Robust Forecast Superiority Testing with an Application to Assessing Pools of Expert Forecasters*

Valentina Corradi[1], Sainan Jin[2], and Norman R. Swanson[3]

[1]University of Surrey, [2]Tsinghua University, and [3]Rutgers University

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Abstract

We develop a forecast superiority testing methodology which is robust to the choice of loss function. Following Jin, Corradi and Swanson (JCS: 2017), we rely on a mapping between generic loss forecast evaluation and stochastic dominance principles. However, unlike JCS tests, which are not uniformly valid, and have correct asymptotic size only under the least favorable case, our tests are uniformly asymptotically valid and non-conservative. These properties are derived by first establishing uniform convergence (over error support) of HAC variance estimators. Monte Carlo experiments indicate good finite sample performance of the new tests, and an empirical illustration suggests that prior forecast accuracy matters in the Survey of Professional Forecasters. Namely, for our longest forecast horizons (4 quarters ahead), selecting pools of expert forecasters based on prior accuracy results in ensemble forecasts that are superior to those based on forming simple averages and medians from the entire panel of experts.

Keywords: Robust Forecast Evaluation, Many Moment Inequalities, Bootstrap, Estimation Error, Combination Forecasts, Survey of Professional Forecasters.

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*Valentina Corradi, School of Economics, University of Surrey, Guildford, Surrey, GU2 7XH, UK, v.corradi@surrey.ac.uk; Sainan Jin, School of Social Sciences and School of Economics and Management, Tsinghua University, Beijing 100084, snjin@smu.edu.sg; and Norman R. Swanson, Department of Economics, Rutgers University, 75 Hamilton Street, New Brunswick, NJ 08901, USA, nswanson@econ.rutgers.edu. We are grateful to the Editor, Barbara Rossi, and two anonymous referees for their very useful and constructive comments. We also thank Kevin Lee, Patrick Marsh, Luis Martins, James Mitchell, Alessia Paccagini, Paulo Parente, Ivan Petrella, Valerio Poti, Simon Van Norden, Claudio Zoli, and to the participants at the 2018 NBER-NSF Times Series Conference, the 2016 European Meeting of the Econometric Society, Conference for 50 years of Keynes College at Kent University, and seminars at Mannheim University, the University of Nottingham, University College Dublin, Instituto Universitário de Lisboa, Universita' di Verona and the Warwick Business School for useful comments and suggestions. Additionally, many thanks are owed to Mingmian Cheng for excellent research assistance.

1 Introduction

Forecast accuracy is typically measured in terms of a given loss function, with quadratic and absolute loss being the most common choices. In recent years, there has been a growing discussion about the choice of the "right" loss function. Gneiting (2011) stresses the importance of matching the quantity to be forecasted and the choice of loss function (or scoring rule). The latter is said to be consistent for a given statistical functional (e.g. the mean or the median), if expected loss is minimized when such a functional is used. In a recent paper, Patton (2019) shows that if forecasts are based on nested information sets and on correctly specified models, then in the absence of estimation error, forecast ranking is robust to the choice of loss function within the class of consistent functions. On the other hand, if any of the above conditions fail, then model ranking is dependent on the specific loss function used. This is an important finding, given that it is natural for researchers to focus on the comparison of multiple misspecified models; immediately implying that model rankings are loss function dependent.

In summary, given the importance of loss function dependence when comparing forecast accuracy, an issue of key concern to empirical economists is the construction of loss function robust forecast accuracy tests. A loss function free forecast evaluation criterion should be based on the distribution of raw forecast errors. Heuristically, one can define the best forecasting model as that producing errors having a step cumulative distribution function that is equal to zero on the negative real line and equal to one on the positive real line. Diebold and Shin (2015, 2017) build on this idea, and suggest choosing the model for which the cumulative distribution of the forecast errors is closest to a step function. This idea is also discussed in Corradi and Swanson (2013). Jin, Corradi and Swanson (JCS: 2017) establish a one to one mapping between generalized loss (GL) forecast superiority and first order stochastic dominance, as well as a one to one mapping between convex loss function (CL) and second order stochastic dominance.[1] In particular, they show that the "best" model (regardless of loss function) according to a GL (CL) function is the one which is first (second) order stochastically dominated on the negative real line and first (second) order stochastically dominant on the positive real line, when comparing forecast errors. In this sense, JCS (2017) establish that loss function free tests for forecast superiority can be framed in terms of tests for stochastic dominance. In this paper, we note that tests for stochastic dominance can be seen as tests for infinitely many moment inequalities. This allows us to utilize tools recently developed by Andrews and Shi (2013, 2017) to derive asymptotically uniformly valid and non conservative forecast superiority tests. Importantly, these tests improve over those introduced in JCS (2017), as the latter were asymptotically non conservative only in the least favorable case under the null (i.e., when all moment weak inequalities hold as equality). Needless to say, controlling for slack inequalities is crucial when there are infinitely many of them.

[1] A loss function is a GL function if it is monotonically non-decreasing as the error moves away from zero. Additionally, CL functions are the subset of convex GL functions.

The implementation of our tests require that sample moments are standardized by an estimator of the standard deviation. Now, forecast errors are typically non martingale difference sequences, either because they are based on dynamically misspecified models or because forecasters do not efficiently use all the available information, in the case of subjective predictions. Hence, we require heteroskedasticity and autocorrelation (HAC) robust variance estimators. In our set-up, each variance estimator depends on a specific point in the forecasting error support. Thus, in order to introduce our new tests for forecast superiority, we must establish the consistency of HAC variance estimators uniformly over the error support. Moreover, in order to carry out inference, using our tests, we also establish uniform convergence of the HAC variance estimator bootstrap counterparts. Because of the presence of the lag truncation parameter, uniform convergence of HAC estimators and of their bootstrap analogs does not follow straightforwardly from uniform convergence of (kernel) nonparametric estimators. To the best of our knowledge this contribution is a novel addition to the vast literature on HAC covariance matrix estimation. In the sequel, we focus on the case of judgmental forecasts, in which there is no parameter estimation error. In a supplemental online appendix, we consider the case of predictions based on estimated models, and extend all of our results to the case of non vanishing estimation error. This is accomplished under a recursive estimation scheme, by extending the recursive block bootstrap introduced in Corradi and Swanson (2007). For the case of fixed estimation scheme, Generalized Moment Selection tests in the presence of non-vanishing estimation error have been considered in Coroneo, Corradi and Santos-Monteiro (2018).

Linton, Song and Whang (2010) also develop tests for stochastic dominance which are correctly asymptotically sized over the boundary of the null, for the pairwise comparison case. A key role in their asymptotic analysis is played by the contact set (i.e., the set of x over which the two CDFs are equal). However, the notion of contact set does not extend straightforwardly to the multiple comparison case considered in this paper. It should also be noted that other papers have addressed the problem of forecast evaluation in the absence of full specification of the loss function. For example, Patton and Timmermann (2007) have studied forecast optimality under only generic assumptions on the loss function. However, they do not address the issue of forecast ranking under (partially) unknown loss. More recently, Barendse and Patton (2019) introduce forecast multiple comparison under loss functions which are specified only up to a shape parameter. Finally, Arvanitis, Post, Poti and Karabati (2021), revisit the set up of JCS and suggest a blockwise Empirical Likelihood type test for testing forecast superiority. Their inference is conservative, since based on majorizing chi-squared critical values.

We assess the forecast superiority testing methodology discussed in this paper via a series of Monte Carlo experiments. Simulation results show that our new tests are in some key cases much more accurately sized and have much higher power than JCS tests. For example, in size experiments where DGPs contain some models which are worse than the benchmark model, our new tests are substantially better sized

than the tests of JCS (2017). Additionally, they exhibit notable power gains, relative to JCS tests, in power experiments where DGPs contain some alternative models that dominate the benchmark, while others are strictly dominated. These findings are as expected, given that JCS tests are undersized, while our new tests are asymptotically non conservative.

In an empirical illustration, we apply our testing procedure to the Survey of Professional Forecasters (SPF) dataset. In the SPF, participants are told which variables to forecast and whether they should provide a point forecast or instead a probability interval, but they are not given a loss function (see Crushore (1993) for a detailed description of the SPF). In the context of analyzing the predictive content of the SPF, many papers find evidence of the usefulness of forecast combinations constructed using individual SPF predictions, under quadratic or absolute loss. For example, Zarnowitz and Braun (1993) find that using the mean or median provides a consensus forecast with lower average errors than most individual forecasts. Aiolfi, Capistrán, and Timmermann (2011) and Genre, Kenny, Meyler, and Timmermann (2013) find that equal weighted averages of SPF and ECB *(European Central Bank)* SPF forecasts often outperform model based forecasts. In our illustration, we depart from these papers by noting that the SPF naturally lends itself to loss function free forecast superiority testing, since participants are not given loss functions. In light of this, we apply our new tests, and show that forecast averages (and medians) from small pools of survey participants ranked according to recent forecast performance are preferred to forecast averages based on the entire pool of experts, for our longest forecast horizon (1-year ahead). We thus conclude that simple average and median forecasts can in some cases be "beaten", regardless of loss function.

The rest of the paper is organized as follows. Section 2 outlines the set-up and introduces our new tests. Section 3 establishes the asymptotic properties of the tests in the context of generalized moment selection. Section 4 contains the results of our Monte Carlo experiments, and Section 5 contains the results of our analysis of GDP growth forecasts from the SPF. Finally, Section 6 provides a number of concluding remarks. Proofs for the case of GL forecast superiority are gathered in an appendix. In Supplement SA1 of the appendix, we report all results concerning CL forecast superiority. In Supplement SA2, we establish the asymptotic properties of our new tests in the context of non-vanishing recursive parameter estimation error, for the recursive estimation schemes. Supplement SA3 provides details of the JCS test used for comparison in the Monte Carlo and empirical sections. Finally, Supplement SA4 and S5 provides additional Monte Carlo and empirical results, respectively.

2 Forecast Superiority Tests

Assume that we have a time series of forecast errors for each model/forecaster. Namely, we observe $e_{j,t}$, for $j = 1, ..., k$ and $t = 1, ...n$, where k denotes the number of models/forecasters, and n denotes the number of observations. As stated earlier, we focus on the case in which we can ignore estimation error,

such as when forecasts are judgmental or subjective. Surveys including the SPF are leading examples of judgmental forecasts. Hereafter, the sequence $e_{1,t}$, $t = 1, ..., n$ is called the "benchmark". In the context of the SPF, an example of a relevant benchmark against which to compare all other sequences is the consensus forecast constructed as the simple arithmetic average of individual forecasts in the survey. Our goal is to test whether there exists some competing forecast that is superior to the benchmark for any loss function, L, satisfying Assumption A0.

Assumption A0 (i) $L \in \mathcal{L}_G$ if $L : \mathbb{R} \to \mathbb{R}^+$ is continuously differentiable, except for finitely many points, with derivative L', such that $L'(z) \leq 0$, for all $z \leq 0$, and $L'(z) \geq 0$, for all $z \geq 0$. (ii) $L \in \mathcal{L}_C$ is a convex function belonging to $\mathcal{L}_G$.

Note that $\mathcal{L}_G$ includes most of the loss functions commonly used by practitioners, including asymmetric loss, and it basically coincides with notion of generalized loss in Granger (1999). The only restriction is that the loss depends solely on the forecast errors. This rules out the class of loss function considered in e.g. Section 3 of Patton and Timmermann (2007).

Hereafter, let $F_j(x)$ denote the cumulative distribution function (CDF) of forecast error e_j. Also, define $sgn(x) = 1 \; if \; x \geq 0$ *and* $sgn(x) = -1 \; if \; x < 0$. Propositions 2.2 and 2.3 in JCS (2017) establish the following results.

1. *For any* $L \in \mathcal{L}_G$, $E(L(e_1)) \leq E(L(e_2))$, *if and only if* $(F_2(x) - F_1(x))sgn(x) \leq 0$, *for all* $x \in \mathcal{X}$.
2. *For any* $L \in \mathcal{L}_C$, $E(L(e_1)) \leq E(L(e_2))$, *if and only if* $\left(\int_{-\infty}^{x}(F_1(t) - F_2(t))dt1(x < 0) + \int_{x}^{\infty}(F_2(t) - F_1(t))dt1(x \geq 0)\right) \leq 0$, *for all* $x \in \mathcal{X}$.

The first statement establishes a mapping between GL forecast superiority and first order stochastic dominance (FOSD). In particular, e_1 is not GL dominated by e_2 if $F_1(x)$ lies below $F_2(x)$ on the negative real line, and lies above $F_2(x)$ on the positive real line. Indeed, this ensures that we choose the forecast whose CDF has larger mass around zero. Likewise, the second statement establishes a mapping between CL superiority and second order stochastic dominance.

In this framework, it follows that testing for loss function robust forecast superiority involves testing:

$$H_0^G : \max_{j=2,...,k} (E(L(e_1)) - E(L(e_k))) \leq 0 \text{ for all } L \in \mathcal{L}_G \tag{2.1}$$

versus

$$H_A^G : \max_{j=2,...,k} (E(L(e_1)) - E(L(e_k))) > 0 \text{ for some } L \in \mathcal{L}_G, \tag{2.2}$$

with H_0^C and H_A^C, defined analogously, replacing $\mathcal{L}_G$ with $\mathcal{L}_C$.

Hereafter, let $\mathcal{X} = \mathcal{X}^- \cup \mathcal{X}^+$ be the union of the support of $(e_1, ..., e_k)$. Given the equivalence between

GL forecast superiority and first order stochastic dominance, we can restate H_0^G and H_A^G as

$$\begin{aligned} H_0^G &= H_0^{G-} \cap H_0^{G+} \\ &: \quad \left(F_1(x) - F_j(x) \leq 0, \text{ for } j = 2, ..., k, \text{ and for all } x \in \mathcal{X}^-\right) \\ &\quad \cap \left(F_j(x) - F_1(x) \leq 0, \text{ for } j = 2, ..., k, \text{ and for all } x \in \mathcal{X}^+\right) \end{aligned}$$

versus

$$\begin{aligned} H_A^G &= H_A^{G-} \cup H_A^{G+} \\ &: \quad \left(F_1(x) - F_j(x) > 0, \text{ for some } j = 2, ..., k, \text{ and for some } x \in \mathcal{X}^-\right) \\ &\quad \cup \left(F_j(x) - F_1(x) > 0, \text{ for some } j = 2, ..., k, \text{ and for some } x \in \mathcal{X}^+\right). \end{aligned}$$

For brevity, hereafter we state all statistics and results for the GL case only, and report the corresponding statistics and results for the CL case in Supplement SA1.

It is immediate to see that H_0^G can be written as the intersection of $(k-1)$ moment inequalities, which have to hold uniformly over $\mathcal{X}$. This gives rise to an infinite number of moment conditions. Andrews and Shi (2013) develop tests for conditional moment inequalities, and as is well known in the literature on consistent specification testing (e.g., see Bierens (1982, 1990)) a finite number of conditional moments can be transformed into an infinite number of unconditional moments. The same is true in the case of weak inequalities. Andrews and Shi (2017) consider tests for conditional stochastic dominance, which are then characterized by an infinite number of conditional moment inequalities and so by a "twice" infinite number of unconditional inequalities. Recalling that our interest is on testing GL or CL forecast superiority as in (2.1) and (2.2), we confine our attention to unconditional testing of stochastic dominance.

Because of the discontinuity at zero in the tests, H_0^{G+} and H_0^{G-} should be tested separately, and then one can use Holm (1979) bounds to control the two resulting p-values (see Rules TG in JCS (2017)). In the sequel, for the sake of brevity, but without loss of generality, we focus our discussion on testing H_0^{G+} versus H_A^{G+}. However, when defining statistics, some discussion of the statistics associated with the case where $x \in \mathcal{X}^-$ is also given, when needed for clarity of exposition.

Let $G^+(x) = \left(G_2^+(x), ..., G_k^+(x)\right)$, with $G_j^+(x) = F_j(x) - F_1(x)$, for $x \geq 0$. Define the empirical analog of $G^+(x)$ as $G_n^+(x) = \left(G_{2,n}^+(x), ..., G_{k,n}^+(x)\right)$, and for $x \geq 0$, let

$$G_{j,n}^+(x) = \widehat{F}_{j,n}(x) - \widehat{F}_{1,n}(x), \tag{2.3}$$

where

$$\widehat{F}_{j,n}(x) = \frac{1}{n} \sum_{t=1}^{n} 1\left\{e_{j,t} \leq x\right\}$$

Further, define

$$\Sigma^{G+}(x, x') = \text{acov}\left(\sqrt{n} G^+(x), \sqrt{n} G^+(x')\right). \tag{2.4}$$

In order to implement the statistic we need only estimators for $\sigma_j^{2,G^+}(x) = \text{avar}\left(\sqrt{n}G_j^+(x)\right)$ for $j = 2, ...k$ and $x \in \mathcal{X}^+$. Letting $\widehat{u}_{j,t}(x) = 1\{e_{j,t} \leq x\} - \frac{1}{n}\sum_{t=1}^n 1\{e_{j,t} \leq x\}$, the HAC estimator for $\sigma_j^{2,G^+}(x)$ reads as

$$\begin{aligned}\widehat{\sigma}_{j,n}^{2,G+}(x) &= \frac{1}{n}\sum_{t=1}^{n}(\widehat{u}_{j,t}(x) - \widehat{u}_{1,t}(x))^2 \\ &\quad +2\frac{1}{n}\sum_{\tau=1}^{l_n}\sum_{t=\tau+1}^{n} w_\tau(\widehat{u}_{j,t}(x) - \widehat{u}_{1,t}(x))(\widehat{u}_{j,t-\tau}(x) - \widehat{u}_{1,t-\tau}(x)), \end{aligned} \tag{2.5}$$

where $w_\tau = 1 - \frac{\tau}{1+l_n}$, with $l_n \to \infty$ as $n \to \infty$. We can now define the statistics for testing GL forecast superiority, namely

$$S_n^{G+} = \int_{x\in\mathcal{X}^+}\sum_{j=2}^{k}\left(\max\left\{0, \sqrt{n}\frac{G_{j,n}(x)}{\overline{\sigma}_{j,n}^G(x)}\right\}\right)^2 dQ(x) \tag{2.6}$$

where

$$\overline{\sigma}_{j,n}^G(x) = \widehat{\sigma}_{j,n}^G(x) + \varepsilon. \tag{2.7}$$

$\varepsilon > 0$, and

$$S_n^{G-} = \int_{x\in\mathcal{X}^-}\sum_{j=2}^{k}\left(\max\left\{0, \sqrt{n}\frac{G_{j,n}(x)}{\overline{\sigma}_{j,n}^G(x)}\right\}\right)^2 dQ(x) \tag{2.8}$$

where Q is a weighting function defined below, $G_{j,n}(x) = G_{j,n}^+(x)$ in (2.8) is the $j-$th component of $G_n^+(x)$ as defined in (2.3). $G_{j,n}(x) = G_{j,n}^-(x)$ is defined similarily. The role of the additional ε term in (2.6) and (2.8) is to correct for the possible singularity of the variance estimator, which occurs when we compare forecast errors from nested models.[2] By scaling moment conditions by $\widehat{\sigma}_2(x) + \varepsilon$ rather than by $\widehat{\sigma}_2(x)$, we have a slightly smaller term and so we may reduce power. However, since we add the same ε in the construction of the bootstrap counterpart of $\widehat{\sigma}_{j,n}^G(x)$, there is no actual loss in power.

Here, S_n^{G+} is a "sum" function, as in equation (3.8) in Andrews and Shi (2013), and satisfies their Assumptions S1-S4, which are required to guarantee that convergence is uniform over the null DGPs.[3],[4] If $k = 2$ and $\widehat{\sigma}_{j,n}^G(x) = 1$, for all j and x (i.e. no standardization), and $\varepsilon = 0$, then S_n^{G+} is the statistic used in Linton, Song and Whang (2010) for testing FOSD.[5]

Of note is that in our context, potential slackness causes a discontinuity in the pointwise asymptotic distribution of the statistic.[6] This is because the pointwise asymptotic distribution is discontinuous,

[2] Pincheira et al. (2020) suggest a randomized version of tests for equal predictive ability in the context of nested models which has an asymptotically normal limiting distribution.

[3] Note that we could have constructed a different "sum" function, using the statistic in (3.9) of Andrews and Shi (2013).

[4] Recall that one main drawback of the $\max_{j=2,\ldots,k}\sup_{x\in\mathcal{X}^+}\sqrt{n}G_n^+$ statistic in JCS (2017) is that it diverges to $-\infty$ under some sequence of probability measures under the null, thus ruling out uniformity.

[5] Andrews and Soares (2010, p.123) suggest standardizing the moment conditions, in order to ensure invariance to rescaling.

[6] By pointwise asymptotic distribution we mean the limiting distribution under a fixed probability measure.

unless all moment conditions hold with equality. On the other hand, the finite sample distribution is not necessarily discontinuous. Thus, in the presence of slackness, the pointwise limiting distribution is not a good approximation of the finite sample distribution, and critical values based on pointwise asymptotics may be invalid. This is why we construct tests that are uniformly asymptotically valid (i.e., this is why we study the limiting distribution of our tests under drifting sequences of probability measures belonging to the null hypothesis). Moreover, in the infinite dimensional case, there is an additional source of discontinuity. In particular, the number of moment inequalities which contributes to the statistic varies across the different values of x. For example, the key difference between the case of $k = 2$ and $k > 2$ is that in the former case, for each value of x there is only one moment inequality which can be binding (or not). On the other hand, if $k = 3$, say, then for each value of x there can be either one or two moment inequalities which may be binding (or not), and whether or not a particular inequality is binding (or not) varies over x. Under this setup, we require the following assumptions in order to analyze the asymptotic behavior of our test statistics.

Assumption A1: For $j = 1, ..., k$, $e_{j,t}$ is strictly stationary and $\beta-$mixing, with mixing coefficients, $a_m = m^{-\beta}$, where $\beta > \frac{6\delta}{1-2\delta}$, $0 < \delta < 1/2$ and $\beta\delta > 1$.

Assumption A2: The union of the supports of $e_1, .., e_k$ is the compact set, $\mathcal{X} = \mathcal{X}^- \cup \mathcal{X}^+$.

Assumption A3: $F_j(x)$ has a continuos bounded density.

Assumption A4: The weighting function Q has full support $\mathcal{X}^+$ (or $\mathcal{X}^-$).

Assumption A1 requires strict stationarity. We require strict stationarity and beta mixing in the proof of Lemma 1. Furthermore, in the proof of Theorem 2 we use the results on bootstrap for empirical processes by Peligrad (1998) which also require strict stationarity.[7]

3 Asymptotic Properties

3.1 Uniform Convergence of the HAC Estimator

If $e_{1,t}, ..., e_{k,t}$ were martingale difference sequences, then we can still use the sample second moment as a variance estimator, and uniform consistency will follow by application of an appropriate uniform law of large numbers. In our set-up, we can assume that $e_1, ..., e_k$ are martingale difference sequences if either: (i) they are judgmental forecasts from professional forecasters, say, who efficiently use all available information at time t (a strong assumption, which is tested in the forecast rationality literature); or (ii) they are prediction errors from one-step ahead forecasts based on dynamically correctly specified models. With respect to (i), it is worth noting that professional forecasters may be rational, ex-post, according to some loss function (see Elliott, Komunjer and Timmermann (2005,2008), although it is not as likely

[7] As with all tests involving out of sample forecast evaluation, our test breaks down in the presence of time instability (see e.g. Giacomini and Rossi (2009) for the discussion of a test for the null of (no) forecast failure in this context.

that they are rational according to a generalized loss function. With respect to (ii), it should be noted that at most one model can be dynamically correctly specified for a given information set, and thus e_j cannot be a martingale difference sequence, for all $j = 1, ..., k$. In light of these facts, we allow for time dependence in the forecast error sequences used in our statistics, and use a HAC variance estimator in (2.6) and (2.8). In order to ensure that the HAC estimators converge uniformly over $\mathcal{X}^+$, it suffices to establish the counterpart of Lemma A1 of Supplement A of Andrews and Shi (2013) to the case of mixing sequences. This is done below.

Lemma 1*: Let Assumptions A1-A3 hold. Then, if as* $n \to \infty$, $\frac{l_n}{n^\delta} \to c$, *with* $0 < c < \infty$, and $0 < \delta < \frac{1}{2}$, *with* δ *defined as in Assumption A1:*

$$\sup_{x \in \mathcal{X}^+} \left| \widehat{\sigma}_{j,n}^{2,G+}(x) - \sigma_j^{2,G+}(x) \right| = o_p(1),$$

with $\sigma_j^{2,G+}(x) = \text{avar}\left(\sqrt{n} G_{j,n}^+(x)\right)$.

Lemma 1 establishes the uniform convergence over $\mathcal{X}^+$ of HAC estimators. Of note is that we require $\beta-$mixing. This differs from the stationary pointwise HAC variance estimator case studied by Andrews (1991), where $\alpha-$mixing suffices, and where the mixing coefficients decline to zero slightly slower than in our Assumption A1. This is because there is a trade-off between the degree of dependence and the rate of growth of the lag truncation parameter in the HAC estimator. Indeed, in the uniform case, the covering number (e.g., see Andrews and Pollard (1994)) grows with both l_n and the degree of dependence, thus leading to a trade-off between the two. For example, in the case of exponential mixing series, δ can be arbitrarily close to 1/2.

Recently, Li and Liao (2020) have established consistency for HAC estimators when the dimension of the long run variance increases with the sample size. On the other hand, our estimator for each $x \in \mathcal{X}^+$ is a scalar, $\widehat{\sigma}_{j,n}^{2,G^+}(x)$ and we show that as $n \to \infty$, $\sup_{x \in \mathcal{X}^+} \left| \widehat{\sigma}_{j,n}^{2,G^+}(x) - \sigma_j^{2,G^+}(x) \right| = o_p(1)$ for $j = 2, ..., J$ with J fixed. Hence, it suffices we show uniform convergence for each single j. If instead, we would let $J = J_n$ with $J_n \to \infty$ as $n \to \infty$ then we would need to combine our approach with Li and Liao (2020).

For carrying out inference on our forecast superiority tests, we require a bootstrap analog of the HAC variance estimator, which can be constructed as follows. Using the block bootstrap, make b_n draws of length l_n from $e_{j,1}, ..., e_{j,n}$, in order to obtain $\left(e_{j,1}^*, ..., e_{j,n}^*\right) = \left(e_{j,I_{1+1}}, ..., e_{j,I_{1+l_n}}, ..., e_{j,I_{b_n+l_n}}\right)$, with $b_n l_n = n$, where the block size, l_n, is equal to the lag truncation parameter in the HAC estimator described above.[8] Now, let $u_{1,t}^*(x) = 1\left\{e_{1,t}^* \leq x\right\} - \frac{1}{n}\sum_{t=1}^n 1\left\{e_{1,t} \leq x\right\}$, $u_{j,t}^*(x) = 1\left\{e_{j,t}^* \leq x\right\} - \frac{1}{n}\sum_{t=1}^n 1\left\{e_{j,t} \leq x\right\}$, and

$$\widehat{\sigma}_{j,n}^{2*G+}(x) = \frac{1}{b_n}\sum_{k=1}^{b_n}\left(\frac{1}{\sqrt{l_n}}\sum_{i=1}^{l_n}\left(u_{j,(k-1)l_n+i}^*(x) - u_{1,(k-1)l_n+i}^*(x)\right)\right)^2. \tag{3.1}$$

[8] We thus use the same notation, l_n, for both the lag truncation parameter and the block length.

Lemma 2: *Let Assumptions A1-A3 hold. Then, if as* $n \to \infty$, $\frac{l_n}{n^\delta} \to c$, *with* $0 < c < \infty$, and $0 < \delta < \frac{1}{2}$, *with* δ *defined as in Assumption A1:*

$$\sup_{x \in \mathcal{X}^+} \left| \widehat{\sigma}_{j,n}^{*G+}(x) - \widehat{\sigma}_{j,n}^{G+}(x) \right| = o_p^*(1),$$

where $o_p^*(1)$ *denotes convergence to zero according to the bootstrap law,* P^*, *conditional on the sample.*

Lemma 2 is a key ingredient for establishing the validity of the boostrap procedure outlined in the sequence.

3.2 Inference Using the Bootstrap and Bounding Limiting Distributions

The statistics S_n^{G+} is highly discontinuous over x. Exactly which moment conditions, and how many of them are binding varies over x. Hence, S_n^{G+} does not necessarily have a well defined limiting distribution; and the continuous mapping theorem cannot be applied. However, following the generalized moment selection (GMS) test approach of Andrews and Shi (2013) we can establish lower and upper bound limiting distributions. Let

$$v^{G+}(.) = (v_2^{G+}(.), ..., v_k^{G+}(.))', \tag{3.2}$$

be a $(k-1)-$dimensional zero mean Gaussian process with covariance kernel defined as in (2.4). Also, let

$$h_{j,A,n}^{G+}(x) = \sigma_j^{G+}(x)^{-1} \sqrt{n} G_j^+(x) \tag{3.3}$$

$$h_{j,B}^{G+}(x) = \sigma_j^{2,G+}(x)^{-1} \left(\sigma_j^{G+}(x) + \varepsilon \right)^2 \tag{3.4}$$

where $h_{A,n}^{G+}(x) = \left(h_{2,A,n}^{G+}(x), ..., h_{k,A,n}^{G+}(x) \right)$ and $h_B^{G+}(x) = \left(h_{2,B}^{G+}(x), ..., h_{k,B}^{G+}(x) \right)$

$$S_n^{\dagger G+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max \left\{ 0, \frac{v_j^{G+}(x) + h_{j,A,n}^{G+}(x)}{\sqrt{h_{j,B}^{G+}(x)}} \right\} \right)^2 \mathrm{d}Q(x), \tag{3.5}$$

and

$$S_\infty^{\dagger G+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max \left\{ 0, \frac{v_j^{G+}(x) + h_{j,A,\infty}^{G+}(x)}{\sqrt{h_{j,B}^{G+}(x)}} \right\} \right)^2 \mathrm{d}Q(x), \tag{3.6}$$

where $h_{j,A,\infty}^{G+}(x) = 0$, if $G_j^+(x) = 0$, and $h_{j,A,\infty}^{G+}(x) = -\infty$, if $G_j^+(x) < 0$. Hereafter let

$$\mathcal{P}_0^{G+} = \left\{ P : \ H_0^{G+} \text{ holds} \right\}$$

so that $\mathcal{P}_0^{G+}$ is the collection of DGPs under which the null hypothesis holds. The following result holds.

Theorem 1: *Let Assumptions A1-A4 hold. Then, under* H_0^{G+}, *there exists* $\delta > 0$ *and* $a^{G+} > 0$, such that

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{G+}} \left[P\left(S_n^{G+} > a^{G+}\right) - P\left(S_n^{\dagger G+} + \delta > a^{G+}\right)\right] \leq 0$$

and

$$\liminf_{n\to\infty} \inf_{P\in\mathcal{P}_0^{G+}} \left[P\left(S_n^{G+} > a^{G+}\right) - P\left(S_n^{\dagger G+} - \delta > a^{G+}\right)\right] \geq 0.$$

Theorem 1 provides upper and lower bounds for $P\left(S_n^{G+} > a^{G+}\right)$, uniformly, over the probabilities under H_0^{G+}. In particular, Theorem 1 establishes lower and upper bound for S_n^{G+} as $n \to \infty$.

Following Andrews and Shi (2013), we can construct bootstrap critical values which properly mimic the critical values of $S_\infty^{\dagger G+}$. We rely on the block bootstrap to capture the dependence in the data when constructing our bootstrap statistics. Let $\left(e_{j,1}^*, ..., e_{j,n}^*\right), b_n$, and l_n be defined as in the previous subsection, and let:

$$G_{j,n}^{*+}(x) = \frac{1}{n}\sum_{i=1}^{n}\left(1\left\{e_{j,i}^* \leq x\right\} - 1\left\{e_{1,i}^* \leq x\right\}\right) \tag{3.7}$$

and $v_n^{*G+}(x) = \left(v_{2,n}^{*G+}(x), ..., v_{k,n}^{*G+}(x)\right)$ with

$$v_{j,n}^{*G+}(x) = \frac{\sqrt{n}\left(G_{j,n}^{*+}(x) - G_{j,n}^{+}(x)\right)}{\widehat{\sigma}_{j,n}^{G+}(x)}. \tag{3.8}$$

Then, define:

$$\xi_{j,n}^{G+}(x) = \kappa_n^{-1} n^{1/2} \frac{G_{j,n}^{+}(x)}{\overline{\sigma}_{j,n}^{G+}(x)}, \tag{3.9}$$

with $\overline{\sigma}_{j,n}^{G+}(x)$ as in (2.7), $\kappa_n \to \infty$, as $n \to \infty$, and

$$\phi_{j,n}^{G+}(x) = c_n 1\left\{\xi_{j,n}^{G+}(x) < -1\right\}, \tag{3.10}$$

and $\phi_n^{G+} = \left(\phi_{2,n}^{G+}, ..., \phi_{k,n}^{G+}\right)$, with c_n a positive sequence, which is bounded away from zero. Thus, $\phi_{j,n}^{G+}(x) = c_n$, when $G_{j,n}^{+}(x) < -\kappa_n n^{-1/2}\overline{\sigma}_{j,n}^{G+}(x)$, i.e., when the $j-$th inequality is slack at x, and is equal to zero otherwise. We say that a competing model is slack at $x > 0$ ($x < 0$), if its CDF evaluated at x is strictly below (strictly above) the CDF of the benchmark model.

It it clear from the selection rule in (3.10), that we do need an estimator of the variance of the moment conditions, despite the fact we use bootstrap critical values. In fact, standardization does not play a crucial role in the statistics, as all positive sample moment conditions matter. On the other hand, without the scaling factor in (3.9), the number of non-slack moment conditions would depend on the scale, and hence our bootstrap critical values would no longer be scale invariant. Let

$$S_n^{*G+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \max\left(\left\{0, \frac{v_{j,n}^{*G+}(x) - \phi_{j,n}^{G+}(x)}{\sqrt{h_{j,B}^{*G+}(x)}}\right\}\right)^2 \mathrm{d}Q(x), \tag{3.11}$$

where

$$h_{j,B}^{*G+}(x) = \overline{\sigma}_{j,n}^{2,*G+}(x)/\widehat{\sigma}_{j,n}^{2,G+}(x), \tag{3.12}$$

with $\overline{\sigma}_{j,n}^{2,*G+} = \left(\widehat{\sigma}_{j,n}^{*G+}(x) + \varepsilon\right)^2$, and $\widehat{\sigma}_{j,n}^{2,*G+}(x)$ defined as in (3.1), and $h_B^{*G+}(x) = \left(h_{2,B}^{*G+}(x), ..., h_{k,B}^{*G+}(x)\right)$. Note that if c_n grows with n, then all slack inequalities are discarded, asymptotically. It is immediate to see that S_n^{*G+} is the bootstrap counterpart of $S_n^{\dagger G+}$ in (3.5), with $\phi_{j,n}^{G+}(x)$ mimicking the contribution of the slackness of inequality j, i.e. $h_{j,A,n}^{G+}(x)$. However, $\phi_{j,n}^{G+}(x)$ is not a consistent estimator of $h_{j,A,n}^{G+}(x)$, since the latter cannot be consistently estimated.

By comparing (3.11) with (2.6), it is immediate to see that $G_{j,n}^{+}(x)$ does not contribute to the test statistic when $G_{j,n}^{+}(x) < 0$, while it does not contribute to the bootstrap statistic when $G_{j,n}^{+}(x) < -\kappa_n n^{-1/2}\overline{\sigma}_{j,n}^{G+}(x)$, with $\kappa_n n^{-1/2} \to 0$. Heuristically, by letting κ_n grow with the sample size, we control the rejection rates in a uniform manner.

It remains to define the GMS bootstrap critical values. Let $c_{n,B,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_B^{*G+}\right)$ be the $(1-\alpha)$-th critical value of S_n^{*G+}, based on B bootstrap replications, with ϕ_n^{G+} defined as in (3.10). The $(1-\alpha)$-th GMS bootstrap critical value, $c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_B^{*G+}\right)$, is defined as:

$$c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_B^{*G+}\right) = \lim_{B\to\infty} c_{n,B,1-\alpha+\eta}^{*G+}\left(\phi_n^{G+}, h_B^{*G+}\right) + \eta,$$

for $\eta > 0$, arbitrarily small.

Here, the constant η is used to guarantee uniformity over the infinite dimensional nuisance parameters, $h_{A,n}^{G+}(.)$ uniformly on $x \in \mathcal{X}^+$, and is termed the infinitesimal uniformity factor by Andrews and Shi (2013).

Finally, let

$$\mathcal{B}^{G+} = \left\{x \in \mathcal{X}^+ \text{ s.t. } h_{A,j,\infty}^{G+}(x) = 0, \text{ for some } j = 2, ..., k\right\} \tag{3.13}$$

where $\mathcal{B}^{G+}$ defines the set over which at least one moment condition holds with strict equality, and this set represents the boundaries of H_0^{G+}.

The following result holds.

Theorem 2: *Let Assumptions A1-A4 hold, and let $l_n \to \infty$ and $l_n n^{\frac{1}{3}-\varepsilon} \to 0$ as $n \to \infty$. Under H_0^{G+} :*

(i) if as $n \to \infty$, $\kappa_n \to \infty$ and $c_n/\kappa_n \to 0$, then

$$\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{G+}} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right) \leq \alpha;$$

and

(ii) if as $n \to \infty$, $\kappa_n \to \infty$, $c_n \to \infty$, $c_n/\kappa_n \to 0$, $\sqrt{n}/\kappa_n \to \infty$, and $Q\left(\mathcal{B}^{G+}\right) > 0$, then

$$\lim_{\eta\to 0}\limsup_{n\to\infty} \sup_{P\in\mathcal{P}_0^{G+}} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right) = \alpha.$$

Note that in Theorem 2 we require l_n to grow slower than $n^{1/3}$, which is a slower rate than that allowed in Lemma 2. This slower rate is required for the bootstrap empirical central limit theorem for a mixing process to hold (see Peligrad (1998)).

Statement (i) of Theorem 2 establishes that inference based on GMS bootstrap critical values is uniformly asymptotically valid. Statement (ii) establishes that inference based on GMS bootstrap critical values is asymptotically non-conservative, whenever $Q(\mathcal{B}^+) > 0$ (i.e., whenever at least one moment condition holds with equality, over a set $x \in \mathcal{X}^+$ with non-zero $Q-$measure). Although the GMS based tests are not similar on the boundary, the degree of non similarity, which is

$$\lim_{\eta \to 0} \lim \sup_{n \to \infty} \sup_{P \in \mathcal{P}_0^{G+}} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right)$$
$$- \lim_{\eta \to 0} \liminf \inf_{P \in \mathcal{P}_0^{G+}} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right),$$

is much smaller than that associated with using the "usual" recentered bootstrap. In the case of pairwise comparison (i.e., $k = 2$), Theorem 2(ii) of Linton, Song and Whang (2010) establishes similarity of stochastic dominance tests on a subset of the boundary.

For implementation of the tests discussed in this paper, it thus follows that one can use Holm bounds as is done in JCS (2017), with modifications due to the presence of the constant η. Estimate bootstrap $p-$values $p_{B,n,S_n^{G+}}^{G+} = \frac{1}{B}\sum_{s=1}^{B} 1\left((S_n^{*G^+} + \eta) \geq S_n^{G^+}\right)$ and $p_{B,n,S_n^{G-}}^{G-} = \frac{1}{B}\sum_{s=1}^{B} 1\left((S_n^{*G^-} + \eta) \geq S_n^{G^-}\right)$. Then, use the following rules (Holm (1979)):

Rule S_n^G: Reject H_0^G at level α, if $\min\left\{p_{B,n,S_n^{G+}}^{G+}, p_{B,n,S_n^{G-}}^{G-}\right\} \leq (\alpha - \eta)/2$.

3.3 Power against Fixed and Local Alternatives

As our statistics are weighted averages over $\mathcal{X}^+$, they have non-trivial power only if the null is violated over a subset of non zero $Q-$measure. This applies to both power against fixed alternative, as well as to power against $\sqrt{n}-$local alternatives. In particular, for power against fixed alternatives, we require the following assumption.

Assumption FA: $Q(B_{FA}^{G+}) > 0$, where $B_{FA}^{G+} = \{x \in \mathcal{X}^+ : G_j(x) > 0 \text{ for some } j = 2, ..., k\}$.

The following result holds.

Theorem 3: *Let Assumptions A0-A4 hold. If Assumption FA holds, then under H_A^{G+} :*

$$\lim_{n \to \infty} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right) = 1.$$

It is immediate to see that we have unit power against fixed alternatives, provided that the null hypothesis is violated, for at least one $j = 2, ..., k$, over a subset of $\mathcal{X}^+$ of non-zero $Q-$measure. Now, if we instead used a Kolmogorov type statistic (i.e., replace the integral over $\mathcal{X}^+$ with the supremum over $\mathcal{X}^+$), then we would not need Assumption FA, and it would suffice to have violation for some x, with possibly zero $Q-$measure, or in general with zero Lebesgue measure.[9] However, as pointed out in Supplement B of

[9] The Kolmorogov versions of S_n^{G+} and S_n^{C+} are:

$$KS_n^{G+} = \max_{x \in \mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{0, \frac{\sqrt{n} G_{j,n}^+(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\}\right)^2$$

Andrews and Shi (2013) the statement in parts (ii) of Theorem 2 does not apply to Kolmogorov tests, and hence asymptotic non-conservativeness would not necessarily hold. This is because the proof of those statements use the bounded convergence theorem, which applies to integrals but not to suprema.

We now consider the following sequences of local alternatives:

$$H_{L,n}^{G+}: G_{Lj}^{+}(x) = G_{j}^{+}(x) + \frac{\delta_j(x)}{\sqrt{n}} + o\left(n^{-1/2}\right), \text{ for } j = 2, ..., k, \ x \in \mathcal{X}^{+},$$

so that $\lim_{n\to\infty} \sqrt{n}\sigma^{2,G+}(x)^{-1/2} G_{Lj}^{+}(x) \to h_{j,A,\infty}^{G+}(x) + \delta_j(x)$. Define,

$$S_{\infty,\delta,LG}^{\dagger,G+} = \int_{\mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{0, \frac{v_j^{G+}(x) + h_{j,A,\infty}^{G+}(x) + \delta_j(x)}{\sqrt{h_{j,B}^{G+}(x)}}\right\}\right)^2 \mathrm{d}Q(x)$$

We require the following assumption.

Assumption LA: $Q(B_{LA}^{G+}) > 0$, where
$B_{LA}^{G+} = \left\{x : \sqrt{n}\sigma_j^{G+}(x)^{-1} G_{Lj}^{+}(x) \to h_{j,A,\infty}^{G+}(x) + \delta_j(x), \ 0 < h_{j,A,\infty}^{G+}(x) + \delta_j(x) < \infty, \text{ for some } j = 2, ..., k\right\}$.

The following result holds.

Theorem 4: *Let Assumptions A1-A4 hold. If Assumption LA(i) holds, then under* $H_{L,n}^{G+}$:

$$\lim_{n\to\infty} P\left(S_n^{G+} \geq c_{0,n,1-\alpha}^{*G+}\left(\phi_n^{G+}, h_{B,n}^{*G+}\right)\right) = P\left(S_{\infty,\delta,LG}^{\dagger G+} \geq c_{LG,1-\alpha}\left(h_{A,\infty}^{G+}, h_{B,\infty}^{G+}\right)\right),$$

with $c_{LG,1-\alpha}\left(h_{A,\infty}^{G+}, h_{B,\infty}^{G+}\right)$ *denoting the* $(1-\alpha)$*-th critical value of* $S_{\infty,\delta,LG}^{\dagger G+}$*, with* $0 < h_{j,A,\infty}^{G+}(x) + \delta_j(x) < \infty$*, for some* $j = 2, ..., k$.

Theorem 4 establishes that our tests have power against $\sqrt{n}-$alternatives, provided that the drifting sequence is bounded away from zero, over a subset of $\mathcal{X}^+$ of non-zero $Q-$measure. Note also that for given loss function, L, the sequence of local alternatives for the White reality check (White 2000)[10] can be defined as:

$$H_{A,n}: \max_{j=2,...,k} \left(E(L(e_1)) - E(L(e_j))\right) = \frac{\lambda}{\sqrt{n}} + o\left(n^{-1/2}\right), \ \lambda > 0. \tag{3.14}$$

For sake of simplicity, suppose that $k = 2$ (this is the well known Diebold and Mariano (1995) test framework). Here,

$$KS_n^{C+} = \max_{x\in\mathcal{X}^+} \sum_{j=2}^{k} \left(\max\left\{0, \frac{\sqrt{n}C_{j,n}^{+}(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\}\right)^2$$

[10] The White reality check null hypothesis states that no competing model outperforms a given benchmark model.

$$
\begin{aligned}
& 0 < \lambda = n^{1/2} E(L(e_1)) - E(L(e_2)) + o(1) \\
= \; & n^{1/2} \int_{-\infty}^{\infty} L(x) \left(f_{1,n}(x) - f_{2,n}(x) \right) \mathrm{d}x \\
= \; & -n^{1/2} \int_{-\infty}^{0} L'(x) \left(F_{1,n}(x) - F_{2,n}(x) \right) \mathrm{d}x \\
& -n^{1/2} \int_{0}^{\infty} L'(x) \left(F_{1,n}(x) - F_{2,n}(x) \right) \mathrm{d}x \\
= \; & n^{1/2} \int_{-\infty}^{0} \left(h^{G-}_{A,\infty}(x) + \delta(x) \right) Q(x) \mathrm{d}x + n^{1/2} \int_{0}^{\infty} \left(h^{G+}_{A,\infty}(x) + \delta(x) \right) Q(x) \mathrm{d}x, \qquad (3.15)
\end{aligned}
$$

where $F_{j,n}(x) = F_j(x) + \frac{\delta_j(x)}{\sqrt{n}}$. Hence, $H_{A,n}$ in (3.14) is equivalent to $H^{G+}_{LA} \cap H^{G-}_{LA}$, whenever Assumption A0 holds and $Q(x) = L'(x) sgn(x)$.

3.4 Implementation

In order to construct the statistic S_n^{G+} as in (2.6), we need to select $\mathcal{X}^+$, the weighting function Q, the lag truncation parameter l_n, and the tuning parameter ε. In order to implement the bootstrap counterpart of S_n^{*G+}, as defined in (3.11), we also need to choose the sequences c_n and κ_n, as well as the uniformity factor η. For S_n^{G-} and S_n^{*G-}, we need to select $\mathcal{X}^-$ instead of $\mathcal{X}^+$. Let $\mathcal{S}_{j,n}$ denote the support of $(e_{j,1}, ..., e_{j,n})$ and note that as n grows, $\mathcal{S}_{j,n}$ gets close to $\mathcal{X}_j$, the support of e_j. Note that $\mathcal{S}_{j,n} = \mathcal{S}^+_{j,n} \cup \mathcal{S}^-_{j,n}$, where $\mathcal{S}^+_{j,n}$ and $\mathcal{S}^-_{j,n}$ denote the positive and negative part of $\mathcal{S}_{j,n}$. Let $\mathcal{S}^{+,tr}_{j,n}$ define the trimmed version of $\mathcal{S}^+_{j,n}$, obtained by trimming the largest 5% or 1% of observations, say, with the degree of trimming being the same across all j, and let $\mathcal{S}^-_{j,n}$ be obtained by trimming the smallest 5% or 1% of observations. Finally, let $\mathcal{S}^{+,tr}_n = \cup_{j=1}^k \mathcal{S}^{+,tr}_{j,n}$ and $\mathcal{S}^{-,tr}_n = \cup_{j=1}^k \mathcal{S}^{-,tr}_{j,n}$. Now, set $\mathcal{X}^+ = \mathcal{S}^{+,tr}_n$. In practice, we need to approximate the integral with a sum, and so we partition $\mathcal{S}^{+,tr}_n$ ($\mathcal{X}^+$) into N equal intervals, say. If we want to set Q to be a uniform weighting matrix, we can simply set $\mathrm{d}Q = Q_\iota - Q_{\iota-1} = N^{-1}$, for $\iota = 1, ..., N$, with $Q_0 = 0$. Hence, for $x_\iota > 0$, we compute S_n^{G+} as

$$
S_n^{G+} = \frac{1}{N} \sum_{\iota=1}^{N} \sum_{j=2}^{k} \left(\max \left\{ 0, \frac{\sqrt{n} G^+_{j,n}(x_\iota)}{\overline{\sigma}^{G+}_{j,n}(x_\iota) + \varepsilon} \right\} \right),
$$

where ε is small positive number, ranging from 0.01 to 0.05, say. As another example, consider the case where we do not care about small prediction errors. Then, choose a $\mathrm{d}Q$ value that is smaller than N^{-1} for small values of ι and a $\mathrm{d}Q$ value that is larger than N^{-1} for large values of ι. With regard to the lag truncation parameter, choice thereof depends on the degree of autocorrelation of the error. In general, l_n can be chosen according to a rule of thumb, i.e. choose a value such that a small increase or decrease does not substantially change the estimator. Even though Lemma 1 requires more stringent conditions on the rate of growth of l_n, one can still use automated procedures, such as Andrews (1991).

Now consider the bootstrap counterpart, S_n^{*G+}, for which rate conditions for the choice of c_n and κ_n are given in Andrews and Shi (2013, 2017). In general, by setting $\kappa_n = c_0 \log(n)$ and $c_n = c_1 \frac{\log(n)}{\log(\log(n))}$, with c_0 and c_1 positive constants, the rate conditions in Theorem 2 are satisfied. To the best of our knowledge, there are no data driven procedures avaiable for choosing c_0 and c_1. Our simulation studies suggest that c_0 and c_1 should be small numbers, typically smaller than 0.5. The choice of the uniformity factor η is more delicate. In order to have an asymptotically non conservative test, we need η very close to zero, though only $\eta > 0$ ensures uniform asymptotic validity. As discussed below, in the simulation study we find that finite sample rejection rates are quite robust to values of η ranging from 0.0015 to 0.003.

4 Monte Carlo Experiments

In this section, we evaluate the finite sample performance of GL and CL forecast superiority tests when there are multiple competing sequences of forecast errors, under stationarity. We analyze the performance of our tests based on S_n^{G+} and S_n^{G-} (GL forecast superiority), as defined in (2.6)-(2.8), and based on S_n^{C+} and S_n^{C-} (CL forecast superiority), as defined in Appendix SA1 and compare their performance with that of the related test statistics from (2.5)-(2.6) in JCS (2017), here called JCS_n^{G+}, JCS_n^{G-}, JCS_n^{C+}, and JCS_n^{C-}.[11] For the sake of brevity, these two classes of tests are called S_n and JCS_n tests, respectively. For each experiment we carry out 1000 Monte Carlo replications, and the number of bootstrap samples is $B = 500$. Additionally, four different values of the smoothing parameter, J_n are examined for the JCS_n tests, including $J_n = \{0.20, 0.35, 0.50, 0.60\}$; and four different values of the uniformity constant, η, are examined for the S_n tests, including $\eta = \{0.0015, 0.002, 0.0025, 0.003\}$. Finally, when implementing the bootstrap counterpart of S_n, we set $\kappa_n = \sqrt{0.3 \log(n)}$ and $c_n = \sqrt{0.4 \log(n)/\log(\log(n))}$, following Andrews and Shi (2013, 2017). Sample sizes of $n \in \{300, 600, 900\}$ are generated using each of the following eight data generating processes (DGPs), with independent forecast errors.

DGP1: $e_{1t} \sim i.i.d.N(0,1)$ and $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3$.

DGP2: $e_{1t} \sim i.i.d.N(0,1)$ and $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3, 4, 5$.

DGP3: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.4^2)$, $k = 4, 5$

DGP4: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.6^2)$, $k = 4, 5$

DGP5: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 1.2^2)$, $k = 4, 5$.

DGP6: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3, 4, 5$ and $e_{kt} \sim i.i.d.N(0, 1.2^2)$, $k = 6, 7, 8, 9$.

DGP7: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 4, 5$.

DGP8: $e_{1t} \sim i.i.d.N(0,1)$, $e_{kt} \sim i.i.d.N(0,1)$, $k = 2, 3$ and $e_{kt} \sim i.i.d.N(0, 0.6^2)$, $k = 4, 5$.

DGP9: $e_{1t} \sim i.i.d.N(0,1)$ and $e_{kt} \sim i.i.d.N(0, 0.8^2)$, $k = 2, 3, 4, 5$.

[11] In JCS (2017), Eqs.(2.5)-(2.6), JCS_n^{G+}, JCS_n^{G-}, JCS_n^{C+}, and JCS_n^{C-} are termed TG_n^+, TG_n^-, TC_n^+ and TG_n^+ respectively. For details on the calculation of these statistics, refer to the supplemental appendix.

DGP10: $e_{1t} \sim i.i.d.N(0,1)$ and $e_{kt} \sim i.i.d.N(0,0.6^2)$, $k = 2,3,4,5$.

Given the importance of the HAC results developed in this paper, we conducted experiments using DGPs specified with autocorrelated errors. Denote $\widetilde{e}_{i,t} = \varrho\widetilde{e}_{i,t-1} + (1-\varrho^2)^{1/2}\eta_{i,t}$,with $\eta_{kt} \sim i.i.d.N(0,1)$ $i = 1,...,5$, and $\varrho = \{0.2, 0.4\}$. The DGPs for these experiments are as follows.

DGP-H1: $e_{1t} = \widetilde{e}_{1,t}$ and $e_{kt} = \widetilde{e}_{k,t}$, $k = 2,3,4,5$.

DGP-H2: $e_{1t} = \widetilde{e}_{1,t}$, $e_{kt} = \widetilde{e}_{k,t}$, $k = 2,3$ and $e_{kt} = 1.4\widetilde{e}_{k,t}$, $k = 4,5$.

DGP-H3: $e_{1t} = \widetilde{e}_{1,t}$, $e_{kt} = 0.8\widetilde{e}_{k,t}$, $k = 2,3$ and $e_{kt} = 1.2\widetilde{e}_{k,t}$, $k = 4,5$.

DGP-H4: $e_{1t} = \widetilde{e}_{1,t}$, $e_{kt} = \widetilde{e}_{k,t}$, $k = 2,3$ and $e_{kt} = 0.6\widetilde{e}_{k,t}$, $k = 4,5$.

In the above setup, DGPs 1-4 and DGPs H1-H2 are used to conduct size experiments, while DGPs 5-10 and DGPs H3-H4 are used to conduct power experiments. In all cases, e_{1t} denote the forecast errors from the benchmark model. Note that DGPs 1-2 correspond to the least favorable elements in the null, while in DGPs 3-4 and DGPs H1-H2, some models underperform the benchmark. This is the case where we expect significant improvement when using our new tests instead of JCS tests. In DGPs 5-6 and DGP-H3, one half of the competing models outperform the benchmark model and the other half underperform. In DGPs 7-8, one half of the competing models outperform, while in DGPs 9-10 and DGP-H4, the competing models all outperform the benchmark model. The above DGPs are similar to those examined in JCS (2017), and are utilized in our experiments because they clearly illustrate the trade-offs associated with using JCS_n and S_n forecast superiority tests.

For the sake of brevity, findings based on DGPs 1-10 are reported in the supplemental online appendix. Additionally, for DGPs H1-H4, we tabulate results for sample sizes of $n \in \{100, 150, 200\}$ in addition to the aforementioned sample sizes of $n \in \{300, 600, 900\}$.

Results reported in Tables 1 (JCS_n tests) and Table 2 (S_n tests) are rejection frequencies based on carrying out the JCS_n and S_n tests using a nominal size equal to 0.1, and for $\varrho = 0.2$ (results based on $\varrho = 0.4$ were qualitatively the same as those reported here). Inspection of the rejection frequencies in Table 1 indicates that $GL - JCS_n$ tests have reasonably good size under DGPs H1-H2 (the least favorable case under the null), across all sample sizes. However, $CL- JCS_n$ tests are undersized under DGP-H2. Also, as reported in the supplemental appendix, both types of tests are undersized for some sample size / J_n permutations when some models are worse than the benchmark (see DGPs 3-4), as should be expected given that the tests are not asymptotically correctly sized under these two DGPs. Moreover, in these cases the empirical size is non monotonic, in paricular for GL forecast superiority. Turning to Table 2, note that S_n tests, which are asymptotically non conservative, often exhibit better size properties under DGPs 3-4 (compare DGPs 3-4 in Tables 1 and 2 in the supplemental appendix) than JCS_n tests. For example, for the CL forecast superiority test the empirical size of the JCS_n test is 0.020 for all values of J_n, when $n = 900$ (see Table 1 in the supplemental appendix). The analogous value based on implementation of the S_n test is 0.083, for all values of η (see Table 2 in the supplemental appendix).

Again, it is worth stressing that this finding comes as no surprise, given that the S_n test is asymptotically non-conservative on the boundary of the null hypotheses, while the JCS_n test is conservative.

Now, from Table 1, we see that the power of the JCS_n test is sometimes quite low relative to that of the S_n test. For example, under DGP-H3, power is 0.385 for the $GL - JCS_n$ test and 0.650 for the $CL - JCS_n$ test, when $n = 300$. Analogous rejection frequencies for the S_n test are 0.896 and 0.952 (see Table 2, DGP-H2, $n = 300$). As expected, thus, S_n tests exhibit improved power relative to JCS_n tests, when some models are worse than the benchmark. Empirical power findings based on DGPs 5-10 are qualitatively the same (see supplemental appendix).

Finally, it should be pointed out that the S_n test is not overly sensitive to the choice of η, and the empirical size of S_n tests appears "best" when η is very small, as should be expected. In conclusion, there is a clear performance improvement when comparing our new robust predictive superiority tests with JCS_n tests.[12]

5 Empirical Illustration: Robust Forecast Evaluation of SPF Expert Pools

In the real-time forecasting literature, predictions from econometric models are often compared with surveys of expert forecasters.[13] Such comparisons are important when assessing the implications associated with using econometric models in policy setting contexts, for example. One key survey dataset collecting expert predictions is the *Survey of Professional Forecasters* (SPF), which is maintained by the Philadelphia Federal Reserve Bank (see Croushore (1993)). This dataset, formerly known as the *American Statistical Association/National Bureau of Economic Research Economic Outlook Survey,* collects predictions on various key economic indicators (including, for example, nominal GDP growth, real GDP growth, prices, unemployment, and industrial production). For further discussion of the variables contained in the SPF, refer to Croushore (1993) and Aiolfi, Capistrán, and Timmermann (2011). The SPF has been examined in numerous papers. For example, Zarnowitz and Braun (1993) comprehensively study the SPF, and find, among other things, that use of the mean or median provides a consensus forecast with lower average errors than most individual forecasts. More recently, Aiolfi, Capistrán, and Timmer-

[12] For a discussion of simulation results based on application of the Diebold and Mariano (DM: 1995) test (in which specific loss functions are utilized) in our experimental setup, refer to JCS(2017). Summarizing from that paper, it is clear that when the loss function is unknown, there is an advantage to using our approach of testing for forecast superiority. However the DM test for pairwise comparison or a reality check test for multiple comparisons might yield improved power, for a given loss function. Indeed, under quadratic loss, JCS (2017) show that when the sample size is small, the DM test has better power performance than JCS_n type tests. When the sample size increases, the power difference between the two tests becomes smaller. This is as expected.

[13] See Fair and Shiller (1990), Swanson and White (1997a,b), Aiolfi, Capistrán and Timmermann (2011), and the references cited therein for further discussion.

mann (2011) consider combinations of SPF survey forecasts, and find that equal weighted averages of survey forecasts outperform model based forecasts, although in some cases these mean forecasts can be improved upon by averaging them with mean econometric model-based forecasts. When utilizing European data from the recently released ECB SPF, Genre, Kenny, Meyler, and Timmermann (2013) again find that it is very difficult to beat the simple average. This well known result pervades the macroeconometric forecasting literature, and reasons for the success of such simple forecast averaging are discussed in Timmermann (2006). He notes that by averaging forecasts, issues such as misspecification and time instability, are attenuated, and this may explain the success of simple model averaging. Our empirical illustration attempts to shed further light on the issue of simple model averaging and its importance in forecasting macroeconomic variables.[14]

Our approach is to address the issue of forecast averaging and combination by viewing the problem through the lens of forecast superiority testing. Our use of loss function robust tests is unique to the SPF literature, to the best of our knowledge. Since we use robust forecast superiority tests, we do not evaluate pooling by using loss function specific tests, such as those discussed in Diebold and Mariano (1995), McCracken (2000), Corradi and Swanson (2003), and Clark and McCracken (2013). Additionally, our approach differs from that taken by Elliott, Timmermann, and Komunjer (2005, 2008), where the rationality of sequences of forecasts is evaluated by determining whether there exists a particular loss function under which the forecasts are rational. We instead evaluate predictive accuracy irrespective of the loss function implicitly used by the forecaster, and determine whether certain forecast combinations are superior when compared against any loss function, regardless of how the forecasts were constructed. In our tests, the benchmarks against which we compare our forecast combinations are simple average and median consensus forecasts. We aim to assess whether the well documented success of these benchmark combinations remains intact when they are compared against other combinations, under generic loss.[15]

In our experiments, we utilize SPF predictions of both nominal and real GDP growth. Results for nominal GDP are reported in the sequel, while those for real GDP are gathered in the supplemental appendix (see Tables 5-8 of the appendix). The SPF is a quarterly survey, and the dataset is available at the Philadelphia Federal Reserve Bank (PFRB) website. The original survey began in 1968:Q4, and PFRB took control of it in 1990:Q2; but from that date, there are only around 100 quarterly observations prior to 2018:Q1, where we end our sample. In our analysis we thus use the entire dataset, which, after

[14] As pointed out by an anonymous referee, and notwithstanding the fact that the forecast averaging methods used in our paper serve to mitigate misspecification and time instability, it remains to analyze our data using different sub-samples in order to ascertain the robustness of our empirical findings. This is left to future research.

[15] For an interesting discussion of machine learning and forecast combination methods, see Lahiri, Peng, and Zhao (2017); and for a discussion of probability forecasting and calibrated combining using the SPF, see Lahiri, Peng, and Zhao (2015). In these papers, various cases where consensus combinations do not "win" are discussed.

trimming to account for differing forecast horizons in our calculations, is 166 observations. [16],[17]

For our analysis, we consider 5 forecast horizons (i.e., $h = 0, 1, 2, 3, 4$). The reason we use $h = 0$ for one of the horizons is that the first horizon for which survey participants predict GDP growth is the quarter in which they are making their predictions. In light of this, forecasts made at $h = 0$ are called nowcasts. Moreover, it is worth noting that nowcasts are very important in policy making settings, since first release GDP data are not available until around the middle of the subsequent quarter. The nominal GDP variable that we examine is called NGDP in the SPF, and the real GDP variable reported on in the supplemental appendix is called RGDP. Consider NGDP growth rate prediction errors. In particular, assume that one survey participant makes a forecast of NGDP, say $y^f_{t+h}|\mathcal{F}_t$.[18] The associated forecast error is:

$$e_t = \{\ln(y_{t+h}) - \ln(y_t)\} - \left\{\ln(y^f_{t+h}|\mathcal{F}_t) - \ln(y_t)\right\} = \ln(y_{t+h}) - \ln(y^f_{t+h}|\mathcal{F}_t),$$

where the actual NGDP value, y_{t+h}, is reported in the SPF, along with the NGDP predictions of each survey participant. Note that when $h = 0$, $\mathcal{F}_t$ does not include y_t. However, for $h > 0$, $\mathcal{F}_t$ includes y_t. As discussed previously, this is due to the release dates associated with the availability of NGDP data. Figure 1 illustrates some of the key properties of the NGDP data that we utilize. Namely, note that the distributions of the expert forecasts vary over time, and exhibit interesting skewness and kurtosis properties (compare Panels A-D of the figure, and the skewness and kurtosis statistics reported below the plots in the figure). Based on examination of the densities in Figure 1, one might wonder whether "trimming" experts from the panel, say those experts that provided the forecasts appearing in the left tails of the distributions, might improve overall predictive accuracy of the panel. Although this question is not directly addressed in our analysis, we do construct and analyze the performance of various "pools" formed by trimming experts that exhibit sub-par predictive accuracy, for example. Note that we cannot distinguish individual from institutions, as individuals from the same institutions have the same identity code. This circumvents the potential issues of unbalanceness in the panel, also individual from the same institutions are likely to be rather homogeneous.

In addition to constructing $S_n^{G+}, S_n^{G-}, S_n^{C-}$, and S_n^{C+} tests in our empirical investigation, we also test for forecast superiority using the JCS_n tests discussed above, which have correct size only under the least favorable case under the null. In particular, we construct $JCS_n^{G+}, JCS_n^{G-}, JCS_n^{C-}$, and JCS_n^{C+} test statistics (see Supplement SA3 for further details). All test statistics are calculated using the same parameter values (for B, J_n, η, l_n, and ε) as used in our Monte Carlo experiments. However, results are

[16] It should be noted that the "timing" of the survey was not known with certainty prior to 1990. However, SPF documentation states that they believe, although are not sure, that the timing of the survey was similar before and after they took control of it.

[17] For further details on the SPF dataset, refer to the documentation at https://www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters.

[18] Here, $\mathcal{F}_t$ denotes the information set available to the expert forecaster at the time their predictions are made.

only reported for $J_n = 0.20$ and $\eta = 0.002$, since our findings remain unchanged when other values of J_n and η from our Monte Carlo experiments are used.

Two different benchmark models are considered, including (i) the arithmetic mean prediction from all participants; and (ii) the median prediction from all participants. Additionally, a variety of alternative model "groups" are considered. In all alternative models, mean and median predictions are again formed, but this time using subsets of the total available panel of experts, chosen in a number of ways, as outlined below.

Group 1 - Experts Chosen Based on Experience: Three expert pools (i.e. three alternative models) consisting of experts with 1, 3, and 5 years of experience.

In all of the remaining groups of combinations, individuals are ranked according to average absolute forecast errors, as well as according to average squared forecast errors. Mean (or median) predictions from these groups are then compared with our benchmark combinations.

Group 2 - Experts Chosen Based on Forecast Accuracy I: Three expert pools consisting of most accurate expert over last 1, 3, and 5 years.

Group 3 - Experts Chosen Based on Forecast Accuracy II: Three expert pools consisting of most accurate group of 3 experts over last 1, 3, and 5 years.

Group 4 - Experts Chosen Based on Forecast Accuracy III: Three expert pools consisting of top 10% most accurate group of experts over last 1, 3, and 5 years.

Group 5 - Experts Chosen Based on Forecast Accuracy III: Three expert pools consisting of top 25% most accurate group of experts over last 1, 3, and 5 years.

Finally, 3 additional groups which combine models from each of Groups 1-5 are analyzed. These include:

Group 6: Five expert pools, including one pool with experts that have 1 year of experience, and 4 additional pools, one from each of Groups 2-5, all defined over the last 1 year.

Group 7: Five expert pools, including one pool with experts that have 3 years of experience, and 4 additional pools, one from each of Groups 2-5, all defined over the last 3 years.

Group 8: Five expert pools, including one pool with experts that have 5 years of experience, and 4 additional pools, one from each of Groups 2-5, all defined over the last 5 years.

As an example of how testing is performed, note that when implementing the S_n^G test using *Group 1,* there are three alternative models. The same is true when implementing tests using *Groups 2-5.* For *Groups 6-8*, tests are implemented using 5 alternative models, where one alternative is taken from each of *Groups 1-5.* Summarizing, we consider: (i) two benchmark models, against which each group of alternatives is compared; (ii) alternative models that are based on either mean or median pooled forecasts for, *Groups 2-8*; (iii) forecast accuracy pools used in *Groups* 1-8 that are based on either average absolute forecast errors or average squared forecast errors; (iv) 5 forecast horizons.

We now discuss our empirical findings. In Tables 3-4, statistics are reported for all forecast superiority tests. Entries are S_n^G, S_n^C, JCS_n^G, and JCS_n^C test statistics reported for forecast horizons $h = 0, 1, 2, 3, 4$. More specifically, $S_n^G = S_n^{G+}$ if $p^{G+}_{B,n,S_n^{G+}} \leq p^{G-}_{B,n,S_n^{G-}}$; otherwise $S_n^G = S_n^{G-}$. The other statistics reported in the tables (i.e., S_n^C, JCS_n^G, and JCS_n^C) are defined analogously. Rejections of the null of no forecast superiority at a 10% level are denoted by a superscript *. In Table 3, the benchmark model is always the arithmetic mean prediction from all participants, and expert pool forecasts are also arithmetic means. Analogously, in Table 4 the benchmark is the median prediction from all participants, and expert pool forecasts are also medians. To understand the layout of the tables, turn to Table 3, and note that for *Group* 1, the 4 statistics defined above (i.e., S_n^G, S_n^C, JCS_n^G, and JCS_n^C) are given, for each forecast horizon, $h = 0, 1, 2, 3$, and 4. Superscripts denote rejection of the null hypothesis based on a particular test. For example, note that application of the JCS_n^G test in *Group* 2 yields a test rejection for horizons $h = 2$ and 4. Turning to the results summarized in the tables, a number of clear conclusions emerge. Summarizing, the results of 160 distinct forecast superiority tests are reported in each of Tables 3 and 4. These tests correspond to eight sets of alternative models against which the benchmark model is tested (i.e., Groups 1-8), five forecast horizons (i.e., $h = 0 - 4$), and 4 distinct tests (i.e., S_n^G, S_n^C, JCS_n^G, and JCS_n^C), for a total of 8x5x4=160 tests.

Our first finding is that the majority of test rejections occur for $h = 4$, as can be seen by inspection of the results in both Tables 3 and 4. In particular, note that for $h = 4$, there are 13 test rejections (of a possible 32) in Table 3 and 11 test rejections (of a possible 32) in Table 4, across *Groups* 1-8. On the other hand, for all other forecast horizons combined (i.e., $h = \{0, 1, 2, 3\}$), there are 11 test rejections (of a possible 128) in Table 3 and 8 test rejections (of a possible 128) in Table 4. This suggests that expert pools which are constructed by "trimming" the least effective experts are most useful for our longest forecast horizon (i.e., $h = 4$). Indeed, expert pools offer little advantage for all horizons less than $h = 4$. These findings make sense if one assumes that it is easier to make short term forecasts than long term forecasts. Namely, some experts are simply not "up to the task" when forecasting at longer horizons. Summarizing, our main finding indicates that simple average or median forecasts can be beaten, in cases where forecasts are more difficult to make (i.e., longer horizons). Second, "experience" as measured by the length of time an expert has taken part in the SPF is not a direct indicator of forecast superiority, since there are no rejections of our tests for *Group* 1, when either mean (see Table 3) or median (see Table 4) forecasts are used in our tests. This does not necessarily mean that experience does not matter, at least indirectly (notice that test rejections sometimes occur for *Groups* 6-8, where experience and accuracy traits are combined.[19] Finally, note that Tables 1 and 2 in the supplemental appendix report root mean square

[19] To explore this finding in more detail, we also constructed additional tables that are closely related to Tables 3 and 4, except that in these tables, RMSFEs are reported for all of the models used in each test (see supplemental appendix, Tables S1 and S2). In these tables, we see that combining experience with prior predictive accuracy can lead to lower RMSFEs, relative to the case where the entire pool of experts is used. However, RMSFEs are even lower for various alternative models

forecast errors (RMSFEs) from the benchmark and competing models utilized when constructing the test statistics discussed above. In these supplemental tables, we see that in the majority of cases considered, forecasts that utilize the mean have lower RMSFEs than when the median is used for constructing combination forecasts. For example, when comparing the benchmark RMSFEs of *Group* 1 that are reported in Tables 1 and 2 in the supplemental appendix, RMSFEs associated with mean combination forecasts are lower for $h = \{0, 2, 3, 4\}$ than the RMSFEs associated with median combination forecasts. This is interesting, given the clear asymmetry and long left tails associated with the distributions of expert forecasts exhibited in Figure 1, and suggests that outlier forecasts from "less accurate" experts are not overly influential when using measures of central tendency as ensemble forecasts.

Summarizing, we have direct evidence that judicious selection of pools of experts can lead to loss function robust forecast superiority. However, it should be stressed that in this illustration of the testing techniques developed in this paper, we do not consider various combination methods, including Bayesian model averaging, for example. Additionally, we only look at GDP, although the SPF has various other variables in it.[20] Extensions such as these are left to future research.

6 Concluding Remarks

We develop uniformly valid forecast superiority tests that are asymptotically non conservative, and that are robust to the choice of loss function. Our tests are based on principles of stochastic dominance, which can be interpreted as tests for infinitely many moment inequalities. In light of this, we use tools from Andrews and Shi (2013, 2017) when developing our tests. The tests build on earlier work due to Jin, Corradi, and Swanson (2017), and are meant to provide a class of predictive accuracy tests that are not reliant on a choice of loss function. In developing the new tests, we establish uniform convergence (over error support) of HAC variance estimators, and of their bootstrap counterparts. In a Supplement, we also extend the theory of generalized moment selection testing to allow for the presence of non-vanishing parameter estimation error. In a series of Monte Carlo experiments, we show that finite sample performance of our tests is quite good, and that the power of our tests dominates those proposed by JCS (2017). Additionally, we carry out an empirical analysis of the well known Survey of Professional Forecasters, and show that utilizing expert pools based on past forecast quality can lead to loss function robust forecast superiority, when compared with pools that include all survey participants. This finding is particularly prevalent for our longest forecast horizon (i.e., 1-year ahead).

for which we only use prior predictive accuracy to select expert pools (compare RMSFEs for *Groups* 3-5 with those for *Groups* 6-8 in the supplemental tables).

[20] Results for RGDP are gathered in Tables 5-8 in the supplemental appendix, and are qualitatively similar to those reported for NGP, although the number of test rejections are fewer.

7 Appendix

Proof of Lemma 1: The proof is the same for all j, so we suppress the subscript. Thus, let $u_t(x) = (1\{e_{j,t} \leq x\} - F_j(x)) - (1\{e_{1,t} \leq x\} - F_1(x))$, and define

$$\widehat{\widehat{\sigma}}_n^{2,G+}(x) = \frac{1}{n}\sum_{t=1}^{n} u_t^2(x) + 2\frac{1}{n}\sum_{\tau=1}^{l_n} w_\tau u_t(x)u_{t-\tau}(x).$$

We first show that

$$\sup_{x\in\mathcal{X}^+}\left|\widehat{\widehat{\sigma}}_n^{2,G+}(x) - \sigma^{2,G+}(x)\right| = o_p(1),$$

and then we show that

$$\sup_{x\in\mathcal{X}^+}\left|\widehat{\widehat{\sigma}}_n^{2,G+}(x) - \widehat{\sigma}_n^{2,G+}(x)\right| = o_p(1). \tag{7.1}$$

and the desired statement follows by triangular inequality. Now, adding and subtracting $\mathrm{E}(u_t(x)u_{t-\tau}(x))$ and $\mathrm{E}\left(u_t^2(x)\right)$

$$\begin{aligned}
&\sup_{x\in\mathcal{X}^+}\left|\widehat{\widehat{\sigma}}_n^{2,G+}(x) - \sigma^{2,G+}(x)\right| \\
\leq\ &\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{t=1}^{n}\left(u_t^2(x) - \mathrm{E}\left(u_t^2(x)\right)\right) + 2\frac{1}{n}\sum_{\tau=1}^{l_n} w_\tau \sum_{t=\tau+1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \\
&+ \sup_{x\in\mathcal{X}^+}\left|\left(\sigma^2(x) - \frac{1}{n}\sum_{t=1}^{n}\mathrm{E}\left(u_t^2(x)\right) + 2\frac{1}{n}\sum_{\tau=1}^{l_n} w_\tau \sum_{t=\tau+1}^{n}\mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right|.
\end{aligned} \tag{7.2}$$

We begin with the first term on the RHS of (7.2). First note that, since $w_\tau \leq 1$,

$$\begin{aligned}
&\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{t=1}^{n}\left(u_t^2(x) - \mathrm{E}\left(u_t^2(x)\right)\right) + 2\frac{1}{n}\sum_{\tau=1}^{l_n} w_\tau \sum_{t=\tau+1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \\
\leq\ &\sup_{x\in\mathcal{X}^+} 2\sum_{\tau=0}^{l_n}\left|\frac{1}{n}\sum_{t=1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right|.
\end{aligned}$$

Now, since the probability of the sum is less than or equal to the sum of the probability,

$$\begin{aligned}
&\Pr\left(\sup_{x\in\mathcal{X}^+} 2\sum_{\tau=0}^{l_n}\left|\frac{1}{n}\sum_{t=\tau+1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \geq \varepsilon\right) \\
\leq\ &2\sum_{\tau=0}^{l_n}\Pr\left(\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{t=\tau+1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \geq \frac{\varepsilon}{l_n}\right),
\end{aligned}$$

so that we need to show that,

$$\Pr\left(\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{t=\tau+1}^{n}\left(u_t(x)u_{t-\tau}(x) - \mathrm{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \geq \frac{\varepsilon}{l_n}\right) < \frac{\delta}{l_n}.$$

Given Assumption A2, WLOG, we can set $\mathcal{X}^+ = [0, \Delta]$, so that it can be covered by a_n^{-1} balls s_j, $j = 1, ..., \Delta a_n^{-1}$, centered at s_j, with radius a_n. Then,

$$\begin{aligned}
& \sup_{x \in \mathcal{X}^+} \left| \frac{1}{n} \sum_{t=\tau+1}^{n} \left(u_t(x) u_{t-\tau}(x) - \mathrm{E}\left(u_t(x) u_{t-\tau}(x) \right) \right) \right| \\
\leq & \max_{j=1,..,\Delta a_n^{-1}} \left| \frac{1}{n} \sum_{t=\tau+1}^{n} \left(u_t(s_j) u_{t-\tau}(s_j) - \mathrm{E}\left(u_t(s_j) u_{t-\tau}(s_j) \right) \right) \right| \\
& + \max_{j=1,..,\Delta a_n^{-1}} \sup_{x \in S_j} 2 \left| \left(\frac{1}{n} \sum_{t=\tau+1}^{n} u_{t-\tau}(s_j) \left(u_t(x) - u_t(s_j) \right) \right) \right. \\
& \left. - \left(\frac{1}{n} \sum_{t=\tau+1}^{n} \mathrm{E}\left(u_{t-\tau}(s_j) \left(u_t(x) - u_t(s_j) \right) \right) \right) \right| \\
= & A_n + B_n.
\end{aligned}$$

Now,

$$\begin{aligned}
B_n \leq & \max_{j=1,..,\Delta a_n^{-1}} \sup_{x \in S_j} \left| \frac{1}{n} \sum_{t=\tau+1}^{n} u_{t-\tau}(s_j) \left(u_t(x) - u_t(s_j) \right) \right| \\
& + \max_{j=1,..,\Delta a_n^{-1}} \sup_{x \in S_j} \left| \frac{1}{n} \sum_{t=\tau+1}^{n} \mathrm{E}\left(u_{t-\tau}(s_j) \left(u_t(x) - u_t(s_j) \right) \right) \right| \\
= & B1_n + B2_n.
\end{aligned}$$

Given Assumption A1, noting that by Cauchy - Schwarz,

$$\begin{aligned}
B2_n \leq & \max_{j=1,..,\Delta a_n^{-1}} \sup_{x \in S_j} \sqrt{\mathrm{E}\left(u_{t-\tau}(s_j) \right)^2} \max_{j=1,..,a_n^{-1}} \sup_{x \in S_j} \sqrt{\mathrm{E}\left(u_t(s_j) - u_t(x) \right)^2} \\
\leq & C a_n^{1/2}
\end{aligned}$$

for some constant C. Recalling given that $u_t(x) = \left(1\left\{ e_{j,t} \leq x \right\} - F_j(x) \right) - \left(1\left\{ e_{1,t} \leq x \right\} - F_1(x) \right)$ and $u_t(s_j)$ stay between -1 and 1

$$\begin{aligned}
& B1_n \leq 2 \max_{j=1,..,\Delta a_n^{-1}} \sup_{x \in S_j} \frac{1}{n} \sum_{t=1}^{n} \left| u_t(s_j) - u_t(x) \right| \\
\leq & \frac{2}{n} \sum_{t=1}^{n} 1\left\{ x - a_n \leq e_{1,t} \leq x + a_n \right\} + \frac{2}{n} \sum_{t=1}^{n} 1\left\{ x - a_n \leq e_{j,t} \leq x + a_n \right\} \\
& + 2 a_n \sup_{x \in \mathcal{X}^+} \left(f_1(x) + f_j(x) \right) \\
= & O_p\left(a_n \right) = o_p(a_n^{1/2})
\end{aligned}$$

Hence, by Chebyshev inequality

$$l_n \Pr\left(B_n > \frac{\varepsilon}{l_n} \right) = O\left(a_n l_n^3 \right) = o(1),$$

for $a_n = o\left(l_n^{-3}\right)$.

Now, consider A_n. Recalling that given Assumption A1, $\text{var}\left(\sum_{t=1}^{m}\left(u_t(s_j)u_{t-\tau}(s_j) - \text{E}\left(u_t(s_j)u_{t-\tau}(s_j)\right)\right)\right) \leq Cm$, for some constant C,

$$\begin{aligned}
& \Pr\left(\max_{j=1,..,a_n^{-1}}\left|\frac{1}{n}\sum_{t=\tau+1}^{n}\left(u_t(s_j)u_{t-\tau}(s_j) - \text{E}\left(u_t(s_j)u_{t-\tau}(s_j)\right)\right)\right| \geq \frac{\varepsilon}{l_n}\right) \\
\leq & \sum_{j=1}^{a_n^{-1}}\Pr\left(\left|\sum_{t=\tau+1}^{n}\left(u_t(s_j)u_{t-\tau}(s_j) - \text{E}\left(u_t(s_j)u_{t-\tau}(s_j)\right)\right)\right| \geq \frac{n\varepsilon}{l_n}\right) \\
\leq & 4a_n^{-1}\left(\exp\left(-\frac{\frac{n^2}{l_n^2}\varepsilon^2}{64Cn + \frac{8}{3}\frac{\Delta n^2}{4l_n^3}}\right) + \frac{16}{\Delta}l_n^2\left(\frac{\Delta}{4}\frac{n}{l_n^2}\right)^{-\beta}\right) \\
= & a_n^{-1}\exp\left(-\frac{1}{64C\frac{l_n^2}{n} + \frac{8}{3}\frac{\Delta}{4l_n}}\right) + 64a_n^{-1}l_n^2 l_n^{2\beta}n^{-\beta} \\
= & o(1) + O\left(n^{\delta(6+2\beta)}n^{-\beta}\right) \\
= & o(1) \text{ for } \beta > \frac{6\delta}{1-2\delta},
\end{aligned}$$

where the RHS of the second inequality follows from the Lemma on page 739 of Hansen (2008), setting $a_n = l_n^{-4}$, $b = 1$, $m = \frac{\Delta n}{4l_n^2}$, and $l_n = n^{\delta}$, with $\delta < 1/2$. Hence, the first term on the RHS of (7.2) is $o_p(1)$ uniformly in $x \in \mathcal{X}^+$.

We now turn to the second term on the RHS of (7.2). Recalling that given stationarity, $\sigma^{2,G+}(x) = \lim_{n\to\infty}\text{E}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{n}u_t(x)\right)^2$

$$\begin{aligned}
& \sup_{x\in\mathcal{X}^+}\left|\left(\sigma^{2,G+}(x) - \frac{1}{n}\sum_{t=1}^{n}\text{E}\left(u_t^2(x)\right) + 2\frac{1}{n}\sum_{\tau=1}^{l_n}w_\tau\sum_{t=\tau+1}^{n}\text{E}\left(u_t(x)u_{t-\tau}(x)\right)\right)\right| \\
\leq & 2\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{\tau=1}^{l_n}(1-w_\tau)\sum_{t=\tau+1}^{n}\text{E}\left(u_t(x)u_{t-\tau}(x)\right)\right| \qquad (7.3) \\
& +2\sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{\tau=l_n+1}^{n}\sum_{t=\tau+1}^{n}\text{E}\left(u_t(x)u_{t-\tau}(x)\right)\right|.
\end{aligned}$$

The first term on the RHS of (7.3) is $o_p(1)$, by the same argument as that used in Theorem 2 of Newey and West (1987). Also, by Lemma 6.17 in White (1984), for $q > 2$,

$$\text{E}\left(u_t(x)u_{t-\tau}(x)\right) \leq C\tau^{-\beta/2-1/q}\text{var}\left(u_t(x)\right)^{1/2}\text{E}\left\|u_t(x)\right\|^q$$

and

$$\begin{aligned}
& \sup_{x\in\mathcal{X}^+}\left|\frac{1}{n}\sum_{\tau=l_n+1}^{n}\sum_{t=\tau+1}^{n}\text{E}\left(u_t(x)u_{t-\tau}(x)\right)\right| \\
\leq & C\sup_{x\in\mathcal{X}^+}\text{var}\left(u_t(x)\right)^{1/2}\text{E}\left\|u_t(x)\right\|^q\sum_{\tau=l_n+1}^{n}\tau^{-\beta/2-1/q} = o(1),
\end{aligned}$$

as $\beta\delta > 1$, given Assumption A1, and noting that q can be taken arbitrarily large because of the boundedness of $u_t(x)$.

Finally, by the same argument as that used in the proof of (7.2), for all j,

$$\sup_{x\in\mathcal{X}^+} \frac{1}{n}\sum_{t=1}^{n} \left(1\left\{e_{j,t}\le x\right\} - F_j(x)\right) = o_p\left(l_n^{-1}\right).$$

The statement in (7.1) follows immediately.

Proof of Lemma 2: Since blocks are identically and indepently distributed conditionally on the sample,

$$\begin{aligned}
& \mathrm{E}^*\left(\widehat{\sigma}^{2*G+}_{j,n}(x)\right) \\
= \;& \mathrm{E}^*\left(\frac{1}{l_n^{1/2}}\sum_{i=1}^{l_n}\left(u^*_{j,1+i}(x) - u^*_{j,1+i}(x)\right)\right)^2 \\
= \;& \frac{1}{l_n}\sum_{i=1}^{l_n}\sum_{s=1}^{l_n}\mathrm{E}^*\left(\left(u^*_{j,1+i}(x) - u^*_{j,1+i}(x)\right)\left(u^*_{j,1+s}(x) - u^*_{j,1+s}(x)\right)\right) \\
= \;& \frac{1}{n}\frac{1}{l_n}\sum_{t=l_n}^{n-l_n}\sum_{i=1}^{l_n}\sum_{s=1}^{l_n}\left(u_{j,t+i}(x) - u_{j,t+i}(x)\right)\left(u_{j,t+s}(x) - u_{j,t+s}(x)\right) + O_p\left(\frac{l_n^2}{n}\right) \\
= \;& \frac{1}{n}\sum_{t=l_n}^{n-l_n}\sum_{i=-l_n}^{l_n}\left(u_{j,t}(x) - u_{j,t}(x)\right)\left(u_{j,t+i}(x) - u_{j,t+i}(x)\right) + O_p\left(\frac{l_n^2}{n}\right) + o_p(1) \\
= \;& \widehat{\sigma}^{2,G+}_{j,n}(x) + o_p(1),
\end{aligned}$$

with the $o_p(1)$ term holding uniformly in $x \in \mathcal{X}^+$. By noting that,

$$\begin{aligned}
& \sup_{x\in\mathcal{X}^+}\left|\widehat{\sigma}^{*G+}_{j,n}(x) - \widehat{\sigma}^{G+}_{j,n}(x)\right| \\
\le \;& \sup_{x\in\mathcal{X}^+}\left|\widehat{\sigma}^{*G+}_{j,n}(x) - \mathrm{E}^*\left(\widehat{\sigma}^*_n(x)\right)\right| + \sup_{x\in\mathcal{X}^+}\left|\mathrm{E}^*\left(\widehat{\sigma}^*_n(x)\right) - \widehat{\sigma}^{G+}_{j,n}(x)\right| \\
= \;& \sup_{x\in\mathcal{X}^+}\left|\widehat{\sigma}^{*G+}_{j,n}(x) - \mathrm{E}^*\left(\widehat{\sigma}^*_n(x)\right)\right| + o_p(1)
\end{aligned}$$

It suffices to show that

$$\sup_{x\in\mathcal{X}^+}\left|\widehat{\sigma}^{*G+}_{j,n}(x) - \mathrm{E}^*\left(\widehat{\sigma}^*_{j,n}(x)\right)\right| = o_{p^*}(1).$$

For brevity, and with a slight abuse of notation, let $u^*_{(k-1)l_n+i}(x) = u^*_{1,(k-1)l_n+i}(x) - u^*_{j,(k-1)l_n+i}(x)$, then

$$\begin{aligned}
& \sup_{x\in\mathcal{X}^+}\left|\widehat{\sigma}^{*2}_{j,n}(x) - \mathrm{E}^*\left(\widehat{\sigma}^*_{j,n}(x)\right)\right| \\
\le \;& \sup_{x\in\mathcal{X}^+}\frac{l_n}{b}\sum_{k=1}^{b}\left|\left(\frac{1}{l_n}\sum_{i=1}^{l_n}u^*_{(k-1)l_n+i}(x)\right)^2 - \mathrm{E}^*\left(\left(\frac{1}{l_n}\sum_{j=1}^{l_n}u^*_{(k-1)l_n+j}(x)\right)^2\right)\right| \\
= \;& \sup_{x\in\mathcal{X}^+}\frac{l_n}{b}\sum_{k=1}^{b}\left|\frac{1}{l_n^2}\sum_{j=1}^{l_n}\sum_{i=1}^{l_n}\left(u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x) - \mathrm{E}^*\left(u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x)\right)\right)\right|
\end{aligned}$$

Now,

$$\Pr\left(\sup_{x\in\mathcal{X}^+}\frac{l_n}{b}\sum_{k=1}^{b}\left|\frac{1}{l_n^2}\sum_{j=1}^{l_n}\sum_{i=1}^{l_n}\left(u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x)-\mathrm{E}^*\left(u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x)\right)\right)\right|\geq\varepsilon_1 a_n\right)$$

$$\leq\ l_n\Pr\left(\sup_{x\in\mathcal{X}^+}\frac{l_n}{b}\sum_{k=1}^{b}\left|\frac{1}{l_n^2}\sum_{j=1}^{l_n}\sum_{i=1}^{l_n}u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x)-\mathrm{E}^*\left(u^*_{(k-1)l_n+j}(x)u^*_{(k-1)l_n+i}(x)\right)\right|\geq\varepsilon_1\frac{a_n}{l_n}\right).$$

It suffices to show that, uniformly in k,

$$\Pr\left(\sup_{x\in\mathcal{X}^+}\left|\frac{1}{l_n^2}\sum_{j=1}^{l_n}\sum_{i=1}^{l_n}u^*_{(k-1)l+j}(x)u^*_{(k-1)l+i}(x)-\mathrm{E}^*\left(u^*_{(k-1)l+j}(x)u^*_{(k-1)l+i}(x)\right)\right|\geq\varepsilon_1\frac{a_n}{l_n}\right)<\frac{\delta}{l_n}.$$

This follows using the same "covering numbers" argument used in the proof of Lemma 1.

Proof of Theorem 1: We need to show that the statement in Lemma A1 in the Supplement Appendix of Andrews and Shi (2013) holds. Then, the proof of the theorem will follow using the same arguments as those used in the proof of their Theorem 1, as the proof is the same for independent and dependent observations. In fact, our set-up differs from Andrews and Shi (2013) only because we have dependent observations, and because we scale the statistic by a Newey-West variance estimator. For the rest of the proof, our set-up is simpler as we can fix their θ_n at a given value, say zero. Let $v_n(x)=v_j^{G+}(x)+h_{j,A,n}^{G+}(x)$, then it suffices to show that:

(i) $v_n(.)\Rightarrow v(.)$, as a process indexed by $x\in\mathcal{X}^+$, where $v(.)$ is a zero-mean $k-1-$dimensional Gaussian process, with covariance kernel given $\Sigma(x,x')$, as defined in (2.4).

(ii) $\sup_{x\in\mathcal{X}^+}\left\|\widehat{\sigma}_{j,n}^{2,G+}(x)-\sigma_j^{2,G+}(x)\right\|=o_p(1)$.

Now, statement (ii) follows directly from Lemma 1. It remains to show that (i) holds. The key difference between the independent and the dependent cases is that in the former we can rely on the concept of manageability, while in the latter we cannot. Nevertheless, (i) follows if we can show that $v_n(.)$ satisfies an empirical process. Given A1-A3, this follows from Lemma A2 in Jin, Corradi and Swanson (2017).

Proof of Theorem 2: (i) The proof of this theorem mirrors the proof of Theorem 2(a) in the Supplement of Andrews and Shi (2013). Let $c_0(h_{A,n},1-\alpha)$ be the $1-\alpha$ critical value of $S_n^{\dagger G^+}$, as defined in (3.5). Given Theorem 1, it follows that for all $\delta>0$,

$$\limsup_{n\to\infty}\sup_{P\in\mathcal{P}_0}P\left(S_n\geq c_0\left(h_{A,n},1-\alpha\right)+\delta\right)\leq\alpha.$$

The statement follows if we can show that

$$\limsup_{n\to\infty}\sup_{P\in\mathcal{P}_0}P\left(c^*_{0,n,1-\alpha}\left(\phi_n,h^{*G+}_{B,n}\right)\leq c_{0,n,1-\alpha}\left(h_{A,n},h^{*G+}_{B,n}\right)\right)=0, \tag{7.4}$$

and that

$$\limsup_{n\to\infty}\sup_{P\in\mathcal{P}_0}P\left(c_{0,n,1-\alpha}\left(h_{A,n},h^{*G+}_{B,n}\right)\leq c_0\left(h_{A,n},1-\alpha\right)\right)=0. \tag{7.5}$$

where $c_{0,n,1-\alpha}\left(h_{A,n}, h_{B,n}^{*G+}\right)$ is defined as $c_0\left(h_{A,n}, 1-\alpha\right)$ with $h_{B,n}^{*G+}(x)$, in (3.12), replacing $h_B(x)$ in (3.4).

For $c_n \to \infty$ and $c_n/\kappa_n \to 0$, $\tau_n \to \infty$ and $\tau_n/\kappa_n \to 0$,

$$
\begin{aligned}
& \sup_{P\in\mathcal{P}_0} P\left(c_{0,n,1-\alpha}^{*}\left(\phi_n, h_{B,n}^{*G+}\right) \leq c_{0,n,1-\alpha}\left(h_{A,n}, h_{B,n}^{*G+}\right)\right) \\
\leq & \sup_{P\in\mathcal{P}_0} P\left(-\phi_{j,n}(x) \leq h_{A,j,n}(x), \text{ for some } x\in\mathcal{X}^+ \text{ and some } j=2,..,k\right) \\
\leq & \sup_{P\in\mathcal{P}_0} P\left(\xi_{j,n}(x) < -1 \text{ AND } -c_n \leq h_{A,j,n}(x), \text{ for some } x\in\mathcal{X}^+ \text{ and } j=2,..,k\right) \\
\leq & \sup_{P\in\mathcal{P}_0} P\left(\overline{\sigma}_{j,n}^{G+}(x)^{-1}\sqrt{n}\left(G_{j,n}^{+}(x)-G_j^{+}(x)\right)+\overline{\sigma}_{j,n}^{G+}(x)^{-1}h_{j,A,n}(x) < -\kappa_n\right. \\
& \left. \text{AND } -c_n \leq h_{A,j,n}(x), \text{ for some } x\in\mathcal{X}^+ \text{ and } j=2,..,k\right) \\
\leq & \sup_{P\in\mathcal{P}_0} P\left(-\tau_n+\overline{\sigma}_{j,n}^{G+}(x)^{-1}h_{j,A,n}(x) < -\kappa_n\right. \\
& \left. \text{AND } -c_n \leq h_{A,j,n}(x), \text{ for some } x\in\mathcal{X}^+ \text{ and } j=2,..,k\right) \\
& + \sup_{P\in\mathcal{P}_0} P\left(\overline{\sigma}_{j,n}^{G+}(x)^{-1}\sqrt{n}\left(G_{j,n}(x)-G_j(x)\right) < -\tau_n, \text{ for some } x\in\mathcal{X}^+ \text{ and } j=2,..,k\right) \\
\leq & \sup_{P\in\mathcal{P}_0} P\left(-\overline{\sigma}_{j,n}^{G+}(x)^{-1}h_{j,A,n}(x) < -\kappa_n + c_n\right. \\
& \left. \text{AND } -c_n \leq h_{A,j,n}(x), \text{ for some } x\in\mathcal{X}^+ \text{ and } j=2,..,k\right) \\
= & \ o(1).
\end{aligned}
$$

This establishes that (7.4) holds. Finally, (7.5) follows from Lemma 1 and Lemma 2.

(ii) Recall that $c_{0,n,1-\alpha}^{*}\left(\phi_n, h_{B,n}\right)$ is the $(1-\alpha)-$ percentile of S_n^*, as defined in (3.11); and define $c_{0,n,1-\alpha}^{GMS}\left(\phi_n, \overline{h}_{B,n}\right)$ to the $(1-\alpha)-$ percentile of S_n^{GMS}, where

$$
S_n^{GMS} = \max_{x\in\mathcal{X}^+}\sum_{j=2}^{k}\max\left(\left\{0, \frac{\overline{v}_{j,n}(x)-\phi_{j,n}(x)}{\sqrt{\overline{h}_{B,j}(x)}}\right\}\right)^2,
$$

with $\overline{h}_{B,j}(x) = \frac{\left(\widehat{\sigma}_{j,n}(x)+\epsilon\right)^2}{\widehat{\sigma}_{j,n}(x)}$, $\overline{v}_n = \left(\overline{v}_{2,n}, ..., \overline{v}_{k,n}\right)'$ is a $k-1$ dimensional Gaussian process, with mean zero and covariance kernel with element given by $\widehat{\sigma}_{i,n}(x)^{-1}\left(\widehat{\sigma}_{i,n}(x)+\epsilon\right)\left(\widehat{\sigma}_{j,n}(x')+\epsilon\right)\widehat{\sigma}_{j,n}(x')^{-1}$ for $i,j = 2,...,k$ and $x, x'\in\mathcal{X}^+$. Finally, let $v = (v_2, ..., v_k)'$ is a $k-1$ dimensional Gaussian process, with mean zero and covariance kernel with element given by $\sigma_i(x)^{-1}\left(\sigma_i(x)+\epsilon\right)\left(\sigma_j(x')+\epsilon\right)\widehat{\sigma}_j(x')^{-1}$ for $i,j = 2,...,k$ and $x, x'\in\mathcal{X}^+$. We first need to show that

$$
c_{0,n,1-\alpha}^{*}\left(\phi_n, h_{B,n}\right) - c_{0,n,1-\alpha}^{GMS}\left(\phi_n, \overline{h}_{B,n}\right) = o_p(1), \tag{7.6}
$$

and then to prove that the statement holds when replacing $c_{0,n,1-\alpha}^{*}\left(\phi_n, h_{B,n}\right)$ with $c_{0,n,1-\alpha}^{GMS}\left(\phi_n, \overline{h}_{B,n}\right)$.

From Lemma 2, $\widehat{\sigma}_{j,n}^{*G+}(x) - \widehat{\sigma}_{j,n}^{G+}(x) = o_p^*(1)$ uniformly in $x \in \mathcal{X}^+$ for all $j = 2, ..., k$. Then, by Theorem 2.3 in Peligrad (1998),

$$v^* \overset{d^*}{\Longrightarrow} v \text{ a.s.-}\omega,$$

where $v^* \overset{*}{\Longrightarrow} v$ denotes weak convergence, conditional on sample. As $\overline{v}_n \Longrightarrow v$, (7.6) follows.

Given Assumption A4, by Lemma B3 in the Supplement of Andrews and Shi (2013), the distribution of $S_\infty^\dagger$, as defined in (3.6), is continuous. It is also strictly increasing and its $(1-\alpha)-$quantile is strictly positive, for all $\alpha < 1/2$. The statement then follows by the same argument as that used in the proof of Theorem 2(b) in the Supplement of Andrews and Shi (2013).

Proof of Theorem 3: **(i)** Without loss of generality, let $B_{FA}^{G+} = \{x \in \mathcal{X}^+ : G_2(x) > 0\}$, and note that for all $x \in B_{FA}^{G+}$, $\max\left\{0, \frac{\sqrt{n}G_{2,n}^+(x)}{\overline{\sigma}_{2,n}^{G+}(x)}\right\} = \frac{\sqrt{n}G_{2,n}^+(x)}{\overline{\sigma}_{2,n}^{G+}(x)}$. Thus,

$$\begin{aligned}
S_n^{G+} &= \int_{B_{FA}^{G+}} \sum_{j=2}^k \left(\max\left\{0, \frac{\sqrt{n}G_{j,n}^+(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\}\right)^2 \mathrm{d}Q(x) + \int_{\mathcal{X}^+ \setminus B_{FA}^{G+}} \sum_{j=2}^k \left(\max\left\{0, \frac{\sqrt{n}G_{j,n}^+(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\}\right)^2 \mathrm{d}Q(x) \\
&= \int_{B_{FA}^{G+}} \left(\frac{\sqrt{n}G_{2,n}^+(x)}{\overline{\sigma}_{2,n}^{G+}(x)}\right)^2 \mathrm{d}Q(x) + \int_{B_{FA}^{G+}} \sum_{j=3}^k \left(\max\left\{0, \frac{\sqrt{n}G_{j,n}^+(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\} - \left(\frac{\sqrt{n}G_{2,n}^+(x)}{\overline{\sigma}_{2,n}^{G+}(x)}\right)\right)^2 \mathrm{d}Q(x) \\
&\quad + \int_{\mathcal{X}^+ \setminus B_{FA}^{G+}} \sum_{j=2}^k \left(\max\left\{0, \frac{\sqrt{n}G_{j,n}^+(x)}{\overline{\sigma}_{j,n}^{G+}(x)}\right\}\right)^2 \mathrm{d}Q(x) \\
&= I_n + II_n + III_n,
\end{aligned}$$

with with $\overline{\sigma}_{j,n}^{2,*G+} = \left(\widehat{\sigma}_{j,n}^{*G+}(x) + \varepsilon\right)^2$. Now, I_n diverges to infinity with probability approaching one. Theorem 1 ensures that II_n and III_n are $O_p(1)$. Thus, S_n^{G+} diverges to infinity. As S_n^{*G+} is $O_{p^*}(1)$, conditional on the sample, the statement follows.

Proof of Theorem 4:

Define, $S_{\infty,LA}^{\dagger G+}$ as in (3.6), but with the vector $h_{j,A,\infty}^{G+}(x)$ having at least one component strictly bounded away above from zero, and finite, for all $x \in B_{LA}^{G+}$. Let $\mathcal{P}_{n,LA}^{G+}$ denote the set of probabilities under the sequence of local alternatives. We have that for all $a > 0$,

$$\limsup_{n\to\infty} \sup_{P \in \mathcal{P}_{n,LA}^{G+}} \left[P\left(S_n^{G+} > a\right) - P\left(S_{\infty,LA}^{\dagger G+} > a\right)\right] = 0,$$

and the distribution of $S_{\infty,LA}^{\dagger G+}$ is continuous at its $(1-\alpha)+\delta$ quintile, for all $0 < \alpha < 1/2$ and $\delta \geq 0$. Also, note that for all $x \in B_{LA}^{G+}$, $\phi_n^{G+} = 0$. The statement then follows by the same argument as that used in the proof of Theorem 2(ii).

8 References

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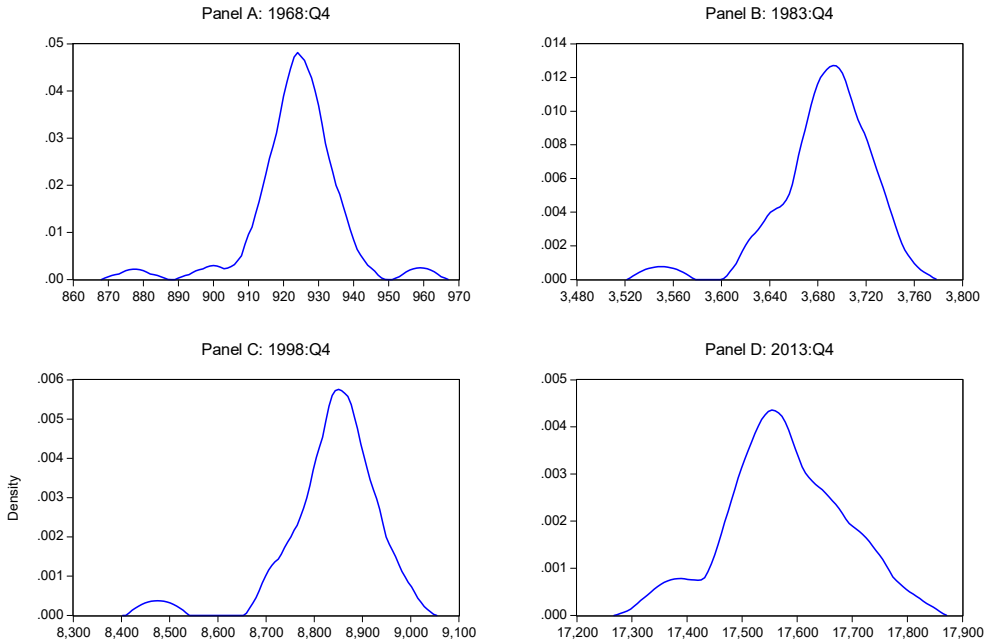
Figure 1: Kernel Densities of Panel Expert Nominal GDP Forecasts*



Summary Statistics

	1968:Q4	1983:Q4	1998:Q4	2013:Q4
Mean	923.7701	3686.444	8837.940	17578.68
Median	925.0000	3687.000	8846.000	17567.91
Maximum	960.0000	3750.000	8984.600	17781.06
Minimum	875.0000	3550.000	8475.000	17358.72
Std. Dev.	12.16333	37.60961	94.13889	100.0608
Skewness	-0.884233	-1.310238	-1.786877	-0.027824
Kurtosis	7.574713	6.090962	8.287103	2.839074
Jarque-Bera	87.20107	24.63142	52.60335	0.049531
Probability	0.000000	0.000004	0.000000	0.975539
Observations	87	36	31	41

* Notes: Figures are kernel density estimates (Epanechnikov kernel) for various 4 quarter ahead predictions made during the fourth quarters of 1968, 1983, 1998, and 2013. The number of experts in each sample ranges from 31 in 1998 to 87 in 1968, as noted in table of summary statistics below the plots in the figure. See Section 5 for further details.

Table 1: Monte Carlo Results for JCS_n^{G+}, JCS_n^{G-}, JCS_n^{C+}, and JCS_n^{C-} Forecast Superiority Tests*

DGP	n	$J_n = 0.20$	$J_n = 0.35$	$J_n = 0.50$	$J_n = 0.65$	$J_n = 0.20$	$J_n = 0.35$	$J_n = 0.50$	$J_n = 0.65$
		GL Forecast Superiority				CL Forecast Superiority			
		Empirical Size							
DGP-H1	100	0.116	0.120	0.112	0.120	0.130	0.126	0.138	0.140
	150	0.112	0.132	0.136	0.146	0.132	0.132	0.134	0.144
	200	0.120	0.128	0.136	0.140	0.114	0.130	0.134	0.146
	300	0.125	0.125	0.135	0.150	0.115	0.135	0.140	0.150
	600	0.115	0.130	0.160	0.155	0.115	0.115	0.130	0.145
	900	0.095	0.115	0.110	0.115	0.125	0.135	0.165	0.170
DGP-H2	100	0.084	0.078	0.086	0.092	0.042	0.040	0.044	0.042
	150	0.078	0.088	0.088	0.096	0.038	0.036	0.044	0.046
	200	0.076	0.090	0.088	0.100	0.032	0.030	0.032	0.040
	300	0.075	0.070	0.075	0.090	0.030	0.030	0.050	0.055
	600	0.085	0.090	0.100	0.110	0.015	0.025	0.030	0.035
	900	0.070	0.065	0.080	0.090	0.020	0.020	0.020	0.025
		Empirical Power							
DGP-H3	100	0.210	0.214	0.234	0.238	0.290	0.300	0.308	0.308
	150	0.240	0.230	0.240	0.274	0.382	0.406	0.430	0.454
	200	0.282	0.308	0.322	0.356	0.488	0.506	0.520	0.550
	300	0.385	0.430	0.460	0.490	0.650	0.660	0.700	0.745
	600	0.720	0.735	0.745	0.780	0.945	0.950	0.960	0.965
	900	0.900	0.915	0.920	0.935	0.995	1.000	1.000	1.000
DGP-H4	100	0.656	0.660	0.682	0.704	0.892	0.914	0.916	0.926
	150	0.840	0.850	0.874	0.880	0.988	0.988	0.994	0.994
	200	0.948	0.958	0.966	0.974	1.000	0.998	1.000	0.998
	300	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

* Notes: Entries denote rejection frequencies of (JCS_n^{G+},JCS_n^{G-}) tests (i.e., GL forecast superiority) and (JCS_n^{C+},JCS_n^{C-}) tests (i.e., CL forecast superiority) under a variety of data generating processes denoted by DGP-H1 - DGP-H4. In DGP-H1 - DGP-H2, no alternative outperforms the benchmark model. In DGPH3 - DGP-H4, at least one alternative model outperfroms the benchmark model. Sample sizes include n=100, 150, 200, 300, 600, and 900 observations, as indicated in the second column of entries in the table. Nominal test size is 10%, and tests are carried out using critical values constructed for values of J_n including 0.20, 0.35, 0.50, and 0.65. See Section 4 for complete details.

Table 2: Monte Carlo Results for S_n^{G+}, S_n^{G-}, S_n^{C+}, and S_n^{C-} Forecast Superiority Tests*

DGP	n	$\eta = 0.0015$	$\eta = 0.002$	$\eta = 0.0025$	$\eta = 0.003$	$\eta = 0.0015$	$\eta = 0.002$	$\eta = 0.0025$	$\eta = 0.003$
		GL Forecast Superiority				CL Forecast Superiority			
		Empirical Size							
DGP-H1	100	0.130	0.128	0.128	0.128	0.123	0.123	0.122	0.122
	150	0.130	0.130	0.130	0.129	0.133	0.133	0.133	0.132
	200	0.124	0.124	0.124	0.124	0.133	0.132	0.131	0.130
	300	0.075	0.075	0.075	0.074	0.082	0.082	0.082	0.082
	600	0.114	0.114	0.113	0.112	0.099	0.099	0.098	0.098
	900	0.123	0.122	0.122	0.120	0.153	0.153	0.152	0.151
DGP-H2	100	0.035	0.035	0.035	0.035	0.031	0.031	0.031	0.031
	150	0.031	0.031	0.031	0.031	0.036	0.036	0.036	0.036
	200	0.033	0.033	0.033	0.032	0.058	0.058	0.058	0.058
	300	0.034	0.033	0.032	0.031	0.040	0.037	0.037	0.036
	600	0.076	0.076	0.076	0.075	0.102	0.102	0.102	0.102
	900	0.092	0.092	0.091	0.091	0.110	0.110	0.110	0.110
		Empirical Power							
DGP-H3	100	0.363	0.363	0.363	0.363	0.429	0.428	0.428	0.427
	150	0.510	0.510	0.510	0.508	0.641	0.639	0.638	0.637
	200	0.710	0.708	0.707	0.705	0.794	0.793	0.793	0.792
	300	0.896	0.895	0.895	0.895	0.952	0.952	0.952	0.952
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
DGP-H4	100	0.967	0.966	0.966	0.966	0.974	0.974	0.974	0.974
	150	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.000
	200	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	300	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	600	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	900	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

* Notes: See notes to Table 1. Entries denote rejection frequencies of (S_n^{G+},S_n^{G-}) tests (i.e., GL forecast superiority) and (S_n^{C+},S_n^{C-}) tests (i.e., CL forecast superiority). Nominal test size is 10%, and tests are carried out using critical values constructed for values of η including 0.0015, 0.002, 0.0025, and 0.0030. See Section 4 for complete details.

Table 3: SPF Forecast Pooling Analysis of Quarterly Nominal GDP Using Mean Benchmark Model and Mean Expert Pool Predictions*

Group	*Statistic*	*Forecast Horizon*				
		$h=0$	$h=1$	$h=2$	$h=3$	$h=4$
Group 1	S_n^G	0.000718	0.000781	0.001284	0.001525	0.024494
	S_n^C	0.000008	0.000020	0.000203	0.000172	0.024470
	JCS_n^G	0.232845	0.232845	0.232845	0.310460	0.024611
	JCS_n^C	0.000357	0.000333	0.000729	0.000745	0.025276
Group 2	S_n^G	0.000820	0.002681	0.002334	0.002451	0.008070
	S_n^C	0.000037	0.001516	0.001983	0.002675	0.003287
	JCS_n^G	0.543305	0.698535	1.164226*	0.698535	1.397071*
	JCS_n^C	0.000115	0.006752	0.009945	0.011544	0.021668
Group 3	S_n^G	0.001975	0.004097	0.005302	0.004784	0.009391
	S_n^C	0.000878	0.001306	0.008277	0.004486	0.014821*
	JCS_n^G	0.465690	0.698535	0.776151	0.620920	1.164226*
	JCS_n^C	0.002955	0.005766	0.011527*	0.010482	0.020510*
Group 4	S_n^G	0.001779	0.005075	0.004359	0.003820	0.008772
	S_n^C	0.000512	0.001760	0.004896	0.004161	0.009259
	JCS_n^G	0.543305	0.853766	0.776151	0.620920	1.241841*
	JCS_n^C	0.002621	0.007716*	0.008856	0.010377	0.022399*
Group 5	S_n^G	0.001540	0.003063	0.005878	0.007226	0.012886*
	S_n^C	0.000507	0.000842	0.008293	0.012770*	0.020682*
	JCS_n^G	0.465690	0.776151	0.620920	0.853766	1.008996*
	JCS_n^C	0.002235	0.004643	0.010388*	0.015376*	0.018873*
Group 6	S_n^G	0.002384	0.007759	0.005390	0.004585	0.012976
	S_n^C	0.000369	0.002493	0.008548	0.007844	0.022130*
	JCS_n^G	0.776151	1.164226*	0.776151	0.698535	1.008996
	JCS_n^C	0.002619	0.008085*	0.009990	0.014823	0.020172
Group 7	S_n^G	0.002284	0.006819	0.008704	0.009044	0.008960
	S_n^C	0.000986	0.002106	0.011060	0.008702	0.007617
	JCS_n^G	0.465690	0.931381*	1.086611*	0.698535	0.931381
	JCS_n^C	0.002803	0.005994*	0.011401*	0.011645	0.012845
Group 8	S_n^G	0.001857	0.004237	0.003911	0.006975	0.016996
	S_n^C	0.000452	0.001066	0.004071	0.007224	0.015145
	JCS_n^G	0.931381	0.776151	0.698535	0.853766	1.397071*
	JCS_n^C	0.002526	0.004420	0.007946	0.008453	0.020293*

* Notes: Entries are S_n^G, S_n^C, JCS_n^G, and JCS_n^C test statistics reported for forecast horizons $h = 0, 1, 2, 3, 4$. More specifically, $S_n^G = S_n^{G+}$ *if* $p^{G+}_{B,n,S_n^{G+}} \leq p^{G-}_{B,n,S_n^{G-}}$; *otherwise* $S_n^G = S_n^{G-}$. S_n^C, JCS_n^G, and JCS_n^C are defined analogously. Rejections of the null of no forecast superiority at a 10% level are denoted by a superscipt *. See Section 5 for complete details.

Table 4: SPF Forecast Pooling Analysis of Quarterly Nominal GDP Using Median Benchmark Model and Median Expert Pool Predictions*

Group	*Statistic*	*Forecast Horizon*				
		$h=0$	$h=1$	$h=2$	$h=3$	$h=4$
Group 1	S_n^G	0.000563	0.001063	0.001381	0.002021	0.001943
	S_n^C	0.000006	0.000152	0.000556	0.001059	0.000194
	JCS_n^G	0.310460	0.232845	0.232845	0.232845	0.388075
	JCS_n^C	0.000178	0.000907	0.000989	0.002319	0.001603
Group 2	S_n^G	0.000715	0.002496	0.002127	0.002244	0.004061
	S_n^C	0.000055	0.001409	0.001548	0.001152	0.001612
	JCS_n^G	0.465690	0.698535	1.086611*	0.931381	1.008996
	JCS_n^C	0.000858	0.007005	0.009498	0.008086	0.012498
Group 3	S_n^G	0.001448	0.003776	0.004268	0.002254	0.010113*
	S_n^C	0.000318	0.001068	0.003558	0.001985	0.014572*
	JCS_n^G	0.620920	0.853766	0.853766	0.465690	1.008996
	JCS_n^C	0.001480	0.005813	0.010052	0.007434	0.017275*
Group 4	S_n^G	0.001730	0.004888	0.002593	0.002697	0.006291
	S_n^C	0.000587	0.002044	0.002279	0.002268	0.006855
	JCS_n^G	0.698535	0.853766	0.776151	0.776151	1.164226*
	JCS_n^C	0.002062	0.007503*	0.007827	0.012697	0.019182*
Group 5	S_n^G	0.001379	0.003470	0.004909	0.005554	0.009491
	S_n^C	0.000534	0.001044	0.007032	0.009277*	0.017460*
	JCS_n^G	0.388075	0.620920	0.776151	0.776151	0.776151
	JCS_n^C	0.001413	0.005926	0.009500*	0.013223*	0.016788*
Group 6	S_n^G	0.001767	0.005915	0.004387	0.005942	0.008262
	S_n^C	0.000177	0.002169	0.006224	0.009615	0.020157*
	JCS_n^G	0.931381*	0.853766	0.698535	0.853766	0.698535
	JCS_n^C	0.001680	0.007416*	0.009586	0.012696	0.018839
Group 7	S_n^G	0.002128	0.007057	0.008938	0.004600	0.009174
	S_n^C	0.001056	0.002847	0.008537	0.005034	0.008895
	JCS_n^G	0.465690	0.931381	1.008996	0.620920	1.086611*
	JCS_n^C	0.002062	0.007216*	0.010561*	0.007966	0.015189
Group 8	S_n^G	0.001997	0.003098	0.002059	0.004183	0.012612
	S_n^C	0.000379	0.000686	0.001185	0.001744	0.010617
	JCS_n^G	0.620920	0.620920	0.388075	0.698535	1.086611*
	JCS_n^C	0.001764	0.003088	0.003562	0.004157	0.016000*

* Notes: See notes to Table 3.