \mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta},\theta\right)\right)\Delta^2.
\end{aligned}$$

Because of Lemma 1,

$$\begin{aligned}
\widehat{\mu}_{3,T,\Delta}=&\frac{1}{T}\sum_{k=1}^{n}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_T}{n}\right)^3\\
&-\frac{1}{T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_T}{n^+}\right)^3 (1+o_p(1)).
\end{aligned}$$

Now, write

$$\begin{aligned}
&\frac{\sqrt{T}}{\Delta}\widehat{\mu}_{3,T,\Delta}\\
&=\frac{\Delta}{\sqrt{T}}\sum_{k=1}^{n}\left(\frac{1}{\Delta^2}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_{n\Delta}-\ln X_{\Delta}}{n}\right)^3-\frac{3}{2}\rho^3\mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta},\theta\right)\right)\right)\\
&-\frac{\sqrt{T}}{T^+}\Delta\sum_{k=1}^{n^+}\left(\frac{1}{\Delta^2}\left(\ln X_{k\Delta}-\ln X_{(k-1)\Delta}-\frac{\ln X_{n^+\Delta}-\ln X_{\Delta}}{n^+}\right)^3-\frac{3}{2}\rho^3\mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta},\theta\right)\right)\right)(1+o_p(1))\\
&=\left(I_{T,\Delta}+II_{T,T^+,\Delta}\right)(1+o_p(1)).
\end{aligned}$$

It is immediate to see that $\mathrm{E}(I_{T,\Delta})=0$. Also, recalling that for m even, the $m-$th central moment of a standard normal is equal to $\frac{m!}{2^{m/2}(m/2)!}$,

$$\begin{aligned}
\omega_0 &= \mathrm{var}\left(I_{T,\Delta}\right)\\
&=\mathrm{var}\left(\frac{1}{\sqrt{T}\Delta}\sum_{k=1}^{n}\left(\left(1-\rho^2\right)^{1/2} V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\right)^3\right)+o(1)\\
&=\mathrm{E}\left(\left(\left(1-\rho^2\right)^{1/2} V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\right)^6\right)\\
&=15\left(1-\rho^2\right)^3\mathrm{E}\left(V_{k\Delta}^3\right)+15\rho^6\mathrm{E}\left(V_{k\Delta}^3\right)+45\left(1-\rho^2\right)^2\rho^2\mathrm{E}\left(V_{k\Delta}^3\right)+45\left(1-\rho^2\right)\rho^4\mathrm{E}\left(V_{k\Delta}^3\right).
\end{aligned}$$

Hence, by the central limit theorem for martingale differences,

$$I_{T,\Delta} \xrightarrow{d} N\left(0, \omega_0\right).$$

Since $T^+/T \rightarrow \infty$, $II_{T,T^+,\Delta}$ is of smaller probability order than $I_{T,\Delta}$, and thus is $o_p(1)$. The statement in Part (i) then follows.

Part (ii): Let

$$\ln X_{(k+1)\Delta} - \ln X_{k\Delta} = \left(\ln X^c_{(k+1)\Delta} - \ln X^c_{k\Delta}\right) + \left(\ln X^d_{(k+1)\Delta} - \ln X^d_{k\Delta}\right),$$

where $\left(\ln X^c_{(k+1)\Delta} - \ln X^c_{k\Delta}\right)$ is defined as in the RHS of (23), and

$$\left(\ln X^d_{(k+1)\Delta} - \ln X^d_{k\Delta}\right) = Z_k 1_{\Delta N_{(k+1)\Delta}},$$

where Z_k denotes a draw from the jump size density, say f_Z, and $1_{\Delta N_{(k+1)\Delta}} = 1$, if $\Delta N_{(k+1)\Delta} = 1$, and equals zero otherwise. Also

$$\frac{\ln X_T}{n} = \frac{\ln X^c_T}{n} + \frac{1}{n}\sum_{i=0}^{N_T} Z_i = \frac{\ln X^c_T}{n} + \Delta\lambda \mathrm{E}\left(Z_k\right) + o_p(1),$$

with $\frac{\ln X^c_T}{n}$ defined as in the RHS of (24). Write,

$$\begin{aligned}
\frac{\sqrt{T}}{\Delta}\widehat{\mu}_{3,T,\Delta} &= \left(\frac{1}{\Delta\sqrt{T}}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\}\right.\\
&\quad\left. - \frac{\sqrt{T}}{\Delta T^+}\sum_{k=1}^{n^+}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n^+\Delta} - \ln X_{\Delta}}{n^+}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\}\right)\\
&\quad + \frac{1}{\Delta\sqrt{T}}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| > \tau\left(\Delta\right)\right\}\\
&= A_{T,T^+,\Delta} + B_{T,\Delta}.
\end{aligned}$$

By Lemma 1,

$$\begin{aligned}
A_{T,T^+,\Delta} &= \frac{1}{\Delta\sqrt{T}}\sum_{k=1}^{n}\left(\left(\ln X^c_{k\Delta} - \ln X^c_{(k-1)\Delta} - \frac{\ln X^c_{n\Delta}}{n}\right)^3 - \frac{3}{2}\rho^3 \mathrm{E}\left(V_{k\Delta}^{1/2} g\left(V_{k\Delta}, \theta\right)\right)\right)\\
&\quad + \left(\frac{1}{\Delta\sqrt{T}}\sum_{k=1}^{n}\left(Z_k 1_{\Delta N_{(k+1)\Delta}} - \Delta\lambda \mathrm{E}\left(Z_k\right)\right)^3 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\}\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sqrt{T}}{\Delta T^{+}}\sum_{k=1}^{n^{+}}\left(Z_k 1_{\Delta N_{(k+1)\Delta}}-\Delta\lambda \mathrm{E}\left(Z_k\right)\right)^3 1\left\{\left|\ln X_{k\Delta}-\ln X_{(k-1)\Delta}\right|\leq \tau\left(\Delta\right)\right\}\Bigg)\\
&+\text{ cross terms}\\
&=A_{1,T,T^{+},\Delta}+A_{2,T,T^{+},\Delta}+\text{ cross terms.}
\end{aligned}
$$

By the same argument as that used in Part (i), $A_{1,T,T^{+},\Delta}\xrightarrow{d}N(0,\omega_0)$, while $A_{2,T,T^{+},\Delta}+$ cross terms is of a smaller probability order than $B_{T,\Delta}$.
Because of Lemma 1,

$$
B_{T,\Delta}=\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}\left(Z_k 1_{\Delta N_{(k+1)\Delta}}-\Delta\lambda \mathrm{E}\left(Z_k\right)\right)^3+o_p(1).
$$

Now,

$$
\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}\left(Z_k 1_{\Delta N_{(k+1)\Delta}}-\Delta\lambda \mathrm{E}\left(Z_k\right)\right)^3\\
&=\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}Z_k^3 1_{\Delta N_{(k+1)\Delta}}-2\lambda \mathrm{E}\left(Z_k\right)\frac{\Delta}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}Z_k^2 1_{\Delta N_{(k+1)\Delta}}\\
&+2\lambda^2\mathrm{E}\left(Z_k\right)^2\frac{\Delta^2}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}Z_k 1_{\Delta N_{(k+1)\Delta}}-\sqrt{T}\Delta\lambda^3\mathrm{E}\left(Z_k\right)^3\\
&=\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\left(\frac{1}{\Delta}Z_k^3 1_{\Delta N_{(k+1)\Delta}}-\lambda \mathrm{E}\left(Z_k^3\right)\right)+\frac{\sqrt{T}}{\Delta}\lambda \mathrm{E}\left(Z_k^3\right)\\
&-2\lambda \mathrm{E}\left(Z_k\right)\frac{\Delta}{\sqrt{T}}\sum_{k=1}^{n}\left(\frac{1}{\Delta}Z_k^2 1_{\Delta N_{(k+1)\Delta}}-\lambda \mathrm{E}\left(Z_k^2\right)\right)-\sqrt{T}2\lambda^2\mathrm{E}\left(Z_k\right)\mathrm{E}\left(Z_k^2\right)+o_p(1),
\end{aligned}
\tag{25}
$$

since $\sqrt{T}\Delta\lambda^3\mathrm{E}\left(Z_k\right)^3=o(1)$ and $\frac{\Delta^2}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}Z_k 1_{\Delta N_{(k+1)\Delta}}=o_p(1)$, for $\sqrt{T}\Delta\to 0$.
From (25), we see that the statistic has $\frac{\sqrt{T}}{\Delta}$ Pitman drift, whenever $\mathrm{E}\left(Z_k^3\right)\neq 0$. The statement in Part (ii) then follows.

Part (iii): When $\mathrm{E}\left(Z_k^3\right)=\mathrm{E}\left(Z_k\right)=0$,

$$
\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}\left(Z_k 1_{\Delta N_{(k+1)\Delta}}-\Delta\lambda \mathrm{E}\left(Z_k\right)\right)^3\\
&=\frac{1}{\sqrt{T}}\sum_{k=1}^{n}\frac{1}{\Delta}Z_k^3 1_{\Delta N_{(k+1)\Delta}}+o_p(1).
\end{aligned}
$$

We now show that $\mathrm{var}\left(S_{T,\Delta}\right)=O\left(\frac{1}{\Delta^2}\right)$, regardless of whether the jump intensity is constant or path dependent.

If $\beta = 0$ (no path dependent intensity), then:

$$\begin{aligned}
&\operatorname{var}\left(S_{T,\Delta}\right) \\
&= \operatorname{var}\left(\frac{1}{\sqrt{T}\Delta}\sum_{k=1}^{n} Z_k^3 1_{\Delta N_{(k+1)\Delta}}\right)(1+o(1)) \\
&= \frac{1}{T\Delta^2}\sum_{k=1}^{n}\operatorname{var}\left(Z_k^3 1_{\Delta N_{(k+1)\Delta}}\right)(1+o(1)) \\
&= \frac{1}{\Delta^3}\operatorname{var}\left(Z_k^3 1_{\Delta N_{(k+1)\Delta}}\right)(1+o(1)) = O\left(\frac{1}{\Delta^2}\right).
\end{aligned}$$

Alternatively, if $\beta > 0$, one must take autocovariance terms into account when carrying out similar calculations. However, given A(iv), the order of magnitude of the variance is still $O\left(\frac{1}{\Delta^2}\right)$. Given that $\sqrt{T}\Delta \to 0$, $S_{T,\Delta}$ is of probability order Δ^{-1}, and the statement in Part (iii) follows.

Proof of Corollary 2:

Part (i). We need to show that $\widehat{\sigma}^2_{\lambda,T,\Delta} - \omega_0 = o_p(1)$, with ω_0 defined as in the statement of Theorem 1. By Lemma 1,

$$\widehat{\sigma}^2_{\lambda,T,\Delta} = \frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(\ln X_{k\Delta} - \ln X_{(k-1)\Delta} - \frac{\ln X_{n\Delta} - \ln X_{\Delta}}{n}\right)^6 + o_p(1).$$

The statement of Part (i) follows directly by the law of large numbers (for iid processes if $\beta = 0$ and for ergodic mixing processes if $\beta > 0$).

Parts (ii)-(iii): We need to show that $\widehat{\sigma}^2_{\lambda,T,\Delta} = O_p(1)$. Now, note that:

$$\begin{aligned}
\widehat{\sigma}^2_{\lambda,T,\Delta} &= \frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(\ln X^c_{k\Delta} - \ln X^c_{(k-1)\Delta} - \frac{\ln X^c_{n\Delta} - \ln X^c_{\Delta}}{n}\right)^6 \\
&+ \frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(Z_k 1_{\Delta N_{(k+1)\Delta}} - \Delta\lambda \mathrm{E}\left(Z\right)\right)^6 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\} \\
&+ \text{ cross terms } + o_p(1). \qquad (26)
\end{aligned}$$

The first term on the RHS of (26) is a consistent estimator of ω_0. It suffices to show that the second term on the RHS of (26) is $O_p(1)$. This follows because the cross term cannot be of a larger order than the second term. Given Lemma 1,

$$\begin{aligned}
&\frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(Z_k 1_{\Delta N_{(k+1)\Delta}} - \Delta\lambda \mathrm{E}\left(Z_k\right)\right)^6 1\left\{\left|\ln X_{k\Delta} - \ln X_{(k-1)\Delta}\right| \leq \tau\left(\Delta\right)\right\} \\
&= O_p(1)\left(\frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(Z_k 1_{\Delta N_{(k+1)\Delta}} - \Delta\lambda \mathrm{E}\left(Z_k\right)\right)^6 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right| \leq \tau\left(\Delta\right)\right\}\right),
\end{aligned}$$

and by Lemma 2,

$$
\begin{aligned}
&P\left(\frac{1}{T\Delta^2}\sum_{k=1}^{n}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}}-\mathrm{E}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right)\right)^6 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right|\leq\tau\left(\Delta\right)\right\}>\varepsilon\right)\\
&\leq\frac{1}{\Delta^2\varepsilon}\mathrm{E}\left(\frac{1}{T}\sum_{k=1}^{n}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}}-\mathrm{E}\left(Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right)\right)^6 1\left\{\left|Z_k 1_{\Delta_{N_{(k+1)\Delta}}}\right|\leq\tau\left(\Delta\right)\right\}\right)\\
&\to 0,
\end{aligned}
$$

provided that $\tau\left(\Delta\right)^7\Delta^{-2}\to 0$.

Proof of Theorem 3:

Part (i): By Lemma 3,

$$
\begin{aligned}
&\sqrt{T}\widehat{\mu}^{Z}_{T,\Delta}\\
&=\frac{1}{\sqrt{T}}\sum_{k=0}^{n}\left(\sqrt{1-\rho^2}V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\right.\\
&\left.+\mu\Delta+Z_k 1_{\Delta N_{(k+1)\Delta}}\right)-\sqrt{T}\mu+o_p(1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sqrt{T}\widehat{\mu}^{Z}_{T,\Delta}\\
&=\frac{1}{\sqrt{T}}\sum_{k=0}^{n}\left(\sqrt{1-\rho^2}V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)+Z_k 1_{\Delta N_{(k+1)\Delta}}\right)+o_p(1)\\
&\xrightarrow{d} N\left(0,\sigma^2_{\mu_Z}\right),
\end{aligned}
$$

with $\sigma^2_{\mu_Z}=\mathrm{E}\left(V_{k\Delta}\right)+\lambda\mathrm{E}\left(Z_k^2\right)$. As $\widehat{\sigma}^2_{\mu_Z}=\sigma^2_{\mu_Z}+o_p(1)$, the statement in Part (i) follows directly.

(ii) The proof is immediate, as

$$
\begin{aligned}
&\sqrt{T}\widehat{\mu}^{Z}_{T,\Delta}\\
&=\frac{1}{\sqrt{T}}\sum_{k=0}^{n}\left(\sqrt{1-\rho^2}V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)+Z_k 1_{\Delta N_{(k+1)\Delta}}\right)\\
&+\lambda\sqrt{T}\mathrm{E}\left(Z_k\right)+o_p(1)
\end{aligned}
$$

Proof of Theorem 4:

Part (i): Note that,

$$
\begin{aligned}
&\sqrt{\frac{T^{+}}{\Delta}}\widehat{\beta}_{T,\Delta} \\
&= \frac{1}{\sqrt{T^{+}\Delta}}\sum_{k=1}^{n^{+}-1}\Big(\Big(\sqrt{1-\rho^{2}}V_{k\Delta}^{1/2}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)+\rho V_{k\Delta}^{1/2}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right) \\
&+Z_{k}1_{\Delta N_{(k+1)\Delta}}-\Delta\lambda \mathrm{E}\left(Z\right)\Big)\left(\sqrt{1-\rho^{2}}V_{k\Delta}^{1/2}\left(W_{1,k\Delta}-W_{1,(k-1)\Delta}\right)\right. \\
&\rho V_{k\Delta}^{1/2}\left(W_{2,k\Delta}-W_{2,(k-1)\Delta}\right)+Z_{k-1}1_{\Delta N_{k\Delta}}-\Delta\lambda\mathrm{E}\left(Z\right)\Big)\Big)+o_{p}\left(1\right) \\
&= \frac{1}{\sqrt{T^{+}\Delta}}\sum_{k=1}^{n^{+}-1}\left(1-\rho^{2}\right)V_{k\Delta}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)\left(W_{1,k\Delta}-W_{1,(k-1)\Delta}\right) \\
&+\frac{2}{\sqrt{T^{+}\Delta}}\sum_{k=1}^{n^{+}-1}\sqrt{1-\rho^{2}}\rho V_{k\Delta}\left(W_{1,(k+1)\Delta}-W_{1,k\Delta}\right)\left(W_{2,k\Delta}-W_{2,(k-1)\Delta}\right) \\
&+\frac{1}{\sqrt{T^{+}\Delta}}\sum_{k=1}^{n^{+}-1}\rho^{2}V_{k\Delta}\left(W_{2,(k+1)\Delta}-W_{2,k\Delta}\right)\left(W_{2,k\Delta}-W_{2,(k-1)\Delta}\right) \\
&+\frac{1}{\sqrt{T^{+}\Delta}}\sum_{k=1}^{n^{+}-1}\left(Z_{k}Z_{k-1}1_{\Delta N_{(k+1)\Delta}}1_{\Delta N_{k\Delta}}-\Delta^{2}\lambda^{2}\mathrm{E}\left(Z\right)^{2}\right)+o_{p}\left(1\right).
\end{aligned}
\tag{27}
$$

Under the null of $\beta=0$,

$$
\begin{aligned}
&\mathrm{E}\left(Z_{k}Z_{k-1}1_{\Delta N_{(k+1)\Delta}}1_{\Delta N_{(k+1)\Delta}}\right) \\
&=\mathrm{E}\left(Z_{k}\right)^{2}\mathrm{E}\left(1_{\Delta N_{(k+1)\Delta}}\right)\mathrm{E}\left(1_{\Delta N_{k\Delta}}\right)=\Delta^{2}\lambda^{2}\mathrm{E}\left(Z_{k}\right)^{2}.
\end{aligned}
$$

Thus, under H_0, all of the terms on the RHS of (27) have zero mean. Also,

$$
\begin{aligned}
\sigma_{\beta}^{2}&=\mathrm{var}\left(\sqrt{\frac{T^{+}}{\Delta}}\widehat{\tau}_{T,\Delta}\right) \\
&=\left(\left(1-\rho^{2}\right)^{2}+4\rho\left(1-\rho^{2}\right)+\rho^{4}\right)\mathrm{E}\left(V_{k\Delta}^{2}\right)+\lambda^{2}\left(\mathrm{E}\left(Z_{k}^{2}\right)\right)^{2},
\end{aligned}
$$

and since $\widehat{\sigma}^{2}_{\beta,T,\Delta}=\sigma^{2}_{\beta,T,\Delta}+o_p(1)$, by the central limit theorem for martingale differences,

$$
\sqrt{\frac{T^{+}}{\Delta}}t_{\beta,T^{+},\Delta}\rightarrow N\left(0,\sigma_{\beta}^{2}\right).
$$

The statement in Part (i) follows from the continuous mapping theorem.

Part (ii): The first three terms on the RHS of (27) are asymptotically normal, under both hypotheses. With regard to the fourth term, note that, under the alternative, from Hawkes (1971),

$$
\begin{aligned}
&\mathrm{E}\left(Z_k Z_{k-1} 1_{\Delta N_{(k+1)\Delta}} 1_{\Delta N_{k\Delta}} - \Delta^2 \lambda^2 \mathrm{E}\left(Z_k\right)^2\right) \\
&= \Delta^2 \frac{\beta\lambda\left(2a-\beta\right)}{2\left(a-\beta\right)} \exp\left(-\left(a-\beta\right)\right) \mathrm{E}\left(Z_k\right)^2,
\end{aligned}
$$

and so the fourth term on the RHS of (27) can be written as:

$$
\begin{aligned}
&\frac{1}{\sqrt{T^+\Delta}} \sum_{k=1}^{n^+-1} \left(\left(Z_k Z_{k-1} 1_{\Delta N_{(k+1)\Delta}} 1_{\Delta N_{k\Delta}} - \Delta^2 \lambda^2 \mathrm{E}\left(Z_k\right)^2\right)\right. \\
&\left. - \Delta^2 \frac{\beta\lambda\left(2a-\beta\right)}{2\left(a-\beta\right)} \exp\left(-\left(a-\beta\right)\right) \mathrm{E}\left(Z_k\right)^2\right) \\
&+ \sqrt{T^+\Delta} \frac{\beta\lambda\left(2a-\beta\right)}{2\left(a-\beta\right)} \exp\left(-\left(a-\beta\right)\right) \mathrm{E}\left(Z_k\right)^2 \\
&= O_p(1) + \sqrt{T^+\Delta} \frac{\beta\lambda\left(2a-\beta\right)}{2\left(a-\beta\right)} \exp\left(-\left(a-\beta\right)\right) \mathrm{E}\left(Z_k\right)^2.
\end{aligned}
$$

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Table 1: **Monte Carlo Experiments - Jump Test (Empirical Size)** *

ζ	Δ	ρ	ω	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
				Case 1: No Misspecification							
0.5	1/78	0		0.092	0.102	0.124	0.146	0.168	0.182	0.188	0.224
		-0.25		0.090	0.116	0.144	0.158	0.170	0.184	0.20	0.218
		-0.5		0.134	0.178	0.238	0.272	0.342	0.410	0.470	0.534
0.25		-0.5		0.038	0.052	0.068	0.092	0.114	0.122	0.160	0.166
		-0.75		0.034	0.064	0.108	0.138	0.206	0.258	0.336	0.412
0.5	1/156	0		0.064	0.058	0.066	0.076	0.070	0.106	0.112	0.124
		-0.25		0.064	0.064	0.074	0.080	0.072	0.078	0.096	0.100
		-0.5		0.086	0.100	0.152	0.176	0.216	0.268	0.308	0.370
0.25		-0.5		0.010	0.018	0.024	0.024	0.038	0.044	0.054	0.072
		-0.75		0.030	0.040	0.060	0.074	0.108	0.130	0.188	0.226
				Case 2: Mean Reversion in Pricing Equation							
0.5	1/78	0		0.056	0.064	0.066	0.076	0.100	0.142	0.154	0.154
		-0.25		0.092	0.126	0.172	0.228	0.298	0.336	0.394	0.440
0.25		0		0.022	0.026	0.034	0.046	0.056	0.056	0.064	0.062
		-0.25		0.034	0.040	0.046	0.064	0.080	0.104	0.122	0.128
0.5	1/156	0		0.038	0.050	0.050	0.068	0.070	0.084	0.094	0.106
		-0.25		0.064	0.064	0.094	0.120	0.156	0.164	0.200	0.226
0.25		0		0.012	0.010	0.020	0.028	0.024	0.034	0.028	0.032
		-0.25		0.022	0.028	0.024	0.038	0.036	0.044	0.048	0.050
0.25		-0.5		0.010	0.018	0.024	0.024	0.038	0.044	0.054	0.072
				Case 3: Microstructure Noise Added to Data Generating Process							
0.5	1/78	0	0.007	0.078	0.104	0.118	0.138	0.146	0.162	0.176	0.204
		0	0.014	0.064	0.068	0.084	0.100	0.128	0.150	0.152	0.174
		-0.25	0.007	0.068	0.072	0.092	0.100	0.126	0.166	0.176	0.208
		-0.25	0.014	0.064	0.074	0.102	0.110	0.114	0.142	0.150	0.162
	1/156	0	0.007	0.054	0.076	0.086	0.086	0.084	0.122	0.116	0.118
		0	0.014	0.030	0.032	0.034	0.048	0.052	0.062	0.064	0.082
		-0.25	0.007	0.044	0.044	0.070	0.064	0.098	0.100	0.094	0.102
		-0.25	0.014	0.020	0.028	0.028	0.026	0.030	0.056	0.046	0.060

* Entries in the table are rejection frequencies for the "jump test" based on $t_{\lambda,T,\Delta}$. Results are tabulated for the following sample size (T) and discretization (Δ) permutations: $\Delta = 1/78$ – $T1$:T=60, $T2$:T=70, $T3$:T=80, $T4$:T=90, $T5$:T=100, $T6$:T=110, $T7$:T=120, $T8$:T=130. For $\Delta = 1/156$ – $T1$:T=160, $T2$:T=180, $T3$:T=200, $T4$:T=220, $T5$:T=240, $T6$:T=260, $T7$:T=280, $T8$:T=300. In our Monte Carlo experiments, ζ is the parameter controlling the strength of the diffusion term in the volatility equation of our DGP, Δ denotes the discretization interval, ρ measures the leverage effect, and ω measures the magnitude of the square root of the noise variance in experiments with microstructure noise. In experiments reported in subsequent tables, additional parameters include: λ_∞, which is the intensity, (a, β), which are parameters controlling the Hawkes diffusion, ς, which is the paremeter of the exponential jump density, and σ, which is the square root of the variance of the normal jump density. In all experiments, we perform 1000 Monte Carlo replications. For complete details, refer to Section 5 of the paper.

Table 2: **Monte Carlo Experiments - Jump Test (Empirical Power)***

λ_∞	(a,β)	ς	σ	ρ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
0.3	(0,0)	2.5		0	0.998	0.998	0.998	1.000	1.000	1.000	1.000	1.000
		5			0.906	0.930	0.954	0.968	0.982	0.988	0.988	0.992
			0.2		0.560	0.580	0.642	0.672	0.666	0.696	0.686	0.712
			0.4		0.920	0.938	0.948	0.948	0.938	0.946	0.952	0.966
	(3,2)	2.5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			0.994	0.996	0.996	0.996	1.000	1.000	1.000	1.000
			0.2		0.758	0.774	0.810	0.818	0.804	0.794	0.816	0.844
			0.4		0.956	0.972	0.962	0.970	0.966	0.962	0.956	0.958
0.7	(0,0)	2.5		0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
			0.2		0.764	0.778	0.804	0.802	0.788	0.818	0.804	0.816
			0.4		0.960	0.956	0.936	0.960	0.966	0.972	0.968	0.974
	(3,2)	2.5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
			0.2		0.824	0.826	0.854	0.874	0.900	0.902	0.900	0.892
			0.4		0.964	0.972	0.984	0.986	0.982	0.988	0.982	0.988
0.3	(3,2)	2.5		-0.5	0.988	0.988	0.994	0.998	1.000	1.000	1.000	1.000
		5			0.878	0.900	0.930	0.934	0.956	0.970	0.982	0.982
			0.2		0.734	0.772	0.762	0.782	0.792	0.804	0.808	0.822
			0.4		0.964	0.962	0.970	0.964	0.964	0.966	0.966	0.982
		2.5		-0.75	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			0.976	0.984	0.996	0.998	1.000	1.000	1.000	1.000
			0.2		0.714	0.752	0.800	0.812	0.830	0.840	0.844	0.860
			0.4		0.966	0.970	0.968	0.968	0.976	0.972	0.968	0.970
0.7	(3,2)	2.5		-0.5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
			0.2		0.852	0.868	0.866	0.878	0.874	0.870	0.864	0.872
			0.4		0.986	0.974	0.970	0.968	0.982	0.974	0.980	0.970
		2.5		-0.75	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		5			1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
			0.2		0.842	0.850	0.876	0.872	0.876	0.852	0.870	0.886
			0.4		0.968	0.966	0.974	0.984	0.984	0.974	0.962	0.968

* See notes to Table 1. In this table, jumps are generated as follows: Either Z_k is $Exp(\varsigma)$ or Z_k is $N(0.0, \sigma^2)$ Additionally, Δ=1/78 and $\zeta = 0.5$.

Table 3: **Monte Carlo Experiments - Test of $E(Z_k) = 0$ ***

λ_∞	(a,β)	ς	σ	Δ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
						EMPIRICAL SIZE						
						Constant Intensity, Z_k ***is*** $N(0.0,\sigma^2)$						
0.3	(0,0)		0.2	1/78	0.070	0.062	0.062	0.078	0.082	0.082	0.086	0.100
				1/156	0.082	0.086	0.076	0.066	0.070	0.068	0.092	0.090
						Self Excitement, Z_k ***is*** $N(0.0,\sigma^2)$						
0.3	(5,4)		0.2	1/78	0.060	0.056	0.068	0.080	0.060	0.078	0.080	0.082
				1/156	0.088	0.088	0.090	0.094	0.094	0.078	0.094	0.082
						EMPIRICAL POWER						
						Constant Intensity, Z_k ***is*** $Exp(\varsigma)$						
0.3	(0,0)	5		1/78	0.182	0.236	0.264	0.272	0.320	0.324	0.334	0.366
				1/156	0.858	0.880	0.928	0.934	0.964	0.972	0.970	0.982
0.5				1/78	0.414	0.458	0.480	0.508	0.562	0.584	0.616	0.650
				1/156	0.998	0.998	0.998	1.000	1.000	1.000	1.000	1.000
0.7				1/78	0.580	0.652	0.716	0.742	0.784	0.814	0.836	0.870
				1/156	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
						Constant Intensity, Z_k ***is*** $N(0.5,\sigma^2)$						
0.3	(0,0)		0.2	1/78	0.138	0.174	0.170	0.190	0.208	0.220	0.208	0.226
				1/156	0.676	0.718	0.728	0.778	0.810	0.868	0.872	0.886
0.5				1/78	0.274	0.310	0.324	0.352	0.394	0.428	0.428	0.454
				1/156	0.954	0.966	0.978	0.986	0.996	0.998	0.998	1.000
0.7				1/78	0.396	0.450	0.502	0.552	0.596	0.624	0.642	0.694
				1/156	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
						Self Excitement, Z_k ***is*** $Exp(\varsigma)$						
0.3	(5,4)	5		1/78	0.650	0.664	0.674	0.690	0.694	0.704	0.708	0.716
				1/156	0.928	0.946	0.968	0.972	0.982	0.990	0.994	0.994
0.5				1/78	0.836	0.846	0.854	0.872	0.872	0.874	0.894	0.904
				1/156	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.7				1/78	0.950	0.954	0.964	0.970	0.972	0.978	0.976	0.980
				1/156	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000
						Self Excitement, Z_k ***is*** $N(0.5,\sigma^2)$						
0.3	(5,4)		0.2	1/78	0.602	0.624	0.628	0.624	0.644	0.638	0.642	0.658
				1/156	0.852	0.870	0.878	0.886	0.906	0.940	0.944	0.958
0.5				1/78	0.784	0.804	0.812	0.820	0.810	0.828	0.828	0.844
				1/156	0.980	0.988	0.992	0.998	1.000	1.000	1.000	1.000
0.7				1/78	0.916	0.928	0.922	0.936	0.942	0.946	0.960	0.956
			0.2	1/156	0.996	0.994	0.998	1.000	1.000	1.000	1.000	1.000

* See notes to Table 1. Entries in the table are rejection frequencies for the zero mean jump test, based on $t_{\mu_Z,T,\Delta} = \sqrt{T}\frac{\widehat{\mu}^Z_{T,\Delta}}{\widehat{\sigma}_{\mu_Z}}$. In the experiments reported on in this table, $\rho = -0.25$. Results are similar for other values of ρ.

Table 4: **Monte Carlo Experiments - $S^{\beta}_{T^{+},\Delta}$ Self Excitement Test (Empirical Size) ***

λ_∞	(a,β)	ς	σ	Δ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
						Z_k ***is*** *Exp*(ς)						
0.3	(0,0)	5		1/78	0.114	0.108	0.120	0.138	0.124	0.146	0.126	0.130
				1/156	0.092	0.094	0.080	0.090	0.082	0.092	0.080	0.076
0.5				1/78	0.088	0.088	0.096	0.098	0.108	0.118	0.122	0.114
				1/156	0.070	0.076	0.068	0.070	0.088	0.078	0.076	0.068
0.7				1/78	0.088	0.098	0.122	0.128	0.108	0.124	0.118	0.122
				1/156	0.076	0.076	0.084	0.074	0.090	0.078	0.082	0.086
						Z_k ***is*** $N(0.5,\sigma^2)$						
0.3	(0,0)		0.1	1/78	0.116	0.124	0.126	0.124	0.136	0.144	0.138	0.146
				1/156	0.084	0.106	0.094	0.092	0.094	0.086	0.078	0.070
			0.2	1/78	0.118	0.122	0.136	0.142	0.136	0.150	0.146	0.134
				1/156	0.102	0.098	0.086	0.088	0.094	0.092	0.084	0.080
			0.4	1/78	0.120	0.116	0.132	0.140	0.130	0.146	0.150	0.132
				1/156	0.082	0.098	0.086	0.094	0.088	0.088	0.076	0.084
0.5			0.1	1/78	0.094	0.098	0.116	0.132	0.134	0.134	0.142	0.136
				1/156	0.082	0.084	0.078	0.090	0.100	0.086	0.088	0.074
			0.2	1/78	0.094	0.094	0.102	0.122	0.116	0.126	0.122	0.124
				1/156	0.086	0.086	0.088	0.088	0.092	0.080	0.076	0.066
			0.4	1/78	0.110	0.096	0.120	0.116	0.118	0.134	0.124	0.122
				1/156	0.096	0.090	0.078	0.092	0.092	0.088	0.090	0.084
0.7			0.1	1/78	0.104	0.112	0.128	0.132	0.130	0.132	0.104	0.112
				1/156	0.086	0.086	0.078	0.082	0.094	0.084	0.094	0.098
			0.2	1/78	0.092	0.102	0.116	0.132	0.118	0.126	0.120	0.126
				1/156	0.072	0.078	0.074	0.084	0.084	0.082	0.092	0.086
			0.4	1/78	0.076	0.094	0.092	0.104	0.110	0.122	0.104	0.098
				1/156	0.078	0.068	0.086	0.072	0.092	0.092	0.098	0.084

* See notes to Table 1. Numerical entries are rejection frequencies for the "self excitement test" of the null of no jump path dependence, based on $S^{\beta}_{T^{+},\Delta} = \max\left\{0, \frac{\sqrt{\frac{T^{+}}{\Delta}}\widehat{\beta}_{T^{+},\Delta}}{\widehat{\sigma}_{\beta,T^{+},\Delta}}\right\}$.

Table 5: **Monte Carlo Experiments - $S^{\beta}_{T^{+},\Delta}$ Self Excitement Test (Empirical Power) ***

λ_∞	(a,β)	ς	σ	Δ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
						Z_k ***is*** $Exp(\varsigma)$						
0.3	(3,2)	5		1/78	0.570	0.568	0.564	0.560	0.558	0.572	0.560	0.554
				1/156	0.566	0.558	0.556	0.552	0.554	0.558	0.548	0.554
	(5,4)			1/78	0.418	0.414	0.426	0.420	0.424	0.430	0.420	0.422
				1/156	0.430	0.432	0.422	0.428	0.436	0.430	0.428	0.424
	(7,5)			1/78	0.332	0.324	0.338	0.342	0.322	0.336	0.316	0.324
				1/156	0.318	0.320	0.324	0.314	0.324	0.318	0.300	0.310
0.5	(3,2)			1/78	0.704	0.708	0.704	0.706	0.706	0.696	0.700	0.700
				1/156	0.704	0.704	0.706	0.708	0.700	0.710	0.710	0.704
	(5,4)			1/78	0.550	0.548	0.544	0.534	0.540	0.536	0.520	0.524
				1/156	0.520	0.520	0.516	0.516	0.530	0.530	0.526	0.524
	(7,5)			1/78	0.446	0.424	0.438	0.444	0.434	0.434	0.430	0.436
				1/156	0.430	0.432	0.452	0.436	0.432	0.432	0.412	0.410
0.7	(3,2)			1/78	0.788	0.782	0.780	0.770	0.776	0.772	0.768	0.772
				1/156	0.780	0.780	0.772	0.770	0.762	0.752	0.760	0.750
	(5,4)			1/78	0.644	0.636	0.648	0.636	0.620	0.616	0.622	0.604
				1/156	0.618	0.612	0.614	0.626	0.616	0.606	0.608	0.620
	(7,5)			1/78	0.510	0.522	0.500	0.498	0.492	0.500	0.486	0.492
				1/156	0.486	0.474	0.468	0.456	0.446	0.438	0.446	0.444
						Z_k ***is*** $N(0.5,\sigma^2)$						
0.3	(3,2)	0.2		1/78	0.560	0.568	0.566	0.560	0.552	0.562	0.548	0.548
				1/156	0.534	0.540	0.534	0.542	0.540	0.544	0.540	0.542
	(5,4)			1/78	0.388	0.394	0.398	0.382	0.388	0.392	0.378	0.372
				1/156	0.396	0.378	0.382	0.384	0.402	0.384	0.380	0.380
	(7,5)			1/78	0.292	0.308	0.302	0.304	0.286	0.292	0.284	0.284
				1/156	0.284	0.278	0.272	0.278	0.268	0.274	0.268	0.262
0.5	(3,2)			1/78	0.676	0.686	0.666	0.674	0.668	0.658	0.642	0.640
				1/156	0.654	0.652	0.638	0.632	0.630	0.624	0.634	0.626
	(5,4)			1/78	0.506	0.510	0.502	0.480	0.464	0.462	0.460	0.442
				1/156	0.476	0.464	0.458	0.450	0.450	0.452	0.454	0.442
	(7,5)			1/78	0.386	0.388	0.380	0.372	0.362	0.356	0.346	0.344
				1/156	0.380	0.390	0.384	0.386	0.390	0.364	0.358	0.356
0.7	(3,2)			1/78	0.706	0.692	0.692	0.686	0.688	0.694	0.682	0.668
				1/156	0.708	0.696	0.694	0.668	0.674	0.662	0.654	0.654
	(5,4)			1/78	0.586	0.576	0.566	0.572	0.560	0.556	0.538	0.530
				1/156	0.560	0.562	0.562	0.556	0.544	0.532	0.530	0.522
	(7,5)			1/78	0.428	0.432	0.428	0.416	0.406	0.402	0.402	0.396
		0.2		1/156	0.420	0.418	0.416	0.420	0.410	0.404	0.390	0.384

* See notes to Table 4.

Table 6: **Monte Carlo Experiments - $\widetilde{S}^{\beta}_{T^{+},\Delta}$ Self Excitement Test (Empirical Size) ***

λ_∞	ς	σ	ρ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
0.3	2.5		0	0.170	0.170	0.160	0.156	0.166	0.176	0.160	0.170
	5			0.670	0.698	0.698	0.716	0.750	0.768	0.784	0.798
		0.2		0.146	0.142	0.150	0.142	0.144	0.150	0.166	0.174
		0.4		0.082	0.084	0.084	0.088	0.072	0.082	0.080	0.088
	2.5		-0.5	0.248	0.220	0.208	0.214	0.202	0.192	0.200	0.200
	5			0.694	0.718	0.742	0.734	0.738	0.764	0.764	0.786
		0.2		0.176	0.182	0.198	0.190	0.192	0.204	0.194	0.188
		0.4		0.116	0.108	0.102	0.098	0.086	0.096	0.096	0.082
0.7	2.5		0	0.026	0.032	0.020	0.020	0.012	0.014	0.012	0.012
	5			0.350	0.370	0.354	0.368	0.378	0.392	0.414	0.422
		0.2		0.016	0.016	0.018	0.012	0.016	0.012	0.008	0.010
		0.4		0.006	0.002	0.002	0.004	0.002	0.004	0.004	0.006
	2.5		-0.5	0.050	0.038	0.034	0.034	0.030	0.028	0.024	0.022
	5			0.342	0.364	0.364	0.380	0.400	0.422	0.402	0.396
		0.2		0.026	0.024	0.026	0.030	0.020	0.016	0.018	0.018
		0.4		0.012	0.010	0.014	0.012	0.008	0.006	0.004	0.004

* See notes to Table 1. In this table, jumps are generated as follows: Either Z_k is $\exp(\varsigma)$ or Z_k is $N(0.5, \sigma^2)$. Numerical entries are rejection frequencies for the "self excitement test" of the null of no jump path dependence, based on $\widetilde{S}^{\beta}_{T^{+},\Delta} = \max\left\{0, \sqrt{\frac{T^{+}}{\Delta}}\frac{\widetilde{\beta}_{T^{+},\Delta}}{\widetilde{\sigma}_{\beta,T^{+},\Delta}}\right\}$. Also, $(a, \beta) = (0, 0)$.

Table 7: **Monte Carlo Experiments - $\widetilde{S}^{\beta}_{T^+,\Delta}$ Self Excitement Test (Empirical Power) ***

λ_∞	(a,β)	ς	σ	Δ	ρ	$T1$	$T2$	$T3$	$T4$	$T5$	$T6$	$T7$	$T8$
0.3	(3,2)	5		1/78	0	0.360	0.346	0.390	0.392	0.422	0.434	0.448	0.454
			0.2			0.216	0.246	0.284	0.322	0.348	0.386	0.412	0.436
		5		1/156		0.218	0.242	0.250	0.270	0.298	0.308	0.336	0.330
			0.2			0.432	0.482	0.538	0.578	0.604	0.652	0.684	0.724
	(7,5)	5		1/78		0.440	0.478	0.494	0.504	0.546	0.574	0.592	0.616
			0.2			0.530	0.590	0.634	0.668	0.726	0.780	0.808	0.832
		5		1/156		0.508	0.540	0.576	0.604	0.632	0.682	0.710	0.742
			0.2			0.938	0.956	0.970	0.982	0.984	0.988	0.992	0.992
0.7	(3,2)	5		1/78		0.148	0.166	0.170	0.208	0.220	0.220	0.238	0.256
			0.2			0.366	0.430	0.468	0.536	0.582	0.630	0.686	0.698
		5		1/156		0.100	0.124	0.124	0.132	0.150	0.160	0.196	0.214
			0.2			0.626	0.680	0.722	0.762	0.816	0.840	0.880	0.898
	(7,5)	5		1/78		0.338	0.384	0.434	0.474	0.502	0.510	0.576	0.630
			0.2			0.810	0.858	0.906	0.930	0.938	0.964	0.976	0.982
		5		1/156		0.592	0.648	0.688	0.714	0.770	0.786	0.812	0.838
			0.2			0.990	0.998	0.998	1.000	1.000	1.000	1.000	1.000
0.3	(3,2)	5		1/78	-0.5	0.368	0.360	0.388	0.396	0.412	0.416	0.442	0.450
			0.2			0.210	0.236	0.262	0.304	0.324	0.370	0.384	0.420
		5		1/156		0.208	0.220	0.246	0.256	0.280	0.300	0.334	0.344
			0.2			0.422	0.472	0.516	0.568	0.608	0.648	0.690	0.712
	(7,5)	5		1/78		0.444	0.458	0.478	0.504	0.526	0.556	0.582	0.582
			0.2			0.526	0.600	0.650	0.706	0.748	0.784	0.810	0.838
		5		1/156		0.480	0.536	0.576	0.608	0.646	0.670	0.712	0.740
			0.2			0.924	0.950	0.958	0.966	0.978	0.988	0.994	0.998
0.7	(3,2)	5		1/78		0.146	0.156	0.174	0.182	0.204	0.220	0.212	0.222
			0.2			0.338	0.386	0.412	0.448	0.494	0.542	0.582	0.630
		5		1/156		0.114	0.132	0.150	0.158	0.166	0.174	0.188	0.204
			0.2			0.662	0.726	0.764	0.788	0.816	0.836	0.862	0.894
	(7,5)	5		1/78		0.362	0.380	0.448	0.488	0.528	0.564	0.598	0.636
			0.2			0.776	0.836	0.900	0.928	0.940	0.960	0.972	0.980
		5		1/156		0.532	0.602	0.668	0.692	0.732	0.756	0.776	0.812
			0.2			0.994	0.998	1.000	1.000	1.000	1.000	1.000	1.000

* See notes to Table 6.

Table 8: **Jump Test Statistics for Various Stocks and Time Spans ***

Time Span	3M	AAPL	AMZN	BAC	COST	GE	GS	IBM	INTC	JNJ	JPM
Time Span = 1 Quarter; 5-Minute Frequency Data											
2004-Q1	-4.03*	16.3*	-0.361	-3.89*	-0.436	-5.74*	-2.17*	-1.47	-2.39*	-7.17*	-2.11*
2004-Q2	-3.99*	0.700	6.87*	-5.34*	1.21	35.1*	3.07*	2.75*	-1.20	37.6*	-5.98*
2004-Q3	-5.21*	6.35*	-0.23	0.938	14.4*	-25.0*	1.30	3.59*	-2.25*	9.63*	6.97*
2004-Q4	-0.172	6.97*	3.31*	10.8*	13.0*	7.74*	4.72*	10.9*	0.652	0.250	-0.455
2013-Q1	3.99*	7.68*	-3.31*	-5.01*	-0.848	-3.71*	1.36	15.0*	0.194	-9.63*	-1.54
2013-Q2	-1.46	3.77*	-3.61*	-4.25*	0.390	-7.63*	-0.513	-22.3*	11.6*	-1.10	-0.671
2013-Q3	11.8*	6.07*	-2.11*	0.853	-3.90*	5.25*	-0.270	5.45*	-1.01	-0.866	4.57*
2013-Q4	19.4*	-11.2*	2.70*	8.20*	-2.59*	4.85*	19.5*	3.49*	4.53*	-0.418	11.2*
Time Span = 2 Quarters; 5-Minute Frequency Data											
2004-H1	-7.59*	19.3*	5.39*	-11.4*	0.580	24.6*	1.06	1.44	-3.79*	31.8*	-8.39*
2004-H2	-4.28*	13.0*	1.96*	13.9*	26.3*	-16.8*	5.09*	13.1*	-2.28*	9.93*	6.62*
2013-H1	-1.01	10.3*	-5.50*	-8.67*	-0.564	-10.6*	-0.693	-20.5*	14.7*	-5.55*	-2.42*
2013-H2	29.9*	-1.22	2.93*	10.4*	-6.86*	11.3*	15.6*	7.43*	2.05*	-1.56	14.3*
Time Span = 4 Quarters; 5-Minute Frequency Data											
2004	-12.3*	34.0*	6.44*	2.30*	25.6*	15.4*	5.05*	12.1*	-5.99*	42.5*	-3.88*
2013	27.2*	13.6*	-4.05*	-8.72*	-6.37*	-2.44*	11.1*	-12.7*	18.7*	-7.17*	7.38*
Time Span = 1 Quarter; 10-Minute Frequency Data											
2004-Q1	-8.88*	17.3*	-8.04*	-20.8*	2.73*	-5.44*	-4.98*	0.228	-6.13*	-6.76*	-4.46*
2004-Q2	-0.773	3.03*	15.5*	-15.4*	0.41	45.6*	0.85	0.208	-0.219	43.0*	-7.78*
2004-Q3	-19.2*	15.4*	-4.76*	0.046	7.27*	-20.5*	1.81*	1.63	1.62	-0.707	0.909
2004-Q4	9.10*	13.1*	2.49*	6.03*	22.4*	8.86*	2.42*	17.0*	2.71*	6.98*	-3.71*
2013-Q1	7.51*	9.35*	9.97*	-4.39*	-2.65*	-21.0*	2.30*	26.5*	4.00*	5.08*	-8.71*
2013-Q2	0.365	17.9*	-2.77*	-7.86*	-2.93*	-25.5*	-14.2*	-19.5*	13.6*	-9.32*	0.463
2013-Q3	7.14*	11.6*	3.19*	2.87*	-1.28	16.9*	4.48*	42.2*	2.37*	1.37	10.4*
2013-Q4	40.0*	-7.10*	4.97*	16.5*	11.4*	9.47*	29.4*	11.3*	8.24*	15.0*	12.1*
Time Span = 2 Quarters; 10-Minute Frequency Data											
2004-H1	-11.8*	21.5*	6.37*	-35.9*	2.84*	34.0*	-4.80*	0.646	-7.90*	43.2*	-13.7*
2004-H2	-7.74*	31.5*	-4.79*	7.18*	28.2*	-14.3*	4.08*	18.1*	3.43*	6.32*	-2.81*
2013-H1	4.19*	25.7*	9.28*	-11.1*	-5.48*	-50.6*	-19.5*	-13.8*	22.2*	-16.5*	-5.23*
2013-H2	52.5*	8.24*	8.82*	22.0*	11.2*	25.7*	31.5*	54.7*	8.92*	15.2*	20.9*
Time Span = 4 Quarters; 10-Minute Frequency Data											
2004	-19.4*	56.4*	0.233	-33.9*	30.7*	34.2*	-4.11*	17.5*	-3.99*	57.8*	-19.7*
2013	52.6*	46.4*	16.7*	-8.08*	0.661	-36.6*	-4.65*	39.5*	36.9*	-1.31	9.34*

* See notes to Table 1. Test implemented is the jump intensity test. Statistics superscripted with a * indicate rejections at a 10% level. Time span ranges from one quarter to one half of a year. Data are sampled at 5-minute frequency. Companies for which statistics are reported include 3M - 3M Company; AAPL - Apple Inc.; AMZN - Amazon.com, Inc.; BAC - Bank of America Corporation; COST - Costco Wholesale Corporation; GE - General Electric Company; GS - The Goldman Sachs Group, Inc.; IBM - International Business Machines Corporation; INTC - Intel Corporation; JNJ - Johnson & Johnson; and JPM - JPMorgan Chase & Co.