Consistent Factor Estimation and Forecasting in Factor-Augmented VAR Models*

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preliminary and incomplete

Abstract

In this paper we establish that conditional mean functions associated with h-step ahead forecasting equations implied by a factor augmented vector autoregressions (FAVARs) can be consistently estimated in the case where factor pervasiveness does not hold. In particular, we begin by stating a common assumption of factor pervasiveness in which all available predictor variables (excepting a negligible subset) load significantly on the underlying factors. We then establish that even when this assumption is relaxed, consistent factor estimation can be achieved if one pre-screens the variables and successfully prunes out the irrelevant ones. Furthermore, use of factors estimated in this manner when constructing h-step ahead forecasting equations implied by FAVAR models enables the consistent estimation of the conditional mean function of said equations, and conditional mean functions constructed used our precedure are consistently estimable in a wide range of situations, including cases where violation of factor pervasiveness is such that consistent estimation is precluded in the absence of variable pre-screening.

Keywords: Factor analysis, factor augmented vector autoregression, forecasting, moderate deviation, principal components, self-normalization, variable selection.

JEL Classification: C32, C33, C38, C52, C53, C55.

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1 Introduction

In economics, three of the key areas in machine learning that have drawn considerable attention in recent years include variable selection, dimension reduction and shrinkage. One reason for this is the availability of new high frequency and high dimensional datasets that are being analyzed in areas ranging from targeted marketing and customer segmentation to forecasting and maroeconomic policy making. This has in turn led to numerous theoretical advances in the areas of estimation, implementation, and inference using techinques such as the least absolute shrinkage operator (lasso) and principal components analysis (PCA). In this paper, we build on pathbreaking work due to Bai and Ng (2002), Stock and Watson (2002a,b), Bai (2003), Forni, Hallin, Lippi, and Reichlin (2005), and Bai and Ng (2008), in which methods for constructing forecasts based on factor-augmented regression models are developed and analyzed. In particular, we establish that latent factors that are critical to the estimation of factor augmented vector autoregressions (FAVARs) can be consistently estimated in cases where factor pervasiveness does not hold, where by factor pervasivness we mean that (almost) all available predictors load significantly on a set of factors that we wish to estimate. To do so, we draw on results of Chao, Qiu, and Swanson (CQS: 2023a), where a completely consistent variable selection procedure useful for specifying FAVAR models is developed. We then establish that the conditional mean of the infeasible h-step ahead forecasting equation implied by an FAVAR can be consistently estimated.

As discussed above, a key assumption commonly used in the factor analysis literature to show consistent factor estimation is that of factor pervasiveness. This assumption presupposes that all available predictor variables in a dataset, with the possible exception of a negligible number of them, load significantly on the underlying factors. Needless to say, this assumption may not be satisfied by many datasets that are available for empirical research. Indeed, a likely scenario is that there is significant underlying heterogeneity, so that some of the available variables are relevant in the sense that they load significantly on the underlying factors, whereas others are irrelevant, in the sense that they do not share any common dynamic structure with each other or with the relevant variables in the dataset. In this paper, we begin by establishing that, under failure of factor pervasiveness in a stylized model with one factor, consistency cannot be achieved, and indeed $\widehat{f}_t \xrightarrow{p} 0$, as $N, T \to \infty$, where f_t is a latent factor, N is the number of variables in the dataset being modelled, and T is the number of time series observations. Findings such as this are the impetus for the work of Chao, Qiu, and Swanson (CQS: 2023a), where a variable selection procedure is developed for pre-selecting relevant predictor variables for use in the consistent estimation of latent factors in an FAVAR model.

Their variable selection procedure is based on the use of easy to construct self normalized statistics measuring the covariation between target variables to be predicted and possible predictor variables to be used in factor estimation. CQS (2023a) show that for their procedure, the probability of Type I and Type II errors goes to zero, asymptotically, implying that the procedure is completely consistent. This property turns out to be important because if one tries to simply control the probability of a Type I error at some predetermined level, which is the typical approach used in multiple hypothesis testing, then one will not in general be able to estimate factors consistently, even up to an invertible matrix transformation. A main result of the current paper is to show that factors estimated using predictor variables selected using the procedure of CQS (2023a) are consistent, up to a rotation. With these results in hand, we then show that by using variables selected via our pre-screening procedure to estimate the underlying factors, and then inserting these factor estimates into h-step ahead forecasting equations implied by a FAVAR model, we can consistently estimate the conditional mean function of the said equations. Importantly, we argue that this result allows the conditional mean function of a factor-augmented forecasting equation to be consistently estimable in a wide range of situations, and in particular in situations where there are violations of factor pervasiveness.

Finally, in order to illustrate the methods discussed in this paper, we analyze a large dataset. This part of the paper is to be completed.

Some of the research reported here is related to the well-known supervised principal components method proposed by Bair, Hastie, Paul, and Tibshirani (2006). Additionally, our research is related to some interesting recent work by Giglio, Xiu, and Zhang (2021), who propose a method for selecting test assets, with the objective of estimating risk premia in a Fama-MacBeth type framework. A crucial difference between the variable selection procedure proposed in our paper and those proposed in these papers is that we use a score statistic that is self-nomalized, whereas the aforementioned papers do not make use of statistics that involve self-normalization. An important advantage of self-normalized statistics is their ability to accommodate a much wider range of possible tail behavior in the underlying distributions, relative to their non-self-normalized counterparts. This makes self-normalized statistics better suited for various types of economic and financial applications, where the data are known not to exhibit the type of exponentially decaying tail behavior assumed in much of the statistics literature on high-dimensional models. In addition, the type of models studied in Bair, Hastie, Paul, and Tibshirani (2006) and Giglio, Xiu, and Zhang (2021) differ significantly from the FAVAR model studied here. In particular, Bair, Hastie, Paul, and Tibshirani (2006) study a one-factor model in an *i.i.d.* Gaussian framework so that complications introduced by dependence and non-normality of distribution are not considered in their paper. Giglio, Xiu, and Zhang (2021) do make certain high-level assumptions which may potentially accommodate some dependence both cross-sectionally and intertemporally, but they do not consider the implications of variable selection and factor estimation for forecasting, and the model that they consider is very different from the type of dynamic vector time series model studied here.

Our research is also closely related to the work of Bai and Ng (2021), who provide results which show that factors can still be estimated consistently in certain situations where the factor loadings are weaker than that implied by the conventional pervasiveness assumption, although in such cases the rate of convergence of the factor estimator is slower and additional assumptions are needed. As discussed in the next section of this paper, their factor consistency result relies on a key condition, and the appropriateness of this condition depends on how severely the condition of factor pervasiveness is violated, which is ultimately an empirical issue.[1]

The rest of the paper is organized as follows. In Section 2 , we provide our counterexample, stated formally as Theorem 2.1, which shows that a latent factor may be inconsistently estimated when the standard assumption of factor pervasiveness does not hold. In Section 3, we discuss the FAVAR model, the variable selection procedure of CQS (2023a), and the assumptions that are required in the sequel. Section 4 gathers our theoretical results on the consistent estimation of latent factors, up to an invertible matrix transformation, as well as results on the consistent estimation of the h-step ahead predictor, based on the FAVAR model. Section 5 presents the results of an empirical illustration where our forecasting approach is compared with related approaches in the literature. Finally, Section 6 offers concluding remarks. Proofs of the main theorems and some supporting lemmas are given in the appendices of this paper.

Before proceeding, we first say a few words about some of the frequently used notation in this paper. Throughout, let $\lambda_{(j)}(A)$, $\lambda_{\max}(A)$, $\lambda_{\min}(A)$, and $tr(A)$ denote, respectively, the j^{th} largest eigenvalue, the maximal eigenvalue, the minimal eigenvalue, and the trace of a square matrix A. Similarly, let $\sigma_{(j)}(B)$, $\sigma_{\max}(B)$, and $\sigma_{\min}(B)$ denote, respectively, the j^{th} largest singular value, the maximal singular value, and the minimal singular value of a matrix B, which is not restricted to be a square matrix. In addition, let $\|a\|_2$ denote the usual Euclidean norm when applied to a (finite-dimensional) vector a. Also, for a matrix A, $\|A\|_2 \equiv \max\left\{\sqrt{\lambda(A'A)} : \lambda(A'A) \text{ is an eigenvalue of } A'A\right\}$ denotes the matrix spectral norm, and $\|A\|_F \equiv \sqrt{tr\{A'A\}}$ denotes the Frobenius norm. For two random variables X and Y, write $X \sim Y$, if $X/Y = O_p(1)$ and $Y/X = O_p(1)$. Furthermore, let $|z|$ denote the absolute value or the modulus of the number z; let $\lfloor \cdot \rfloor$ denote the floor function, so that $\lfloor x \rfloor$ gives the integer part of the real number x, and let $\iota_p = (1, 1, ..., 1)'$ denote a $p \times 1$ vector

[1] Various authors have documented cases in economics-related research where empirical results suggest that the underlying factors may be quite weak, so that the rate condition given in Bai and Ng (2021) may not be appropriate. See, for example, the discussions in Jagannathan and Wang (1998), Kan and Zhang (1999), Harding (2008), Kleibergen (2009), Ontaski (2012), Bryzgalova (2016), Burnside (2016), Gospodinov, Kan, and Robotti (2017), Anatolyev and Mikusheva (2021), and Freyaldenhoven (2021a,b).

of ones. Finally, the abbreviation w.p.a.1 stands for "with probability approaching one".

2 Inconsistency in High-Dimensional Factor Estimation

To provide some motivation for the problem we will be studying in this paper, consider the following simple, stylized one-factor model:

$$\underset{N\times 1}{Z_t} = \underset{N\times 1}{\gamma}\underset{1\times 1}{f_t} + \underset{N\times 1}{u_t}, \; t = 1, ..., T \tag{1}$$

for which we make the following assumption.

Assumption 2-1: (a) $\{u_t\} \equiv i.i.d.N(0, I_N)$;(b) $\{f_t\} \equiv i.i.d.N(0,1)$;and (c) u_s and f_t are independent for all t, s.

Much of the literature on factor analysis focuses on the case where the factors are pervasive. In the special case of the simple one factor model given in expression (1) above, pervasiveness means that:

$$\frac{\|\gamma\|_2^2}{N} \to c,$$

for some constant c such that $0 < c < \infty$, where $\|\gamma\|_2 = \sqrt{\gamma'\gamma}$. In practice, however, one may have a high-dimensional data vector Z_t such that not all of the components of Z_t load significantly on the underlying factor, f_t. In particular, let $\mathcal{P}$ be a permutation matrix which reorders the components of Z_t, so that $\mathcal{P}Z_t$ can be partitioned as follows:

$$\mathcal{P}Z_t = \begin{pmatrix} \underset{N_1\times 1}{Z_t^{(1)}} \\ \underset{N_2\times 1}{Z_t^{(2)}} \end{pmatrix},$$

where $Z_t^{(1)} = \gamma^{(1)} f_t + u_t^{(1)}$ and $Z_t^{(2)} = u_t^{(2)}$ and where all components of the $N_1 \times 1$ vector $\gamma^{(1)}$ are different from zero, so that the components of $Z_t^{(1)}$ all load significantly on f_t, whereas the components of $Z_t^{(2)}$ do not. Of course, an empirical researcher will not typically have à priori knowledge as to which components of Z_t will load significantly on f_t and which will not. The following result shows that if one proceeds with factor estimation assuming that the factor is pervasive, then the usual estimator of a factor based on principal component methods may be inconsistent and may, in fact, behave in a rather pathological manner in large samples. To consider this possibility, assume the following condition, which implies a violation of the pervasiveness assumption.

Assumption 2-2: As $N, T \to \infty$, let $\|\gamma\|_2 \to \infty$ such that:

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right),$$

for some constant c, such that $0 < c < \infty$, and for some constant κ, such that $0 < \kappa < 1$. Note that under Assumption 2-2:

$$\frac{\|\gamma\|_2^2}{N} \sim (TN^{\kappa})^{-\frac{1}{(1+\kappa)}} \to 0 \text{ as } N, T \to \infty,$$

so that the factor does not satisfy the pervasiveness assumption. This can, of course, occur if a significant proportion of the components of γ are zero or are very small. Next, let $\widehat{\pi}_1 / \|\widehat{\pi}_1\|_2$ denote the (normalized) eigenvector associated with the largest eigenvalue of the sample covariance matrix, $\widehat{\Sigma}_Z = \mathbf{Z}'\mathbf{Z}/T$, where $\mathbf{Z} = (Z_1, ..., Z_T)'$. Then, the usual principal component estimator of f_t is given by:

$$\widehat{f}_t = \frac{\langle \widehat{\pi}_1, Z_t \rangle}{\sqrt{N} \|\widehat{\pi}_1\|_2}.$$

The following theorem characterizes the asymptotic behavior of this estimator under the assumptions given above.

Theorem 2.1: *Suppose that Assumptions 2-1 and 2-2 hold. Then, for all t:* $\widehat{f}_t \xrightarrow{p} 0$, as $N, T \to \infty$.

It is well-known that without further identifying assumptions, such as those given in Assumption F1 of Stock and Watson (2002a), factors can only be estimated consistently up to an invertible matrix transformation. However, even in cases where we are not willing to specify enough conditions so as to fully identify the factors, estimating the factors consistently up to an invertible matrix transformation will often suffice for many purposes. One such case is when we are trying to forecast using a factor-augmented vector autoregression (FAVAR). As we will show in results given in Section 4 of this paper, point forecasts constructed using factors which are estimated consistently up to an invertible matrix transformation will nevertheless converge in probability to the desired infeasible forecast (i.e., the conditional mean of the FAVAR), that in turn depends on the true unobserved factors. On the other hand, the problem illustrated by the result given in Theorem 1 is different and is in some sense more problematic and pathological. The estimated factor in Theorem 1 converges to zero regardless of what happens to be the realized value of the true latent factor. In this case, one clearly cannot consistently estimate the conditional mean of the FAVAR.

Theorem 1 is related to results previously given in the statistics literature showing the possible inconsistency of sample eigenvectors as estimators of population eigenvectors in high dimensional situations. See, for example, Paul (2007), Johnstone and Lu (2009), Shen, Shen, Zhu, and Marron

(2016), and Johnstone and Paul (2018). However, most of the results in the statistics literature are not explicitly framed in the setting of a factor model, but are instead derived for the related spiked covariance model. Theorem 1 is intended to give an inconsistency result of this type, but in a context that may be more familiar to researchers in economics.

It should also be noted that, in an interesting and thought-provoking recent paper, Bai and Ng (2021) provide results which show that factors can still be estimated consistently in certain situations where the factor loadings are weaker than that implied by the conventional pervasiveness assumption, but that in such cases the rate of convergence is slower and additional assumptions are needed. To understand the relationship between their results and the example given above, note that a key condition for the consistency result given in their paper, when expressed in terms of our notation, is the assumption that $N/\left(T\left\|\gamma\right\|_2^2\right) \to 0$[2]. On the other hand, if $N/\left(T\left\|\gamma\right\|_2^2\right) \to c_1$, for some positive constant c_1, or even worse, if $N/\left(T\left\|\gamma\right\|_2^2\right) \to \infty$, which is essentially what is specified in Assumption 2-2 above, then consistent factor estimation cannot be achieved[3]. Hence, whether or not consistent factor estimation can be attained depends on how nonpervasive the factors are, which is ultimately an empirical question, and which depends on the application and on the dataset employed. Moreover, various authors have now documented cases where empirical results suggest that the underlying factors may be quite weak, so that the rate condition given in Bai and Ng (2021) may not be appropriate, at least for some of the situations for which factor modeling is of interest. For example, see Jagannathan and Wang (1998), Kan and Zhang (1999), Harding (2008), Kleibergen (2009), Ontaski (2012), Bryzgalova (2016), Burnside (2016), Gospodinov, Kan, and Robotti (2017), Anatolyev and Mikusheva (2021), and Freyaldenhoven (2021a,b). In such cases, it is of interest to explore the possibility that the weakness in the loadings is not uniform across all variables, but rather is due to the fact that only a small percentage of the variables loads significantly on the underlying factors. Furthermore, even if the empirical situation of interest is one where, strictly speaking, the condition $N/\left(T\left\|\gamma\right\|_2^2\right) \to 0$ does hold, it may still be beneficial in some such instances to do variable pre-screening. This is particularly true in situations where the condition $N/\left(T\left\|\gamma\right\|_2^2\right) \to 0$ is "barely" satisfied, in which case one would expect to pay a rather hefty finite sample price for not pruning out variables that do not load significantly on the underlying factors, since these variables will add unwanted noise to the estimation process. For all these reasons, there is a clear need to develop methods that will enable empirical researchers

[2]See Assumption A4 of Bai and Ng (2021). Note that Bai and Ng (2021) state this condition in the form $N/\left(TN^{\alpha}\right) \to 0$, for some $\alpha \in (0,1]$, but since part (ii) of their Assumption A2, when specialized to the one factor model studied here, simplifies to the condition that $\lim_{N\to\infty} \left\|\gamma\right\|_2^2/N^{\alpha} = \sigma_{\Lambda} > 0$, it is easy to see that their Assumption A4 is equivalent to the condition that $N/\left(T\left\|\gamma\right\|_2^2\right) \to 0$.

[3]Note that Assumption 2-2 is actually stronger than required in order to show inconsistency, but that we impose this condition to highlight the fact that, in this case, not only is the estimator of the factor inconsistent but it actually converges to zero.

to pre-screen the components of Z_t, so that variables which are informative and helpful to the estimation process can be properly identified.

3 Model, Assumptions, and Variable Selection in High Dimensions

Following CQS (2023a), we begin by considering the following p^{th}-order factor-augmented vector autoregression (FAVAR):

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}, \tag{2}$$

where

$$\begin{aligned}
\underset{(d+K)\times 1}{W_{t+1}} &= \begin{pmatrix} \underset{d\times 1}{Y_{t+1}} \\ \underset{K\times 1}{F_{t+1}} \end{pmatrix}, \ \underset{(d+K)\times 1}{\varepsilon_{t+1}} = \begin{pmatrix} \underset{d\times 1}{\varepsilon^Y_{t+1}} \\ \underset{K\times 1}{\varepsilon^F_{t+1}} \end{pmatrix}, \ \underset{(d+K)\times 1}{\mu} = \begin{pmatrix} \underset{d\times 1}{\mu_Y} \\ \underset{K\times 1}{\mu_F} \end{pmatrix}, \text{ and} \\
\underset{(d+K)\times(d+K)}{A_g} &= \begin{pmatrix} \underset{d\times d}{A_{YY,g}} & \underset{d\times K}{A_{YF,g}} \\ \underset{K\times d}{A_{FY,g}} & \underset{K\times K}{A_{FF,g}} \end{pmatrix}, \text{ for } g = 1, ..., p.
\end{aligned}$$

This system of equations, where Y_t denotes the vector of observable economic variables, and F_t is a vector of unobserved (latent) factors can also be written is several alternative ways, the following two of which are are variously used throughout this paper. Namely:

$$Y_{t+1} = \mu_Y + A_{YY}\underline{Y}_t + A_{YF}\underline{F}_t + \varepsilon^Y_{t+1}, \tag{3}$$
$$F_{t+1} = \mu_F + A_{FY}\underline{Y}_t + A_{FF}\underline{F}_t + \varepsilon^F_{t+1}, \tag{4}$$

where

$$\begin{aligned}
\underset{d\times dp}{A_{YY}} &= \begin{pmatrix} A_{YY,1} & A_{YY,2} & \cdots & A_{YY,p} \end{pmatrix}, \ \underset{d\times Kp}{A_{YF}} = \begin{pmatrix} A_{YF,1} & A_{YF,2} & \cdots & A_{YF,p} \end{pmatrix}, \\
\underset{K\times dp}{A_{FY}} &= \begin{pmatrix} A_{FY,1} & A_{FY,2} & \cdots & A_{FY,p} \end{pmatrix}, \ \underset{K\times Kp}{A_{FF}} = \begin{pmatrix} A_{FF,1} & A_{FF,2} & \cdots & A_{FF,p} \end{pmatrix}, \\
\underset{dp\times 1}{\underline{Y}_t} &= \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \text{ and } \underset{Kp\times 1}{\underline{F}_t} = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix},
\end{aligned} \tag{5}$$

and

$$\underset{(d+K)p\times 1}{\underline{W}_t} = \alpha + A\underline{W}_{t-1} + E_t,$$

where $\underline{W}_t = \begin{pmatrix} W_t' & W_{t-1}' & \cdots & W_{t-p+2}' & W_{t-p+1}' \end{pmatrix}'$ and where

$$\alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & 0 & 0 \\ 0 & I_{d+K} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}, \text{ and } E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

The companion form given in equation (6) is convenient for establishing certain moment conditions on $\underline{Y}_t$ and $\underline{F}_t$, given a moment condition on ε_t, and for establishing certain mixing properties of the FAVAR model, as shown in the proofs of Lemmas C-4 and C5 in Appendix C below. It remains to define the relationship between the F_t and the variables used to extract these factors. To do this, we assume that:

$$\underset{N\times 1}{Z_t} = \underset{N\times Kp}{\Gamma}\underline{F}_t + u_t, \tag{7}$$

where the properties of u_t are given in Assumptions 3-3 and 3-4, below. Following Chao, Qiu, and Swanson (2023a), we assume that not all components of Z_t provide useful information for estimating $\underline{F}_t$, implying that the $N \times Kp$ parameter matrix Γ may have some rows whose elements are all zero. More precisely, let the $1 \times Kp$ vector, γ_i', denote the i^{th} row of Γ, and assume that the rows of the matrix Γ can be divided into two classes:

$$H = \{k \in \{1, ..., N\} : \gamma_k = 0\} \text{ and} \tag{8}$$

$$H^c = \{k \in \{1, ..., N\} : \gamma_k \neq 0\}. \tag{9}$$

Thus, there exists a permutation matrix $\mathcal{P}$ such that $\mathcal{P}Z_t = \begin{pmatrix} Z_t^{(1)\prime} & Z_t^{(2)\prime} \end{pmatrix}'$, where

$$\underset{N_1\times 1}{Z_t^{(1)}} = \Gamma_1\underline{F}_t + u_t^{(1)} \tag{10}$$

$$\underset{N_2\times 1}{Z_t^{(2)}} = u_t^{(2)}. \tag{11}$$

In this way, the components of $Z_t^{(1)}$ can be interpreted as "information" variables that are useful for estimating $\underline{F}_t$. On the other hand, for the purpose of factor estimation, the components of the subvector $Z_t^{(2)}$ are pure "noise" variables, as they do not load on the underlying factors and only add noise if they are included in the factor estimation process. Given that an empirical researcher will often not have prior knowledge as to which variables are elements of $Z_t^{(1)}$ and which are elements of $Z_t^{(2)}$, Theorem 2.1 suggests the need for a variable selection procedure which will allow us to properly identify the components of of $Z_t^{(1)}$ and to use only these variables when we try to estimate $\underline{F}_t$; for, if we unknowingly include too many components of $Z_t^{(2)}$ in the estimation process, then inconsistent estimation in the sense described in the previous section can result.[4] As discussed in CQS (2023a), there is an important related paper by Bai and Ng (2021) that establishes factor estimator consistency for cases where $N/(TN_1) \to 0$. For cases where $N/(TN_1) \to c$, or $N/(TN_1) \to \infty$, where c is a constant, their result does not hold. In this paper, we establish that consistency can be achieved in our context even if $N/(TN_1) \nrightarrow 0$, if one pre-screens variables using the self-normalized statistics outlined below. This is important because the degree of factor pervasiveness is ultimately data dependent, and one way to estimate N_1 invloves utilizing the variable screening statistic that is discussed in the sequel.

In the sequel, we require the following assumptions.

Assumption 3-1: Suppose that:

$$\det\left\{I_{(d+K)} - A_1 z - \cdots - A_p z^p\right\} = 0, \text{ implies that } |z| > 1. \tag{12}$$

Assumption 3-2: Let ε_t satisfy the following set of conditions: (a) $\{\varepsilon_t\}$ is an independent sequence of random vectors with $E[\varepsilon_t] = 0 \ \forall t$; (b) there exists a positive constant C such that $\sup_t E\|\varepsilon_t\|_2^6 \leq C < \infty$; (c) ε_t admits a density g_{ε_t} such that, for some positive constant $M < \infty, \sup_t \int |g_{\varepsilon_t}(\upsilon - u) - g_{\varepsilon_t}(\upsilon)| \, d\varepsilon \leq M|u|$, whenever $|u| \leq \overline{\kappa}$ for some constant $\overline{\kappa} > 0$; and (d) there exists a constant $\underline{C} > 0$ such that $\inf_t \lambda_{\min}\{E[\varepsilon_t \varepsilon_t']\} \geq \underline{C} > 0$.

Assumption 3-3: Let $u_{i,t}$ be the i^{th} element of the error vector u_t in expression (7), and we assume that it satisfies the following conditions: (a) $E[u_{i,t}] = 0$ for all i and t; (b) there exists a positive constant $\overline{C}$ such that $\sup_{i,t} E|u_{i,t}|^7 \leq \overline{C} < \infty$, and there exists a constant $\underline{C} > 0$ such that $\inf_{i,t} E\left[u_{i,t}^2\right] \geq \underline{C}$; (c) define $\mathcal{F}_{i,-\infty}^t = \sigma(....,u_{i,t-2}, u_{i,t-1}, u_t)$, $\mathcal{F}_{i,t+m}^\infty = \sigma(u_{i,t+m}, u_{i,t+m+1}, u_{i,t+m+2}, ...)$, and

$$\beta_i(m) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{i,-\infty}^t\right) - P(B)\right| : B \in \mathcal{F}_{i,t+m}^\infty\right\}\right].$$

[4] In statistics, there is a growing literature on the potential inconsistency of sample eigenvectors in high dimensional problems, as discussed in Paul (2007), Johnstone and Lu (2009), Shen Shen, Zhu, and Marron (2016), and Johnstone and Paul (2018).

Assume that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}, \text{ for all } i;$$

and (d) there exists a positive constant C such that $\sup_t \left(\frac{1}{N_1} \sum_{i \in H^c} \sum_{k \in H^c} |E[u_{i,t} u_{k,t}]| \right) \leq C < \infty$ for every positive integer N_1, where H^c is defined in expression (9) above.

Assumption 3-4: ε_t and $u_{i,s}$ are independent, for all i, t, and s.

Assumption 3-5: There exists a positive constant $\overline{C}$, such that $\sup_{i \in H^c} \|\gamma_i\|_2 \leq \overline{C} < \infty$ and $\|\mu\|_2 \leq \overline{C} < \infty$, where $\mu = (\mu_Y', \mu_F')'$.

Assumption 3-6: There exists a positive constant $\overline{C}$, such that:

$$0 < \frac{1}{\overline{C}} \leq \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \leq \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right) \leq \overline{C} < \infty \text{ for all } N_1,\ N_2 \text{ sufficiently large,}$$

where N_1 is the number of components of the subvector $Z_t^{(1)}$ and N_2 is the number of components of the subvector $Z_t^{(2)}$, as previously defined in expressions (10) and (11).

Assumption 3-7: Let A be as defined in expression (6) above, and let the eigenvalues of the matrix $I_{(d+K)p} - A$ be sorted so that:

$$\left|\lambda_{(1)}\left(I_{(d+K)p} - A\right)\right| \geq \left|\lambda_{(2)}\left(I_{(d+K)p} - A\right)\right| \geq \cdots \geq \left|\lambda_{((d+K)p)}\left(I_{(d+K)p} - A\right)\right| = \overline{\phi}_{\min}.$$

Suppose that there is a constant $\underline{C} > 0$ such that

$$\sigma_{\min}\left(I_{(d+K)p} - A\right) \geq \underline{C}\,\overline{\phi}_{\min} \tag{13}$$

In addition, there exists a positive constant $\overline{C} < \infty$ such that, for all positive integer j,

$$\sigma_{\max}\left(A^j\right) \leq \overline{C} \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}. \tag{14}$$

Assumption 3-1 is the stability condition that one typically assumes for a stationary VAR process, although we allow for possible heterogeneity in the distribution of ε_t across time, so that our FAVAR process is not necessarily a strictly stationary process. Under Assumption 3-1, there exists a vector moving average representation for the FAVAR process. Assumption 3-1 is a well known assumption that is equivalent to the condition that $\det\left\{I_{(d+K)} - Az\right\} = 0$ implies that $|z| > 1$.

Since the factor loading matrix Γ is an $N \times Kp$ matrix, where $N = N_1 + N_2$, the matrix $\Gamma'\Gamma$ will have order of magnitude equal to N if the factors are pervasive. Much of the factor analysis

literature in both econometrics and statistics has studied the case where factors are pervasive in this sense. For example, see Bai and Ng (2002), Stock and Watson (2002a), Bai (2003), and Fan, Liao, and Mincheva (2011, 2013). Assumption 3-6 allows for possible violations of this conventional pervasiveness assumption, which will occur in our setup when $N_1/N \to 0$.
Finally, Assumption 3-7 imposes a condition whereby the extreme singular values of the matrices A^j and $I_{(d+K)p} - A$ have bounds that depend on the extreme eigenvalues of these matrices. For further discussion of this Assumption, see CQS (2023a).

Note that Assumptions 3-1, 3-2(a)-(c), and 3-7 are sufficient to prove Lemma C-5 in Appendix C[5], which states that the process $\{W_t\}$ generated by the FAVAR model given in expression (2) is a β-mixing process with β-mixing coefficient satisfying:

$$\beta_W(m) \leq a_1 \exp\{-a_2 m\},$$

for some positive constants a_1 and a_2, with

$$\beta_W(m) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{A}_{-\infty}^t\right) - P(B)\right| : B \in \mathcal{A}_{t+m}^\infty\right\}\right],$$

and with $\mathcal{A}_{-\infty}^t = \sigma(..., W_{t-2}, W_{t-1}, W_t)$ and $\mathcal{A}_{t+m}^\infty = \sigma(W_{t+m}, W_{t+m+1}, W_{t+m+2}, ...)$. Note that Assumption 3-2 (c) rules out situations such as that given in the famous counterexample presented by Andrews (1984) which shows that a first-order autoregression with errors having a discrete Bernoulli distribution is not α-mixing, even if it satisfies the stability condition. Conditions similar to Assumption 3-2(c) have also appeared in previous papers, such as Gorodetskii (1977) and Pham and Tran (1985), which seek to provide sufficient conditions for establishing the α or β mixing properties of linear time series processes.

Prior to presenting the main theorems of this paper, we first summarize the variable selection procedure based on self-normalized statistics that is outlined in CQS (2023a), and draws on path-breaking moderate deviation results from Chen, Shao, Wu, and Xu (2016). To accommodate data dependence, consider self-nomalized statistics that are constructed from observations which are first split into blocks in a manner similar to the kind of construction one would employ in implementing a block bootstrap or in proving a central limit theorem using the blocking technique. One such statistic has the form of an ℓ_∞ norm and is given by:

$$\max_{1\leq \ell \leq d} |S_{i,\ell,T}| = \max_{1\leq \ell \leq d} \left| \frac{\overline{S}_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}} \right|, \tag{15}$$

[5]For a statement and proof of Lemma C-5, see Appendix C below.

where

$$\overline{S}_{i,\ell,T} = \sum_{r=1}^{q} \sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it} y_{\ell,t+1} \text{ and} \tag{16}$$

$$\overline{V}_{i,\ell,T} = \sum_{r=1}^{q} \left[\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} Z_{it} y_{\ell,t+1} \right]^2. \tag{17}$$

Here, Z_{it} denotes the i^{th} component of Z_t , $y_{\ell,t+1}$ denotes the ℓ^{th} component of Y_{t+1}, $\tau_1 = \lfloor T_0^{\alpha_1} \rfloor$, and $\tau_2 = \lfloor T_0^{\alpha_2} \rfloor$, where $1 > \alpha_1 \geq \alpha_2 > 0$, $\tau = \tau_1 + \tau_2$, $q = \lfloor T_0/\tau \rfloor$, and $T_0 = T - p + 1$. Note that the statistic given in expression (15) can be interpreted as the maximum of the (self-normalized) sample covariances between the i^{th} component of Z_t and the components of Y_{t+1}. A second statistic has the form of a pseudo-L_1 norm and is given by:

$$\sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right| = \sum_{\ell=1}^{d} \varpi_\ell \left| \frac{\overline{S}_{i,\ell,T}}{\sqrt{\overline{V}_{i,\ell,T}}} \right|,$$

where $\overline{S}_{i,\ell,T}$ and $\overline{V}_{i,\ell,T}$ are as defined in expressions (16) and (17) above and where $\{\varpi_\ell : \ell = 1,..,d\}$ denotes pre-specified weights, such that $\varpi_\ell \geq 0$, for every $\ell \in \{1,...,d\}$ and $\sum_{\ell=1}^{d} \varpi_\ell = 1$. In order to keep the effects of dependence under control, the construction of these statistics is based only on observations in every other block. In order to consistently estimate the factors up to an invertible matrix transformation, the variable selection procedure here must be such that the the probability of a false positive and the probability of a false negative converge to zero as N_1, N_2, $T \to \infty$[6]. This is different from the typical multiple hypothesis testing approach whereby one tries to control the familywise error rate (or, alternatively, the false discovery rate), so that it is no greater than 0.05, say, but does not try to ensure that this probability goes to zero as the sample size grows.

In order to implement this procedure, it reamins only to determine whether the i^{th} component of Z_t is a relevant variable for the purpose of factor estimation. Define $i \in \widehat{H}^c$ to indicate that Z_{it} is a relevant variable and $i \in \widehat{H}$ to indicate that Z_{it} is an irrelevant variable, for factor estiamtion. Now, let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right|$ or the statistic $\sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$. The variable selection procedure is based on the decision rule:

$$i \in \begin{cases} \widehat{H}^c & \text{if } \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \\ \widehat{H} & \text{if } \mathbb{S}_{i,T}^{+} < \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \end{cases}, \tag{18}$$

[6]Here, a false positive refers to mis-classifying a variable, Z_{it}, as a relevant variable for the purpose of factor estimation when its factor loading $\gamma_i' = 0$, whereas a false negative refers to the opposite case, where $\gamma_i' \neq 0$, but the variable Z_{it} is mistakenly classified as irrelevant.

where $\Phi^{-1}(\cdot)$ denotes the quantile function or the inverse of the cumulative distribution function of the standard normal random variable, and where φ is a tuning parameter which may depend on N. Some conditions on φ will be given in Assumptions 3-11 and 3-11* below. For a discussion of the use of the quantile function of the standard normal as the threshold function, refer to CQS (2023a), and note that the threshold function used here is related to the one employed in Belloni, Chen, Chernozhukov, and Hansen (2012).

In the sequel, we further require the following assumptions.

Assumption 3-8: There exists a positive constant, $\underline{c}$, such that for T sufficiently large:

$$\min_{1\leq\ell\leq d}\min_{i\in H}\min_{r\in\{1,\ldots,q\}} E\left\{\left[\frac{1}{\sqrt{\tau_1}}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1} y_{\ell,t+1}u_{it}\right]^2\right\}\geq\underline{c},$$

where, as defined earlier,

$$\tau_1=\lfloor T_0^{\alpha_1}\rfloor, \tau_2=\lfloor T_0^{\alpha_2}\rfloor \text{ for } 1>\alpha_1\geq\alpha_2>0 \text{ and } q=\left\lfloor\frac{T_0}{\tau_1+\tau_2}\right\rfloor,$$

and $T_0=T-p+1$.

Assumption 3-9: Let $i\in H^c=\{k\in\{1,....,N\}:\gamma_k\neq 0\}$.Suppose that there exists a positive constant, $\underline{c}$, such that, for all N_1, N_2,and T sufficiently large:

$$\begin{aligned}
&\min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{\mu_{i,\ell,T}}{q\tau_1}\right| \\
=&\min_{1\leq\ell\leq d}\min_{i\in H^c}\left|\frac{1}{q}\sum_{r=1}^{q}\frac{1}{\tau_1}\sum_{t=(r-1)\tau+p}^{(r-1)\tau+\tau_1+p-1}\gamma_i'\left\{E\left[\underline{F}_t\right]\mu_{Y,\ell}+E\left[\underline{F}_t\underline{Y}_t'\right]\alpha_{YY,\ell}+E\left[\underline{F}_t\underline{F}_t'\right]\alpha_{YF,\ell}\right\}\right| \\
\geq&\ \underline{c}>0,
\end{aligned}$$

where $\mu_{Y,\ell}=e_{\ell,d}'\mu_Y$, $\alpha_{YY,\ell}=A_{YY}'e_{\ell,d}$, and $\alpha_{YF,\ell}=A_{YF}'e_{\ell,d}$. Here, $e_{\ell,d}$ is a $d\times 1$ elementary vector whose ℓ^{th} component is 1 and all other components are 0.

Assumption 3-10: Suppose that, as N_1, N_2, and $T\to\infty$, the following rate conditions hold:

(a)

$$\frac{\sqrt{\ln N}}{T^{\min\left\{\frac{1-\alpha_1}{6},\frac{\alpha_2}{2}\right\}}}\to 0$$

where $1>\alpha_1\geq\alpha_2>0$ and $N=N_1+N_2$.

(b)

$$\frac{N_1}{T^{3\alpha_1}}\to 0 \text{ where } 1>\alpha_1>0.$$

Assumption 3-11: Let φ satisfy the following two conditions: (a) $\varphi \to 0$ as $N_1, N_2 \to \infty$, and (b) there exists some constant $a > 0$, such that $\varphi \geq \frac{1}{N^a}$, for all N_1, N_2 sufficiently large.

Note that Assumption 3-9 is a fairly mild condition which allows us to differentiate the alternative hypothesis, $i \in H^c$, from the null hypothesis, $i \in H$. For further discussion of Assumptions 3-8 - 3-11, refer to CQS (2023a). Given the above assumptions, Theorem 1 of CQS (2023a) shows that the probability of a false positive, i.e., the probability that $i \in \widehat{H}^c$, even though $\gamma_i = 0$, approaches zero, as $N, T \to \infty$, and Theorem 2 of the same paper shows that the probability of a false negative, i.e., the probability that $i \in \widehat{H}$ even though $\gamma_i \neq 0$, also approaches zero, as $N, T \to \infty$. Together, these two theorems show that our variable selection procedure is (completely) consistent in the sense that the probability of committing a misclassification error vanishes as $N, T \to \infty$. CQS (2023a) also note that the above variable selection procedure provides us with a consistent estimate $\widehat{N}_1$ of the unobserved quantity N_1, where the latter, in light of Assumption 3-6, can be interpreted as giving the order of magnitude of $\Gamma'\Gamma$ and is, thus, a measure of the overall pervasiveness of the factors in a given application. Finally, note that knowledge of the number of factors is not needed to implement the above variable selection procedure. Hence, in the case where the number of factors needs to be determined empirically, an applied researcher could first use our procedure to properly select the relevant variables and then apply an information criterion such as that proposed in Bai and Ng (2002) to estimate the number of factors.

Before presenting the main theoretical results proven in this paper, it is worth making a final comment about variable selection. In particualr, note that Bai and Ng (2008) address the important issue of choosing predictor variables Z_{it} based on their predictability for Y_{t+1}. While we agree with this viewpoint, it is worth stressing that in our setup, whether Z_{it} helps to predict Y_{t+h} depends on two things: (i) whether Z_{it} loads significantly on the underlying factors $\underline{F}_t$ (i.e., whether $\gamma_i \neq 0$ or not) and (ii) whether at least some components of $\underline{F}_t$ are helpful for predicting certain components of Y_{t+h}. The variable selection procedure which we propose here focuses on the first issue but not the second. This is because, in our view, it is important to first obtain good factor estimates with certain desirable asymptotic properties before trying to assess which factor may or may not be useful for predicting Y_{t+h}. It is important to distinguish between these two things because, if we try to do too much at the variable selection stage and end up excluding a significant number of (predictor) variables that load strongly on at least some of the factors, then, this can lead to the factor vector $\underline{F}_t$ being inconsistently estimated, and this is true even if the variables do not individually help to predict Y_{t+h}, but instead are crucial for the consistent estimation of the factor, which in turn is useful for predicting Y_{t+h}.

4 Consistent Estimation of Factors and the h-Step Ahead Predictor Based on the FAVAR Model

In this section, we provide our main theoretical results on factor estimation and on the estimation of the h-step predictor implied by the FAVAR model. To obtain these results, we need to impose a further rate condition on the tuning parameter, φ (see part (c) of Assumption 3-11*).

Assumption 3-11*: Let φ satisfy the following three conditions: (a) $\varphi \to 0$ as $N_1, N_2 \to \infty$, (b) there exists some constant $a > 0$, such that $\varphi \geq \frac{1}{N^a}$ for all N_1, N_2 sufficiently large, and (c)

$$\max\left\{\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}, \frac{N^{\frac{1}{3}}\varphi}{N_1 T}\right\} \to 0 \text{ as } N_1, N_2, T \to \infty.$$

Remark 4.1: Note that the rate condition given in part (c) of Assumption 3-11* depends on N_1. However, if we choose φ so that:

$$\varphi N^{\frac{2}{5}} = O\left(1\right),$$

then

$$\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} = O\left(\frac{1}{N_1}\right) = o\left(1\right) \text{ and } \frac{N^{\frac{1}{3}}\varphi}{N_1 T} = O\left(\frac{1}{N_1 N^{\frac{1}{15}} T}\right) = o\left(\frac{1}{N_1}\right).$$

Hence, with this choice of φ, Assumption 3-11* part (c) will be satisfied as long as $N_1 \to \infty$, and there is no need to impose any further condition on the rate at which N_1 grows. Requiring that $N_1 \to \infty$ is a minimal condition, since if $N_1 \nrightarrow \infty$; then consistent factor estimation, even up to an invertible matrix transformation, is impossible. Additionally, Monte Carlo results reported in Section 3 of CQS (2023a) show that the variable selection procedure discussed above performs very well in finite samples, under the tuning parameter choice $\varphi = N^{-\frac{2}{5}}$, both in terms of controlling the probability of a false positive (or Type I) error and in terms of controlling the probability of a false negative (or Type II) error.

Next, consider the post-variable-selection principal component estimator of $\underline{F}_t = \left(F_t', F_{t-1}', ..., F_{t-p+1}'\right)$:

$$\widehat{\underline{F}}_t = \frac{\widehat{\Gamma}' Z_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1}, \tag{19}$$

where

$$Z_{t,N}\left(\widehat{H^c}\right) = \begin{bmatrix} Z_{1,t}\mathbb{I}\left\{1 \in \widehat{H^c}\right\} & Z_{2,t}\mathbb{I}\left\{2 \in \widehat{H^c}\right\} & \cdots & Z_{N,t}\mathbb{I}\left\{N \in \widehat{H^c}\right\} \end{bmatrix}',$$

with

$$\mathbb{I}\left\{i \in \widehat{H^c}\right\} = \begin{cases} 1 & \text{if } i \in \widehat{H^c}, \text{ i.e., if } \mathbb{S}_{i,T}^{+} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \\ 0 & \text{if } i \in \widehat{H}, \text{ i.e., if } \mathbb{S}_{i,T}^{+} \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \end{cases},$$

and where $\widehat{N}_1 = \#\left(\widehat{H^c}\right)$, i.e., the cardinality of the set $\widehat{H^c}$. Here, $\widehat{\Gamma}$ denotes the principal component estimator of the loading matrix Γ constructed from taking $\sqrt{\widehat{N}_1}$ times the matrix whose columns are the eigenvectors of the post-variable-selection sample covariance matrix $\widehat{\Sigma}\left(\widehat{H^c}\right)$ associated with the Kp largest eigenvalues of this matrix, where, in this case,

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} = \frac{1}{\widehat{N}_1 T_0} \sum_{t=p}^{T} Z_{t,N}\left(\widehat{H^c}\right) Z_{t,N}\left(\widehat{H^c}\right)',$$

with $T_0 = T - p + 1$.

Our next result shows that the estimator given in expression (19) consistently estimates the unobserved factors $\underline{F}_t$,up to an invertible $Kp \times Kp$ matrix transformation.

Theorem 4.1: *Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, and 3-10 hold. Let $\widehat{\underline{F}}_t$ be as defined in expression (19). Assume further that the specification of the tuning parameter, φ, in the decision rule (18) satisfies Assumption 3-11*. Then,*

$$\left\|\widehat{\underline{F}}_t - Q'\underline{F}_t\right\|_2 = o_p(1), \text{ for all fixed } t,$$

where

$$Q = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \Xi \widehat{V},$$

and where $\widehat{V}$ is the $Kp \times Kp$ orthogonal matrix given in Lemma D-14, and Ξ is a $Kp \times Kp$ orthogonal matrix whose columns are the eigenvectors of the matrix

$$M_{FF}^{*} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}.$$

If we examine the proof of Theorem 4.1 in Appendix 1 as well as the supporting arguments given in the proof of Lemma D-15 in Appendix D below, we see that two of the key components of the proof involve showing that:

$$\left\|\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right\|_2 \xrightarrow{p} 0$$

and that

$$\frac{\widehat{N}_1 - N_1}{N_1} \xrightarrow{p} 0.$$

This is one of the reasons why we argue that initial variable selection should focus on determining which variables load strongly on the factors without worrying specifically at that stage about the related issues of predictability or, for that matter, any other issue. By contrast, if we make our initial variable selection based on some more stringent criterion that takes into consideration not only variable relevance but also other concerns such as predictability, then, we may end up with a much smaller set $\widetilde{H}^c$ of selected variables relative to the set $\widehat{H^c}$ selected under our procedure. In particular, in this case, it may be possible that even in large samples a significant number of rows of $\Gamma\left(\widetilde{H^c}\right)$ may contain only zero elements even though the corresponding row of Γ is not a zero vector, so that the result:

$$\left\| \frac{\Gamma\left(\widetilde{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0$$

may not hold. For the same reason, if we let $\widetilde{N}_1$ denote the cardinality of the set of selected indices based on an alternative, more stringent variable selection procedure, then, the result:

$$\frac{\widetilde{N}_1 - N_1}{N_1} \xrightarrow{p} 0$$

also may not hold, since, by definition, N_1 is the number of rows of Γ which have at least one non-zero element.

Although Theorem 4.1 shows that, without further identifying assumptions, we can only estimate the factors $\underline{F}_t$ consistently up to an invertible $Kp \times Kp$ matrix transformation, this result turns out to be sufficient for us to estimate the h-step ahead predictor consistently. More specifically, in Appendix D below, we show that for h-step ahead forecasts associated with the (infeasible) forecasting equation implied by the FAVAR model (2), we have the form

$$Y_{t+h} = \beta_0 + B_1' \underline{Y}_t + B_2' \underline{F}_t + \eta_{t+h}, \tag{20}$$

where $\underline{Y}_t$ and $\underline{F}_t$ are as defined in expression (5) above and where:

$$\begin{aligned} \beta_0 &= \sum_{j=0}^{h-1} J_d A^j \alpha,\ B_1' = J_d A^h \mathcal{P}'_{(d+K)p} S_d,\ B_2' = J_d A^h \mathcal{P}'_{(d+K)p} S_K \text{ and} \\ \eta_{t+h} &= \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}. \end{aligned} \tag{21}$$

Here, α and A are, respectively, the intercept (vector) and the coefficient matrix of the companion form defined in expression (6) above, $\mathcal{P}_{(d+K)p}$ is a permutation matrix such that:

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix},$$

and

$$\begin{aligned} S_d &= \begin{pmatrix} I_{dp} \\ \underset{Kp\times dp}{0} \end{pmatrix}, S_K = \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_{Kp} \end{pmatrix}, \underset{d\times(d+K)p}{J_d} = \begin{bmatrix} I_d & 0 & \cdots & 0 \end{bmatrix}, \text{ and} \\ \underset{(d+K)\times(d+K)p}{J_{d+K}} &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}. \end{aligned}$$

See the beginning of Appendix D for a derivation of the equation given in expression (20). The reason expression (20) is called an infeasible forecasting equation is, of course, because $\underline{F}_t$ is not observed, so to obtain a feasible version of this forecasting equation, we must replace $\underline{F}_t$ in equation (20) with the estimate $\widehat{\underline{F}}_t$ given in expression (19). Doing so, we arrive at a feasible h-step ahead forecasting equation of the form:

$$\begin{aligned} Y_{t+h} &= \beta_0 + \sum_{g=1}^{p} B_{1,g}' Y_{t-g+1} + \sum_{g=1}^{p} B_{2,g}' \widehat{F}_{t-g+1} + \widehat{\eta}_{t+h} \\ &= \beta_0 + B_1'\underline{Y}_t + B_2'\widehat{\underline{F}}_t + \widehat{\eta}_{t+h}, \end{aligned} \tag{22}$$

where $\widehat{\eta}_{t+h} = \eta_{t+h} - B_2'\left(\widehat{\underline{F}}_t - \underline{F}_t\right)$, with $\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J_{d+K}' \varepsilon_{t+h-j}$.

One can interpret expression (22) as a "reduced form" formulation of the forecasting equation where the reduced form parameters β_0, B_1, and B_2 are nonlinear functions of the parameters $(\mu, A_1, ..., A_p)$ of the FAVAR model, in the case where $h > 1$. For forecasting purposes, while it is possible to estimate the conditional mean of the forecasting equation (22) by estimating the underlying parameters directly by nonlinear least squares, here we choose instead to estimate the conditional mean by estimating the reduced form parameters β_0, B_1, and B_2 via linear least squares. An important reason why we choose this latter approach is due to complications that arise both because we are forecasting with a FAVAR which contains unobserved factors that must first be estimated and because we do not make enough identifying assumptions so that the factors can only be estimated consistently up to an invertible $Kp \times Kp$ matrix transformation. In fact, it turns out that estimating the underlying parameters $\mu, A_1, ..., A_p$ by nonlinear least squares and constructing an estimator of the conditional mean of the forecasting equation based on these estimates will not lead to a consistently estimated h-step predictor, unless further identifying assumptions are made.

On the other hand, as we will show in Theorem 5 below, estimating the reduced form parameters β_0, B_1, and B_2 by linear least squares does allow us to construct a consistent estimator of the conditional mean, even in the absence of additional identifying assumptions.

More precisely, let $\underline{\widehat{F}}_t$ denotes the factor estimates given in expression (19). Our procedure minimizes the least squares criterion function:

$$\begin{aligned} Q(\beta_0, B_1, B_2) &= \sum_{t=p}^{T-h} \left\| Y_{t+h} - \beta_0 - B_1' \underline{Y}_t - B_2' \underline{\widehat{F}}_t \right\|_2^2 \\ &= \sum_{t=p}^{T-h} \left\| Y_{t+h} - \beta_0 - \sum_{g=1}^{p} B_{1,g}' Y_{t-g+1} - \sum_{g=1}^{p} B_{2,g}' \widehat{F}_{t-g+1} \right\|_2^2 \end{aligned} \tag{23}$$

with respect to the parameters β_0, B_1, and B_2, and delivers the OLS estimates $\widehat{\beta}_0$, $\widehat{B}_1$, and $\widehat{B}_2$. We then forecast Y_{T+h} using the h-step predictor:

$$\widehat{Y}_{T+h} = \widehat{\beta}_0 + \widehat{B}_1' \underline{Y}_T + \widehat{B}_2' \underline{\widehat{F}}_T. \tag{24}$$

The following result shows that $\widehat{Y}_{T+h}$ is a consistent estimator of the conditional mean of the infeasible forecast equation (20).

Theorem 4.2: *Let $\widehat{Y}_{T+h}$ be as defined in expression (24). Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then,*

$$\widehat{Y}_{T+h} - \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T\right) \xrightarrow{p} 0 \text{ as } N_1, N_2, T \to \infty.$$

5 Empirical Illustration

To be completed.

6 Conclusion

In this paper, we study the problem of consistently estimating the conditional mean of a factor-augmented forecasting equation based on the FAVAR model. When the underlying dynamic factor model generating the latent factors is high-dimensional, we show that it is important to pre-screen the variables in terms of their association with the underlying factors prior to estimation, particularly in cases where one suspects that the conventional assumption of factor pervasiveness may not hold. For this purpose, we utiluize a new variable selection procedure based on a self-normalized score statistic (see Chao, Qiu, and Swanson (CQS: 2023a) that correctly identifies the set of variables which load significantly on the underlying factors, with probability approaching one, as the sample sizes go to infinity. Furthermore, CQS(2023a) show that estimating the factors using only those variables selected by their method allows factors to be consistently estimated, up to an invertible matrix transformation, even in certain situations where the standard pervasiveness assumption does not hold, provided that the number of relevant variables is sufficiently large. Using the factors estimated in such a manner, we show that the conditional mean function of a factor-augmented forecasting equation can be consistently estimated, even for the case of multi-step ahead forecasts.

7 Appendix A: Proofs of Theorems 2.1, 4.1, and 4.2

Proof of Theorem 2.1:

The proof of Theorem 2.1 requires a long series of calculations. Hence, we have divided this proof into six different steps.

Step 1:

In step 1, we shall transform the simple factor model

$$\underset{N\times 1}{Z_t} = \underset{N\times 1}{\gamma}\underset{1\times 1}{f_t} + \underset{N\times 1}{u_t},\ t = 1, ..., T \tag{25}$$

into a more convenient form. Let Π denote an $N \times N$ orthogonal matrix whose columns are the eigenvectors of the covariance matrix $\Sigma_Z = E\left[Z_t Z_t'\right]$. Without loss of generality, we can partition Π as

$$\underset{N\times N}{\Pi} = \left[\begin{array}{cc} \underset{N\times 1}{\pi_1} & \underset{N\times(N-1)}{\Pi_2} \end{array}\right]$$

where π_1 is the eigenvector associated with the largest eigenvalue of $\Sigma_Z = E\left[Z_t Z_t'\right]$, i.e., $\lambda_{(1)}\left(\Sigma_Z\right)$. By the result of Lemma B-8, we know that

$$\pi_1 = \frac{\gamma}{\|\gamma\|_2} \text{ and } \lambda_{(1)}\left(\Sigma_Z\right) = \|\gamma\|_2^2 + 1.$$

Next, we define

$$
\begin{aligned}
W_t &= \Pi' Z_t \\
&= \Pi' \left(\gamma f_t + u_t\right) \\
&= \|\gamma\|_2 \, \Pi' \frac{\gamma}{\|\gamma\|_2} f_t + \Pi' u_t \\
&= \|\gamma\|_2 \, f_t \Pi' \pi_1 + \Pi' u_t \quad \left(\text{since } \pi_1 = \frac{\gamma}{\|\gamma\|_2}\right) \\
&= \|\gamma\|_2 \, f_t \begin{pmatrix} \pi_1' \\ \Pi_2' \end{pmatrix} \pi_1 + \Pi' u_t \\
&= \|\gamma\|_2 \, f_t \mathbf{e}_{1,N} + \eta_t
\end{aligned} \tag{26}
$$

where $\mathbf{e}_{1,N}$ is an elementary vector whose first component is 1 and all remaining components are 0 and where $\eta_t = \Pi' u_t$. Moreover, note that $\{\eta_t\} \equiv i.i.d.N\left(0, I_N\right)$ since Π is an orthogonal matrix and $\eta_t = \Pi' u_t$ with $\{u_t\} \equiv i.i.d.N\left(0, I_N\right)$. We can write out the covariance matrix of W_t as

$$
\begin{aligned}
& \Sigma_W \\
= \; & E\left[W_t W_t'\right] \\
= \; & E\left[\left(\|\gamma\|_2 \, f_t \mathbf{e}_{1,N} + \eta_t\right)\left(\|\gamma\|_2 \, f_t \mathbf{e}_{1,N} + \eta_t\right)'\right] \\
= \; & \|\gamma\|_2^2 \, E\left[f_t^2\right] \mathbf{e}_{1,N} \mathbf{e}_{1,N}' + \|\gamma\|_2 \, E\left[\eta_t f_t\right] \mathbf{e}_{1,N}' + \|\gamma\|_2 \, \mathbf{e}_{1,N} E\left[f_t \eta_t'\right] + E\left[\eta_t \eta_t'\right] \\
= \; & \|\gamma\|_2^2 \, \mathbf{e}_{1,N} \mathbf{e}_{1,N}' + I_N \\
= \; & \begin{pmatrix}
\|\gamma\|_2^2 + 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\end{aligned}
$$

from which it is easily seen that $\lambda_{(1)}\left(\Sigma_W\right) = \|\gamma\|_2^2 + 1$ and $\lambda_{(2)}\left(\Sigma_W\right) = \lambda_{(3)}\left(\Sigma_W\right) = \cdots = \lambda_{(N)}\left(\Sigma_W\right) = 1$, where we let $\lambda_{(j)}\left(\Sigma_W\right)$ denote the j^{th} largest eigenvalue of Σ_W. In addition, the eigenvector associated with $\lambda_{(j)}\left(\Sigma_W\right)$ is $\mathbf{e}_{j,N}$, an elementary vector whose j^{th} component is 1 and all other components are 0.

Note further that we can also write W_t in the alternative form

$$\begin{aligned} W_t &= \begin{pmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{N,t} \end{pmatrix} \\ &= \begin{pmatrix} \|\gamma\|_2 \, f_t \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \eta_{1t} \\ \eta_{2,t} \\ \vdots \\ \eta_{N,t} \end{pmatrix} \\ &= \begin{pmatrix} \|\gamma\|_2 \, \zeta_{1,t} \\ \zeta_{2,t} \\ \vdots \\ \zeta_{N,t} \end{pmatrix} \\ &= \sum_{j=1}^{N} \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N} \end{aligned} \tag{27}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1} \eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, ..., N$ and where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, ..., N$. In fact, this is the representation of W_t that is given in Lemma B-10. (See Appendix B below).

Step 2:

Define $\underset{N\times T}{\mathbf{W}} = (W_1, ..., W_T)$, where W_t is as defined in expression (26) in step 1 above. Partition $\mathbf{W}$ as follows

$$\underset{N\times T}{\mathbf{W}} = \begin{bmatrix} \underset{1\times T}{\mathbf{W}_1'} \\ \underset{(N-1)\times T}{\mathbf{W}_2'} \end{bmatrix} = \begin{bmatrix} \underset{1\times T}{\pi_1'\mathbf{Z}} \\ \underset{(N-1)\times T}{\Pi_2'\mathbf{Z}} \end{bmatrix},$$

where $\underset{N\times T}{\mathbf{Z}} = (Z_1, ..., Z_T)$ with Z_t as defined in expression (25). Note that the first row of $\mathbf{W}$, i.e., $\mathbf{W}_1'$, contains the "signal" observations with the elevated variance $\lambda_1 = \|\gamma\|_2^2 + 1$ and where the remaining $N-1$ rows contain the elements of the $(N-1)\times T$ matrix $\mathbf{W}_2'$ which contain only the noise variables. Now, define the sample covariance matrix

$$\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T}\mathbf{W}\mathbf{W}' = \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2 \\ T^{-1}\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\mathbf{W}_2'\mathbf{W}_2 \end{pmatrix}$$

In this step, we shall further transform $\widehat{\Sigma}_{\mathbf{W}}$ into the so-called arrowhead matrix. To proceed, consider the spectral decomposition

$$\frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}_2'$$

where $\widetilde{\Lambda} = diag\left(\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(N)}\right)$ with $\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(N)}$ being the $N-1$ eigenvalues of $\mathbf{W}_2'\mathbf{W}_2/T$ and $\widetilde{\mathbf{B}}_2$ is an $(N-1)\times(N-1)$ orthogonal matrix whose columns are the eigenvectors of $\mathbf{W}_2'\mathbf{W}_2/T$. Note that, without loss of generality, we can assume that the eigenvalues are ordered so that $\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$. Next, create the modified data matrix

$$\underset{N\times T}{\widetilde{\mathbf{W}}} = \begin{bmatrix} \underset{1\times T}{\mathbf{W}_1'} \\ \underset{(N-1)\times T}{\widetilde{\mathbf{B}}_2'\mathbf{W}_2'} \end{bmatrix}$$

The sample covariance matrix based on the modified data matrix is then given by

$$\begin{aligned}
\underset{N\times N}{\widetilde{\Sigma}_{\mathbf{W}}} &= \frac{\widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'}{T} \\
&= \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \\ T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \end{pmatrix} \\
&= \begin{pmatrix} s & \upsilon' \\ \upsilon & \widetilde{\Lambda} \end{pmatrix} \\
&= \begin{pmatrix} s & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_N \\ \upsilon_2 & \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ \upsilon_3 & 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \upsilon_N & 0 & \cdots & 0 & \widetilde{\lambda}_{(N)} \end{pmatrix}
\end{aligned}$$

where $\underset{1\times 1}{s} = \mathbf{W}_1'\mathbf{W}_1/T$ and

$$\underset{(N-1)\times 1}{\upsilon} = \begin{pmatrix} \upsilon_2 \\ \vdots \\ \upsilon_N \end{pmatrix} = \frac{\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1}{T}. \tag{28}$$

Note that the non-zero entries of $\widetilde{\Sigma}_{\mathbf{W}}$ form the shape of an arrow, and so such matrices have been referred to in the linear algebra literature as an "arrowhead matrix".

An advantage of this arrowhead form is that it allows us to obtain a useful representation for the top eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}}$. This part of step 2 comes from Johnstone and Paul (2018) following an approach originally due to Nadler (2008), but for completeness we provide some details of the argument here. To proceed, let $\widehat{\lambda}_{(1)}$ denote the largest eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}}$ and let $\widetilde{\mathbf{v}}_{(1)}$ be the associated eigenvector, where, following Johnstone and Paul (2018), we will normalize $\widetilde{\mathbf{v}}_{(1)}$ to have the form $\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 & \widetilde{v}_{(1),2} & \cdots & \widetilde{v}_{(1),N} \end{pmatrix}'$, i.e., we normalize $\widetilde{\mathbf{v}}_{(1)}$ so that its first component is 1. The eigen-equation $\widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(1)} = \widehat{\lambda}_{(1)}\widetilde{\mathbf{v}}_{(1)}$ can then be written out more explicitly as

$$\begin{pmatrix} s & \upsilon_2 & \upsilon_3 & \cdots & \upsilon_N \\ \upsilon_2 & \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ \upsilon_3 & 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \upsilon_N & 0 & \cdots & 0 & \widetilde{\lambda}_{(N)} \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \widetilde{v}_{(1),3} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \widehat{\lambda}_{(1)} \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \widetilde{v}_{(1),3} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} \tag{29}$$

Solving this system of equations, we see that

$$\widetilde{v}_{(1),j} = \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}} \text{ for } j = 2, ..., N; \tag{30}$$

where υ_j is the j^{th} component of υ as defined in expression (28). Hence,

$$\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ \upsilon_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)}\right) \\ \vdots \\ \upsilon_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)}\right) \end{pmatrix} \tag{31}$$

Moreover, since expression (29) implies that

$$\widehat{\lambda}_{(1)} = s + \upsilon_2 \widetilde{v}_{(1),2} + \cdots + \upsilon_N \widetilde{v}_{(1),N}$$

It follows from substituting the right-hand side of equation (30) for $j = 2, ..., N$ into the above expression that

$$\widehat{\lambda}_{(1)} = s + \sum_{j=2}^{N} \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}} = \frac{\mathbf{W}_1'\mathbf{W}_1}{T} + \sum_{j=2}^{N} \frac{\upsilon_j}{\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)}}. \tag{32}$$

Finally, in this step, we shall relate the eigenvalues and eigenvectors of $\widetilde{\Sigma}_{\mathbf{W}}$ to that of the pre-transformed sample covariance matrix of our simple factor model, i.e.,

$$\widehat{\Sigma}_Z = \frac{\mathbf{ZZ}'}{T} = \frac{1}{T}\sum_{t=1}^{T} Z_t Z_t' \text{ where } \underset{N\times T}{\mathbf{Z}} = (Z_1, ..., Z_T).$$

Understanding this relationship then allows us to derive asymptotic properties of quantities involving the leading eigenvector of $\widehat{\Sigma}_Z$ using the explicit representation of $\widetilde{\mathbf{v}}_1$ and $\widehat{\lambda}_1$ given in expressions (31) and (32), respectively. To proceed, we first relate the eigenvalues and eigenvectors of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'/T$ to that of $\widehat{\Sigma}_{\mathbf{W}}= \mathbf{WW}'/T$. Define

$$\underset{N\times N}{\widetilde{\mathbf{B}}} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix}$$

Now, since $\widetilde{\mathbf{B}}_2$ is an orthogonal matrix, it follows that

$$\widetilde{\mathbf{B}}'\widetilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2'\widetilde{\mathbf{B}}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

and

$$\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}' = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2\widetilde{\mathbf{B}}_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{N-1} \end{pmatrix} = I_N$$

so that $\widetilde{\mathbf{B}}$ is an orthogonal matrix as well. Next, note that

$$\begin{aligned}
&\frac{\widetilde{\mathbf{B}}'\mathbf{WW}'\widetilde{\mathbf{B}}}{T} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2' \end{pmatrix}\begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2 \\ T^{-1}\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\mathbf{W}_2'\mathbf{W}_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \\
&= \begin{pmatrix} T^{-1}\mathbf{W}_1'\mathbf{W}_1 & T^{-1}\mathbf{W}_1'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \\ T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1 & T^{-1}\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_2\widetilde{\mathbf{B}}_2 \end{pmatrix} \\
&= \frac{\widetilde{\mathbf{W}}\widetilde{\mathbf{W}}'}{T} \\
&= \widetilde{\Sigma}_{\mathbf{W}}
\end{aligned}$$

Hence, to relate the eigenvalues and eigenvectors of $\widehat{\Sigma}_{\mathbf{W}}= \mathbf{WW}'/T$ to those of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{WW}'\widetilde{\mathbf{B}}/T$, we note that the eigenvalues of the $\widetilde{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$\begin{aligned} 0 &= \det\left\{\frac{\widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}}{T} - \lambda I_N\right\} \\ &= \det\left\{\widetilde{\mathbf{B}}'\right\}\det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\right\}\det\left\{\widetilde{\mathbf{B}}\right\} \\ &= \det\left\{\widetilde{\mathbf{B}}'\right\}\det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\}\det\left\{\widetilde{\mathbf{B}}\right\} \quad \left(\text{since } \widetilde{\mathbf{B}} \text{ is an orthogonal matrix}\right) \\ &= \det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\} \end{aligned}$$

where the last equality holds because $\det\left\{\widetilde{\mathbf{B}}'\right\} = \det\left\{\widetilde{\mathbf{B}}\right\} = \pm 1$ given that $\widetilde{\mathbf{B}}$ is an orthogonal matrix. It follows that $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$ have the same set of eigenvalues. Moreover, let $\widehat{\lambda}_{(j)}$ be the j^{th} largest eigenvalue of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$, which is of course also the j^{th} largest eigenvalue of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. Also, let $\widetilde{\mathbf{v}}_{(j)}$ be an eigenvector of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$ associated with $\widehat{\lambda}_{(j)}$. Define $\mathbf{v}_{(j)} \equiv \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)}$ for $j = 1, ..., N$, and note that, since $\widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\widetilde{\mathbf{v}}_{(j)}$, we have

$$\begin{aligned} \widetilde{\mathbf{B}}'\widehat{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} &= \left(\frac{\widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}}{T}\right)\widetilde{\mathbf{v}}_{(j)} \\ &= \widetilde{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{v}}_{(j)} \\ &= \widehat{\lambda}_{(j)}\widetilde{\mathbf{v}}_{(j)} \end{aligned}$$

which implies that

$$\widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)} = \widehat{\Sigma}_{\mathbf{W}}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)} = \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}$$

so that $\mathbf{v}_{(j)} = \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(j)}$ is an eigenvector of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ associated with $\widehat{\lambda}_{(j)}$. Note, further that, previously, we have normalized the first element of $\widetilde{\mathbf{v}}_{(1)}$ to be 1. This, in turn, implies that the first element of $\mathbf{v}_{(1)}$ will be 1 as well since

$$\begin{aligned}
\mathbf{v}_{(1)} &= \widetilde{\mathbf{B}}\widetilde{\mathbf{v}}_{(1)} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \widetilde{\mathbf{B}}_2 \widetilde{\mathbf{v}}_{(1)}^{(2)} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \mathbf{v}_{(1)}^{(2)} \end{pmatrix} \qquad (33)
\end{aligned}$$

where we let $\underset{(N-1)\times 1}{\widetilde{\mathbf{v}}_{(1)}^{(2)}} = \begin{pmatrix} \widetilde{v}_{(1),2} & \widetilde{v}_{(1),3} & \cdots & \widetilde{v}_{(1),N} \end{pmatrix}'$ and $\underset{(N-1)\times 1}{\mathbf{v}_{(1)}^{(2)}} = \widetilde{\mathbf{B}}_2 \widetilde{\mathbf{v}}_{(1)}^{(2)} = \begin{pmatrix} v_{(1),2} & v_{(1),3} & \cdots & v_{(1),N} \end{pmatrix}'$.

In a similar manner, we can relate the eigenvalues and eigenvectors of $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ to those of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also to those of $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. In this case, note that the eigenvalues of the $\widehat{\Sigma}_{\mathbf{W}}$ are the solutions of the determinantal equation

$$\begin{aligned}
0 &= \det\left\{\frac{\mathbf{W}\mathbf{W}'}{T} - \lambda I_N\right\} \\
&= \det\left\{\frac{\Pi'\mathbf{Z}\mathbf{Z}'\Pi}{T} - \lambda I_N\right\} \quad \left(\text{ since } \mathbf{W} = \Pi'\mathbf{Z}\right) \\
&= \det\left\{\mathbf{\Pi}'\right\} \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda \mathbf{\Pi}\mathbf{\Pi}'\right\} \det\left\{\mathbf{\Pi}\right\} \\
&= \det\left\{\mathbf{\Pi}'\right\} \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N\right\} \det\left\{\mathbf{\Pi}\right\} \\
&\quad \left(\text{since } \mathbf{\Pi} \text{ is an orthogonal matrix whose columns are the eigenvectors of } \Sigma_Z = E\left[Z_t Z_t'\right]\right) \\
&= \det\left\{\frac{\mathbf{Z}\mathbf{Z}'}{T} - \lambda I_N\right\}
\end{aligned}$$

where the last equality holds because $\det\{\mathbf{\Pi}'\} = \det\{\mathbf{\Pi}\} = \pm 1$ given that $\mathbf{\Pi}$ is an orthogonal matrix. It follows that $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$ has the same set of eigenvaluses as $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ and, thus, also the same set of eigenvalues as $\widetilde{\Sigma}_{\mathbf{W}} = \widetilde{\mathbf{B}}'\mathbf{W}\mathbf{W}'\widetilde{\mathbf{B}}/T$. Using the same notation as above, we will then also let $\widehat{\lambda}_{(j)}$ to denote the j^{th} largest eigenvalue of $\widehat{\Sigma}_Z = \mathbf{Z}\mathbf{Z}'/T$. Moreover, as before, let $\mathbf{v}_j$ denote an eigenvector of $\widehat{\Sigma}_{\mathbf{W}} = \mathbf{W}\mathbf{W}'/T$ associated with $\widehat{\lambda}_{(j)}$. Now, define $\widehat{\pi}_{(j)} \equiv \Pi\mathbf{v}_{(j)}$, and note that since $\widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}$, we have, for $j = 1, ..., N$,

$$\begin{aligned}
\Pi'\widehat{\Sigma}_Z\Pi\mathbf{v}_{(j)} &= \left(\frac{\Pi'\mathbf{Z}\mathbf{Z}'\Pi}{T}\right)\mathbf{v}_{(j)} \\
&= \widehat{\Sigma}_{\mathbf{W}}\mathbf{v}_{(j)} \\
&= \widehat{\lambda}_{(j)}\mathbf{v}_{(j)}
\end{aligned}$$

which implies that

$$\widehat{\Sigma}_Z\widehat{\pi}_{(j)} = \widehat{\Sigma}_Z\Pi\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\Pi\mathbf{v}_{(j)} = \widehat{\lambda}_{(j)}\widehat{\pi}_{(j)}$$

so that

$$\widehat{\pi}_{(j)} = \Pi\mathbf{v}_{(j)} \tag{34}$$

is an eigenvector of $\widehat{\Sigma}_Z$ associated with the eigenvalue $\widehat{\lambda}_{(j)}$.

Step 3:

For the simple factor model given in expression (25), i.e.,

$$\begin{aligned}
Z_t &= \gamma f_t + u_t \\
&= \|\gamma\|_2 \, \pi_{(1)} f_t + u_t \text{ for } t = 1, ..., T;
\end{aligned}$$

with $\pi_1 = \gamma / \|\gamma\|_2$; the principal-component estimator of the latent factor f_t can be written as

$$\begin{aligned}
\widehat{f}_t &= \frac{1}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, Z_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\widehat{\pi}_{(1)}}{\left\|\widehat{\pi}_{(1)}\right\|_2}, u_t \right\rangle \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2}, \pi_1 \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\Pi \mathbf{v}_{(1)}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2}, u_t \right\rangle \quad \text{(making use of expression (34) in step 2)} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' \pi_1}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' u_t}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_1' \mathbf{e}_{1,N}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \Pi' u_t}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} \\
&\quad \left(\text{since } \Pi' \pi_{(1)} = \begin{pmatrix} \pi_{(1)}' \\ \Pi_{(2)}' \end{pmatrix} \pi_{(1)} = \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} = \mathbf{e}_{1,N} \text{ given that } \Pi \text{ is an orthogonal matrix} \right) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \mathbf{e}_{1,N}}{\left\|\Pi \mathbf{v}_{(1)}\right\|_2} + \frac{1}{\sqrt{N}} \frac{\mathbf{v}_{(1)}' \eta_t}{\|\Pi \mathbf{v}_1\|_2} \quad \left(\text{since, by definition, } \eta_t = \Pi' u_t\right) \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \eta_t \right\rangle
\end{aligned}$$

where the notation $\langle y, x \rangle = y'x$ denotes the dot product of the vectors y and x and where the last equality above follows from the fact that

$$\left\|\Pi \mathbf{v}_{(1)}\right\|_2 = \sqrt{\mathbf{v}_{(1)}' \Pi' \Pi \mathbf{v}_{(1)}} = \sqrt{\mathbf{v}_{(1)}' \mathbf{v}_{(1)}} = \left\|\mathbf{v}_{(1)}\right\|_2 .$$

Next, given expression (33) in step 2, we see that

$$\begin{aligned}
\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle &= \frac{\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\mathbf{e}_{1,N}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \ \left(\text{since } \widetilde{\mathbf{v}}_{(1)} = \widetilde{\mathbf{B}}'\mathbf{v}_{(1)}\right) \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \begin{pmatrix} 1 & \mathbf{v}^{(2)\prime}_{(1)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \widetilde{\mathbf{B}}_2 \end{pmatrix} \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \begin{pmatrix} 1 & \mathbf{v}^{(2)}_{(1)} \end{pmatrix} \begin{pmatrix} 1 \\ \underset{(N-1)\times 1}{0} \end{pmatrix} \\
&= \frac{1}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \left\langle \mathbf{v}_{(1)}, \mathbf{e}_{1,N} \right\rangle \\
&= \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle,
\end{aligned}$$

where the last line follows from the fact that

$$\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2 = \sqrt{\widetilde{\mathbf{v}}'_{(1)}\widetilde{\mathbf{v}}_{(1)}} = \sqrt{\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\mathbf{v}_{(1)}} = \sqrt{\mathbf{v}'_{(1)}\mathbf{v}_{(1)}} = \left\|\mathbf{v}_{(1)}\right\|_2$$

since $\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}' = I_N$. In addition, let $\widetilde{\eta}_t = \widetilde{\mathbf{B}}'\eta_t$, and note that

$$\begin{aligned}
\left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \eta_t \right\rangle &= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\mathbf{v}'_{(1)}\eta_t \\
&= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\mathbf{v}'_{(1)}\widetilde{\mathbf{B}}\widetilde{\mathbf{B}}'\eta_t \\
&= \frac{1}{\left\|\mathbf{v}_{(1)}\right\|_2}\widetilde{\mathbf{v}}'_{(1)}\widetilde{\eta}_t \\
&= \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \widetilde{\eta}_t \right\rangle \ \left(\text{given that } \left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2 = \left\|\mathbf{v}_{(1)}\right\|_2 \right).
\end{aligned}$$

Since

$$\{\eta_t\} \equiv i.i.d.N\left(0, I_N\right)$$

and $\widetilde{\mathbf{B}}$ is an orthogonal matrix, we also have

$$\{\widetilde{\eta}_t\} \equiv i.i.d.N\left(0, I_N\right).$$

Using these calculations, we can then rewrite the expression for $\widehat{f}_t$ in terms of $\widetilde{\mathbf{v}}_{(1)}$ and $\widetilde{\eta}_t$ as follows.

$$\begin{aligned}
\widehat{f}_t &= \frac{\langle \widehat{\pi}_{(1)}, Z_t \rangle}{\sqrt{N} \left\| \widehat{\pi}_{(1)} \right\|_2} \\
&= \frac{\|\gamma\|_2 f_t}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle + \frac{1}{\sqrt{N}} \left\langle \frac{\mathbf{v}_{(1)}}{\left\|\mathbf{v}_{(1)}\right\|_2}, \eta_t \right\rangle \\
&= \frac{\|\gamma\|_2}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle f_t + \frac{1}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \widetilde{\eta}_t \right\rangle .
\end{aligned} \tag{35}$$

Given the requirement in Assumption 2-2 that

$$\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ as } N, T \to \infty,$$

for constants c and κ such that $0 < c < \infty$ and $0 < \kappa < 1$; it is easily seen that

$$\frac{\|\gamma\|_2}{\sqrt{N}} = O\left(\left(\frac{1}{TN^{\kappa}}\right)^{\frac{1}{2(1+\kappa)}}\right) = o(1). \tag{36}$$

In the next two steps of this proof, we will show that

$$\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0 \text{ and } \frac{1}{\sqrt{N}} \left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \widetilde{\eta}_t \right\rangle \xrightarrow{p} 0.$$

Step 4:

We will first show that

$$\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_1\right\|_2}, \mathbf{e}_{1,N} \right\rangle \xrightarrow{p} 0.$$

. To proceed, note that, from expression (31) in step 2, $\widetilde{\mathbf{v}}_1$ has the explicit form

$$\widetilde{\mathbf{v}}_{(1)} = \begin{pmatrix} 1 \\ \widetilde{v}_{(1),2} \\ \vdots \\ \widetilde{v}_{(1),N} \end{pmatrix} = \begin{pmatrix} 1 \\ v_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)}\right) \\ \vdots \\ v_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)}\right) \end{pmatrix}$$

It follows that

$$\begin{aligned}
&\frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle^2}{\left\| \widetilde{\mathbf{v}}_{(1)} \right\|^2} \\
= \quad & \left[1 + \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2} \right]^{-1} \\
& \left(\text{since } \left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle = \begin{bmatrix} 1 & \upsilon_2 / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(2)} \right) & \cdots & \upsilon_N / \left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(N)} \right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1 \right) \\
= \quad & \frac{1}{1+\tau^2}
\end{aligned}$$

where

$$\tau^2 = \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2}.$$

Next, write

$$\begin{aligned}
\tau^2 &= \sum_{j=2}^{N} \frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)} - \widetilde{\lambda}_{(j)} \right)^2} \\
&= \frac{N \left\| \gamma \right\|_2^2}{T} \frac{1}{\left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 \upsilon_j^2 / \left(N \left\| \gamma \right\|_2^2 \right)}{\left(1 - \widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right)^2} \\
&= \frac{N}{T \left\| \gamma \right\|_2^{2(1+2\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 \upsilon_j^2 / \left(N \left\| \gamma \right\|_2^2 \right)}{\left(1 - \widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right)^2}
\end{aligned}$$

Recall from step 2 that $\widehat{\lambda}_{(1)}$ is the largest eigenvalue of the sample covariance matrix

$$\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \mathbf{W} \mathbf{W}' = \begin{pmatrix} T^{-1} \mathbf{W}_1' \mathbf{W}_1 & T^{-1} \mathbf{W}_1' \mathbf{W}_2 \\ T^{-1} \mathbf{W}_2' \mathbf{W}_1 & T^{-1} \mathbf{W}_2' \mathbf{W}_2 \end{pmatrix}$$

while $\widetilde{\lambda}_{(j)}$ (for $j = 2, ..., N$) is the $(j-1)^{th}$ largest eigenvalue of the submatrix $T^{-1} \mathbf{W}_2' \mathbf{W}_2$. Applying Lemma B-9 and noting that $\widehat{\Sigma}_{\mathbf{W}}$ and $T^{-1} \mathbf{W}_2' \mathbf{W}_2$ are positive semidefinite matrices whose elements are continuous random variables, we see that

$$0 \leq \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} < 1 \; a.s. \text{ for } j = 2, ..., N.$$

Note also that, by part (a) of Lemma B-5, $\widetilde{\lambda}_{(j)} = 0$ for $j = T+2, ..., N$. Hence, we can further write

$$\tau^2 \leq \frac{N}{T \left\| \gamma \right\|_2^{2(1+2\kappa)}} \frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} \left(1 - \max_{2 \leq j \leq T+1} \frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \right)^{-2} \frac{1}{T} \sum_{j=2}^{N} \frac{T^2 v_j^2}{N \left\| \gamma \right\|_2^2} \tag{37}$$

To analyze the asymptotic behavior of τ^2, note first that we can apply the result of Lemma B-10 in Appendix B below to obtain

$$\begin{aligned} \frac{\widehat{\lambda}_{(1)}^2}{\left\| \gamma \right\|_2^{4(1+\kappa)}} &= \left[\frac{\widehat{\lambda}_{(1)}}{\left\| \gamma \right\|_2^{2(1+\kappa)}} \right]^2 \\ &= \left[c + \frac{1}{\left\| \gamma \right\|_2^{2\kappa}} + o_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right]^2 \\ &= c^2 \left[1 + O_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right]. \end{aligned}$$

from which it follows that

$$\frac{1}{\widehat{\lambda}_{(1)}^2 / \left\| \gamma \right\|_2^{4(1+\kappa)}} = \frac{1}{c^2} \left[1 + O_p \left(\frac{1}{\left\| \gamma \right\|_2^{2\kappa}} \right) \right] \tag{38}$$

where $0 < 1/c^2 < \infty$ given that $0 < c < \infty$.

Next, consider $\left(1 - \max_{2 \leq j \leq T+1} \left[\widetilde{\lambda}_{(j)} / \widehat{\lambda}_{(1)} \right] \right)^{-2}$. To analyze its asymptotic behavior, we make use of Assumption 2-2, part (b) of Lemma B-5, and Lemma B-10 to obtain

$$
\begin{aligned}
& \max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} \\
= \; & \frac{N-1}{T\left\Vert \gamma\right\Vert_2^{2(1+\kappa)}}\frac{1}{\widehat{\lambda}_{(1)}/\left\Vert \gamma\right\Vert_2^{2(1+\kappa)}}\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} \\
= \; & \left[c+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\left[c+\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]^{-1}\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[c+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\left[c+\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right]^{-1}\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[c+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\frac{1}{c}\left[1+\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}\right]^{-1}\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\left[1-\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}+O\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{4\kappa}}\right)\right]\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}+O\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{4\kappa}}\right)+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}\right]\left[1+O\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{4\kappa}}\right)+o\left(\frac{1}{\left\Vert \gamma\right\Vert_2^2}\right)\right]\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]\left[1+O_p\left(\sqrt{\frac{T}{N}}\right)\right] \\
= \; & \left[1-\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}\right]\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right]
\end{aligned}
$$

so that

$$
\begin{aligned}
1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}} &= 1-\left[1-\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}\right]\left[1+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right)\right] \\
&= 1-1+\frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right) \\
&= \frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}+o_p\left(\frac{1}{\left\Vert \gamma\right\Vert_2^{2\kappa}}\right) \\
&= \frac{1}{c\left\Vert \gamma\right\Vert_2^{2\kappa}}\left[1+o_p\left(1\right)\right]
\end{aligned}
$$

and, thus,

$$
\left(1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-2}=c^2\left\Vert \gamma\right\Vert_2^{4\kappa}\left[1+o_p\left(1\right)\right]. \tag{39}
$$

Now, consider $T^{-1}\sum_{j=2}^{N} T^2 v_j^2 / \left(N \|\gamma\|_2^2\right)$. To proceed, note first that

$$W_{1,t} = \|\gamma\|_2 f_t + \eta_{1t} = \|\gamma\|_2 f_t + \mathbf{e}_{1,N}' \Pi' u_t$$

so that, given Assumption 2-1 and given the fact that Π is an orthogonal matrix, we have that

$$\{W_{1,t}\} \equiv i.i.d.\ N\left(0, \|\gamma\|_2^2 + 1\right)$$

from which we further deduce that

$$\frac{\mathbf{W}_1}{\|\gamma\|_2} = \begin{pmatrix} W_{1,1}/\|\gamma\|_2 \\ W_{1,2}/\|\gamma\|_2 \\ \vdots \\ W_{1,T}/\|\gamma\|_2 \end{pmatrix} \sim N\left(0, \left\{1 + \frac{1}{\|\gamma\|_2^2}\right\} I_T\right)$$

Moreover, note that

$$\mathbf{W}_{2,t} = \begin{pmatrix} \eta_{2t} \\ \vdots \\ \eta_{Nt} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_{2,N}' \Pi' u_t \\ \vdots \\ \mathbf{e}_{N,N}' \Pi' u_t \end{pmatrix}$$

so that, under Assumption 2-1,

$$\{\mathbf{W}_{2,t}\} \equiv i.i.d.N\left(0, I_{N-1}\right)$$

By direct calculation, we have for $j = 2, ..., T + 1$

$$\begin{aligned}
E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right] &= T^2\frac{\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2E\left[\mathbf{W}_1\mathbf{W}'_1|\mathbf{W}_2\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{N\left\|\gamma\right\|_2^2T^2} \\
&\quad\left(\text{since}\underset{(N-1)\times 1}{\upsilon} = \frac{\widetilde{\mathbf{B}}'_2\mathbf{W}'_2\mathbf{W}_1}{T}\right) \\
&= \frac{T}{N}\frac{\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2E\left[\mathbf{W}_1\mathbf{W}'_1\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{\left\|\gamma\right\|_2^2T} \\
&\quad\text{(by independence of }\mathbf{W}_1\text{ and }\mathbf{W}_2\text{)} \\
&= \frac{T}{N}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}}{T} \\
&= \frac{T}{N}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1} \\
&\quad\left(\text{since }\frac{\mathbf{W}'_2\mathbf{W}_2}{T} = \widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\right) \\
&= \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}
\end{aligned}$$

In addition, by straightforward calculation, we also get for $j = 2, ..., T + 1$

$$
\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] \\
= \; & \frac{T^4}{N^2} E\left\{\left(\frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{\|\gamma\|_2^2 T^2}\right)^2 | \mathbf{W}_2\right\} \\
= \; & \frac{T^4}{N^2 T^4} \sum_{r=1}^{T} \sum_{s=1}^{T} \sum_{t=1}^{T} \sum_{\upsilon=1}^{T} \left\{ E\left[\frac{W_{1,r}}{\|\gamma\|_2} \frac{W_{1,s}}{\|\gamma\|_2} \frac{W_{1,t}}{\|\gamma\|_2} \frac{W_{1,\upsilon}}{\|\gamma\|_2} | \mathbf{W}_2\right] \left(\mathbf{W}_{2,r}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \left. \times \left(\mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \left(\mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \left(\mathbf{W}_{2,\upsilon}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right\} \\
= \; & \frac{T^4}{N^2 T^4} \sum_{t=1}^{T} E\left[\frac{W_{1,t}^4}{\|\gamma\|_2^4}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right)^2 \\
& + \frac{3T^4}{N^2 T^4} \left\{ \sum_{t=1}^{T} E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \qquad \left. \times \sum_{s \neq t} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,s} \mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right\} \\
= \; & 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \frac{T^4}{N^2 T^2} \left(\sum_{t=1}^{T} \frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{T}\right)^2 \\
& \left(\text{since } \frac{W_{1,t}}{\|\gamma\|_2} = f_t + \|\gamma\|_2^{-1} \eta_{1t} \sim N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right)\right) \\
= \; & 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T}{N} \widetilde{\lambda}_{(j)}\right)^2
\end{aligned}
$$

On the other hand, for $j = T+2, .., N-1$, we have

$$
E\left[\frac{T^2 \upsilon_j^2}{N \|\gamma\|_2^2} | \mathbf{W}_2\right] = \left(1 + \frac{1}{\|\gamma\|_2^2}\right) \frac{T \widetilde{\lambda}_{(j)}}{N} = 0
$$

and

$$
E\left[\frac{T^4 \upsilon_j^4}{N^2 \|\gamma\|_2^4} | \mathbf{W}_2\right] = 3\left(1 + \frac{1}{\|\gamma\|_2^2}\right)^2 \left(\frac{T}{N} \widetilde{\lambda}_{(j)}\right)^2 = 0
$$

since $\widetilde{\lambda}_{(j)} = 0$ for $j > T+1$ by part (a) of Lemma B-5.

Next, we show that

$$E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\frac{1}{T}\sum_{j=2}^{N}E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right]\right)^2|\mathbf{W}_2\right\}=O_{a.s.}\left(\frac{1}{T}\right)$$

To proceed, write

$$\begin{aligned}
& E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\frac{1}{T}\sum_{j=2}^{N}E\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}|\mathbf{W}_2\right]\right)^2|\mathbf{W}_2\right\} \\
= \; & E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\right)^2|\mathbf{W}_2\right\} \\
= \; & \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
& +\frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\right. \\
& \qquad\qquad \left.\times\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \qquad (40)
\end{aligned}$$

Consider the second term on the right-hand side of expression (40)

$$\begin{aligned}
& \frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \\
= \; & \frac{1}{T^2}\sum_{j\neq k}E\left[\frac{T^4\upsilon_j^2\upsilon_k^2}{N^2\left\|\gamma\right\|_2^4}|\mathbf{W}_2\right]-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \qquad (41)
\end{aligned}$$

For the first term in expression (41), note that

$$
\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^2 \upsilon_k^2}{N^2 \left\|\gamma\right\|_2^4} | \mathbf{W}_2\right] \\
= \quad & \frac{T^4}{N^2} E\left\{\left(\frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{\left\|\gamma\right\|_2^2 T^2}\right)\right. \\
& \qquad \left. \times \left(\frac{\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}}{\left\|\gamma\right\|_2^2 T^2}\right) | \mathbf{W}_2 \right\} \\
= \quad & \frac{T^4}{N^2 T^4} \sum_{r=1}^{T} \sum_{s=1}^{T} \sum_{t=1}^{T} \sum_{\upsilon=1}^{T} \left\{ E\left[\frac{W_{1,r}}{\left\|\gamma\right\|_2} \frac{W_{1,s}}{\left\|\gamma\right\|_2} \frac{W_{1,t}}{\left\|\gamma\right\|_2} \frac{W_{1,\upsilon}}{\left\|\gamma\right\|_2} | \mathbf{W}_2\right] \left(\mathbf{W}_{2,r}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \left. \times \left(\mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \left(\mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \left(\mathbf{W}_{2,\upsilon}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right\} \\
= \quad & \frac{T^4}{N^2 T^4} \sum_{t=1}^{T} E\left\{ \left[\frac{W_{1,t}^4}{\left\|\gamma\right\|_2^4}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \qquad\qquad \left. \times \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right\} \\
& + \frac{T^4}{N^2 T^4} \left\{ \sum_{s=1}^{T} E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,s} \mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}\right) \right. \\
& \qquad\qquad \left. \times \sum_{t \neq s} E\left[\frac{W_{1,t}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{k-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right\} \\
& + \frac{T^4}{N^2 T^4} \left\{ \sum_{t=1}^{T} E\left[\frac{W_{1,t}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right. \\
& \qquad\qquad \left. \times \sum_{s \neq t} E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,s} \mathbf{W}_{2,s}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right\} \\
& + \frac{T^4}{N^2 T^4} \left\{ \sum_{r=1}^{T} E\left[\frac{W_{1,r}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,r} \mathbf{W}_{2,r}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right. \\
& \qquad\qquad \left. \times \sum_{t \neq r} E\left[\frac{W_{1,s}^2}{\left\|\gamma\right\|_2^2}\right] \left(\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_{2,t} \mathbf{W}_{2,t}' \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1}\right) \right\} \qquad (42)
\end{aligned}
$$

Calculating the expectation for the first term on the right-hand side of expression (42) above, we have

$$
\begin{aligned}
& \frac{T^4}{N^2T^4} \sum_{t=1}^{T} \left\{ E\left[\frac{W_{1,t}^4}{\|\gamma\|_2^4} \right] \left(\mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \qquad\qquad \left. \times \left(\mathbf{e}'_{k-1,N-1} \widetilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\} \\
= \; & \frac{3T^4}{N^2T^4} \left(1 + \frac{1}{\|\gamma\|_2^2} \right)^2 \sum_{t=1}^{T} \left\{ \left(\mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \right) \right. \\
& \qquad\qquad \left. \times \left(\mathbf{e}'_{k-1,N-1} \widetilde{\mathbf{B}}'_2 \mathbf{W}_{2,t} \mathbf{W}'_{2,t} \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \right) \right\}
\end{aligned}
$$

Moreover, using the fact that

$$
\begin{aligned}
\mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{B}}'_2 \sum_{s=1}^{T} \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} &= \mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{B}}'_2 \widetilde{\mathbf{B}}_2 \widetilde{\Lambda} \widetilde{\mathbf{B}}'_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1} \widetilde{\Lambda} \mathbf{e}_{j-1,N-1} \\
&= \widetilde{\lambda}_{(j)}
\end{aligned}
$$

and, for $j \neq k$,

$$
\begin{aligned}
\mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{V}}'_2 \sum_{s=1}^{T} \frac{\mathbf{W}_{2,s} \mathbf{W}'_{2,s}}{T} \widetilde{\mathbf{V}}_2 \mathbf{e}_{k-1,N-1} &= \mathbf{e}'_{j-1,N-1} \widetilde{\mathbf{B}}'_2 \widetilde{\mathbf{B}}_2 \widetilde{\Lambda} \widetilde{\mathbf{B}}'_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{k-1,N-1} \\
&= \mathbf{e}'_{j-1,N-1} \widetilde{\Lambda} \mathbf{e}_{k-1,N-1} \\
&= 0
\end{aligned}
$$

we further obtain

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{s=1}^{T} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,s}\mathbf{W}_{2,s}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad \left.\times \sum_{t\neq s} E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left\{\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}_{2,s}'}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \left.\times\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}_{2,t}'}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad \left.\times\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & \left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad \left.\times\left(\mathbf{e}_{k-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_{2,t}\mathbf{W}_{2,t}'\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{t=1}^{T} E\left[\frac{W_{1,t}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\sum_{s\neq t} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,s}\mathbf{W}'_{2,s}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}'_{2,t}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& \times\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{s=1}^{T}\frac{\mathbf{W}_{2,s}\mathbf{W}'_{2,s}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{T^4}{N^2T^4}\left\{\sum_{r=1}^{T} E\left[\frac{W_{1,r}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,r}\mathbf{W}'_{2,r}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right. \\
& \qquad\qquad \left.\times\sum_{t\neq r} E\left[\frac{W_{1,s}^2}{\|\gamma\|_2^2}\right]\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & \frac{T^4}{N^2T^2}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\sum_{r=1}^{T}\frac{\mathbf{W}_{2,r}\mathbf{W}'_{2,r}}{T}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right) \\
& \times\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{V}}'_2\sum_{t=1}^{T}\frac{\mathbf{W}_{2,t}\mathbf{W}'_{2,t}}{T}\widetilde{\mathbf{V}}_2\mathbf{e}_{k-1,N-1}\right) \\
& -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\} \\
= & -\frac{T^4}{N^2T^4}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^2\sum_{t=1}^{T}\left\{\left(\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right)\right. \\
& \qquad\qquad\qquad \left.\times\left(\mathbf{e}'_{k-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}_{2,t}\mathbf{W}'_{2,t}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\right\}
\end{aligned}
$$

It follows from these calculations that, for $j \neq k$

$$\begin{aligned}
& E\left[\frac{T^4 \upsilon_j^2 \upsilon_k^2}{N^2 \left\|\gamma\right\|_2^4} | \mathbf{W}_2\right] \\
= & \frac{3T^4}{N^2 T^4}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right) \\
& \qquad\qquad\qquad \times\left(\mathbf{e}_{k-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\Big\} \\
& +\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
& -\frac{T^4}{N^2 T^4}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right) \\
& \qquad\qquad\qquad \times\left(\mathbf{e}_{k-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\Big\} \\
& -\frac{T^4}{N^2 T^4}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right) \\
& \qquad\qquad\qquad \times\left(\mathbf{e}_{k-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\Big\} \\
& -\frac{T^4}{N^2 T^4}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2 \sum_{t=1}^{T}\Big\{\left(\mathbf{e}_{j-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{j-1,N-1}\right) \\
& \qquad\qquad\qquad \times\left(\mathbf{e}_{k-1,N-1}^{\prime}\widetilde{\mathbf{B}}_2^{\prime}\mathbf{W}_{2,t}\mathbf{W}_{2,t}^{\prime}\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1}\right)\Big\} \\
= & \left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right)
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{1}{T^2}\sum_{j\neq k} E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \\
= & \frac{1}{T^2}\sum_{j\neq k} E\left[\frac{T^4\upsilon_j^2\upsilon_k^2}{N^2\left\|\gamma\right\|_2^4}|\mathbf{W}_2\right]-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
= & \frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right)-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j\neq k}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)\left(\frac{T\widetilde{\lambda}_{(k)}}{N}\right) \\
= & 0
\end{aligned}$$

Hence,

$$
\begin{aligned}
& E\left\{\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2|\mathbf{W}_2\right\} \\
= \;& \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
& +\frac{1}{T^2}\sum_{j\neq k}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]\left[\frac{T^2\upsilon_k^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(k)}}{N}\right]|\mathbf{W}_2\right\} \\
= \;& \frac{1}{T^2}\sum_{j=2}^{N}E\left\{\left[\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{T\widetilde{\lambda}_{(j)}}{N}\right]^2|\mathbf{W}_2\right\} \\
= \;& \frac{1}{T^2}\sum_{j=2}^{N}E\left[\frac{T^4\upsilon_j^4}{N^2\left\|\gamma\right\|_2^4}|\mathbf{W}_2\right]-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \\
= \;& \frac{3}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2-\frac{1}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \\
= \;& \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{N}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2 \\
= \;& \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}\right)^2 \quad \left(\text{since } \widetilde{\lambda}_{(j)}=0 \text{ for } j>T+1\right) \\
\leq \;& \frac{2}{T^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{N-1}{N}\right)^2 T\left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \\
& \left(\text{since } \widetilde{\lambda}_{(j)}\geq 0 \text{ for } j=2,..,T+1\right) \\
= \;& \frac{2}{T}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)^2\left(\frac{N-1}{N}\right)^2\left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \\
= \;& O_{a.s.}\left(\frac{1}{T}\right) \quad \left(\text{by Lemma B-7 and by the fact that } \left\|\gamma\right\|_2^2\rightarrow\infty \text{ under Assumption 2-2}\right) \\
= \;& o_{a.s.}(1)
\end{aligned}
$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995), we see that there exists a constant $\overline{C} < \infty$ such that for all n sufficiently large

$$
\begin{aligned}
& E\left\{T\left(\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \; & E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \; & E_{\mathbf{W}_2}\left[E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 |\mathbf{W}_2\right\}\right] \\
\leq \; & \overline{C}.
\end{aligned}
$$

Now, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{\overline{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$
\begin{aligned}
& \Pr\left\{\sqrt{T}\left|\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right| \geq C_\epsilon\right\} \\
= \; & \Pr\left\{\left(\frac{\sqrt{T}}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{\sqrt{T}}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2 \geq C_\epsilon^2\right\} \\
\leq \; & \frac{1}{C_\epsilon^2}E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
= \; & \frac{\epsilon}{\overline{C}}E\left\{\left(\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{\sqrt{T}}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N}\right)^2\right\} \\
\leq \; & \epsilon
\end{aligned}
$$

which shows that

$$
\begin{aligned}
& \frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{N}\frac{T\widetilde{\lambda}_{(j)}}{N} \\
= \; & \frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\left\|\gamma\right\|_2^2}-\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} \quad \left(\text{since } \widetilde{\lambda}_{(j)}=0 \text{ for } j>T+1\right) \\
= \; & O_p\left(\frac{1}{\sqrt{T}}\right) = o_p\left(1\right) \qquad (43)
\end{aligned}
$$

In addition, note that

$$\begin{aligned}
\left|\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right)\right| &\leq \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right| \\
&\leq \left(1+\frac{1}{\|\gamma\|_2^2}\right)\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right| \overset{a.s.}{\to} 0 \\
&\quad \text{(by Lemma B-7)}
\end{aligned}$$

Making use of this result and the Slutsky's theorem, we obtain

$$\begin{aligned}
&\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} \\
= &\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left[\frac{T\widetilde{\lambda}_{(j)}}{N}-1+1\right] \\
= &\left(1+\frac{1}{\|\gamma\|_2^2}\right)+\left(1+\frac{1}{\|\gamma\|_2^2}\right)\left[\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}-1\right] \\
= &\left(1+\frac{1}{\|\gamma\|_2^2}\right)+\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right) \overset{a.s.}{\to} 1 \qquad (44) \\
&\text{(since } \|\gamma\|_2 \to \infty)
\end{aligned}$$

from which we further deduce, in light of expression (43), that

$$\frac{1}{T}\sum_{j=2}^{N}\frac{T^2 v_j^2}{N\|\gamma\|_2^2} = \left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N} + O_p\left(\frac{1}{\sqrt{T}}\right) \overset{p}{\to} 1 \text{ as } N,T\to\infty. \qquad (45)$$

Putting together the results given in expressions (37), (38), (39), and (45); we see that as $N,T\to\infty$ such that $T/N\to 0$

$$
\begin{aligned}
&\tau^2 \\
\leq\ & \frac{N}{T\left\|\gamma\right\|_2^{2(1+2\kappa)}}\frac{1}{\widehat{\lambda}_{(1)}^2/\left\|\gamma\right\|_2^{4(1+\kappa)}}\left(1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-2}\frac{1}{T}\sum_{j=2}^{N}\frac{T^2\upsilon_j^2}{N\left\|\gamma\right\|_2^2} \\
=\ & \frac{1}{\left\|\gamma\right\|_2^{2\kappa}}\frac{N}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{1}{c^2}c^2\left\|\gamma\right\|_2^{4\kappa}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}\left[1+o_p\left(1\right)\right] \\
=\ & \frac{N}{T\left\|\gamma\right\|_2^2}\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}\left[1+o_p\left(1\right)\right] \\
=\ & O_p\left(\frac{N}{T\left\|\gamma\right\|_2^2}\right) \qquad (46)\\
& \left(\text{since }\left(1+\frac{1}{\left\|\gamma\right\|_2^2}\right)\frac{1}{T}\sum_{j=2}^{T+1}\frac{T\widetilde{\lambda}_{(j)}}{N}\overset{p}{\to}1\text{ by expression (44)}\right)
\end{aligned}
$$

Moreover, since Assumption 2-2 implies that $N/\left(T\left\|\gamma\right\|_2^2\right)\to\infty$ as $N,T\to\infty$ such that $T/N\to 0$, we further deduce that

$$\tau^2\to\infty \text{ w.p.a.1.} \qquad (47)$$

Finally, we note that expression (47) further implies that

$$\frac{\left\langle\widetilde{\mathbf{v}}_{(1)},\mathbf{e}_{1,N}\right\rangle^2}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|^2}=\frac{1}{1+\tau^2}\overset{p}{\to}0 \qquad (48)$$

as $N,T\to\infty$ such that $T/N\to 0$.

Step 5:

In this step, we will show that

$$\frac{1}{\sqrt{N}}\left\langle\frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2},\widetilde{\eta}_t\right\rangle\overset{p}{\to}0.$$

To proceed, write

$$\begin{aligned}
\frac{1}{\sqrt{N}}\left\langle\frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2},\widetilde{\eta}_t\right\rangle &= \frac{1}{\sqrt{N}}\frac{\left\langle\widetilde{\mathbf{v}}_{(1)},\widetilde{\eta}_t\right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \\
&= \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left[\widetilde{\eta}_{1t}+\sum_{j=2}^{N}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)}\right]
\end{aligned}$$

From the result given in expression (46) of Step 4 above, we have

$$\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}=\tau^2=O_p\left(\frac{N}{T\left\|\gamma\right\|_2^2}\right)$$

where $N/\left(T\left\|\gamma\right\|_2^2\right)\to\infty$ under our Assumption 2-2. This implies that

$$\frac{T\left\|\gamma\right\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}=O_p\left(1\right). \tag{49}$$

Next, note that

$$\begin{aligned}
\sum_{j=2}^{N}\upsilon_j\widetilde{\eta}_{jt} &= \sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}+\sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt} \\
&= \sum_{j=2}^{T+1}\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\widetilde{\eta}_{jt}}{T}+\sum_{j=T+2}^{N}\frac{\mathbf{e}_{j-1,N-1}'\widetilde{\mathbf{B}}_2'\mathbf{W}_2'\mathbf{W}_1\widetilde{\eta}_{jt}}{T}
\end{aligned} \tag{50}$$

Recall that $\{\widetilde{\eta}_t\}\equiv i.i.d.N\left(0,I_N\right)$ so that $\left\{\widetilde{\eta}_{j,t}\right\}\equiv i.i.d.N\left(0,1\right)$ across both j and t. Recall also that $\{f_t\}\equiv i.i.d.N\left(0,1\right)$ and f_t and $\widetilde{\eta}_s$ are independent for all s and t. In addition, since

$$\underset{T\times 1}{\mathbf{W}_1}=\begin{pmatrix}\left\|\gamma\right\|_2\left(f_1+\left\|\gamma\right\|_2^{-1}\eta_{1,1}\right)\\ \left\|\gamma\right\|_2\left(f_2+\left\|\gamma\right\|_2^{-1}\eta_{1,2}\right)\\ \vdots\\ \left\|\gamma\right\|_2\left(f_T+\left\|\gamma\right\|_2^{-1}\eta_{1,T}\right)\end{pmatrix} \text{ and } \underset{T\times(N-1)}{\mathbf{W}_2}=\begin{pmatrix}\eta_{2,1} & \eta_{3,1} & \cdots & \eta_{N-1,1}\\ \eta_{2,2} & \eta_{3,2} & \cdots & \eta_{N-1,2}\\ \vdots & \vdots & & \vdots\\ \eta_{2,T} & \eta_{3,T} & \cdots & \eta_{N-1,T}\end{pmatrix},$$

it follows that $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent. Now, focusing first on the term $\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}$ on the right-hand side of expression (50) above, note that

$$
\begin{aligned}
& E\left[\left(\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2|\mathbf{W}_2\right] \\
= \;& \frac{1}{T^2}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1}\widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2E\left[\mathbf{W}_1\mathbf{W}'_1|\mathbf{W}_2\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= \;& \frac{1}{T^2}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1}\widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\mathbf{W}'_2E\left[\mathbf{W}_1\mathbf{W}'_1\right]\mathbf{W}_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= \;& \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1}\widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\left(\frac{\mathbf{W}'_2\mathbf{W}_2}{T}\right)\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= \;& \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1}\widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\widetilde{\Lambda}\widetilde{\mathbf{B}}'_2\widetilde{\mathbf{B}}_2\mathbf{e}_{k-1,N-1} \\
= \;& \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\sum_{k=2}^{T+1}\widetilde{\eta}_{jt}\widetilde{\eta}_{kt}\mathbf{e}'_{j-1,N-1}\widetilde{\Lambda}\mathbf{e}_{k-1,N-1} \\
= \;& \frac{\left(\|\gamma\|_2^2+1\right)}{T}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^2\widetilde{\lambda}_{(j)}
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& E\left[\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2|\mathbf{W}_2\right] \\
= \;& \frac{\left(\|\gamma\|_2^2+1\right)}{T\|\gamma\|_2^2}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^2\frac{T}{N-1}\widetilde{\lambda}_{(j)} \\
\leq \;& \frac{\left(\|\gamma\|_2^2+1\right)}{\|\gamma\|_2^2}\sqrt{\frac{1}{T}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^4}\sqrt{\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N-1}\widetilde{\lambda}_{(j)}\right)^2} \\
= \;& O_{a.s.}(1)
\end{aligned}
$$

given that, as $N,T\to\infty$,

$$\frac{1}{T}\sum_{j=2}^{T+1}\widetilde{\eta}_{jt}^4\overset{a.s.}{\to}3$$

and, by Lemma B-7,

$$
\begin{aligned}
\frac{1}{T}\sum_{j=2}^{T+1}\left(\frac{T}{N-1}\widetilde{\lambda}_{(j)}\right)^2 &\leq \left(\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}\right)^2 \quad \left(\text{since } \widetilde{\lambda}_{(j)}\geq 0 \text{ for } j=2,..,T+1\right) \\
&= \left(\frac{T}{N-1}\widetilde{\lambda}_{(2)}\right)^2 \overset{a.s.}{\to} 1.
\end{aligned}
$$

Applying the law of iterated expectations as well as part (i) of Theorem 16.1 of Billingsley (1995), we see that there exists a constant $\overline{C}<\infty$ such that for all n sufficiently large

$$
\begin{aligned}
E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\} &= E_{\mathbf{W}_2}\left[E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2 |\mathbf{W}_2\right\}\right] \\
&\leq \overline{C}.
\end{aligned}
$$

Now, for any $\epsilon>0$, set $C_\epsilon=\sqrt{\overline{C}/\epsilon}$, and the Markov's inequality then implies that, for all n sufficiently large,

$$
\begin{aligned}
\Pr\left\{\left|\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|\right|\geq C_\epsilon\right\} &= \Pr\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\geq C_\epsilon^2\right\} \\
&\leq \frac{1}{C_\epsilon^2}E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\} \\
&= \frac{\epsilon}{\overline{C}}E\left\{\left(\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right)^2\right\} \\
&\leq \epsilon
\end{aligned}
$$

which shows that

$$
\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right| = O_p(1). \tag{51}
$$

Next, consider the second term on the right-hand side of expression (50). Define

$$
\underset{T\times(N-1)}{\widetilde{D}} = \left[\begin{array}{cc} \underset{T\times T}{\widetilde{\Lambda}_1} & \underset{T\times(N-T-1)}{0}\end{array}\right]
$$

where

$$\widetilde{\Lambda}_1 = \begin{pmatrix} \widetilde{\lambda}_{(2)} & 0 & \cdots & 0 \\ 0 & \widetilde{\lambda}_{(3)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{\lambda}_{(T+1)} \end{pmatrix}$$

Given that $N-1 > T$ for N, T sufficiently large and given that $\widetilde{\lambda}_{(j)} = 0$ for $j > T+1$, we have the following singular-value decomposition of $\mathbf{W}_2$:

$$\mathbf{W}_2 = \mathbb{O}\widetilde{D}\widetilde{\mathbf{B}}_2'$$

where $\mathbb{O}$ is a $T \times T$ orthogonal matrix and $\widetilde{\mathbf{B}}_2$ is as defined previously. Making use of this decomposition, we see that

$$\begin{aligned} \sum_{j=T+2}^{N} \upsilon_j \widetilde{\eta}_{jt} &= \sum_{j=T+2}^{N} \frac{\mathbf{e}_{j-1,N-1}' \widetilde{\mathbf{B}}_2' \mathbf{W}_2' \mathbf{W}_1 \widetilde{\eta}_{jt}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt} \mathbf{W}_1' \mathbf{W}_2 \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt} \mathbf{W}_1' \mathbb{O}\widetilde{D}\widetilde{\mathbf{B}}_2' \widetilde{\mathbf{B}}_2 \mathbf{e}_{j-1,N-1}}{T} \\ &= \sum_{j=T+2}^{N} \frac{\widetilde{\eta}_{jt} \mathbf{W}_1' \mathbb{O}\widetilde{D} \mathbf{e}_{j-1,N-1}}{T} \\ &= 0 \end{aligned}$$

Putting things together, we have

$$
\begin{aligned}
& \frac{1}{\sqrt{N}}\left|\left\langle \frac{\widetilde{\mathbf{v}}_{(1)}}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}, \widetilde{\eta}_t \right\rangle\right| \\
= \ & \frac{1}{\sqrt{N}}\left|\frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \widetilde{\eta}_t \right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2}\right| \\
= \ & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\sum_{j=2}^{N}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)}\right| \\
= \ & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt}\right| \\
& \left(\text{noting that } \widetilde{\lambda}_j = 0 \text{ for } j > T+1\right) \\
= \ & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}\right| \\
& \left(\text{since } \sum_{j=T+2}^{N}\upsilon_j\widetilde{\eta}_{jt}=0\right) \\
= \ & \frac{1}{\sqrt{N}}\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\left|\widetilde{\eta}_{1t}+\frac{1}{\widehat{\lambda}_{(1)}/\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{1}{\left\|\gamma\right\|_2^{2(1+\kappa)}}\sum_{j=2}^{T+1}\frac{\upsilon_j\widetilde{\eta}_{jt}}{\left(1-\widetilde{\lambda}_{(j)}/\widehat{\lambda}_{(1)}\right)}\right| \\
\leq \ & \frac{1}{\sqrt{N}}\left\{\left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2}\right. \\
& \quad \left.\times\left[\left|\widetilde{\eta}_{1t}\right|+\frac{1}{\widehat{\lambda}_{(1)}/\left\|\gamma\right\|_2^{2(1+\kappa)}}\frac{1}{\left\|\gamma\right\|_2^{2(1+\kappa)}}\left(1-\max_{2\leq j\leq T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-1}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad\times\left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\left(c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\right)^{-1}\frac{\|\gamma\|_2}{\sqrt{N}}\frac{c\,\|\gamma\|_2^{2\kappa}}{\|\gamma\|_2\,\|\gamma\|_2^{2(1+\kappa)}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&\quad\left(\text{given that } \frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}}=c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)\text{ for } 0<\kappa<1,\right. \\
&\quad\left.\text{and }\left(1-\max_{2\le j\le T+1}\frac{\widetilde{\lambda}_{(j)}}{\widehat{\lambda}_{(1)}}\right)^{-1}=c\,\|\gamma\|_2^{2\kappa}\left[1+o_p(1)\right]\right) \\
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad\times\left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\sqrt{\frac{N-1}{T}}\frac{\|\gamma\|_2^{2\kappa}}{\|\gamma\|_2^{2(1+\kappa)}}\frac{\|\gamma\|_2}{\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= \left[1+\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad\times\left[\frac{|\widetilde{\eta}_{1t}|}{\sqrt{N}}+\sqrt{\frac{N-1}{T}}\frac{\|\gamma\|_2}{\|\gamma\|_2^2\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= \left[\frac{T\,\|\gamma\|_2^2}{N}+\frac{T\,\|\gamma\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2}\right]^{-1/2} \\
&\quad\times\left[\sqrt{\frac{T\,\|\gamma\|_2^2}{N}}\frac{|\widetilde{\eta}_{1t}|}{N}+\sqrt{\frac{N-1}{N}}\frac{1}{\sqrt{N}}\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right|(1+o_p(1))\right] \\
&= o_p(1), \qquad (52)
\end{aligned}
$$

where the last line follows from the fact that

$$
\begin{aligned}
\frac{1}{\|\gamma\|_2}\sqrt{\frac{T}{N-1}}\left|\sum_{j=2}^{T+1}\upsilon_j\widetilde{\eta}_{jt}\right| &= O_p(1)\quad\text{(by expression (51))} \\
\frac{T\,\|\gamma\|_2^2}{N}\sum_{j=2}^{N}\frac{\upsilon_j^2}{\left(\widehat{\lambda}_{(1)}-\widetilde{\lambda}_{(j)}\right)^2} &= O_p(1)\quad\text{(by expression (49))}
\end{aligned}
$$

and the fact that

$$\|\gamma\|_2^2 \to \infty \text{ and } \frac{T\,\|\gamma\|_2^2}{N} \to 0 \text{ (by Assumption 2-2)}.$$

Step 6:

Finally, in this last step, we bring everything together. Combining the results given in expressions (36) of step 3, (48) of step 4, and (52) of step 5 and noting the fact that $f_t = O_p(1)$, we can apply the Slutsky's theorem to deduce that

$$\widehat{f}_t = \frac{\|\gamma\|_2}{\sqrt{N}} \frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \mathbf{e}_{1,N} \right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} f_t + \frac{1}{\sqrt{N}} \frac{\left\langle \widetilde{\mathbf{v}}_{(1)}, \widetilde{\eta}_t \right\rangle}{\left\|\widetilde{\mathbf{v}}_{(1)}\right\|_2} \xrightarrow{p} 0 \text{ as } N, T \to \infty$$

which is the required result. □

Proof of Theorem 4.1:

To proceed, note first that the principal component estimator of $\underline{F}_t$ can be written as

$$\widehat{\underline{F}}_t = \frac{\widehat{\Gamma}' Z_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1}$$

where $\widehat{\Gamma} = \sqrt{\widehat{N}_1}\widehat{B}$ and where the columns of the matrix $\widehat{B}$ are the eigenvectors associated with the Kp largest eigenvalues of the (post-variable-selection) sample covariance matrix

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0}.$$

Moreover, by the result of part (d) of Lemma D-14, the matrix $\widehat{B}$ has the representation

$$\widehat{B} = \widehat{G}_1 \widehat{V}$$

where $\widehat{G}_1$ is an $N \times Kp$ matrix, whose columns define an orthonormal basis for an invariant subspace of $\widehat{\Sigma}\left(\widehat{H^c}\right)$ and where $\widehat{V}$ is a $Kp \times Kp$ orthogonal matrix as defined in expression (116) in part (c) of Lemma D-14. (See Lemma D-14 and also Lemma D-13 for additional discussion on the origin of this representation). Making use of this representation, we can further write

$$
\begin{aligned}
\widehat{\underline{F}}_t - Q'\underline{F}_t &= \frac{\sqrt{\widehat{N}_1}\widehat{V}'\widehat{G}_1' Z_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1} - Q'\underline{F}_t \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q' &= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} - Q' \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \\
&= \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} + \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}} &= \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right)
\end{aligned}
$$

so that

$$\begin{aligned}
&\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} \\
&= Q'\underline{F}_t + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}}\right)\underline{F}_t \\
&= Q'\underline{F}_t + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right)\underline{F}_t + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\underline{F}_t \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right)\underline{F}_t
\end{aligned}$$

It follows that

$$\begin{aligned}
\underline{\widehat{F}}_t - Q'\underline{F}_t &= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right)\underline{F}_t + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}}\underline{F}_t \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}}\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right)\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Hence, applying the triangle inequality as well as parts (a)-(c), (g), and (i) of Lemma D-15 along with the Slutsky's theorem, we obtain

$$
\begin{aligned}
& \left\| \widehat{\underline{F}}_t - Q' \underline{F}_t \right\|_2 \\
\leq \; & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \left\| \underline{F}_t \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \left\| \underline{F}_t \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \widehat{V}' \widehat{G}_1' \right\|_2 \left\| \frac{\Gamma \left(\widehat{H^c} \right) - \Gamma}{\sqrt{N_1}} \right\|_2 \left\| \underline{F}_t \right\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{t,N} \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
= \; & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \left\| \underline{F}_t \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \left\| \underline{F}_t \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\| \frac{\Gamma \left(\widehat{H^c} \right) - \Gamma}{\sqrt{N_1}} \right\|_2 \left\| \underline{F}_t \right\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{t,N} \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
& \left(\text{since } \left\| \widehat{V}' \widehat{G}_1' \right\|_2 = \lambda_{\max} \left(\widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}_1' \right) = \lambda_{\max} \left(\widehat{V}' \widehat{G}_1' \widehat{G}_1 \widehat{V} \right) = \lambda_{\max} \left(I_{Kp} \right) = 1 \right) \\
= \; & o_p(1) O_p(1) + o_p(1) O_p(1) O_p(1) + O_p(1) o_p(1) O_p(1) + o_p(1) \\
= \; & o_p(1) . \; \square
\end{aligned}
$$

Proof of Theorem 4.2:

To proceed, note that for any $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we have

$$
\begin{aligned}
& \left| a' \widehat{Y}_{T+h} - a' \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) \right| \\
= \; & \left| a' \left(\widehat{\beta}_0 + \widehat{B}_1' \underline{Y}_T + \widehat{B}_2' \widehat{\underline{F}}_T \right) - a' \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) \right| \\
= \; & \Big| a' \left(\widehat{\beta}_0 - \beta_0 \right) + a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \\
& + a' \left(\widehat{B}_2 - Q^{-1} B_2 + Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T + Q' \underline{F}_T \right) - a' B_2' \underline{F}_T \Big| \\
\leq \; & \left| a' \left(\widehat{\beta}_0 - \beta_0 \right) \right| + \left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
& + \left| a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right| + \left| a' B_2' Q^{-1\prime} Q' \underline{F}_T - a' B_2' \underline{F}_T \right| \\
= \; & \left| a' \left(\widehat{\beta}_0 - \beta_0 \right) \right| + \left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
& + \left| a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right|
\end{aligned}
$$

Lemma D-18 and Slutsky's theorem directly imply that

$$\left|a'\left(\widehat{\beta}_0-\beta_0\right)\right|=o_p(1)$$

Now, applying the CS inequality, we obtain

$$\begin{aligned}\left|a'\left(\widehat{B}_1-B_1\right)'\underline{Y}_T\right| &\leq \sqrt{a'\left(\widehat{B}_1-B_1\right)'\left(\widehat{B}_1-B_1\right)a}\sqrt{\underline{Y}_T'\underline{Y}_T} \\ &= \sqrt{a'\left(\widehat{B}_1-B_1\right)'\left(\widehat{B}_1-B_1\right)a}\,\|\underline{Y}_T\|_2^2,\end{aligned}$$

and

$$\begin{aligned}&\left|a'\left(\widehat{B}_2-Q^{-1}B_2\right)'Q'\underline{F}_T\right| \\ \leq\ &\sqrt{a'\left(\widehat{B}_2-Q^{-1}B_2\right)'\left(\widehat{B}_2-Q^{-1}B_2\right)a}\sqrt{\underline{F}_T'QQ'\underline{F}_T} \\ =\ &\sqrt{a'\left(\widehat{B}_2-Q^{-1}B_2\right)'\left(\widehat{B}_2-Q^{-1}B_2\right)a}\sqrt{\underline{F}_T'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\Xi\widehat{V}\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\underline{F}_T} \\ =\ &\sqrt{a'\left(\widehat{B}_2-Q^{-1}B_2\right)'\left(\widehat{B}_2-Q^{-1}B_2\right)a}\sqrt{\underline{F}_T'\left(\frac{\Gamma'\Gamma}{N_1}\right)\underline{F}_T} \\ \leq\ &\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)}\sqrt{a'\left(\widehat{B}_2-Q^{-1}B_2\right)'\left(\widehat{B}_2-Q^{-1}B_2\right)a}\,\|\underline{F}_T\|_2^2 \\ \leq\ &\overline{C}\sqrt{a'\left(\widehat{B}_2-Q^{-1}B_2\right)'\left(\widehat{B}_2-Q^{-1}B_2\right)a}\,\|\underline{F}_T\|_2^2\end{aligned}$$

Moreover, note that

$$\begin{aligned}E\left[\|\underline{Y}_T\|_2^2\right] &\leq \left(E\,\|\underline{Y}_T\|_2^6\right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\ &\leq \overline{C}^{\frac{1}{3}}=C<\infty \quad \text{(by Lemma C-4)}\end{aligned}$$

and

$$\begin{aligned}E\left[\|\underline{F}_T\|_2^2\right] &\leq \left(E\,\|\underline{F}_T\|_2^6\right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\ &\leq \overline{C}^{\frac{1}{3}}=C<\infty \quad \text{(by Lemma C-4)}\end{aligned}$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \sqrt{C/\epsilon}$, and Markov's inequality then implies that, for all $T > p-1$,

$$\Pr\left\{\|\underline{Y}_T\|_2 \geq C_\epsilon\right\} = \Pr\left\{\|\underline{Y}_T\|_2^2 \geq C_\epsilon^2\right\} \leq \frac{E\left[\|\underline{Y}_T\|_2^2\right]}{C_\epsilon^2} = \frac{\epsilon E\left[\|\underline{Y}_T\|_2^2\right]}{C} \leq \epsilon$$

from which it follows that

$$\|\underline{Y}_T\|_2 = O_p\left(1\right).$$

In a similar way, we can also show that

$$\|\underline{F}_T\|_2 = O_p\left(1\right).$$

Application of the result given in Lemma D-18 then allows us to deduce that

$$\left|a'\left(\widehat{B}_1 - B_1\right)'\underline{Y}_T\right| \leq \sqrt{a'\left(\widehat{B}_1 - B_1\right)'\left(\widehat{B}_1 - B_1\right)a}\,\|\underline{Y}_T\|_2^2 = o_p\left(1\right)$$

and

$$\begin{aligned}
&\left|a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'Q'\underline{F}_T\right| \\
\leq\ & \sqrt{a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\widehat{B}_2 - Q^{-1}B_2\right)a}\sqrt{\underline{F}_T'QQ'\underline{F}_T} \\
\leq\ & \sqrt{a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\widehat{B}_2 - Q^{-1}B_2\right)a}\sqrt{\lambda_{\max}\left(QQ'\right)}\,\|\underline{F}_T\|_2 \\
=\ & \sqrt{a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\widehat{B}_2 - Q^{-1}B_2\right)a}\sqrt{\lambda_{\max}\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}}\Xi\widehat{V}\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}}\right\}}\,\|\underline{F}_T\|_2 \\
=\ & \sqrt{a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\widehat{B}_2 - Q^{-1}B_2\right)a}\sqrt{\lambda_{\max}\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)\right\}}\,\|\underline{F}_T\|_2 \\
& \left(\text{since } \widehat{V}\widehat{V}' = I_{Kp} \text{ and } \Xi\Xi' = I_{Kp}\right) \\
\leq\ & \sqrt{\overline{C}}\sqrt{a'\left(\widehat{B}_2 - Q^{-1}B_2\right)'\left(\widehat{B}_2 - Q^{-1}B_2\right)a}\,\|\underline{F}_T\|_2 \quad \text{(by Assumption 3-6)} \\
=\ & o_p\left(1\right)
\end{aligned}$$

In addition, we can apply the CS inequality to get

$$
\begin{aligned}
& \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
\leq \; & \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \sqrt{\left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)} \\
\leq \; & \sqrt{a' \left(\widehat{B}_1 - B_1 \right)' \left(\widehat{B}_1 - B_1 \right) a} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
= \; & o_p(1) \quad \text{(by Lemma D-18 and part (j) of Lemma D-15 in Appendix D)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left| a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
\leq \; & \sqrt{a' B_2' Q^{-1\prime} Q^{-1} B_2 a} \sqrt{\left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right)} \\
= \; & \sqrt{a' B_2' Q^{-1\prime} Q^{-1} B_2 a} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq \; & \sqrt{\left[\lambda_{\min} \left(\frac{\Gamma' \Gamma}{N_1} \right) \right]^{-1} \lambda_{\max} \left(B_2' B_2 \right)} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq \; & \sqrt{C^*} \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \quad \text{(for some positive constant } C^* \text{ as shown in} \\
& \text{expression (135) in Appendix D. See the proof of part (d) of Lemma D-17)} \\
= \; & o_p(1) \quad \text{(by part (j) of Lemma D-15)}
\end{aligned}
$$

Putting everything together and applying Slutsky's theorem, we then obtain

$$
\begin{aligned}
& \left| a' \widehat{Y}_{T+h} - a' \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) \right| \\
\leq \; & \left| a' \left(\widehat{\beta}_0 - \beta_0 \right) \right| + \left| a' \left(\widehat{B}_1 - B_1 \right)' \underline{Y}_T \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| \\
& + \left| a' B_2' Q^{-1\prime} \left(\widehat{\underline{F}}_T - Q' \underline{F}_T \right) \right| + \left| a' \left(\widehat{B}_2 - Q^{-1} B_2 \right)' Q' \underline{F}_T \right| \\
= \; & o_p(1).
\end{aligned}
$$

Since the above argument holds for all $a \in \mathbb{R}^d$ such that $\|a\|_2 = 1$, we further deduce that

$$
\widehat{Y}_{T+h} - \left(\beta_0 + B_1' \underline{Y}_T + B_2' \underline{F}_T \right) = o_p(1).
$$

as required. □

8 Appendix B: Supporting Lemmas Used in the Proof of Theorem 2.1

In this appendix, we first state and prove a number of lemmas which are used in the proof of Theorem 2.1.

Lemma B-1 (Weyl's inequality): Let A, B be real, symmetric $T \times T$ matrices and let the eigenvalues $\lambda_{(i)}(A)$, $\lambda_{(i)}(B)$, and $\lambda_{(i)}(A+B)$ be arranged in decreasing (or, more generally, non-increasing) order, so that

$$\begin{aligned}
\lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(T)}(A), \\
\lambda_{(1)}(B) &\geq \lambda_{(2)}(B) \geq \cdots \geq \lambda_{(T)}(B), \\
\lambda_{(1)}(A+B) &\geq \lambda_{(2)}(A+B) \geq \cdots \geq \lambda_{(T)}(A+B).
\end{aligned}$$

Then, for each $j = 1, 2, ..., T$, we have

$$\lambda_{(j)}(A) + \lambda_{(T)}(B) \leq \lambda_{(j)}(A+B) \leq \lambda_{(j)}(A) + \lambda_{(1)}(B).$$

Proof of Lemma B-1: This inequality is well-known, and its proof can be found in many linear algebra textbooks. See, for example, Theorem 4.3.1 and its proof on pages 181-182 of Horn and Johnson (1985). Hence, we shall not provide an explicit proof here. $\square$

Lemma B-2: Suppose that $\|\gamma\|_2^2 \to \infty$ as $N \to \infty$, and suppose that, given N,

$$\{\zeta_{1,t,N}\} \equiv i.i.d.N\left(0, 1 + \frac{1}{\|\gamma\|_2^2}\right) \text{ for } t = 1, ..., T.$$

Let $\zeta_{1,N} = \begin{pmatrix} \zeta_{1,1,N} & \zeta_{1,2,N} & \cdots & \zeta_{1,T,N} \end{pmatrix}'$ and $\underset{T\times T}{A} = T^{-1}\|\gamma\|_2^2 \zeta_{1,N}\zeta_{1,N}'$. Then, as $N, T \to \infty$ such that $T/N \to 0$, we have

$$\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

where $\lambda_{(1)}(A)$ denotes the largest eigenvalue of the matrix A.

Proof of Lemma B-2:

Note that, since $A = \|\gamma\|_2^2 \zeta_{1,N}\zeta_{1,N}'/T$, we can write its dual a_D as

$$\underset{1\times 1}{a_D} = \frac{1}{T}\|\gamma\|_2^2 \zeta_{1,N}'\zeta_{1,N}$$

Next, write

$$\frac{1}{T}\zeta_{1,N}'\zeta_{1,N}=\frac{1}{T}\sum_{t=1}^{T}\zeta_{1,t,N}^{2}=\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{t=1}^{T}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^{2}$$

where, by assumption,

$$\left\{\zeta_{1,t,N}\right\}\equiv i.i.d.N\left(0,1+\frac{1}{\|\gamma\|_2^2}\right)\text{ for each }N.$$

This implies that

$$\begin{aligned}\left\{\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1/2}\zeta_{1,t,N}\right\} &\equiv i.i.d.N\left(0,1\right)\text{ and}\\ \left\{\mathcal{X}_{t,N}^{*}\right\} &\equiv i.i.d.\chi_1^2\end{aligned}$$

where

$$\mathcal{X}_{t,N}^{*}=\left[\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1/2}\zeta_{1,t,N}\right]^{2}=\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^{2}$$

and where χ_1^2 denotes a chi-square random variable with one degree of freedom. Hence, by direct calculation, we get

$$\begin{aligned}&E\left(\frac{1}{T}\zeta_{1\cdot,N}'\zeta_{1\cdot,N}-\left[1+\frac{1}{\|\gamma\|_2^2}\right]\right)^{2}\\ =\;&E\left[\left(1+\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\sum_{t=1}^{T}\left(\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^{2}-1\right)\right]^{2}\\ =\;&\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{2}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left\{\left[\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^{2}-1\right]\left[\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,s,N}^{2}-1\right]\right\}\\ =\;&\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{2}\frac{1}{T^2}\sum_{t=1}^{T}E\left\{\left[\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{-1}\zeta_{1,t,N}^{2}-1\right]^{2}\right\}\\ =\;&\frac{2}{T}\left(1+\frac{1}{\|\gamma\|_2^2}\right)^{2}\;\left(\text{since }E\left[\chi_1^2\right]=1\text{ and }Var\left(\chi_1^2\right)=2\right)\\ =\;&O\left(\frac{1}{T}\right)\end{aligned}$$

Applying Markov's inequality, we then obtain

$$\frac{1}{T}\zeta_{1\cdot,N}'\zeta_{1\cdot,N} = 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

Hence, as $N, T \to \infty$

$$\begin{aligned}
\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} &= \frac{a_D}{\|\gamma\|_2^2} \\
&= \left(\frac{1}{\|\gamma\|_2^2}\right)\frac{1}{T}\|\gamma\|_2^2\,\zeta_{1\cdot,N}'\zeta_{1\cdot,N} \\
&= \frac{1}{T}\zeta_{1\cdot,N}'\zeta_{1\cdot,N} \\
&= 1 + \frac{1}{\|\gamma\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

where the first equality above follows from the fact that $\lambda_{(1)}(A) = \lambda_{\max}(A) = \lambda_{\max}(a_D) = a_D$ given that a_D is a scalar. This proves Lemma B-2. $\square$

Lemma B-3: Let $X_1, X_2, ..., X_N$ be N independent T dimensional sub-Gaussian random vectors with zero mean vector and identity covariance matrix and the sub-Gaussian norms bounded by a constant C_0. Then, for every $\tau \geq 0$, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\},$$

one has

$$\begin{aligned}
\overline{w} - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N}\sum_{i=1}^{N} w_i X_i X_i'\right) \\
&\leq \lambda_{(1)}\left(\frac{1}{N}\sum_{i=1}^{N} w_i X_i X_i'\right) \\
&= \overline{w} + \max\left\{\delta, \delta^2\right\}
\end{aligned}$$

where

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}}$$

for constants $C, c > 0$, depending on C_0. Here, $|w_i|$ is bounded for all i and

$$\overline{w} = \frac{1}{N}\sum_{i=1}^{N} w_i.$$

Remark: Lemma B-3 is Lemma A.1 given in Appendix A of Wang and Fan (2017), and so we state this result here without proof. As discussed there, this lemma is an extension of the classical Davidson-Szarek bound. See Davidson and Szarek (2001) and Vershynin (2010) for additional discussion.

Lemma B-4: Suppose that

$$\left\{\zeta_{i,t}\right\} \equiv i.i.d.N\left(0,1\right) \text{ for } i=2,...,N;\, t=1,...,T$$

Let $\zeta_i = \begin{pmatrix} \zeta_{i,1} & \zeta_{i,2} & \cdots & \zeta_{i,T} \end{pmatrix}'$. Also, let

$$\underset{T\times T}{B} = \frac{1}{T}\sum_{i=2}^{N}\zeta_i\zeta_i'$$

and let

$$\lambda_{(1)}\left(B\right) \geq \lambda_{(2)}\left(B\right) \geq \cdot\cdot\cdot \geq \lambda_{(T)}\left(B\right)$$

denote the eigenvalues of B. Then, for $k = 1, ..., T$;

$$\frac{T}{N-1}\lambda_{(k)}\left(B\right) = 1 + O_p\left(\sqrt{\frac{T}{N}}\right) = 1 + o_p\left(1\right),$$

as $N, T \to \infty$ such that $T/N \to 0$.

Proof of Lemma B-4:

Applying Lemma B-3 above for the case where $\tau = \sqrt{T}$ and where $w_i = 1$ for all i, we see that, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\} = 1 - 2\exp\left\{-cT\right\},$$

the following inequality holds for any $k \in \{1, ..., T\}$

$$\begin{aligned} 1 - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &\leq \lambda_{(k)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &\leq \lambda_{(1)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \\ &= 1 + \max\left\{\delta, \delta^2\right\}. \end{aligned}$$

Since in this case

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = (1+C)\sqrt{\frac{T}{N}},$$

the above inequality relationship simplifies to

$$1-(1+C)\sqrt{\frac{T}{N}} \leq \lambda_{(k)}\left(\frac{1}{N-1}\sum_{j=2}^{N}\zeta_j\zeta_j'\right) \leq 1+(1+C)\sqrt{\frac{T}{N}}$$

or

$$1-(1+C)\sqrt{\frac{T}{N}} \leq \frac{T}{N-1}\lambda_{(k)}\left(\frac{1}{T}\sum_{j=2}^{N}\zeta_{j\cdot}\zeta_{j\cdot}'\right) = \frac{T}{N-1}\lambda_{(k)}(B) \leq 1+(1+C)\sqrt{\frac{T}{N}}$$

This shows that, as $N,T\to\infty$ such that $T/N\to 0$,

$$\frac{T}{N-1}\lambda_{(k)}(B) = 1+O_p\left(\sqrt{\frac{T}{N}}\right) = 1+o_p(1)$$

for $k=1,...,T$. □

Lemma B-5: Suppose that $\{\mathbf{W}_{2,t}\}\equiv i.i.d.N(0,I_{N-1})$. Now, let

$$\underset{(N-1)\times T}{\mathbf{W}_2'} = \begin{pmatrix} \underset{(N-1)\times 1}{\mathbf{W}_{2,1}} & \underset{(N-1)\times 1}{\mathbf{W}_{2,2}} & \cdots & \underset{(N-1)\times 1}{\mathbf{W}_{2,T}} \end{pmatrix}$$

and let

$$\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$$

be the $N-1$ eigenvalues of

$$\widehat{\Sigma}_{\mathbf{W_2}} = \frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{W}_{2,t}\mathbf{W}_{2,t}'.$$

Then, the following results hold as $N,T\to\infty$ such that $T/N\to 0$.

(a)

$$\widetilde{\lambda}_{(j)} = 0 \text{ for } j=T+2,...,N$$

(b)

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} = 1+O_p\left(\sqrt{\frac{T}{N}}\right) = 1+o_p(1).$$

Proof of Lemma B-5:

To show part (a), note that, by assumption, for N, T sufficiently large, we have $N-1 > T$, so that $\widehat{\Sigma}_{\mathbf{W_2}} = \mathbf{W}_2'\mathbf{W}_2/T$ is a $(N-1)\times(N-1)$ matrix with rank less than or equal to T, from which it follows trivially that

$$\widetilde{\lambda}_{(j)} = 0 \text{ for } j = T+2, ..., N.$$

Next, to show part (b), first write

$$\underset{T\times(N-1)}{\mathbf{W}_2} = \begin{pmatrix} \underline{W}_{2,1} & \underline{W}_{2,2} & \cdots & \underline{W}_{2,N-1} \end{pmatrix}$$

so that $\underline{W}_{2,i}$ denotes the i^{th} column of $\mathbf{W}_2$ for $i = 1, ..., N-1$. Note that, by Sylvester's determinantal identity, the non-zero eigenvalues of $\widehat{\Sigma}_{\mathbf{W_2}} = \mathbf{W}_2'\mathbf{W}_2/T$ (i.e., $\widetilde{\lambda}_{(2)}, ..., \widetilde{\lambda}_{(T+1)}$) are the same as those of the dual matrix

$$\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W_2},D}} = \frac{\mathbf{W}_2\mathbf{W}_2'}{T} = \frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'$$

Now, under our assumptions, $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\{0,1\}$ for $t = 1, ..., T$ and $i = 1, ..., N-1$ where $\mathbf{W}_{2,t,i}$ denotes the $(t,i)^{th}$ element of $\mathbf{W}_2$. Applying Lemma B-3 above with $\tau = \sqrt{T}$, we see that, with probability at least

$$1 - 2\exp\left\{-c\tau^2\right\} = 1 - 2\exp\left\{-cT\right\},$$

the following inequality holds for any $j \in \{2, ..., T+1\}$

$$\begin{aligned} 1 - \max\left\{\delta, \delta^2\right\} &\leq \lambda_{(T)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &\leq \lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &\leq \lambda_{(1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\ &= 1 + \max\left\{\delta, \delta^2\right\} \end{aligned}$$

where

$$\delta = C\sqrt{\frac{T}{N}} + \frac{\tau}{\sqrt{N}} = (1+C)\sqrt{\frac{T}{N}}$$

Moreover, by our definition,

$$\widetilde{\lambda}_{(j)} = \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}'_{2,i} \right),$$

so that, by multiplying and dividing by T, we see that

$$\begin{aligned} 1 - (1 + C) \sqrt{\frac{T}{N}} &\leq \frac{T}{N-1} \lambda_{(j-1)} \left(\frac{1}{T} \sum_{i=1}^{N-1} \underline{W}_{2,i} \underline{W}'_{2,i} \right) = \frac{T}{N-1} \widetilde{\lambda}_{(j)} \\ &\leq 1 + (1 + C) \sqrt{\frac{T}{N}} \end{aligned}$$

Furthermore, since the above inequality relationship above holds for any $j \in \{2, ..., T+1\}$, it must be that

$$1 - (1 + C) \sqrt{\frac{T}{N}} \leq \frac{T}{N-1} \max_{2 \leq j \leq T+1} \widetilde{\lambda}_{(j)} \leq 1 + (1 + C) \sqrt{\frac{T}{N}}$$

It follows that, as $N, T \to \infty$ such that $T/N \to 0$,

$$\frac{T}{N-1} \max_{2 \leq j \leq T+1} \widetilde{\lambda}_{(j)} = 1 + O_p \left(\sqrt{\frac{T}{N}} \right) = 1 + o_p(1). \ \square$$

Lemma B-6: Let X be a $N \times T$ random matrix, and let X_{it} be the $(i, t)^{th}$ element of X. Suppose that

$$\{X_{it}\} \equiv i.i.d. (0, 1)$$

and suppose that $E\left[X_{it}^4\right] < \infty$. Moreover, let

$$B = \frac{1}{N} X'X.$$

Then, as $N, T \to \infty$ such that $T/N \to c \in [0, 1)$,

$$\begin{aligned} \lambda_{\min}(B) &\overset{a.s.}{\to} \left(1 - \sqrt{c}\right)^2, \\ \lambda_{\max}(B) &\overset{a.s.}{\to} \left(1 + \sqrt{c}\right)^2. \end{aligned}$$

Remark: Lemma B-6 is a special case of Lemma 1 given in Shen, Shen, Zhu, and Marron (2016) and is a slightly extended version of Theorem 2 of Bai and Yin (1993). Hence, we state this result here without proof.

Lemma B-7: Suppose that $\{\mathbf{W}_{2,t}\} \equiv i.i.d.N\left(0, I_{N-1}\right)$. Let

$$\widetilde{\lambda}_{(2)} \geq \widetilde{\lambda}_{(3)} \geq \cdots \geq \widetilde{\lambda}_{(N)}$$

be the $N-1$ eigenvalues of

$$\widehat{\Sigma}_{\mathbf{W_2}} = \frac{\mathbf{W}_2'\mathbf{W}_2}{T} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{W}_{2,t}\mathbf{W}_{2,t}'.$$

where $\underset{T\times(N-1)}{\mathbf{W}_2} = \begin{pmatrix} \underset{(N-1)\times 1}{\mathbf{W}_{2,1}} & \underset{(N-1)\times 1}{\mathbf{W}_{2,2}} & \cdots & \underset{(N-1)\times 1}{\mathbf{W}_{2,T}} \end{pmatrix}'$. Then, as $N, T \to \infty$ such that $T/N \to 0$,

$$\frac{T}{N-1}\widetilde{\lambda}_{(j)} \overset{a.s.}{\to} 1 \text{ for any } j \in \{2, ..., T+1\}.$$

In particular,

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} \overset{a.s.}{\to} 1$$

and

$$\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)} - 1\right| \overset{a.s.}{\to} 0.$$

Proof of Lemma B-7:

To proceed, first define the dual matrix of $\widehat{\Sigma}_{\mathbf{W_2}}$ given by

$$\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W_2},D}} = \frac{\mathbf{W}_2\mathbf{W}_2'}{T} = \frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'$$

where $\underline{W}_{2,i}$ denotes the i^{th} column of $\mathbf{W}_2$ for $i = 1, ..., N-1$. Now, since $T/\left(N-1\right) \to 0$ and since $\{\mathbf{W}_{2,t,i}\} \equiv i.i.d.N\{0,1\}$ for $t = 1, ..., T$ and $i = 1, ..., N-1$; it follows from applying Lemma B-6 that

$$\begin{aligned}
\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} &= \frac{T}{N-1}\max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\widehat{\Sigma}_{\mathbf{W_2},D}\right) \\
&= \frac{T}{N-1}\max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\
&= \max_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \overset{a.s.}{\to} 1 \text{ as } N, T \to \infty \qquad (53)
\end{aligned}$$

and

$$\begin{aligned}
\frac{T}{N-1}\min_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)} &= \frac{T}{N-1}\min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\widehat{\Sigma}_{\mathbf{W_2},D}\right) \\
&= \frac{T}{N-1}\min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{T}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right) \\
&= \min_{2\leq j\leq T+1}\lambda_{(j-1)}\left(\frac{1}{N-1}\sum_{i=1}^{N-1}\underline{W}_{2,i}\underline{W}_{2,i}'\right)\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty. \qquad (54)
\end{aligned}$$

Expressions (53) and (54) then imply that, for any $j\in\{2,...,T+1\}$,

$$\frac{T}{N-1}\widetilde{\lambda}_{(j)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty,$$

so that

$$\frac{T}{N-1}\max_{2\leq j\leq T+1}\widetilde{\lambda}_{(j)}=\frac{T}{N-1}\widetilde{\lambda}_{(2)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty.$$

In addition, note that, for any $j\in\{2,...,T+1\}$,

$$\frac{T}{N}\widetilde{\lambda}_{(j)}=\frac{N-1}{N}\frac{T}{N-1}\widetilde{\lambda}_{(j)}\overset{a.s.}{\to} 1 \text{ as } N,T\to\infty$$

from which it further follows that

$$\max_{2\leq j\leq T+1}\left|\frac{T}{N}\widetilde{\lambda}_{(j)}-1\right|\leq\left|\frac{T}{N}\widetilde{\lambda}_{(1)}-1\right|+\left|\frac{T}{N}\widetilde{\lambda}_{(T+1)}-1\right|\overset{a.s.}{\to} 0. \ \square$$

Lemma B-8: Consider the simple factor model

$$\underset{N\times 1}{Z_t}=\underset{N\times 1}{\gamma}\underset{1\times 1}{f_t}+\underset{N\times 1}{u_t},\ t=1,...,T;$$

where we assume that $\{u_t\}\equiv i.i.d.N(0,I_N)$, $\{f_t\}\equiv i.i.d.N(0,1)$, and u_s and f_t are independent for all t,s. Let $\Sigma_Z=E[Z_tZ_t']$; then, the eigenvalues of Σ_Z are given by

$$\lambda_{(1)}=\|\gamma\|_2^2+1 \text{ and } \lambda_{(j)}=1 \text{ for } j=2,...,N.$$

Moreover, let $\pi_{(1)}$ $(N\times 1)$ be the eigenvector assocated with the top eigenvalue $\lambda_{(1)}$; then,

$$\pi_{(1)}=\frac{\gamma}{\|\gamma\|_2}.$$

Proof of Lemma B-8: To show part (a), note first that

$$\begin{aligned} \Sigma_Z &= E\left[Z_t Z_t'\right] \\ &= E\left[\left(\gamma f_t + u_t\right)\left(\gamma' f_t + u_t'\right)\right] \\ &= \gamma\gamma' + I_N \end{aligned}$$

Consider the determinantal equation

$$\begin{aligned} 0 &= \det\left\{\lambda I_N - \left(\gamma\gamma' + I_N\right)\right\} \\ &= \det\left\{\left(\lambda - 1\right) I_N - \gamma\gamma'\right\} \\ &= \det\left\{\kappa I_N - \gamma\gamma'\right\} \quad \text{(where } \kappa = \lambda - 1\text{)} \\ &= \kappa^N \det\left\{I_N - \kappa^{-1}\gamma\gamma'\right\} \\ &= \kappa^N\left(1 - \kappa^{-1}\gamma'\gamma\right) \quad \text{(by Sylvester's determinantal theorem)} \\ &= \kappa^{N-1}\left(\kappa - \gamma'\gamma\right) \end{aligned}$$

so the roots of this equation are

$$\kappa_{(1)} = \gamma'\gamma = \|\gamma\|_2^2, \; \kappa_{(2)} = 0, ..., \kappa_{(N)} = 0$$

and, thus,

$$\lambda_{(1)} = \gamma'\gamma + 1 = \|\gamma\|_2^2 + 1, \lambda_{(2)} = 1, ..., \lambda_{(N)} = 1.$$

Next, note that

$$\begin{aligned} \left(\gamma\gamma' + I_N\right)\gamma &= \|\gamma\|_2^2 \gamma + \gamma \\ &= \left(\|\gamma\|_2^2 + 1\right)\gamma \end{aligned}$$

so that γ is an (unnormalized) eigenvector of the matrix $\gamma\gamma' + I_N$ associated with the eigenvalue $\lambda_{(1)} = \|\gamma\|_2^2 + 1$. It follows that we can take

$$\pi_{(1)} = \gamma / \|\gamma\|_2$$

to be the (normalized) eigenvector of $\Sigma_Z = E\left[Z_t Z_t'\right] = \gamma\gamma' + I_N$ associated with the eigenvalue

$$\lambda_{(1)} = \|\gamma\|_2^2 + 1. \; \square$$

Lemma B-9: Let $A \in M_n$ be a Hermetian matrix, let r be an integer with $1 \leq r \leq n$, and let A_r denote any $r \times r$ principal submatrix of A (obtained by deleting $n - r$ rows and the corresponding columns of A). Let the eigenvalues of A and A_r be ordered as follows

$$\begin{aligned} \lambda_{(1)}(A) &\geq \lambda_{(2)}(A) \geq \cdots \geq \lambda_{(n)}(A), \\ \lambda_{(1)}(A_r) &\geq \lambda_{(2)}(A_r) \geq \cdots \geq \lambda_{(r)}(A_r). \end{aligned}$$

Then, for each integer k such that $1 \leq k \leq r$, we have

$$\lambda_{(k)}(A) \geq \lambda_{(k)}(A_r) \geq \lambda_{(n-[r-k])}(A)$$

so that for $r = n - 1$, we have

$$\lambda_{(1)}(A) \geq \lambda_{(1)}(A_{n-1}) \geq \lambda_{(2)}(A) \geq \lambda_{(2)}(A_{n-1}) \geq \cdots \geq \lambda_{(n-1)}(A) \geq \lambda_{(n-1)}(A_{n-1}) \geq \lambda_{(n)}(A)$$

Proof of Lemma B-9: This result is essentially Theorem 4.3.15 in Horn and Johnson (1985), except that we use different notations here. A proof of this lemma can be obtained by a slight adaptation of the proof given in Horn and Johnson (1985) for Theorem 4.3.15 using our notations here.

Lemma B-10: Let

$$\underset{N\times 1}{W_t} = \sum_{j=1}^{N} \sqrt{\ell_j}\zeta_{j,t}\mathbf{e}_{j,N}$$

where $\zeta_{1,t} = f_t + \|\gamma\|_2^{-1}\eta_{1t}$ and $\zeta_{j,t} = \eta_{j,t}$ for $j = 2, ..., N$; where $\ell_1 = \|\gamma\|_2^2$ and $\ell_j = 1$ for $j = 2, ..., N$; and where $\mathbf{e}_{j,N}$ is an $N \times 1$ elementary vector whose j^{th} component is 1 and all remaining components are 0. Suppose that $\{\eta_t\} \equiv i.i.d.N(0, I_N)$, $\{f_t\} \equiv i.i.d.N(0,1)$, and f_t and η_s are independent for all t, s. In addition, suppose that the following assumptions hold.

(i) As $N \to \infty$

$$\|\gamma\|_2 \to \infty.$$

(ii) As $N, T \to \infty$

$$\frac{N}{T\|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty$$

for some κ such that $0 < \kappa < 1$.

Moreover, let $\widehat{\lambda}_{(1)}$ denote the largest eigenvalue of the sample covariance matrix

$$\widehat{\Sigma}_{\mathbf{W}} = \frac{1}{T} \sum_{t=1}^{T} W_t W_t',$$

where $\underset{N \times T}{\mathbf{W}} = (W_1, ..., W_T)$. Then, as $N, T \to \infty$ such that $T/N \to 0$; the largest sample eigenvalue $\widehat{\lambda}_{(1)}$ satisfy

$$\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}} = c + \frac{1}{\|\gamma\|_2^{2\kappa}} + o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \text{ for } 0 < \kappa < 1.$$

Proof of Lemma B-10:

Following Shen, Shen, Zhu, and Marron (2016), we shall study the sample eigenvalue properties via the dual matrix

$$\underset{T \times T}{\widehat{\Sigma}_{\mathbf{W},D}} = \frac{1}{T} \mathbf{W}'\mathbf{W}$$

which shares the same nonzero eigenvalues with the sample covariance matrix

$$\underset{N \times N}{\widehat{\Sigma}_{\mathbf{W}}} = \frac{1}{T} \mathbf{W}\mathbf{W}'.$$

Define $\zeta_{j\cdot} = \begin{pmatrix} \zeta_{j,1} & \zeta_{j,2} & \cdots & \zeta_{j,T} \end{pmatrix}'$. Since $W_t = \sum_{j=1}^{N} \sqrt{\ell_j} \zeta_{j,t} \mathbf{e}_{j,N}$, we can write

$$\frac{1}{T} W_t' W_s = \sum_{k=1}^{N} \sum_{\ell=1}^{N} \ell_k^{1/2} \ell_\ell^{1/2} \zeta_{k,t} \zeta_{\ell,s} e_{k,N}^T e_{\ell,N} = \sum_{k=1}^{N} \ell_k \zeta_{k,t} \zeta_{k,s}$$

where

$$\begin{aligned} \ell_1 &= \|\gamma\|_2^2, \ \ell_2 = \cdots = \ell_N = 1 \\ \zeta_{1,t} &= f_t + \frac{1}{\|\gamma\|_2} \eta_{1t}, \ \zeta_{2,t} = \eta_{2t}, ..., \zeta_{N,t} = \eta_{Nt}. \end{aligned}$$

so that

$$
\begin{aligned}
& \widehat{\Sigma}_{\mathbf{W},D} \\
= \ & \frac{1}{T} \underset{T\times N}{\mathbf{W}'} \underset{N\times T}{\mathbf{W}} = \frac{1}{T} \begin{pmatrix} W_1' \\ W_2' \\ \vdots \\ W_T' \end{pmatrix} \begin{pmatrix} W_1 & W_2 & \cdots & W_T \end{pmatrix} \\
= \ & \frac{1}{T} \begin{pmatrix} W_1'W_1 & W_1'W_2 & \cdots & W_1'W_T \\ W_2'W_1 & W_2'W_2 & \cdots & W_2'W_T \\ \vdots & \vdots & & \vdots \\ W_T'W_1 & W_T'W_2 & \cdots & W_T'W_T \end{pmatrix} = \frac{1}{T} \sum_{k=1}^{N} \ell_k \begin{pmatrix} \zeta_{k,1}^2 & \zeta_{k,1}\zeta_{k,2} & \cdots & \zeta_{k.1k,T} \\ \zeta_{k,2}\zeta_{k,1} & \zeta_{k,2}^2 & \cdots & \zeta_{k,2}\zeta_{k,T} \\ \vdots & \vdots & & \vdots \\ \zeta_{k,T}\zeta_{k,1} & \zeta_{k,T}\zeta_{k,2} & \cdots & \zeta_{k,T}^2 \end{pmatrix} \\
= \ & \frac{1}{T} \sum_{k=1}^{N} \ell_k \begin{pmatrix} \zeta_{k,1} \\ \zeta_{k,2} \\ \vdots \\ \zeta_{k,T} \end{pmatrix} \begin{pmatrix} \zeta_{k,1} & \zeta_{k,2} & \cdots & \zeta_{k,T} \end{pmatrix} = \frac{1}{T} \sum_{k=1}^{N} \ell_k \zeta_{k\cdot} \zeta_{k\cdot}'.
\end{aligned}
$$

which can be decomposed into sum of two matrices as follows

$$
\underset{T\times T}{\widehat{\Sigma}_{\mathbf{W},D}} = A + B
$$

where

$$
\underset{T\times T}{A} = \frac{1}{T} \ell_1 \underset{T\times 1}{\zeta_{1\cdot}} \underset{1\times T}{\zeta_{1\cdot}'} = \frac{1}{T} \|\gamma\|_2^2 \zeta_{1\cdot}\zeta_{1\cdot}' \text{ and } B = \frac{1}{T} \sum_{k=2}^{N} \zeta_{k\cdot}\zeta_{k\cdot}'.
$$

Next, we apply Weyl's inequality (given in Lemma B-1 above) to obtain

$$
\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} \leq \frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^2} = \frac{\lambda_{(1)}(A+B)}{\|\gamma\|_2^2} \leq \frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2} + \frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2}
$$

Moreover, as $N, T \to \infty$, $\|\gamma\|_2^2 \to \infty$ under Assumption (i); whereas Assumption (ii) states that

$$
\frac{N}{T \|\gamma\|_2^{2(1+\kappa)}} = c + o\left(\frac{1}{\|\gamma\|_2^2}\right), \text{ with } 0 < c < \infty
$$

from which it follows that

$$\begin{aligned}
\frac{N-1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} &= \frac{N}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} + O\left(\frac{1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\right) \\
&= c + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right) + O\left(\frac{1}{T\left\|\gamma\right\|_2^{2(1+\kappa)}}\right) \\
&= c + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right) \qquad (55)
\end{aligned}$$

In addition, recall that the result of Lemma B-4 shows that, as $N, T \to \infty$,

$$\frac{T\lambda_{(1)}(B)}{(N-1)} = 1 + O_p\left(\sqrt{\frac{T}{N}}\right) \text{ and } \frac{T\lambda_{(T)}(B)}{N-1} = 1 + O_p\left(\sqrt{\frac{T}{N}}\right)$$

Hence, applying Lemma B-4 and Assumpton (ii); we obtain, as $N, T \to \infty$

$$\begin{aligned}
\frac{1}{\left\|\gamma\right\|_2^{2\kappa}} \frac{\lambda_{(1)}(B)}{\left\|\gamma\right\|_2^2} &= \frac{(N-1)}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} \frac{T\lambda_{(1)}(B)}{(N-1)} \\
&= \left[c + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)\right]\left(1 + O_p\left(\sqrt{\frac{T}{N}}\right)\right) \\
&= c + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right) \\
\frac{1}{\left\|\gamma\right\|_2^{2\kappa}} \frac{\lambda_{(T)}(B)}{\left\|\gamma\right\|_2^2} &= \frac{(N-1)}{T\left\|\gamma\right\|_2^{2(1+\kappa)}} \frac{T\lambda_{(T)}(B)}{(N-1)} \\
&= \left[c + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)\right]\left(1 + O_p\left(\sqrt{\frac{T}{N}}\right)\right) \\
&= c + O_p\left(\sqrt{\frac{T}{N}}\right) + o\left(\frac{1}{\left\|\gamma\right\|_2^2}\right)
\end{aligned}$$

which, together with the inequality relationship

$$\frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2} + \frac{\lambda_{(T)}(B)}{\left\|\gamma\right\|_2^2} \leq \frac{\widehat{\lambda}_{(1)}}{\left\|\gamma\right\|_2^2} \leq \frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2} + \frac{\lambda_{(1)}(B)}{\left\|\gamma\right\|_2^2}$$

and the fact that, by Lemma B-2,

$$\frac{\lambda_{(1)}(A)}{\left\|\gamma\right\|_2^2} = 1 + \frac{1}{\left\|\gamma\right\|_2^2} + O_p\left(\frac{1}{\sqrt{T}}\right)$$

imply that

$$\begin{aligned}
\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2}+\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(T)}(B)}{\|\gamma\|_2^2} &= \frac{1}{\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right)+O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}\sqrt{T}}\right)+c \\
&\quad +O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right) \\
\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(A)}{\|\gamma\|_2^2}+\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\lambda_{(1)}(B)}{\|\gamma\|_2^2} &= \frac{1}{\|\gamma\|_2^{2\kappa}}+O\left(\frac{1}{\|\gamma\|_2^{2(1+\kappa)}}\right)+O_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}\sqrt{T}}\right)+c \\
&\quad +O_p\left(\sqrt{\frac{T}{N}}\right)+o\left(\frac{1}{\|\gamma\|_2^2}\right) \\
&= c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right)
\end{aligned}$$

so that

$$\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^{2(1+\kappa)}}=\frac{1}{\|\gamma\|_2^{2\kappa}}\frac{\widehat{\lambda}_{(1)}}{\|\gamma\|_2^2}=c+\frac{1}{\|\gamma\|_2^{2\kappa}}+o_p\left(\frac{1}{\|\gamma\|_2^{2\kappa}}\right).$$

9 Appendix C: Lemmas Used in the Proofs of the Key Supporting Lemmas Given in Appendix D.

Lemmas C-1, C-2, C-3, C-4, and C-5 correspond, respectively, to Lemmas OA-1, OA-2, OA-3, OA-5, and OA-11 in Chao, Qiu, and Swanson (2023b). However, since these lemmas are used to prove key supporting lemmas given in Appendix D below, for readers' convenience, we restate these results here.

Lemma C-1: Let a and θ be real numbers such that $a > 0$ and $\theta \geq 1$. Also, let G be a finite non-negative integer. Then,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

Proof of Lemma C-1: By the integral test,

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty \text{ for finite non-negative integer } G$$

if

$$\int_1^\infty x^G \exp\left\{-ax^\theta\right\} dx < \infty \text{ for finite non-negative integer } G$$

In addition, note that since, by assumption, $a > 0$ and $\theta \geq 1$, we have

$$\int_1^\infty x^G \exp\left\{-ax^\theta\right\} dx \leq \int_1^\infty x^G \exp\{-ax\} \, dx$$

We will first consider the case where $G = 0$. In this case, note that

$$\int_1^\infty x^0 \exp\{-ax\} \, dx = \int_1^\infty \exp\{-ax\} \, dx$$

Let $u = -ax$, so that $-\frac{du}{a} = dx$; and we have

$$\begin{aligned}
\int_1^\infty \exp\{-ax\} \, dx &= -\frac{1}{a}\int_{-a}^{-\infty} \exp\{u\} \, du \\
&= \frac{1}{a}\int_{-\infty}^{-a} \exp\{u\} \, du \\
&= \frac{\exp\{-a\}}{a} \\
&< \infty \text{ for any } a > 0.
\end{aligned} \tag{56}$$

Next, consider the case where G is an integer such that $G \geq 1$. Here, we will show that

$$\int_1^\infty x^G \exp\{-ax\} \, dx = \left[\frac{1}{a} + \sum_{k=1}^{G} \frac{1}{a}\left(\prod_{j=0}^{k-1} \frac{G-j}{a}\right)\right] \exp\{-a\} < \infty$$

using mathematical induction. To proceed, first consider the case where $G = 1$. Let

$$\begin{aligned}
u &= x, \; du = dx \\
dv &= \exp\{-ax\} \, dx, \; v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned}
\int_1^\infty x\exp\{-ax\}\,dx &= -\frac{x}{a}\exp\{-ax\}\Big|_1^\infty + \int_1^\infty \frac{1}{a}\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} - \frac{1}{a^2}\exp\{-ax\}\Big|_1^\infty \\
&= \frac{1}{a}\exp\{-a\} + \frac{1}{a^2}\exp\{-a\} \\
&= \left(\frac{1}{a}+\frac{1}{a^2}\right)\exp\{-a\} \\
&= \left\{\frac{1}{a}+\sum_{k=1}^{1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{1-j}{a}\right)\right\}\exp\{-a\} < \infty
\end{aligned}$$

Next, for $G = 2$, let

$$\begin{aligned}
u &= x^2,\ du = 2xdx \\
dv &= \exp\{-ax\}\,dx,\ v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned}
\int_1^\infty x^2\exp\{-ax\}\,dx &= -\frac{x^2}{a}\exp\{-ax\}\Big|_1^\infty + \frac{2}{a}\int_1^\infty x\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} + \frac{2}{a}\left(\frac{1}{a}+\frac{1}{a^2}\right)\exp\{-a\} \\
&= \frac{1}{a}\exp\{-a\} + 2\left(\frac{1}{a^2}+\frac{1}{a^3}\right)\exp\{-a\} \\
&= \left(\frac{1}{a}+\frac{2}{a^2}+\frac{2}{a^3}\right)\exp\{-a\} \\
&= \left[\frac{1}{a}+\sum_{k=1}^{2}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{2-j}{a}\right)\right]\exp\{-a\} \\
&< \infty
\end{aligned}$$

Now, suppose that, for some $G \geq 2$,

$$\int_1^\infty x^{G-1}\exp\{-ax\}\,dx = \left[\frac{1}{a}+\sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\};$$

then, let

$$
\begin{aligned}
u &= x^G,\ du = Gx^{G-1}dx \\
dv &= \exp\{-ax\}\,dx,\ v = -\frac{1}{a}\exp\{-ax\};
\end{aligned}
$$

and, using integration-by-parts, we have

$$
\begin{aligned}
\int_1^\infty x^G \exp\{-ax\}\,dx &= -\frac{x^G}{a}\exp\{-ax\}\Big|_1^\infty + \frac{G}{a}\int_1^\infty x^{G-1}\exp\{-ax\}\,dx \\
&= \frac{1}{a}\exp\{-a\} + \frac{G}{a}\left[\frac{1}{a} + \sum_{k=1}^{G-1}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-1-j}{a}\right)\right]\exp\{-a\} \\
&= \frac{1}{a}\exp\{-a\} + \left[\frac{G}{a^2} + \sum_{k=1}^{G-1}\frac{1}{a}\frac{G}{a}\left(\prod_{j=0}^{k-1}\frac{G-(j+1)}{a}\right)\right]\exp\{-a\} \\
&= \left\{\frac{1}{a} + \frac{G}{a^2} + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right) + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\right. \\
&\quad \left. + \cdots + \frac{1}{a}\frac{G}{a}\left(\frac{G-1}{a}\right)\left(\frac{G-2}{a}\right)\times\cdots\times\left(\frac{1}{a}\right)\right\}\exp\{-a\} \\
&= \left\{\frac{1}{a} + \sum_{k=1}^{G}\frac{1}{a}\left(\prod_{j=0}^{k-1}\frac{G-j}{a}\right)\right\}\exp\{-a\} \\
&< \infty. \qquad (57)
\end{aligned}
$$

In view of expressions (56) and (57), it then follows by the integral test for series convergence that

$$\sum_{m=1}^{\infty} m^G \exp\left\{-am^\theta\right\} < \infty$$

for any finite non-negative integer G and for any constants a and θ such that $a > 0$ and $\theta \geq 1$. $\square$

Lemma C-2: Let $\{V_t\}$ be a sequence of random variables (or random vectors) defined on some probability space $(\Omega, \mathcal{F}, P)$, and let

$$X_t = g\left(V_t, V_{t-1}, ..., V_{t-\varkappa}\right)$$

be a measurable function for some finite positive integer $\varkappa$. In addition, defne $\mathcal{G}_{-\infty}^t = \sigma\left(...., X_{t-1}, X_t\right)$, $\mathcal{G}_{t+m}^\infty = \sigma\left(X_{t+m}, X_{t+m+1}, ...\right)$, $\mathcal{F}_{-\infty}^t = \sigma\left(...., V_{t-1}, V_t\right)$, and $\mathcal{F}_{t+m-\varkappa}^\infty = \sigma\left(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...\right)$. Under this setting, the following results hold.

(a) Let

$$\begin{aligned}\beta_{V,m-\varkappa} &= \sup_t \beta\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^t\right) - P(B)\right| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\right\}\right], \\ \beta_{X,m} &= \sup_t \beta\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P(H)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right].\end{aligned}$$

If $\{V_t\}$ is β-mixing with

$$\beta_{V,m-\varkappa} \le \overline{C}_1 \exp\{-C_2(m-\varkappa)\}$$

for all $m \ge \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also β-mixing with β-mixing coefficient satisfying

$$\beta_{X,m} \le C_1 \exp\{-C_2 m\} \text{ for all } m \ge \varkappa,$$

where C_1 is a positive constant such that $C_1 \ge \overline{C}_1 \exp\{C_2\varkappa\}$.

(b) Let

$$\begin{aligned}\alpha_{V,m-\varkappa} &= \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m-\varkappa}^\infty\right) = \sup_t \sup_{G\in\mathcal{F}_{-\infty}^t, H\in\mathcal{F}_{t+m-\varkappa}^\infty} \left|P(G\cap H) - P(G)P(H)\right|, \\ \alpha_{X,m} &= \sup_t \alpha\left(\mathcal{G}_{-\infty}^t, \mathcal{G}_{t+m}^\infty\right) = \sup_t \sup_{G\in\mathcal{G}_{-\infty}^t, H\in\mathcal{G}_{t+m}^\infty} \left|P(G\cap H) - P(G)P(H)\right|\end{aligned}$$

If $\{V_t\}$ is α-mixing with

$$\alpha_{V,m-\varkappa} \le \overline{C}_1 \exp\{-C_2(m-\varkappa)\}$$

for all $m \ge \varkappa$ and for some positive constants $\overline{C}_1$ and C_2; then X_t is also α-mixing with α-mixing coefficient satisfying

$$\alpha_{X,m} \le C_1 \exp\{-C_2 m\} \text{ for all } m \ge \varkappa,$$

where C_1 is a positive constant such that $C_1 \ge \overline{C}_1 \exp\{C_2\varkappa\}$.

Proof of Lemma C-2:

To show part (a), note first that it is well known that

$$\begin{aligned}\beta_{X,m} &= \sup_t E\left[\sup\left\{\left|P\left(H|\mathcal{G}_{-\infty}^t\right) - P(H)\right| : H \in \mathcal{G}_{t+m}^\infty\right\}\right] \\ &= \sup_t \left\{\frac{1}{2}\sup \sum_{i=1}^I \sum_{j=1}^J \left|P(G_i\cap H_j) - P(G_i)P(H_j)\right|\right\}\end{aligned}$$

where the second supremum on the last line above is taken over all pairs of finite partitions $\{G_1, ..., G_I\}$ and $\{H_1, ..., H_J\}$ of Ω such that $G_i \in \mathcal{G}_{-\infty}^t$ for $i = 1, ..., I$ and $H_j \in \mathcal{G}_{t+m}^\infty$ for $j = 1, ..., J$. See, for example, Borovkova, Burton, and Dehling (2001). Similarly,

$$\begin{aligned}\beta_{V,m-\varkappa} &= \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{F}_{-\infty}^t\right) - P(B)\right| : B \in \mathcal{F}_{t+m-\varkappa}^\infty\right\}\right] \\ &= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P(A_i \cap B_j) - P(A_i)P(B_j)\right|\right\}\end{aligned}$$

where, similar to the definition of $\beta_{X,m}$, the second supremum on the last line above is taken over all pairs of finite partitions $\{A_1, ..., A_L\}$ and $\{B_1, ..., B_M\}$ of Ω such that $A_i \in \mathcal{F}_{-\infty}^t$ for $i = 1, ..., I$ and $B_j \in \mathcal{F}_{t+m-\varkappa}^\infty$ for $j = 1, ..., M$. Moreover, since X_t is measurable on any σ-field on which $V_t, V_{t-1}, ..., V_{t-\varkappa}$ are measurable, we also have

$$\mathcal{G}_{-\infty}^t = \sigma(...., X_{t-1}, X_t) \subseteq \sigma(...., V_{t-1}, V_t) = \mathcal{F}_{-\infty}^t$$

and

$$\mathcal{G}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, ...) \subseteq \sigma(V_{t+m-\varkappa}, V_{t+m+1-\varkappa}, ...) = \mathcal{F}_{t+m-\varkappa}^\infty.$$

It, thus, follows that, for all $m \geq \varkappa$,

$$\begin{aligned}\beta_{X,m} &= \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{I}\sum_{j=1}^{J}\left|P(G_i \cap H_j) - P(G_i)P(H_j)\right|\right\} \\ &\leq \sup_t \left\{\frac{1}{2}\sup\sum_{i=1}^{L}\sum_{j=1}^{M}\left|P(A_i \cap B_j) - P(A_i)P(B_j)\right|\right\} \\ &= \beta_{V,m-\varkappa} \\ &\leq \overline{C}_1 \exp\{-C_2(m-\varkappa)\} \\ &= \overline{C}_1 \exp\{C_2\varkappa\}\exp\{-C_2 m\} \\ &\leq C_1 \exp\{-C_2 m\}\end{aligned}$$

for some positive constant $C_1 \geq \overline{C}_1 \exp\{C_2\varkappa\}$ which exists given that $\varkappa$ is fixed. Moreover, we have

$$\beta_{X,m} \leq C_1 \exp\{-C_2 m\} \to 0 \text{ as } m \to \infty,$$

which establishes the required result for part (a).

Part (b) can be shown in a manner similar to part (a), so to avoid redundancy, we do not include an explicit proof here. □

Lemma C-3: Let $\{X_t\}$ be a sequence of random variables that is α-mixing. Let $p > 1$ and $r \geq p/(p-1)$, and let $q = \max\{p, r\}$. Suppose that, for all t,

$$\|X_t\|_q = (E|X_t|^q)^{\frac{1}{q}} < \infty$$

Then,

$$|Cov(X_t, X_{t+m})| \leq 2\left(2^{1-1/p} + 1\right)\alpha_m^{1-1/p-1/r}\|X_t\|_p\|X_{t+m}\|_r$$

where

$$\alpha_m = \sup_t \alpha\left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty\right) = \sup_{G\in\mathcal{F}_{-\infty}^t, H\in\mathcal{F}_{t+m}^\infty}|P(G\cap H) - P(G)P(H)|.$$

Remark: This is Corollary 14.3 of Davidson (1994). For a proof, see pages 212-213 of Davidson (1994).

Lemma C-4: Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, there exists a positve constant $\overline{C}$ such that

$$E\|\underline{W}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t$$

and, thus,

$$E\|\underline{Y}_t\|_2^6 \leq \overline{C} < \infty \text{ and } E\|\underline{F}_t\|_2^6 \leq \overline{C} < \infty \text{ for all } t,$$

where

$$\underset{dp\times 1}{\underline{Y}_t} = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \text{ and } \underset{Kp\times 1}{\underline{F}_t} = \begin{pmatrix} F_t \\ F_{t-1} \\ \vdots \\ F_{t-p+1} \end{pmatrix}.$$

Proof of Lemma C-4:

To proceed, note that, given Assumption 3-1, we can write the vector moving-average (VMA) representation of the companion form of the FAVAR model as

$$
\begin{aligned}
\underline{W}_t &= \left(I_{(d+K)p} - A\right)^{-1} \alpha + \sum_{j=0}^{\infty} A^j E_{t-j} \\
&= \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' J_{d+K} \alpha + \sum_{j=0}^{\infty} A^j J_{d+K}' J_{d+K} E_{t-j} \\
&= \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu + \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}, \qquad (58)
\end{aligned}
$$

where

$$
\underline{W}_t = \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \ E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},
$$

$$
\underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.
$$

By the triangle inequality,

$$
\left\|\underline{W}_t\right\|_2 \leq \left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2 + \left\| \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j} \right\|_2
$$

Moreover, using the inequality $\left|\sum_{i=1}^{m} a_i\right|^r \leq m^{r-1} \sum_{i=1}^{m} |a_i|^r$ for $r \geq 1$, we get

$$
\left\|\underline{W}_t\right\|_2^6 \leq 2^5 \left(\left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 + \left\| \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j} \right\|_2^6 \right)
$$

so that

$$
E\left\|\underline{W}_t\right\|_2^6 \leq 32 \left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 + 32 E \left\| \sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j} \right\|_2^6 \qquad (59)
$$

Focusing first on the first term on the right-hand side of the inequality (59), we note that

$$\begin{aligned}
\left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 &= \left(\mu' J_{d+K} \left(I_{(d+K)p} - A\right)^{-1\prime} \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu\right)^3 \\
&= \left(\mu' J_{d+K} \left[\left(I_{(d+K)p} - A\right) \left(I_{(d+K)p} - A\right)'\right]^{-1} J_{d+K}' \mu\right)^3 \\
&\leq \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu' J_{d+K} J_{d+K}' \mu\right)^3 \\
&= \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu' \mu\right)^3
\end{aligned}$$

Now, by Assumption 3-7, there exists a constant $\underline{C} > 0$ such that

$$\begin{aligned}
\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\} &= \lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)'\left(I_{(d+K)p} - A\right)\right\} \\
&= \sigma_{\min}^2 \left(I_{(d+K)p} - A\right) \\
&\geq \underline{C} \lambda_{\min}^2 \left(I_{(d+K)p} - A\right) \\
&\geq \underline{C} \left[1 - \phi_{\max}\right]^2 \\
&> 0
\end{aligned}$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. It follows by Assumption 3-5 that, there exists a positive constant $\overline{C}_1$ such that

$$\begin{aligned}
\left\| \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu \right\|_2^6 &\leq \left(\frac{1}{\lambda_{\min}\left\{\left(I_{(d+K)p} - A\right)\left(I_{(d+K)p} - A\right)'\right\}}\right)^3 \left(\mu' \mu\right)^3 \\
&\leq \frac{\|\mu\|_2^6}{\underline{C}^3 \left[1 - \phi_{\max}\right]^6} \leq \overline{C}_1 < \infty.
\end{aligned}$$

To show the boundedness of the second term on the right-hand side of the inequality (59), let $e_{g,(d+K)p}$ be a $(d+K)\,p \times 1$ elementary vector whose g^{th} component is 1 and all other components are 0 for $g \in \{1, 2, ..., (d+K)\,p\}$, and note that

$$\left\Vert \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\Vert_2^2 = \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^2$$
$$= \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} \left(A'\right)^k e_{g,(d+K)p}$$

from which we obtain, by applying the inequality $\left|\sum_{i=1}^{m} a_i\right|^r \leq m^{r-1} \sum_{i=1}^{m} |a_i|^r$ for $r \geq 1$

$$\left\Vert \sum_{j=0}^{\infty} A^j J'_{d+K} \varepsilon_{t-j} \right\Vert_2^6$$
$$= \left[\sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^2 \right]^3$$
$$\leq [(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \right)^6$$
$$= [(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \left\{ \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} e'_{g,(d+K)p} A^j J'_{d+K} \varepsilon_{t-j} \varepsilon'_{t-k} J_{d+K} \left(A'\right)^k e_{g,(d+K)p} \right.$$
$$\left. \times e'_{g,(d+K)p} A^i J'_d \varepsilon_{t-i} \varepsilon'_{t-\ell} J_{d+K} \left(A'\right)^\ell e_{g,(d+K)p} e'_{g,(d+K)p} A^r J'_{d+K} \varepsilon_{t-r} \varepsilon'_{t-s} J_d \left(A'\right)^s e_{g,(d+K)p} \right\}$$

Hence,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}\right\|_2^6 \\
\leq\ & \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^6 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{3} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^3\right)^2 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{2}\binom{4}{2} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^2\right)^3 \\
& + \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \binom{6}{4} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^4 \sum_{k=0}^{\infty} E\left|e_{g,(d+K)p}' A^k J_{d+K}' \varepsilon_{t-k}\right|^2 \\
=\ & \left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^6 \\
& +20\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^3\right)^2 \\
& +90\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^2\right)^3 \\
& +15\left[(d+K)\,p\right]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} E\left|e_{g,(d+K)p}' A^j J_{d+K}' \varepsilon_{t-j}\right|^4 \sum_{k=0}^{\infty} E\left|e_{g,(d+K)p}' A^k J_{d+K}' \varepsilon_{t-k}\right|^2
\end{aligned}
$$

Next, applying the Cauchy-Schwarz inequality, we further obtain

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}' \varepsilon_{t-j}\right\|_2^6 \\
\leq\ & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j J_{d+K}' J_{d+K} \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad \left. \times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k J_{d+K}' J_{d+K} \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\} \\
\leq\ & [(d+K)p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left(\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\left[(d+K)p\right]^2 \sum_{g=1}^{(d+K)p} \left\{\sum_{j=0}^{\infty} \left[e_{g,(d+K)p}' A^j \left(A^j\right)' e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad \left. \times \sum_{k=0}^{\infty} \left[e_{g,(d+K)p}' A^k \left(A^k\right)' e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\}
\end{aligned}
$$

In addition, observe that, for every $g \in \{1, 2, ..., (d+K)p\}$

$$
\begin{aligned}
& e'_{g,(d+K)p} A^j \left(A^j\right)' e_{g,(d+K)p} \\
\leq \; & \lambda_{\max} \left\{ A^j \left(A^j\right)' \right\} \\
= \; & \lambda_{\max} \left\{ \left(A^j\right)' A^j \right\} \\
= \; & \sigma^2_{\max} \left(A^j\right) \\
\leq \; & C \max \left\{ \left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2 \right\} \text{ (by Assumption 3-7)} \\
= \; & C \max \left\{ \left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j} \right\} \\
= \; & C \phi^{2j}_{\max}
\end{aligned}
$$

where $\phi_{\max} = \max \{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. Now, in light of Assumption 3-2(b), we can set $C \geq 1$ to be a constant such that $E \|\varepsilon_{t-j}\|_2^6 \leq C < \infty$, so that, by Liapunov's inequality,

$$
\begin{aligned}
E \|\varepsilon_{t-j}\|_2^2 & \leq \left(E \|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{3}} \leq C^{\frac{1}{3}}, \; E \|\varepsilon_{t-j}\|_2^3 \leq \left(E \|\varepsilon_{t-j}\|_2^6\right)^{\frac{1}{2}} \leq C^{\frac{1}{2}}, \\
E \|\varepsilon_{t-j}\|_2^4 & \leq \left(E \|\varepsilon_{t-j}\|_2^6\right)^{\frac{2}{3}} \leq C^{\frac{2}{3}},
\end{aligned}
$$

and, thus,

$$
\begin{aligned}
& E\left\|\sum_{j=0}^{\infty} A^j J_{d+K}^{\prime} \varepsilon_{t-j}\right\|_2^6 \\
\leq\; & [(d+K)\,p]^2 \sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j \left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^3 E\left\|\varepsilon_{t-j}\right\|_2^6 \\
& +20\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j \left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^{\frac{3}{2}} E\left\|\varepsilon_{t-j}\right\|_2^3\right)^2 \\
& +90\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j \left(A^j\right)^{\prime} e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-j}\right\|_2^2\right)^3 \\
& +15\,[(d+K)\,p]^2 \sum_{g=1}^{(d+K)p}\left\{\sum_{j=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^j \left(A^j\right)^{\prime} e_{g,(d+K)p}\right]^2 E\left\|\varepsilon_{t-j}\right\|_2^4\right. \\
& \qquad\qquad \left.\times \sum_{k=0}^{\infty}\left[e_{g,(d+K)p}^{\prime} A^k \left(A^k\right)^{\prime} e_{g,(d+K)p}\right] E\left\|\varepsilon_{t-k}\right\|_2^2\right\} \\
\leq\; & C\,[(d+K)\,p]^2\left\{\sum_{g=1}^{(d+K)p} \sum_{j=0}^{\infty} \phi_{\max}^{6j}+20 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{3j}\right)^2+90 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{2j}\right)^3\right. \\
& \qquad\qquad \left.+15 \sum_{g=1}^{(d+K)p}\left(\sum_{j=0}^{\infty} \phi_{\max}^{4j}\right)\left(\sum_{k=0}^{\infty} \phi_{\max}^{2k}\right)\right\} \\
\leq\; & C\,[(d+K)\,p]^3 \\
& \times\left\{\frac{1}{1-\phi_{\max}^6}+20\left(\frac{1}{1-\phi_{\max}^3}\right)^2+90\left(\frac{1}{1-\phi_{\max}^2}\right)^3+15\left(\frac{1}{1-\phi_{\max}^4}\right)\left(\frac{1}{1-\phi_{\max}^2}\right)\right\} \\
\leq\; & \overline{C}_2<\infty
\end{aligned}
$$

for some constant such that

$$
\begin{aligned}
& \overline{C}_2 \\
\geq\; & C\,[(d+K)\,p]^3 \\
& \times\left\{\frac{1}{1-\phi_{\max}^6}+20\left(\frac{1}{1-\phi_{\max}^3}\right)^2+90\left(\frac{1}{1-\phi_{\max}^2}\right)^3+15\left(\frac{1}{1-\phi_{\max}^4}\right)\left(\frac{1}{1-\phi_{\max}^2}\right)\right\}.
\end{aligned}
$$

Putting everything together, we see that

$$\begin{aligned}E\left\|\underline{W}_t\right\|_2^6 &\leq 32\left\|\left(I_{(d+K)p}-A\right)^{-1}J'_{d+K}\mu\right\|_2^6+32E\left\|\sum_{j=0}^{\infty}A^jJ'_{d+K}\varepsilon_{t-j}\right\|_2^6 \\ &\leq 32\left(\overline{C}_1+\overline{C}_2\right) \\ &\leq \overline{C}<\infty\end{aligned}$$

for a constant $\overline{C}$ such that $0<32\left(\overline{C}_1+\overline{C}_2\right)\leq\overline{C}<\infty$.

In addition, define$\mathcal{P}_{(d+K)p}$ to be the $(d+K)\,p\times(d+K)\,p$ permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t=\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}; \tag{60}$$

and let $S'_d=\begin{pmatrix}I_{dp} & \underset{dp\times Kp}{0}\end{pmatrix}$ and $S'_K=\begin{pmatrix}\underset{Kp\times dp}{0} & I_{Kp}\end{pmatrix}$. Note that

$$\begin{aligned}S'_d\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix}I_{dp} & \underset{dp\times Kp}{0}\end{pmatrix}\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}=\underline{Y}_t, \\ S'_K\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix}\underset{Kp\times dp}{0} & I_{Kp}\end{pmatrix}\begin{pmatrix}\underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t}\end{pmatrix}=\underline{F}_t.\end{aligned}$$

so that

$$\begin{aligned}\left\|\underline{Y}_t\right\|_2 &\leq \left\|S'_d\right\|_2\left\|\mathcal{P}_{(d+K)p}\right\|_2\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S_dS'_d\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}'_{(d+K)p}\mathcal{P}_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(S'_dS_d\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \sqrt{\lambda_{\max}\left(I_{dp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)}\left\|\underline{W}_t\right\|_2 \\ &= \left\|\underline{W}_t\right\|_2\end{aligned}$$

and

$$\begin{aligned}
\left\|\underline{F}_t\right\|_2 &\leq \left\|S_K'\right\|_2 \left\|\mathcal{P}_{(d+K)p}\right\|_2 \left\|\underline{W}_t\right\|_2 \\
&= \sqrt{\lambda_{\max}\left(S_K S_K'\right)}\sqrt{\lambda_{\max}\left(\mathcal{P}_{(d+K)p}'\mathcal{P}_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\
&= \sqrt{\lambda_{\max}\left(S_K' S_K\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\
&= \sqrt{\lambda_{\max}\left(I_{Kp}\right)}\sqrt{\lambda_{\max}\left(I_{(d+K)p}\right)} \left\|\underline{W}_t\right\|_2 \\
&= \left\|\underline{W}_t\right\|_2
\end{aligned}$$

It further follows that

$$E\left\|\underline{Y}_t\right\|_2^6 \leq E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty \text{ and } E\left\|\underline{F}_t\right\|_2^6 \leq E\left\|\underline{W}_t\right\|_2^6 \leq \overline{C} < \infty. \ \square$$

Lemma C-5: Let $W_t = (Y_t', F_t')'$ be generated by the factor-augmented VAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}$$

described in section 3 of the main paper. Under Assumptions 3-1, 3-2(a)-(c), and 3-7; $\{W_t\}$ is a β-mixing process with β-mixing coefficient $\beta_W(m)$ such that

$$\beta_W(m) \leq C_1 \exp\{-C_2 m\}$$

for some positive constants C_1 and C_2. Here,

$$\beta_W(m) = \sup_t E\left[\sup\left\{\left|P\left(B|\mathcal{A}_{-\infty}^t\right) - P(B)\right| : B \in \mathcal{A}_{t+m}^{\infty}\right\}\right]$$

with $\mathcal{A}_{-\infty}^t = \sigma(..., W_{t-2}, W_{t-1}, W_t)$ and $\mathcal{A}_{t+m}^{\infty} = \sigma(W_{t+m}, W_{t+m+1}, W_{t+m+2}, ...)$.

Proof of Lemma C-5:

To prove this lemma, we shall verify the conditions of Lemma OA-8 of Chao, Qiu, and Swanson (2023b) for the vector moving-average representation of W_t, i.e.,

$$W_t = J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} J_{d+K} A^j J_{d+K}' \varepsilon_{t-j} = \mu_* + \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j},$$

where

$$
\begin{aligned}
\mu_* &= J_{d+K}\left(I_{(d+K)p} - A\right)^{-1} J'_{d+K}\mu,\ \Psi_j = J_{d+K}A^j J'_{d+K}, \\
\underset{(d+K)\times(d+K)p}{J_{d+K}} &= \left[\begin{array}{ccccc} I_{d+K} & 0 & \cdots & 0 & 0 \end{array}\right], \text{ and } A = \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}
\end{aligned}
$$

To proceed, set

$$
\xi_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j} \tag{61}
$$

and note first that, setting $\delta = 5$ in Lemma OA-8 of Chao, Qiu, and Swanson (2023b), and we see that Assumptions (i) and (ii) of this lemma are the same as the conditions specified in Assumption 3-2 (a)-(c). Next, note that, since in this case $\Psi_j = J_{d+K}A^j J'_{d+K}$, we have

$$
\begin{aligned}
\left\|\Psi_j\right\|_2 &\leq \left\|J_{d+K}\right\|_2 \left\|A^j\right\|_2 \left\|J'_{d+K}\right\|_2 \\
&\leq \sqrt{\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)}\left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right)\sqrt{\lambda_{\max}\left(J_{d+K}J'_{d+K}\right)} \\
&= \lambda_{\max}\left(J_{d+K}J'_{d+K}\right)\left(\sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}}\right) \\
&= \sqrt{\lambda_{\max}\left\{\left(A^j\right)' A^j\right\}} \\
&= \sigma_{\max}\left(A^j\right) \\
&\leq C\left[\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}\right] \quad \text{(by Assumption 3-7)} \\
&= C\left[\max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}\right]^j \\
&= C\phi_{\max}^j
\end{aligned}
$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since, by Assumption 3-1, all eigenvalues of A have modulus less than 1. It follows that

$$
\sum_{j=0}^{\infty} \left\|\Psi_j\right\|_2 \leq C\sum_{j=0}^{\infty} \phi_{\max}^j = \frac{C}{1-\phi_{\max}} < \infty.
$$

Moreover, by Assumption 3-1,

$$
\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \neq 0 \text{ for all } z \text{ such that } |z| \leq 1
$$

and, by definition,

$$\sum_{j=0}^{\infty} \Psi_j z^j = \Psi(z) = \left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)^{-1} \text{ for all } z \text{ such that } |z| \leq 1$$

so that

$$\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right) = I_{(d+K)p} \text{ for all } z \text{ such that } |z| \leq 1$$

In addition, since

$$\begin{aligned}
& \det\{\Psi(z)\} \det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\} \\
= \; & \det\left\{\Psi(z)\left(I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right)\right\} \\
= \; & \det\left\{I_{(d+K)p}\right\} \\
= \; & 1,
\end{aligned}$$

and since

$$\left|\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\}\right| < \infty \text{ for all } z \text{ such that } |z| \leq 1,$$

it follows that

$$\begin{aligned}
\det\left\{\sum_{j=0}^{\infty} \Psi_j z^j\right\} &= \det\{\Psi(z)\} \\
&= \frac{1}{\det\left\{I_{(d+K)p} - A_1 z - \cdots - A_p z^p\right\}} \\
&\neq 0 \text{ for all } z \text{ such that } |z| \leq 1.
\end{aligned}$$

Finally, note that, setting $\delta = 5$,

$$
\begin{aligned}
& \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{\delta}{1+\delta}} \\
= & \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
\leq & \sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
= & C^{\frac{5}{6}}\sum_{j=0}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
\leq & C^{\frac{5}{6}}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^{k} \\
& \left(\text{by the inequality } \left|\sum_{i=1}^{\infty}a_i\right|^{r} \leq \sum_{i=1}^{\infty}|a_i|^{r} \text{ for } r \leq 1\right) \\
= & C^{\frac{5}{6}}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^{j} \\
= & C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \quad \text{(by Lemma OA-10 of Chao, Qiu, and Swanson (2023b))} \\
< & \infty \quad \left(\text{since } 0 < \phi_{\max}^{\frac{5}{6}} < 1 \text{ given that } 0 < \phi_{\max} < 1\right).
\end{aligned}
$$

Hence, all conditions of Lemma OA-8 of Chao, Qiu, and Swanson (2023b) are fulfilled. Applying this lemma, we then obtain that there exists a constant $\overline{C}$ such that

$$\begin{aligned}
\beta_{\xi}(m) &\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\|\Psi_k\|_2\right)^{\frac{5}{6}} \\
&\leq \overline{C}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}C\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&= \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\left(\sum_{k=j}^{\infty}\phi_{\max}^{k}\right)^{\frac{5}{6}} \\
&\leq \overline{C}C^{\frac{5}{6}}\sum_{j=m}^{\infty}\sum_{k=j}^{\infty}\left(\phi_{\max}^{\frac{5}{6}}\right)^{k} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\sum_{j=0}^{\infty}(j+1)\left(\phi_{\max}^{\frac{5}{6}}\right)^{j} \\
&= \overline{C}C^{\frac{5}{6}}\left(\phi_{\max}^{\frac{5}{6}}\right)^{m}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \\
&= \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2}\exp\left\{-\left[\frac{5}{6}\left|\ln\phi_{\max}\right|\right]m\right\} \quad (\text{since } 0<\phi_{\max}<1) \\
&\leq C_1\exp\{-C_2 m\}\to 0 \text{ as } m\to\infty.
\end{aligned}$$

for some positive constants C_1 and C_2 such that

$$C_1 \geq \overline{C}C^{\frac{5}{6}}\left[1-\phi_{\max}^{\frac{5}{6}}\right]^{-2} \text{ and } C_2 \leq \frac{5}{6}\left|\ln\phi_{\max}\right|$$

It follows that the process $\{\xi_t\}$ (as defined in expression (61)) is β mixing with beta coefficient $\beta_{\xi}(m)$ satisfying

$$\beta_{\xi}(m) \leq C_1\exp\{-C_2 m\}.$$

Since

$$W_t = \mu_* + \sum_{j=0}^{\infty}\Psi_j\varepsilon_{t-j} = \mu_* + \xi_t$$

and since μ_* is a nonrandom parameter, we can then apply part (a) of Lemma C-2 to deduce that $\{W_t\}$ is a β mixing process with β coefficient $\beta_W(m)$ satisfying the inequality

$$\beta_W(m) \leq C_1\exp\{-C_2 m\}. \ \square$$

.

10 Appendix D: Key Supporting Lemmas Used in the Proofs of Theorems 4.1 and 4.2

Derivation of the h-step Ahead Forecasting Equation Given in Expression (22) of the Main Paper:

Consider the FAVAR process

$$W_{t+1} = \mu + A_1 W_t + \cdots + A_p W_{t-p+1} + \varepsilon_{t+1}, \tag{62}$$

where $W_t = (Y_t', F_t')'$. Suppose that equation (62) satisfies Assumptions 3-1 and 3-2 of the main paper. Then, similar to a VAR process, we can rewrite this model in the companion form

$$\underline{W}_t = \alpha + A\underline{W}_{t-1} + E_t$$

where

$$\begin{aligned}
\underline{W}_t &= \begin{pmatrix} W_t \\ W_{t-1} \\ \vdots \\ W_{t-p+2} \\ W_{t-p+1} \end{pmatrix}, \; W_t = \begin{pmatrix} Y_t \\ F_t \end{pmatrix}, \; E_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; \alpha = \begin{pmatrix} \mu \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \text{ and} \\
A &= \begin{pmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \\ I_{d+K} & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_{d+K} & 0 \end{pmatrix}.
\end{aligned} \tag{63}$$

Successive substitution for the lagged $\underline{W}_t$'s gives

$$\underline{W}_{t+h} = \sum_{j=0}^{h-1} A^j \alpha + A^h \underline{W}_t + \sum_{j=0}^{h-1} A^j E_{t+h-j}$$

Let

$$\underset{d\times(d+K)p}{J_d} = \left[\begin{array}{cccc} I_d & 0 & \cdots & 0 \end{array}\right] \text{ and } \underset{(d+K)\times(d+K)p}{J_{d+K}} = \left[\begin{array}{cccc} I_{d+K} & 0 & \cdots & 0 \end{array}\right]$$

and note that

$$J_d \underline{W}_{t+h} = Y_{t+h}, \;\; J_{d+K} E_{t+h-j} = \varepsilon_{t+h-j},$$

and

$$J'_{d+K} J_{d+K} E_{t+h-j} = \begin{pmatrix} I_{d+K} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{t+h-j} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Hence,

$$\begin{aligned} Y_{t+h} &= J_d \underline{W}_{t+h} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} J_{d+K} E_{t+h-j} \\ &= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \end{aligned} \tag{64}$$

Furthermore, let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p} \underline{W}_t = \begin{pmatrix} \underline{Y}_t \\ \underline{F}_t \end{pmatrix}, \text{ where } \underline{Y}_t = \begin{pmatrix} Y_t \\ \vdots \\ Y_{t-p+1} \end{pmatrix} \text{ and } \underline{F}_t = \begin{pmatrix} F_t \\ \vdots \\ F_{t-p+1} \end{pmatrix}. \tag{65}$$

and note that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix, so that $\mathcal{P}'_{(d+K)p}\mathcal{P}_{(d+K)p} = I_{(d+K)p} = \mathcal{P}_{(d+K)p}\mathcal{P}'_{(d+K)p}$. Next, for $g = 1, ..., p$, let $e_{g,p}$ be a $p \times 1$ elementary vector whose g^{th} component is 1 and all other components are 0; and define

$$
\begin{aligned}
\underset{(d+K)p\times d}{S_{d,g}} &= \begin{pmatrix} e_{g,p}\otimes I_d \\ \underset{Kp\times d}{0} \end{pmatrix}, \quad \underset{(d+K)p\times K}{S_{K,g}} = \begin{pmatrix} \underset{dp\times K}{0} \\ e_{g,p}\otimes I_K \end{pmatrix}, \\
\underset{(d+K)p\times dp}{S_d} &= \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} \\
&= \begin{pmatrix} e_{1,p}\otimes I_d & e_{2,p}\otimes I_d & \cdots & e_{p,p}\otimes I_d \\ \underset{Kp\times d}{0} & \underset{Kp\times d}{0} & \cdots & \underset{Kp\times d}{0} \end{pmatrix} \\
&= \begin{pmatrix} I_p\otimes I_d \\ \underset{Kp\times dp}{0} \end{pmatrix} = \begin{pmatrix} I_{dp} \\ \underset{Kp\times dp}{0} \end{pmatrix} \\
\underset{(d+K)p\times Kp}{S_K} &= \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} \\
&= \begin{pmatrix} \underset{dp\times K}{0} & \underset{dp\times K}{0} & \cdots & \underset{dp\times K}{0} \\ e_{1,p}\otimes I_K & e_{2,p}\otimes I_K & \cdots & e_{p,p}\otimes I_K \end{pmatrix} \\
&= \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_p\otimes I_K \end{pmatrix} = \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_{Kp} \end{pmatrix}
\end{aligned}
$$

It follows that

$$
\underset{(d+K)p\times(d+K)p}{S} = \begin{pmatrix} \underset{(d+K)p\times dp}{S_d} & \underset{(d+K)p\times Kp}{S_K} \end{pmatrix} = \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \\ \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix} = I_{(d+K)p} \tag{66}
$$

In addition, using these notations, it is easy to see that

$$
S'_{d,g}\mathcal{P}_{(d+K)p}\underline{W}_t = Y_{t-g+1} \text{ for } g = 1, ..., p \tag{67}
$$

and, similarly,

$$
S'_{K,g}\mathcal{P}_{(d+K)p}\underline{W}_t = F_{t-g+1} \text{ for } g = 1, ..., p. \tag{68}
$$

Hence, making use of expressions (64) and (66) and the fact that $\mathcal{P}_{(d+K)p}$ is an orthogonal matrix, we can write

$$
\begin{aligned}
Y_{t+h} &= J_d\underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + J_d A^h \mathcal{P}'_{(d+K)p} S S' \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} \left(S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g} \right) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}
\end{aligned}
$$

so that, in light of expressions (67) and (68), we further deduce that

$$
\begin{aligned}
Y_{t+h} &= J_d\underline{W}_{t+h} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} \left(S_{d,g} S'_{d,g} + S_{K,g} S'_{K,g} \right) \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} S'_{d,g} \mathcal{P}_{(d+K)p} \underline{W}_t + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} S'_{K,g} \mathcal{P}_{(d+K)p} \underline{W}_t \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \sum_{j=0}^{h-1} J_d A^j \alpha + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} Y_{t-g+1} + \sum_{g=1}^{p} J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} F_{t-g+1} \\
&\quad + \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j} \\
&= \beta_0 + \sum_{g=1}^{p} B'_{1,g} Y_{t-g+1} + \sum_{g=1}^{p} B'_{2,g} F_{t-g+1} + \eta_{t+h}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_0 &= \sum_{j=0}^{h-1} J_d A^j \alpha, \; \eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J'_{d+K} \varepsilon_{t+h-j}, \\
B'_{1,g} &= J_d A^h \mathcal{P}'_{(d+K)p} S_{d,g} \text{ and } B'_{2,g} = J_d A^h \mathcal{P}'_{(d+K)p} S_{K,g} \text{ for } g = 1, ..., p.
\end{aligned} \tag{69}
$$

Next, define $B'_1 = \begin{pmatrix} B'_{1,1} & B'_{1,2} & \cdots & B'_{1,p} \end{pmatrix}$ and $B'_2 = \begin{pmatrix} B'_{2,1} & B'_{2,2} & \cdots & B'_{2,p} \end{pmatrix}$, and note that, by expression (69) above,

$$\begin{aligned}
B_1' &= J_d A^h \mathcal{P}_{(d+K)p}' \begin{pmatrix} S_{d,1} & S_{d,2} & \cdots & S_{d,p} \end{pmatrix} = J_d A^h \mathcal{P}_{(d+K)p}' S_d \\
B_2' &= J_d A^h \mathcal{P}_{(d+K)p}' \begin{pmatrix} S_{K,1} & S_{K,2} & \cdots & S_{K,p} \end{pmatrix} = J_d A^h \mathcal{P}_{(d+K)p}' S_K.
\end{aligned}$$

Finally, let $\underline{Y}_t$ and $\underline{F}_t$ be as defined in expression (65), and we can write the h-step ahead forecast equation more succinctly as

$$\begin{aligned}
Y_{t+h} &= \beta_0 + \sum_{g=1}^{p} B_{1,g}' Y_{t-g+1} + \sum_{g=1}^{p} B_{2,g}' F_{t-g+1} + \eta_{t+h} \\
&= \beta_0 + B_1' \underline{Y}_t + B_2' \underline{F}_t + \eta_{t+h}. \ \square
\end{aligned}$$

Lemma D-1: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7 hold. Then, the following statements are true.

(a) There exists a positive constant $\underline{c}$ such that

$$\lambda_{\min} \left\{ \frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' \right\} \geq \underline{c} > 0,$$

where A is the coefficient matrix of the companion form given in expression (63) and where

$$\underset{(d+K)\times(d+K)p}{J_{d+K}} = \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 \end{bmatrix}. \tag{70}$$

(b) The matrix

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t \underline{Y}_t'\right] & E\left[\underline{Y}_t \underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t \underline{Y}_t'\right] & E\left[\underline{F}_t \underline{F}_t'\right] \end{pmatrix}$$

is non-singular for all $T > h + p - 1$.

Proof of Lemma D-1:

For part (a), we prove by contradiction. To proceed, let

$$J_{d+K,r} = e_{r,p}' \otimes I_{d+K} \text{ for } r \in \{1, ..., p\}$$

where $e_{r,p}$ is a $p \times 1$ elementary vector whose r^{th} component is equal to 1 and all other components are equal to 0. Note that, under this definition, $J_{d+K,1} = J_{d+K}$, where J_{d+K} is as defined previously in expression (70). Suppose that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)'$$

is singular; then, there exists $b \in \mathbb{R}^{(d+K)p} \setminus \{0\}$ such that

$$\sum_{j=0}^{\infty} b' A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)' b = 0$$

This, in turn, implies that $J_{d+K,1} \left(A^j\right)' b = 0$ for all j. Now, partition

$$b = \begin{pmatrix} \underset{(d+K)\times 1}{b_1} \\ \underset{(d+K)\times 1}{b_2} \\ \vdots \\ \underset{(d+K)\times 1}{b_p} \end{pmatrix}$$

Note that, for $j = 0$, let $L_0 = I_{d+K}$, and it is easily seen that

$$\begin{aligned} 0 &= J_{d+K,1} \left(A^0\right)' b \\ &= J_{d+K,1} b \\ &= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\ &= b_1 \ \ (= L_0 b_1) \end{aligned}$$

Now, for $j=1$, define $\overline{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_{p-1} & A_p \end{bmatrix}$, and note that

$$\begin{aligned}
0 &= J_{d+K,1}A'b \\
&= \begin{bmatrix} I_{d+K} & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{pmatrix} A_1' & I_{d+K} & 0 & \cdots & 0 \\ A_2' & 0 & I_{d+K} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{p-1}' & \vdots & 0 & \ddots & I_{d+K} \\ A_p' & 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} \\
&= J_{d+K,1} \begin{bmatrix} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' \end{bmatrix} b \\
&= \left[J_{d+K,1}\overline{A}' J_{d+K,1} + J_{d+K,2} \right] b \\
&= \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] b \\
&= L_1 b_1 + L_0 b_2
\end{aligned}$$

where $L_1 = J_{d+K,1}\overline{A}' = A_1'$. Since previously we have shown that $b_1 = 0$, it follows that

$$b_2 = L_1 b_1 + L_0 b_2 = 0.$$

Moreover, for $j=2$, using the fact that $J_{d+K,r}J_{d+K,r}' = I_{d+K}$ and $J_{d+K,r}J_{d+K,s}' = 0$ for $r \neq s$, we obtain

$$\begin{aligned}
0 &= J_{d+K,1}\left(A'\right)^2 b \\
&= J_{d+K,1} \begin{bmatrix} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' & J_{d+K,p}' \end{bmatrix}^2 b \\
&= \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \begin{bmatrix} \overline{A}' & J_{d+K,1}' & J_{d+K,2}' & \cdots & J_{d+K,p-1}' & J_{d+K,p}' \end{bmatrix} b \\
&= \left(\left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \overline{A}' J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3} \right) b \\
&= \left(L_2 J_{d+K,1} + L_1 J_{d+K,2} + L_0 J_{d+K,3} \right) b \\
&= L_2 b_1 + L_1 b_2 + L_0 b_3
\end{aligned}$$

where

$$L_2 = \left[L_1 J_{d+K,1} + L_0 J_{d+K,2} \right] \overline{A}'$$

Given that $b_1 = 0$ and $b_2 = 0$, as we have previously shown, it then follows that

$$b_3 = L_2 b_1 + L_1 b_2 + L_0 b_3 = 0 \ \ (\text{since } L_0 = I_{d+K})$$

We will show by mathematical induction that, in fact, $b_r = 0$ for every $r \in \{1, ..., p\}$. To proceed, suppose that $b_1 = b_2 = \cdots = b_j = 0$ and $0 = J_{d+K,1} (A')^j b$. By straightforward calculations, one can show (in a manner similar to the case where $j = 0$, 1, or 2 given earlier) that $J_{d+K,1} (A')^j b$ has the representation

$$J_{d+K,1} \left(A'\right)^j b = L_j b_1 + L_{j-1} b_2 + \cdots L_1 b_j + L_0 b_{j+1}$$

for coefficients $L_j, L_{j-1}, ..., L_1$, and L_0 where $L_0 = I_{d+K}$. It follows from the induction hypotheses that

$$\begin{aligned} b_{j+1} &= L_j b_1 + L_{j-1} b_2 + \cdots L_1 b_j + L_0 b_{j+1} \\ &= J_{d+K,1} \left(A'\right)^j b \\ &= 0. \end{aligned}$$

Hence, by mathematical induction, we conclude that $b_r = 0$ for every $r \in \{1, ..., p\}$, but this implies that

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{p-1} \\ b_p \end{pmatrix} = \underset{(d+K)p \times 1}{0}$$

which contradicts our initial assumption that $b \neq 0$. It then follows that the matrix

$$\sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)'$$

is positive definite and, thus, also non-singular, so that there exists a positive constant C_* such that

$$\lambda_{\min} \left\{ \sum_{j=0}^{\infty} A^j J'_{d+K,1} J_{d+K,1} \left(A^j\right)' \right\} \geq C_* > 0$$

Moreover, in light of Assumption 3-2(d), this further implies that

$$\begin{aligned}
& \lambda_{\min}\left\{\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
= & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K,1}' \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K,1}\left(A^j\right)'\right\} \text{ (since } J_{d+K,1} = J_{d+K}) \\
\geq & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K,1}' J_{d+K,1}\left(A^j\right)'\right\}\lambda_{\min}\left\{\frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K,1}' J_{d+K,1}\left(A^j\right)'\right\}\inf_t \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq & C_* \underline{C} \\
\geq & \underline{c} > 0 \quad \left(\text{by choosing } \underline{c} \leq C_* \underline{C}\right).
\end{aligned}$$

where the second inequality above follows from the fact that

$$\begin{aligned}
\lambda_{\min}\left\{\sum_{t=p}^{T-h}\frac{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]}{T_h}\right\} & \geq \sum_{t=p}^{T-h}\lambda_{\min}\left\{\frac{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]}{T_h}\right\} \\
& = \frac{1}{T_h}\sum_{t=p}^{T-h}\lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
& \geq \inf_t \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\}.
\end{aligned}$$

Now, to show part (b), note first that expression (58) in the proof of Lemma C-4 in Appendix C above gives a vector moving-average representation for $\underline{W}_t$ of the form

$$\underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} A^j J_{d+K}'\varepsilon_{t-j},$$

where $J_{d+K} = J_{d+K,1} = \left[\begin{array}{ccccc} I_{d+K} & 0 & \cdots & 0 & 0 \end{array}\right]$. Now, let

$$\underset{(d+K)p\times dp}{S_d} = \begin{pmatrix} I_{dp} \\ \underset{Kp\times dp}{0} \end{pmatrix} \text{ and } \underset{(d+K)p\times Kp}{S_K} = \begin{pmatrix} \underset{dp\times Kp}{0} \\ I_{Kp} \end{pmatrix},$$

and let $\mathcal{P}_{(d+K)p}$ be a permutation matrix such that

$$\mathcal{P}_{(d+K)p}\underline{W}_t = \left(\begin{array}{c} \underline{Y}_t \\ \underline{F}_t \end{array} \right).$$

It follows that

$$\begin{aligned}
\underline{Y}_t &= S_d'\mathcal{P}_{(d+K)p}\underline{W}_t \\
&= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} S_d'\mathcal{P}_{(d+K)p}A^j J_{d+K}'\varepsilon_{t-j}
\end{aligned}$$

and

$$\begin{aligned}
\underline{F}_t &= S_K'\mathcal{P}_{(d+K)p}\underline{W}_t \\
&= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} S_K'\mathcal{P}_{(d+K)p}A^j J_{d+K}'\varepsilon_{t-j}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& E\left[\underline{Y}_t\underline{Y}_t'\right] \\
= \; & E\Bigg\{\left(S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu + \sum_{j=0}^{\infty} S_d'\mathcal{P}_{(d+K)p}A^j J_{d+K}'\varepsilon_{t-j}\right) \\
& \times \left(\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_d + \sum_{k=0}^{\infty}\varepsilon_{t-k}' J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_d\right)\Bigg\} \\
= \; & S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_d \\
& + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} S_d'\mathcal{P}_{(d+K)p}A^j J_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-k}'\right] J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_d \\
= \; & S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_d \\
& + \sum_{j=0}^{\infty} S_d'\mathcal{P}_{(d+K)p}A^j J_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_d,
\end{aligned}$$

$$
\begin{aligned}
& E\left[\underline{F}_t\underline{F}_t'\right] \\
= \ & E\left\{\left(S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu+\sum_{j=0}^{\infty}S_K'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'\varepsilon_{t-j}\right)\right. \\
& \left.\times\left(\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K+\sum_{k=0}^{\infty}\varepsilon_{t-k}'J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K\right)\right\} \\
= \ & S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \\
& +\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_K'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-k}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K \\
= \ & S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \\
& +\sum_{j=0}^{\infty}S_K'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K,
\end{aligned}
$$

and

$$
\begin{aligned}
& E\left[\underline{Y}_t\underline{F}_t'\right] \\
= \ & E\left\{\left(S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu+\sum_{j=0}^{\infty}S_d'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'\varepsilon_{t-j}\right)\right. \\
& \left.\times\left(\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K+\sum_{k=0}^{\infty}\varepsilon_{t-k}'J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K\right)\right\} \\
= \ & S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \\
& +\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}S_d'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-k}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K \\
= \ & S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \\
& +\sum_{j=0}^{\infty}S_d'\mathcal{P}_{(d+K)p}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'S_K,
\end{aligned}
$$

In addition, since

$$
\begin{aligned}
E\left[\underline{W}_t\right] &= \left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu \text{ and} \\
E\left[\underline{W}_t\underline{W}_t'\right] &= \left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu\mu'J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1} \\
&\quad +\sum_{j=0}^{\infty}A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'
\end{aligned}
$$

and since

$$\left[\begin{array}{cc} S_d & S_K \end{array}\right] = \left(\begin{array}{cc} I_{dp} & \underset{dp\times Kp}{0} \\ \underset{Kp\times dp}{0} & I_{Kp} \end{array}\right) = I_{(d+K)p}$$

it is easy to see that

$$\begin{aligned}
& \left(\begin{array}{cc} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{array}\right) \\
= & \left(\begin{array}{c} S_d' \\ S_K' \end{array}\right) \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
& + \left(\begin{array}{c} S_d' \\ S_K' \end{array}\right) \sum_{j=0}^{\infty} \mathcal{P}_{(d+K)p}A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
= & \mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}' \\
& + \sum_{j=0}^{\infty} \mathcal{P}_{(d+K)p}A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\mathcal{P}_{(d+K)p}' \\
= & \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'
\end{aligned}$$

and

$$\begin{aligned}
& \left(\begin{array}{cc} E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \end{array}\right) \\
= & \left(\begin{array}{cc} \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_d & \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'S_K \end{array}\right) \\
= & \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}'\left(\begin{array}{cc} S_d & S_K \end{array}\right) \\
= & \mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\mathcal{P}_{(d+K)p}' \\
= & E\left[\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'
\end{aligned}$$

Making use of these expressions, we can then write

$$\begin{aligned}
\left(\begin{array}{ccc} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{array}\right) &= \left(\begin{array}{cc} 1 & E\left[\underline{W}_t'\right]\mathcal{P}_{(d+K)p}' \\ \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\right] & \mathcal{P}_{(d+K)p}E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}' \end{array}\right) \\
&= \left(\begin{array}{cc} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{array}\right)\left(\begin{array}{cc} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{array}\right)\left(\begin{array}{cc} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p}' \end{array}\right).
\end{aligned}$$

Next, note that

$$
\begin{aligned}
& \det \begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t \underline{W}_t'\right] \end{pmatrix} \\
= \;& \det(1) \det\left\{E\left[\underline{W}_t \underline{W}_t'\right] - E\left[\underline{W}_t\right] E\left[\underline{W}_t'\right]\right\} \\
= \;& \det\left\{E\left[\underline{W}_t \underline{W}_t'\right] - E\left[\underline{W}_t\right] E\left[\underline{W}_t'\right]\right\} \\
= \;& \det\Big\{\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu\mu' J_{d+K} \left(I_{(d+K)p} - A'\right)^{-1} \\
& + \sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' \\
& - \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu\mu' J_{d+K} \left(I_{(d+K)p} - A'\right)^{-1}\Big\} \\
= \;& \det\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)'\right\}
\end{aligned}
$$

Now, by Assumption 3-2(d) and by the same argument as that used to prove part (a) above, we see that there exists a constant $\underline{c}$ such that

$$
\begin{aligned}
& \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)'\right\} \\
\geq \;& \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' J_{d+K} \left(A^j\right)'\right\} \inf_j \lambda_{\min}\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\} \\
\geq \;& \underline{c} > 0
\end{aligned}
$$

for all t, which, in turn, implies that in this case

$$
\begin{aligned}
\det \begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t \underline{W}_t'\right] \end{pmatrix} &= \det\left\{\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)'\right\} \\
&\geq \underline{c}^{(d+K)p} > 0
\end{aligned}
$$

for all t. Furthermore, since the matrix

$$
\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix}
$$

is nonsingular, it follows that the matrix

$$\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h}\begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} \\
= &\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p} \end{pmatrix}\frac{1}{T_h}\sum_{t=p}^{T-h}\begin{pmatrix} 1 & E\left[\underline{W}_t'\right] \\ E\left[\underline{W}_t\right] & E\left[\underline{W}_t\underline{W}_t'\right] \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_{(d+K)p}' \end{pmatrix}
\end{aligned}$$

will be nonsingular and, thus, positive definite as required. □

Lemma D-2: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-5, and 3-7 hold. Then, the following statements are true.

(a)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

where

$$\underline{W}_t = \begin{pmatrix} W_t \\ \vdots \\ W_{t-p+1} \end{pmatrix} \text{ and } W_t = \begin{bmatrix} Y_t \\ F_t \end{bmatrix}.$$

(b)

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right) \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right), \text{ and} \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{F}_t'\right] &= O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

where $\underline{Y}_t$ and $\underline{F}_t$ are as defined in expression (65).

(c)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

(d)

$$\begin{aligned}\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t &= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu+O_p\left(\frac{1}{\sqrt{T}}\right),\\ \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t &= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu+O_p\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

(e)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\eta_{t+h}'=O_p\left(\frac{1}{\sqrt{T}}\right),\text{ where }\eta_{t+h}=\sum_{j=0}^{h-1}J_dA^jJ_{d+K}'\varepsilon_{t+h-j}$$

with $\underset{d\times(d+K)p}{J_d}=\left[\begin{array}{cccc} I_d & 0 & \cdots & 0\end{array}\right]$ and $\underset{(d+K)\times(d+K)p}{J_{d+K}}=\left[\begin{array}{cccc} I_{d+K} & 0 & \cdots & 0\end{array}\right]$.

(f)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\eta_{t+h}'=O_p\left(\frac{1}{\sqrt{T}}\right)\text{ and }\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\eta_{t+h}'=O_p\left(\frac{1}{\sqrt{T}}\right),$$

where η_{t+h} is as defined in part (e) above.

(g)

$$\frac{\mathfrak{H}'\iota_{T_h}}{T_h}=\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}=O_p\left(\frac{1}{\sqrt{T}}\right)=o_p\left(1\right).$$

(h)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}'-\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right]=O_p\left(\frac{1}{\sqrt{T}}\right),$$

where η_{t+h} is as defined in part (e) above.

Proof of Lemma D-2:

To show part (a), we note that for $a, b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we can write

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\right]^2 \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] \\
& +\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\}
\end{aligned}
$$

Note first that

$$
\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] &= \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\underline{W}_t\underline{W}_t'b\right)^2 - \frac{1}{T_h^2}\sum_{t=p}^{T-h}\left(E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2 \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'a\right)\left(b'\underline{W}_t\underline{W}_t'b\right)\right] \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left(a'\underline{W}_t\underline{W}_t'a\right)^2}\sqrt{E\left(b'\underline{W}_t\underline{W}_t'b\right)^2} \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left\|\underline{W}_t\right\|_2^4 \\
&\leq \frac{C}{T_h} = O\left(\frac{1}{T}\right)
\end{aligned}
$$

where the fourth inequality above follows from applying Liapunov's inequality and the result given in Lemma C-4.

Next, note that, by Lemma C-5, $\{W_t\}$ is β-mixing with β mixing coefficient satisfying $\beta_W(m) \leq C_1 \exp\{-C_2 m\}$. Since $\alpha_{W,m} \leq \beta_W(m)$, it follows that W_t is α-mixing as well, with α mixing coefficient satisfying $\alpha_{W,m} \leq C_1 \exp\{-C_2 m\}$. Moreover, by applying part (b) of Lemma C-2, we further deduce that $X_t = a'\underline{W}_t\underline{W}_t'b$ is also α-mixing with α mixing coefficient satisfying

$$
\begin{aligned}
\alpha_{X,m} &\leq C_1 \exp\{-C_2(m-p+1)\} \\
&\leq C_1^* \exp\{-C_2 m\}
\end{aligned}
$$

for some positive constant $C_1^* \geq C_1 \exp\{C_2(p-1)\}$. Hence, we can apply Lemma C-3 with $p = 2$

and $r = 3$ to obtain

$$\begin{aligned}
& \left|E\left\{\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\}\right| \\
\leq\ & 2\left(2^{\frac{1}{2}}+1\right)\alpha_{X,m}^{\frac{1}{6}}\sqrt{E\left(a'\underline{W}_t\underline{W}_t'b\right)^2}\left(E\left|a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right|^3\right)^{1/3}
\end{aligned}$$

where $\alpha_{X,m}$ denotes the α mixing coefficient for the process $X_t = a'\underline{W}_t\underline{W}_t'b$ and where, by our previous calculations,

$$\alpha_{X,m}^{\frac{1}{6}} \leq (C_1^*)^{\frac{1}{6}} \exp\left\{-\frac{C_2 m}{6}\right\} \text{ for all } m \text{ sufficiently large.}$$

It further follows that there exists a positive constant C_3 such that

$$\begin{aligned}
\sum_{m=1}^{\infty} \alpha_{X,m}^{\frac{1}{6}} &\leq (C_1^*)^{\frac{1}{6}} \sum_{m=1}^{\infty} \exp\left\{-\frac{C_2 m}{6}\right\} \\
&\leq (C_1^*)^{\frac{1}{6}} \sum_{m=0}^{\infty} \exp\left\{-\frac{C_2 m}{6}\right\} \\
&\leq (C_1^*)^{\frac{1}{6}} \left[1 - \exp\left\{-\frac{C_2}{6}\right\}\right]^{-1} \\
&\leq C_3
\end{aligned}$$

where the last inequality stems from the fact that $\sum_{m=0}^{\infty} \exp\left\{-\left(C_2 m/6\right)\right\}$ is a convergent geometric series given that $0 < \exp\{-(C_2/6)\} < 1$ for $C_2 > 0$. Hence,

$$\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
\leq\ & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
\leq\ & \frac{4}{T_h^2}\left(2^{\frac{1}{2}}+1\right) \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \alpha_{X,m}^{\frac{1}{6}} \sqrt{E\left(a'\underline{W}_t\underline{W}_t'b\right)^2}\left(E\left|a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right|^3\right)^{1/3} \\
\leq\ & 4\left(\sqrt{2}+1\right)\frac{1}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left\{ \alpha_{X,m}^{\frac{1}{6}} \left[E\left(a'\underline{W}_t\right)^4\right]^{1/4}\left[E\left(b'\underline{W}_t\right)^4\right]^{1/4}\left[E\left(a'\underline{W}_{t+m}\right)^6\right]^{\frac{1}{6}} \right. \\
& \qquad\qquad\qquad\qquad \left. \times \left[E\left(b'\underline{W}_{t+m}\right)^6\right]^{\frac{1}{6}} \right\} \\
\leq\ & 4\left(\sqrt{2}+1\right)\left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}}\left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} \frac{1}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\infty}\alpha_{X,m}^{\frac{1}{6}} \\
\leq\ & 4\left(\sqrt{2}+1\right)\left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}}\left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} \frac{1}{T_h^2}\sum_{t=p}^{T-h} C_3 \\
\leq\ & \frac{\overline{C}}{T_h} = O\left(\frac{1}{T}\right) \quad \left(\text{where } \overline{C} \geq 4\left(\sqrt{2}+1\right)\left(\sup_t E\left[\|\underline{W}_t\|_2^4\right]\right)^{\frac{1}{2}}\left(\sup_t E\left[\|\underline{W}_t\|_2^6\right]\right)^{\frac{1}{3}} C_3\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\right]^2 \\
\leq\ & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)^2\right] \\
& + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left\{ \left(a'\underline{W}_t\underline{W}_t'b - E\left[a'\underline{W}_t\underline{W}_t'b\right]\right)\left(a'\underline{W}_{t+m}\underline{W}_{t+m}'b - E\left[a'\underline{W}_{t+m}\underline{W}_{t+m}'b\right]\right)\right\} \right| \\
=\ & O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t\underline{W}_t'b - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[a'\underline{W}_t\underline{W}_t'b\right] = O_p\left(\frac{1}{\sqrt{T}}\right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

To show part (b), note first that

$$\begin{aligned}
S_d'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S_K'\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

By the result given in part (a) above, it follows from applying Slutsky's theorem that

$$\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right] \\
=\ & S_d'\mathcal{P}_{(d+K)p}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right]\right)\mathcal{P}_{(d+K)p}S_d \\
=\ & O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right] \\
=\ & S_K'\mathcal{P}_{(d+K)p}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right]\right)\mathcal{P}_{(d+K)p}S_K \\
=\ & O_p\left(\frac{1}{\sqrt{T}}\right),
\end{aligned}$$

and

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{F}_t'\right] \\
= \;& S_d'\mathcal{P}_{(d+K)p}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t\underline{W}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{W}_t\underline{W}_t'\right]\right)\mathcal{P}_{(d+K)p}S_K \\
= \;& O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}
$$

To show part (c), let $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, and write

$$
\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}a'\underline{W}_t &= \frac{1}{T_h}\sum_{t=p}^{T-h}\left\{a'\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu + \sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}\right\} \\
&= a'\left(I_{(d+K)p} - A\right)^{-1}J_{d+K}'\mu + \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'\varepsilon_{t-j}\right]^2 &= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{s-k}'\right]J_{d+K}\left(A^k\right)'a \\
&= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'a \\
&\quad + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^{m+j}\right)'a \\
&= \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'a \\
&\quad + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty}a'A^jJ_{d+K}'E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'a
\end{aligned}
$$

Now,

$$\begin{aligned}
& \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a'A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' a \\
\leq\ & \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} E\left\|\varepsilon_{t-j}\right\|_2^2 a'A^j J_{d+K}' J_{d+K}\left(A^j\right)' a \\
\leq\ & \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \left(E\left\|\varepsilon_{t-j}\right\|_2^6\right)^{\frac{1}{3}} a'A^j \left(A^j\right)' a \\
& \left(\text{by Liapunov's inequality and } \lambda_{\max}\left(J_{d+K}' J_{d+K}\right) = 1\right) \\
\leq\ & \overline{C}^{\frac{1}{3}}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a'A^j \left(A^j\right)' a \\
& \left(\text{where } \overline{C} \geq 1 \text{ is a constant such that } E\left\|\varepsilon_{t-j}\right\|_2^6 \leq \overline{C} < \infty \text{ by Assumption 3-2(b)}\right) \\
\leq\ & \overline{C}^{\frac{1}{3}}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \lambda_{\max}\left\{A^j\left(A^j\right)'\right\} a'a \\
=\ & \overline{C}^{\frac{1}{3}}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \\
& \left(\text{since } \lambda_{\max}\left\{A^j\left(A^j\right)'\right\} = \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \text{ and } a'a = 1\right) \\
=\ & \overline{C}^{\frac{1}{3}}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \sigma_{\max}^2\left(A^j\right) \\
\leq\ & \overline{C}^{\frac{1}{3}}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} C\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \quad \text{(by Assumption 3-7)} \\
=\ & \overline{C}^{\frac{1}{3}}C\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2j}, \left|\lambda_{\min}\left(A\right)\right|^{2j}\right\} \\
=\ & \overline{C}^{\frac{1}{3}}C\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} \phi_{\max}^{2j}
\end{aligned}$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}\left(A\right)\right|, \left|\lambda_{\min}\left(A\right)\right|\right\}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a'A^j J'_{d+K} E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right] J_{d+K}\left(A^j\right)' a &\leq \overline{C}^{\frac{1}{3}} C \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}\phi_{\max}^{2j} \\
&= \overline{C}^{\frac{1}{3}} C \frac{T-h-p+1}{T_h^2}\frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}} C \frac{1}{T_h}\frac{1}{1-\phi_{\max}^2} \\
&\quad (\text{since } T_h = T-h-p+1) \\
&= O\left(\frac{1}{T}\right)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty} a'A^j J'_{d+K} E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' a\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\left|\sum_{j=0}^{\infty} a'A^j J'_{d+K} E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' a\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\left\{\sqrt{\sum_{j=0}^{\infty} a'A^j J'_{d+K} E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right] E\left[\varepsilon_{t-j}\varepsilon'_{t-j}\right] J_{d+K}\left(A^j\right)' a}\right. \\
&\left.\times\sqrt{\sum_{j=0}^{\infty}\sum_{m_1=1}^{T-h-t} a'A^{m_1}A^j J'_{d+K} J_{d+K}\left(A^j\right)' \sum_{m_2=1}^{T-h-t}\left(A^{m_2}\right)' a}\right\}
\end{aligned}$$

Observe that

$$
\begin{aligned}
& \sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' a \\
\leq\ & \sum_{j=0}^{\infty} \lambda_{\max}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) a' A^j J_{d+K}' J_{d+K}\left(A^j\right)' a \\
\leq\ & \sum_{j=0}^{\infty} \lambda_{\max}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) C\phi_{\max}^{2j} \\
=\ & C\sum_{j=0}^{\infty} \lambda_{\max}^{2}\left(E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right) \phi_{\max}^{2j} \\
\leq\ & C\sum_{j=0}^{\infty} \left(tr\left\{E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right]\right\}\right)^2 \phi_{\max}^{2j} \\
=\ & C\sum_{j=0}^{\infty} \left(E\left\|\varepsilon_{t-j}\right\|_2^2\right)^2 \phi_{\max}^{2j} \\
\leq\ & C\sum_{j=0}^{\infty} \left(E\left\|\varepsilon_{t-j}\right\|_2^6\right)^{\frac{2}{3}} \phi_{\max}^{2j} \quad \text{(by Liapunov's inequality)} \\
\leq\ & \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J_{d+K}' J_{d+K} \left(A^j\right)' \sum_{m_2=1}^{T-h-t} \left(A^{m_2}\right)' a \\
\leq \quad & \sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} a' A^{m_1} A^j \left(A^j\right)' \left(A^{m_2}\right)' a \\
\leq \quad & C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \left| a' A^{m_1} \left(A^{m_2}\right)' a \right| \\
\leq \quad & C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{a' A^{m_1} \left(A^{m_1}\right)' a} \sqrt{a' A^{m_2} \left(A^{m_2}\right)' a} \\
\leq \quad & C \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{C \phi_{\max}^{2m_1}} \sqrt{C \phi_{\max}^{2m_2}} \\
\leq \quad & C^2 \sum_{j=0}^{\infty} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \phi_{\max}^{m_1} \sum_{m_2=1}^{T-h-t} \phi_{\max}^{m_2} \\
\leq \quad & C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' \sum_{m=1}^{T-h-t} \left(A^m\right)' a \right| \\
\leq \quad & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \left\{ \sqrt{ \sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] E\left[\varepsilon_{t-j} \varepsilon_{t-j}'\right] J_{d+K} \left(A^j\right)' a } \right. \\
& \qquad \left. \times \sqrt{ \sum_{j=0}^{\infty} \sum_{m_1=1}^{T-h-t} a' A^{m_1} A^j J_{d+K}' J_{d+K} \left(A^j\right)' \sum_{m_2=1}^{T-h-t} \left(A^{m_2}\right)' a } \right\} \\
\leq \quad & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sqrt{ \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2} } \sqrt{ C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2 } \\
\leq \quad & 2 \overline{C}^{\frac{1}{3}} C^{\frac{3}{2}} \frac{1}{T_h} \left(\frac{1}{1-\phi_{\max}^2} \right) \left(\frac{1}{1-\phi_{\max}} \right) \\
= \quad & O\left(\frac{1}{T} \right)
\end{aligned}
$$

Putting these results together, we obtain

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j}\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' a \\
& + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\infty} a' A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' a \\
= & O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, upon applying Markov's inequality, we get

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

from which we further deduce, upon applying Slutsky's theorem, that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h} a' \underline{W}_t &= a'\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu + \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty} a' A^j J_{d+K}' \varepsilon_{t-j} \\
&= a'\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu + O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

Since the above result holds for all $a \in \mathbb{R}^{(d+K)p}$ such that $\|a\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t = \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}' \mu + O_p\left(\frac{1}{\sqrt{T}}\right).$$

To show part (d), note again that

$$\begin{aligned}
S_d' \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix} \begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S_K' \mathcal{P}_{(d+K)p} \underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix} \begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

By the result given in part (c) above, it follows by Slutsky's theorem that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t &= S_d'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t \\
&= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + S_d'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}A^jJ_{d+K}'\varepsilon_{t-j} \\
&= S_d'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t &= S_K'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{W}_t \\
&= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + S_K'\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{\infty}A^jJ_{d+K}'\varepsilon_{t-j} \\
&= S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p}-A\right)^{-1}J_{d+K}'\mu + O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Turning our attention to part (e), let $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; and, by direct calculation, we obtain

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}a'\underline{W}_t\eta_{t+h}'b\right]^2 \\
= \;& \frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[\left(a'\underline{W}_t\right)^2\left(\eta_{t+h}'b\right)^2\right] + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left\{\left(a'\underline{W}_t\eta_{t+h}'b\right)\left(a'\underline{W}_{t+m}\eta_{t+m+h}'b\right)\right\}
\end{aligned}$$

Let $\sigma_{\max}\left(A^j\right)$ denotes the max singular value of A^j and let $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$, and note first that

$$\begin{aligned}
E\left(b'\eta_{t+h}\right)^4 &= E\left(\sum_{j=0}^{h-1} b' J_d A^j J_{d+K}' \varepsilon_{t+h-j}\right)^4 \\
&\le h^3 \sum_{j=0}^{h-1} E\left[\left(b' J_d A^j J_{d+K}' \varepsilon_{t+h-j}\right)^4\right] \quad \text{(by Loève's } c_r \text{ inequality)} \\
&\le h^3 \sum_{j=0}^{h-1} E\left[\left(b' J_d A^j J_{d+K}' J_{d+K} \left(A'\right)^j J_d' b\right)^2 \left(\varepsilon_{t+h-j}' \varepsilon_{t+h-j}\right)^2\right] \\
&= h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j J_{d+K}' J_{d+K} \left(A^j\right)' J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\le h^3 \sum_{j=0}^{h-1} \left(b' J_d A^j \left(A^j\right)' J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\le h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4\left(A^j\right) \left(b' J_d J_d' b\right)^2 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&= h^3 \sum_{j=0}^{h-1} \sigma_{\max}^4\left(A^j\right) E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\le h^3 \sum_{j=0}^{h-1} \overline{C}\left[\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|, \left|\lambda_{\min}\left(A^j\right)\right|\right\}\right]^4 E\left\|\varepsilon_{t+h-j}\right\|_2^4 \quad \text{(by Assumption 3-7)} \\
&= h^3 \sum_{j=0}^{h-1} \overline{C} \phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \\
&\le C^{\frac{2}{3}} \overline{C} h^3 \sum_{j=0}^{h-1} \phi_{\max}^{4j} \\
&\le C^* \qquad (71)
\end{aligned}$$

where the next to last inequality follows from the fact that $E\left\|\varepsilon_{t+h-j}\right\|_2^4 \le \left(\sup_t E\left\|\varepsilon_t\right\|^6\right)^{\frac{2}{3}} \le C^{\frac{2}{3}}$ by Liapunov's inequality and by application of Assumption 3-2(b) and where the last inequality follows from the fact that h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. Applying the Cauchy-Schwarz inequality and the existence of moment result given in Lemma C-4, it then follows that

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\right)^2\left(\eta_{t+h}'b\right)^2\right] &\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left(a'\underline{W}_t\underline{W}_t'a\right)^2}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
&\leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left\|\underline{W}_t\right\|_2^4}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
&\leq \frac{C}{T_h} = \frac{C}{T-h-p+1} = O\left(\frac{1}{T}\right)
\end{aligned}$$

Next, observe that

$$\begin{aligned}
& E\left\{\left(a'\underline{W}_t\eta_{t+h}'b\right)\left(a'\underline{W}_{t+m}\eta_{t+m+h}'b\right)\right\} \\
= \;& E\left\{\left(a'\underline{W}_t\eta_{t+h}'b\right)\left(a'\underline{W}_{t+m}\eta_{t+m+h}'b\right)\right\} \\
= \;& E\left\{a'\underline{W}_t\underline{W}_{t+m}'a\sum_{j=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+h-j}\sum_{k=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+m+h-k}\right\} \\
= \;& E\left\{a'\underline{W}_t\underline{W}_{t+m}'a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+h-j}\varepsilon_{t+m+h-k}'J_{d+K}\left(A^j\right)'J_d'b\right\},
\end{aligned}$$

so that, for $m \geq h$, we have

$$\begin{aligned}
& E\left\{\left(a'\underline{W}_t\eta_{t+h}'b\right)\left(a'\underline{W}_{t+m}\eta_{t+m+h}'b\right)\right\} \\
= \;& E\left\{a'\underline{W}_t\underline{W}_{t+m}'a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+h-j}\varepsilon_{t+m+h-k}'J_{d+K}\left(A^j\right)'J_d'b\right\} \\
= \;& E\left\{a'\underline{W}_t\underline{W}_{t+m}'a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+h-j}E\left[\varepsilon_{t+m+h-k}'|\mathcal{F}_{-\infty}^{t+m}\right]J_{d+K}\left(A^j\right)'J_d'b\right\} \\
= \;& E\left\{a'\underline{W}_t\underline{W}_{t+m}'a\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}b'J_dA^jJ_{d+K}'\varepsilon_{t+h-j}E\left[\varepsilon_{t+m+h-k}'\right]J_{d+K}\left(A^j\right)'J_d'b\right\} \\
= \;& 0
\end{aligned}$$

Hence, defining $\sum_{m=1}^{0} E\left|\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right| = 0$, we have

$$\begin{aligned}
& \left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\}\right| \\
= & \left|\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\}\right| \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} E\left|\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right| \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(a'\underline{W}_t\underline{W}'_{t+m}a\right)^2}\sqrt{E\left(b'\eta_{t+h}\eta'_{t+m+h}b\right)^2} \\
= & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(a'\underline{W}_t\underline{W}'_t a a'\underline{W}_{t+m}\underline{W}'_{t+m}a\right)}\sqrt{E\left\{\left(b'\eta_{t+h}\right)^2\left(b'\eta_{t+m+h}\right)^2\right\}} \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \sqrt{E\left(\left\|\underline{W}_t\right\|_2^2\left\|\underline{W}_{t+m}\right\|_2^2\right)}\sqrt{E\left\{\left(b'\eta_{t+h}\right)^2\left(b'\eta_{t+m+h}\right)^2\right\}} \\
\leq & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{\min\{h-1,T-h-t\}} \left(E\left\|\underline{W}_t\right\|_2^4\right)^{\frac{1}{4}}\left(E\left\|\underline{W}_{t+m}\right\|_2^4\right)^{\frac{1}{4}}\left(E\left(b'\eta_{t+h}\right)^4\right)^{\frac{1}{4}}\left(E\left(b'\eta_{t+m+h}\right)^4\right)^{\frac{1}{4}} \\
\leq & \frac{2\left(T-h-p\right)\left(h-1\right)}{T_h^2}\overline{C} \quad \text{(applying Lemma C-4 and expression (71) above)} \\
< & \frac{2\left(h-1\right)\overline{C}}{T_h} \quad \text{(since } T_h = T-h-p+1\text{)} \\
= & O\left(\frac{1}{T}\right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t\eta'_{t+h}b\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\underline{W}_t\right)^2\left(\eta'_{t+h}b\right)^2\right] + \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t} E\left\{\left(a'\underline{W}_t\eta'_{t+h}b\right)\left(a'\underline{W}_{t+m}\eta'_{t+m+h}b\right)\right\} \\
= & O\left(\frac{1}{T}\right)
\end{aligned}$$

so that, applying Markov's inequality, we get

$$\frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{W}_t\eta'_{t+h}b = O_p\left(\frac{1}{\sqrt{T}}\right)$$

Since this result holds for every $a \in \mathbb{R}^{(d+K)p}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Now, for part (f), note that

$$\begin{aligned}
S'_d\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} I_{dp} & \underset{dp\times Kp}{0} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{Y}_t, \\
S'_K\mathcal{P}_{(d+K)p}\underline{W}_t &= \begin{pmatrix} \underset{Kp\times dp}{0} & I_{Kp} \end{pmatrix}\begin{pmatrix} \underset{dp\times 1}{\underline{Y}_t} \\ \underset{Kp\times 1}{\underline{F}_t} \end{pmatrix} = \underline{F}_t
\end{aligned}$$

Hence, it follows by applying the result given in part (e) above and the Slutsky's theorem that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{Y}_t\eta'_{t+h} &= S'_d\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right) \text{ and} \\
\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{F}_t\eta'_{t+h} &= S'_K\mathcal{P}_{(d+K)p}\frac{1}{T_h}\sum_{t=p}^{T-h} \underline{W}_t\eta'_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

To show part (g), let $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$ and write

$$
\begin{aligned}
E\left(\frac{b'\mathfrak{H}'\iota_{T_h}}{\sqrt{T_h}}\right)^2 &= E\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h}\right)^2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{j=0}^{h-1}\sum_{k=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{s+h-k}\right]J_{d+K}\left(A^k\right)'J'_db \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
&\quad +\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^{m+j}\right)'J'_db \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
&\quad +\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'J'_db
\end{aligned}
$$

Now,

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b' J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' J_d' b \\
\leq\ & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} E\left\|\varepsilon_{t+h-j}\right\|_2^2 b' J_d A^j J_{d+K}' J_{d+K}\left(A^j\right)' J_d' b \\
\leq\ & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^6\right)^{\frac{1}{3}} b' J_d A^j \left(A^j\right)' J_d' b \\
& \left(\text{by Liapunov's inequality and the fact that } \lambda_{\max}\left(J_{d+K}' J_{d+K}\right) = 1\right) \\
\leq\ & \overline{C}^{\frac{1}{3}} \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b' J_d A^j \left(A^j\right)' J_d' b \\
& \left(\text{where } \overline{C} \geq 1 \text{ is a constant such that } E\left\|\varepsilon_{t-j}\right\|_2^6 \leq \overline{C} < \infty \text{ by Assumption 3-2(b)}\right) \\
\leq\ & \overline{C}^{\frac{1}{3}} \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \lambda_{\max}\left\{A^j\left(A^j\right)'\right\} b' J_d J_d' b \\
=\ & \overline{C}^{\frac{1}{3}} \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \lambda_{\max}\left\{\left(A^j\right)' A^j\right\} \\
& \left(\text{since } \lambda_{\max}\left\{A^j\left(A^j\right)'\right\} = \lambda_{\max}\left\{\left(A^j\right)' A^j\right\},\ \lambda_{\max}\left(J_d J_d'\right) = 1, \text{ and } b'b = 1\right) \\
=\ & \overline{C}^{\frac{1}{3}} \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \sigma_{\max}^2\left(A^j\right) \\
\leq\ & \overline{C}^{\frac{1}{3}} \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} C \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \quad \text{(by Assumption 3-7)} \\
=\ & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \\
=\ & \overline{C}^{\frac{1}{3}} C \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} \phi_{\max}^{2j}
\end{aligned}
$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where $0 < \phi_{\max} < 1$ since Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. It follows that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] J_{d+K}\left(A^j\right)' J'_d b &\leq \overline{C}^{\frac{1}{3}} C \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1}\phi_{\max}^{2j} \\
&\leq \overline{C}^{\frac{1}{3}} C \frac{T-h-p+1}{T_h}\frac{1}{1-\phi_{\max}^2} \\
&= \overline{C}^{\frac{1}{3}} C \frac{1}{1-\phi_{\max}^2} \\
&\quad \text{(since } T_h = T-h-p+1\text{)} \\
&= O(1)
\end{aligned}$$

Moreover, write

$$\begin{aligned}
&\left|\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' J'_d b\right| \\
\leq\ & \frac{2}{T_h}\sum_{t=p}^{T-h-1}\left|\sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] J_{d+K}\left(A^j\right)' \sum_{m=1}^{T-h-t}\left(A^m\right)' J'_d b\right| \\
\leq\ & \frac{2}{T_h}\sum_{t=p}^{T-h-1}\left\{\sqrt{\sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right] J_{d+K}\left(A^j\right)' J'_d b}\right. \\
&\left.\times\sqrt{\sum_{j=0}^{\max\{0,h-2\}}\sum_{m_1=1}^{T-h-t}\sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K}\left(A^j\right)'\left(A^{m_2}\right)' J'_d b}\right\}
\end{aligned}$$

Similar to the argument given previously, we have

$$
\begin{aligned}
& \sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J_{d+K}' E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] J_{d+K}\left(A^j\right)' J_d' b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}\left(E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right]\right) b' J_d A^j J_{d+K}' J_{d+K}\left(A^j\right)' J_d' b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}\left(E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right] E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right]\right) C\phi_{\max}^{2j} \\
= & \; C \sum_{j=0}^{\max\{0,h-2\}} \lambda_{\max}^2\left(E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right]\right) \phi_{\max}^{2j} \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(tr\left\{E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right]\right\}\right)^2 \phi_{\max}^{2j} \\
= & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^2\right)^2 \phi_{\max}^{2j} \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \left(E\left\|\varepsilon_{t+h-j}\right\|_2^6\right)^{\frac{2}{3}} \phi_{\max}^{2j} \\
\leq & \; \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2}
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} \left(A^j\right)' \left(A^{m_2}\right)' J'_d b \\
\leq & \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j \left(A^j\right)' \left(A^{m_2}\right)' J'_d b \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \left| b' J_d A^{m_1} \left(A^{m_2}\right)' J'_d b \right| \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{b' J_d A^{m_1} \left(A^{m_1}\right)' J'_d b} \sqrt{b' J_d A^{m_2} \left(A^{m_2}\right)' J'_d b} \\
\leq & \; C \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} \sqrt{C \phi_{\max}^{2m_1}} \sqrt{C \phi_{\max}^{2m_2}} \\
\leq & \; C^2 \sum_{j=0}^{\max\{0,h-2\}} \phi_{\max}^{2j} \sum_{m_1=1}^{T-h-t} \phi_{\max}^{m_1} \sum_{m_2=1}^{T-h-t} \phi_{\max}^{m_2} \\
\leq & \; C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \left| \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sum_{j=0}^{h-2} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] J_{d+K} \left(A^j\right)' \sum_{m=1}^{T-h-t} \left(A^m\right)' J'_d b \right| \\
\leq & \; \frac{2}{T_h} \sum_{t=p}^{T-h-1} \left\{ \sqrt{ \sum_{j=0}^{\max\{0,h-2\}} b' J_d A^j J'_{d+K} E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] E\left[\varepsilon_{t+h-j} \varepsilon'_{t+h-j}\right] J_{d+K} \left(A^j\right)' J'_d b } \right. \\
& \left. \times \sqrt{ \sum_{j=0}^{\max\{0,h-2\}} \sum_{m_1=1}^{T-h-t} \sum_{m_2=1}^{T-h-t} b' J_d A^{m_1} A^j J'_{d+K} J_{d+K} \left(A^j\right)' \left(A^{m_2}\right)' J'_d b } \right\} \\
\leq & \; \frac{2}{T_h} \sum_{t=p}^{T-h-1} \sqrt{ \overline{C}^{\frac{2}{3}} C \frac{1}{1-\phi_{\max}^2} } \sqrt{ C^2 \frac{1}{1-\phi_{\max}^2} \left(\frac{1}{1-\phi_{\max}} \right)^2 } \\
= & \; 2 \overline{C}^{\frac{1}{3}} C^{\frac{3}{2}} \frac{T-h-p+1}{T_h} \left(\frac{1}{1-\phi_{\max}^2} \right) \left(\frac{1}{1-\phi_{\max}} \right) \\
= & \; O(1)
\end{aligned}
$$

Putting these results together, we obtain

$$\begin{aligned}
& E\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h}\right)^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{j=0}^{h-1} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'J'_db \\
& +\frac{2}{T_h}\sum_{t=p}^{T-h-1}\sum_{j=0}^{\max\{0,h-2\}} b'J_dA^jJ'_{d+K}E\left[\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right]J_{d+K}\left(A^j\right)'\sum_{m=1}^{T-h-t}\left(A^m\right)'J'_db \\
= & O\left(1\right)
\end{aligned}$$

so that, upon applying Markov's inequality, we get

$$\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h} b'\eta_{t+h} = O_p\left(1\right).$$

Since the above result holds for all $b \in \mathbb{R}^d$ such that $\left\|b\right\|_2 = 1$, we further deduce that

$$\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h}\eta_{t+h} = O_p\left(1\right)$$

and that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h} = \frac{1}{\sqrt{T_h}}\left(\frac{1}{\sqrt{T_h}}\sum_{t=p}^{T-h}\eta_{t+h}\right) = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p\left(1\right).$$

Lastly, to show part (h), let $a, b \in \mathbb{R}^d$ such that $\left\|a\right\|_2 = \left\|b\right\|_2 = 1$; and write

$$\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\right]^2 \\
= & \frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)^2\right] \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left\{\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\right. \\
& \qquad \left.\times\left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\}
\end{aligned}$$

Making use of the Cauchy-Schwarz inequality, we then have

$$\begin{aligned}
& \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2\right] \\
= \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\eta_{t+h}\eta_{t+h}'b\right)^2 - \frac{1}{T_h^2}\sum_{t=p}^{T-h}\left(E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2 \\
\leq \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left(a'\eta_{t+h}\eta_{t+h}'b\right)^2 \\
\leq \; & \frac{1}{T_h^2}\sum_{t=p}^{T-h} \sqrt{E\left(a'\eta_{t+h}\right)^4}\sqrt{E\left(b'\eta_{t+h}\right)^4}
\end{aligned}$$

In the proof of part (e) of this lemma, we have shown that, given Assumptions 3-2(b) and 3-7, there exists positive constants C and $\overline{C}$ such that

$$E\left(b'\eta_{t+h}\right)^4 \leq h^3\sum_{j=0}^{h-1} C\phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \leq \overline{C} < \infty.$$

where $\phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}$ and where h is a fixed integer and $0 < \phi_{\max} < 1$ in light of Assumption 3-1. In a similar manner, we can also show that

$$E\left(a'\eta_{t+h}\right)^4 \leq h^3\sum_{j=0}^{h-1} C\phi_{\max}^{4j} E\left\|\varepsilon_{t+h-j}\right\|_2^4 \leq \overline{C} < \infty.$$

It follows that

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2\right] & \leq \frac{1}{T_h^2}\sum_{t=p}^{T-h} \sqrt{E\left(a'\eta_{t+h}\right)^4}\sqrt{E\left(b'\eta_{t+h}\right)^4} \\
& \leq \frac{1}{T_h^2}\sum_{t=p}^{T-h}\overline{C} \\
& = \overline{C}\frac{T-h-p+1}{T_h^2} \\
& = \frac{\overline{C}}{T_h} \quad (\text{since } T_h = T-h-p+1) \\
& = O\left(\frac{1}{T}\right) \qquad (72)
\end{aligned}$$

Next, observe that

$$
\begin{aligned}
& a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right] \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1} a'J_dA^jJ_{d+K}'\left(\varepsilon_{t+h-j}\varepsilon_{t+h-k}' - E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-k}'\right]\right)J_{d+K}\left(A^k\right)'J_d'b \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)\left\{vec\left(\varepsilon_{t+h-j}\varepsilon_{t+h-k}'\right) - vec\left(E\left[\varepsilon_{t+h-j}\varepsilon_{t+h-k}'\right]\right)\right\} \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)\left\{\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left[\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right] \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1} a'J_dA^\ell J_{d+K}'\left(\varepsilon_{t+m+h-\ell}\varepsilon_{t+m+h-r}' - E\left[\varepsilon_{t+m+h-\ell}\varepsilon_{t+m+h-r}'\right]\right)J_{d+K}\left(A^r\right)'J_d'b \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left(b'J_dA^rJ_{d+K}' \otimes a'J_dA^\ell J_{d+K}'\right)\left\{vec\left(\varepsilon_{t+m+h-\ell}\varepsilon_{t+m+h-r}'\right) - vec\left(E\left[\varepsilon_{t+m+h-\ell}\varepsilon_{t+m+h-r}'\right]\right)\right\} \\
= & \sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left(b'J_dA^rJ_{d+K}' \otimes a'J_dA^\ell J_{d+K}'\right)\left\{\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left[\varepsilon_{t+m+h-k}\otimes\varepsilon_{t+m+h-r}\right]\right\}
\end{aligned}
$$

Moreover, note that, for $m \geq h$

$$
\begin{aligned}
& E\left\{\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)\left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\} \\
= & \sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)\right. \\
& \times E\left(\left[\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)\right]\right. \\
& \qquad \left.\times\left[\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)\right]'\right) \\
& \left.\times\left(J_{d+K}\left(A^r\right)'J_d'b \otimes J_{d+K}\left(A^\ell\right)'J_d'a\right)\right\} \\
= & \; 0
\end{aligned}
$$

Note further that, when $h = 1$, we will always have $m \geq h$, given that by definition m is an integer ≥ 1. This implies we need to distinguish between the case where $h = 1$ from the case where $h \geq 2$.

Consider first the case where $h = 1$. In this case, we have, for all $m \geq 1$

$$
\begin{aligned}
& E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} \\
= & \left(b'J_d A^0 J_{d+K}' \otimes a'J_d A^0 J_{d+K}'\right) \\
& \times E\left(\left[\left(\varepsilon_{t+1} \otimes \varepsilon_{t+1}\right) - E\left(\varepsilon_{t+1} \otimes \varepsilon_{t+1}\right)\right]\left[\left(\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1}\right) - E\left(\varepsilon_{t+m+1} \otimes \varepsilon_{t+m+1}\right)\right]'\right) \\
& \times \left(J_{d+K}\left(A^0\right)' J_d' b \otimes J_{d+K}\left(A^0\right)' J_d' a\right) \\
= & \ 0
\end{aligned}
$$

so that, in this case, we have

$$
\begin{aligned}
& E\left[\frac{1}{T_1}\sum_{t=p}^{T-1}\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\right]^2 \\
= & \frac{1}{T_1^2}\sum_{t=p}^{T-1} E\left[\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)^2\right] \\
& + \frac{2}{T_1^2}\sum_{t=p}^{T-1}\sum_{m=1}^{T-1-t} E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\right. \\
& \qquad \left. \times \left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} \\
= & \frac{1}{T_1^2}\sum_{t=p}^{T-1} E\left[\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)^2\right] \\
= & O\left(\frac{1}{T}\right) \quad \text{(as shown previously in expression (72))}
\end{aligned} \tag{73}
$$

Consider next the case where $h \geq 2$. In this case,

$$
E\left\{\left(a'\eta_{t+1}\eta_{t+1}'b - E\left[a'\eta_{t+1}\eta_{t+1}'b\right]\right)\left(a'\eta_{t+m+1}\eta_{t+m+1}'b - E\left[a'\eta_{t+m+1}\eta_{t+m+1}'b\right]\right)\right\} = 0
$$

for all $m \geq h$ as previously shown; however, for $1 \leq m \leq h-1$, we have

$$
\begin{aligned}
& \left|E\left\{\left(a'\eta_{t+h}\eta'_{t+h}b - E\left[a'\eta_{t+h}\eta'_{t+h}b\right]\right)\left(a'\eta_{t+m+h}\eta'_{t+m+h}b - E\left[a'\eta_{t+m+h}\eta'_{t+m+h}b\right]\right)\right\}\right| \\
= \; & \left|\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)\right.\right. \\
& \times E\left(\left[\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right) - E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)\right]\right. \\
& \left.\times\left[\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right) - E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)\right]'\right) \\
& \left.\left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\}\right| \\
= \; & \left|\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+m+h-r}\right)\right.\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\} \\
& -\sum_{j=0}^{h-1}\sum_{k=0}^{h-1}\sum_{\ell=0}^{h-1}\sum_{r=0}^{h-1}\left\{\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-r}\right)'\right. \\
& \qquad\qquad \left.\left.\times\left(J_{d+K}\left(A^r\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right\}\right| \\
\leq \; & \sum_{j=0}^{h-1}\left|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right. \\
& \quad \left.\times\left(J_{d+K}\left(A^j\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right| \\
& +\sum_{j=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+h-k}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^k\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right| \\
& +\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^kJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right| \\
& +\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\varepsilon'_{t+h-k}\right)\right. \\
& \qquad\qquad \left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^k\right)'J'_da\right)\right|
\end{aligned}
$$

$$+\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)'\right.$$
$$\left.\times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\right|.$$

Analyzing each term on the majorant side of the function above, we have

$$\begin{aligned}
&\sum_{j=0}^{h-1}\left|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right.\\
&\qquad\left.\times\left(J_{d+K}\left(A^j\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right|\\
\leq\;&\overline{C}\sum_{j=0}^{h-1}\left(b'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_db\right)\left(a'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_da\right)\\
=\;&\overline{C}\sum_{j=0}^{h-1}\left[b'J_dA^j\left(A^j\right)'J'_db\right]\left[a'J_dA^j\left(A^j\right)'J'_da\right]\quad\left(\text{since }\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)=1\right)\\
\leq\;&\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]^2\left[b'J_dJ'_db\right]\left[a'J_dJ'_da\right]\\
=\;&\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]^2\quad\left(\text{since }J_dJ'_d=I_d\text{ and }a'a=b'b=1\right)\\
=\;&\overline{C}\sum_{j=0}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]^2\\
=\;&\overline{C}\sum_{j=0}^{h-1}\sigma^4_{\max}\left(A^j\right)\\
\leq\;&\overline{C}\sum_{j=0}^{h-1}C^*\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^4,\left|\lambda_{\min}\left(A^j\right)\right|^4\right\}\;\;(\text{by Assumption 3-7})\\
=\;&\overline{C}\sum_{j=0}^{h-1}C^*\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{4j},\left|\lambda_{\min}\left(A\right)\right|^{4j}\right\}\\
=\;&\overline{C}\sum_{j=0}^{h-1}C^*\phi^{4j}_{\max}\;\;(\text{where }0<\phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}<1)\\
\leq\;&\overline{C}hC^*\quad\left(\text{since }0<\phi_{\max}<1\text{ and }\phi^0_{\max}=1\right)\\
\leq\;&C\quad\left(\text{for }\overline{C}hC^*\leq C<\infty\right),
\end{aligned}$$

$$
\begin{aligned}
& \sum_{j=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left|\left(b'J_dA^kJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-k}\varepsilon'_{t+h-k}\otimes\varepsilon_{t+h-j}\varepsilon'_{t+h-j}\right)\right.\\
&\qquad\left.\times\left(J_{d+K}\left(A^k\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\right|\\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(b'J_dA^kJ'_{d+K}J_{d+K}\left(A^k\right)'J'_db\right)\left(a'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_da\right)\\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[b'J_dA^k\left(A^k\right)'J'_db\right]\left[a'J_dA^j\left(A^j\right)'J'_da\right]\quad\left(\text{since }\lambda_{\max}\left(J'_{d+K}J_{d+K}\right)=1\right)\\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\left[b'J_dJ'_db\right]\left[a'J_dJ'_da\right]\\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^k\left(A^k\right)'\right\}\right]\quad\left(\text{since }J_dJ'_d=I_d\text{ and }a'a=b'b=1\right)\\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]\left[\lambda_{\max}\left\{\left(A^k\right)'A^k\right\}\right]\\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\sigma^2_{\max}\left(A^j\right)\sigma^2_{\max}\left(A^k\right)\\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2,\left|\lambda_{\min}\left(A^j\right)\right|^2\right\}\max\left\{\left|\lambda_{\max}\left(A^k\right)\right|^2,\left|\lambda_{\min}\left(A^k\right)\right|^2\right\}\\
& \text{(by Assumption 3-7)}\\
=\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2j},\left|\lambda_{\min}\left(A\right)\right|^{2j}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\}\\
\leq\ & \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{k=0\\k\neq j}}^{h-1}\left(C^*\right)^2\phi^{2j}_{\max}\phi^{2k}_{\max}\ \left(\text{where }\phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right)\\
=\ & \overline{C}h^2\left(C^*\right)^2\ \ \left(\text{since }0<\phi_{\max}<1\text{ given Assumption 3-1}\right)\\
\leq\ & C\quad\left(\text{for }\overline{C}h^2\left(C^*\right)^2\leq C<\infty\right),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right| \\
\leq & \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left\{ \left[\left(b' J_d A^k J'_{d+K} \otimes a' J_d A^k J'_{d+K} \right) E \left(\varepsilon_{t+h-k} \varepsilon'_{t+h-k} \otimes \varepsilon_{t+h-k} \varepsilon'_{t+h-k} \right) \right. \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^k \right)' J'_d b \otimes J_{d+K} \left(A^k \right)' J'_d a \right) \right]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J'_{d+K} \otimes a' J_d A^\ell J'_{d+K} \right) E \left(\varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell} \varepsilon'_{t+m+h-\ell} \right) \right. \\
& \qquad \left. \left. \times \left(J_{d+K} \left(A^\ell \right)' J'_d b \otimes J_{d+K} \left(A^\ell \right)' J'_d a \right) \right]^{1/2} \right\} \\
\leq & \ \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left(b' J_d A^k J'_{d+K} J_{d+K} \left(A^k \right)' J'_d b \right) \left(a' J_d A^k J'_{d+K} J_{d+K} \left(A^k \right)' J'_d a \right) } \\
& \qquad \times \sqrt{ \left(b' J_d A^\ell J'_{d+K} J_{d+K} \left(A^\ell \right)' J'_d b \right) \left(a' J_d A^\ell J'_{d+K} J_{d+K} \left(A^\ell \right)' J'_d a \right) } \\
= & \ \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left[b' J_d A^k \left(A^k \right)' J'_d b \right] \left[a' J_d A^k \left(A^k \right)' J'_d a \right] } \sqrt{ \left[b' J_d A^\ell \left(A^\ell \right)' J'_d b \right] \left[a' J_d A^\ell \left(A^\ell \right)' J'_d a \right] } \\
& \left(\text{since } \lambda_{\max} \left(J'_{d+K} J_{d+K} \right) = 1 \right) \\
\leq & \ \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \left[b' J_d J'_d b \right] \left[a' J_d J'_d a \right] \\
= & \ \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \quad \left(\text{since } J_d J'_d = I_d \text{ and } a'a = b'b = 1 \right) \\
= & \ \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ \left(A^k \right)' A^k \right\} \right] \left[\lambda_{\max} \left\{ \left(A^\ell \right)' A^\ell \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\sigma_{\max}^{2}\left(A^{k}\right)\sigma_{\max}^{2}\left(A^{\ell}\right)\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A^{k}\right)\right|^{2},\left|\lambda_{\min}\left(A^{k}\right)\right|^{2}\right\}\max\left\{\left|\lambda_{\max}\left(A^{\ell}\right)\right|^{2},\left|\lambda_{\min}\left(A^{\ell}\right)\right|^{2}\right\}\\
&\quad \text{(by Assumption 3-7)}\\
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2\ell},\left|\lambda_{\min}\left(A\right)\right|^{2\ell}\right\}\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\phi_{\max}^{2k}\phi_{\max}^{2\ell}\ \ \left(\text{since } \phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right)\\
&= \overline{C}h^{2}\left(C^{*}\right)^{2}\ \ \left(\text{since } 0<\phi_{\max}<1 \text{ and } \phi_{\max}^{0}=1\right)\\
&\leq C\ \ \ \left(\text{for } \overline{C}h^{2}\left(C^{*}\right)^{2}\leq C<\infty\right),
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left| \left(b' J_d A^k J_{d+K}' \otimes a' J_d A^\ell J_{d+K}' \right) E \left(\varepsilon_{t+h-k} \varepsilon_{t+m+h-\ell}' \otimes \varepsilon_{t+m+h-\ell} \varepsilon_{t+h-k}' \right) \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^\ell \right)' J_d' b \otimes J_{d+K} \left(A^k \right)' J_d' a \right) \right| \\
\leq\ & \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left\{ \left[\left(b' J_d A^k J_{d+K}' \otimes a' J_d A^\ell J_{d+K}' \right) E \left(\varepsilon_{t+h-k} \varepsilon_{t+h-k}' \otimes \varepsilon_{t+m+h-\ell} \varepsilon_{t+m+h-\ell}' \right) \right. \right. \\
& \qquad \left. \times \left(J_{d+K} \left(A^k \right)' J_d' b \otimes J_{d+K} \left(A^\ell \right)' J_d' a \right) \right]^{1/2} \\
& \quad \times \left[\left(b' J_d A^\ell J_{d+K}' \otimes a' J_d A^k J_{d+K}' \right) E \left(\varepsilon_{t+m+h-\ell} \varepsilon_{t+m+h-\ell}' \otimes \varepsilon_{t+h-k} \varepsilon_{t+h-k}' \right) \right. \\
& \qquad \left. \left. \times \left(J_{d+K} \left(A^\ell \right)' J_d' b \otimes J_{d+K} \left(A^k \right)' J_d' a \right) \right]^{1/2} \right\} \\
\leq\ & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left(b' J_d A^k J_{d+K}' J_{d+K} \left(A^k \right)' J_d' b \right) \left(a' J_d A^\ell J_{d+K}' J_{d+K} \left(A^\ell \right)' J_d' a \right) } \\
& \qquad \times \sqrt{ \left(b' J_d A^\ell J_{d+K}' J_{d+K} \left(A^\ell \right)' J_d' b \right) \left(a' J_d A^k J_{d+K}' J_{d+K} \left(A^k \right)' J_d' a \right) } \\
=\ & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \sqrt{ \left[b' J_d A^k \left(A^k \right)' J_d' b \right] \left[a' J_d A^\ell \left(A^\ell \right)' J_d' a \right] } \sqrt{ \left[b' J_d A^\ell \left(A^\ell \right)' J_d' b \right] \left[a' J_d A^k \left(A^k \right)' J_d' a \right] } \\
& \left(\text{since } \lambda_{\max} \left(J_{d+K}' J_{d+K} \right) = 1 \right) \\
\leq\ & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \left[b' J_d J_d' b \right] \left[a' J_d J_d' a \right] \\
=\ & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ A^k \left(A^k \right)' \right\} \right] \left[\lambda_{\max} \left\{ A^\ell \left(A^\ell \right)' \right\} \right] \quad \left(\text{since } J_d J_d' = I_d \text{ and } a'a = b'b = 1 \right) \\
=\ & \overline{C} \sum_{k=0}^{h-1} \sum_{\substack{\ell=0 \\ \ell \neq k+m}}^{h-1} \left[\lambda_{\max} \left\{ \left(A^k \right)' A^k \right\} \right] \left[\lambda_{\max} \left\{ \left(A^\ell \right)' A^\ell \right\} \right]
\end{aligned}
$$

$$
\begin{aligned}
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\sigma_{\max}^{2}\left(A^{k}\right)\sigma_{\max}^{2}\left(A^{\ell}\right)\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A^{k}\right)\right|^{2},\left|\lambda_{\min}\left(A^{k}\right)\right|^{2}\right\}\max\left\{\left|\lambda_{\max}\left(A^{\ell}\right)\right|^{2},\left|\lambda_{\min}\left(A^{\ell}\right)\right|^{2}\right\}\\
&\quad\text{(by Assumption 3-7)}\\
&= \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2k},\left|\lambda_{\min}\left(A\right)\right|^{2k}\right\}\max\left\{\left|\lambda_{\max}\left(A\right)\right|^{2\ell},\left|\lambda_{\min}\left(A\right)\right|^{2\ell}\right\}\\
&\leq \overline{C}\sum_{k=0}^{h-1}\sum_{\substack{\ell=0\\ \ell\neq k+m}}^{h-1}\left(C^{*}\right)^{2}\phi_{\max}^{2k}\phi_{\max}^{2\ell}\ \ \left(\text{since } \phi_{\max}=\max\left\{\left|\lambda_{\max}\left(A\right)\right|,\left|\lambda_{\min}\left(A\right)\right|\right\}\right)\\
&= \overline{C}h^{2}\left(C^{*}\right)^{2}\ \ \left(\text{since } 0<\phi_{\max}<1 \text{ and } \phi_{\max}^{0}=1\right)\\
&\leq C\ \ \ \left(\text{for } \overline{C}h^{2}\left(C^{*}\right)^{2}\leq C<\infty\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\Big|\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)' \\
& \qquad\qquad \times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\Big| \\
\leq\ & \sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\Big\{\Big[\left(b'J_dA^jJ'_{d+K}\otimes a'J_dA^jJ'_{d+K}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+h-j}\otimes\varepsilon_{t+h-j}\right)' \\
& \qquad\quad \times\left(J_{d+K}\left(A^j\right)'J'_db\otimes J_{d+K}\left(A^j\right)'J'_da\right)\Big]^{1/2} \\
& \quad\times\Big[\left(b'J_dA^\ell J'_{d+K}\otimes a'J_dA^\ell J'_{d+K}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)E\left(\varepsilon_{t+m+h-\ell}\otimes\varepsilon_{t+m+h-\ell}\right)' \\
& \qquad\qquad \times\left(J_{d+K}\left(A^\ell\right)'J'_db\otimes J_{d+K}\left(A^\ell\right)'J'_da\right)\Big]^{1/2}\Big\} \\
\leq\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sqrt{\left(b'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_db\right)\left(a'J_dA^jJ'_{d+K}J_{d+K}\left(A^j\right)'J'_da\right)} \\
& \qquad\qquad \times\sqrt{\left(b'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_db\right)\left(a'J_dA^\ell J'_{d+K}J_{d+K}\left(A^\ell\right)'J'_da\right)} \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sqrt{\left[b'J_dA^j\left(A^j\right)'J'_db\right]\left[a'J_dA^j\left(A^j\right)'J'_da\right]}\sqrt{\left[b'J_dA^\ell\left(A^\ell\right)'J'_db\right]\left[a'J_dA^\ell\left(A^\ell\right)'J'_da\right]} \\
& \left(\text{since } \lambda_{\max}\left(J'_{d+K}J_{d+K}\right)=1\right) \\
\leq\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\left[b'J_dJ'_db\right]\left[a'J_dJ'_da\right] \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{A^j\left(A^j\right)'\right\}\right]\left[\lambda_{\max}\left\{A^\ell\left(A^\ell\right)'\right\}\right]\quad\left(\text{since } J_dJ'_d=I_d \text{ and } a'a=b'b=1\right) \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left[\lambda_{\max}\left\{\left(A^j\right)'A^j\right\}\right]\left[\lambda_{\max}\left\{\left(A^\ell\right)'A^\ell\right\}\right] \\
=\ & \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\sigma^2_{\max}\left(A^j\right)\sigma^2_{\max}\left(A^\ell\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}(C^*)^2 \max\left\{\left|\lambda_{\max}\left(A^j\right)\right|^2, \left|\lambda_{\min}\left(A^j\right)\right|^2\right\} \max\left\{\left|\lambda_{\max}\left(A^\ell\right)\right|^2, \left|\lambda_{\min}\left(A^\ell\right)\right|^2\right\} \\
&\quad \text{(by Assumption 3-7)} \\
&= \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}(C^*)^2 \max\left\{\left|\lambda_{\max}(A)\right|^{2j}, \left|\lambda_{\min}(A)\right|^{2j}\right\} \max\left\{\left|\lambda_{\max}(A)\right|^{2\ell}, \left|\lambda_{\min}(A)\right|^{2\ell}\right\} \\
&\leq \overline{C}\sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}(C^*)^2 \phi_{\max}^{2j}\phi_{\max}^{2\ell} \;\; (\text{since } \phi_{\max} = \max\left\{\left|\lambda_{\max}(A)\right|, \left|\lambda_{\min}(A)\right|\right\}) \\
&= \overline{C}h^2(C^*)^2 \;\; \left(\text{since } 0 < \phi_{\max} < 1 \text{ and } \phi_{\max}^0 = 1\right) \\
&\leq C \quad \left(\text{for } \overline{C}h^2(C^*)^2 \leq C < \infty\right),
\end{aligned}
$$

where upper bounds given above have made use of the fact that for all t and s

$$
\begin{aligned}
& E\left[\varepsilon_t\varepsilon_t' \otimes \varepsilon_s\varepsilon_s'\right] \\
=\; & E\left[(\varepsilon_t \otimes \varepsilon_s)(\varepsilon_t \otimes \varepsilon_s)'\right] \\
\leq\; & tr\left\{E\left[(\varepsilon_t \otimes \varepsilon_s)(\varepsilon_t \otimes \varepsilon_s)'\right]\right\} \cdot I_{(d+K)^2} \\
& \text{(where the inequality holds in positive semi-definite sense)} \\
=\; & E\left[tr\left\{(\varepsilon_t \otimes \varepsilon_s)(\varepsilon_t \otimes \varepsilon_s)'\right\}\right] \cdot I_{(d+K)^2} \\
=\; & E\left[tr\left\{(\varepsilon_t \otimes \varepsilon_s)'(\varepsilon_t \otimes \varepsilon_s)\right\}\right] \cdot I_{(d+K)^2} \\
=\; & E\left[\varepsilon_t'\varepsilon_t\varepsilon_s'\varepsilon_s\right] \cdot I_{(d+K)^2} \\
=\; & E\left[\left\|\varepsilon_t\right\|_2^2\left\|\varepsilon_s\right\|_2^2\right] \cdot I_{(d+K)^2} \\
\leq\; & \sup_t E\left[\left\|\varepsilon_t\right\|_2^4\right] \cdot I_{(d+K)^2} \\
\leq\; & \overline{C} \cdot I_{d^2} \;\; \text{(by Assumption 3-2(b))}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(\varepsilon_t \otimes \varepsilon_t\right) E\left(\varepsilon_t \otimes \varepsilon_t\right)' &\leq tr\left\{E\left(\varepsilon_t \otimes \varepsilon_t\right) E\left(\varepsilon_t \otimes \varepsilon_t\right)'\right\} \cdot I_{(d+K)^2} \\
&\quad \text{(where the inequality holds in positive semi-definite sense)} \\
&= E\left(\varepsilon_t \otimes \varepsilon_t\right)' E\left(\varepsilon_t \otimes \varepsilon_t\right) \cdot I_{(d+K)^2} \\
&= \sum_{g=1}^{d} \sum_{\ell=1}^{d} \left(E\left[\varepsilon_{gt}\varepsilon_{\ell t}\right]\right)^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^{d} \sum_{\ell=1}^{d} \left(E\left|\varepsilon_{gt}\varepsilon_{\ell t}\right|\right)^2 \cdot I_{(d+K)^2} \\
&\leq \sum_{g=1}^{d} \sum_{\ell=1}^{d} E\left[\varepsilon_{gt}^2\right] E\left[\varepsilon_{\ell t}^2\right] \cdot I_{(d+K)^2} \\
&= E\left[\sum_{g=1}^{d} \varepsilon_{gt}^2\right] E\left[\sum_{\ell=1}^{d} \varepsilon_{\ell t}^2\right] \cdot I_{(d+K)^2} \\
&= \left(E\left\|\varepsilon_t\right\|_2^2\right)^2 \cdot I_{(d+K)^2} \\
&\leq \overline{C} \cdot I_{(d+K)^2} \ \text{ (by Assumption 3-2(b))}
\end{aligned}
$$

for some positive constant $\overline{C}$. It follows from these calculations that, for $1 \leq m \leq h-1$ where

$h \geq 2$, we have

$$
\begin{aligned}
& \left|E\left\{\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)\left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\}\right| \\
\leq & \sum_{j=0}^{h-1}\left|\left(b'J_dA^jJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-j}\varepsilon_{t+h-j}' \otimes \varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right)\right. \\
& \qquad \left. \times \left(J_{d+K}\left(A^j\right)'J_d'b \otimes J_{d+K}\left(A^j\right)'J_d'a\right)\right| \\
& + \sum_{j=0}^{h-1}\sum_{\substack{k=0 \\ k\neq j}}^{h-1}\left|\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-k}\varepsilon_{t+h-k}' \otimes \varepsilon_{t+h-j}\varepsilon_{t+h-j}'\right)\right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^k\right)'J_d'b \otimes J_{d+K}\left(A^j\right)'J_d'a\right)\right| \\
& + \sum_{k=0}^{h-1}\sum_{\substack{\ell=0 \\ \ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^kJ_{d+K}'\right)E\left(\varepsilon_{t+h-k}\varepsilon_{t+m+h-\ell}' \otimes \varepsilon_{t+h-k}\varepsilon_{t+m+h-\ell}'\right)\right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J_d'b \otimes J_{d+K}\left(A^\ell\right)'J_d'a\right)\right| \\
& + \sum_{k=0}^{h-1}\sum_{\substack{\ell=0 \\ \ell\neq k+m}}^{h-1}\left|\left(b'J_dA^kJ_{d+K}' \otimes a'J_dA^\ell J_{d+K}'\right)E\left(\varepsilon_{t+h-k}\varepsilon_{t+m+h-\ell}' \otimes \varepsilon_{t+m+h-\ell}\varepsilon_{t+h-k}'\right)\right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J_d'b \otimes J_{d+K}\left(A^k\right)'J_d'a\right)\right| \\
& + \sum_{j=0}^{h-1}\sum_{\ell=0}^{h-1}\left|\left(b'J_dA^jJ_{d+K}' \otimes a'J_dA^jJ_{d+K}'\right)E\left(\varepsilon_{t+h-j} \otimes \varepsilon_{t+h-j}\right)E\left(\varepsilon_{t+m+h-\ell} \otimes \varepsilon_{t+m+h-\ell}\right)'\right. \\
& \qquad\qquad \left. \times \left(J_{d+K}\left(A^\ell\right)'J_d'b \otimes J_{d+K}\left(A^\ell\right)'J_d'a\right)\right| \\
\leq & \; 5C
\end{aligned}
$$

so that, when $h \geq 2$,

$$
\begin{aligned}
& \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right) \left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\} \right| \\
= & \left| \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} E\left\{ \left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right) \right.\right. \\
& \qquad \left.\left. \times \left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\} \right| \\
\leq & \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\min\{h-1, T-h-t\}} \left| E\left\{ \left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right) \right.\right. \\
& \qquad \left.\left. \times \left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\} \right| \\
\leq & \frac{2}{T_h} \frac{T-h-p}{T_h} (h-1) 5C \\
< & \frac{10(h-1)C}{T_h} \quad (\text{ since } T_h = T-h-p+1) \\
= & O\left(\frac{1}{T}\right)
\end{aligned}
$$

Putting everything together for the case where $h \geq 2$, we see that

$$
\begin{aligned}
& E\left[\frac{1}{T_h} \sum_{t=p}^{T-h} \left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right) \right]^2 \\
= & \frac{1}{T_h^2} \sum_{t=p}^{T-h} E\left[\left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right)^2 \right] \\
& + \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} E\left\{ \left(a'\eta_{t+h}\eta_{t+h}'b - E\left[a'\eta_{t+h}\eta_{t+h}'b\right]\right) \left(a'\eta_{t+m+h}\eta_{t+m+h}'b - E\left[a'\eta_{t+m+h}\eta_{t+m+h}'b\right]\right)\right\} \\
= & O\left(\frac{1}{T}\right) + O\left(\frac{1}{T}\right) \\
= & O\left(\frac{1}{T}\right) \qquad (74)
\end{aligned}
$$

In light of the results given in expressions (73) and (74), we can apply Markov's inequality to show that regardless of whether $h = 1$ or $h \geq 2$

$$
\frac{1}{T_h} \sum_{t=p}^{T-h} a'\eta_{t+h}\eta_{t+h}'b - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[a'\eta_{t+h}\eta_{t+h}'b\right] = O_p\left(\frac{1}{\sqrt{T}}\right).
$$

Moreover, since the above result holds for all $a, b \in \mathbb{R}^d$ such that $\|a\|_2 = \|b\|_2 = 1$, we further deduce that for all (fixed) positive integer h

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right] = O_p\left(\frac{1}{\sqrt{T}}\right). \ \square$$

Lemma D-3: Suppose that A is an $N \times N$ symmetric matrix which we can partition as

$$A = \begin{pmatrix} \underset{r\times r}{A_{11}} & \underset{r\times(N-r)}{A_{12}} \\ \underset{(N-r)\times r}{A_{21}} & \underset{(N-r)\times(N-r)}{A_{22}} \end{pmatrix}$$

Then,

$$\|A_{21}\|_2 \leq \|A\|_2 .$$

Proof of Lemma D-3: Define

$$\underset{N\times r}{B_1} = \begin{pmatrix} I_r \\ 0 \end{pmatrix}.$$

Let $\overline{\upsilon} \in \mathbb{R}^r$ be such that $\|\overline{\upsilon}\|_2 = 1$ and

$$\overline{\upsilon}' A_{21}' A_{21} \overline{\upsilon} = \max_{\|\upsilon\|_2 = 1} \upsilon' A_{21}' A_{21} \upsilon$$

It follows that

$$\begin{aligned}
\|A_{21}\|_2 &= \sqrt{\overline{\upsilon}' A_{21}' A_{21} \overline{\upsilon}} \\
&\leq \sqrt{\overline{\upsilon}' A_{11}' A_{11} \overline{\upsilon} + \overline{\upsilon}' A_{21}' A_{21} \overline{\upsilon}} \\
&= \sqrt{\overline{\upsilon}' B_1' A' A B_1 \overline{\upsilon}} \\
&\leq \sqrt{\max_{\|\upsilon\|_2=1} \upsilon' A' A \upsilon} \ \left(\text{noting that } \|B_1\overline{\upsilon}\|_2 = \sqrt{\overline{\upsilon}' B_1' B_1 \overline{\upsilon}} = \sqrt{\overline{\upsilon}'\overline{\upsilon}} = 1\right) \\
&= \|A\|_2 . \ \square
\end{aligned}$$

Remark: This is a well-known linear algebraic result. A similar result has also been given in the beginning of section 6 of Johnstone and Lu (2009).

Lemma D-4: Let

$$M_{FF} = \frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \tag{75}$$

where $T_0 = T - p + 1$. Then, under Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, and 3-7; there exists a positive constant $\underline{C}$ such that

$$\lambda_{\min}\left\{M_{FF}\right\} \geq \underline{C} > 0$$

for all $T > p - 1$.

Proof of Lemma D-4:

To proceed, note that we can write

$$\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} = \mathcal{P}_{(d+K)p}\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'$$

from which it follows that

$$\begin{aligned}
\lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix}\right\} &= \lambda_{\min}\left\{\mathcal{P}_{(d+K)p}\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\mathcal{P}_{(d+K)p}'\right\} \\
&\geq \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\}\lambda_{\min}\left\{\mathcal{P}_{(d+K)p}\mathcal{P}_{(d+K)p}'\right\} \\
&= \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\}\lambda_{\min}\left\{I_{(d+K)p}\right\} \\
&\quad \left(\text{since } \mathcal{P}_{(d+K)p} \text{ is an orthogonal matrix}\right) \\
&= \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\}
\end{aligned}$$

Next, note that

$$\begin{aligned}
\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right] &= \frac{1}{T_0}\sum_{t=p}^{T}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1} \\
&\quad + \frac{1}{T_0}\sum_{t=p}^{T}\sum_{j=0}^{\infty} A^j J_{d+K}' E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)' \\
&= \left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p} - A'\right)^{-1} \\
&\quad + \sum_{j=0}^{\infty} A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'
\end{aligned}$$

so that there exists a positive constant $\underline{C}$ such that

$$
\begin{aligned}
& \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\} \\
\geq\ & \lambda_{\min}\left\{\left(I_{(d+K)p}-A\right)^{-1} J_{d+K}'\mu\mu' J_{d+K}\left(I_{(d+K)p}-A'\right)^{-1}\right\} \\
& +\lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
& \text{(by Weyl's Theorem (see Theorem 4.3.1 of Horn and Johnson, 1985))} \\
\geq\ & \lambda_{\min}\left\{\sum_{j=0}^{\infty} A^j J_{d+K}'\frac{1}{T_0}\sum_{t=p}^{T} E\left[\varepsilon_{t-j}\varepsilon_{t-j}'\right] J_{d+K}\left(A^j\right)'\right\} \\
\geq\ & \underline{c}>0 \text{ for all } T>p-1 \ \text{(by the result given in part (a) of Lemma D-1)}
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
& \lambda_{\min}\left\{M_{FF}\right\} \\
=\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{F}_t\underline{F}_t'\right]\right\} \\
\geq\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T}\begin{pmatrix} E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right]\end{pmatrix}\right\} \\
& \text{(by the Poincaré separation theorem (see Corollary 4.3.16 of Horn and Johnson, 1985))} \\
\geq\ & \lambda_{\min}\left\{\frac{1}{T_0}\sum_{t=p}^{T} E\left[\underline{W}_t\underline{W}_t'\right]\right\} \\
\geq\ & \underline{C}>0 \text{ for all } T>p-1,
\end{aligned}
$$

as required. □

Lemma D-5: Let $T_h = T-h-p+1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumption 3-3 hold. Then,

(a)

$$\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| = O(1)$$

(b)

$$\frac{1}{T_h}\sum_{\substack{t,s,g=p\\t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|=O\left(1\right)$$

(c)

$$\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right|=O\left(1\right)$$

Proof of Lemma D-5:

To show part (a), first write

$$\frac{1}{T_h}\sum_{\substack{v,w=p\\v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right|=\frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^2\right]+\frac{1}{T_h}\sum_{\substack{v,w=p\\v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| \tag{76}$$

Consider now the first term on the right-hand side of expression (76). Note that, trivially, by Assumption 3-3(b),

$$\frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^2\right]\leq C=O\left(1\right) \tag{77}$$

For the second term on the right-hand side of expression (76), note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right)\leq a_1\exp\left\{-a_2m\right\}.$$

for every i. Since $\alpha_{i,m}\leq\beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m}\leq a_1\exp\left\{-a_2m\right\}\text{ for every }i.$$

Hence, in this case, we can apply Lemma C-3 with $p=6$ and $r=5/4$ to obtain

$$\begin{aligned}&\frac{1}{T_h}\sum_{\substack{v,w=p\\v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right|\\ \leq\ &\frac{1}{T_h}\sum_{\substack{v,w=p\\v<w}}^{T-h}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\end{aligned}$$

Application of Liapunov's inequality then gives us

$$\begin{aligned}
\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \\
&\leq \left(\sup_{t} E\left|u_{it}\right|^{6}\right)^{\frac{1}{3}} \\
&= C^{\frac{1}{3}} < \infty \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho = w - v$, so that $w = v + \varrho$. Using these notations and the boundedness of $\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| \\
& \frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{iw}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\ & \frac{C^{\frac{1}{3}}}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^{*}}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
& \left(\text{for some constant } C^{*} \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{3}}a_1^{\frac{1}{30}} \leq C^{*} < \infty\right) \\
\leq\ & \frac{C^{*}}{T_h}\sum_{v=p}^{T-h}\sum_{\varrho=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\ & C^{*}\sum_{\varrho_1=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho\right\} \\
=\ & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (78)
\end{aligned}$$

It follows from expressions (76), (77), and (78) that

$$\begin{aligned}
\frac{1}{T_h}\sum_{\substack{v,w=p \\ v\leq w}}^{T-h}\left|E\left[u_{iv}u_{iw}\right]\right| &= \frac{1}{T_h}\sum_{v=p}^{T-h}E\left[u_{iv}^{2}\right]+\frac{1}{T_h}\sum_{\substack{v,w=p \\ v<w}}^{T-h}\left|E\left[u_{ig}u_{ih}\right]\right| \\
&= O\left(1\right)+O\left(1\right) \\
&= O\left(1\right).
\end{aligned}$$

To show part (b), first write

$$\begin{aligned}\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| &= \frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3+\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s, s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\ &\quad +\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \qquad (79)\end{aligned}$$

For the first term on the right-hand side of expression (79) above, note that, trivially, we can apply Assumption 3-3(b) to obtain

$$\frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3\leq C=O\left(1\right). \qquad (80)$$

Next, note that, for the second term on the right-hand side of expression (79) above, we can apply Lemma C-3 with $p=6$ and $r=5/4$ to obtain

$$\begin{aligned}&\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s, s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\ \leq\ & \frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s, s-t>0}}^{T-h}2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(s-t\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\end{aligned}$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} &\leq \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{ig}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}} \\ &= \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{ig}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}} \\ &\leq \left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}} \\ &\qquad \text{(by Liapunov's inequality)} \\ &= C^{\frac{1}{2}}<\infty\ \text{ (by Assumption 3-3(b))}\end{aligned}$$

Moreover, let $\varrho_1=s-t$ and $\varrho_2=g-s$, so that $s=t+\varrho_1$ and $g=s+\varrho_2=t+\varrho_1+\varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ s-t>g-s, s-t>0}}^{T-h} \left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\; & \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ s-t>g-s, s-t>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(s-t\right)\right\}\right]^{1-\frac{1}{6}-\frac{4}{5}}\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}u_{ig}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \\
\leq\; & \frac{C^{\frac{1}{2}}}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ s-t>g-s, s-t>0}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(s-t\right)\right\}\right]^{\frac{1}{30}} \\
\leq\; & \frac{C^{*}}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ s-t>g-s, s-t>0}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
& \left(\text{for some constant } C^{*} \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{2}}a_1^{\frac{1}{30}} \leq C^{*} < \infty\right) \\
\leq\; & \frac{C^{*}}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \sum_{\varrho_2=0}^{\varrho_1-1} \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
\leq\; & \frac{C^{*}}{T_h} \sum_{t=p}^{T-h} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\; & C^{*} \sum_{\varrho_1=1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{30}\varrho_1\right\} \\
=\; & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (81)
\end{aligned}
$$

Similarly, for the third term on the right-hand side of expression (79), we can apply Lemma C-3 with $p = 6$ and $r = 5/4$, we have

$$
\begin{aligned}
& \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} \left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\; & \frac{1}{T_h} \sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}
\end{aligned}
$$

Next, applying Hölder's inequality, we have

$$\begin{aligned}
\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} &\leq \left(\left(E\left|u_{it}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{1}{2}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&= \left(E\left|u_{it}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{is}\right|^{\frac{5}{2}}\right)^{\frac{2}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&\leq \left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
&\quad \text{(by Liapunov's inequality)} \\
&= C^{\frac{1}{2}} < \infty \ \ \text{(by Assumption 3-3(b))}
\end{aligned}$$

Moreover, let $\varrho_1 = s - t$ and $\varrho_2 = g - s$, so that $s = t + \varrho_1$ and $g = s + \varrho_2 = t + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}$ as shown above, we can further write

$$\begin{aligned}
& \frac{1}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
\leq\ & \frac{1}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{1-\frac{1}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{4}{5}-\frac{1}{6}}\left(E\left|u_{it}u_{is}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}} \\
\leq\ & \frac{C^{\frac{1}{2}}}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} 2\left(2^{\frac{5}{6}}+1\right)\left[a_1\exp\left\{-a_2\left(g-s\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C^{*}}{T_h}\sum_{\substack{t,s,g=p \\ t\leq s\leq g \\ g-s\geq s-t,\ g-s>0}}^{T-h} \exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
& \left(\text{for some constant } C^{*} \text{ such that } 2\left(2^{\frac{5}{6}}+1\right)C^{\frac{1}{2}}a_1^{\frac{1}{30}} \leq C^{*} < \infty\right) \\
\leq\ & \frac{C^{*}}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & \frac{C^{*}}{T_h}\sum_{t=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)\exp\left\{-\frac{a_2}{30}\varrho_2\right\} \\
=\ & C^{*}\left[\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{30}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{30}\varrho_2\right\}\right] \\
=\ & O\left(1\right) \quad \text{(given Lemma C-1)} \qquad (82)
\end{aligned}$$

It follows from expressions (**??**), (**??**), (**??**), and (**??**) that

$$\begin{aligned}
\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| &= \frac{1}{T_h}\sum_{t=p}^{T-h}E\left|u_{it}\right|^3+\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ s-t>g-s,s-t>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
&\quad +\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g\\ g-s\geq s-t,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right| \\
&= O\left(1\right)+O\left(1\right)+O\left(1\right) \\
&= O\left(1\right).
\end{aligned}$$

Finally, to show part (c), we first write

$$\begin{aligned}
&\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right| \\
=\ &\frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{iv}\right)\right| \\
=\ &\frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)+E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)+E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
\leq\ &\frac{1}{T_h^2}\sum_{\substack{t,s=p\\ t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
&+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-g\leq g-s,\ g-s>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right|+\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\ t\leq s\leq g\leq v\\ v-s>0}}^{T-h}\left|E\left(u_{it}u_{is}\right)\right|\left|E\left(u_{ig}u_{iv}\right)\right| \qquad (83)
\end{aligned}$$

For the first term on the right-hand side of expression (83) above, note that, by Jensen's inequality, the Cauchy-Schwarz inequality, and Assumption 3-3(b); we have

$$\begin{aligned}
\frac{1}{T_h^2}\sum_{\substack{t,s=p\\t\leq s}}^{T-h}\left|E\left[u_{it}u_{is}^3\right]\right| &\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\t\leq s}}^{T-h}E\left[\left|u_{it}u_{is}^3\right|\right] \\
&\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\t\leq s}}^{T-h}\sqrt{E\left|u_{it}\right|^2}\sqrt{E\left|u_{is}\right|^6} \\
&\leq \frac{1}{T_h^2}\sum_{\substack{t,s=p\\t\leq s}}^{T-h}\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\sqrt{E\left|u_{is}\right|^6} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{C^{\frac{2}{3}}T_h^2}{T_h^2}\ \text{(by Assumption 3-3(b))} \\
&= O(1) \qquad (84)
\end{aligned}$$

Next, for the second term on the right-hand side of expression (83), we can apply Lemma C-3 with $p = 4/3$ and $r = 6$ to obtain

$$\begin{aligned}
&\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v\\v-g>g-s,\ v-g>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}u_{iv}\right]\right| \\
\leq\ &\frac{1}{T_h^2}\sum_{\substack{t,s,g,v=p\\t\leq s\leq g\leq v\\v-g>g-s,\ v-g>0}}^{T-h}\left\{2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1\exp\left\{-a_2\left(v-g\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}}\right. \\
&\qquad\left.\times\left(E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}}\left(E\left|u_{iv}\right|^6\right)^{\frac{1}{6}}\right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality,

$$\begin{aligned}
E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}\right|^{\frac{4}{3}} &\leq \left[E\left(u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right)^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{it}u_{is}\right|^{\frac{12}{7}}+\left|E\left[u_{it}u_{is}\right]\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq \left[2^{\frac{5}{7}}\left(E\left|u_{it}u_{is}\right|^{\frac{12}{7}}+E\left|u_{it}u_{is}\right|^{\frac{12}{7}}\right)\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\quad \text{(by Jensen's inequality)} \\
&= \left[2^{\frac{12}{7}}E\left|u_{it}u_{is}\right|^{\frac{12}{7}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{\frac{24}{7}}\right)^{\frac{1}{2}}\right]^{\frac{7}{9}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&= 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\left(E\left|u_{is}\right|^{\frac{24}{7}}\right)^{\frac{7}{24}}\right]^{\frac{4}{3}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{4}{3}}\left[E\left|u_{ig}\right|^{6}\right]^{\frac{2}{9}} \\
&\leq 2^{\frac{4}{3}}\left(C\right)^{\frac{2}{9}}\left(C\right)^{\frac{2}{9}}\left(C\right)^{\frac{2}{9}} \quad \text{(by Assumption 3-3(b))} \\
&= 2^{\frac{4}{3}}C^{\frac{2}{3}}
\end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s+ \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}u_{ig}\right|^{\frac{4}{3}}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\} u_{ig}u_{iv} \right] \right| \\
\leq\ & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \left\{ 2\left(2^{1-\frac{3}{4}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-g\right)\right\}\right]^{1-\frac{3}{4}-\frac{1}{6}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\} u_{ig}\right|^{\frac{4}{3}}\right)^{\frac{3}{4}} \left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{6}} \right\} \\
\leq\ & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g>g-s,\ v-g>0}}^{T-h} 2\left(2^{\frac{1}{4}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-g\right)\right\}\right]^{\frac{1}{12}} \left(2^{\frac{4}{3}}C^{\frac{2}{3}}\right)^{\frac{3}{4}} \left(C\right)^{\frac{1}{6}} \\
\leq\ & \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{4}}+1\right) C^{\frac{2}{3}} a_1^{\frac{1}{12}} \leq C^* < \infty\right) \\
\leq\ & \frac{C^*}{T_h^2} \sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2-1} \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
=\ & C^* \sum_{\rho_2=1}^{\infty} \varrho_2 \exp\left\{-\frac{a_2}{12}\varrho_2\right\} \\
=\ & O\left(1\right) \quad \text{(given Lemma C-1)}
\end{aligned} \tag{85}
$$

Similarly, for the third term on the right-hand side of expression (83) above, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\} u_{ig}u_{iv} \right] \right| \\
\leq\ & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\}\right|^{2}\right)^{\frac{1}{2}} \left(E\left|u_{ig}u_{iv}\right|^{3}\right)^{\frac{1}{3}} \right\}
\end{aligned}
$$

Next, applications of Hölder's inequality yield

$$\begin{aligned} E\left|u_{ig}u_{iv}\right|^{3} &\leq \left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{6}\right)^{\frac{1}{2}} \\ &\leq (C)^{\frac{1}{2}}(C)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\ &= C<\infty \end{aligned}$$

and

$$\begin{aligned} E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}\right|^{2} &\leq 2\left(E\left|u_{it}u_{is}\right|^{2}+\left|E\left[u_{it}u_{is}\right]\right|^{2}\right) \quad \text{(by Loève's } c_r \text{ inequality)} \\ &\leq 2\left(E\left|u_{it}u_{is}\right|^{2}+E\left|u_{it}u_{is}\right|^{2}\right) \quad \text{(by Jensen's inequality)} \\ &= 4E\left|u_{it}u_{is}\right|^{2} \\ &\leq 4\left[\left(E\left|u_{it}\right|^{4}\right)^{\frac{1}{4}}\left(E\left|u_{is}\right|^{4}\right)^{\frac{1}{4}}\right]^{2} \\ &\leq 4\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\right]^{2} \quad \text{(by Liapunov's inequality)} \\ &\leq 4\left(\sup_{t} E\left|u_{it}\right|^{6}\right)^{\frac{2}{3}} \\ &\leq 4(C)^{\frac{2}{3}}<\infty \quad \text{(by Assumption 3-3(b))} \end{aligned}$$

Moreover, let $\varrho_1 = g - s$ and $\varrho_2 = v - g$ so that $g = s+ \varrho_1$ and $v = g + \varrho_2 = s + \varrho_1 + \varrho_2$. Using these notations and the boundedness of $E\left|u_{ig}u_{iv}\right|^{3}$ and $E\left|\left\{u_{it}u_{is}-E\left(u_{it}u_{is}\right)\right\}\right|^{2}$ as shown above, we can further write

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left| E\left[\left\{ u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\} u_{ig}u_{iv} \right] \right| \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{1-\frac{1}{2}-\frac{1}{3}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{it}u_{is} - E\left(u_{it}u_{is}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{ig}u_{iv}\right|^3\right)^{\frac{1}{3}} \right\} \\
\leq & \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} 2\left(2^{\frac{1}{2}}+1\right)\left[a_1 \exp\left\{-a_2\left(g-s\right)\right\}\right]^{\frac{1}{6}} \left(4C^{\frac{2}{3}}\right)^{\frac{1}{2}} \left(C\right)^{\frac{1}{3}} \\
\leq & \frac{C^*}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
& \left(\text{for some constant } C^* \text{ such that } 4\left(2^{\frac{1}{2}}+1\right)C^{\frac{2}{3}}a_1^{\frac{1}{6}} \leq C^* < \infty\right) \\
\leq & \frac{C^*}{T_h^2} \sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{\varrho_1=1}^{\infty}\sum_{\varrho_2=0}^{\varrho_1} \exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & C^* \sum_{\rho_1=1}^{\infty} \left(\varrho_1+1\right)\exp\left\{-\frac{a_2}{6}\varrho_1\right\} \\
= & O\left(1\right) \quad \text{(given Lemma C-1)}
\end{aligned} \tag{86}
$$

Finally, consider the fourth term on the right-hand side of expression (83) above. For this term, we apply the result given in part (a) to obtain

$$
\begin{aligned}
\frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t\leq s\leq g\leq v \\ v-s>0}}^{T-h} \left|E\left(u_{it}u_{is}\right)\right|\left|E\left(u_{ig}u_{iv}\right)\right| & \leq \left(\frac{1}{T_h}\sum_{\substack{t,s=p \\ t\leq s}}^{T-h}\left|E\left(u_{it}u_{is}\right)\right|\right)\left(\frac{1}{T_h}\sum_{\substack{g,v=p \\ g\leq v}}^{T-h}\left|E\left(u_{ig}u_{iv}\right)\right|\right) \\
& = O\left(1\right).
\end{aligned} \tag{87}
$$

It follows from expressions (83)-(87) that

$$
\begin{aligned}
& \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v}}^{T-h} \left| E\left(u_{it} u_{is} u_{ig} u_{iv}\right)\right| \\
\leq \;& \frac{1}{T_h^2} \sum_{\substack{t,s=p \\ t \leq s}}^{T-h} \left| E\left[u_{it} u_{is}^3\right]\right| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g>g-s,\ v-g>0}}^{T-h} \left| E\left[\left\{u_{it} u_{is} - E\left(u_{it} u_{is}\right)\right\} u_{ig} u_{iv}\right]\right| \\
& + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-g\leq g-s,\ g-s>0}}^{T-h} \left| E\left[\left\{u_{it} u_{is} - E\left(u_{it} u_{is}\right)\right\} u_{ig} u_{iv}\right]\right| + \frac{1}{T_h^2} \sum_{\substack{t,s,g,v=p \\ t \leq s \leq g \leq v \\ v-s>0}}^{T-h} \left| E\left(u_{it} u_{is}\right)\right| \left| E\left(u_{ig} u_{iv}\right)\right| \\
= \;& O(1). \ \square
\end{aligned}
$$

Lemma D-6: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(b), 3-5, and 3-7 hold. Then, as $N_1, N_2, T \to \infty$,

$$
\max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h} \right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T} \right).
$$

Proof of Lemma D-6:

To proceed, we first show the boundedness of the quantity

$$
\frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E\left(\sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t} \right)^6
$$

Note first that there exist a constant $C_1 > 1$ such that

$$
\begin{aligned}
& \frac{1}{N_1} \sum_{k \in H^c} \sum_{i \in H} \frac{1}{N_2 T_h^3} E\left(\sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t} \right)^6 \\
\leq \;& \frac{C_1}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w}}^{T-h} \left\{ \left| E\left[u_{it} u_{is} u_{ig} u_{i\ell} u_{iv} u_{iw}\right]\right| \right. \\
& \qquad \left. \times \left| E\left[\left(\gamma_k' \underline{F}_t\right)\left(\gamma_k' \underline{F}_s\right)\left(\gamma_k' \underline{F}_g\right)\left(\gamma_k' \underline{F}_\ell\right)\left(\gamma_k' \underline{F}_v\right)\left(\gamma_k' \underline{F}_w\right)\right]\right| \right\}
\end{aligned}
$$

Next, note that, by repeated application of Hölder's inequality, we have by Assumption 3-5 and

Lemma C-4 that there exists a positive constant C such that

$$\begin{aligned}
& \left|E\left[\left(\gamma_k'\underline{F}_t\right)\left(\gamma_k'\underline{F}_s\right)\left(\gamma_k'\underline{F}_g\right)\left(\gamma_k'\underline{F}_\ell\right)\left(\gamma_k'\underline{F}_v\right)\left(\gamma_k'\underline{F}_w\right)\right]\right| \\
\leq\ & E\left[\left|\gamma_k'\underline{F}_t\right|\left|\gamma_k'\underline{F}_s\right|\left|\gamma_k'\underline{F}_g\right|\left|\gamma_k'\underline{F}_\ell\right|\left|\gamma_k'\underline{F}_v\right|\left|\gamma_k'\underline{F}_w\right|\right] \\
\leq\ & \|\gamma_k\|_2^6\, E\left[\|\underline{F}_t\|_2\,\|\underline{F}_s\|_2\,\left\|\underline{F}_g\right\|_2\,\|\underline{F}_\ell\|_2\,\|\underline{F}_v\|_2\,\|\underline{F}_w\|_2\right] \\
\leq\ & \|\gamma_k\|_2^6 \left(E\left[\|\underline{F}_t\|_2^2\,\|\underline{F}_s\|_2^2\,\left\|\underline{F}_g\right\|_2^2\right]\right)^{\frac{1}{2}} \left(E\left[\|\underline{F}_\ell\|_2^2\,\|\underline{F}_v\|_2^2\,\|\underline{F}_w\|_2^2\right]\right)^{\frac{1}{2}} \\
\leq\ & \|\gamma_k\|_2^6 \left(\left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{3}} \left(E\left[\|\underline{F}_s\|_2^3\,\left\|\underline{F}_g\right\|_2^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
& \times \left(\left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{3}} \left(E\left[\|\underline{F}_v\|_2^3\,\|\underline{F}_w\|_2^3\right]\right)^{\frac{2}{3}}\right)^{\frac{1}{2}} \\
\leq\ & \|\gamma_k\|_2^6 \left(\left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{3}} \left\{E\left[\|\underline{F}_s\|_2^6\right]\right\}^{\frac{1}{3}} \left\{E\left[\left\|\underline{F}_g\right\|_2^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
& \times \left(\left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{3}} \left\{E\left[\|\underline{F}_v\|_2^6\right]\right\}^{\frac{1}{3}} \left\{E\left[\|\underline{F}_w\|_2^6\right]\right\}^{\frac{1}{3}}\right)^{\frac{1}{2}} \\
\leq\ & \|\gamma_k\|_2^6 \left\{E\left[\|\underline{F}_t\|_2^6\right]\right\}^{\frac{1}{6}} \left\{E\left[\|\underline{F}_s\|_2^6\right]\right\}^{\frac{1}{6}} \left\{E\left[\left\|\underline{F}_g\right\|_2^6\right]\right\}^{\frac{1}{6}} \\
& \times \left\{E\left[\|\underline{F}_\ell\|_2^6\right]\right\}^{\frac{1}{6}} \left\{E\left[\|\underline{F}_v\|_2^6\right]\right\}^{\frac{1}{6}} \left\{E\left[\|\underline{F}_w\|_2^6\right]\right\}^{\frac{1}{6}} \\
\leq\ & \|\gamma_k\|_2^6 \sup_t E\left[\|\underline{F}_t\|_2^6\right] \\
\leq\ & C < \infty
\end{aligned}$$

Hence, we can write

$$
\begin{aligned}
& \frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_2 T_h^3}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^6 \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ w-v\geq \max\{v-\ell,\ell-g\},w-v>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ w-v\geq \max\{v-\ell,\ell-g\},w-v>0}}^{T-h}\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left[\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h} \left|E\left[\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\} u_{i\ell}u_{iv}u_{iw}\right]\right| \\
& +\frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ \ell-g\geq \max\{w-v,v-\ell\},\ell-g>0}}^{T-h} \left|E\left(u_{it}u_{is}u_{ig}\right)\right| \left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right| \\
= \; & \mathcal{TT}_1 + \mathcal{TT}_2 + \mathcal{TT}_3 + \mathcal{TT}_4 + \mathcal{TT}_5 + \mathcal{TT}_6, \;\; (say).
\end{aligned}
$$

Consider first $\mathcal{TT}_1$. Note that

$$
\begin{aligned}
\mathcal{TT}_1 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \left|E\left[u_{it}u_{is}u_{ig}^4\right]\right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} E\left[\left|u_{it}u_{is}u_{ig}^4\right|\right] \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \left(E\left[\left|u_{it}u_{is}\right|^3\right]\right)^{\frac{1}{3}} \left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \text{(by Hölder's inequality)} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \left(\left[E\left\{\left|u_{it}\right|^6\right\}\right]^{\frac{1}{2}} \left[E\left\{\left|u_{is}\right|^6\right\}\right]^{\frac{1}{2}}\right)^{\frac{1}{3}} \left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\quad \text{(by further application of Hölder's inequality)} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \left(E\left\{\left|u_{it}\right|^6\right\}\right)^{\frac{1}{6}} \left(E\left\{\left|u_{is}\right|^6\right\}\right)^{\frac{1}{6}} \left(E\left[\left|u_{ig}\right|^6\right]\right)^{\frac{2}{3}} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \left(\sup_t E\left\{\left|u_{it}\right|^7\right\}\right)^{\frac{6}{7}} \\
&\quad \text{(using Liapunov's inequality)} \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g=p \\ t\leq s\leq g}}^{T-h} \overline{C}^{\frac{6}{7}} \;\; \text{(by Assumption 3-3(b))} \\
&\leq C_1 C \overline{C}^{\frac{6}{7}} \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \\
&= C_1 C \overline{C}^{\frac{6}{7}} = O\left(1\right) \qquad (88)
\end{aligned}
$$

Next, consider $\mathcal{TT}_2$. For this term, note first that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, in this case, we can apply Lemma C-3 with $p = 5/4$ and $r = 6$ to obtain

$$\begin{aligned}
& \mathcal{TT}_2 \\
= \; & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} |E[u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}u_{iw}]| \\
\leq \; & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{1-\frac{4}{5}}+1\right)\left[a_1 \exp\{-a_2(w-v)\}\right]^{1-\frac{4}{5}-\frac{1}{6}} \right. \\
& \left. \times \left(E|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E|u_{iw}|^6\right)^{\frac{1}{6}} \right\} \\
= \; & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\{-a_2(w-v)\}\right]^{\frac{1}{30}} \right. \\
& \left. \times \left(E|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E|u_{iw}|^6\right)^{\frac{1}{6}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we have

$$
\begin{aligned}
& E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}} \\
\leq\ & \left[E\left|u_{it}u_{is}u_{ig}\right|^{\frac{25}{12}}\right]^{\frac{3}{5}}\left[E\left|u_{i\ell}u_{iv}\right|^{\frac{25}{8}}\right]^{\frac{2}{5}} \\
\leq\ & \left[\left(E\left|u_{it}u_{is}\right|^{\frac{150}{47}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{i\ell}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq\ & \left[\left(\sqrt{E\left|u_{it}\right|^{\frac{300}{47}}}\sqrt{E\left|u_{is}\right|^{\frac{300}{47}}}\right)^{\frac{47}{72}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{25}{72}}\right]^{\frac{3}{5}}\left[\left(E\left|u_{i\ell}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{2}}\right]^{\frac{2}{5}} \\
\leq\ & \left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{141}{720}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{15}{72}}\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}}\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{1}{5}} \\
=\ & \left[\left(E\left|u_{it}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\left(E\left|u_{is}\right|^{\frac{300}{47}}\right)^{\frac{47}{300}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{\frac{25}{4}}\right)^{\frac{4}{25}}\right]^{\frac{5}{4}} \\
\leq\ & \left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{i\ell}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}}\left[\left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{5}{4}} \\
& \text{(by Liapunov's inequality)} \\
\leq\ & \left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}}\left(\overline{C}\right)^{\frac{5}{28}} \quad \text{(by Assumption 3-3(b))} \\
=\ & \overline{C}^{\frac{25}{28}}
\end{aligned}
$$

By Liapunov's inequality and Assumption 3-3(b), we also obtain

$$\left(E\left|u_{iw}\right|^{6}\right)^{\frac{1}{6}} \leq \left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{7}} \leq \overline{C}^{\frac{1}{7}}.$$

Moreover, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$.. Using these notations and the boundedness of $E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}}$

as shown above, we can further write

$$
\begin{aligned}
& \mathcal{TT}_2 \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \right. \\
& \qquad \left. \times \left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}u_{iv}\right|^{\frac{5}{4}}\right)^{\frac{4}{5}} \left(E\left|u_{iw}\right|^6\right)^{\frac{1}{6}} \right\} \\
\leq\ & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \left(\overline{C}^{\frac{25}{28}}\right)^{\frac{4}{5}} \overline{C}^{\frac{1}{7}} \\
\leq\ & \frac{C_1 C \overline{C}^{\frac{6}{7}}}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} 2\left(2^{\frac{1}{5}}+1\right)\left[a_1 \exp\left\{-a_2\left(w-v\right)\right\}\right]^{\frac{1}{30}} \\
\leq\ & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ w-v \geq \max\{v-\ell,\ell-g\}, w-v>0}}^{T-h} \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
& \left(\text{for some constant } C_1^* \text{ such that } 2\left(2^{\frac{1}{5}}+1\right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{30}} \leq C_1^* < \infty\right) \\
\leq\ & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} \sum_{\rho_1=0}^{\rho_3} \sum_{\rho_2=0}^{\rho_3} \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
\leq\ & \frac{C_1^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\rho_3=1}^{\infty} \left(\rho_3+1\right)^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} \\
=\ & C_1^* \frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \left[\sum_{\rho_3=1}^{\infty} \rho_{3\backslash}^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} + 2\sum_{\rho_3=1}^{\infty} \rho_{3\backslash} \exp\left\{-\frac{a_2}{30}\rho_3\right\} + \sum_{\rho_3=1}^{\infty} \exp\left\{-\frac{a_2}{30}\rho_3\right\}\right] \\
\leq\ & C_1^* \overline{\overline{C}}_1 \qquad (89)
\end{aligned}
$$

for some positive constant

$$
\overline{\overline{C}}_1 \geq \sum_{\rho_3=1}^{\infty} \rho_{3\backslash}^2 \exp\left\{-\frac{a_2}{30}\rho_3\right\} + 2\sum_{\rho_3=1}^{\infty} \rho_{3\backslash} \exp\left\{-\frac{a_2}{30}\rho_3\right\} + \sum_{\rho_3=1}^{\infty} \exp\left\{-\frac{a_2}{30}\rho_3\right\}.
$$

which exists in light of Lemma C-1.

Now, consider $\mathcal{TT}_3$. Here, we apply Lemma C-3 with $p = 3/2$ and $r = 7/2$ to obtain

$$\begin{aligned}
\mathcal{TT}_3 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left| E\left[\left\{u_{it}u_{is}u_{ig}u_{i\ell} - E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\} u_{iv}u_{iw}\right]\right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left\{ 2\left(2^{1-\frac{2}{3}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{1-\frac{2}{3}-\frac{2}{7}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell} - E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}} \right\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ v-\ell \geq \max\{w-v,\ell-g\}, v-\ell>0}}^{T-h} \left\{ 2\left(2^{\frac{1}{3}}+1\right)\left[a_1 \exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{\frac{1}{21}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell} - E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}} \right\}
\end{aligned}$$

Next, observe that by applying of Hölder's inequality, we have

$$\begin{aligned}
E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}} &\leq \left(E\left|u_{iv}\right|^{7}\right)^{\frac{1}{2}} \left(E\left|u_{iw}\right|^{7}\right)^{\frac{1}{2}} \\
&\leq \left(\overline{C}\right)^{\frac{1}{2}} \left(\overline{C}\right)^{\frac{1}{2}} \quad \text{(by Assumption 3-3(b))} \\
&= \overline{C} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}} &\leq 2^{\frac{1}{2}}\left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}+\left|E\left[u_{it}u_{is}u_{ig}u_{i\ell}\right]\right|^{\frac{3}{2}}\right) \\
&\quad \text{(by Loève's } c_r \text{ inequality)} \\
&\leq 2^{\frac{1}{2}}\left(E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}+E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}}\right) \\
&\quad \text{(by Jensen's inequality)} \\
&\leq 2^{\frac{3}{2}}E\left|u_{it}u_{is}u_{ig}u_{i\ell}\right|^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}}\left(E\left|u_{it}u_{is}\right|^{3}\right)^{\frac{1}{2}}\left(E\left|u_{ig}u_{i\ell}\right|^{3}\right)^{\frac{1}{2}} \\
&\leq 2^{\frac{3}{2}}\left(\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\left(\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{2}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{is}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^{6}\right)^{\frac{1}{6}}\left(E\left|u_{i\ell}\right|^{6}\right)^{\frac{1}{6}}\right]^{\frac{3}{2}} \\
&\leq 2^{\frac{3}{2}}\left[\left(E\left|u_{it}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{is}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^{7}\right)^{\frac{1}{7}}\left(E\left|u_{i\ell}\right|^{7}\right)^{\frac{1}{7}}\right]^{\frac{3}{2}} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq 2^{\frac{3}{2}}\left[\left(\sup_{t}E\left|u_{it}\right|^{7}\right)^{\frac{4}{7}}\right]^{\frac{3}{2}} \\
&= 2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

Again, let $\rho_1 = \ell - g$, $\rho_2 = v - \ell$, and $\rho_3 = w - v$, so that $\ell = g + \rho_1$, $v = \ell + \rho_2 = g + \rho_1 + \rho_2$, $w = v + \rho_3 = g + \rho_1 + \rho_2 + \rho_3$. Using these notations and the boundedness of $E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}$ as shown above, we can further write

$$\begin{aligned}
&\mathcal{TT}_3 \\
= \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h} \left|E\left[\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}u_{iv}u_{iw}\right]\right| \\
\leq \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h} \left\{2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{\frac{1}{21}}\right. \\
&\left. \times\left(E\left|\left\{u_{it}u_{is}u_{ig}u_{i\ell}-E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right\}\right|^{\frac{3}{2}}\right)^{\frac{2}{3}}\left(E\left|u_{iv}u_{iw}\right|^{\frac{7}{2}}\right)^{\frac{2}{7}}\right\} \\
\leq \;& \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h} 2\left(2^{\frac{1}{3}}+1\right)\left[a_1\exp\left\{-a_2\left(v-\ell\right)\right\}\right]^{\frac{1}{21}}\left(2^{\frac{3}{2}}\overline{C}^{\frac{6}{7}}\right)^{\frac{2}{3}}\left(\overline{C}\right)^{\frac{2}{7}} \\
\leq \;& \frac{C_2^*}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t\leq s\leq g\leq \ell\leq v\leq w \\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h} \exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
&\left(\text{for some constant } C_2^* \text{ such that } 4\left(2^{\frac{1}{3}}+1\right)C_1 C\overline{C}^{\frac{6}{7}}a_1^{\frac{1}{21}}\leq C_2^*<\infty\right) \\
\leq \;& \frac{C_2^*}{N_1 N_2 T_h^3} \sum_{k\in H^c}\sum_{i\in H}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}\sum_{g=p}^{T-h}\sum_{\varrho_2=1}^{\infty}\sum_{\varrho_1=0}^{\varrho_2}\sum_{\varrho_3=0}^{\varrho_2}\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
= \;& C_2^*\frac{N_1 N_2 T_h^3}{N_1 N_2 T_h^3}\sum_{\varrho_2=1}^{\infty}\left(\varrho_2+1\right)^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\} \\
= \;& C_2^*\left[\sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}\right] \\
\leq \;& C_2^*\overline{\overline{C}}_2 \qquad (90)
\end{aligned}$$

for some positive constant

$$\overline{\overline{C}}_2 \geq \sum_{\varrho_2=1}^{\infty}\varrho_2^2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+2\sum_{\varrho_2=1}^{\infty}\varrho_2\exp\left\{-\frac{a_2}{21}\varrho_2\right\}+\sum_{\varrho_2=1}^{\infty}\exp\left\{-\frac{a_2}{21}\varrho_2\right\}$$

which exists in light of Lemma C-1.

Turning our attention to the term $\mathcal{TT}_4$, note that, from the upper bounds given in the proofs of parts (a) and (c) of Lemma D-5, it is clear that there exists a positive constant C^{**} such that, for all i and for all T sufficiently large,

$$\frac{1}{T_h}\sum_{\substack{v,w=p\\ v\leq w}}^{T-h}\left|E\left(u_{iv}u_{iw}\right)\right|\leq C_1^{**}$$

and

$$\frac{1}{T_h^2}\sum_{\substack{t,s,g,\ell=p\\ t\leq s\leq g\leq \ell}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\leq C_1^{**}$$

from which it follows that

$$\begin{aligned}
\mathcal{TT}_4 &= \frac{C_1 C}{N_1 N_2 T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ v-\ell\geq \max\{w-v,\ell-g\},v-\ell>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\left|E\left(u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1 C}{N_1 N_2}\sum_{k\in H^c}\sum_{i\in H}\left(\frac{1}{T_h^2}\sum_{\substack{t,s,g,\ell=p\\ t\leq s\leq g\leq \ell}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}u_{i\ell}\right)\right|\right)\left(\frac{1}{T_h}\sum_{\substack{v,w=p\\ v\leq w}}^{T-h}\left|E\left(u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1 C}{N_1 N_2}\sum_{k\in H^c}\sum_{i\in H}\left(C_1^{**}\right)^2 \\
&= C_1 C\left(C_1^{**}\right)^2\frac{N_1 N_2}{N_1 N_2} \\
&= C_1 C\left(C_1^{**}\right)^2 \qquad (91)
\end{aligned}$$

Consider now $\mathcal{TT}_5$. In this case, we apply Lemma C-3 with $p = 2$ and $r = 9/4$ to obtain

$$\begin{aligned}
\mathcal{TT}_5 &= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left| E\left[\left\{u_{it} u_{is} u_{ig} - E\left(u_{it} u_{is} u_{ig}\right)\right\} u_{i\ell} u_{iv} u_{iw}\right]\right| \\
&\leq \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left\{ 2\left(2^{1-\frac{1}{2}} + 1\right) \left[a_1 \exp\left\{-a_2 (\ell - g)\right\}\right]^{1-\frac{1}{2}-\frac{4}{9}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it} u_{is} u_{ig} - E\left(u_{it} u_{is} u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{i\ell} u_{iv} u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}} \right\} \\
&= \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left\{ 2\left(2^{\frac{1}{2}} + 1\right) \left[a_1 \exp\left\{-a_2 (\ell - g)\right\}\right]^{\frac{1}{18}} \right. \\
&\qquad \left. \times \left(E\left|\left\{u_{it} u_{is} u_{ig} - E\left(u_{it} u_{is} u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{i\ell} u_{iv} u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}} \right\}
\end{aligned}$$

Next, by repeated application of Hölder's inequality, we obtain

$$\begin{aligned}
& E\left|u_{i\ell} u_{iv} u_{iw}\right|^{\frac{9}{4}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[E\left|u_{iv} u_{iw}\right|^{\frac{63}{19}}\right]^{\frac{19}{28}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{1}{2}}\right]^{\frac{19}{28}} \\
=\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{56}} \\
=\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}} \left(E\left|u_{iw}\right|^{\frac{126}{19}}\right)^{\frac{19}{126}}\right]^{\frac{9}{4}} \\
\leq\ & \left[E\left|u_{i\ell}\right|^7\right]^{\frac{9}{28}} \left[\left(E\left|u_{iv}\right|^7\right)^{\frac{1}{7}} \left(E\left|u_{iw}\right|^7\right)^{\frac{1}{7}}\right]^{\frac{9}{4}} \quad \text{(by Liapunov's inequality)} \\
\leq\ & \left(\sup_t E\left|u_{it}\right|^7\right)^{\frac{27}{28}} \\
\leq\ & \overline{C}^{\frac{27}{28}} \quad \text{(by Assumption 3-3(b))}
\end{aligned}$$

and

$$
\begin{aligned}
E\left|\{u_{it}u_{is}u_{ig}-E\left(u_{it}u_{is}u_{ig}\right)\}\right|^2 &\leq 2\left(E\left|u_{it}u_{is}u_{ig}\right|^2+\left|E\left[u_{it}u_{is}u_{ig}\right]\right|^2\right)\\
&\quad \text{(by Loève's } c_r \text{ inequality)}\\
&\leq 2\left(E\left|u_{it}u_{is}u_{ig}\right|^2+E\left|u_{it}u_{is}u_{ig}\right|^2\right)\\
&\quad \text{(by Jensen's inequality)}\\
&\leq 4E\left|u_{it}u_{is}u_{ig}\right|^2\\
&\leq 4\left(E\left|u_{it}\right|^6\right)^{\frac{1}{3}}\left(E\left|u_{is}u_{ig}\right|^3\right)^{\frac{2}{3}}\\
&\leq 4\left(E\left|u_{it}\right|^6\right)^{\frac{1}{3}}\left(\sqrt{E\left|u_{is}\right|^6}\sqrt{E\left|u_{ig}\right|^6}\right)^{\frac{2}{3}}\\
&= 4\left[\left(E\left|u_{it}\right|^6\right)^{\frac{1}{6}}\right]^2\left[\left(E\left|u_{is}\right|^6\right)^{\frac{1}{6}}\left(E\left|u_{ig}\right|^6\right)^{\frac{1}{6}}\right]^2\\
&\leq 4\left[\left(E\left|u_{it}\right|^7\right)^{\frac{1}{7}}\right]^2\left[\left(E\left|u_{is}\right|^7\right)^{\frac{1}{7}}\left(E\left|u_{ig}\right|^7\right)^{\frac{1}{7}}\right]^2\\
&\quad \text{(by Liapunov's inequality)}\\
&\leq 4\left[\left(\sup_t E\left|u_{it}\right|^7\right)^{\frac{1}{7}}\right]^6\\
&\leq 4\overline{C}^{\frac{6}{7}} \ \text{(by Assumption 3-3(b))}
\end{aligned}
$$

Define again $\rho_1=\ell-g$, $\rho_2=v-\ell$, and $\rho_3=w-v$, so that $\ell=g+\rho_1$, $v=\ell+\rho_2=g+\rho_1+\rho_2$, $w=v+\rho_3=g+\rho_1+\rho_2+\rho_3$. Using these notations and the boundedness of $E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}$ and

$E\left|\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2$ as shown above, we can further write

$$
\begin{aligned}
& \mathcal{TT}_5 \\
\leq \quad & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \left\{ 2\left(2^{\frac{1}{2}} + 1\right)\left[a_1 \exp\left\{-a_2\left(\ell - g\right)\right\}\right]^{\frac{1}{18}} \right. \\
& \qquad \left. \times \left(E\left|\left\{u_{it}u_{is}u_{ig} - E\left(u_{it}u_{is}u_{ig}\right)\right\}\right|^2\right)^{\frac{1}{2}} \left(E\left|u_{i\ell}u_{iv}u_{iw}\right|^{\frac{9}{4}}\right)^{\frac{4}{9}} \right\} \\
\leq \quad & \frac{C_1 C}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} 2\left(2^{\frac{1}{2}} + 1\right)\left[a_1 \exp\left\{-a_2\left(\ell - g\right)\right\}\right]^{\frac{1}{18}} \left(4\overline{C}^{\frac{6}{7}}\right)^{\frac{1}{2}} \left(\overline{C}^{\frac{27}{28}}\right)^{\frac{4}{9}} \\
\leq \quad & \frac{C_3^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{\substack{t,s,g,\ell,v,w=p \\ t \leq s \leq g \leq \ell \leq v \leq w \\ \ell - g \geq \max\{w-v, v-\ell\}, \ell - g > 0}}^{T-h} \exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
& \left(\text{for some constant } C_3^* \text{ such that } 4\left(2^{\frac{1}{2}} + 1\right) C_1 C \overline{C}^{\frac{6}{7}} a_1^{\frac{1}{18}} \leq C_3^* < \infty\right) \\
\leq \quad & \frac{C_3^*}{N_1 N_2 T_h^3} \sum_{k \in H^c} \sum_{i \in H} \sum_{t=p}^{T-h} \sum_{s=p}^{T-h} \sum_{g=p}^{T-h} \sum_{\varrho_1 = 1}^{\infty} \sum_{\varrho_2 = 0}^{\varrho_1} \sum_{\varrho_3 = 0}^{\varrho_1} \exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq \quad & \frac{C_3^* N_1 N_2 T_h^3}{N_1 N_2 T_h^3} \sum_{\varrho_1 = 1}^{\infty} \left(\varrho_1 + 1\right)^2 \exp\left\{-\frac{a_2}{18}\varrho_1\right\} \\
\leq \quad & C_3^* \left[\sum_{\varrho_1 = 1}^{\infty} \varrho_1^2 \exp\left\{-\frac{a_2}{18}\varrho_1\right\} + 2\sum_{\varrho_1 = 1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{18}\varrho_1\right\} + \sum_{\varrho_1 = 1}^{\infty} \exp\left\{-\frac{a_2}{18}\varrho_1\right\} \right] \\
\leq \quad & C_3^* \overline{\overline{C}}_3 \qquad (92)
\end{aligned}
$$

for some positive constant

$$
\overline{\overline{C}}_3 \geq \sum_{\varrho_1 = 1}^{\infty} \varrho_1^2 \exp\left\{-\frac{a_2}{18}\varrho_1\right\} + 2\sum_{\varrho_1 = 1}^{\infty} \varrho_1 \exp\left\{-\frac{a_2}{18}\varrho_1\right\} + \sum_{\varrho_1 = 1}^{\infty} \exp\left\{-\frac{a_2}{18}\varrho_1\right\}
$$

which exists in light of Lemma C-1.

Finally, consider $\mathcal{TT}_6$. Note that, from the upper bounds given in the proofs of part (b) of Lemma D-5, it is clear that there exists a positive constant C_2^{**} such that, for all i and for all T

sufficiently large,

$$\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\leq C_2^{**}$$

and

$$\frac{1}{T_h}\sum_{\substack{\ell,v,w=p\\ \ell\leq v\leq w}}^{T-h}\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right|\leq C_2^{**}$$

from which it follows that

$$\begin{aligned}
\mathcal{TT}_6 &= \frac{C_1C}{N_1N_2T_h^3}\sum_{k\in H^c}\sum_{i\in H}\sum_{\substack{t,s,g,\ell,v,w=p\\ t\leq s\leq g\leq \ell\leq v\leq w\\ \ell-g\geq\max\{w-v,v-\ell\},\ell-g>0}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right| \\
&\leq \frac{C_1C}{N_1N_2T_h}\sum_{k\in H^c}\sum_{i\in H}\left(\frac{1}{T_h}\sum_{\substack{t,s,g=p\\ t\leq s\leq g}}^{T-h}\left|E\left(u_{it}u_{is}u_{ig}\right)\right|\right)\left(\frac{1}{T_h}\sum_{\substack{\ell,v,w=p\\ \ell\leq v\leq w}}^{T-h}\left|E\left(u_{i\ell}u_{iv}u_{iw}\right)\right|\right) \\
&\leq \frac{C_1C}{N_1N_2T_h}\sum_{k\in H^c}\sum_{i\in H}\left(C^{**}\right)^2 \\
&= C_1C\left(C_2^{**}\right)^2\frac{N_1N_2}{N_1N_2T_h} \\
&= \frac{C_1C\left(C_2^{**}\right)^2}{T_h}=O\left(\frac{1}{T}\right).
\end{aligned} \tag{93}$$

It follows from expressions (88)-(93) that, for all N_1, N_2, and T sufficiently large,

$$\begin{aligned}
&\frac{1}{N_1}\sum_{k\in H^c}\sum_{i\in H}\frac{1}{N_1^3T_h^3}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_tu_{i,t}\right)^6 \\
&\leq \mathcal{TT}_1+\mathcal{TT}_2+\mathcal{TT}_3+\mathcal{TT}_4+\mathcal{TT}_5+\mathcal{TT}_6 \\
&\leq C_1C\overline{C}^{\frac{6}{7}}+C_1^*\overline{\overline{C}}_1+C_2^*\overline{\overline{C}}_2+C_1C\left(C_1^{**}\right)^2+C_3^*\overline{\overline{C}}_3+\frac{C_1C\left(C_2^{**}\right)^2}{T_h} \\
&\leq \widetilde{C}
\end{aligned}$$

for some positive constant $\widetilde{C}$ such that

$$\widetilde{C}\geq C_1C\overline{C}^{\frac{6}{7}}+C_1^*\overline{\overline{C}}_1+C_2^*\overline{\overline{C}}_2+C_1C\left(C_1^{**}\right)^2+C_3^*\overline{\overline{C}}_3+\frac{C_1C\left(C_2^{**}\right)^2}{T_h}.$$

Hence, for any $\epsilon > 0$, set $C_\epsilon = \left(\widetilde{C}/\epsilon\right)^{\frac{1}{3}}$, and note that

$$\begin{aligned}
& \Pr\left\{\frac{N_1 T_h}{N_2^{\frac{1}{3}}} \max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 \geq C_\epsilon\right\} \\
= \;& \Pr\left\{\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^2 \geq C_\epsilon\right\} \\
= \;& \Pr\left\{\max_{i\in H} \left[\frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^2\right]^3 \geq C_\epsilon^3\right\} \\
\leq \;& \Pr\left\{\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \geq C_\epsilon^3\right\} \text{ (by Jensen's inequality)} \\
\leq \;& \Pr\left\{\frac{1}{N_1} \sum_{k\in H^c} \sum_{i\in H} \left(\frac{1}{N_2^{\frac{1}{6}}\sqrt{T_h}} \sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \geq C_\epsilon^3\right\} \\
\leq \;& \frac{\epsilon}{\overline{C}} \frac{1}{N_1} \sum_{k\in H^c} \sum_{i\in H} \frac{1}{N_2 T_h^3} E\left(\sum_{t=p}^{T-h} \gamma_k' \underline{F}_t u_{i,t}\right)^6 \\
\leq \;& \frac{\epsilon}{\widetilde{C}} \widetilde{C} \\
= \;& \epsilon
\end{aligned}$$

This shows that

$$\max_{i\in H} \frac{1}{N_1} \sum_{k\in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T_h}\right) = O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T}\right). \;\square$$

Before stating the next lemma, we first introduce some more notations. Let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the statistic $\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, and define

$$\widehat{H^c} = \left\{i \in \{1,....,N\} : \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}, \tag{94}$$

$$\widehat{H} = \left\{i \in \{1,....,N\} : \mathbb{S}_{i,T}^{+} < \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}, \tag{95}$$

$$\widehat{N}_1 = \#\left(\widehat{H^c}\right), \text{ i.e., the cardinality of the set } \widehat{H^c}, \tag{96}$$

$$\Gamma\left(\widehat{H^c}\right) = \begin{pmatrix} \gamma_1\left(\widehat{H^c}\right)' \\ \gamma_2\left(\widehat{H^c}\right)' \\ \vdots \\ \gamma_N\left(\widehat{H^c}\right)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\left\{1\in\widehat{H^c}\right\}\gamma_1' \\ \mathbb{I}\left\{2\in\widehat{H^c}\right\}\gamma_2' \\ \vdots \\ \mathbb{I}\left\{N\in\widehat{H^c}\right\}\gamma_N' \end{pmatrix}, \text{ and}$$

$$U\left(\widehat{H^c}\right) = \begin{pmatrix} u_{1\cdot}\left(\widehat{H^c}\right)' \\ u_{2\cdot}\left(\widehat{H^c}\right)' \\ \vdots \\ u_{N\cdot}\left(\widehat{H^c}\right)' \end{pmatrix} = \begin{pmatrix} \mathbb{I}\left\{1\in\widehat{H^c}\right\}u_{1\cdot}' \\ \mathbb{I}\left\{2\in\widehat{H^c}\right\}u_{2\cdot}' \\ \vdots \\ \mathbb{I}\left\{N\in\widehat{H^c}\right\}u_{N\cdot}' \end{pmatrix}, \tag{97}$$

where $u_{i\cdot} = (u_{i,p}, u_{i,p+1}, ..., u_{i,T-h})'$ for $i = 1, ..., N$.

Lemma D-7: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11 hold. Then, as $N_1, N_2, T \to \infty$, the following statements are true.

(a)

$$\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\} = O_p(\varphi)$$

(b)

$$\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 = O_p\left(\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}\right).$$

(c)

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 = O_p\left(\frac{1}{T}\right)$$

Proof of Lemma D-7:

To show part (a), let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or the statistic $\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$. Following arguments similar to that given in the proof of part (a) of Theorem 1 (see Chao, Qiu, and and Swanson (2023b)), we see that there exists a constant $C > 2d$ such that

$$\begin{aligned}
\sum_{i \in H} E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] &= \sum_{i \in H} \Pr\left(i \in \widehat{H^c}\right) \\
&= \sum_{i \in H} \Pr\left\{\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\} \\
&\leq C\frac{N_2 \varphi}{N} \\
&\leq C\varphi
\end{aligned}$$

for all N_1, N_2, and T sufficiently large. Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}
\Pr\left\{\frac{1}{\varphi}\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \geq C_\epsilon\right\} &\leq \frac{1}{C_\epsilon \varphi}\sum_{i \in H} E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad \text{(by Markov's inequality)} \\
&\leq \frac{\epsilon}{C\varphi} C\varphi \\
&= \epsilon
\end{aligned}$$

which shows that

$$\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} = O_p\left(\varphi\right)$$

Next, to show part (b), we combine the result given in part (a) of this lemma with the result of Lemma D-6 to obtain

$$\begin{aligned}
&\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 \\
\leq\ & \max_{i \in H} \frac{1}{N_1} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 \left[\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad \text{(by Hölder's inequality)} \\
=\ & O_p\left(\frac{N_2^{\frac{1}{3}}}{N_1 T}\right) O_p\left(\varphi\right) \\
=\ & O_p\left(\frac{N_2^{\frac{1}{3}} \varphi}{N_1 T}\right).
\end{aligned}$$

Finally, to show part (c), note first that

$$\frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{\sqrt{N_1} T_h}\right)^2 \leq \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \left(\frac{\gamma_k' \underline{F}' u_{i\cdot}}{T_h}\right)^2$$

Moreover, write

$$\begin{aligned}
0 &\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2 \\
&= \frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2 \\
&= \frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}E\left\{\gamma_k'\underline{F}_s u_{i,s}u_{i,t}\underline{F}_t'\gamma_k\right\} \\
&= \frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k'E_F\left[\underline{F}_tE\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad +\frac{2}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E_F\left[\gamma_k'\underline{F}_tE\left(u_{i,t}u_{i,t+m}\right)\underline{F}_{t+m}'\gamma_k\right] \\
&= \frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k'E_F\left[\underline{F}_tE\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad +\frac{2}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left(u_{i,t}u_{i,t+m}\right)E_F\left[\gamma_k'\underline{F}_t\underline{F}_{t+m}'\gamma_k\right] \\
&\leq \frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k'E_F\left[\underline{F}_tE\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k \\
&\quad +\frac{2}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left(u_{i,t}u_{i,t+m}\right)\right|\left|\gamma_k'E_F\left[\underline{F}_t\underline{F}_{t+m}'\right]\gamma_k\right|
\end{aligned}$$

Note that by Assumption 3-3(c), $\{u_{i,t}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i\left(m\right)\leq a_1\exp\left\{-a_2m\right\}$$

for every i. Since $\alpha_{i,m}\leq\beta_i\left(m\right)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m}\leq a_1\exp\left\{-a_2m\right\}\text{ for every } i.$$

Hence, applying Lemma C-3 with $p=3$ and $r=3$ as well as Assumptions 3-3(b) and 3-5 and

Lemma C-4; we get

$$\begin{aligned}
&\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left[\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2\right]\\
\leq\;&\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k'E_F\left[\underline{F}_tE\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k\\
&+\frac{2}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left(u_{i,t}u_{i,t+m}\right)\right|\left|\gamma_k'E_F\left[\underline{F}_t\underline{F}_{t+m}'\right]\gamma_k\right|\\
\leq\;&\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}\gamma_k'E_F\left[\underline{F}_tE\left(u_{i,t}^2\right)\underline{F}_t'\right]\gamma_k\\
&+\frac{2}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left(u_{i,t}u_{i,t+m}\right)\right|E\left|\gamma_k'\underline{F}_t\underline{F}_{t+m}'\gamma_k\right|\\
\leq\;&\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}E\left(u_{i,t}^2\right)\left\|\gamma_k\right\|_2^2E\left[\left\|\underline{F}_t\right\|_2^2\right]\\
&+\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}2\left(2^{\frac{2}{3}}+1\right)2\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left\{\alpha_m^{\frac{1}{3}}\left(E\left|u_{i,t}\right|^3\right)^{\frac{1}{3}}\left(E\left|u_{i,t+m}\right|^3\right)^{\frac{1}{3}}\right.\\
&\qquad\left.\times\sqrt{\gamma_k'E\left[\underline{F}_t\underline{F}_t'\right]\gamma_k}\sqrt{\gamma_k'E\left[\underline{F}_{t+m}\underline{F}_{t+m}'\right]\gamma_k}\right\}\\
\leq\;&\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}\sum_{t=p}^{T-h}E\left(u_{i,t}^2\right)\left\|\gamma_k\right\|_2^2E\left[\left\|\underline{F}_t\right\|_2^2\right]\\
&+\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}4\left(2^{\frac{2}{3}}+1\right)\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left\{\alpha_m^{\frac{1}{3}}\left(E\left|u_{i,t}\right|^3\right)^{\frac{1}{3}}\left(E\left|u_{i,t+m}\right|^3\right)^{\frac{1}{3}}\right.\\
&\qquad\left.\times\left\|\gamma_k\right\|_2^2\sqrt{E\left\|\underline{F}_t\right\|_2^2}\sqrt{E\left\|\underline{F}_{t+m}\right\|_2^2}\right\}\\
\leq\;&\frac{C_1}{T_h}+C_2\frac{1}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}a_1^{\frac{1}{3}}\exp\left\{-\frac{a_2}{3}m\right\}\\
\leq\;&\frac{C_1}{T_h}+C_2a_1^{\frac{1}{3}}\frac{1}{T_h}\sum_{m=1}^{\infty}\exp\left\{-\frac{a_2}{3}m\right\}\\
\leq\;&\frac{\overline{C}}{T_h}
\end{aligned}$$

for some positive constant

$$\overline{C}\geq C_1+C_2a_1^{\frac{1}{3}}\sum_{m=1}^{\infty}\exp\left\{-\frac{a_2}{3}m\right\}$$

which exists in light of Lemma C-1. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and note that

$$
\begin{aligned}
& \Pr\left\{\frac{T_h}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 \geq C_\epsilon\right\} \\
\leq\ & \Pr\left\{\frac{T_h}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{T_h}\right)^2 \geq C_\epsilon\right\} \\
\leq\ & \frac{T_h}{C_\epsilon}\frac{1}{N_1^2T_h^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left[\left(\sum_{t=p}^{T-h}\gamma_k'\underline{F}_t u_{i,t}\right)^2\right] \\
\leq\ & \frac{\epsilon}{\overline{C}}T_h\frac{\overline{C}}{T_h} \\
=\ & \epsilon
\end{aligned}
$$

which shows that

$$
\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_h}\right)^2 = O_p\left(\frac{1}{T_h}\right) = O_p\left(\frac{1}{T}\right). \ \square
$$

Lemma D-8: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3(a)-(c), 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)
$$
\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p\left(1\right).
$$

(b)
$$
\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) = o_p\left(1\right).
$$

Proof of Lemma D-8:

To show part (a), note first that

$$\begin{aligned}
\frac{1}{N_1}\sum_{i\in H^c} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] &\leq \frac{1}{N_1}\sum_{i\in H^c} E\left[\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c} E\left[\frac{1}{T_h}\sum_{t=p}^{T-h} u_{i,t}^2\right] \\
&\leq \frac{1}{N_1}\sum_{i\in H^c} \frac{1}{T_h}\sum_{t=p}^{T-h} \sup_{i,t} E\left[u_{i,t}^2\right] \\
&\leq \frac{1}{N_1}\sum_{i\in H^c} \frac{1}{T_h}\sum_{t=p}^{T-h} C \\
&= C
\end{aligned}$$

for some positive constant $C \geq \sup_{i,t} E\left[u_{i,t}^2\right]$ which exists in light of Assumption 3-3(b). Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$, and note that

$$\begin{aligned}
&\Pr\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) \geq C_\epsilon\right\} \\
\leq\ & \frac{1}{C_\epsilon}\frac{1}{N_1}\sum_{i\in H^c} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \quad \text{(by Markov's inequality)} \\
\leq\ & \frac{\epsilon}{C}C \\
=\ & \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right) = O_p(1).$$

Next, to show part (b), note that

$$
\begin{aligned}
& \frac{1}{N_1}\sum_{i\in H} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] \\
\leq \;& \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{i\in\widehat{H^c}\right\}\right)^{\frac{5}{7}}\left(E\left[\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)^{\frac{7}{2}}\right]\right)^{\frac{2}{7}} \text{ (by Hölder's inequality)} \\
= \;& \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{i\in\widehat{H^c}\right\}\right)^{\frac{5}{7}}\left(E\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2\right)^{\frac{7}{2}}\right]\right)^{\frac{2}{7}} \\
\leq \;& \frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\sup_{i,t}E\left[|u_{i,t}|^7\right]\right)^{\frac{2}{7}} \\
\leq \;& C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}}
\end{aligned}
$$

for some positive constant $C_1 \geq \sup_{i,t} E\left[|u_{i,t}|^7\right]$ which exists in light of Assumption 3-3(b). Now, let $\mathbb{S}_{i,T}^{+}$ denote either the statistic $\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the statistic $\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$; and, following arguments similar to that given in the proof of part (a) of Theorem 1 (see Chao, Qiu, and Swanson (2023b)), we see that, for any $i\in H$, there exists a constant $C_2 > 2d$ such that

$$
\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\leq C_2\frac{\varphi}{N}
$$

for all N_1, N_2, and T sufficiently large, from which it follows that

$$
\begin{aligned}
\frac{1}{N_1}\sum_{i\in H} E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right] &\leq C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}\left(\Pr\left\{\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right)^{\frac{5}{7}} \\
&\leq C_1^{\frac{2}{7}}\frac{1}{N_1}\sum_{i\in H}C_2^{\frac{5}{7}}\left(\frac{\varphi}{N}\right)^{\frac{5}{7}} \\
&= C_1^{\frac{2}{7}}C_2^{\frac{5}{7}}\frac{N_2\varphi^{\frac{5}{7}}}{N_1N^{\frac{5}{7}}} \\
&\leq C_3\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}
\end{aligned}
$$

for all N_1, N_2, and T sufficiently large and for some positive constant $C_3 \geq C_1^{\frac{2}{7}}C_2^{\frac{5}{7}}$. Hence, for any

$\epsilon > 0$, set $C_\epsilon = C_3/\epsilon$, and note that

$$\begin{aligned}
& \Pr\left\{ \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{1}{N_1} \sum_{i\in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{i\cdot}}{T_h} \right) \geq C_\epsilon \right\} \\
\leq & \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{1}{C_\epsilon} \frac{1}{N_1} \sum_{i\in H} E\left[\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{i\cdot}}{T_h} \right) \right] \quad \text{(by Markov's inequality)} \\
\leq & \frac{N_1}{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}} \frac{\epsilon}{C_3} C_3 \frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \\
= & \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1} \sum_{i\in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{i\cdot}}{T_h} \right) = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1). \ \square$$

Lemma D-9: Let $T_h = T - h - p + 1$ where h is a (fixed) non-negative integer and p is a (fixed) positive integer. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10(a) and 3-11* hold. Then, the following statements are true.

(a)

$$\mathcal{T}_1 = \frac{1}{N_1} \sum_{i\in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k\in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_h} \right)^2 = O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) = o_p(1).$$

(b)

$$\mathcal{T}_2 = \frac{1}{N_1} \sum_{i\in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k\in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(c)

$$\mathcal{T}_3 = \frac{1}{N_1} \sum_{i\in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k\in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1} \right) = o_p(1).$$

(d)

$$\mathcal{T}_4 = \frac{1}{N_1} \sum_{i\in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k\in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_h} \right)^2 = O_p\left(\frac{N^{\frac{4}{7}}\varphi^{\frac{10}{7}}}{N_1^2} \right) = o_p(1).$$

Proof of Lemma D-9:

To show part (a), note that

$$
\begin{aligned}
0 &\leq \mathcal{T}_1 \\
&= \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sum_{s=p}^{T-h}u_{i,t}u_{k,t}u_{i,s}u_{k,s} \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}u_{i,t}^2u_{k,t}^2 \\
&\quad +\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}
\end{aligned}
$$

From the non-negativity of $\mathcal{T}_1$, we get

$$
\begin{aligned}
E\left|\mathcal{T}_1\right| &= E\left[\mathcal{T}_1\right] \\
&\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right] \\
&\quad +\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}\right]
\end{aligned}
$$

Now,

$$
\begin{aligned}
\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right] &\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}\sqrt{E\left[u_{i,t}^4\right]}\sqrt{E\left[u_{k,t}^4\right]} \\
&\leq \left(\sup_{i,t}E\left[u_{i,t}^4\right]\right)\frac{1}{T_h} \\
&\leq \frac{C_1}{T_h}
\end{aligned}
$$

for some positive constant $C_1 \geq \sup_{i,t} E\left[u_{i,t}^4\right]$ which exists in light of Assumption 3-3(b). Moreover,

$$\begin{aligned}
& \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}\right] \\
= & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[\left(u_{i,t}u_{k,t}-E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m}-E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right] \\
& +\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[u_{i,t}u_{k,t}\right]E\left[u_{i,t+m}u_{k,t+m}\right] \\
\leq & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left[\left(u_{i,t}u_{k,t}-E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m}-E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right]\right| \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right|
\end{aligned}$$

Consider the first term on the right-hand side above. Note that by Assumption 3-3(c), $\{u_{it}\}_{t=-\infty}^{\infty}$ is β-mixing with β mixing coefficient satisfying

$$\beta_i(m) \leq a_1 \exp\{-a_2 m\}$$

for every i. Since $\alpha_{i,m} \leq \beta_i(m)$, it follows that $\{u_{it}\}_{t=-\infty}^{\infty}$ is α-mixing as well, with α mixing coefficient satisfying

$$\alpha_{i,m} \leq a_1 \exp\{-a_2 m\} \text{ for every } i.$$

Hence, we can apply Lemma C-3 with $p = 2$ and $r = 3$ to obtain

$$
\begin{aligned}
& \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} \left| E\left[\left(u_{i,t} u_{k,t} - E\left[u_{i,t} u_{k,t} \right] \right) \left(u_{i,t+m} u_{k,t+m} - E\left[u_{i,t+m} u_{k,t+m} \right] \right) \right] \right| \\
\leq \; & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{2}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} 2 \left(\sqrt{2} + 1 \right) \alpha_m^{\frac{1}{6}} \sqrt{E\left[u_{i,t}^2 u_{k,t}^2 \right]} \left(E\left| u_{i,t+m} u_{k,t+m} \right|^3 \right)^{\frac{1}{3}} \\
\leq \; & \frac{1}{N_1^2} \sum_{i \in H^c} \sum_{k \in H^c} \frac{4\left(\sqrt{2} + 1 \right)}{T_h^2} \sum_{t=p}^{T-h-1} \sum_{m=1}^{T-h-t} a_1^{\frac{1}{6}} \exp\left\{ -\frac{a_2}{6} m \right\} \sqrt{E\left[u_{i,t}^2 u_{k,t}^2 \right]} \left(E\left| u_{i,t+m} u_{k,t+m} \right|^3 \right)^{\frac{1}{3}} \\
\leq \; & \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \frac{4 a_1^{\frac{1}{6}} \left(\sqrt{2} + 1 \right) \left(E\left[u_{i,t}^4 \right] \right)^{\frac{1}{4}} \left(E\left[u_{k,t}^4 \right] \right)^{\frac{1}{4}} \left(E\left[u_{i,t+m}^6 \right] E\left[u_{k,t+m}^6 \right] \right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
\leq \; & \sum_{i \in H^c} \sum_{k \in H^c} \sum_{t=p}^{T-h-1} \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \frac{4 a_1^{\frac{1}{6}} \left(\sqrt{2} + 1 \right) \left(E\left[u_{i,t}^6 \right] \right)^{\frac{1}{6}} \left(E\left[u_{k,t}^6 \right] \right)^{\frac{1}{6}} \left(E\left[u_{i,t+m}^6 \right] E\left[u_{k,t+m}^6 \right] \right)^{\frac{1}{6}}}{N_1^2 T_h^2} \\
\leq \; & \frac{4 \overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} \left(\sup_{i,t} E\left[u_{i,t}^6 \right] \right)^{\frac{2}{3}}}{T_h} \\
& \left(\text{for some positive constant } \overline{C} \text{ such that } \overline{C} \; \geq \sum_{m=1}^{\infty} \exp\left\{ -\frac{a_2}{6} m \right\} \right) \\
\leq \; & \frac{4 \overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} C^{\frac{2}{3}}}{T_h} \\
& \left(\text{by Assumption 3-3(b), there exists positive constant } C \text{ such that } \sup_{i,t} E\left| u_{i,t} \right|^6 \leq C < \infty \right) \\
\leq \; & \frac{C_2}{T_h} \quad \left(\text{setting } C_2 \geq 4 \overline{C} \left(\sqrt{2} + 1 \right) a_1^{\frac{1}{6}} C^{\frac{2}{3}} \right) \\
= \; & O\left(\frac{1}{T} \right)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right| \\
\leq\ & \frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\sqrt{E\left[u_{i,t+m}^2\right]}\sqrt{E\left[u_{k,t+m}^2\right]} \\
\leq\ & \left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right| \\
\leq\ & \frac{2}{N_1}\left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\sup_t\left(\frac{1}{N_1}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\right) \\
\leq\ & \frac{C_3}{N_1}.
\end{aligned}
$$

for some positive constant C_3 such that

$$
2\left(\sup_{i,t}E\left[u_{i,t}^2\right]\right)\sup_t\left(\frac{1}{N_1}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\right)\leq C_3<\infty
$$

which exists in light of Assumptions 3-3(b) and 3-3(d). It follows from these results that

$$
\begin{aligned}
& E\left|\mathcal{T}_1\right| \\
=\ & E\left[\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right] \\
\leq\ & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right]+\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}E\left[u_{i,t}u_{k,t}u_{i,t+m}u_{k,t+m}\right] \\
\leq\ & \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{1}{T_h^2}\sum_{t=p}^{T-h}E\left[u_{i,t}^2u_{k,t}^2\right] \\
& +\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\left|E\left[\left(u_{i,t}u_{k,t}-E\left[u_{i,t}u_{k,t}\right]\right)\left(u_{i,t+m}u_{k,t+m}-E\left[u_{i,t+m}u_{k,t+m}\right]\right)\right]\right| \\
& +\frac{2}{T_h^2}\sum_{t=p}^{T-h-1}\sum_{m=1}^{T-h-t}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left|E\left[u_{i,t}u_{k,t}\right]\right|\left|E\left[u_{i,t+m}u_{k,t+m}\right]\right| \\
\leq\ & \frac{C_1}{T_h}+\frac{C_2}{T_h}+\frac{C_3}{N_1} \\
\leq\ & \frac{\overline{C}}{\min\left\{N_1,T_h\right\}}
\end{aligned}
$$

for some positive constant $\overline{C} \geq C_1 + C_2 + C_3$. Hence, for any $\epsilon > 0$, set $C_\epsilon = \overline{C}/\epsilon$, and applying Markov's inequality, we obtain

$$\begin{aligned}
& \Pr\left(\min\{N_1, T_h\}|\mathcal{T}_1| \geq C_\epsilon\right) \\
= \; & \Pr\left(\min\{N_1, T_h\}\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right| \geq C_\epsilon\right) \\
\leq \; & \frac{\min\{N_1, T_h\}}{C_\epsilon}E\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2\right\} \\
\leq \; & \min\{N_1, T_h\}\frac{\epsilon}{\overline{C}}\frac{\overline{C}}{\min\{N_1, T_h\}} \\
= \; & \epsilon
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{T}_1 &= \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= O_p\left(\frac{1}{\min\{N_1, T_h\}}\right) = O_p\left(\frac{1}{\min\{N_1, T\}}\right) = O_p\left(\max\left\{\frac{1}{N_1}, \frac{1}{T}\right\}\right).
\end{aligned}$$

Next, to show part (b), we apply parts (a) and (b) of Lemma D-8 to obtain

$$\begin{aligned}
\mathcal{T}_2 &= \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right) \text{ (by CS inequality)} \\
&= \left[\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right]\left[\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right)\right] \\
&= O_p(1)\,O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) \\
&= O_p\left(\frac{N^{\frac{2}{7}}\varphi^{\frac{5}{7}}}{N_1}\right) = o_p(1)
\end{aligned}$$

Part (c) can be shown in the same way as part (b) above. Hence, to avoid redundancy, we do not give an explicit proof here.

Finally, to show part (d), we apply part (b) of Lemma D-8 to obtain

$$
\begin{aligned}
\mathcal{T}_4 &= \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&= \frac{1}{N_1^2}\sum_{i\in H}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{k\cdot}}{T_h}\right)^2 \\
&\leq \frac{1}{N_1^2}\sum_{i\in H}\sum_{k\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\left(\frac{u_{k\cdot}'u_{k\cdot}}{T_h}\right) \text{ (by CS inequality)} \\
&= \left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\frac{u_{i\cdot}'u_{i\cdot}}{T_h}\right)\right]^2 \\
&= O_p\left(\frac{N^{\frac{4}{7}}\varphi^{\frac{10}{7}}}{N_1^2}\right) = o_p(1). \;\square
\end{aligned}
$$

Lemma D-10: Let

$$
\widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)'Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} \tag{98}
$$

where $T_0 = T - p + 1$, where $\widehat{H^c}$ and $\widehat{N}_1$ are as defined, respectively, in expressions (94) and (96) above, and where

$$
\underset{T_0\times N}{Z\left(\widehat{H^c}\right)} = \left[\begin{array}{cccc} Z_{1\cdot}\mathbb{I}\left\{1\in\widehat{H^c}\right\} & Z_{2\cdot}\mathbb{I}\left\{2\in\widehat{H^c}\right\} & \cdots & Z_{N\cdot}\mathbb{I}\left\{N\in\widehat{H^c}\right\}\end{array}\right] \tag{99}
$$

with $Z_{i\cdot} = (Z_{i,p}, Z_{i,p+1}, ..., Z_{i,T})'$ for $i = 1, ..., N$. Suppose that Assumptions 3-1, 3-2(a)-(c), 3-3, 3-4, 3-5, 3-7, 3-8, 3-10, and 3-11* hold.

Under the assumed conditions,

$$
\left\|\widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_2 = o_p(1) \text{ as } N_1,\ N_2,\ T\to\infty,
$$

where

$$
M_{FF} = \frac{1}{T_0}\sum_{t=p}^{T}E\left[\underline{F}_t\underline{F}_t'\right].
$$

Proof of Lemma D-10:

To proceed, note that we can write

$$
Z\left(\widehat{H^c}\right) = \underline{F}\Gamma\left(\widehat{H^c}\right)' + U\left(\widehat{H^c}\right),
$$

so that

$$
\begin{aligned}
& \widehat{\Sigma}\left(\widehat{H^c}\right)-\frac{\Gamma M_{FF}\Gamma'}{N_1} \\
= \quad & \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0}-\frac{\Gamma M_{FF}\Gamma'}{N_1} \\
= \quad & \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0}-\frac{\Gamma M_{FF}\Gamma'}{N_1} \\
= \quad & \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\frac{\Gamma\left(\widehat{H^c}\right) \underline{F}'\underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0}+\frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0}\right. \\
& \qquad\qquad\qquad \left.+\frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0}+\frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0}\right\}-\frac{\Gamma M_{FF}\Gamma'}{N_1} \\
= \quad & -\left(\frac{\widehat{N}_1-N_1}{\widehat{N}_1}\right) \frac{\Gamma M_{FF}\Gamma'}{N_1}+\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\frac{1}{N_1} \Gamma\left(\widehat{H^c}\right)\left[\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right] \Gamma\left(\widehat{H^c}\right)\right. \\
& +\frac{1}{N_1}\left(\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'-\Gamma M_{FF}\Gamma'\right)+\frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \\
& \left.+\frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0}+\frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0}\right\} \qquad (100)
\end{aligned}
$$

where M_{FF} is as defined in (75), where $\Gamma\left(\widehat{H^c}\right)$ and $U\left(\widehat{H^c}\right)$ are as defined in (97), and where $Z\left(\widehat{H^c}\right)$ is as defined in expression (99).

Consider first the term $-\left[\left(\widehat{N}_1-N_1\right)/\widehat{N}_1\right]\left(\Gamma M_{FF}\Gamma'/N_1\right)$. Note that, for some positive constant $\overline{C}$ such that

$$
\begin{aligned}
\|M_{FF}\|_F &= \left\| \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \right\|_F \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} \left\| E\left[\underline{F}_t \underline{F}_t'\right] \right\|_F \\
&\quad \text{(by the homogeneity of matrix norm and the triangle inequality)} \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} E\left\| \underline{F}_t \underline{F}_t' \right\|_F \text{ (by the Jensen's inequality)} \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\left[\sqrt{tr\left\{\underline{F}_t \underline{F}_t' \underline{F}_t \underline{F}_t'\right\}} \right] \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_0} \sum_{t=p}^{T} E\left[\|\underline{F}_t\|_2^2 \right] \\
&\leq \frac{1}{T_0} \sum_{t=p}^{T} \left(E\left[\|\underline{F}_t\|_2^6 \right] \right)^{\frac{1}{3}} \text{ (by Liapunov's inequality)} \\
&\leq \overline{C}^{\frac{1}{3}} \text{ (by Lemma C-4)} \qquad (101) \\
&< \infty
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F &= \sqrt{tr\left\{ \frac{\Gamma M_{FF} \Gamma'}{N_1} \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right) tr\left\{ \frac{\Gamma M_{FF}^2 \Gamma'}{N_1} \right\}} \\
&= \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right) tr\left\{ \frac{M_{FF} \Gamma' \Gamma M_{FF}}{N_1} \right\}} \\
&\leq \sqrt{\lambda_{\max}^2\left(\frac{\Gamma'\Gamma}{N_1}\right) tr\left\{ M_{FF}^2 \right\}} \\
&= \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right) \|M_{FF}\|_F \\
&\leq C^* \overline{C}^{\frac{1}{3}} < \infty \text{ for all } N_1,\ N_2 \text{ sufficiently large,}
\end{aligned}
$$

since, by Assumption 3-6, there exists some positive constant C^* such that $\lambda_{\max}(\Gamma'\Gamma/N_1) \leq C^* < \infty$ for all N_1, N_2 sufficiently large. Moreover, applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left| -\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right) \right| = \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| = \left| \frac{\widehat{N}_1 - N_1}{N_1} \right| \left| \frac{1}{\left(\widehat{N}_1 - N_1 \right) / N_1 + 1} \right| \xrightarrow{p} 0$$

so that by a further application of the Slutsky's theorem, we can deduce that

$$\left\| -\left(\frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right) \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F = \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F \xrightarrow{p} 0. \tag{102}$$

Consider now the other terms on the right-hand side of expression (100). To proceed, we first note that, by applying part (a) of Lemma D-15 and the Slutsky's theorem, we have

$$\left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-1} \right| = \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \xrightarrow{p} 1.$$

Next, note that

$$\begin{aligned} & \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right\|_F^2 \\ = & \sum_{i=1}^{N} \sum_{k=1}^{N} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \gamma_i' M_{FF} \gamma_k - \gamma_i' M_{FF} \gamma_k \right)^2 \\ = & \sum_{i \in H^c} \sum_{k \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \gamma_i' M_{FF} \gamma_k - \gamma_i' M_{FF} \gamma_k \right)^2 \end{aligned}$$

where $H^c = \{k \in \{1, ..., N\} : \gamma_k \neq 0\}$, where $\widehat{H^c} = \left\{ i \in \{1, ..., N\} : \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \right\}$, and where $\mathbb{S}_{i,T}^{+}$ denotes either the statistic $\sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the statistic $\max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right|$. Note that

$$\sum_{i \in H^c} \sum_{k \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \gamma_i' M_{FF} \gamma_k - \gamma_i' M_{FF} \gamma_k \right)^2 = 0 \text{ if } \mathbb{I}\left\{ i \in \widehat{H^c} \right\} = 1 \text{ for every } i \in H^c,$$

so that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{\left\|\frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \geq \epsilon\right\} &\subseteq \left\{i \notin \widehat{H^c} \text{ for at least one } i \in H^c\right\} \\
&= \bigcup_{i\in H^c}\left\{i \notin \widehat{H^c}\right\} \\
&= \bigcup_{i\in H^c}\left\{\mathbb{S}^{+}_{i,T} < \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\} \\
&= \left\{\bigcap_{i\in H^c}\left\{\mathbb{S}^{+}_{i,T} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right\}^c
\end{aligned}$$

Hence, applying either part (a) or part (b) of Theorem 2 in Chao, Qiu, and Swanson (2023a) depending on whether $\mathbb{S}^{+}_{i,T} = \sum_{\ell=1}^{d} \varpi_\ell \left|S_{i,\ell,T}\right|$ or $\mathbb{S}^{+}_{i,T} = \max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we obtain

$$\begin{aligned}
&\Pr\left(\left\|\frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \geq \epsilon\right) \\
\leq\ & 1 - \Pr\left(\bigcap_{i\in H^c}\left\{\mathbb{S}^{+}_{i,T} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right\}\right) \\
=\ & 1 - \Pr\left(\min_{i\in H^c}\mathbb{S}^{+}_{i,T} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\to\ & 1 - 1 = 0 \text{ as } N_1, N_2, T \to \infty,
\end{aligned}$$

so that

$$\left\|\frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F = o_p(1) \tag{103}$$

Now, consider the term $\Gamma\left(\widehat{H^c}\right)\left[\frac{F'F}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'/N_1$. For this term, note first that, by sub-multiplicativity of matrix norms, we have that

$$\begin{aligned}
\left\|\frac{\Gamma\left(\widehat{H^c}\right)\left[\frac{F'F}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'}{N_1}\right\|_F &\leq \left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F \left\|\frac{F'F}{T_0} - M_{FF}\right\|_F \left\|\frac{\Gamma\left(\widehat{H^c}\right)'}{\sqrt{N_1}}\right\|_F \\
&= \left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F^2 \left\|\frac{F'F}{T_0} - M_{FF}\right\|_F
\end{aligned}$$

Note that

$$\begin{aligned}
\left\| \frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_F^2 &= tr\left\{ \frac{\Gamma\left(\widehat{H^c}\right)' \Gamma\left(\widehat{H^c}\right)}{N_1} \right\} \\
&= tr\left\{ \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \gamma_i \gamma_i' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} tr\left\{ \gamma_i \gamma_i' \right\} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \|\gamma_i\|_2^2 \, \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&\leq \sup_i \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&= \sup_{i \in \in H^c} \|\gamma_i\|_2^2 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \\
&\quad \text{(since } \gamma_i = 0 \text{ for all } \gamma_i \in H\text{)} \\
&\leq C_1 \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\}
\end{aligned}$$

for some positive constant $C_1 \geq \sup_i \|\gamma_i\|_2^2 = \sup_{i \in \in H^c} \|\gamma_i\|_2^2$ which exists in light of Assumption 3-5. Moreover, write

$$\frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} = \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \tag{104}$$

For the first term on the right-hand side of expression (104) above, we can apply part (a) of Lemma D-7 to obtain

$$\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} = O_p\left(\frac{\varphi}{N_1} \right) = o_p(1).$$

With regard to the second term on the right-hand side of expression (104), note that

$$\frac{1}{N_1} \sum_{i \in H^c} E\left[\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right] \leq 1$$

since, by definition, N_1 is the cardinality of the set$\{i \in \{1, ..., N\} : i \in H^c\}$. Hence, for any $\epsilon > 0$, set $C_\epsilon = C/\epsilon$ for any positive constant $C \geq 1$, and note that

$$\begin{aligned}
\Pr\left\{\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\geq C_\epsilon\right\} &\leq \frac{1}{C_\epsilon}\frac{1}{N_1}\sum_{i\in H^c}E\left[\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right] \quad \text{(by Markov's inequality)} \\
&\leq \frac{\epsilon}{C}C \\
&= \epsilon
\end{aligned}$$

which shows that

$$\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\} = O_p\left(1\right).$$

It follows that

$$\begin{aligned}
\left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F^2 &\leq C_1\frac{1}{N_1}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\} \\
&= \frac{C_1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}+\frac{C_1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\} \\
&= O_p\left(\frac{\varphi}{N_1}\right)+O_p\left(1\right) \\
&= O_p\left(1\right).
\end{aligned}$$

In addition, applying the result of part (b) of Lemma D-2, we have that

$$\left\|\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right\|_F = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p\left(1\right)$$

from which we further deduce that

$$\begin{aligned}
\left\|\frac{\Gamma\left(\widehat{H^c}\right)\left[\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'}{N_1}\right\|_F &\leq \left\|\frac{\Gamma\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_F^2\left\|\frac{\underline{F}'\underline{F}}{T_0}-M_{FF}\right\|_F \\
&= O_p\left(1\right)O_p\left(\frac{1}{\sqrt{T}}\right) \\
&= O_p\left(\frac{1}{\sqrt{T}}\right). \qquad (105)
\end{aligned}$$

Turning our attention to the term $U\left(\widehat{H^c}\right)'\underline{F}\Gamma\left(\widehat{H^c}\right)'/\left(N_1T_0\right)$, we first write

$$
\begin{aligned}
& \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F^2 \\
= \;& \sum_{i=1}^{N}\sum_{k=1}^{N}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\frac{u_{i\cdot}'\underline{F}\gamma_k}{N_1 T_0}\right)^2 \\
= \;& \frac{1}{N_1^2 T_0^2}\sum_{i=1}^{N}\sum_{k=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\gamma_k'\underline{F}'u_{i\cdot}u_{i\cdot}'\underline{F}\gamma_k \\
= \;& \frac{1}{N_1^2 T_0^2}\sum_{i=1}^{N}\sum_{k\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\gamma_k'\underline{F}'u_{i\cdot}u_{i\cdot}'\underline{F}\gamma_k \\
= \;& \frac{1}{N_1}\sum_{k\in H^c}\mathbb{I}\left\{k\in\widehat{H^c}\right\}\frac{1}{N_1 T_0^2}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
\leq \;& \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
= \;& \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 + \frac{1}{N_1}\sum_{k\in H^c}\frac{1}{N_1 T_0^2}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\left(\gamma_k'\underline{F}'u_{i\cdot}\right)^2 \\
= \;& \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 + \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2
\end{aligned}
$$

Applying parts (b) and (c) of Lemma D-7, we obtain

$$
\begin{aligned}
& \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F^2 \\
\leq \;& \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{1}{N_1}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 + \frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\sum_{k\in H^c}\left(\frac{\gamma_k'\underline{F}'u_{i\cdot}}{\sqrt{N_1}T_0}\right)^2 \\
= \;& O_p\left(\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}\right) + O_p\left(\frac{1}{T}\right) \\
= \;& O_p\left(\max\left\{\frac{N_2^{\frac{1}{3}}\varphi}{N_1 T}, \frac{1}{T}\right\}\right) \\
= \;& o_p(1) \quad \text{(by Assumption 3-11*)}
\end{aligned}
$$

so that

$$\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma \left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F = O_p\left(\max\left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\}\right) = o_p(1). \tag{106}$$

Since

$$\left\| \frac{\Gamma \left(\widehat{H^c}\right) \underline{F}' U \left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F = \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma \left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F$$

it follows immediately also that

$$\left\| \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma \left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F = O_p\left(\max\left\{ N_2^{\frac{1}{6}} \sqrt{\frac{\varphi}{N_1 T}}, \frac{1}{\sqrt{T}} \right\}\right) = o_p(1). \tag{107}$$

Finally, consider the term $\left\| U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right) / N_1 T_0 \right\|_F^2$, where

$$U\left(\widehat{H^c}\right) = \left[\begin{array}{cccc} u_{1\cdot} \mathbb{I}\left\{1 \in \widehat{H^c}\right\} & u_{2\cdot} \mathbb{I}\left\{2 \in \widehat{H^c}\right\} & \cdots & u_{N\cdot} \mathbb{I}\left\{N \in \widehat{H^c}\right\} \end{array} \right].$$

Given that

$$\begin{aligned} & U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right) \\ = & \begin{pmatrix} u_{1\cdot}' u_{1\cdot} \mathbb{I}\left\{1 \in \widehat{H^c}\right\} & \cdots & u_{1\cdot}' u_{N\cdot} \mathbb{I}\left\{1 \in \widehat{H^c}\right\} \mathbb{I}\left\{N \in \widehat{H^c}\right\} \\ u_{1\cdot}' u_{2\cdot} \mathbb{I}\left\{1 \in \widehat{H^c}\right\} \mathbb{I}\left\{2 \in \widehat{H^c}\right\} & \cdots & u_{2\cdot}' u_{N\cdot} \mathbb{I}\left\{2 \in \widehat{H^c}\right\} \mathbb{I}\left\{N \in \widehat{H^c}\right\} \\ \vdots & & \vdots \\ u_{1\cdot}' u_{N\cdot} \mathbb{I}\left\{1 \in \widehat{H^c}\right\} \mathbb{I}\left\{N \in \widehat{H^c}\right\} & \cdots & u_{N\cdot}' u_{N\cdot} \mathbb{I}\left\{N \in \widehat{H^c}\right\} \end{pmatrix}, \end{aligned}$$

we can write

$$
\begin{aligned}
\left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F^2 &= \sum_{i=1}^N \sum_{k=1}^N \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \frac{u_{i\cdot}' u_{k\cdot}}{N_1 T_0} \right)^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k=1}^N \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(u_{i\cdot}' u_{k\cdot} \right)^2 \\
&= \frac{1}{N_1} \sum_{i=1}^N \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k=1}^N \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H^c} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&\quad + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{1}{N_1} \sum_{k \in H} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(\frac{u_{i\cdot}' u_{k\cdot}}{T_0} \right)^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \;\; (say),
\end{aligned}
$$

where the order of magnitude in probability of the terms $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, and $\mathcal{T}_4$ are given in parts (a)-(d) of Lemma D-9. It, thus, follows by applying parts (a)-(d) of Lemma D-9 with $h = 0$ that

$$
\begin{aligned}
&\left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F^2 \\
&= \frac{1}{N_1^2 T_0^2} \sum_{i=1}^N \sum_{k=1}^N \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ k \in \widehat{H^c} \right\} \left(u_{i\cdot}' u_{k\cdot} \right)^2 \\
&= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 \\
&= O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T} \right\} \right) + O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p\left(\frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right) + O_p\left(\frac{N^{\frac{4}{7}} \varphi^{\frac{10}{7}}}{N_1^2} \right) \\
&= O_p\left(\max\left\{ \frac{1}{N_1}, \frac{1}{T}, \frac{N^{\frac{2}{7}} \varphi^{\frac{5}{7}}}{N_1} \right\} \right) \\
&= o_p(1).
\end{aligned}
$$

from which we further deduce that

$$\left\|\frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0}\right\|_F = O_p\left(\max\left\{\frac{1}{\sqrt{N_1}}, \frac{1}{\sqrt{T}}, \frac{N^{\frac{1}{7}}\varphi^{\frac{5}{14}}}{\sqrt{N_1}}\right\}\right) = o_p(1) \text{ as } N_1, N_2, T \to \infty. \tag{108}$$

Expressions (102)-(108) together imply that

$$\begin{aligned}
& \left\|\widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \\
= & \left\|\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \\
\leq & \left|\frac{\widehat{N}_1 - N_1}{\widehat{N}_1}\right| \left\|\frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F + \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\|_F \\
& + \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\frac{1}{N_1}\Gamma\left(\widehat{H^c}\right)\left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF}\right]\Gamma\left(\widehat{H^c}\right)'\right\|_F \\
& + \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0}\right\|_F \\
& + \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\frac{\Gamma\left(\widehat{H^c}\right)\underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0}\right\|_F + \left|1 + \frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0}\right\|_F \\
= & \; o_p(1) \text{ as } N_1, N_2, T \to \infty.
\end{aligned}$$

Since $\|A\|_2 \leq \|A\|_F$, we also have

$$\begin{aligned}
& \|E\|_2 \\
= & \left\| \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
= & \left\| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-1} \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
\leq & \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right| \left\| \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1}\Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF}\right] \Gamma\left(\widehat{H^c}\right)' \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_2 \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_2 + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_2 \\
\leq & \left| \frac{\widehat{N}_1 - N_1}{\widehat{N}_1} \right|^{-1} \left\| \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) M_{FF}\Gamma\left(\widehat{H^c}\right)'}{N_1} - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{1}{N_1}\Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF}\right] \Gamma\left(\widehat{H^c}\right)' \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' \underline{F}\Gamma\left(\widehat{H^c}\right)'}{N_1 T_0} \right\|_F \\
& + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F + \left| 1 + \frac{\widehat{N}_1 - N_1}{N_1} \right|^{-1} \left\| \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{N_1 T_0} \right\|_F \\
= & \; o_p(1) \text{ as } N_1, N_2, T \to \infty. \; \square
\end{aligned}$$

Lemma D-11: Let

$$\underset{N \times N}{A} = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

where

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \text{ with } T_0 = T - p + 1.$$

Suppose that Assumptions 3-1, 3-2(a)-(b), 3-2(d), 3-5, 3-6 and 3-7 hold; and let G be an $N \times N$ orthogonal matrix whose columns are the eigenvectors of A. Under the assumed conditions, the following statements are true.

(a) $Rank\,(A) = Kp$ for all N_1, N_2 sufficiently large, and, hence, 0 is an eigenvlaue of A with algebraic multiplicity equaling $N - Kp$.

(b) Partition G as follows:

$$\underset{N\times N}{G} = \left[\begin{array}{cc} \underset{N\times Kp}{G_1} & \underset{N\times(N-Kp)}{G_2} \end{array} \right]$$

Without loss of generality, suppose that the columns of G_1 are eigenvectors associated with the non-zero eigenvalues of A, whereas G_2 contains the eigenvectors associated with the zero eigenvalue. Then, the matrix $G'AG$ can be partitioned as follows:

$$G'AG = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{\Lambda_2} \end{pmatrix} = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{0} \end{pmatrix}. \tag{109}$$

where Λ_1 is a diagonal matrix whose diagonal elements are the non-zero eigenvalues of A and where $\Lambda_2 = 0$.

(c) Define the separation measure

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \min_{X \neq 0} \frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F};$$

then, there exists a positive constant $\underline{c}$ such that

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \text{sep}\,(\Lambda_1, 0) = \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \geq \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \geq \underline{c} > 0.$$

Proof of Lemma D-11: To show part (a), note first that, by the result of Lemma D-4 above, there exists a positive constant $\underline{C}$ such that

$$\lambda_{\min}\left\{M_{FF}\right\} \geq \underline{C} > 0$$

for all $T > p - 1$; and, by Assumption 3-6, we have,

$$\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \frac{1}{\overline{C}} \text{ for } N_1, N_2 \text{ sufficiently large.}$$

for some constant $\overline{C}$ such that $0 < \overline{C} < \infty$. Combining these two inequalities, we see that

$$\begin{aligned} \lambda_{\min}\left\{\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} &\geq \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\lambda_{\min}\left\{M_{FF}\right\} \\ &\geq \frac{\underline{C}}{\overline{C}} > 0 \text{ for all } N_1,\ N_2, \text{ and } T \text{ sufficiently large.} \end{aligned}$$

This implies that the $Kp \times Kp$ matrix

$$\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}$$

is a positive definite (and, therefore, also non-singular) for N_1, N_2, and T sufficiently large. Moreover, observe that

$$\begin{aligned} &\det\left\{\lambda I_N - \frac{\Gamma M_{FF}\Gamma'}{N_1}\right\} \\ = \;& \lambda^N \det\left\{I_N - \lambda^{-1}\frac{\Gamma M_{FF}\Gamma'}{N_1}\right\} \\ = \;& \lambda^N \det\left\{I_{Kp} - \lambda^{-1}\ \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} \text{ (by Sylvester's determinantal theorem)} \\ = \;& \lambda^{N-Kp}\det\left\{\lambda I_{Kp} - \ \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} \end{aligned} \tag{110}$$

Hence, the non-zero eigenvalues of the matrix $\Gamma M_{FF}\Gamma'/N_1$ correspond exactly to the eigenvalues of the positive definite matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$, from which we further deduce that the matrix

$$A = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

must be of rank Kp for N_1, N_2, T sufficiently large. Since A is an $N \times N$ matrix with $N = N_1 + N_2$, it follows immediately that 0 is an eigenvalue of A with algebraic multiplicity equaling $N - Kp$ for N_1, N_2, T sufficiently large.

To show part (b), let $\Lambda_1 = diag\left(\lambda_{1,1}, ..., \lambda_{1,Kp}\right)$, whose diagonal elements $\lambda_{1,i} > 0$, for $i \in \{1, ..., Kp\}$, denote the non-zero eigenvalues of A (which must all be positive given that they correspond to the eigenvalues of the positive definite matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$ as shown in the proof of part (a)). Moreover, let

$$\Lambda_2 = \underset{(N-Kp)\times(N-Kp)}{0}$$

whose diagonal elements are the $N-Kp$ zero eigenvalues of A.Since A is a symmetric matrix, the representation given in expression (109) follows immediately from the usual spectral decomposition.

Finally, to show part (c), note that for any $Kp\times(N-Kp)$ matrix $X\neq 0$, we have

$$\begin{aligned}
\|\Lambda_1 X - X\Lambda_2\|_F &= \|\Lambda_1 X\|_F \quad (\text{since } \Lambda_2 = 0) \\
&= \sqrt{tr\left\{X'\Lambda_1'\Lambda_1 X\right\}} \\
&\geq \lambda_{\min}(\Lambda_1)\sqrt{tr\left\{X'X\right\}} \\
&= \lambda_{\min}(\Lambda_1)\|X\|_F
\end{aligned}$$

It follows that

$$\begin{aligned}
\operatorname{sep}(\Lambda_1,\Lambda_2) &= \min_{X\neq 0}\frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F} \\
&= \min_{X\neq 0}\frac{\|\Lambda_1 X\|_F}{\|X\|_F} \quad (\text{since } \Lambda_2 = 0 \text{ in this case}) \\
&\geq \frac{\lambda_{\min}(\Lambda_1)\|X\|_F}{\|X\|_F} \\
&= \lambda_{\min}(\Lambda_1)
\end{aligned}$$

Furthermore, in light of expression (110), the diagonal elements of Λ_1, being the non-zero eigenvalues of A, must all be the solutions of the determinantal equation

$$\det\left\{\lambda I_{Kp} - \frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right\} = 0$$

so that, as noted in the proof of part (a) above, they are also the eigenvalues of the dual matrix $M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}/N_1$. It follows from the proof of part (a) that there exists a positive constant $\underline{c}$ such that for all N_1, N_2, and T sufficiently large.

$$\begin{aligned}
\operatorname{sep}(\Lambda_1,\Lambda_2) &= \operatorname{sep}(\Lambda_1, 0) \\
&\geq \lambda_{\min}(\Lambda_1) \\
&= \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \\
&\geq \underline{c} > 0. \ \square
\end{aligned}$$

Lemma D-12: Suppose that A and E are both $n \times n$ symmetric matrices and that

$$G = \left[\underset{n\times r}{G_1} \quad \underset{n\times(n-r)}{G_2} \right]$$

is an orthogonal matrix such that

$$\text{ran}\,(G_1) = \{y \in \mathbb{R}^n : y = G_1 x \text{ for some } x \in \mathbb{R}^r\}$$

is an invariant subspace for A, i.e., for any $\widetilde{q} \in \text{ran}(G_1)$ and let $q^* = A\widetilde{q}$; then $q^* \in \text{ran}(G_1)$. Partition the matrices $G'AG$ and $G'EG$ as follows:

$$G'AG = \begin{pmatrix} \underset{r\times r}{\Lambda_1} & \underset{r\times(n-r)}{0} \\ \underset{(N-r)\times r}{0} & \underset{(n-r)\times(n-r)}{\Lambda_2} \end{pmatrix} \text{ and } G'EG = \begin{pmatrix} \underset{r\times r}{E_{11}} & \underset{r\times(n-r)}{E'_{21}} \\ \underset{(n-r)\times r}{E_{21}} & \underset{(n-r)\times(n-r)}{E_{22}} \end{pmatrix}.$$

If

$$\text{sep}\,(\Lambda_1, \Lambda_2) = \min_{X\neq 0} \frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F} > 0 \tag{111}$$

and if

$$\begin{aligned} \|E\|_2 &\leq \frac{\text{sep}\,(\Lambda_1, \Lambda_2)}{5} \\ &= \frac{1}{5} \min_{X\neq 0} \frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F}, \end{aligned} \tag{112}$$

then, there exists a matrix $R \in \mathbb{R}^{(n-r)\times r}$ satisfying

$$\begin{aligned} \|R\|_2 &\leq \frac{4}{\text{sep}\,(\Lambda_1, \Lambda_2)} \|E_{21}\|_2 \\ &= 4 \left(\min_{X\neq 0} \frac{\|\Lambda_1 X - X\Lambda_2\|_F}{\|X\|_F} \right)^{-1} \|E_{21}\|_2 \end{aligned}$$

such that the columns of

$$\widehat{G}_1 = (G_1 + G_2 R) \left(I_r + R'R\right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$.

Remark: Lemma D-12 is a well-known result in linear algebra restated here in our notations. It is given in Golub and van Loan (1996) as Theorem 8.1.10. As noted in Golub and van Loan (1996), this result is also a slight adaptation of Theorem 4.11 in Stewart (1973), which could be consulted for proof details.

Lemma D-13: Let $\mathcal{X}$ be an invariant subspace of A, and let the columns of X form a basis for $\mathcal{X}$. Then, there is a unique matrix L such that

$$AX = XL.$$

The matrix L is the representation of A on $\mathcal{X}$ with respect to the basis X. In particular, (v, λ) is an eigenpair of L if and only if (Xv, λ) is an eigenpair of A.

Proof of Lemma D-13: This is Theorem 3.9 of Stewart and Sun (1990). For a proof of this theorem, see Stewart and Sun (1990).

A straightforward application of Lemma D-12 (or Theorem 8.1.10 of Golub and van Loan, 1996) to our setting here leads to the following lemma.

Lemma D-14: Let $\widehat{\Sigma}\left(\widehat{H^c}\right)$ be the post-variable-selection sample covariance matrix as defined in expression (98) in Lemma D-10. Decompose $\widehat{\Sigma}\left(\widehat{H^c}\right)$ as follows:

$$\widehat{\Sigma}\left(\widehat{H^c}\right) = A + E,$$

where

$$A = \frac{\Gamma M_{FF} \Gamma'}{N_1} \tag{113}$$

and where

$$\begin{aligned} E &= \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF} \Gamma'}{N_1} \\ &= \left(\frac{\Gamma\left(\widehat{H^c}\right) M_{FF} \Gamma\left(\widehat{H^c}\right)'}{\widehat{N}_1} - \frac{\Gamma M_{FF} \Gamma'}{N_1} \right) + \frac{1}{\widehat{N}_1} \Gamma\left(\widehat{H^c}\right) \left[\frac{\underline{F}'\underline{F}}{T_0} - M_{FF} \right] \Gamma\left(\widehat{H^c}\right)' \\ &\quad + \frac{U\left(\widehat{H^c}\right)' \underline{F} \Gamma\left(\widehat{H^c}\right)'}{\widehat{N}_1 T_0} + \frac{\Gamma\left(\widehat{H^c}\right) \underline{F}' U\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0} + \frac{U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)}{\widehat{N}_1 T_0}, \end{aligned} \tag{114}$$

with $T_0 = T - p + 1$ and

$$M_{FF} = \frac{1}{T_0} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right].$$

Suppose that Assumptions 3-1, 3-2, 3-3, 3-4 3-5, 3-6, 3-7, 3-8, 3-10, and 3-11* hold, and define

$$\underset{N\times N}{G} = \left[\begin{array}{cc} \underset{N\times Kp}{G_1} & \underset{N\times(N-Kp)}{G_2} \end{array} \right]$$

to be an orthogonal matrix whose columns are the eigenvectors of the matrix A. Without loss of generality, suppose that the columns of G_1 are the eigenvectors associated with the non-zero eigenvalues of A, whereas G_2 contains the eigenvectors associated with the zero eigenvalue which has an algebraic multiplicity of $N - Kp$ in this case[7]. Partition the matrices $G'AG$ and $G'EG$ as follows:

$$
\begin{aligned}
G'AG &= \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{\Lambda_2} \end{pmatrix} = \begin{pmatrix} \underset{Kp\times Kp}{\Lambda_1} & \underset{Kp\times(N-Kp)}{0} \\ \underset{(N-Kp)\times Kp}{0} & \underset{(N-Kp)\times(N-Kp)}{0} \end{pmatrix} \text{ and} \\
G'EG &= \begin{pmatrix} \underset{Kp\times Kp}{E_{11}} & \underset{Kp\times(N-Kp)}{E'_{21}} \\ \underset{(N-Kp)\times Kp}{E_{21}} & \underset{(N-Kp)\times(N-Kp)}{E_{22}} \end{pmatrix},
\end{aligned}
$$

where Λ_1 is a diagonal matrix whose diagonal elements are the Kp largest eigevalues of the matrix A.[8]

Under the assumed conditions, the following statements are true.

(a) There exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = (G_1 + G_2R)\left(I_{Kp} + R'R\right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $\widehat{\Sigma}\left(\widehat{H^c}\right) = A + E$. Moreover,

$$\|R\|_2 = o_p(1) \text{ as } N_1, N_2, \text{ and} T \to \infty$$

(b) $\left\|\widehat{G}_1 - G_1\right\|_2 = o_p(1)$ as N_1, N_2, and $T \to \infty$

(c) The exists a unique symmetric matrix L such that

$$(A + E)\,\widehat{G}_1 = \widehat{G}_1 L.$$

Moreover, let

$$\widehat{\Lambda} = diag\left(\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right) \tag{115}$$

[7] That 0 is an eigenvalue of the matrix

$$A = \frac{\Gamma M_{FF}\Gamma'}{N_1}$$

with algebraic multiplicity equaling $N - Kp$ has already been shown previously in Lemma D-11.

[8] We have also previously shown in Lemma D-11 that $G'AG$ can be partitioned in the manner given here.

denote a diagonal matrix whose diagonal elements are the eigenvalues of the matrix L, and let

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} \tag{116}$$

be a $Kp \times Kp$ matrix whose ℓ^{th} column (i.e., $\widehat{v}_\ell$) is an eigenvector of L associated with the eigenvalue $\widehat{\lambda}_\ell$ for $\ell = 1, ..., Kp$. Then, $\widehat{V}$ is an orthogonal matrix and $\left(\widehat{G}_1 \widehat{v}_\ell, \widehat{\lambda}_\ell\right)$ is an eigenpair for the matrix $A + E$ for $\ell = 1, ..., Kp$.

(d) The columns of the matrix

$$\widehat{G}_1 \widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1 \widehat{v}_1 & \widehat{G}_1 \widehat{v}_2 & \cdots & \widehat{G}_1 \widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the post-variable-selection sample covariance matrix

$$A + E = \widehat{\Sigma}\left(\widehat{H^c}\right).$$

Proof of Lemm D-14:

To show part (a), we first verify that the conditions (111) and (112) of Lemma D-12 are satisfied here. To proceed, let $\text{ran}(G_1)$ denote the range space of G_1, i.e.,

$$\text{ran}\left(G_1\right) = \left\{g \in \mathbb{R}^N : g = G_1 b \text{ for some } b \in \mathbb{R}^{Kp}\right\}$$

and, by definition, Λ_1 is a $Kp \times Kp$ diagonal matrix whose diagonal elements are the non-zero eigenvalues of the matrix $A = \Gamma M_{FF} \Gamma' / N_1$. Now, for any $\widetilde{g} \in \text{ran}(G_1)$, note that

$$\begin{aligned} g^* &= A\widetilde{g} \\ &= \left(\frac{\Gamma M_{FF} \Gamma'}{N_1}\right) G_1 b \\ &= G_1 \Lambda_1 b \\ &= G_1 b^* \text{ where } b^* = \Lambda_1 b. \end{aligned}$$

from which it follows that $g^* \in \text{ran}(G_1)$, so that $\text{ran}(G_1)$ is an invariant subspace of A. Next, by applying the result of Lemma D-11, we have

$$\begin{aligned}
\text{sep}\left(\Lambda_1, \Lambda_2\right) &= \text{sep}\left(\Lambda_1, 0\right) \\
&= \min_{X \neq 0} \frac{\|\Lambda_1 X\|_F}{\|X\|_F} \\
&\geq \lambda_{\min}\left(\Lambda_1\right) \\
&= \lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right) \\
&\geq \underline{c} > 0 \text{ for } N_1 \text{ and } N_2 \text{ sufficiently large,}
\end{aligned}$$

so that condition (111) of Lemma D-12 is fulfilled. Next, note that, from the result of Lemma D-10, we have

$$\|E\|_2 = \left\| \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} \right\|_2 = o_p\left(1\right) \quad \text{as } N_1,\ N_2, \text{ and } T \to 0;$$

from which it follows that

$$\|E\|_2 \leq \frac{\text{sep}\left(\Lambda_1, 0\right)}{5} \text{ w.p.a.1 as } N_1,\ N_2, \text{ and } T \to 0.$$

so that condition (112) of Lemma D-12 is also satisfied here w.p.a.1. Hence, application of Lemma D-12 allows us to conclude that there exists a $(N - Kp) \times Kp$ matrix R such that the columns of the matrix

$$\widehat{G}_1 = \left(G_1 + G_2 R\right)\left(I_{Kp} + R'R\right)^{-1/2}$$

define an orthonormal basis for a subspace that is invariant for $A + E$. In addition,

$$\begin{aligned}
\|R\|_2 &\leq \frac{4}{\text{sep}\left(\Lambda_1, 0\right)} \|E\|_2 \\
&= 4\left[\lambda_{\min}\left(\frac{M_{FF}^{1/2}\Gamma'\Gamma M_{FF}^{1/2}}{N_1}\right)\right]^{-1} \|E\|_2 \\
&\leq \frac{4}{\underline{c}} \|E\|_2 \quad \text{(for some } \underline{c} > 0 \text{ by Assumption 3-6 and Lemma D-4)} \\
&= o_p\left(1\right),
\end{aligned}$$

which shows result (a).

To show that $\left\|\widehat{G}_1 - G_1\right\|_2 = o_p\left(1\right)$, we first show that an explicit representation for G_1 can be given as

$$G_1 = \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi = \Gamma\left(\Gamma'\Gamma\right)^{-1/2} \Xi$$

where Ξ is an orthogonal matrix whose columns are eigenvectors of the matrix

$$\underset{Kp\times Kp}{M^*_{FF}} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}$$

To see that this representation satisfies the various properties we require of G_1, note first that

$$G_1'G_1 = \Xi' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \frac{\Gamma'\Gamma}{N_1} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \ = I_{Kp};$$

hence, G_1 so represented does have orthonormal columns. Moreover, note that

$$\begin{aligned}
\frac{\Gamma M_{FF}\Gamma'}{N_1}G_1 &= \frac{\Gamma}{\sqrt{N_1}} M_{FF} \frac{\Gamma'\Gamma}{\sqrt{N_1}} \left(\Gamma'\Gamma\right)^{-1/2} \Xi \\
&= \frac{\Gamma}{\sqrt{N_1}} M_{FF} \frac{\Gamma'\Gamma}{N_1} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \\
&= \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \frac{\Gamma'\Gamma}{N_1} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \\
&= \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} M^*_{FF}\Xi \\
&= \Gamma \left(\Gamma'\Gamma\right)^{-1/2} \Xi\Lambda_1 \\
&= G_1\Lambda_1 \qquad (117)
\end{aligned}$$

where Λ_1 is a $Kp \times Kp$ diagonal matrix whose diagonal elements are the eigenvalues of the matrix M^*_{FF}, which also happen to be the non-zero eigenvalues of the matrix $A = \Gamma M_{FF}\Gamma'/N_1$. Premultiplying the above equation by G_1', we obtain

$$G_1'\frac{\Gamma M_{FF}\Gamma'}{N_1}G_1 = G_1'G_1\Lambda_1 = \Lambda_1.$$

Since equation (117) shows that the columns of $\Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$ are indeed the eigenvectors of the matrix $A = \Gamma M_{FF}\Gamma'/N_1$, by the argument given previously in the proof of part (a) above, we can then deduce that ran(G_1), the range space of G_1 with $G_1 = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$, is an invariant subspace of A. It follows that setting

$$G_1 = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi$$

fulfills all the required properties of G_1 specified in Lemma D-12 above.

Next, write

$$\begin{aligned}
\widehat{G}_1 - G_1 &= (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2} - G_1 \\
&= G_1\left[\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right] + G_2 R\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi\left[\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right] + G_2 R\left(I_{Kp} + R'R\right)^{-1/2}
\end{aligned}$$

Applying the submultiplicative property of matrix norms and the triangle inequality, we obtain

$$\begin{aligned}
& \left\|\widehat{G}_1 - G_1\right\|_2 \\
\leq & \left\|\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\|_2 \|\Xi\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right\|_2 \\
& + \|G_2\|_2 \|R\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \\
= & \left\|\left(I_{Kp} + R'R\right)^{-1/2} - I_{Kp}\right\|_2 + \|R\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2
\end{aligned}$$

where the last equality follows from the fact that

$$\begin{aligned}
\|\Xi\|_2 &= \sqrt{\lambda_{\max}\left(\Xi'\Xi\right)} = \sqrt{\lambda_{\max}\left(I_{Kp}\right)} = 1, \\
\left\|G_2'\right\|_2 &= \sqrt{\lambda_{\max}\left(G_2 G_2'\right)} = \sqrt{\lambda_{\max}\left(G_2' G_2\right)} = \sqrt{\lambda_{\max}\left(I_{N-Kp}\right)} = 1, \text{ and} \\
\left\|\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\|_2 &= \sqrt{\lambda_{\max}\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'\Gamma}{N_1}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\}} = \sqrt{\lambda_{\max}\left\{I_{Kp}\right\}} = 1.
\end{aligned}$$

Now, if (λ, ρ) is an eigen-pair of $R'R$ so that

$$R'R\rho = \lambda\rho \text{ with } \lambda \geq 0 \text{ given that } R'R \text{ is positive semidefinite;}$$

then,

$$\begin{aligned}
\left(I_{Kp} + R'R\right)\rho &= (1 + \lambda)\rho, \\
\left(I_{Kp} + R'R\right)^{-1/2}\rho &= \frac{1}{\sqrt{1+\lambda}}\rho, \text{ and} \\
\left[I_{Kp} - \left(I_{Kp} + R'R\right)^{-1/2}\right]\rho &= \left(I_{Kp} - \frac{1}{\sqrt{1+\lambda}} I_{Kp}\right)\rho \\
&= \frac{\sqrt{1+\lambda} - 1}{\sqrt{1+\lambda}}\rho
\end{aligned}$$

since

$$\frac{1}{\sqrt[]{1+\lambda}} \text{ is an eigenvalue of } \left(I_{Kp}+R'R\right)^{-1/2} \text{ associated with the eigenvector } \rho$$

and

$$\frac{\sqrt{1+\lambda}-1}{\sqrt[]{1+\lambda}} \text{ is an eigenvalue of } I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2} \text{ associated with the eigenvector } \rho$$

Moreover, let

$$g(\lambda)=\frac{\sqrt{1+\lambda}-1}{\sqrt[]{1+\lambda}}$$

and note that

$$\begin{aligned} g'(\lambda) &= \frac{1}{2}\frac{1}{1+\lambda}-\frac{1}{2}\frac{\sqrt{1+\lambda}-1}{(1+\lambda)^{3/2}} \\ &= \frac{1}{2}\frac{\sqrt{1+\lambda}-\sqrt{1+\lambda}+1}{(1+\lambda)^{3/2}} \\ &= \frac{1}{2(1+\lambda)^{3/2}}>0 \end{aligned}$$

so that, in particular, $g(\lambda)$ is an increasing function of λ for $\lambda \geq 0$. It follows that

$$\begin{aligned} & \left\|\widehat{G}_1-G_1\right\|_2 \\ \leq & \left\|\left(I_{Kp}+R'R\right)^{-1/2}-I_{Kp}\right\|_2+\|R\|_2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \\ = & \left\|I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2+\|R\|_2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \\ = & \sqrt{\lambda_{\max}\left(\left[I_{Kp}-(I_{Kp}+R'R)^{-1/2}\right]'\left[I_{Kp}-(I_{Kp}+R'R)^{-1/2}\right]\right)} \\ & +\|R\|_2\sqrt{\lambda_{\max}\left((I_{Kp}+R'R)^{-1/2\prime}(I_{Kp}+R'R)^{-1/2}\right)} \\ = & \lambda_{\max}\left[I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2}\right]+\|R\|_2\,\lambda_{\max}\left[\left(I_{Kp}+R'R\right)^{-1/2}\right] \\ & \left(\text{since } I_{Kp}-\left(I_{Kp}+R'R\right)^{-1/2} \text{ and } \left(I_{Kp}+R'R\right)^{-1/2} \text{ are both symmetric and positive semidefinite}\right) \\ \leq & \frac{\sqrt{1+\lambda_{\min}(R'R)}-1}{\sqrt[]{1+\lambda_{\min}(R'R)}}+\frac{\|R\|_2}{\sqrt[]{1+\lambda_{\min}(R'R)}} \\ \leq & \sqrt{1+\|R\|_2^2}-1+\|R\|_2 \quad \left(\text{since } \lambda_{\min}\left(R'R\right)\geq 0 \text{ given that } R'R \text{ is positive semi-definite}\right) \\ = & o_p(1) \text{ as } N_1,\ N_2, \text{ and } T\rightarrow\infty \ \left(\text{since } \|R\|_2=o_p(1)\right). \end{aligned}$$

This shows result (b).

To show part (c), note that, by the result given in part (a) above, the columns of $\widehat{G}_1 = (G_1 + G_2 R)(I_r + R'R)^{-1/2}$ form an orthonormal basis for a subspace that is invariant for $A + E$. It then follows immediately from Lemma D-13 that there exists a unique matrix L such that

$$\begin{aligned}
(A+E)\widehat{G}_1 &= (A+E)(G_1 + G_2 R)\left(I_r + R'R\right)^{-1/2} \\
&= (G_1 + G_2 R)\left(I_r + R'R\right)^{-1/2} L \\
&= \widehat{G}_1 L.
\end{aligned}$$

Note further that

$$\begin{aligned}
\widehat{G}_1'\widehat{G}_1 &= \left(I_{Kp} + R'R\right)^{-1/2}\left(G_1' + R'G_2'\right)(G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \left(I_{Kp} + R'R\right)^{-1/2}\left(G_1'G_1 + R'G_2'G_1 + G_1'G_2 R + R'G_2'G_2 R\right)\left(I_{Kp} + R'R\right)^{-1/2} \\
&= \left(I_{Kp} + R'R\right)^{-1/2}\left(I_{Kp} + R'R\right)\left(I_{Kp} + R'R\right)^{-1/2} \\
&\quad \left(\text{since by assumption } G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \text{ is an orthogonal matrix}\right) \\
&= I_{Kp}
\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}
\widehat{G}_1'(A+E)\widehat{G}_1 &= \widehat{G}_1'\left(\frac{\Gamma M_{FF}\Gamma'}{N_1} + E\right)\widehat{G}_1 = \widehat{G}_1'\widehat{G}_1 L \\
&= L
\end{aligned}$$

so that L must be symmetric since, in our situation here,

$$A + E = \frac{\Gamma M_{FF}\Gamma'}{N_1} + \widehat{\Sigma}\left(\widehat{H^c}\right) - \frac{\Gamma M_{FF}\Gamma'}{N_1} = \widehat{\Sigma}\left(\widehat{H^c}\right) = \frac{Z\left(\widehat{H^c}\right)' Z\left(\widehat{H^c}\right)}{N_1 T_0}$$

is a symmetric matrix. Now, let $\widehat{\Lambda} = diag\left(\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right)$ and

$$\widehat{V} = \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix}$$

be as defined in expressions (115) and (116). The fact that L is symmetric implies that $\widehat{V}$ is an orthogonal matrix. In addition, further application of Lemma D-13 shows that $\left(\widehat{G}_1\widehat{v}_g, \widehat{\lambda}_g\right)$ is an eigenpair for the matrix $A + E$ for $g = 1, ..., Kp$.

Finally, to show part (d), let $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, and note that, by assumption,

$$G'AG = \begin{pmatrix} G_1'AG_1 & G_1'AG_2 \\ G_2'AG_1 & G_2'AG_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} = \Lambda$$

where $\Lambda_1 = diag\left(\lambda_{1,1.}, ..., \lambda_{1,Kp}\right)$ contains the Kp largest eigenvalues of A. Without loss of generality, we can further assume that $\lambda_{1,1.}, ..., \lambda_{1,Kp}$ are ordered, so that $\lambda_{1,j} = \lambda_{(j)}\left(A\right)$, i.e., $\lambda_{1,j}$ is the j^{th} largest eigenvalue of A.[9] Given that, $G'G = GG' = I_N$, we have

$$\begin{pmatrix} AG_1 & AG_2 \end{pmatrix} = AG = G\Lambda = \begin{pmatrix} G_1\Lambda_1 & 0 \end{pmatrix}$$

from which it follows that

$$AG_1G_1'\widehat{G}_1\widehat{v}_\ell = G_1\Lambda_1G_1'\widehat{G}_1\widehat{v}_\ell, \text{ for } \ell \in \{1, ..., Kp\}. \tag{118}$$

Now, the result of part (c) above shows $\left(\widehat{G}_1\widehat{v}_\ell, \widehat{\lambda}_\ell\right)$ to be an eigenpair of the matrix $A + E$ for $\ell \in \{1, ..., Kp\}$, so that

$$\left(A + E\right)\widehat{G}_1\widehat{v}_\ell = \widehat{\lambda}_\ell\widehat{G}_1\widehat{v}_\ell \text{ for } \ell \in \{1, ..., Kp\} \tag{119}$$

where $\widehat{G}_1 = \left(G_1 + G_2R\right)\left(I_{Kp} + R'R\right)^{-1/2}$ as given in the result for part (a). Multiplying both sides of expression (119) by $\widehat{v}_\ell'\widehat{G}_1'G_1G_1'$, we get

$$\begin{aligned} \widehat{\lambda}_\ell\widehat{v}_\ell'\widehat{G}_1'G_1G_1'\widehat{G}_1\widehat{v}_\ell &= \widehat{v}_\ell'\widehat{G}_1'G_1G_1'\left(A + E\right)\widehat{G}_1\widehat{v}_\ell \\ &= \widehat{v}_\ell'\widehat{G}_1'G_1G_1'A\widehat{G}_1\widehat{v}_\ell + \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell \end{aligned} \tag{120}$$

Since $A = \Gamma M_{FF}\Gamma'/N_1$ is symmetric, it further follows by expression (118) that

$$\widehat{v}_\ell'\widehat{G}_1'G_1G_1'A = \widehat{v}_\ell'\widehat{G}_1'G_1G_1'A' = \widehat{v}_\ell'\widehat{G}_1'G_1\Lambda_1G_1' \tag{121}$$

[9] If this is not the case; then, we can always define a permutation matrix $\mathcal{P}$ such that

$$\Lambda^* = \mathcal{P}'\Lambda\mathcal{P}$$

results in a diagonal matrix whose diagonal elements are repermutated in such a way, so that the required ordering of the eigenvalues is satisfied. Moreover, since $\mathcal{P}$ is an orthogonal matrix, it further follows that

$$A = G\mathcal{P}\mathcal{P}'\Lambda\mathcal{P}\mathcal{P}'G' = G\mathcal{P}\Lambda^*\mathcal{P}'G'.$$

Now, define $\widetilde{G} = G\mathcal{P}$, and note that $\widetilde{G}$ is an orthogonal matrix whose columns are just the columns of G repermutated. Hence, we can simply proceed with our analysis using $\widetilde{G}$ in lieu of G, and the associated eigenvalues will be in the order which we have assumed.

Moreover, note that

$$
\begin{aligned}
0 &\leq \left(\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell\right)^2 \\
&\leq \left(\widehat{v}_\ell'\widehat{G}_1'G_1G_1'G_1G_1'\widehat{G}_1\widehat{v}_\ell\right)\left(\widehat{v}_\ell'\widehat{G}_1'E'E\widehat{G}_1\widehat{v}_\ell\right) \text{ (by CS inequality)} \\
&= \left(\widehat{v}_\ell'\widehat{G}_1'G_1G_1'\widehat{G}_1\widehat{v}_\ell\right)\left(\widehat{v}_\ell'\widehat{G}_1'E'E\widehat{G}_1\widehat{v}_\ell\right) \text{ (since } G_1'G_1 = I_{Kp}) \\
&= \left[\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1/2}\left(G_1'+R'G_2'\right)G_1G_1'\left(G_1+G_2R\right)\left(I_{Kp}+R'R\right)^{-1/2}\widehat{v}_\ell\right]\left(\widehat{v}_\ell'\widehat{G}_1'E'E\widehat{G}_1\widehat{v}_\ell\right) \\
&= \left[\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell\right]\left(\widehat{v}_\ell'\widehat{G}_1'E'E\widehat{G}_1\widehat{v}_\ell\right) \\
&\leq \left[\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell\right]\lambda_{\max}\left(E'E\right)
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
-\sqrt{\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell}\,\|E\|_2 &= -\sqrt{\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell}\sqrt{\lambda_{\max}\left(E'E\right)} \\
&\leq -\sqrt{\left(\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell\right)^2} \\
&\leq -\left|\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell\right| \\
&\leq \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell \qquad (122)
\end{aligned}
$$

where the last inequality follows from the fact that

$$\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell > -\left|\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell\right| \text{ if } \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell > 0$$

whereas

$$\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell = -\left|\widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell\right| \text{ if } \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell \leq 0$$

Combining expressions (120), (121), and (122), we see that

$$
\begin{aligned}
\widehat{\lambda}_\ell\widehat{v}_\ell'\widehat{G}_1'G_1G_1'\widehat{G}_1\widehat{v}_\ell &= \widehat{v}_\ell'\widehat{G}_1'G_1G_1'A\widehat{G}_1\widehat{v}_\ell + \widehat{v}_\ell'\widehat{G}_1'G_1G_1'E\widehat{G}_1\widehat{v}_\ell \\
&\geq \widehat{v}_\ell'\widehat{G}_1'G_1\Lambda_1G_1'\widehat{G}_1\widehat{v}_\ell - \sqrt{\widehat{v}_\ell'\left(I_{Kp}+R'R\right)^{-1}\widehat{v}_\ell}\,\|E\|_2 \qquad (123)
\end{aligned}
$$

for $\ell \in \{1, ..., Kp\}$. In addition, note that

$$\begin{aligned}
\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell &= \widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \left(G_1 + G_2 R\right) \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \widehat{G}_1' G_1 \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1/2} \left(G_1' + R'G_2'\right) G_1 \left(I_{Kp} + R'R\right)^{-1/2} \widehat{v}_\ell \\
&= \widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell \\
&> 0
\end{aligned}$$

Hence, dividing both sides of expression (123) by $\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell$, we obtain

$$\begin{aligned}
\widehat{\lambda}_\ell &\geq \frac{\widehat{v}_\ell' \widehat{G}_1' G_1 \Lambda_1 G_1' \widehat{G}_1 \widehat{v}_\ell}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} - \frac{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}\, \|E\|_2}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} \\
&= \widetilde{v}_\ell' \Lambda_1 \widetilde{v}_\ell - \frac{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}\, \|E\|_2}{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell} \\
&= \widetilde{v}_\ell' \Lambda_1 \widetilde{v}_\ell - \frac{\|E\|_2}{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{\|E\|_2}{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}}
\end{aligned}$$

where

$$\widetilde{v}_\ell = \frac{G_1' \widehat{G}_1 \widehat{v}_\ell}{\widehat{v}_\ell' \widehat{G}_1' G_1 G_1' \widehat{G}_1 \widehat{v}_\ell} \text{ so that } \|\widetilde{v}_\ell\|_2^2 = \sum_{\ell=1}^{Kp} \widetilde{v}_{\ell,j}^2 = 1.$$

Note also that

$$\begin{aligned}
\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell &\geq \lambda_{\min}\left\{\left(I_{Kp} + R'R\right)^{-1}\right\} \widehat{v}_\ell' \widehat{v}_\ell \\
&= \lambda_{\min}\left\{\left(I_{Kp} + R'R\right)^{-1}\right\} \quad \left(\text{since } \|\widehat{v}_\ell\|_2^2 = 1\right) \\
&= \frac{1}{\lambda_{\max}\left(I_{Kp} + R'R\right)} \\
&\geq \frac{1}{1 + \lambda_{\max}\left(R'R\right)} \\
&= \frac{1}{1 + \|R\|_2^2} \\
&\geq \left[1 + \frac{16\,\|E_{21}\|_2^2}{\left(\mathrm{sep}\left(\Lambda_1, \Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-12)} \\
&\geq \left[1 + \frac{16\,\|E\|_2^2}{\left(\mathrm{sep}\left(\Lambda_1, \Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-3)} \\
&\geq \left[1 + \frac{16\left(\mathrm{sep}\left(\Lambda_1, \Lambda_2\right)\right)^2/25}{\left(\mathrm{sep}\left(\Lambda_1, \Lambda_2\right)\right)^2}\right]^{-1} \quad \text{(by Lemma D-12)} \\
&= \frac{25}{41}
\end{aligned}$$

Making use of this lower bound, we obtain

$$\begin{aligned}
\widehat{\lambda}_\ell &\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{\|E\|_2}{\sqrt{\widehat{v}_\ell' \left(I_{Kp} + R'R\right)^{-1} \widehat{v}_\ell}} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{25}{41}\,\|E\|_2 .
\end{aligned}$$

Next, recall the notations we have introduced previously on the ordering of the eigenvalues of the matrices $A + E$ and A, i.e.,

$$\begin{aligned}
\lambda_{(1)}\left(A + E\right) &\geq \cdots \geq \lambda_{(Kp)}\left(A + E\right) \geq \lambda_{(Kp+1)}\left(A + E\right) \geq \cdots \geq \lambda_{(N)}\left(A + E\right), \\
\lambda_{(1)}\left(A\right) &\geq \cdots \geq \lambda_{(Kp)}\left(A\right) \geq \lambda_{(Kp+1)}\left(A\right) \geq \cdots \geq \lambda_{(N)}\left(A\right)
\end{aligned}$$

Since $A = \Gamma M_{FF} \Gamma' /\!\!/ N_1$ and since part (a) of Lemma D-11 shows that $Rank\left(A\right) = Kp$ for all N_1, N_2, and T sufficiently large; it follows that

$$\lambda_{(Kp+1)}\left(A\right) = \cdots = \lambda_{(N)}\left(A\right) = 0. \tag{124}$$

In addition, by Corollary 8.1.6 of Golub and van Loan (1996), we have the inequality.

$$\lambda_{(Kp+1)}(A+E) \leq \lambda_{(Kp+1)}(A) + \|E\|_2 . \tag{125}$$

Making use of expressions (124) and (125); we see that, for any $\ell \in \{1, ..., Kp\}$,

$$\begin{aligned}
\widehat{\lambda}_\ell - \lambda_{(Kp+1)}(A+E) &\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{25}{41}\|E\|_2 - \left\{\lambda_{(Kp+1)}(A) + \|E\|_2\right\} \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{1,j} - \frac{66}{41}\|E\|_2 \quad \left(\text{since } \lambda_{(Kp+1)}(A) = 0 \text{ here}\right) \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41}\|E\|_2 \\
&\quad \left(\text{since } \lambda_{1,j} = \lambda_{(j)}(A) \text{ as discussed previously}\right) \\
&\geq \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{41}\frac{\operatorname{sep}(\Lambda_1,\Lambda_2)}{5} \qquad (\text{by Lemma D-12}) \\
&= \sum_{j=1}^{Kp} \widetilde{v}_{\ell,j}^2 \lambda_{(j)}(A) - \frac{66}{205}\operatorname{sep}(\Lambda_1, 0) \quad (\text{since } \Lambda_2 = 0 \text{ here}) \\
&\geq \lambda_{\min}(\Lambda_1) - -\frac{66}{205}\operatorname{sep}(\Lambda_1, 0) \quad \left(\text{since } \Lambda_1 = diag\left(\lambda_{(1)}(A), ..., \lambda_{(Kp)}(A)\right)\right) \\
&= \frac{139}{205}\operatorname{sep}(\Lambda_1, 0) \\
&\quad (\text{since } \operatorname{sep}(\Lambda_1, 0) = \lambda_{\min}(\Lambda_1) \text{ by Theorem 3.1 of Stewart and Sun (1990)}) \\
&\geq \frac{139}{205}\underline{c} > 0 \quad (\text{by part (c) of Lemma D-11}).
\end{aligned}$$

This shows that the set $\left\{\widehat{\lambda}_1, ..., \widehat{\lambda}_{Kp}\right\}$ contains the Kp largest eigenvalues of the matrix $A+E$. It further follows from the result given in part (c) that the columns of the matrix

$$\widehat{G}_1\widehat{V} = \widehat{G}_1 \begin{pmatrix} \widehat{v}_1 & \widehat{v}_2 & \cdots & \widehat{v}_{Kp} \end{pmatrix} = \begin{pmatrix} \widehat{G}_1\widehat{v}_1 & \widehat{G}_1\widehat{v}_2 & \cdots & \widehat{G}_1\widehat{v}_{Kp} \end{pmatrix}$$

are the eigenvectors associated with the Kp largest eigenvalues of the matrix $A+E$. □

Lemma D-15: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$\frac{\widehat{N}_1 - N_1}{N_1} \xrightarrow{p} 0$$

(b)

$$\left\| \frac{\Gamma\left(\widehat{H}^c\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0$$

(c) Let

$$\widehat{G}_1 = \left(G_1 + G_2 R\right)\left(I_{Kp} + R'R\right)^{-1/2}$$

where G_1, G_2, and R are as defined in Lemma D-14 above. Also, let $\widehat{V}$ be the $Kp \times Kp$ orthogonal matrix given in expression (116) of Lemma D-14. Then, there exists some positive constant $\overline{C}$ such that

$$\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \leq \overline{C} < \infty$$

for N_1, N_2, and T sufficiently large. In addition,

$$\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0$$

where

$$Q = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \Xi \widehat{V},$$

with Ξ being the $Kp \times Kp$ orthogonal matrix whose columns are the eigenvectors of the matrix

$$M_{FF}^* = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} M_{FF} \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2} \frac{1}{T-p+1} \sum_{t=p}^{T} E\left[\underline{F}_t \underline{F}_t'\right] \left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}.$$

(d) For all fixed index t

$$\left\| \frac{G_1' U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2 = o_p(1).$$

(e) For all fixed index t

$$\left\| \frac{U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2^2 = O_p(1).$$

(f) For all fixed index t,

$$\left\| \frac{G_2' U_{t,N}\left(\widehat{H}^c\right)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

(g) Let

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

where G_1, G_2, and R are as defined in Lemma D-14 above. Also, let $\widehat{V}$ be the $Kp \times Kp$ orthogonal matrix given in expression (116) of Lemma D-14. Then, for all fixed index t,

$$\left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1,\ N_2, \text{ and } T \to \infty.$$

(h)

$$\left\| \frac{\Gamma\left(\widehat{H^c}\right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 = \left\| \frac{\widehat{V}'\widehat{G}_1' \Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 = o_p(1) \text{ as } N_1, N_2, T \to \infty.$$

where Q is as defined in part (c) above.

(i)

$$\|\underline{F}_t\|_2 = O_p(1) \text{ for all } t.$$

(j)

$$\left\|\widehat{\underline{F}}_T - Q'\underline{F}_T\right\|_2 = o_p(1) \text{ as } N_1,\ N_2, \text{ and } T \to \infty$$

where $\widehat{\underline{F}}_T$ denotes the principal component estimator of the factor vector $\underline{F}_T$ obtained after the variables have been pre-screened based on the decision rule

$$i \in \begin{cases} \widehat{H}^c & \text{if } \mathbb{S}^+_{i,T} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \\ \widehat{H} & \text{if } \mathbb{S}^+_{i,T} \leq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right) \end{cases},$$

as described in section 3. Here, $\mathbb{S}^+_{i,T}$ may be either the statistic $\sum_{\ell=1}^{d} \varpi_\ell |S_{i,\ell,T}|$or the statistic $\max_{1\leq \ell \leq d} |S_{i,\ell,T}|$.

Proof of Lemma D-15:

To show part (a), note first that, for any $\epsilon > 0$,

$$\begin{aligned}
\left\{\left|\frac{\widehat{N}_1 - N_1}{N_1}\right| \geq \epsilon\right\} &= \left\{\left|\frac{1}{N_1}\sum_{i=1}^{N}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \epsilon\right\} \\
&= \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} + \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \epsilon\right\} \\
&\subseteq \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| + \left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \epsilon\right\} \\
&\subseteq \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\} \\
&= \left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\}
\end{aligned}$$

By Markov's inequality, we have

$$\begin{aligned}
&\Pr\left(\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right) \\
=\ &\Pr\left(\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right|^2 \geq \frac{\epsilon^2}{4}\right) \\
\leq\ &\frac{4}{\epsilon^2}E\left\{\left|\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right|^2\right\} \\
=\ &\frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}E\left[\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\left(\mathbb{I}\left\{k \in \widehat{H^c}\right\} - 1\right)\right] \\
=\ &\frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left(E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\mathbb{I}\left\{k \in \widehat{H^c}\right\}\right] - E\left[\mathbb{I}\left\{k \in \widehat{H^c}\right\}\right] - E\left[\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] + 1\right) \\
=\ &\frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{\Pr\left(\left\{i \in \widehat{H^c}\right\} \cap \left\{k \in \widehat{H^c}\right\}\right) - \Pr\left(k \in \widehat{H^c}\right)\right\} \\
&+\frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{1 - \Pr\left(i \in \widehat{H^c}\right)\right\} \\
\leq\ &\frac{4}{\epsilon^2}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{k\in H^c}\left\{\Pr\left(k \in \widehat{H^c}\right) - \Pr\left(k \in \widehat{H^c}\right)\right\} + \frac{4}{\epsilon^2}\frac{1}{N_1}\sum_{i\in H^c}\left\{1 - \Pr\left(i \in \widehat{H^c}\right)\right\} \\
\leq\ &\frac{4}{\epsilon^2}\frac{1}{N_1}\sum_{i\in H^c}\left\{1 - \min_{i\in H^c}\Pr\left(i \in \widehat{H^c}\right)\right\} \to 0 \text{ as } N_1, N_2, \text{ and } T \to \infty.
\end{aligned}$$

where the last line above follows from the fact that, for $i \in H^c$ and for either the case where

$\mathbb{S}^{+}_{i,T} = \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the case where $\mathbb{S}^{+}_{i,T} = \max_{1\leq \ell \leq d} \left| S_{i,\ell,T} \right|$, we can apply the results of Theorem 2 in Chao, Qiu, and Swanson (2023a) to obtain

$$\begin{aligned}
\min_{i \in H^c} \Pr\left(i \in \widehat{H^c}\right) &\geq \Pr\left(\bigcap_{i \in H^c} \left\{\mathbb{S}^{+}_{i,T} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right\}\right) \\
&= P\left(\min_{i \in H^c} \mathbb{S}^{+}_{i,T} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
&\to 1
\end{aligned}$$

Also, making use of Markov's inequality, we obtain, for either the case where $\mathbb{S}^{+}_{i,T} = \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the case where $\mathbb{S}^{+}_{i,T} = \max_{1\leq \ell \leq d} \left| S_{i,\ell,T} \right|$,

$$\begin{aligned}
& \Pr\left(\left| \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right| \geq \frac{\epsilon}{2}\right) \\
= & \Pr\left(\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \geq \frac{\epsilon}{2}\right) \\
\leq & \frac{2}{\epsilon} E\left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \\
= & \frac{2}{\epsilon} \frac{1}{N_1} \sum_{i \in H} \Pr\left(i \in \widehat{H^c}\right) \\
= & \frac{2}{\epsilon} \frac{1}{N_1} \sum_{i \in H} \Pr\left(\mathbb{S}^{+}_{i,T} > \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
\leq & \frac{2}{\epsilon} \frac{d N_2 \varphi}{N N_1} \left[1 + o\left(1\right)\right] \\
& \text{(following an argument similar to that given in the proof of Theorem 1 in} \\
& \text{Chao, Qiu, and Swanson (2023a)}^{10}\text{)} \\
\to & \; 0 \; \left(\text{since } \frac{\varphi}{N_1} \to 0 \text{ and } \frac{N_2}{N} = O\left(1\right)\right).
\end{aligned}$$

Combining these results, we have that

$$\begin{aligned}
& \Pr\left(\left|\frac{\widehat{N}_1 - N_1}{N_1}\right| \geq \epsilon\right) \\
\leq\ & \Pr\left(\left\{\left|\frac{1}{N_1}\sum_{i \in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right\} \cup \left\{\left|\frac{1}{N_1}\sum_{i \in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right\}\right) \\
\leq\ & \Pr\left(\left|\frac{1}{N_1}\sum_{i \in H^c}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\} - 1\right)\right| \geq \frac{\epsilon}{2}\right) + \Pr\left(\left|\frac{1}{N_1}\sum_{i \in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\right| \geq \frac{\epsilon}{2}\right) \\
& \text{(by the union bound)} \\
\to\ & 0
\end{aligned}$$

For part (b), note that

$$\begin{aligned}
\left\|\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}}\right\|_F^2 &= \frac{1}{N_1} tr\left\{\left(\Gamma\left(\widehat{H^c}\right) - \Gamma\right)'\left(\Gamma\left(\widehat{H^c}\right) - \Gamma\right)\right\} \\
&= \frac{1}{N_1}\sum_{i=1}^{N} tr\left\{\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)'\right\} \\
&= \frac{1}{N_1}\sum_{i=1}^{N}\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right)'\left(\mathbb{I}\left\{i \in \widehat{H^c}\right\}\gamma_i - \gamma_i\right) \\
&= \frac{1}{N_1}\sum_{i=1}^{N}\gamma_i'\gamma_i\left[1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \\
&= \frac{1}{N_1}\sum_{i \in H^c}\gamma_i'\gamma_i\left[1 - \mathbb{I}\left\{i \in \widehat{H^c}\right\}\right] \quad (\text{since } \gamma_i = 0 \text{ for } i \in H)
\end{aligned}$$

Applying Markov's inequality, we have, for any $\epsilon > 0$,

$$
\begin{aligned}
& \Pr\left(\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_F^2 \geq \epsilon\right) \\
\leq \quad & \frac{1}{\epsilon} E\left\{\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i\left[1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\right\} \\
= \quad & \frac{1}{\epsilon}\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i\left[1-\Pr\left(i\in\widehat{H^c}\right)\right] \\
\leq \quad & \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\frac{1}{N_1}\sum_{i\in H^c}\gamma_i'\gamma_i \\
\leq \quad & \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\left(\sup_{i\in H^c}\|\gamma_i\|_2\right)^2 \\
\leq \quad & \frac{1}{\epsilon}\left[1-\min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\right]\overline{C}^2 \quad \text{(by Assumption 3-5)} \\
\to \quad & 0 \left(\text{since } \min_{i\in H^c}\Pr\left(i\in\widehat{H^c}\right)\to 1 \text{ for } i\in H^c \text{ by Theorem 2 in Chao, Qiu, and Swanson (2023a)}\right)
\end{aligned}
$$

from which we further deduce that

$$\left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_2 \leq \left\|\frac{\Gamma\left(\widehat{H^c}\right)-\Gamma}{\sqrt{N_1}}\right\|_F \xrightarrow{p} 0.$$

Turning our attention to part (c), note that since, by definition,

$$\widehat{G}_1 = (G_1 + G_2 R)\left(I_{Kp} + R'R\right)^{-1/2}$$

where $G_1'G_1 = I_{Kp}$, $G_2'G_2 = I_{N-Kp}$, and $G_1'G_2 = 0$; it follows that

$$
\begin{aligned}
\widehat{G}_1'\widehat{G}_1 &= \left(I_{Kp}+R'R\right)^{-1/2}\left(G_1'+R'G_2'\right)(G_1+G_2R)\left(I_{Kp}+R'R\right)^{-1/2} \\
&= \left(I_{Kp}+R'R\right)^{-1/2}\left(I_{Kp}+R'R\right)\left(I_{Kp}+R'R\right)^{-1/2} \\
&= I_{Kp}
\end{aligned}
$$

Hence, by Assumption 3-6,

$$\begin{aligned}
\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 &\leq \left\| \widehat{V}'\widehat{G}_1' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&= \sqrt{\lambda_{\max}\left(\widehat{G}_1\widehat{V}\widehat{V}'\widehat{G}_1'\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}\left(\widehat{V}'\widehat{G}_1'\widehat{G}_1\widehat{V}\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \\
&= \sqrt{\lambda_{\max}\left(I_{Kp}\right)}\sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \quad \left(\text{since } \widehat{V} \text{ is an orthogonal matrix}\right) \\
&= \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \leq \overline{C} < \infty \text{ for } N_1, N_2 \text{ sufficiently large}
\end{aligned}$$

Now, to show the second result in part (c), note that, since

$$Q = \left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \Xi\widehat{V} \text{ and } G_1 = \frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi = \Gamma\left(\Gamma'\Gamma\right)^{-1/2}\Xi\ ,$$

we can write

$$\begin{aligned}
\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' &= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{\frac{1}{2}} \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'\Gamma}{N_1} \\
&= \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - \frac{\widehat{V}'G_1'\Gamma}{\sqrt{N_1}} \\
&= \widehat{V}'\left(\widehat{G}_1 - G_1\right)'\frac{\Gamma}{\sqrt{N_1}}
\end{aligned}$$

from which it follows that

$$
\begin{aligned}
\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 &\leq \left\| \widehat{V}' \right\|_2 \left\| \left(\widehat{G}_1 - G_1 \right)' \right\|_2 \left\| \frac{\Gamma}{\sqrt{N_1}} \right\|_2 \\
&= \sqrt{\lambda_{\max}\left(\widehat{V}\widehat{V}' \right)} \sqrt{\lambda_{\max}\left\{ \left(\widehat{G}_1 - G_1 \right) \left(\widehat{G}_1 - G_1 \right)' \right\}} \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right)} \\
&= \sqrt{\lambda_{\max}\left(I_{Kp} \right)} \sqrt{\lambda_{\max}\left\{ \left(\widehat{G}_1 - G_1 \right)' \left(\widehat{G}_1 - G_1 \right) \right\}} \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1} \right)} \\
&\quad \left(\text{since } \widehat{V} \text{ is an orthogonal matrix and since } \lambda_{\max}\left(AA' \right) = \lambda_{\max}\left(A'A \right) \right) \\
&\leq \sqrt{\overline{C}} \left\| \widehat{G}_1 - G_1 \right\|_2 \quad \text{(by Assumption 3-6)} \\
&= o_p(1) \quad \text{as } N_1,\ N_2, \text{ and } T \to \infty \ \text{(by part (b) of Lemma D-14)}.
\end{aligned}
$$

Next, to show part (d), we first write

$$
\begin{aligned}
\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c} \right)}{\sqrt{N_1}} \right\|_2^2 &= \sum_{k=1}^{Kp} \left(\sum_{i=1}^{N} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} + \sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&\leq 2 \sum_{k=1}^{K} \left(\sum_{i \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 + 2 \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \frac{g_{1,ik} u_{i,t}}{\sqrt{N_1}} \right)^2 \\
&= \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&\quad + \frac{2}{N_1} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \qquad (126)
\end{aligned}
$$

where $g_{1,ik}$ denotes the $(i,k)^{th}$ element of G_1. Now, consider the first term on the right-hand side of expression (126). Write

$$\begin{aligned}
& \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1+1\right)\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1+1\right)g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}g_{1,jk}u_{j,t} \\
& +\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
= & \mathcal{E}_{1,1,t}+\mathcal{E}_{1,2,t}+\mathcal{E}_{1,3,t}+\mathcal{E}_{1,4,t}
\end{aligned}$$

Focusing first on the term $\mathcal{E}_{1,1,t}$, we have

$$\begin{aligned}
& \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}\left(\mathbb{I}\left\{j\in\widehat{H^c}\right\}-1\right)g_{1,jk}u_{j,t} \\
= & \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right)^2 \\
\leq & \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\left|\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right|\right)^2 \\
\leq & 2\sum_{k=1}^{Kp}\left(\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2\right)\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-2\mathbb{I}\left\{i\in\widehat{H^c}\right\}+1\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
= & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right)
\end{aligned}$$

Now, for either the case where $\mathbb{S}_{i,T}^{+}=\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+}=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have

$$
\begin{aligned}
0 &\leq E\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c}\left[1-\Pr\left(i\in\widehat{H^c}\right)\right] \\
&= \frac{1}{N_1}\sum_{i\in H^c}\left[1-P\left(\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\right] \\
&\leq 1-P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
&\quad \text{(given that } N_1=\#\left\{H^c\right\}\text{, where } \#\left\{H^c\right\} \text{ denotes the cardinality of the set } H^c\text{)} \\
&\to 0\ ,
\end{aligned}
$$

since, by Theorem 2 in Chao, Qiu, and Swanson (2023a), $P\left(\min_{i\in H^c}\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right)\to 1$. Moreover, by part (b) of Assumption 3-3, we have

$$
E\left[\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right]=\sum_{i\in H^c}g_{1,ik}^2E\left[u_{i,t}^2\right]\leq C\sum_{i=1}^{N}g_{1,ik}^2\leq C
$$

It follows by Markov's inequality that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)=o_p\left(1\right)\ \text{and}\ \sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2=O_p\left(1\right)
$$

from which we deduce that

$$
\begin{aligned}
\mathcal{E}_{1,1,t} &= \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right)^2 \\
&\leq 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\left(\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2\right) \\
&= o_p\left(1\right)
\end{aligned}
$$

Consider next the term $\mathcal{E}_{1,2,t}$. To proceed, let $U_{t,N}\left(H^c\right)$ denote an $N\times 1$ vector whose i^{th} component $U_{i,t,N}\left(H^c\right)$ is given by

$$
U_{i,t,N}\left(H^c\right)=\begin{cases} u_{i,t} & \text{if } i\in H^c \\ 0 & \text{if } i\in H \end{cases}.
$$

and we can write

$$
\begin{aligned}
\mathcal{E}_{1,2,t} &= \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&= 2\left\|\frac{G_1' U_{t,N}\left(H^c\right)}{\sqrt{N_1}}\right\|_2^2 \\
&\leq 2tr\left\{\frac{G_1' U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)' G_1}{N_1}\right\} \\
&= 2tr\left\{\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'}{\sqrt{N_1}}\frac{U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'}{N_1}\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi\right\} \\
&= 2tr\left\{\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma'}{\sqrt{N_1}}\frac{U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'}{N_1}\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right\} \\
&= 2tr\left\{\frac{\Gamma_*' U_{t,N}\left(H^c\right)U_{t,N}\left(H^c\right)'\Gamma_*}{N_1^2}\right\}\quad\left(\text{where }\Gamma_*=\Gamma\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right) \\
&= \frac{2}{N_1^2}U_{t,N}\left(H^c\right)'\Gamma_*\Gamma_*' U_{t,N}\left(H^c\right) \\
&= \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\gamma_{*,i}'\gamma_{*,j}u_{i,t}u_{j,t}
\end{aligned}
$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma\left(\Gamma'\Gamma/N_1\right)^{-1/2}$. Hence,

$$\begin{aligned}
0 &\leq E\left[\mathcal{E}_{1,2,t}\right] \\
&= \frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_{*,i}\gamma_{*,j}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j E\left[u_{i,t}u_{j,t}\right] \\
&\leq \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\left|\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j\right|\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{1}{N_1^2}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\quad \Bigg(\text{since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \sup_{i\in H^c}\left\|\gamma_i\right\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\Bigg) \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1} \to 0 \text{ as } N_1 \to \infty. \Bigg(\text{since, under Assumption 3-3(d), there exists a positive constant } \overline{C} \\
&\qquad \text{such that } \sup_t \frac{1}{N_1}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \leq \overline{C} < \infty\Bigg)
\end{aligned}$$

It follows by Markov's inequality that

$$\mathcal{E}_{1,s,t} = o_p\left(1\right).$$

Now, for $\mathcal{E}_{1,3,t}$, write

$$
\begin{aligned}
& \left|\mathcal{E}_{1,3,t}\right| \\
= & \left|\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}g_{1,jk}u_{j,t}\right| \\
= & \left|\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{j\in H^c}g_{1,jk}u_{j,t}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right| \\
\leq & \frac{2}{N_1}\sum_{k=1}^{Kp}\left|\sum_{j\in H^c}g_{1,jk}u_{j,t}\right|\left|\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right| \\
\leq & \frac{2}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\sqrt{\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2}\left|\sum_{j\in H^c}g_{1,jk}u_{j,t}\right| \\
\leq & \frac{1}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
& +\frac{1}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2 \\
& \left(\text{by the inequality } |XY|\leq\frac{1}{2}X^2+\frac{1}{2}Y^2\right) \\
= & \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
& +\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= & \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sum_{j\in H^c}\sum_{\ell\in H^c} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= & \frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sum_{k=1}^{Kp} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= & \frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\frac{e_{j,N}'\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\sum_{k=1}^{Kp}\Xi e_{k,Kp}e_{k,Kp}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\frac{\Gamma' e_{\ell,N}}{\sqrt{N_1}}E\left[u_{j,t}u_{\ell,t}\right] \\
& \left(\text{since } G_1=\frac{\Gamma}{\sqrt{N_1}}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\Xi\right) \\
= & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Xi\Xi'\Gamma_*' e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \quad \left(\text{where } \Gamma_*=\Gamma\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right) \\
= & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Gamma_*' e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
& (\text{since } \Xi \text{ is an orthogonal matrix}) \\
= & \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right]
\end{aligned}
$$

where we take

$$\gamma_{*,j}=\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j,$$

Applying the triangle and Cauchy-Schwarz inequalities, we further obtain

$$
\begin{aligned}
& E\left[\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right] \\
= \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell E\left[u_{j,t}u_{\ell,t}\right] \\
\leq \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell\right|\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \;& \frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sqrt{\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_j}\sqrt{\gamma_\ell'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_\ell}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \;& \frac{\overline{c}}{\underline{C}}\frac{1}{\sqrt{N_1}}\frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
& \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
& \left.\sup_{i\in H^c}\|\gamma_i\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\geq \underline{C} > 0\right) \\
\leq \;& \frac{\overline{c}}{\underline{C}}\frac{\overline{C}}{\sqrt{N_1}} \;\to 0 \text{ as } N_1\to\infty. \text{(since, under Assumption 3-3(d) that there exists a} \\
& \qquad\qquad \left.\text{positive constant } \overline{C} \text{ such that } \sup_t \frac{1}{N_1}\sum_{j\in H}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}
$$

from which we further deduce, upon applying Markov's inequality, that

$$\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2 = o_p(1).$$

Moreover, since we have previously shown that

$$\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right) = o_p(1) \text{ and } \sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1),$$

it follows from these calculations that

$$
\begin{aligned}
\left|\mathcal{E}_{1,3,t}\right| &\leq \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
&\quad+\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2 \\
&= o_p(1).
\end{aligned}
$$

In a similar way, we can also show that

$$
\left|\mathcal{E}_{1,4,t}\right| = o_p(1).
$$

Finally, application of the Slutsky's theorem then allows us to deduce that

$$
\begin{aligned}
\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} &= \mathcal{E}_{1,1,t}+\mathcal{E}_{1,2,t}+\mathcal{E}_{1,3,t}+\mathcal{E}_{1,4,t} \\
&= o_p(1)+o_p(1)+o_p(1)+o_p(1) \\
&= o_p(1).
\end{aligned}
$$

Next, consider the second term on the right-hand side of expression (126). In this case, write

$$
\begin{aligned}
&\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&= \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}g_{1,ik}u_{i,t}\right)^2 \\
&= \frac{2}{N_1}\sum_{k=1}^{Kp}\left(\left|\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}g_{1,ik}u_{i,t}\right|\right)^2 \\
&\leq 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\left[\sum_{i\in H}g_{1,ik}^2u_{i,t}^2\right]
\end{aligned}
$$

Note that, for either the case where $\mathbb{S}_{i,T}^{+}=\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+}=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have, by applying an argument similar to that given in the proof of Theorem 1 in Chao, Qiu, and Swanson (2023b),

$$\begin{aligned}
0 &\leq E\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right] \\
&= \frac{1}{N_1}\sum_{i\in H}\Pr\left(i\in\widehat{H^c}\right) \\
&= \frac{1}{N_1}\sum_{i\in H}P\left(\mathbb{S}^+_{i,T}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
&\leq \frac{N_2\varphi}{NN_1}\left\{1+2^2AT_0^{-(1-\alpha_1)\frac{1}{2}}+2^2A\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)^3T_0^{-(1-\alpha_1)\frac{1}{2}}\right\} \\
&= \frac{N_2\varphi}{N_1\left(N_1+N_2\right)}\left[1+o\left(1\right)\right] \\
&\to 0 \text{ as } N_1,N_2,T\to\infty
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$E\left[\sum_{i\in H}g^2_{1,ik}u^2_{i,t}\right]=\sum_{i\in H}g^2_{1,ik}E\left[u^2_{i,t}\right]\leq C\sum_{i=1}^N g^2_{1,ik}\leq C.$$

It follows by Markov's inequality that

$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}=o_p\left(1\right) \text{ and } \sum_{i\in H}g^2_{1,ik}u^2_{i,t}=O_p\left(1\right)$$

from which we deduce that

$$\begin{aligned}
&\frac{2}{N_1}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
\leq\ & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\left[\sum_{i\in H}g^2_{1,ik}u^2_{i,t}\right] \\
=\ & o_p\left(1\right).
\end{aligned}$$

Combining these results and using the inequality $\sqrt{a_1+a_2}\leq\sqrt{a_1}+\sqrt{a_2}$, we further obtain, for all t,

$$\begin{aligned}
\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 &\leq \sqrt{\frac{2}{N_1} \sum_{k=1}^{K} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
&\quad + \sqrt{\frac{2}{N_1} \sum_{k=1}^{K} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t}} \\
&= o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}$$

For part (e), write

$$\begin{aligned}
\left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &= \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{N_1} \sum_{i=1}^{N} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&= \frac{1}{N_1} \sum_{i \in H^c} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&\leq \frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 + \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2
\end{aligned}$$

Note that, by Assumption 3-3(b),

$$E\left[\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2\right] = \frac{1}{N_1} \sum_{i \in H^c} E\left[u_{i,t}^2\right] \leq C \ \ (\text{since } N_1 = \#\{H^c\})$$

so that, by applying Markov's inequality, we obtain

$$\frac{1}{N_1} \sum_{i \in H^c} u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i \in H} \left\{i \notin \widehat{H^c}\right\} \subseteq \left\{ \frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 < \epsilon \right\}$$

so that by DeMorgan's law

$$\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2\geq\epsilon\right\}\subseteq\left\{\bigcap_{i\in H}\left\{i\notin\widehat{H^c}\right\}\right\}^c=\bigcup_{i\in H}\left\{i\in\widehat{H^c}\right\}$$

Hence, for either the case where $\mathbb{S}_{i,T}^{+}=\sum_{\ell=1}^{d}\varpi_\ell\left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}_{i,T}^{+}=\max_{1\leq\ell\leq d}\left|S_{i,\ell,T}\right|$, we have, by applying an argument similar to that given in the proof of Theorem 1 as shown in Chao, Qiu, and Swanson (2023b),

$$\begin{aligned}
& \Pr\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2\geq\epsilon\right\} \\
\leq\ & \Pr\left\{\bigcup_{i\in H}\left\{i\in\widehat{H^c}\right\}\right\} \\
\leq\ & \sum_{i\in H}\Pr\left\{i\in\widehat{H^c}\right\} \\
=\ & \sum_{i\in H}P\left(\mathbb{S}_{i,T}^{+}\geq\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq\ & \frac{N_2\varphi}{N}\left\{1+2^2AT_0^{-(1-\alpha_1)\frac{1}{2}}+2^2A\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)^3T_0^{-(1-\alpha_1)\frac{1}{2}}\right\} \\
=\ & \frac{N_2\varphi}{N_1+N_2}\left[1+o\left(1\right)\right] \\
\to\ & 0\text{ as }N_1,N_2,T\to\infty
\end{aligned}$$

Hence,

$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2=o_p\left(1\right)$$

from which it further follows that

$$\begin{aligned}
\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{1}{N_1}\sum_{i\in H^c}u_{i,t}^2+\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}u_{i,t}^2 \\
&= O_p\left(1\right)+o_p\left(1\right) \\
&= O_p\left(1\right).
\end{aligned}$$

Turning our attention to part (f), note first that since $G=\left[\begin{array}{cc}G_1 & G_2\end{array}\right]$ is an orthogonal matrix, we have $I_N = GG' = G_1G_1' + G_2G_2'$ or $G_2G_2' = I_N - G_1G_1'$. Hence, we can write

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &= \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} - \frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&\leq \frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} + \frac{U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Applying the results from parts (d) and (e) above, we then obtain

$$\begin{aligned}
\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 &\leq \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

so that

$$\left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1).$$

Now, to show part (g), first write

$$\begin{aligned}
\frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} &= \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1 - N_1 + N_1\right)/N_1}} \\
&= \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \frac{\widehat{V}' \widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}
\end{aligned}$$

Note that

$$\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
= & \left\| \frac{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}\left[G_1' U_{t,N}\left(\widehat{H^c}\right)+R'G_2' U_{t,N}\left(\widehat{H^c}\right)\right]}{\sqrt{N_1}} \right\|_2 \\
\leq & \left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
& + \left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
= & \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
& \left(\text{since } \widehat{V}'\widehat{V} = I_{Kp} \text{ so that } \left\|\widehat{V}\right\|_2 = 1\right)
\end{aligned}$$

It follows that

$$
\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
= & \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \\
\leq & \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \left\| \left(I_{Kp} + R'R\right)^{-1/2} \right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \left\| \left(I_{Kp} + R'R\right)^{-1/2} \right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
\leq & \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \frac{1}{\sqrt{1 + \lambda_{\min}\left(R'R\right)}} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \frac{\|R\|_2}{\sqrt{1 + \lambda_{\min}\left(R'R\right)}} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
\leq & \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\} \\
= & o_p(1)
\end{aligned}
$$

where the last line follows from the fact that

$$
\|R\|_2 \xrightarrow{p} 0, \ \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \xrightarrow{p} 1, \ \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \text{ and } \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1)
$$

as shown in part (a) in Lemma D-14 and in parts (a), (d), and (f) of this lemma.

Turning our attention to part (h), we write

$$
\begin{aligned}
&\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
= \;& Q' + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{\widehat{N}_1}} - Q'\right) + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{\widehat{N}_1}}\right) \\
= \;& Q' + \left(\left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right) \\
& + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 + 1\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right) \\
= \;& Q' + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q'\right) + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \\
& + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1\right]\widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right) + \widehat{V}'\widehat{G}_1'\left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N}_1}\right)
\end{aligned}
$$

so that, by the triangle inequality

$$
\begin{aligned}
& \left\| \frac{\Gamma\left(\widehat{H^c}\right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 \\
= & \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
\leq & \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left\| \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right] \widehat{V}'\widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 + \left\| \widehat{V}'\widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \right\|_2 \\
\leq & \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \widehat{V}'\widehat{G}_1' \right\|_2 \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \widehat{V}'\widehat{G}_1' \right\|_2 \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \\
= & \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2
\end{aligned}
$$

where the last equality follows from the fact that

$$\left\| \widehat{V}'\widehat{G}_1' \right\|_2 = \left\| \widehat{G}_1 \widehat{V} \right\|_2 = \sqrt{\lambda_{\max}\left(\widehat{V}'\widehat{G}_1'\widehat{G}_1\widehat{V}\right)} = \sqrt{\lambda_{\max}\left(I_{Kp}\right)} = 1.$$

Now, by parts (a), (b), and (c) of this lemma, we have that

$$\left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \xrightarrow{p} 0, \quad \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0, \left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right\|_2 \xrightarrow{p} 0, \text{ and}$$

and

$$\left\| \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \right\|_2 \leq \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)} \leq \overline{C} < \infty \text{ for all } N_1, N_2 \text{ sufficiently large.}$$

It follows that

$$\begin{aligned}
\left\| \frac{\Gamma\left(\widehat{H^c}\right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right\|_2 &= \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \\
&\leq \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \\
&\quad + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 + \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \\
&= o_p(1).
\end{aligned}$$

To show part (i), let $\overline{C}$ be the positive constant given in Lemma C-4 such that

$$E \left\| \underline{F}_t \right\|_2^6 \leq \overline{C} < \infty \text{ for all } t;$$

and, for any $\epsilon > 0$, we let $C_\epsilon = \overline{C}^{\frac{1}{6}} / \sqrt{\epsilon}$. Applying Markov's inequality, we see that

$$\begin{aligned}
\Pr\left(\left\| \underline{F}_t \right\|_2 \geq C_\epsilon\right) &\leq \Pr\left(\left\| \underline{F}_t \right\|_2^2 \geq C_\epsilon^2\right) \\
&\leq \frac{1}{C_\epsilon^2} E \left\| \underline{F}_t \right\|_2^2 \\
&\leq \frac{1}{C_\epsilon^2} \left(E \left\| \underline{F}_t \right\|_2^6\right)^{\frac{1}{3}} \\
&\quad \text{(by Liapunov's inequality)} \\
&\leq \frac{\epsilon}{\overline{C}^{\frac{1}{3}}} \overline{C}^{\frac{1}{3}} \\
&\leq \epsilon
\end{aligned}$$

from which it follows that $\left\| \underline{F}_t \right\|_2 = O_p(1)$ for all t.

Lastly, to show part (j), note that, similar to the derivation given in the proof of Theorem 4.1, except that we replace the fixed index t with the sample size T, we can write

$$\begin{aligned}
\widehat{\underline{F}}_T - Q'\underline{F}_T &= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q' \right) \underline{F}_T + \frac{\widehat{V}'\widehat{G}_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \\
&= \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} - Q' \right) \underline{F}_T + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} - 1 \right] \frac{\widehat{V}'\widehat{G}_1'\Gamma}{\sqrt{N_1}} \underline{F}_T \\
&\quad + \left[\left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right] \widehat{V}'\widehat{G}_1' \left(\frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right) \underline{F}_T + \frac{\widehat{V}'\widehat{G}_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}
\end{aligned}$$

Next, note that, by following the same derivation as that given for the proof of part (g), we can show that

$$\begin{aligned}
&\left\| \frac{\widehat{V}'\widehat{G}_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
\leq\ & \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1} \right)^{-\frac{1}{2}} \right| \left\{ \left\| \frac{G_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \|R\|_2 \left\| \frac{G_2'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right\}
\end{aligned}$$

Moreover, by argument similar to that given for parts (d) and (f) of this lemma, we can show that, as N_1, N_2, and $T \to \infty$;

$$\left\| \frac{G_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \xrightarrow{p} 0 \tag{127}$$

and

$$\left\| \frac{G_2'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1). \tag{128}$$

It follows from applying expressions (127) and (128), part (a) of this lemma, and part (a) of Lemma D-14 that

$$\left\| \frac{\widehat{V}'\widehat{G}_1'U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \xrightarrow{p} 0 \text{ as } N_1, N_2, \text{ and } T \to \infty. \tag{129}$$

In addition, note that by applying Lemma C-4 and the Markov's inequality in a way similar to the argument given for the proof of part (i) above, we can show that

$$\|\underline{F}_T\|_2 = O_p(1). \tag{130}$$

Making use of the results given in expressions (129) and (130) and applying the triangle inequality as well as parts (a)-(c) of this lemma, expression (130), and the Slutsky's theorem; we then obtain, as N_1, N_2, and $T \to \infty$;

$$\begin{aligned}
& \left\| \widehat{\underline{F}}_T - Q' \underline{F}_T \right\|_2 \\
\leq & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|\underline{F}_T\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{F}_T\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\| \widehat{V}' \widehat{G}_1' \right\|_2 \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{F}_T\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
= & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} - Q' \right\|_2 \|\underline{F}_T\|_2 + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} - 1 \right| \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{F}_T\|_2 \\
& + \left| \left(1 + \frac{\widehat{N}_1 - N_1}{N_1}\right)^{-\frac{1}{2}} \right| \left\| \frac{\Gamma\left(\widehat{H^c}\right) - \Gamma}{\sqrt{N_1}} \right\|_2 \|\underline{F}_T\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U_{T,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
& \left(\text{again since } \left\| \widehat{V}' \widehat{G}_1' \right\|_2 = \lambda_{\max}\left(\widehat{G}_1 \widehat{V} \widehat{V}' \widehat{G}_1'\right) = \lambda_{\max}\left(\widehat{V}' \widehat{G}_1' \widehat{G}_1 \widehat{V}\right) = \lambda_{\max}\left(I_{Kp}\right) = 1\right) \\
= & \; o_p(1) O_p(1) + o_p(1) O_p(1) O_p(1) + O_p(1) o_p(1) O_p(1) + o_p(1) \\
= & \; o_p(1) . \; \square
\end{aligned}$$

Lemma D-16: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true as $N_1, N_2, T \to \infty$.

(a)
$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = o_p(1), \text{ where } T_h = T - h - p + 1.$$

(b)
$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 = O_p(1).$$

(c)
$$, \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 = O_p(1)$$

(d)

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\right\|_2^2 = O_p\left(1\right) \text{ and } \left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 = O_p\left(1\right)$$

(e)

$$\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}\right\|_2 = o_p\left(1\right)$$

(f)

$$\left\|\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 = o_p\left(1\right)$$

(g)

$$\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{F}_t - Q'\underline{F}_t\right)\left(\widehat{F}_t - Q'\underline{F}_t\right)'\right\|_2 = o_p\left(1\right).$$

Proof of Lemma D-16:

For part (a), first write

$$\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i=1}^{N}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}} + \sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
\leq & \frac{2}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 + \frac{2}{T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\frac{g_{1,ik}u_{i,t}}{\sqrt{N_1}}\right)^2 \\
= & \frac{2}{N_1T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
& +\frac{2}{N_1T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \qquad (131)
\end{aligned}$$

where $g_{1,ik}$ denotes the $(i,k)^{th}$ element of

$$G_1 = \frac{\Gamma_* \Xi}{\sqrt{N_1}} = \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma' \Gamma}{N_1} \right)^{-1/2} \Xi$$

Now, where

$$\begin{aligned}
& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \mathbb{I}\left\{ j \in \widehat{H^c} \right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 + 1 \right) \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 + 1 \right) g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 \right) g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 \right) g_{1,jk} u_{j,t} \\
& + \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \underline{\mathcal{E}}_{1,1} + \underline{\mathcal{E}}_{1,2} + \underline{\mathcal{E}}_{1,3} + \underline{\mathcal{E}}_{1,4}
\end{aligned}$$

Focusing first on the term $\underline{\mathcal{E}}_{1,1}$, we have

$$\begin{aligned}
& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \sum_{j \in H^c} \left(\mathbb{I}\left\{ j \in \widehat{H^c} \right\} - 1 \right) g_{1,jk} u_{j,t} \\
= \;& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \right)^2 \\
\leq \;& \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right) g_{1,ik} u_{i,t} \right| \right)^2 \\
\leq \;& 2 \sum_{k=1}^{Kp} \left(\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 1 \right)^2 \right) \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
= \;& 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(\mathbb{I}\left\{ i \in \widehat{H^c} \right\} - 2\mathbb{I}\left\{ i \in \widehat{H^c} \right\} + 1 \right) \right] \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right) \\
= \;& 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right) \right] \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\sum_{i \in H^c} g_{1,ik}^2 u_{i,t}^2 \right)
\end{aligned}$$

Now, for either the case where $\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right|$, we have, by applying Theorem 2 in Chao, Qiu, and Swanson (2023a),

$$\begin{aligned}
0 \;\leq\;& E\left[\frac{1}{N_1} \sum_{i \in H^c} \left(1 - \mathbb{I}\left\{ i \in \widehat{H^c} \right\} \right) \right] \\
=\;& \frac{1}{N_1} \sum_{i \in H^c} \left[1 - \Pr\left(i \in \widehat{H^c} \right) \right] \\
=\;& \frac{1}{N_1} \sum_{i \in H^c} \left[1 - P\left(\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N} \right) \right) \right] \\
\leq\;& 1 - P\left(\min_{i \in H^c} \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N} \right) \right) \\
& \text{(given that } N_1 = \#\left\{ H^c \right\}, \text{ where } \#\left\{ H^c \right\} \text{ denotes the cardinality of the set } H^c \text{)} \\
\to\;& 0 \; \left(\text{since } P\left(\min_{i \in H^c} \mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N} \right) \right) \to 1 \right).
\end{aligned}$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2\right] &= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 E\left[u_{i,t}^2\right] \\
&\leq C\frac{T-h-p+1}{T_h}\sum_{i=1}^{N} g_{1,ik}^2 \\
&\leq C \quad \left(\text{since } \sum_{i=1}^{N} g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1\right)
\end{aligned}$$

It follows by Markov's inequality that

$$\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right) = o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2 = O_p(1)$$

from which we deduce that

$$\begin{aligned}
\underline{\mathcal{E}}_{1,1} &= \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left(\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right) g_{1,ik} u_{i,t}\right)^2 \\
&\leq 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\right]\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{i\in H^c} g_{1,ik}^2 u_{i,t}^2\right) \\
&= o_p(1).
\end{aligned}$$

Next, consider the term $\underline{\mathcal{E}}_{1,2}$. To proceed, write

$$
\begin{aligned}
& \left|\underline{\mathcal{E}}_{1,2}\right| \\
= & \left|\frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\sum_{j\in H^c}g_{1,jk}u_{j,t}\right| \\
= & \left|\frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{j\in H^c}g_{1,jk}u_{j,t}\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right| \\
\leq & \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\left|\sum_{j\in H^c}g_{1,jk}u_{j,t}\right|\left|\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)g_{1,ik}u_{i,t}\right| \\
\leq & \frac{2}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2}\left|\sum_{j\in H^c}g_{1,jk}u_{j,t}\right| \\
\leq & \frac{1}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
& +\frac{1}{N_1}\sum_{k=1}^{Kp}\sqrt{\sum_{i\in H^c}\left(\mathbb{I}\left\{i\in\widehat{H^c}\right\}-1\right)^2}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2 \\
& \left(\text{by the inequality } |XY|\leq\frac{1}{2}X^2+\frac{1}{2}Y^2\right) \\
= & \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2 \\
& +\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sum_{j\in H^c}\sum_{\ell\in H^c} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sum_{k=1}^{Kp} g_{1,jk}g_{1,\ell k}E\left[u_{j,t}u_{\ell,t}\right] \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{j\in H^c}\sum_{\ell\in H^c}\frac{e_{j,N}'\Gamma_*}{\sqrt{N_1}}\sum_{k=1}^{Kp}\Xi e_{k,Kp}e_{k,Kp}'\Xi'\frac{\Gamma_*'e_{\ell,N}}{\sqrt{N_1}}E\left[u_{j,t}u_{\ell,t}\right] \\
& \left(\text{since } G_1=\frac{\Gamma_*\Xi}{\sqrt{N_1}} \text{ with } \Gamma_*=\Gamma\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\right) \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Xi\Xi'\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} e_{j,N}'\Gamma_*\Gamma_*'e_{\ell,N}E\left[u_{j,t}u_{\ell,t}\right] \\
& (\text{since } \Xi \text{ is an orthogonal matrix}) \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c} \gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right]
\end{aligned}
$$

where $\gamma_{*,j}=\left(\Gamma'\Gamma/N_1\right)^{-1/2}\gamma_j$. Applying the triangle and Cauchy-Schwarz inequalities, we further obtain

$$
\begin{aligned}
& E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2\right] \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_{*,j}'\gamma_{*,\ell}E\left[u_{j,t}u_{\ell,t}\right] \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell E\left[u_{j,t}u_{\ell,t}\right] \\
\leq \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_\ell\right|\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
\leq \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{N_1^{\frac{3}{2}}}\sum_{j\in H^c}\sum_{\ell\in H^c}\sqrt{\gamma_j'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_j}\sqrt{\gamma_\ell'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_\ell}\left|E\left[u_{j,t}u_{\to,t}\right]\right| \\
\leq \;& \frac{\overline{c}}{\underline{C}}\frac{1}{\sqrt{N_1}T_h}\sum_{t=p}^{T-h}\frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \\
& \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
& \left.\sup_{i\in H^c}\|\gamma_i\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\right) \\
\leq \;& \frac{\overline{c}}{\underline{C}}\frac{\overline{C}}{\sqrt{N_1}} \;\to 0 \text{ as } N_1 \to \infty. \text{ (since, under Assumption 3-3(d), there exists a} \\
& \left.\text{positive constant } \overline{C} \text{ such that } \sup_t \frac{1}{N_1}\sum_{j\in H^c}\sum_{\ell\in H^c}\left|E\left[u_{j,t}u_{\ell,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}
$$

from which we further deduce, upon applying Markov's inequality, that

$$
\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c} g_{1,jk}u_{j,t}\right)^2 = o_p(1)
$$

Moreover, since we have previously shown that

$$
\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right) = o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c} g_{1,ik}^2u_{i,t}^2 = O_p(1),
$$

it follows from these calculations that

$$\begin{aligned}\left|\underline{\mathcal{E}}_{1,2}\right| &\leq \frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H^c}g_{1,ik}^2u_{i,t}^2} \\ &\quad+\sqrt{\frac{1}{N_1}\sum_{i\in H^c}\left(1-\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\frac{1}{\sqrt{N_1}}\sum_{k=1}^{Kp}\left(\sum_{j\in H^c}g_{1,jk}u_{j,t}\right)^2 \\ &= o_p(1).\end{aligned}$$

In a similar way, we can also show that

$$\left|\underline{\mathcal{E}}_{1,3}\right| = o_p(1).$$

Finally, let $U_{t,N}(H^c)$ denote an $N\times 1$ vector whose i^{th} component $U_{i,t,N}(H^c)$ is given by

$$U_{i,t,N}(H^c) = \begin{cases} u_{i,t} & \text{if } i\in H^c \\ 0 & \text{if } i\in H \end{cases}.$$

and we can write

$$
\begin{aligned}
\underline{\mathcal{E}}_{1,4} &= \frac{2}{N_1 T_h} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H^c} \sum_{j \in H^c} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(H^c\right)}{\sqrt{N_1}} \right\|_2^2 \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} tr \left\{ \frac{G_1' U_{t,N}\left(H^c\right) U_{t,N}\left(H^c\right)' G_1}{N_1} \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} tr \left\{ \Xi' \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}\left(H^c\right) U_{t,N}\left(H^c\right)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \Xi \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} tr \left\{ \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \frac{\Gamma'}{\sqrt{N_1}} \frac{U_{t,N}\left(H^c\right) U_{t,N}\left(H^c\right)'}{N_1} \frac{\Gamma}{\sqrt{N_1}} \left(\frac{\Gamma'\Gamma}{N_1} \right)^{-1/2} \right\} \\
&= \frac{2}{T_h} \sum_{t=p}^{T-h} tr \left\{ \frac{\Gamma_*' U_{t,N}\left(H^c\right) U_{t,N}\left(H^c\right)' \Gamma_*}{N_1^2} \right\} \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} U_{t,N}\left(H^c\right)' \Gamma_* \Gamma_*' U_{t,N}\left(H^c\right) \\
&= \frac{2}{T_h N_1^2} \sum_{t=p}^{T-h} \sum_{i \in H^c} \sum_{j \in H^c} \gamma_{*,i}' \gamma_{*,j} u_{i,t} u_{j,t}
\end{aligned}
$$

where $\gamma'_{*,i}$ denotes the i^{th} row of $\Gamma_* = \Gamma\left(\Gamma'\Gamma/N_1\right)^{-1/2}$. Taking expectation, we then obtain

$$\begin{aligned}
0 &\leq E\left[\underline{\mathcal{E}}_{1,4}\right] \\
&= \frac{2}{N_1 T_h}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c} g_{1,ik}g_{1,jk}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_{*,i}\gamma_{*,j}E\left[u_{i,t}u_{j,t}\right] \\
&= \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j E\left[u_{i,t}u_{j,t}\right] \\
&\leq \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\left|\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2}\gamma_j\right|\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\sqrt{\gamma'_i\left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1}\gamma_i}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{1}{T_h N_1^2}\sum_{t=p}^{T-h}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \\
&\quad \text{(since, under Assumptions 3-5 and 3-6, there exist positive constants } \overline{c} \text{ and } \underline{C} \text{ such that} \\
&\quad \left.\sup_{i\in H^c}\left\|\gamma_i\right\|_2 \leq \overline{c} < \infty \text{ and } \lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right) \geq \underline{C} > 0\right) \\
&\leq \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1}\frac{T-h-p+1}{T_h} \;= \frac{2\overline{c}}{\underline{C}}\frac{\overline{C}}{N_1} \to 0 \text{ as } N_1, T \to \infty. \\
&\quad \Big(\text{since, under Assumption 3-3(d), there exist a positive constant } \overline{C} \\
&\qquad \left.\text{such that } \sup_t \frac{1}{N_1}\sum_{i\in H^c}\sum_{j\in H^c}\left|E\left[u_{i,t}u_{j,t}\right]\right| \leq \overline{C} < \infty\right)
\end{aligned}$$

It follows by Markov's inequality that

$$\underline{\mathcal{E}}_{1,4} = o_p(1).$$

Application of the Slutsky's theorem then allows us to deduce that

$$\begin{aligned}
\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\}g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} &= \underline{\mathcal{E}}_{1,1}+\underline{\mathcal{E}}_{1,2}+\underline{\mathcal{E}}_{1,3}+\underline{\mathcal{E}}_{1,4} \\
&= o_p(1)+o_p(1)+o_p(1)+o_p(1) \\
&= o_p(1).
\end{aligned}$$

Consider now the second term on the extreme right-hand side of expression (131)

$$
\begin{aligned}
& \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \sum_{i \in H} \sum_{j \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \mathbb{I}\left\{j \in \widehat{H^c}\right\} g_{1,ik} g_{1,jk} u_{i,t} u_{j,t} \\
= \; & \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right)^2 \\
= \; & \frac{2}{T_h N_1} \sum_{t=p}^{T-h} \sum_{k=1}^{Kp} \left(\left| \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} g_{1,ik} u_{i,t} \right| \right)^2 \\
\leq \; & 2 \sum_{k=1}^{Kp} \left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right] \left[\frac{1}{T_h} \sum_{t=p}^{T-h} \sum_{i \in H} g_{1,ik}^2 u_{i,t}^2 \right]
\end{aligned}
$$

Note that, for either the case where $\mathbb{S}_{i,T}^{+} = \sum_{\ell=1}^{d} \varpi_\ell \left| S_{i,\ell,T} \right|$ or the case where $\mathbb{S}_{i,T}^{+} = \max_{1 \leq \ell \leq d} \left| S_{i,\ell,T} \right|$, we have

$$
\begin{aligned}
0 \quad \leq \quad & E\left[\frac{1}{N_1} \sum_{i \in H} \mathbb{I}\left\{i \in \widehat{H^c}\right\} \right] \\
= \quad & \frac{1}{N_1} \sum_{i \in H} \Pr\left(i \in \widehat{H^c}\right) \\
= \quad & \frac{1}{N_1} \sum_{i \in H} P\left(\mathbb{S}_{i,T}^{+} \geq \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)\right) \\
\leq \quad & \frac{N_2 \varphi}{N_1 N} \left\{ 1 + 2^{1+\delta} A T_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta} A \Phi^{-1}\left(1 - \frac{\varphi}{2N}\right)^{2+\delta} T_0^{-(1-\alpha_1)\frac{\delta}{2}} \right\} \\
= \quad & \frac{N_2 \varphi}{N_1 N} \left[1 + o\left(1\right)\right] \\
& \text{(following an argument similar to that given in the proof of Theorem 1} \\
& \text{as shown in Chao, Qiu, and Swanson (2023b))} \\
\to \quad & 0 \text{ as } N_1, N_2, T \to \infty
\end{aligned}
$$

Moreover, making use of part (b) of Assumption 3-3, we have

$$
\begin{aligned}
E\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2\right] &= \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 E\left[u_{i,t}^2\right] \\
&\leq C\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i=1}^{N} g_{1,ik}^2 \\
&\leq C\frac{T-h-p+1}{T_h} \\
&\quad \left(\text{given that } \sum_{i=1}^{N} g_{1,ik}^2 = 1 \text{ and } T_h = T-h-p+1\right) \\
&\leq C < \infty
\end{aligned}
$$

It follows by Markov's inequality that

$$
\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\} = o_p(1) \text{ and } \frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2 = O_p(1)
$$

from which we deduce that

$$
\begin{aligned}
&\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
\leq\ & 2\sum_{k=1}^{Kp}\left[\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\right]\left[\frac{1}{T_h}\sum_{t=p}^{T-h}\sum_{i\in H} g_{1,ik}^2 u_{i,t}^2\right] \\
=\ & o_p(1)
\end{aligned}
$$

Combining these results, we further obtain

$$
\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H^c}\sum_{j\in H^c}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&\quad +\frac{2}{T_h N_1}\sum_{t=p}^{T-h}\sum_{k=1}^{Kp}\sum_{i\in H}\sum_{j\in H}\mathbb{I}\left\{i\in\widehat{H^c}\right\}\mathbb{I}\left\{j\in\widehat{H^c}\right\} g_{1,ik}g_{1,jk}u_{i,t}u_{j,t} \\
&= o_p(1) + o_p(1) \\
&= o_p(1).
\end{aligned}
$$

To show part (b), write

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &= \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)' U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i=1}^{N}\mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}\mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 + \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\} u_{i,t}^2 \\
&\leq \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c} u_{i,t}^2 + \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h} u_{i,t}^2
\end{aligned}$$

Next, note that, by making use of part (b) of Assumption 3-3, we have

$$\begin{aligned}
E\left[\frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c} u_{i,t}^2\right] &= \frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c} E\left[u_{i,t}^2\right] \\
&\leq C\frac{T-h-p+1}{T_h} \text{ (since } N_1 = \#\{H\}, \\
&\quad \text{where } \#\{H\} \text{ denotes the cardinality of the set } H) \\
&\leq C \text{ (since } T_h = T-h-p+1)
\end{aligned}$$

so that, by Markov's inequality,

$$\frac{1}{T_h N_1}\sum_{t=p}^{T-h}\sum_{i\in H^c} u_{i,t}^2 = O_p(1).$$

Moreover, note that, for any $\epsilon > 0$,

$$\bigcap_{i\in H}\left\{i \notin \widehat{H^c}\right\} \subseteq \left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h} u_{i,t}^2 < \epsilon\right\}$$

so that, applying DeMorgan's law, we obtain

$$\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h} u_{i,t}^2 \geq \epsilon\right\} \subseteq \left\{\bigcap_{i\in H}\left\{i \notin \widehat{H^c}\right\}\right\}^c = \bigcup_{i\in H}\left\{i \in \widehat{H^c}\right\}$$

It follows that, for any $\epsilon > 0$ and for either the case where $\mathbb{S}_{i,T}^+ = \sum_{\ell=1}^d \varpi_\ell \left|S_{i,\ell,T}\right|$ or the case where $\mathbb{S}^{+}_{i,T} = \max_{1 \leq \ell \leq d} |S_{i,\ell,T}|$, we have

$$\begin{aligned}
& \Pr\left\{\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2 \geq \epsilon\right\} \\
\leq \quad & \Pr\left\{\bigcup_{i\in H}\left\{i \in \widehat{H^c}\right\}\right\} \\
\leq \quad & \sum_{i\in H}\Pr\left\{i \in \widehat{H^c}\right\} \\
= \quad & \sum_{i\in H}P\left(\mathbb{S}^{+}_{i,T} \geq \Phi^{-1}\left(1-\frac{\varphi}{2N}\right)\right) \\
\leq \quad & \frac{N_2\varphi}{N}\left\{1 + 2^{1+\delta}AT_0^{-(1-\alpha_1)\frac{\delta}{2}} + 2^{1+\delta}A\Phi^{-1}\left(1-\frac{\varphi}{2N}\right)^{2+\delta}T_0^{-(1-\alpha_1)\frac{\delta}{2}}\right\} \\
= \quad & \frac{N_2\varphi}{N}\left[1 + o\left(1\right)\right] \\
& \text{(following an argument similar to that given in the proof of Theorem 1} \\
& \text{as shown in Chao, Qiu, and Swanson (2023b))} \\
\to \quad & 0 \text{ as } N_1, N_2, T \to \infty
\end{aligned}$$

Hence,

$$\frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2 = o_p\left(1\right)$$

from which it we further deduce that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{1}{T_hN_1}\sum_{t=p}^{T-h}\sum_{i\in H^c}u_{i,t}^2 + \frac{1}{N_1}\sum_{i\in H}\mathbb{I}\left\{i \in \widehat{H^c}\right\}\frac{1}{T_h}\sum_{t=p}^{T-h}u_{i,t}^2 \\
&= O_p\left(1\right) + o_p\left(1\right) \\
&= O_p\left(1\right).
\end{aligned}$$

Now, for part (c), note first that since $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ is an orthogonal matrix, we have

$I_N = GG' = G_1G_1' + G_2G_2'$ or $G_2G_2' = I_N - G_1G_1'$. Hence, we can write

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &= \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)'U_{t,N}\left(\widehat{H^c}\right)}{N_1} - \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)'G_1G_1'U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)'U_{t,N}\left(\widehat{H^c}\right)}{N_1} + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{U_{t,N}\left(\widehat{H^c}\right)'G_1G_1'U_{t,N}\left(\widehat{H^c}\right)}{N_1} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2
\end{aligned}$$

Applying the results from parts (a) and (b) of this lemma, we then obtain

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&= O_p(1) + o_p(1) \\
&= O_p(1).
\end{aligned}$$

Next, to show part (d), let $\overline{C}$ be the constant given in Lemma C-4 such that

$$E\left\|\underline{F}_t\right\|_2^6 \leq \overline{C} < \infty \text{ for all } t.$$

Now, for any $\epsilon > 0$, let $C_\epsilon^* = \overline{C}^{\frac{1}{3}}/\epsilon$; then, upon application of Markov's inequality, we have

$$\begin{aligned}
\Pr\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\right\|_2^2 \geq C_\epsilon^*\right) &\leq \frac{1}{C_\epsilon^*}\frac{1}{T_h}\sum_{t=p}^{T-h}E\left\|\underline{F}_t\right\|_2^2 \\
&\leq \frac{1}{C_\epsilon^*}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(E\left\|\underline{F}_t\right\|_2^6\right)^{\frac{1}{3}} \text{ (by Liapunov's inequality)} \\
&= \frac{\epsilon}{\overline{C}^{\frac{1}{3}}}\frac{1}{T_h}\sum_{t=p}^{T-h}\overline{C}^{\frac{1}{3}} \\
&= \epsilon\frac{T-h-p+1}{T_h} \\
&\leq \epsilon \ \text{ (since } T_h = T-h-p+1\text{)}
\end{aligned}$$

so that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 = O_p\left(1\right)$$

In addition, note that

$$\begin{aligned}
\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\underline{F}_t'\right\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^2\,\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^2\,\lambda_{\max}\left(\underline{F}_t'\underline{F}_t\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 \\
&= O_p\left(1\right)
\end{aligned}$$

Turning our attention to part (e), write

$$
\begin{aligned}
& \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1'\widehat{V}}{T_h\widehat{N}_1} \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'\mathbf{e}_{t,T}}{\sqrt{\widehat{N}_1}}\frac{\mathbf{e}_{t,T}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1-N_1+N_1\right)/N_1}}\frac{U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{N_1}\sqrt{\left(\widehat{N}_1-N_1+N_1\right)/N_1}} \\
= \;& \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\frac{1}{N_1T_h}\sum_{t=p}^{T-h}\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V} \\
= \;& \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\frac{1}{N_1T_h}\sum_{t=p}^{T-h}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}\left[G_1'+R'G_2'\right]U_{t,N}\left(\widehat{H^c}\right)\right. \\
& \qquad\qquad \left.\times U_{t,N}\left(\widehat{H^c}\right)'\left[G_1+G_2R\right]\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
= \;& \left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}G_1'\right. \\
& \left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_1\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}G_1'\right. \\
& \quad \left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_2R\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}R'G_2'\right. \\
& \quad \left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_1\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\} \\
& +\left(1+\frac{\widehat{N}_1-N_1}{N_1}\right)^{-1}\left\{\widehat{V}'\left(I_{Kp}+R'R\right)^{-1/2}R'G_2'\right. \\
& \quad \left.\times\frac{1}{N_1T_h}\sum_{t=p}^{T-h}U_{t,N}\left(\widehat{H^c}\right)U_{t,N}\left(\widehat{H^c}\right)'G_2R\left(I_{Kp}+R'R\right)^{-1/2}\widehat{V}\right\}
\end{aligned}
$$

To analyze the four terms on the right-hand side of the expression above, note first that, by the homogeneity of matrix norm and the triangle inequality,

$$\begin{aligned}
\left\| G_1' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1 \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left\{ \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right)^2 \right\}} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}^2 \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right)} \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \lambda_{\max} \left(\frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1}{N_1} \right) \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
& \left\| G_1' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2 \right\|_2 \\
\leq\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right\|_2 \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left\{ \left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_1 G_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1^2} \right) \right\}} \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \sqrt{\left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \lambda_{\max}\left\{ \left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right) \right\}} \\
=\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \sqrt{\lambda_{\max}\left(\frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right)} \\
\leq\ & \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2
\end{aligned}
$$

and

$$
\begin{aligned}
\left\| G_2' \frac{1}{N_1 T_h} \sum_{t=p}^{T-h} U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2 \right\|_2 &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}\left(\widehat{H^c}\right)' G_2}{N_1} \right\|_2 \\
&= \frac{1}{T_h} \sum_{t=p}^{T-h} \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left\|\frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1'\widehat{V}}{T_h\widehat{N}_1}\right\|_2 \\
\leq\ & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\widehat{V}\right\|_2^2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\widehat{V}\right\|_2^2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\|R\|_2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \\
&+\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\widehat{V}\right\|_2^2\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
=\ & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2\|R\|_2\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2 \\
&+\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&\left(\text{since } \widehat{V}'\widehat{V}=I_{Kp} \text{ so that } \left\|\widehat{V}\right\|_2=1\right) \\
\leq\ & 2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\left\| \frac{\widehat{V}'\widehat{G}_1' U\left(\widehat{H^c}\right)' U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
\leq\;& 2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left\|\left(I_{Kp}+R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq\;& 2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left(\frac{1}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\right)^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left(\frac{\|R\|_2}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\right)^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
=\;& 2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \frac{1}{1+\lambda_{\min}\left(R'R\right)} \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
&+2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \frac{\|R\|_2^2}{1+\lambda_{\min}\left(R'R\right)} \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq\;& 2\left|1+\frac{\widehat{N}_1 - N_1}{N_1}\right|^{-1} \left[\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 + \|R\|_2^2 \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right] \\
=\;& o_p(1) \quad \text{(applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
&\qquad\qquad \text{parts (a) and (c) of this lemma, and Slutsky's theorem)}
\end{aligned}
$$

To show part (f), first write

$$
\begin{aligned}
\left\| \frac{\underline{F}' U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}} \right\|_2 &= \left\| \frac{1}{T_h}\sum_{t=p}^{T-h} \frac{\underline{F}_t U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \frac{1}{T_h}\sum_{t=p}^{T-h} \left\| \frac{\underline{F}_t U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h} \sqrt{\lambda_{\max}\left(\frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)\underline{F}_t'\underline{F}_t U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{\lambda_{\max}\left(\frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right) U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{\widehat{N}_1} \right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \sqrt{ \frac{U_{t,N}'\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\widehat{N}_1} } \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h} \|\underline{F}_t\|_2 \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2 \\
&\leq \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} \|\underline{F}_t\|_2^2} \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} \right\|_2^2}
\end{aligned}
$$

Next, note that

$$
\begin{aligned}
& \left\| \frac{\widehat{V}'\widehat{G}_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 \\
= & \left(\left\| \frac{\widehat{V}' \left(I_{Kp} + R'R\right)^{-1/2} \left[G_1' + R'G_2'\right] U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
\leq & \left(\left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right. \\
& \left. + \left\|\widehat{V}\right\|_2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
= & \left(\left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 + \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2 \|R\|_2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2 \right)^2 \\
& \left(\text{since } \widehat{V}'\widehat{V} = I_{Kp} \text{ so that } \left\|\widehat{V}\right\|_2 = 1\right) \\
\leq & \; 2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2^2 \left\| \frac{G_1' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2 + 2 \left\|\left(I_{Kp} + R'R\right)^{-1/2}\right\|_2^2 \|R\|_2^2 \left\| \frac{G_2' U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}} \right\|_2^2
\end{aligned}
$$

from which we obtain

$$\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2^2 \\
= & \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}\sqrt{\left(N_1+\widehat{N}_1-N_1\right)/N_1}}\right\|_2^2 \\
= & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2 \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{2\left\|\left(I_K+R'R\right)^{-1/2}\right\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right. \\
& \left.+2\left\|\left(I_K+R'R\right)^{-1/2}\right\|_2^2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{\frac{2}{1+\lambda_{\min}\left(R'R\right)}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right. \\
& \left.+\frac{2\|R\|_2^2}{\sqrt{1+\lambda_{\min}\left(R'R\right)}}\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
\leq & \left|1+\frac{\widehat{N}_1-N_1}{N_1}\right|^{-1}\left\{\frac{2}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2+2\|R\|_2^2\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{G_2'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{N_1}}\right\|_2^2\right\} \\
= & o_p(1) \quad \text{(applying part (a) of Lemma D-14, part (a) of Lemma D-15,} \\
& \qquad \text{parts (a) and (c) of this lemma, and Slutsky's theorem)}
\end{aligned}$$

It then follows from part (d) of this lemma and the Slutsky's theorem that

$$\begin{aligned}
\left\|\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}}\right\|_2 & \leq \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right\|_2^2} \\
& = O_p(1)\,o_p(1) \\
& = o_p(1)
\end{aligned}$$

Lastly, to show part (g), first write

$$
\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\left\{\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right)\right. \\
& \qquad \left.\times\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t + \frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\right)\right\}' \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)\frac{U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)\underline{F}_t}{\sqrt{\widehat{N}_1}} - Q'\underline{F}_t\right)' \\
& + \frac{1}{T_h}\sum_{t=p}^{T-h}\frac{\widehat{V}'\widehat{G}_1'U_{t,N}\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}}\frac{U_{t,N}\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} \\
= \; & \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} - Q\right) \\
& + \left(\frac{\widehat{V}'\widehat{G}_1'\Gamma\left(\widehat{H^c}\right)}{\sqrt{\widehat{N}_1}} - Q'\right)\frac{\underline{F}'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\sqrt{\widehat{N}_1}} \\
& + \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'\underline{F}}{T_h\sqrt{\widehat{N}_1}}\left(\frac{\Gamma\left(\widehat{H^c}\right)'\widehat{G}_1\widehat{V}}{\sqrt{\widehat{N}_1}} - Q\right) + \frac{\widehat{V}'\widehat{G}_1'U\left(\widehat{H^c}\right)'U\left(\widehat{H^c}\right)\widehat{G}_1\widehat{V}}{T_h\widehat{N}_1}
\end{aligned}
$$

where $U_{t,N}\left(\widehat{H^c}\right) = U\left(\widehat{H^c}\right)'\mathbf{e}_{t,T} = \begin{pmatrix} \mathbb{I}\left\{1 \in \widehat{H^c}\right\}u_{1,t} & \mathbb{I}\left\{2 \in \widehat{H^c}\right\}u_{2,t} & \cdots & \mathbb{I}\left\{N \in \widehat{H^c}\right\}u_{N,t}\end{pmatrix}'$. Applying part (h) of Lemma D-15 and parts (d), (e), and (f) of this lemma and the Slutsky's theorem, we obtain

$$
\begin{aligned}
& \left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{F}_t - Q' \underline{F}_t \right) \left(\widehat{F}_t - Q' \underline{F}_t \right)' \right\|_2 \\
\leq \; & \left\| \left(\frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \left(\frac{\Gamma \left(\widehat{H^c} \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right) \right\|_2 \\
& + \left\| \left(\frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{\underline{F}' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 \\
& + \left\| \frac{\widehat{V}' \widehat{G}_1' U \left(\widehat{H^c} \right)' \underline{F}}{T_h \sqrt{\widehat{N}_1}} \left(\frac{\Gamma \left(\widehat{H^c} \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right) \right\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U \left(\widehat{H^c} \right)' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
= \; & \left\| \left(\frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' \left(\frac{\Gamma \left(\widehat{H^c} \right)' \widehat{G}_1 \widehat{V}}{\sqrt{\widehat{N}_1}} - Q \right) \right\|_2 \\
& + 2 \left\| \left(\frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right) \frac{\underline{F}' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 + \left\| \frac{\widehat{V}' \widehat{G}_1' U \left(\widehat{H^c} \right)' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
\leq \; & \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2^2 \left\| \frac{1}{T_h} \sum_{t=p}^{T} \underline{F}_t \underline{F}_t' \right\|_2 + 2 \left\| \frac{\widehat{V}' \widehat{G}_1' \Gamma \left(\widehat{H^c} \right)}{\sqrt{\widehat{N}_1}} - Q' \right\|_2 \left\| \frac{\underline{F}' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \sqrt{\widehat{N}_1}} \right\|_2 \\
& + \left\| \frac{\widehat{V}' \widehat{G}_1' U \left(\widehat{H^c} \right)' U \left(\widehat{H^c} \right) \widehat{G}_1 \widehat{V}}{T_h \widehat{N}_1} \right\|_2 \\
= \; & o_p(1) O_p(1) + o_p(1) o_p(1) + o_p(1) \\
= \; & o_p(1). \; \square
\end{aligned}
$$

Lemma D-17: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then, the following statements are true.

(a)

$$
\frac{\widehat{\underline{F}}' \widehat{\underline{F}}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q = o_p(1), \text{ where } T_h = T - h - p + 1.
$$

(b)

$$\frac{\widehat{\underline{F}}'\underline{Y}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{Y}_t'\right] = o_p\left(1\right)$$

(c)

$$\frac{\widehat{\underline{F}}'\iota_{T_h}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] = o_p\left(1\right),$$

where $\iota_{T_h} = (1, 1, ..., 1)'$ is a $T_h \times 1$ vector.

(d)

$$\frac{\widehat{\underline{F}}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p\left(1\right)$$

(e)

$$\frac{\underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p\left(1\right)$$

(f)

$$\frac{\iota_{T_h}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p\left(1\right)$$

(g)

$$\frac{\widehat{\underline{F}}'\mathfrak{H}}{T_h} = \frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{\underline{F}}_t\eta_{t+h}' = o_p\left(1\right)$$

Proof of Lemma D-17:

To show part (a), first write

$$\begin{aligned}
&\frac{\underline{\widehat{F}}'\underline{\widehat{F}}}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} \underline{\widehat{F}}_t\underline{\widehat{F}}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)' - \frac{1}{T_h}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right]Q \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' + Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \\
&+ \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\underline{F}_t'Q + Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{F}_t'\right]\right)Q
\end{aligned}$$

Now, by part (g) of Lemma D-16, we have that

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)' \xrightarrow{p} 0$$

Moreover, for any $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$

$$\begin{aligned}
&\left|a'Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'b\right| \\
\leq \;& \sqrt{a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right)Qa}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'b} \\
\leq \;& \sqrt{a'Q'Qa\lambda_{\max}\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right)}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'b} \\
= \;& \sqrt{a'Q'Qa\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)'b}
\end{aligned}$$

(since, for a symmetric psd matrix A,
$\|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} = \lambda_{\max}(A)\Big)$

Now, by Assumption 3-6, there exists a positive constant C such that

$$\begin{aligned}
a'Q'Qa &= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\left(\frac{\Gamma'\Gamma}{N_1}\right)^{1/2}\Xi\widehat{V}a \\
&= a'\widehat{V}'\Xi'\left(\frac{\Gamma'\Gamma}{N_1}\right)\Xi\widehat{V}a \\
&\leq \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)a'\widehat{V}'\Xi'\Xi\widehat{V}a \\
&= \lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)\quad\left(\text{since } \Xi'\Xi = I_{Kp},\ \widehat{V}'\widehat{V} = I_{Kp}, \text{ and } a'a = 1\right) \\
&\leq C \text{ for all } N_1, N_2 \text{ sufficiently large.} \qquad (132)
\end{aligned}$$

while, applying the triangle inequality and part (d) of Lemma D-16 allow us to show that

$$\begin{aligned}
\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2 &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left\|\underline{F}_t\underline{F}_t'\right\|_2 \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\underline{F}_t\underline{F}_t'\right)} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\left[\lambda_{\max}\left(\underline{F}_t\underline{F}_t'\right)\right]^2} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\sqrt{\|\underline{F}_t\|_2^4} \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h}\|\underline{F}_t\|_2^2 \\
&= O_p(1)
\end{aligned}$$

Combining this result with part (g) of Lemma D-16 and the Slutsky's Theorem, we deduce that

$$\begin{aligned}
&\left|a'Q'\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'b\right| \\
\leq\ & \sqrt{a'Q'Qa\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{F}_t'\right\|_2}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}b'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'b} \\
=\ & o_p(1)
\end{aligned}$$

Since this argument holds for all $a, b \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = \|b\|_2 = 1$, we further obtain

$$Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' = o_p(1)$$

Now, given that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \underline{F}_t' Q = \left[Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' \right]',$$

a similar argument also shows that

$$\frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \underline{F}_t' Q = o_p(1).$$

Making use of part (b) of Lemma D-2 and the Slutsky's theorem, we also see that

$$Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{F}_t' \right] \right) Q \xrightarrow{p} 0$$

Putting these results together and apply Slutsky's theorem, we then obtain

$$\begin{aligned}
& \frac{\underline{\widehat{F}}' \underline{\widehat{F}}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E\left[\underline{F}_t \underline{F}_t' \right] Q \\
= \; & \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' + Q' \frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' \\
& + \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \underline{F}_t' Q + Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{F}_t' \right] \right) Q \\
= \; & o_p(1)
\end{aligned}$$

To show part (b), first write, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$,

$$\begin{aligned}
& \frac{a'\underline{\widehat{F}}'\underline{Y}b}{T_h} - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{\widehat{F}}_t\underline{Y}_t'b - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\underline{Y}_t'b - \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t\underline{Y}_t'\right]b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\underline{\widehat{F}}_t - Q'\underline{F}_t\right)\underline{Y}_t'b + a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{F}_t\underline{Y}_t'\right]\right)b
\end{aligned}$$

Focusing first on the first term on last line above, we note that,

$$
\begin{aligned}
& \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \underline{Y}_t' b \right| \\
\leq & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' \underline{Y}_t \underline{Y}_t' b} \\
= & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \\
& \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b + \frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
\leq & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} b' E \left[\underline{Y}_t \underline{Y}_t' \right] b} \\
& \left(\text{since } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \right) \\
\leq & \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\underline{Y}_t \underline{Y}_t' \right] \right) b} \\
& + \sqrt{a' \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right) \left(\underline{\widehat{F}}_t - Q' \underline{F}_t \right)' a} \sqrt{\frac{1}{T_h} \sum_{t=r+1}^{T-h} E \left\| \underline{Y}_t \right\|_2^2} \\
& \left(\text{since } b' E \left[\underline{Y}_t \underline{Y}_t' \right] b = E \left[\left(b' \underline{Y}_t \right)^2 \right] \leq E \left[b' b \underline{Y}_t' \underline{Y}_t \right] = E \left[\left\| \underline{Y}_t \right\|_2^2 \right] \right) \\
= & \; o_p (1)
\end{aligned}
$$

by part (b) of Lemma D-2 and parts (d) and (g) of Lemma D-16. In addition, note that, by making use of part (b) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
& \left| a'Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right) b \right| \\
\leq & \sqrt{a'Q'Qa} \sqrt{ b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right)' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right) b } \\
\leq & \sqrt{ \lambda_{\max} \left(\frac{\Gamma'\Gamma}{N_1} \right) } \sqrt{ b' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right)' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right) b } \\
= & o_p(1).
\end{aligned}$$

Combining these results, we then get

$$\begin{aligned}
& \left| \frac{a'\widehat{\underline{F}}'\underline{Y}b}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} a'Q'E\left[\underline{F}_t \underline{Y}_t' \right] b \right| \\
\leq & \left| \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q'\underline{F}_t \right) \underline{Y}_t' b \right| + \left| a'Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{F}_t \underline{Y}_t' \right] \right) b \right| \\
= & o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{dp}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\widehat{\underline{F}}'\underline{Y}}{T_h} - \frac{1}{T_h} \sum_{t=p}^{T-h} Q'E\left[\underline{F}_t \underline{Y}_t' \right] = o_p(1).$$

To show part (c), first write, for any $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$,

$$\begin{aligned}
& \frac{a'\widehat{\underline{F}}'\iota_{T_h}}{T_h} - a'Q'S_K'\mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}'\mu \\
= & \frac{1}{T_h} \sum_{t=p}^{T-h} a'\widehat{\underline{F}}_t - a'Q'S_K'\mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}'\mu \\
= & \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t \right) - a'Q'S_K'\mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}'\mu \\
= & \frac{1}{T_h} \sum_{t=p}^{T-h} a' \left(\widehat{\underline{F}}_t - Q'\underline{F}_t \right) + a'Q' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{F}_t - S_K'\mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}'\mu \right)
\end{aligned}$$

Focusing first on the first term on last line above, we note that,

$$\begin{aligned}
\left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| &\leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left|a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| \quad \text{(by triangle inequality)} \\
&\leq \sqrt{a'\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a} \quad \text{(by Liapunov's inequality)} \\
&\leq \sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2} \\
&= o_p(1)
\end{aligned}$$

by part (g) of Lemma D-16 and Slutsky's theorem. In addition, note that, by making use of part (d) of Lemma D-2, Assumption 3-6, and Slutsky's theorem; we obtain

$$\begin{aligned}
&\left|a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)\right| \\
\leq\ & \sqrt{a'Q'Qa}\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)'\right. \\
&\qquad \left.\times\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)\right]^{1/2} \\
\leq\ & \sqrt{\lambda_{\max}\left(\frac{\Gamma'\Gamma}{N_1}\right)}\left[\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)'\right. \\
&\qquad \left.\times\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)\right]^{1/2} \\
=\ & o_p(1).
\end{aligned}$$

Combining these results and applying Slutsky's theorem, we then get

$$\begin{aligned}
&\left|\frac{a'\widehat{\underline{F}}'\iota_{T_h}}{T_h} - a'Q'S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right| \\
\leq\ & \left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\right| + \left|a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t - S_K'\mathcal{P}_{(d+K)p}\left(I_{(d+K)p} - A\right)^{-1} J_{d+K}'\mu\right)\right| \\
=\ & o_p(1)
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ such that $\|a\|_2 = 1$; we further deduce that

$$\frac{\underline{\widehat{F}}' \iota_{T_h}}{T_h} - Q' S_K' \mathcal{P}_{(d+K)p} \left(I_{(d+K)p} - A \right)^{-1} J_{d+K}' \mu = o_p \left(1 \right).$$

Turning our attention to part (d), note that for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{a'\widehat{\underline{F}}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 b}{T_h} \right| \\
= \quad & \left| \frac{1}{T_h}\sum_{t=p}^{T-h} a'\widehat{\underline{F}}_t\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b \right| \\
\leq \quad & \sqrt{a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{F}_t\widehat{F}_t'\right)a}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
\leq \quad & \sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{F}_t\widehat{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q\right)a\right| + \frac{1}{T_h}\sum_{t=p}^{T-h}a'Q'E\left[\underline{F}_t\underline{F}_t'\right]Qa} \\
& \times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
\leq \quad & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{F}_t\widehat{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q\right)a\right|}\right. \\
& \left.\times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b}\right\} \\
& +\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}a'Q'E\left[\underline{F}_t\underline{F}_t'\right]Qa}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
& \left(\text{using the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0\right) \\
\leq \quad & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{F}_t\widehat{F}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q\right)a\right|}\right. \\
& \left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b}\right\} \\
& +\sqrt{a'Q'Qa\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\|\underline{F}_t\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b} \\
& \left(\text{since for a symmetric psd matrix } A,\ \|A\|_2 = \sqrt{\lambda_{\max}\left(A'A\right)} = \sqrt{\lambda_{\max}\left(A^2\right)} = \sqrt{\left[\lambda_{\max}\left(A\right)\right]^2}\right. \\
& \left. = \lambda_{\max}\left(A\right) \text{ and since } a'Q'E\left[\underline{F}_t\underline{F}_t'\right]Qa = E\left[\left(a'Q'\underline{F}_t\right)^2\right] \leq E\left[a'Q'Qa\underline{F}_t'\underline{F}_t\right] = a'Q'QaE\left[\|\underline{F}_t\|_2^2\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \Bigg\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right|} \\
&\quad \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \Bigg\} \\
&\quad + \sqrt{a' Q' Q a \frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\| \underline{F}_t \|_2^2 \right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b}
\end{aligned}
$$

Now, by part (a) of this lemma and Slutsky's theorem, we have

$$
\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \widehat{F}_t \widehat{F}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} Q' E \left[\underline{F}_t \underline{F}_t' \right] Q \right) a \right| = o_p(1) \tag{133}
$$

Note also that

$$
\begin{aligned}
\frac{1}{T_h} \sum_{t=p}^{T-h} E \left[\| \underline{F}_t \|_2^2 \right] &\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(E \left[\| \underline{F}_t \|_2^6 \right] \right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
&\leq \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\overline{C} \right)^{\frac{1}{3}} \quad \text{(by Lemma C-4)} \\
&= \left(\overline{C} \right)^{\frac{1}{3}}. \qquad (134)
\end{aligned}
$$

In addition, note that, by Assumption 3-7, there exists a positive constant C such that

$$\begin{aligned}
& \lambda_{\max}\left(B_2' B_2\right) \\
= \; & \lambda_{\max}\left(J_d A^h \mathcal{P}_{(d+K)p}' S_K S_K' \mathcal{P}_{(d+K)p}\left(A^h\right)' J_d'\right) \\
\leq \; & \lambda_{\max}\left(S_K S_K'\right) \lambda_{\max}\left(\mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p}\right) \lambda_{\max}\left\{A^h\left(A^h\right)'\right\} \lambda_{\max}\left(J_d J_d'\right) \\
= \; & \lambda_{\max}\left(S_K S_K'\right) \lambda_{\max}\left\{A^h\left(A^h\right)'\right\} \quad \Big(\text{since } \mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p} = I_{(d+K)p} \text{ and } J_d J_d' = I_d \\
& \qquad\qquad\qquad \text{so } \lambda_{\max}\left(\mathcal{P}_{(d+K)p}' \mathcal{P}_{(d+K)p}\right) = \lambda_{\max}\left(J_d J_d'\right) = 1\Big) \\
= \; & \lambda_{\max}\left(S_K' S_K\right) \lambda_{\max}\left\{\left(A^h\right)' A^h\right\} \\
= \; & \lambda_{\max}\left\{\left(A^h\right)' A^h\right\} \quad \left(\text{since } S_K' S_K = I_{Kp} \text{ so } \lambda_{\max}\left(S_K' S_K\right) = 1\right) \\
= \; & \sigma_{\max}^2\left(A^h\right) \\
\leq \; & C \max\left\{\left|\lambda_{\max}\left(A^h\right)\right|^2, \left|\lambda_{\min}\left(A^h\right)\right|^2\right\} \quad (\text{by Assumption 3-7}) \\
= \; & C \max\left\{\left|\lambda_{\max}(A)\right|^{2h}, \left|\lambda_{\min}(A)\right|^{2h}\right\} \\
= \; & C \phi_{\max}^{2h} \\
< \; & C \text{ for integer } h \geq 1,
\end{aligned}$$

where $\phi_{\max} = \max\{|\lambda_{\max}(A)|, |\lambda_{\min}(A)|\}$ and where the last equality follows from the fact that $0 < \phi_{\max} < 1$ given that Assumption 3-1 implies that all eigenvalues of A have modulus less than 1. The boundedness of $\lambda_{\max}(B_2' B_2)$ allows us to further deduce that

$$\begin{aligned}
& b' B_2' Q'^{-1} Q^{-1} B_2 b \\
= \; & b' B_2' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} \Xi \widehat{V} \widehat{V}' \Xi' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1/2} B_2 b \\
= \; & b' B_2' \left(\frac{\Gamma'\Gamma}{N_1}\right)^{-1} B_2 b \\
\leq \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} b' B_2' B_2 b \\
\leq \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}\left(B_2' B_2\right) b' b \\
= \; & \left[\lambda_{\min}\left(\frac{\Gamma'\Gamma}{N_1}\right)\right]^{-1} \lambda_{\max}\left(B_2' B_2\right) \\
\leq \; & C^* < \infty
\end{aligned} \tag{135}$$

for some positive constant C^{*} in light of Assumption 3-6. It follows by applying expression (132) in the proof for part (a), expressions (133)-(135) here, as well as the result given in part (g) of Lemma D-16 and the Slutsky' theorem that

$$
\begin{aligned}
& \left|\frac{a'\underline{\widehat{F}}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2 b}{T_h}\right| \\
\leq\ & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{F}_t\widehat{F}_t'-\frac{1}{T_h}\sum_{t=p}^{T-h}Q'E\left[\underline{F}_t\underline{F}_t'\right]Q\right)a\right|}\right. \\
& \left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t-Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t-Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b}\right\} \\
& +\sqrt{a'Q'Qa\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\left\|\underline{F}_t\right\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}_t-Q'\underline{F}_t\right)\left(\underline{\widehat{F}}_t-Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2 b} \\
=\ & o_p(1).
\end{aligned}
$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^{d}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$
\frac{\underline{\widehat{F}}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1).
$$

To show part (e), note that for any $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^{d}$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{a'\underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 b}{T_h} \right| \\
= \;& \left| \frac{1}{T_h}\sum_{t=p}^{T-h} a'\underline{Y}_t\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b \right| \\
\leq \;& \sqrt{a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t'\right)a}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
\leq \;& \left\{ \sqrt{\left| a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a \right| + \left| \frac{1}{T_h}\sum_{t=p}^{T-h}a'E\left[\underline{Y}_t\underline{Y}_t'\right]a \right|} \right. \\
& \left. \times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \right\} \\
\leq \;& \left\{ \sqrt{\left| a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a \right|} \right. \\
& \left. \times\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \right\} \\
& +\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}a'E\left[\underline{Y}_t\underline{Y}_t'\right]a}\sqrt{b'B_2'Q'^{-1}\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)Q^{-1}B_2 b} \\
& \left(\text{using the inequality } \sqrt{a_1+a_2} \leq \sqrt{a_1}+\sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0\right)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right|}\right. \\
& \left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\right\} \\
& + \sqrt{a'a\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\left\|\underline{Y}_t\right\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b} \\
& \Bigg(\text{since for a symmetric psd matrix } A,\ \|A\|_2 = \sqrt{\lambda_{\max}\left(A'A\right)} = \sqrt{\lambda_{\max}\left(A^2\right)} = \sqrt{\left[\lambda_{\max}\left(A\right)\right]^2} \\
& = \lambda_{\max}\left(A\right) \text{ and since } a'E\left[\underline{Y}_t\underline{Y}_t'\right]a = E\left[\left(a'\underline{Y}_t\right)^2\right] \leq E\left[a'a\underline{Y}_t'\underline{Y}_t\right] = E\left[\left\|\underline{Y}_t\right\|_2^2\right]\Bigg) \\
= & \left\{\sqrt{\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right|}\right. \\
& \left.\times\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b}\right\} \\
& + \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\left\|\underline{Y}_t\right\|_2^2\right]}\sqrt{\left\|\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right\|_2 b'B_2'Q'^{-1}Q^{-1}B_2b} \\
& \left(\text{since } \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right)'\left(\widehat{\underline{F}}_t' - \underline{F}_t'Q\right) = \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)'\right)
\end{aligned}
$$

Now, by part (b) of Lemma D-2 and Slutsky's theorem, we have

$$
\left|a'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\underline{Y}_t' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\underline{Y}_t\underline{Y}_t'\right]\right)a\right| = O_p\left(\frac{1}{\sqrt{T}}\right) = o_p\left(1\right) \tag{136}
$$

Note also that

$$
\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\left\|\underline{Y}_t\right\|_2^2\right] & \leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left(E\left[\left\|\underline{Y}_t\right\|_2^6\right]\right)^{\frac{1}{3}} \quad \text{(by Liapunov's inequality)} \\
& \leq \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\overline{C}\right)^{\frac{1}{3}} \quad \text{(by Lemma C-4)} \\
& = \left(\overline{C}\right)^{\frac{1}{3}}.
\end{aligned} \tag{137}
$$

It follows by applying expressions (135), (136), and (137) as well as the result given in part (g) of

Lemma D-16 and the Slutsky' theorem that

$$\begin{aligned}
& \left| \frac{a'\underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 b}{T_h} \right| \\
\leq \; & \left\{ \sqrt{\left| a' \left(\frac{1}{T_h} \sum_{t=p}^{T-h} \underline{Y}_t \underline{Y}_t' - \frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\underline{Y}_t \underline{Y}_t'\right] \right) a \right|} \right. \\
& \left. \times \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \right\} \\
& + \sqrt{\frac{1}{T_h} \sum_{t=p}^{T-h} E\left[\left\|\underline{Y}_t\right\|_2^2\right]} \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
= \; & o_p(1).
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{dp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we further deduce that

$$\frac{\underline{Y}'\left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2}{T_h} = o_p(1).$$

To show part (f), note that for any $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we can write

$$
\begin{aligned}
& \left| \frac{\iota_{T_h}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| \\
= & \left| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b \right| \\
\leq & \frac{1}{T_h} \sum_{t=p}^{T-h} \left| \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b \right| \quad \text{(by triangle inequality)} \\
\leq & \sqrt{b' B_2' Q'^{-1} \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right) Q^{-1} B_2 b} \quad \text{(by Liapunov's inequality)} \\
\leq & \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right) \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \ \Bigg(\text{since for a symmetric} \\
& \text{psd matrix } A, \ \|A\|_2 = \sqrt{\lambda_{\max}(A'A)} = \sqrt{\lambda_{\max}(A^2)} = \sqrt{[\lambda_{\max}(A)]^2} = \lambda_{\max}(A) \Bigg) \\
= & \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
& \left(\text{since } \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right)' \left(\widehat{\underline{F}}_t' - \underline{F}_t' Q \right) = \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right)
\end{aligned}
$$

It follows by applying expression (135), the result given in part (g) of Lemma D-16, and the Slutsky' theorem that

$$
\begin{aligned}
\left| \frac{\iota_{T_h}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2 b}{T_h} \right| & \leq \sqrt{\left\| \frac{1}{T_h} \sum_{t=p}^{T-h} \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right) \left(\widehat{\underline{F}}_t - Q' \underline{F}_t \right)' \right\|_2 b' B_2' Q'^{-1} Q^{-1} B_2 b} \\
& = o_p(1).
\end{aligned}
$$

Since the above argument holds for all $b \in \mathbb{R}^d$ such that $\|b\|_2 = 1$, we further deduce that

$$
\frac{\iota_{T_h}' \left(\widehat{\underline{F}} - \underline{F} Q \right) Q^{-1} B_2}{T_h} = o_p(1).
$$

For part (g), note that, for any $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$, we have, by direct calculation,

$$
\begin{aligned}
&\frac{a'\widehat{\underline{F}}'\mathfrak{H}b}{T_h} \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} a'\widehat{\underline{F}}_t \eta_{t+h}' b \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t + Q'\underline{F}_t\right)\eta_{t+h}' b \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}' b + \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t \eta_{t+h}' b
\end{aligned}
$$

Focusing first on the first term on last line above, we note that

$$
\begin{aligned}
&\left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}' b\right| \\
\leq \;& \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{b'\frac{1}{T_h}\sum_{t=p}^{T-h} \eta_{t+h}\eta_{t+h}' b} \\
\leq \;& \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a} \\
& \times\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right| + \left|\frac{1}{T_h}\sum_{t=p}^{T-h} b'E\left[\eta_{t+h}\eta_{t+h}'\right]b\right|} \\
\leq \;& \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right|} \\
& + \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b'E\left[\eta_{t+h}\eta_{t+h}'\right]b} \\
& \left(\text{by the inequality } \sqrt{a_1 + a_2} \leq \sqrt{a_1} + \sqrt{a_2} \text{ for } a_1 \geq 0 \text{ and } a_2 \geq 0\right)
\end{aligned}
$$

Note that, by part (g) of Lemma D-16, we have

$$
\frac{1}{T_h}\sum_{t=p}^{T-h}\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' = o_p(1).
$$

and, by part (h) of Lemma D-2,

$$\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right] = O_p\left(\frac{1}{\sqrt{T}}\right).$$

Moreover, note that

$$\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J_{d+K}' \varepsilon_{t+h-j}$$

and, using expression (71) given in the proof of part (e) of Lemma D-2 and Assumption 3-2(b), we see that there exists a positive constant C^* such that

$$\begin{aligned}
& \frac{1}{T_h}\sum_{t=p}^{T-h} b' E\left[\eta_{t+h}\eta_{t+h}'\right] b \\
= \; & \frac{1}{T_h}\sum_{t=p}^{T-h} E\left[\left(b'\eta_{t+h}\right)^2\right] \\
\leq \; & \frac{1}{T_h}\sum_{t=p}^{T-h} \left(E\left[\left(b'\eta_{t+h}\right)^4\right]\right)^{\frac{1}{2}} \\
\leq \; & \frac{1}{T_h}\sum_{t=p}^{T-h} \left(\, C^*\right)^{\frac{1}{2}} \\
& \text{(for some positive constant } C^* \text{ as shown in expression (71))} \\
\leq \; & \left(\, C^*\right)^{\frac{1}{2}} < \infty
\end{aligned}$$

Making use of these calculations and applying Slutsky's theorem, we deduce that

$$\begin{aligned}
& \left|\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}' b\right| \\
\leq \; & \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\left|b'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}\eta_{t+h}' - \frac{1}{T_h}\sum_{t=p}^{T-h}E\left[\eta_{t+h}\eta_{t+h}'\right]\right)b\right|} \\
& + \sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)' a}\sqrt{\frac{1}{T_h}\sum_{t=p}^{T-h} b' E\left[\eta_{t+h}\eta_{t+h}'\right] b} \\
= \; & o_p(1).
\end{aligned}$$

Next, note that, by part (f) of Lemma D-2 and Slutsky's theorem, we see that

$$\begin{aligned}
\frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t\eta_{t+h}'b &= a'Q'\left(\frac{1}{T_h}\sum_{t=p}^{T-h}\underline{F}_t\eta_{t+h}'\right)b \\
&= O_p\left(\frac{1}{\sqrt{T}}\right) = o_p(1)
\end{aligned}$$

Putting everything together and applying Slutsky's theorem once more, we then obtain

$$\begin{aligned}
\frac{a'\widehat{\underline{F}}'\mathfrak{H}b}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h} a'\widehat{\underline{F}}_t\eta_{t+h}'b \\
&= \frac{1}{T_h}\sum_{t=p}^{T-h} a'\left(\widehat{\underline{F}}_t - Q'\underline{F}_t\right)\eta_{t+h}'b + \frac{1}{T_h}\sum_{t=p}^{T-h} a'Q'\underline{F}_t\eta_{t+h}'b \\
&= o_p(1).
\end{aligned}$$

Since the above argument holds for all $a \in \mathbb{R}^{Kp}$ and $b \in \mathbb{R}^d$ such that $\|a\|_2 = 1$ and $\|b\|_2 = 1$; we further deduce that

$$\frac{\widehat{\underline{F}}'\mathfrak{H}}{T_h} = \frac{1}{T_h}\sum_{t=p}^{T-h}\widehat{\underline{F}}_t\eta_{t+h}' = o_p(1). \ \square$$

Lemma D-18: Suppose that Assumptions 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11* hold. Then,

$$\begin{pmatrix} \widehat{\beta}_0' - \beta_0' \\ \widehat{B}_1 - B_1 \\ \widehat{B}_2 - Q^{-1}B_2 \end{pmatrix} = o_p(1).$$

Here, $\widehat{\beta}_0$, $\widehat{B}_1$, and $\widehat{B}_2$ denote the OLS estimators of the coefficient parameters in the (feasible) h-step ahead forecast equation

$$\begin{aligned}
Y_{t+h} &= \beta_0 + \sum_{g=1}^{p} B_{1,g}'Y_{t-g+1} + \sum_{g=1}^{p} B_{2,g}'\widehat{F}_{t-g+1} + \widehat{\eta}_{t+h} \\
&= \beta_0 + B_1'\underline{Y}_t + B_2'\widehat{\underline{F}}_t + \widehat{\eta}_{t+h},
\end{aligned}$$

for $t = p, ..., T-h$, where the unobserved factor vector $\underline{F}_t$ is replaced by the estimate $\widehat{\underline{F}}_t$ and where $\widehat{\eta}_{t+h} = \eta_{t+h} - B_2'\left(\widehat{\underline{F}}_t - \underline{F}_t\right)$ with $\eta_{t+h} = \sum_{j=0}^{h-1} J_d A^j J_{d+K}'\varepsilon_{t+h-j}$ as previously defined.

Proof of Lemma D-18: To proceed, we first stack the observations to obtain the representation

$$\underset{T_h\times d}{Y(h)} = \underset{T_h\times 1}{\iota_{T_h}} \underset{1\times d}{\beta_0'} + \underset{T_h\times dp}{\underline{Y}} \underset{dp\times d}{B_1} + \underset{T_h\times Kp}{\widehat{\underline{F}}} \underset{Kp\times d}{B_2} + \underset{T_h\times d}{\widehat{\mathfrak{H}}} \tag{138}$$

where $T_h = T - h - p + 1$ and where

$$\underset{T_h\times d}{Y(h)} = \begin{pmatrix} Y_{h+p}' \\ \vdots \\ Y_T' \end{pmatrix}, \underset{T_h\times dp}{\underline{Y}} = \begin{pmatrix} \underline{Y}_p' \\ \vdots \\ \underline{Y}_{T-h}' \end{pmatrix}, \underset{T_h\times Kp}{\widehat{\underline{F}}} = \begin{pmatrix} \widehat{\underline{F}}_p' \\ \vdots \\ \widehat{\underline{F}}_{T-h}' \end{pmatrix}, \text{ and } \underset{T_h\times d}{\widehat{\mathfrak{H}}} = \begin{pmatrix} \widehat{\eta}_{h+p}' \\ \vdots \\ \widehat{\eta}_T' \end{pmatrix}.$$

It is easily seen from expression (138) that the OLS estimators of the coefficients β_0, B_1, and B_2 are given by

$$\begin{pmatrix} \widehat{\beta}_0' \\ \widehat{B}_1 \\ \widehat{B}_2 \end{pmatrix} = \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\widehat{\underline{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\widehat{\underline{F}} \\ \widehat{\underline{F}}'\iota_{T_h} & \widehat{\underline{F}}'\underline{Y} & \widehat{\underline{F}}'\widehat{\underline{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'Y(h) \\ \underline{Y}'Y(h) \\ \widehat{\underline{F}}'Y(h) \end{bmatrix}.$$

Now, rewrite expression (138) as

$$\begin{aligned} Y(h) &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}B_2 + \widehat{\mathfrak{H}} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}B_2 + \mathfrak{H} - \left(\widehat{\underline{F}} - \underline{F}\right)B_2 \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \underline{F}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \underline{F}QQ^{-1}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \left(\widehat{\underline{F}} + \underline{F}Q - \widehat{\underline{F}}\right)Q^{-1}B_2 + \mathfrak{H} \\ &= \iota_{T_h}\beta_0' + \underline{Y}B_1 + \widehat{\underline{F}}Q^{-1}B_2 - \left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 + \mathfrak{H} \\ &= \begin{bmatrix} \iota_{T_h} & \underline{Y} & \widehat{\underline{F}} \end{bmatrix} \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} - \left(\widehat{\underline{F}} - \underline{F}Q\right)Q^{-1}B_2 + \mathfrak{H}, \end{aligned}$$

and it follows that

$$
\begin{aligned}
&\begin{pmatrix} \widehat{\beta}_0' - \beta_0' \\ \widehat{B}_1 - B_1 \\ \widehat{B}_2 - Q^{-1}B_2 \end{pmatrix} \\
= &\begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \underline{\widehat{F}}' \end{bmatrix} Y(h) - \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
= &\begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \underline{\widehat{F}}' \end{bmatrix} \begin{bmatrix} \iota_{T_h} & \underline{Y} & \underline{\widehat{F}} \end{bmatrix} \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} - \begin{pmatrix} \beta_0' \\ B_1 \\ Q^{-1}B_2 \end{pmatrix} \\
&- \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \underline{\widehat{F}}' \end{bmatrix} \left(\underline{\widehat{F}} - \underline{F}Q\right) Q^{-1}B_2 \\
&+ \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}' \\ \underline{Y}' \\ \underline{\widehat{F}}' \end{bmatrix} \mathfrak{H}. \\
= &- \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'\left(\underline{\widehat{F}} - \underline{F}Q\right) Q^{-1}B_2 \\ \underline{Y}'\left(\underline{\widehat{F}} - \underline{F}Q\right) Q^{-1}B_2 \\ \underline{\widehat{F}}'\left(\underline{\widehat{F}} - \underline{F}Q\right) Q^{-1}B_2 \end{bmatrix} \\
&+ \begin{pmatrix} T_h & \iota_{T_h}'\underline{Y} & \iota_{T_h}'\underline{\widehat{F}} \\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}} \\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}} \end{pmatrix}^{-1} \begin{bmatrix} \iota_{T_h}'\mathfrak{H} \\ \underline{Y}'\mathfrak{H} \\ \underline{\widehat{F}}'\mathfrak{H} \end{bmatrix}
\end{aligned}
$$

Next, applying parts (b) and (d) of Lemma D-2 and parts (a), (b), (c), and (d) of Lemma D-17, we obtain

$$
\begin{aligned}
& \begin{pmatrix} 1 & \iota_{T_h}'\underline{Y}/T_h & \iota_{T_h}'\underline{\widehat{F}}/T_h \\ \underline{Y}'\iota_{T_h}/T_h & \underline{Y}'\underline{Y}/T_h & \underline{Y}'\underline{\widehat{F}}/T_h \\ \underline{\widehat{F}}'\iota_{T_h}/T_h & \underline{\widehat{F}}'\underline{Y}/T_h & \underline{\widehat{F}}'\underline{\widehat{F}}/T_h \end{pmatrix} \\
& - \begin{pmatrix} 1 & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{F}_t'\right] Q \\ T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{F}_t'\right] Q \\ T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right] Q \end{pmatrix} \\
= \;& o_p\left(1\right).
\end{aligned}
$$

Moreover, note that

$$
\begin{aligned}
& \begin{pmatrix} 1 & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{F}_t'\right] Q \\ T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} E\left[\underline{Y}_t\underline{F}_t'\right] Q \\ T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{Y}_t'\right] & T_h^{-1}\sum_{t=p}^{T-h} Q'E\left[\underline{F}_t\underline{F}_t'\right] Q \end{pmatrix} \\
= \;& \frac{1}{T_h}\sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] Q \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] Q \\ Q'E\left[\underline{F}_t\right] & Q'E\left[\underline{F}_t\underline{Y}_t'\right] & Q'E\left[\underline{F}_t\underline{F}_t'\right] Q \end{pmatrix} \\
= \;& \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q' \end{pmatrix} \frac{1}{T_h}\sum_{t=p}^{T-h} \begin{pmatrix} 1 & E\left[\underline{Y}_t'\right] & E\left[\underline{F}_t'\right] \\ E\left[\underline{Y}_t\right] & E\left[\underline{Y}_t\underline{Y}_t'\right] & E\left[\underline{Y}_t\underline{F}_t'\right] \\ E\left[\underline{F}_t\right] & E\left[\underline{F}_t\underline{Y}_t'\right] & E\left[\underline{F}_t\underline{F}_t'\right] \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{dp} & 0 \\ 0 & 0 & Q \end{pmatrix}
\end{aligned}
$$

which is non-singular and, therefore, also positive definite for all T sufficiently large in light of the result given in part (b) of Lemma D-1.

In addition, applying parts (f) and (g) of Lemma D-2 and parts (d), (e), (f), and (g) of Lemma D-17, we have

$$
\begin{aligned}
\frac{\iota_{T_h}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right), \\
\frac{\underline{Y}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\eta_{t+h}' = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\iota'_{T_h}\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\left(\underline{\widehat{F}}'_t-\underline{F}'_tQ\right)Q^{-1}B_2=o_p(1),\\
\frac{\underline{Y}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{Y}_t\left(\underline{\widehat{F}}'_t-\underline{F}'_tQ\right)Q^{-1}B_2=o_p(1),\\
\frac{\underline{\widehat{F}}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{\widehat{F}}_t\left(\underline{\widehat{F}}'_t-\underline{F}'_tQ\right)Q^{-1}B_2=o_p(1),\\
\frac{\underline{\widehat{F}}'\mathfrak{H}}{T_h} &= \frac{1}{T_h}\sum_{t=p}^{T-h}\underline{\widehat{F}}_t\eta'_{t+h}=o_p(1)
\end{aligned}$$

Putting everything together and applying the Slutsky's theorem

$$\begin{aligned}
&\begin{pmatrix}\widehat{\beta}'_0-\beta'_0\\ \widehat{B}_1-B_1\\ \widehat{B}_2-Q^{-1}B_2\end{pmatrix}\\
= & -\begin{pmatrix}T_h & \iota'_{T_h}\underline{Y} & \iota'_{T_h}\underline{\widehat{F}}\\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}}\\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}}\end{pmatrix}^{-1}\begin{bmatrix}\iota'_{T_h}\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\\ \underline{Y}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\\ \underline{\widehat{F}}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\end{bmatrix}\\
& +\begin{pmatrix}T_h & \iota'_{T_h}\underline{Y} & \iota'_{T_h}\underline{\widehat{F}}\\ \underline{Y}'\iota_{T_h} & \underline{Y}'\underline{Y} & \underline{Y}'\underline{\widehat{F}}\\ \underline{\widehat{F}}'\iota_{T_h} & \underline{\widehat{F}}'\underline{Y} & \underline{\widehat{F}}'\underline{\widehat{F}}\end{pmatrix}^{-1}\begin{bmatrix}\iota'_{T_h}\mathfrak{H}\\ \underline{Y}'\mathfrak{H}\\ \underline{\widehat{F}}'\mathfrak{H}\end{bmatrix}\\
= & -\begin{pmatrix}1 & T_h^{-1}\iota'_{T_h}\underline{Y} & T_h^{-1}\iota'_{T_h}\underline{\widehat{F}}\\ T_h^{-1}\underline{Y}'\iota_{T_h} & T_h^{-1}\underline{Y}'\underline{Y} & T_h^{-1}\underline{Y}'\underline{\widehat{F}}\\ T_h^{-1}\underline{\widehat{F}}'\iota_{T_h} & T_h^{-1}\underline{\widehat{F}}'\underline{Y} & T_h^{-1}\underline{\widehat{F}}'\underline{\widehat{F}}\end{pmatrix}^{-1}\begin{bmatrix}T_h^{-1}\iota'_{T_h}\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\\ T_h^{-1}\underline{Y}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\\ T_h^{-1}\underline{\widehat{F}}'\left(\underline{\widehat{F}}-\underline{F}Q\right)Q^{-1}B_2\end{bmatrix}\\
& +\begin{pmatrix}1 & T_h^{-1}\iota'_{T_h}\underline{Y} & T_h^{-1}\iota'_{T_h}\underline{\widehat{F}}\\ T_h^{-1}\underline{Y}'\iota_{T_h} & T_h^{-1}\underline{Y}'\underline{Y} & T_h^{-1}\underline{Y}'\underline{\widehat{F}}\\ T_h^{-1}\underline{\widehat{F}}'\iota_{T_h} & T_h^{-1}\underline{\widehat{F}}'\underline{Y} & T_h^{-1}\underline{\widehat{F}}'\underline{\widehat{F}}\end{pmatrix}^{-1}\begin{bmatrix}T_h^{-1}\iota'_{T_h}\mathfrak{H}\\ T_h^{-1}\underline{Y}'\mathfrak{H}\\ T_h^{-1}\underline{\widehat{F}}'\mathfrak{H}\end{bmatrix}\\
= & \ o_p(1).\ \square
\end{aligned}$$

References

[1] Anatolyev, S. and A. Mikusheva (2021): “Factor Models with Many Assets: Strong Factors, Weak Factors, and the Two-Pass Procedure,” *Journal of Econometrics*, forthcoming.

[2] Andrews, D.W.K. (1984): “Non-strong Mixing Autoregressive Processes,” *Journal of Applied Probability*, 21, 930-934.

[3] Bai, J. and S. Ng (2002): “Determining the Number of Factors in Approximate Factor Models,” *Econometrica*, 70, 191-221.

[4] Bai, J. (2003): “Inferential Theory for Factor Models of Large Dimensions,” *Econometrica*, 71, 135-171.

[5] Bai, J. and S. Ng (2008): “Forecasting Economic Time Series Using Targeted Predictors,” *Journal of Econometrics*, 146, 304-317.

[6] Bai, J. and S. Ng (2021): “Approximate Factor Models with Weaker Loading,” Working Paper, Columbia University.

[7] Bai, Z. D. and Y. Q. Yin (1993): “Limit of the Smallest Eigenvalue of a Large Dimensional Sample Covariance Matrix," *Annals of Probability*, 21, 1275-1294.

[8] Bair, E., T. Hastie, D. Paul, and R. Tibshirani (2006): “Prediction by Supervised Principal Components,” *Journal of the American Statistical Association*, 101, 119-137.

[9] Belloni, A., D. Chen, V. Chernozhukov, and C. Hansen (2012): “Sparse Models and Methods for Optimal Instruments with an Application to Eminent Domain,” *Econometrica*, 80, 2369-2429.

[10] Billingsley, P. (1995): *Probability and Measure*. New York: John Wiley & Sons.

[11] Borovkova, S., R. Burton, and H. Dehling (2001): “Limit Theorems for Functionals of Mixing Processes to U-Statistics and Dimension Estimation,” *Transactions of the American Mathematical Society*, 353, 4261-4318.

[12] Bryzgalova, S. (2016): “Spurious Factors in Linear Asset Pricing Models,” Working Paper, Stanford Graduate School of Business.

[13] Burnside, C. (2016): “Identification and Inference in Linear Stochastic Discount Factor Models with Excess Returns,” *Journal of Financial Econometrics*, 14, 295-330.

[14] Chao, J. C., K. Qiu, and N. R. Swanson (2023a): "Selecting the Relevant Variables for Factor Estimation in Factor Estimation in FAVAR Models, With An Application to Forecasting the Yield Curve," Working Paper, Rutgers University and University of Maryland.

[15] Chao, J.C., K. Qiu, and N. R. Swanson (2023b): "Online Supplement: Selecting the Relevant Variables for Factor Estimation in FAVAR Models, With An Application to Forecasting the Yield Curve," Working Paper, Rutgers University and University of Maryland.

[16] Chen, X., Q. Shao, W. B. Wu, and L. Xu (2016): "Self-normalized Cramér-type Moderate Deviations under Dependence," *Annals of Statistics*, 44, 1593-1617.

[17] Davidson. J. (1994): *Stochastic Limit Theory: An Introduction for Econometricians*. New York: Oxford University Press.

[18] Davidson, K. R. and S. J. Szarek (2001): "Local Operator Theory, Random Matrices and Banach Spaces." In *Handbook of the Geometry of Banach Spaces*, 1, 317-366. Amsterdam: North-Holland.

[19] Fan, J., Y. Liao, and M. Mincheva (2011): "High-dimensional Covariance Matrix Estimation in Approximate Factor Models," *Annals of Statistics*, 39, 3320-3356.

[20] Fan, J., Y. Liao, and M. Mincheva (2013): "Large Covariance Estimation by Thresholding Principal Orthogonal Complements," *Journal of the Royal Statistical Society, Series B*, 75, 603-680.

[21] Freyaldenhoven, S. (2021a): "Factor Models with Local Factors - Determining the Number of Relevant Factors," *Journal of Econometrics*, forthcoming.

[22] Freyaldenhoven, S. (2021b): "Identification through Sparsity in Factor Models: The ℓ_1-Rotation Criterion," Working Paper, Federal Reserve Bank of Philadelphia.

[23] Giglio, S., D. Xiu, and D. Zhang (2021): "Test Assets and Weak Factors," Working Paper, Yale School of Management and the Booth School of Business, University of Chicago.

[24] Golub, G. H. and C. F. van Loan (1996): *Matrix Computations*, 3rd Edition. Baltimore: The Johns Hopkins University Press.

[25] Goroketskii, V. V. (1977): "On the Strong Mixing Property for Linear Sequences," *Theory of Probability and Applications*, 22, 411-413.

[26] Gospodinov, N., R. Kan, and C. Robotti (2017): "Spurious Inference in Reduced-Rank Asset Pricing Models," *Econometrica*, 85, 1613-1628.

[27] Harding, M. C. (2008): "Explaining the Single Factor Bias of Arbitrage Pricing Models in Finite Samples," *Economics Letters*, 99, 85-88.

[28] Horn, R. and C. Johnson (1985): *Matrix Analysis*. Cambridge University Press.

[29] Jagannathan, R. and Z. Wang (1998): "An Asymptotic Theory for Estimating Beta-Pricing Models Using Cross-Sectional Regression," *Journal of Finance*, 53, 1285-1309.

[30] Johnstone, I. M. and A. Lu (2009): "On Consistency and Sparsity for Principal Components Analysis in High Dimensions," *Journal of the American Statistical Association*, 104, 682-697.

[31] Johnstone, I. M. and D. Paul (2018): "PCA in High Dimensions: An Orientation," *Proceedings of the IEEE*, 106, 1277-1292.

[32] Kan, R. and C. Zhang (1999): "Two-Pass Tests of Asset Pricing Models with Useless Factors," *Journal of Finance*, 54, 203-235.

[33] Kleibergen, F. (2009): "Tests of Risk Premia in Linear Factor Models," *Journal of Econometrics*, 149, 149-173.

[34] Lütkepohl, H. (2005): *New Introduction to Multiple Time Series Analysis*. New York: Springer.

[35] Nadler, B. (2008): "Finite Sample Approximation Results for Principal Component Analysis: A Matrix Perturbation Approach," *Annals of Statistics*, 36, 2791-2817.

[36] Onatski, A. (2012): "Asymptotics of the Principal Components Estimator of Large Factor Models with Weakly Influential Factors," *Journal of Econometrics*, 168, 244-258.

[37] Paul, D. (2007): "Asymptotics of Sample Eigenstructure for a Large Dimensional Spiked Covariance Model," *Statistica Sinica*, 17, 1617-1642.

[38] Pham, T. D. and L. T. Tran (1985): "Some Mixing Properties of Time Series Models," *Stochastic Processes and Their Applications*, 19, 297-303.

[39] Ruhe, A. (1975): "On the Closeness of Eigenvalues and Singular Values for Almost Normal Matrices," *Linear Algebra and Its Applications*, 11, 87-94.

[40] Shen, D., H. Shen, H. Zhu, J.S. Marron (2016): "The Statistics and Mathematics of High Dimension Low Sample Size Asymptotics," *Statistica Sinica*, 26, 1747-1770.

[41] Stewart, G.W. (1973): "Error and Perturbation Bounds for Subspaces Associated with Certain Eigenvalue Problems," *SIAM Review*, 15, 727-764.

[42] Stewart, G.W. and J. Sun (1990): *Matrix Perturbation Theory*. Boston: Academic Press.

[43] Stock, J. H. and M. W. Watson (2002a): "Forecasting Using Principal Components from a Large Number of Predictors," *Journal of the American Statistical Association*, 97, 1167-1179.

[44] Stock, J. H. and M. W. Watson (2002b): "Macroeconomic Forecasting Using Diffusion Indexes," *Journal of Business and Economic Statistics*, 20, 147-162.

[45] Vershynin, R. (2012): "Introduction to the Non-asymptotic Analysis of Random Matrices," In *Compressed Sensing, Theory and Applications,* 210-268. Cambridge University Press.

[46] Wang, W. and J. Fan (2017): "Asymptotics of Empirical Eigenstructure for High Dimensional Spiked Covariance," *Annals of Statistics*, 45, 1342-1374.