

# Supplemental Appendix for “Jackknife Estimation of a Cluster-Sample IV Regression Model with Many Weak Instruments”<sup>1</sup>

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## Abstract

This Supplemental Appendix is comprised of two sub-appendices. Appendix S1 provides proofs for Theorems 2 and 3 of the main paper. Appendix S2 states additional supporting lemmas used to prove the main theorems of the paper. Proofs for these additional lemmas are reported in a separate Online Appendix which can be viewed at the URL:

[http://econweb.umd.edu/~chao/Research/research\\_files/Additional\\_](http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf)

[Online\\_Appendix\\_Jackknife\\_Estimation\\_Cluster\\_Sample\\_IV\\_Model\\_December\\_20\\_2022.pdf](http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf)

## Appendix S1: Proof of Theorems 2 and 3

**Proof of Theorem 2:** Define  $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$ . Note that, by the result of Lemma S2-9 given in Appendix S2 below, we have that  $D_\mu^{-1} \hat{\Delta}(\delta_0) = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1) = \mathcal{Y}_n + o_p(1)$ .

We now establish the asymptotic normality of  $\mathcal{Y}_n$ , upon appropriate standardization, in the case where  $K_{2,n} / (\mu_n^{\min})^2 = O(1)$ . To proceed, let  $a \in \mathbb{R}^d$  such that  $\|a\|_2 = 1$  and define  $b_{1n} = \Sigma_n^{-1/2} a$  and  $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1} \Sigma_n^{-1/2} a$ , where  $\Sigma_n = VC(\mathcal{Y}_n | \mathcal{F}_n^Z) = \Sigma_{1,n} + \Sigma_{2,n}$ , with  $\Sigma_{1,n} = VC(\Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)$  and  $\Sigma_{2,n} = VC(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z)$ . Now, let  $\mathcal{L}_{(i,t),n}$

$$= b_{1n}' \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \varepsilon_{(i,t)} / \sqrt{n} \text{ and } \mathcal{N}_{(i,t),n} = K_{2,n}^{-1/2} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{2,(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{2,(j,s),n} \varepsilon_{(i,t)} \right],$$

where  $\underline{u}_{2,(i,t),n} = b_{2n}' \underline{U}_{(i,t)}$ , with  $\underline{u}_{2,(j,s),n}$  similarly defined, and where  $e_{(i,t)}$  denotes an  $m_n \times 1$  elementary vector whose  $(i,t)^{th}$  component is 1 and all other components are 0. Using these notations, note that we can write  $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{L}_{(1,1),n} + \sum_{(i,t)=2}^{m_n} \{ \mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n} \}$ . Next, observe that,

$$\begin{aligned} E \left[ \mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] &= E \left[ \varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z \right] \frac{\left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right]^2}{n} \\ &\leq E \left[ \varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z \right] a' \Sigma_n^{-1} a \left( \frac{\left\| \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right\|_2}{\sqrt{n}} \right)^2 \quad (\text{by CS inequality}) \end{aligned}$$

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$$\begin{aligned}
&\leq \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right) a' \Sigma_n^{-1} a \left( \frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \\
&= o_p(1)
\end{aligned}$$

in light of Assumptions 2(i) and 7 and part (d) of Lemma S2-3<sup>2</sup>. Moreover, under Assumptions 2 and 3(iii), there exists a positive constant  $C^*$  such that

$$\begin{aligned}
E_Z \left\{ \left( E \left[ \mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\} &= \frac{E_Z \left\{ \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right]^4 \left( E \left[ \varepsilon_{(1,1)}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\}}{n^2} \\
&\leq \frac{C}{n^2} E \left( \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \right]^4 \right) \quad (\text{by Assumption 2(i)}) \\
&\leq C E \left( \frac{a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \right)^2 \quad (\text{by CS inequality}) \\
&\leq C \overline{C} = C^* < \infty \quad (\text{by Assumption 3(iii) and Lemma S2-3(d)})
\end{aligned}$$

Since the upper bound above does not depend on  $n$ , we further deduce that

$\sup_n E_Z \left\{ \left( E \left[ \mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right)^2 \right\} < \infty$ . It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that  $E \left( \mathcal{L}_{(1,1),n}^2 \right) = E_Z \left( E \left[ \mathcal{L}_{(1,1),n}^2 | \mathcal{F}_n^Z \right] \right) \rightarrow 0$ . Application of Markov's inequality then allows us to deduce that  $\mathcal{L}_{(1,1),n} = b_{1n}' \Upsilon' Z_2' M^{(Z_1, Q)} e_{(1,1)} \varepsilon_{(1,1)} / \sqrt{n} = o_p(1)$ , from which we obtain the representation  $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{V}_n + o_p(1)$ , where  $\mathcal{V}_n = \sum_{(i,t)=2}^{m_n} \mathcal{V}_{(i,t),n}$  with  $\mathcal{V}_{(i,t),n} = \mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n}$ . Note we can also write  $\mathcal{V}_n = \mathcal{L}_n + \mathcal{N}_n$ , where  $\mathcal{L}_n = \sum_{(i,t)=2}^{m_n} \mathcal{L}_{(i,t),n}$  and  $\mathcal{N}_n = \sum_{(i,t)=2}^{m_n} \mathcal{N}_{(i,t),n}$ .

Next, define the  $\sigma$ -fields  $\mathcal{F}_{(i,t),n} = \sigma \left( \{ \varepsilon_{(k,v)}, U_{(k,v)} \}_{(k,v)=1}^{(i,t)}, Z \right)$  for  $(i,t) = 1, 2, \dots, m_n$ , note that by construction  $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$  for  $(i,t) = 2, \dots, m_n$  and  $\mathcal{V}_{(i,t),n}$  is  $\mathcal{F}_{(i,t),n}$ -measurable. Note also that, under Assumption 1, it is easily seen that  $E \left[ \mathcal{V}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] = 0$ . In addition, note that, by part (d) of Lemma S2-3 and Lemma S2-6, and Assumption 2(i);

$$\begin{aligned}
E \left[ \mathcal{L}_{2,(i,t),n}^2 | \mathcal{F}_n^Z \right] &\leq (b_{2n}' b_{2n}) \max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] \\
&\leq \frac{K_{2,n}}{(\mu_n^{\min})^2} a' \Sigma_n^{-1} a \max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] = O_{a.s.}(1) \quad (1)
\end{aligned}$$

since, for this theorem, we assume that  $K_{2,n} / (\mu_n^{\min})^2 = O(1)$ . It follows then from straightforward calculations, from applying the triangle and CS inequalities, as well as from expression (1), part

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<sup>2</sup>Lemma S2-3 is stated in Appendix S2 below. A proof of this lemma is provided in section 1 of the Additional Online Appendix which can be viewed at the URL:  
[http://econweb.umd.edu/~chao/Research/research\\_files/Additional\\_Online\\_Appendix\\_Jackknife\\_Estimation\\_Cluster\\_Sample\\_IV\\_Model\\_December\\_20\\_2022.pdf](http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf)

(d) of Lemma S2-1, part (d) of Lemma S2-3, and Assumptions 2(i) and 3(iii) that

$$\begin{aligned}
& \text{Var}(\mathcal{V}_{(i,t),n} | \mathcal{F}_n^Z) \\
&= E \left[ \mathcal{L}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] + E \left[ \mathcal{N}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\
&\leq \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right) a' \Sigma_n^{-1} a \lambda_{\max} \left( \frac{\Upsilon' Z_2' Z_2 \Upsilon}{n} \right) \\
&\quad + \frac{4}{K_{2,n}} \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right) \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{2,(i,t),n}^2 | \mathcal{F}_n^Z \right] \right) \left( \max_{1 \leq (i,t) \leq m_n} \sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 \right) \\
&= O_{a.s.}(1) + O_{a.s.} \left( \frac{K_{2,n}}{(\mu_n^{\min})^2 n} \right) = O_{a.s.}(1)
\end{aligned}$$

By the law of iterated expectations and Theorem 16.1 of Billingsley (1995), there exists a constant  $\overline{C}$  such that  $\text{Var}(\mathcal{V}_{(i,t),n}) = E(\mathcal{V}_{(i,t),n}^2) = E_Z \left[ E(\mathcal{V}_{(i,t),n}^2 | \mathcal{F}_n^Z) \right] \leq \overline{C} < \infty$  for all  $n$  sufficiently large. These results show that  $\{\mathcal{V}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n, n \geq 1\}$  forms a square-integrable martingale difference array.

To show the asymptotic normality of  $\mathcal{V}_n$ , we verify the conditions of the central limit theorem for martingale difference arrays given in Lemma S2-15. To proceed, first consider condition (22), which, as noted in the remark following Lemma S2-15, is a sufficient condition for condition (20) of Lemma S2-15. We shall verify (22) for the case where  $\delta = 2$ . Note first that, by applying Loève's  $c_r$  inequality, we get

$$\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{V}_{(i,t),n}^4 \right] = \sum_{(i,t)=2}^{m_n} E \left[ (\mathcal{L}_{(i,t),n} + \mathcal{N}_{(i,t),n})^4 \right] \leq 8 \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 \right] + 8 \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{N}_{(i,t),n}^4 \right]$$

Hence, to verify condition (22), it suffices to show that  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 \right] = o(1)$  and

$\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{N}_{(i,t),n}^4 \right] = o(1)$ . To do this, we first focus on a conditional expectation analogue of  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 \right]$ . Note that

$$\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \\
&= \frac{1}{n^2} \sum_{(i,t)=2}^{m_n} \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^4 E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \\
&\leq a' \Sigma_n^{-1} a \frac{1}{n} \sum_{(i,t)=2}^{m_n} \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^2 \left( \frac{\| \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \|_2}{\sqrt{n}} \right)^2 E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \\
&\quad (\text{by CS inequality})
\end{aligned}$$

$$\begin{aligned}
&\leq a' \Sigma_n^{-1} a \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right) \left( \frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \\
&\quad \times \frac{1}{n} a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e'_{(i,t)} M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \\
&\leq a' \Sigma_n^{-1} a \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right) \left( \frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \\
&\quad \times \frac{a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \\
&\leq (a' \Sigma_n^{-1} a)^2 \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right) \left( \frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 \lambda_{\max} \left( \frac{\Upsilon' Z_2' Z_2 \Upsilon}{n} \right) \\
&\leq C \left( \frac{\max_{1 \leq (i,t) \leq m_n} \|\Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)}\|_2}{\sqrt{n}} \right)^2 = o_p(1)
\end{aligned}$$

where the last line above follows from Assumptions 2(i), 3(iii), and 7 and by Lemma S2-3(d). Next, note that, under Assumptions 2 and 3(iii), there exists a positive constant  $C^*$  such that

$$\begin{aligned}
&E_Z \left( \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \right)^2 \\
&= \frac{1}{n^4} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=2}^{m_n} E_Z \left( \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^4 \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(j,s)} \right]^4 \right. \\
&\quad \left. \times E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^4 | \mathcal{F}_n^Z \right] \right) \\
&\leq \frac{C}{n^4} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=2}^{m_n} E_Z \left( \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(i,t)} \right]^4 \left[ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} e_{(j,s)} \right]^4 \right) \\
&\leq \frac{C}{n^4} E_Z \left\{ a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} \sum_{(i,t)=1}^{m_n} e_{(i,t)} e'_{(i,t)} M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} \right. \\
&\quad \left. \times \sum_{(j,s)=1}^{m_n} e_{(j,s)} e'_{(j,s)} M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \left( a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a \right)^2 \right\} \\
&= C E_Z \left( \frac{a' \Sigma_n^{-1/2} \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon \Sigma_n^{-1/2} a}{n} \right)^4 \\
&\leq C \bar{C} = C^* < \infty \quad (\text{by Assumption 3(iii) and Lemma S2-3(d)})
\end{aligned}$$

where the second inequality above follows from applying the CS inequality. Since the upper bound above does not depend on  $n$ , we further deduce that

$\sup_n E_Z \left( \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \right)^2 < \infty$ . It follows by the law of iterated expectations and by Theorem 25.12 of Billingsley (1995) that  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{L}_{(i,t),n}^4 \right] = \sum_{(i,t)=2}^{m_n} E_Z \left( E \left[ \mathcal{L}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \right) \rightarrow 0$ .

Turning our attention to the bilinear term, note that by Loève's  $c_r$  inequality we have  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{N}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq \mathcal{R}_1 + \mathcal{R}_2$ , where

$$\mathcal{R}_1 = \sum_{(i,t)=2}^{m_n} (8/K_{2,n}^2) E \left[ \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 | \mathcal{F}_n^Z \right] \text{ and}$$

$$\mathcal{R}_2 = \sum_{(i,t)=2}^{m_n} (8/K_{2,n}^2) E \left[ \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right)^4 | \mathcal{F}_n^Z \right].$$

Focusing first on the term  $\mathcal{R}_1$ , note that, by straightforward calculations as well as by making use of Assumptions 2(i) and 5(ii), parts (b) and (c) of Lemma S2-1, part (d) of Lemma S2-3, and Lemma S2-6; we deduce that, there exists a positive constant  $\overline{C}$  such that

$$\begin{aligned} \frac{(\mu_n^{\min})^4}{K_{2,n}^2} \mathcal{R}_1 &\leq 24n (a' \Sigma_n^{-1} a)^2 \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^Z \right] \right) \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right) \\ &\quad \times \left[ \frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &\leq \overline{C} n \left[ \frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &= O_{a.s.} \left( \frac{K_{2,n}}{n} \right) + O_{a.s.}(1) = O_{a.s.}(1). \end{aligned}$$

Applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we then have

$$\begin{aligned} &\frac{(\mu_n^{\min})^4}{K_{2,n}^2} E_Z(\mathcal{R}_1) \\ &\leq \overline{C} n E_Z \left[ \frac{1}{K_{2,n}^2} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \\ &= O(1) \end{aligned}$$

from which we further deduce that

$$E_Z(\mathcal{R}_1) = \sum_{(i,t)=2}^{m_n} \frac{8}{K_{2,n}^2} E \left[ \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} \right)^4 \right] = O \left( \frac{K_{2,n}^2}{(\mu_n^{\min})^4 n} \right) = o(1)$$

In a similar way, we can also show that

$$E_Z(\mathcal{R}_2) = (8/K_{2,n}^2) E \left[ \sum_{(i,t)=2}^{m_n} \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(j,s),(i,t)} \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right)^4 \right] = o(1). \text{ It follows that}$$

$$\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{N}_{(i,t),n}^4 \right] \leq E_Z(\mathcal{R}_1) + E_Z(\mathcal{R}_2) = o(1). \text{ This verifies condition (22).}$$

Next, we verify condition (21) of Lemma S2-15. To proceed, first let  $s_Z^2 = \text{Var} [\mathcal{V}_n | \mathcal{F}_n^Z] = \text{Var} \left( \sum_{(i,t)=2}^{m_n} \mathcal{V}_{(i,t),n} | \mathcal{F}_n^Z \right)$ , and note that

$$s_Z^2 = \text{Var} \left( \frac{b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon}{\sqrt{n}} + \frac{b'_{2n} \underline{U}' A \varepsilon}{\sqrt{K_{2,n}}} | \mathcal{F}_n^Z \right) + o_p(1) = a' \Sigma_n^{-1/2} \Sigma_n \Sigma_n^{-1/2} a + o_p(1) = 1 + o_p(1) \quad (2)$$

On the other hand, by straightforward calculation, we can write

$$\begin{aligned} s_Z^2 &= \frac{1}{n} \sum_{(i,t)=2}^{m_n} \left[ b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} \right]^2 E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + \frac{1}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left\{ E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] + E \left[ \underline{u}_{2,(j,s)}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right\} \\ &\quad + \frac{2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] E \left[ \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \end{aligned} \quad (3)$$

Making use of expression (3), we obtain, after some further calculations,

$$\begin{aligned} &\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - s_Z^2 \\ &= \frac{2}{\sqrt{n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[ b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} \right] \frac{A_{(i,t),(j,s)}}{\sqrt{K_{2,n}}} \left\{ \varepsilon_{(j,s)} E \left[ \varepsilon_{(i,t)} \underline{u}_{2,(i,t)} | \mathcal{F}_n^Z \right] + \underline{u}_{2,(j,s)} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \right\} \\ &\quad + \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left( \varepsilon_{(j,s)}^2 - E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left( \underline{u}_{2,(j,s)}^2 - E \left[ \underline{u}_{2,(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ &\quad + 2 \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left( \varepsilon_{(j,s)} \underline{u}_{2,(j,s)} - E \left[ \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \\ &\quad + 2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \left\{ \underline{u}_{2,(j,s)} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{2,(k,v)} \right\} \end{aligned}$$

$$\begin{aligned}
& +2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \\
& +2 \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} \frac{A_{(i,t),(j,s)} A_{(i,t),(k,v)}}{K_{2,n}} \underline{u}_{2,(j,s)} \underline{u}_{2,(k,v)} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\
& = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7, \quad (\text{say})
\end{aligned}$$

Note first that, by applying parts (a)-(c) of Lemma S2-14, we have  $\mathcal{T}_1 \xrightarrow{p} 0$ ,  $\mathcal{T}_2 \xrightarrow{p} 0$ , and  $\mathcal{T}_3 \xrightarrow{p} 0$ . Consider next the term

$$\mathcal{T}_4 = 2 \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \frac{A_{(i,t),(j,s)}^2}{K_{2,n}} \left( \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} - E \left[ \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right].$$

In this case, we apply part (a) of Lemma S2-8 with  $u_{(j,s)} = \underline{u}_{2,(j,s)}$ ,  $\bar{\psi}_{(j,s)} = E \left[ \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right]$ , and  $\phi_{(i,t)} = E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ . Note that, in this case,  $\left\{ \left( \underline{u}_{2,(i,t)}, \varepsilon_{(i,t)} \right) \right\}_{(i,t)=1}^{m_n}$  is independent conditional on  $\mathcal{F}_n^Z$ , and  $\max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s. by Assumptions 1 and 2(i), respectively. Moreover, note that Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that  $K_{2,n} / (\mu_n^{\min})^2 = O(1)$  in this case together imply that there exists a constant  $C \geq 1$  such that  $E \left[ \underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq \left[ K_{2,n}^2 / (\mu_n^{\min})^4 \right] E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^4 | \mathcal{F}_n^Z \right] (a' \Sigma_n^{-1} a)^2 \leq C < \infty$  a.s. for all  $(i,t) \in \{1, 2, \dots, m_n\}$  and for all  $n$  sufficiently large, so that

$\max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s.n. Finally, using the upper bound derived in expression (29) in the proof of part (a) of Lemma S2-14<sup>3</sup>, we obtain  $\max_{1 \leq (i,t) \leq m_n} \left| \phi_{(i,t)} \right| \leq \max_{1 \leq (i,t) \leq m_n} E \left[ \left| \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} \right| | \mathcal{F}_n^Z \right] \leq C$  a.s.n. and  $\max_{1 \leq (j,s) \leq m_n} \left| \bar{\psi}_{(j,s)} \right| \leq \max_{1 \leq (j,s) \leq m_n} E \left[ \left| \underline{u}_{2,(j,s)} \varepsilon_{(j,s)} \right| | \mathcal{F}_n^Z \right] \leq C$  a.s.n. It follows by part (a) of Lemma S2-8 that  $\mathcal{T}_4 \xrightarrow{p} 0$ .

Now, consider  $\mathcal{T}_5$ . Here, we apply part (b) of Lemma S2-8 with  $u_{(j,s)} = \underline{u}_{2,(j,s)}$  and  $\phi_{(i,t)} = E \left[ \underline{u}_{2,(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ . Note again that  $\left\{ \left( \underline{u}_{2,(i,t)}, \varepsilon_{(i,t)} \right) \right\}_{(i,t)=1}^{m_n}$  is independent conditional on  $\mathcal{F}_n^Z$ , and  $\max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s. by Assumptions 1 and 2(i), respectively. Moreover, previously, we have shown that  $E \left[ \underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s.n. and  $\max_{1 \leq (i,t) \leq m_n} \left| \phi_{(i,t)} \right| \leq C$  a.s.n. Hence, applying part (b) of Lemma S2-8, we deduce that  $\mathcal{T}_5 \xrightarrow{p} 0$ .

Turning our attention to  $\mathcal{T}_6$ , we note that, for this term, we can apply part (c) of Lemma S2-8 with  $\phi_{(i,t)} = E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right]$ . From (1), there exists a positive constant  $C$  such that  $E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C < \infty$  a.s. for all  $(i,t) \in \{1, 2, \dots, m_n\}$  and for all  $n$  sufficiently large, so that  $\max_{1 \leq (i,t) \leq m_n} \left| \phi_{(i,t)} \right| = \max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{2,(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C$  a.s.n. Hence, applying part (c) of Lemma

<sup>3</sup> A proof of Lemma S2-14 is given in section 1 of the Additional Online Appendix which can be viewed at the URL:

[http://econweb.umd.edu/~chao/Research/research\\_files/Additional\\_Online\\_Appendix\\_Jackknife\\_Estimation\\_Cluster\\_Sample\\_IV\\_Model\\_December\\_20\\_2022.pdf](http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf)

S2-8, we obtain  $\mathcal{T}_6 \xrightarrow{p} 0$ .

Finally, consider  $\mathcal{T}_7$ . In this case, we apply part (d) of Lemma S2-8 with  $u_{(j,s)} = \underline{u}_{2,(j,s)}$ ,  $u_{(k,v)} = \underline{u}_{2,(k,v)}$ , and  $\phi_{(i,t)} = E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$ . Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain  $E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq \left( E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right)^{1/2} \leq C < \infty$  a.s. for all  $(i,t) \in \{1, 2, \dots, m_n\}$  and for all  $n$ , so that  $\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t)}|$   
 $= \max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C$  a.s. Moreover, as noted previously, Assumption 2, part (d) of Lemma S2-3, Lemma S2-6, and the fact that  $K_{2,n} / (\mu_n^{\min})^2 = O(1)$  together imply that  
 $\max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{2,(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s.n. It follows by applying part (d) of Lemma S2-8 that  $\mathcal{T}_7 \xrightarrow{p} 0$ .

The above argument shows that  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - s_Z^2 = \sum_{k=1}^7 \mathcal{T}_k = o_p(1)$ . On the other hand, expression (2) above implies that  $s_Z^2 - 1 = o_p(1)$ . Putting these two results together, we obtain  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{V}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - 1 = o_p(1)$ , which establishes condition (21) of Lemma S2-15.

It now follows from Lemma S2-15 that

$$\mathcal{V}_n = \sum_{(i,t)=2}^{m_n} \left\{ b'_{1n} \Upsilon' Z'_2 M^Q e_{(i,t)} \varepsilon_{(i,t)} / \sqrt{n} + \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{2,(i,t)} \varepsilon_{(j,s)} + \underline{u}_{2,(j,s)} \varepsilon_{(i,t)} \right] \right\} \xrightarrow{d} N(0, 1).$$

Since, previously, we have shown that  $a' \Sigma_n^{-1/2} \mathcal{Y}_n = \mathcal{V}_n + o_p(1)$ , this further implies that  $a' \Sigma_n^{-1/2} \mathcal{Y}_n \xrightarrow{d} N(0, 1)$ . Given that this result holds for all  $a \in \mathbb{R}^d$  such that  $\|a\|_2 = 1$ , we can then apply the Cramér-Wold device to obtain

$$\Sigma_n^{-1/2} \mathcal{Y}_n = \Sigma_n^{-1/2} \left( \frac{\Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} \underline{U}' A \varepsilon \right) \xrightarrow{d} N(0, I_d) \quad (4)$$

Next, let  $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n$ ,  $\Lambda_{I,n} = H_n^{-1} \Sigma_n H_n^{-1}$ , and  $\mathcal{Y}_n = \Upsilon' Z'_2 M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$ , as given above. Consider first  $\hat{\delta}_{L,n}$ . Theorem 1 has already shown that  $\hat{\delta}_{L,n} \xrightarrow{p} \delta_0$ . To show asymptotic normality of  $\hat{\delta}_L$ , note first that, by Lemma S2-11,  $\hat{\delta}_{L,n}$  satisfies the set of (normalized) first-order conditions  $\hat{\Delta}(\hat{\delta}_{L,n}) = 0$ , where

$\hat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1, Q)} (y - X\delta) / 2] [\partial \hat{Q}_{FELIM}(\delta) / \partial \delta]$ . Applying the mean-value theorem to each component of  $\hat{\Delta}(\delta)$  and expanding it around the point  $\delta = \delta_0$ , we obtain  $0 = \hat{\Delta}(\hat{\delta}_{L,n}) = \hat{\Delta}(\delta_0) + \left( \partial \hat{\Delta}(\bar{\delta}_n) / \partial \delta' \right) (\hat{\delta}_{L,n} - \delta_0)$ , with  $\bar{\delta}_n$  lying on the line segment between  $\hat{\delta}_{L,n}$  and  $\delta_0$ . Multiplying both sides of this equation by  $D_\mu^{-1}$ , we further obtain

$$0 = D_\mu^{-1} \hat{\Delta}(\delta_0) + D_\mu^{-1} \frac{\partial \hat{\Delta}(\bar{\delta}_n)}{\partial \delta'} (\hat{\delta}_{L,n} - \delta_0) = D_\mu^{-1} \hat{\Delta}(\delta_0) + D_\mu^{-1} \frac{\partial \hat{\Delta}(\bar{\delta}_n)}{\partial \delta'} D_\mu^{-1} D_\mu (\hat{\delta}_{L,n} - \delta_0) \quad (5)$$

From the result of Lemma S2-10, we have  $-D_\mu^{-1} \left( \partial \hat{\Delta}(\bar{\delta}_n) / \partial \delta' \right) D_\mu^{-1} = H_n + o_p(1)$ , where  $H_n = \Upsilon' Z'_2 M^{(Z_1, Q)} Z_2 \Upsilon / n$  is a positive definite matrix a.s.n. by Assumption 3(iii), which, in turn, implies that  $D_\mu^{-1} \left( \partial \hat{\Delta}(\bar{\delta}_n) / \partial \delta' \right) D_\mu^{-1}$  is nonsingular and, thus, invertible w.p.a.1. It follows that, for all  $n$



sufficiently large, we can solve for  $D_\mu (\widehat{\delta}_{L,n} - \delta_0)$  in (5) above to get

$$\begin{aligned} D_\mu (\widehat{\delta}_{L,n} - \delta_0) &= - \left[ D_\mu^{-1} \left( \frac{\partial \widehat{\Delta}(\bar{\delta}_n)}{\partial \delta'} \right) D_\mu^{-1} \right]^{-1} D_\mu^{-1} \widehat{\Delta}(\delta_0) \\ &= H_n^{-1} \left( \frac{\Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon}{\sqrt{n}} + D_\mu^{-1} \underline{U}' A \varepsilon \right) [1 + o_p(1)], \end{aligned} \quad (6)$$

where the last equality follows by applying Lemma S2-9. By part (d) of Lemma S2-3,  $\Sigma_n$  is positive definite *a.s.n.*, so that  $\Sigma_n^{-1}$  is well-defined for all  $n$  sufficiently large, and both  $\Sigma_n^{1/2}$  and  $\Sigma_n^{-1/2}$  can be taken to be symmetric matrices. Since  $H_n$  is also symmetric, it further follows that  $\Lambda_{I,n} = H_n^{-1} \Sigma_n H_n^{-1}$  is symmetric and positive definite *a.s.n.*, and both  $\Lambda_{I,n}^{-1} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1}$  and  $\Lambda_{I,n}^{-1/2} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2}$  are well-defined for all  $n$  sufficiently large. Multiplying both sides of the equation above by  $\Lambda_{I,n}^{-1/2}$ , we then get  $\Lambda_{I,n}^{-1/2} D_\mu (\widehat{\delta}_{L,n} - \delta_0) = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \mathcal{Y}_n [1 + o_p(1)]$ , where  $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$ . Let  $R_{W,n} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \Sigma_n^{1/2}$ , and note that  $R_{W,n} R_{W,n}' = I_d$  for all  $n$  sufficiently large. It, thus, follows from the result given in (4) above and the continuous mapping theorem that  $\Lambda_{I,n}^{-1/2} D_\mu (\widehat{\delta}_{L,n} - \delta_0) \xrightarrow{d} N(0, I_d)$ , as  $n \rightarrow \infty$ , as required.

Turning our attention now to  $\widehat{\delta}_{F,n}$ , note that we can write this estimator, appropriately standardized, as

$$D_\mu (\widehat{\delta}_{F,n} - \delta_0) = \left( D_\mu^{-1} X' \left[ A - \widehat{\ell}_{F,n} M^{(Z_1, Q)} \right] X D_\mu^{-1} \right)^{-1} D_\mu^{-1} X' \left[ A - \widehat{\ell}_{F,n} M^{(Z_1, Q)} \right] (y - X \delta_0) \quad (7)$$

so that, multiplying by  $\Lambda_{I,n}^{-1/2} = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2}$  and applying Lemmas S2-12 and S2-13, we obtain  $\Lambda_{I,n}^{-1/2} D_\mu (\widehat{\delta}_{F,n} - \delta_0) = (H_n^{-1} \Sigma_n H_n^{-1})^{-1/2} H_n^{-1} \mathcal{Y}_n [1 + o_p(1)]$ . It follows from the result given in (4) above and the continuous mapping theorem that  $\Lambda_{I,n}^{-1/2} D_\mu (\widehat{\delta}_{F,n} - \delta_0) \xrightarrow{d} N(0, I_d)$ , as  $n \rightarrow \infty$ , as required.  $\square$

**Proof of Theorem 3:** To proceed, note that, in this case,  $(\mu_n^{\min}) / \sqrt{K_{2,n}} = o(1)$  but  $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$ , so that, by the result given in Lemma S2-9, we have

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu^{-1} \widehat{\Delta}(\delta_0) = \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1) \quad (8)$$

where  $\underline{U} = U - \varepsilon \rho'$ . Again, let  $H_n = \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n$ , and  $\Sigma_{2,n} = VC(D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z) = D_\mu^{-1} VC(\underline{U}' A \varepsilon | \mathcal{F}_n^Z) D_\mu^{-1}$ . Now, by assumption,  $\widetilde{L}_n$  can be any sequence of bounded  $(l \times d)$  non-random matrices such that  $\lambda_{\min} \left( (\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right) \geq \underline{C}$  *a.s.n.* for some constant  $\underline{C} > 0$ . It follows that  $(\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n}$  is positive definite *a.s.n.*, so that, with probability one,  $\left( (\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)^{-1/2}$  is well-defined for all  $n$  sufficiently large. Hence, we can let

$\widetilde{N}_n = \left( (\mu_n^{\min})^2 \widetilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \widetilde{L}_n' / K_{2,n} \right)^{-1/2} \widetilde{L}_n H_n^{-1} (\mu_n^{\min} / \sqrt{K_{2,n}}) D_\mu^{-1} \underline{U}' A \varepsilon$  and construct the

linear combination  $\mathcal{J}_n = a' \tilde{\mathcal{N}}_n$  for any  $a \in \mathbb{R}^d$  such that  $\|a\|_2 = 1$ . Next, define  $\underline{u}_{(i,t),n} = a' \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} U_{(i,t)}$ , with  $\underline{u}_{(j,s),n}$  similarly defined, and we can write  $\mathcal{J}_n = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right]$   
 $= \sum_{(i,t)=2}^{m_n} \mathcal{J}_{(i,t),n}$ , where  $\mathcal{J}_{(i,t),n} = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right]$ .  
Again, define the  $\sigma$ -fields  $\mathcal{F}_{(i,t),n} = \sigma \left( \left\{ \varepsilon_{(k,v)}, U_{(k,v)} \right\}_{(k,v)=1}^{(i,t)}, Z \right)$  for  $(i,t) = 1, 2, \dots, m_n$ , noting that by construction  $\mathcal{F}_{(i,t)-1,n} \subseteq \mathcal{F}_{(i,t),n}$  for  $(i,t) = 2, \dots, m_n$  and  $\mathcal{J}_{(i,t),n}$  is  $\mathcal{F}_{(i,t),n}$ -measurable. In addition, note that, using Assumption 1, it is easily seen that  $E \left[ \underline{u}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] = 0$  and  $E \left[ \varepsilon_{(i,t)} | \mathcal{F}_{(i,t)-1,n} \right] = 0$ , from which it follows that  $E \left[ \mathcal{J}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] = (\mu_n^{\min} / \sqrt{K_{2,n}}) \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left\{ \varepsilon_{(j,s)} E \left[ \underline{u}_{(i,t),n} | \mathcal{F}_{(i,t)-1,n} \right] + \underline{u}_{(j,s),n} E \left[ \varepsilon_{(i,t)} | \mathcal{F}_{(i,t)-1,n} \right] \right\} = 0$ . Moreover, applying the CS inequality and making use of the fact that

$$E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq \frac{\max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right] \left\| \tilde{L}_n \right\|_F^2}{\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) [\lambda_{\min}(H_n)]^2} \left( \frac{1}{\mu_n^{\min}} \right)^2 = O_{a.s.} \left( \frac{1}{(\mu_n^{\min})^2} \right) \quad (9)$$

and that  $E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq \bar{C}$  a.s. by Assumption 2(i), we see that

$$\begin{aligned} & Var \left( \mathcal{J}_{(i,t),n} | \mathcal{F}_n^Z \right) \\ & \leq \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] + E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] E \left[ \underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] \right. \\ & \quad \left. + 2 \sqrt{E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]} \sqrt{E \left[ \underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right]} \right) \\ & \leq \frac{4\bar{C}^2}{(\mu_n^{\min})^2} \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 = \frac{4\bar{C}^2}{K_{2,n}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \quad a.s.n. \end{aligned} \quad (10)$$

Hence, applying the law of iterated expectations, part (d) of Lemma S2-1, and Theorem 16.1 of Billingsley (1995), we further deduce that  $Var \left( \mathcal{J}_{(i,t),n} \right) = E_Z \left[ E \left( \mathcal{J}_{(i,t),n}^2 | \mathcal{F}_n^Z \right) \right]$   
 $\leq \left( 4\bar{C}^2 / K_{2,n} \right) \sum_{(j,s)=1}^{(i,t)-1} E_Z \left[ A_{(i,t),(j,s)}^2 \right] \leq C$  for some positive constant  $C$  for all  $n$  sufficiently large. These results show that  $\left\{ \mathcal{J}_{(i,t),n}, \mathcal{F}_{(i,t),n}, 1 \leq (i,t) \leq m_n, n \geq 1 \right\}$  forms a square-integrable martingale difference array.

Next, we verify condition (22) of the central limit theorem for martingale difference arrays given

in Lemma S2-15 below. By Loève's  $c_r$  inequality we have

$$\begin{aligned}
& \sum_{(i,t)=2}^{m_n} E \left[ \left( \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right] \right)^4 \middle| \mathcal{F}_n^Z \right] \\
& \leq 8 \sum_{(i,t)=2}^{m_n} \frac{(\mu_n^{\min})^4}{K_{2,n}^2} E \left[ \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 \middle| \mathcal{F}_n^Z \right] \\
& \quad + 8 \sum_{(i,t)=2}^{m_n} \frac{(\mu_n^{\min})^4}{K_{2,n}^2} E \left[ \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right)^4 \middle| \mathcal{F}_n^Z \right] \\
& = \mathcal{E}_1 + \mathcal{E}_2, \quad (say).
\end{aligned} \tag{11}$$

Focusing first on  $\mathcal{E}_1$ , it is easy to see that there exists some positive constant  $C$  such that

$$\begin{aligned}
& \mathcal{E}_1 \\
& = \frac{8(\mu_n^{\min})^4}{K_{2,n}^2} E \left[ \sum_{(i,t)=2}^{m_n} \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 \middle| \mathcal{F}_n^Z \right] \\
& \leq \frac{8(\mu_n^{\min})^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s)=1 \\ (j,s) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^4 E \left[ \underline{u}_{(i,t),n}^4 \middle| \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^4 \middle| \mathcal{F}_n^Z \right] \\
& \quad + \frac{24(\mu_n^{\min})^4}{K_{2,n}^2} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t) \\ (j,s) \neq (k,v)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 E \left[ \underline{u}_{(i,t),n}^4 \middle| \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(j,s)}^2 \middle| \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(k,v)}^2 \middle| \mathcal{F}_n^Z \right] \\
& \leq \frac{C}{K_{2,n}} \left[ \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (j,s) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right]
\end{aligned}$$

where the second inequality above follows from Assumption 2(i) and from an upper bound on the conditional fourth moment of

$\underline{u}_{(i,t),n} = a' \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$  given by

$$\begin{aligned}
E \left[ \underline{u}_{(i,t),n}^4 \middle| \mathcal{F}_n^Z \right] & \leq \frac{1}{(\mu_n^{\min})^4} \left( \max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^4 \middle| \mathcal{F}_n^Z \right] \right) \frac{1}{[\lambda_{\min}(H_n)]^4} \\
& \quad \times \left\| \tilde{L}_n \right\|_F^4 \left( \frac{1}{\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right)} \right)^2 \\
& \leq \frac{C^*}{(\mu_n^{\min})^4} \quad a.s.n., \text{ for some constant } C^* > 0.
\end{aligned} \tag{12}$$

Note also that, in deriving the upper bound given in (12), we have applied Assumption 3(iii), Lemma S2-6, the boundedness of  $\|\tilde{L}_n\|_F^2$ , and the assumption that

$\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) \geq \underline{C} > 0$  a.s.n. Moreover, by parts (b) and (c) of Lemma S2-1, we have that  $K_{2,n}^{-1} \sum_{(i,t),(j,s)=1, (i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.} (K_{2,n}^2 / n^2)$  and  $K_{2,n}^{-1} \sum_{(i,t)=1}^{m_n} \sum_{(j,s),(k,v)=1, (j,s) \neq (i,t), (k,v) \neq (i,t)}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 = O_{a.s.} (K_{2,n} / n)$ . from which it follows that  $n\mathcal{E}_1 = O_{a.s.} (1)$  in light of Assumption 5(ii). Hence, by applying the law of iterated expectations and Theorem 16.1 of Billingsley (1995), we obtain, for all  $n$  sufficiently large,

$$\begin{aligned} & nE_Z [\mathcal{E}_1] \\ &= \frac{8n (\mu_n^{\min})^4}{K_{2,n}^2} E_Z \left\{ E \left[ \sum_{(i,t)=2}^{m_n} \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \underline{u}_{(i,t),n} \varepsilon_{(j,s)} \right)^4 \middle| \mathcal{F}_n^Z \right] \right\} \\ &\leq \frac{Cn}{K_{2,n}} \left\{ E_Z \left[ \frac{1}{K_{2,n}} \sum_{\substack{(i,t),(j,s)=1 \\ (i,t) \neq (j,s)}}^{m_n} A_{(i,t),(j,s)}^4 + \frac{1}{K_{2,n}} \sum_{(i,t)=1}^{m_n} \sum_{\substack{(j,s),(k,v)=1 \\ (j,s) \neq (i,t), (k,v) \neq (i,t)}}^{m_n} A_{(i,t),(j,s)}^2 A_{(i,t),(k,v)}^2 \right] \right\} \\ &= O(1), \end{aligned}$$

which shows that  $E_Z [\mathcal{E}_1] = O(1/n) = o(1)$ . In a similar way, we can also show that

$$E_Z [\mathcal{E}_2] = 8 \left[ (\mu_n^{\min})^4 / K_{2,n}^2 \right] E \left[ \sum_{(i,t)=2}^{m_n} \left( \sum_{(j,s)=1}^{(i,t)-1} A_{(j,s),(i,t)} \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right)^4 \right] = o(1). \quad \text{Condition (22) of Lemma S2-15 then follows from these calculations since}$$

$$\sum_{(i,t)=2}^{m_n} E \left[ \left( \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)} \left[ \underline{u}_{(i,t),n} \varepsilon_{(j,s)} + \underline{u}_{(j,s),n} \varepsilon_{(i,t)} \right] \right)^4 \right] \leq E_Z [\mathcal{E}_1] + E_Z [\mathcal{E}_2] = o(1)$$

Next, we verify condition (21) of Lemma S2-15. Note first that, by construction,  $Var(\mathcal{J}_n | \mathcal{F}_n^Z) = a' \left( \tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \right)^{-1/2} \tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \left( \tilde{L}_n \Lambda_{II,n} \tilde{L}'_n \right)^{-1/2} a = 1$ , with  $\Lambda_{II,n} = \left[ (\mu_n^{\min})^2 / K_{2,n} \right] H_n^{-1} \Sigma_{2,n} H_n^{-1}$ . This, in turn, implies that  $Var(\mathcal{J}_n) = E_Z [E(\mathcal{J}_n^2 | \mathcal{F}_n^Z)] = E_Z [Var(\mathcal{J}_n | \mathcal{F}_n^Z)] = 1$ . On the other hand, by direct calculation, we obtain

$$\begin{aligned} 1 &= Var(\mathcal{J}_n | \mathcal{F}_n^Z) \\ &= \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\ &\quad + \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[ \underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \end{aligned}$$

$$+2 \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 E \left[ \underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] E \left[ \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \quad (13)$$

Making use of expression (13), we obtain, after some further calculations,

$$\begin{aligned} & \sum_{(i,t)=2}^{m_n} E \left[ \mathcal{J}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - 1 \\ = & \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \varepsilon_{(j,s)}^2 - E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \underline{u}_{(j,s),n}^2 - E \left[ \underline{u}_{(j,s),n}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \underline{u}_{(j,s),n} \varepsilon_{(j,s)} - E \left[ \underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} E \left[ \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \left\{ \underline{u}_{(j,s),n} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{(k,v),n} \right\} \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \\ & + \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \underline{u}_{(j,s),n} \underline{u}_{(k,v),n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\ = & \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6 \end{aligned} \quad (14)$$

To analyze the terms  $\mathcal{T}\mathcal{T}_k$  ( $k = 1, \dots, 6$ ), note first that, by applying parts (b) and (a) of Lemma S2-16, we obtain  $\mathcal{T}\mathcal{T}_1 \xrightarrow{P} 0$  and  $\mathcal{T}\mathcal{T}_2 \xrightarrow{P} 0$ , respectively. Consider now the term

$$\mathcal{T}\mathcal{T}_3 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \underline{u}_{(j,s),n} \varepsilon_{(j,s)} - E \left[ \underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$$

In this case, we apply part (a) of Lemma S2-8 with  $u_{(j,s),n} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$ ,  $\bar{\psi}_{(j,s)} = E \left[ (\mu_n^{\min}) \underline{u}_{(j,s),n} \varepsilon_{(j,s)} | \mathcal{F}_n^Z \right]$ , and  $\phi_{(i,t)} = E \left[ (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ . Note that, in this case,  $\left\{ (u_{(i,t),n}, \varepsilon_{(i,t)}) \right\}_{(i,t)=1}^{m_n}$  is independent conditional on  $\mathcal{F}_n^Z = \sigma(Z)$ , and

$\max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s. by Assumptions 1(i) and 2(i), respectively. Moreover, the upper bound given by (12) implies that there exists a constant  $C^* > 0$  such that

$\max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] = \max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[ \underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq (\mu_n^{\min})^4 C^* / (\mu_n^{\min})^4 = C^*$  a.s.n. Finally, note that, by using the fact that

$\underline{u}_{(i,t),n} = a' \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}_{(i,t)}$  and by applying Assumption

2(i), Lemma S2-6, and the assumption that

$\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right) \geq \underline{C} > 0$  a.s.n.; we can show that there exists a constant  $C > 0$  such that

$$\begin{aligned}
& E \left[ \left| (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right| | \mathcal{F}_n^Z \right] \\
&= (\mu_n^{\min}) E \left[ \left| \varepsilon_{(i,t)} \underline{U}'_{(i,t)} D_\mu^{-1} H_n^{-1} \tilde{L}'_n \left( \frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} a \right| | \mathcal{F}_n^Z \right] \\
&\leq (\mu_n^{\min}) \sqrt{E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]} \left[ a' \left( \frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \right. \\
&\quad \times E \left[ \underline{U}_{(i,t)} \underline{U}'_{(i,t)} | \mathcal{F}_n^Z \right] D_\mu^{-1} H_n^{-1} \tilde{L}'_n \left( \frac{(\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n}{K_{2,n}} \right)^{-1/2} a \left. \right]^{1/2} \quad (\text{by CS inequality}) \\
&\leq (\mu_n^{\min}) \sqrt{E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]} \frac{1}{(\mu_n^{\min})} \left( \sqrt{\max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^2 | \mathcal{F}_n^Z \right]} \right) \\
&\quad \times \frac{1}{\lambda_{\min} (\Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n)} \left\| \tilde{L}_n \right\|_F \left( \frac{1}{\sqrt{\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}'_n / K_{2,n} \right)}} \right) \\
&\leq C < \infty \quad \text{a.s. for all } (i, t) \in \{1, 2, \dots, m_n\} \text{ and for all } n \text{ sufficiently large} \quad (15)
\end{aligned}$$

from which we further deduce that  $\max_{(i,t)} \left| \phi_{(i,t)} \right| \leq \max_{(i,t)} E \left[ \left| (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} \right| | \mathcal{F}_n^Z \right] \leq C$  a.s.n. and also that  $\max_{(j,s)} \left| \bar{\psi}_{(j,s)} \right| \leq \max_{(j,s)} E \left[ \left| (\mu_n^{\min}) \underline{u}_{(j,s),n} \varepsilon_{(j,s)} \right| | \mathcal{F}_n^Z \right] \leq C$  a.s.n. Hence, applying part (a) of Lemma S2-8, we have  $\mathcal{T}\mathcal{T}_3 \xrightarrow{p} 0$ .

Next, consider the term

$$\mathcal{T}\mathcal{T}_4 = \frac{2 (\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} E \left[ \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right] \left\{ \underline{u}_{(j,s),n} \varepsilon_{(k,v)} + \varepsilon_{(j,s)} \underline{u}_{(k,v),n} \right\}$$

Here, we apply part (b) of Lemma S2-8 with  $u_{(j,s),n} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$  and

$\phi_{(i,t)} = E \left[ (\mu_n^{\min}) \underline{u}_{(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ . Note that  $\{(u_{(i,t),n}, \varepsilon_{(i,t)})\}_{(i,t)=1}^{m_n}$  is independent conditional on  $\mathcal{F}_n^Z = \sigma(Z)$ , and

$\max_{1 \leq (i,t) \leq m_n} E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s. by Assumptions 1 and 2(i), respectively. Moreover, from calculations given previously, we have  $\max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[ \underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s.n. and  $\max_{(i,t)} \left| \phi_{(i,t)} \right| \leq C$  a.s.n. Hence, by applying part (b) of Lemma S2-8, we deduce that  $\mathcal{T}\mathcal{T}_4 \xrightarrow{p} 0$ .

Turning our attention to the term

$$\mathcal{TT}_5 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \varepsilon_{(j,s)} \varepsilon_{(k,v)} E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right]$$

For this term, we apply part (c) of Lemma S2-8 with  $\phi_{(i,t)} = E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right]$  and  $u_{(i,t),n} = (\mu_n^{\min}) \underline{u}_{(i,t),n}$ . From (9), there exists a positive constant  $C$  such that  $E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] = (\mu_n^{\min})^2 E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq C < \infty$  a.s. for all  $(i,t) \in \{1, 2, \dots, m_n\}$  and for all  $n$  sufficiently large, so that  $\max_{(i,t)} |\phi_{(i,t)}| = \max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{(i,t),n}^2 | \mathcal{F}_n^Z \right] \leq C$  a.s. Hence, applying part (c) of Lemma S2-8, we obtain  $\mathcal{TT}_5 \xrightarrow{p} 0$ .

Finally, consider the term

$$\mathcal{TT}_6 = \frac{2(\mu_n^{\min})^2}{K_{2,n}} \sum_{(i,t)=3}^{m_n} \sum_{(j,s)=2}^{(i,t)-1} \sum_{(k,v)=1}^{(j,s)-1} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \underline{u}_{(j,s),n} \underline{u}_{(k,v),n} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$$

In this case, we apply part (d) of Lemma S2-8 with  $u_{(j,s)} = (\mu_n^{\min}) \underline{u}_{(j,s),n}$ ,  $u_{(k,v)} = (\mu_n^{\min}) \underline{u}_{(k,v),n}$ , and  $\phi_{(i,t)} = E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$ . Using a conditional version of Liapounov's inequality and Assumption 2(i), we obtain  $E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq \left( E \left[ \varepsilon_{(i,t)}^4 | \mathcal{F}_n^Z \right] \right)^{1/2} \leq C < \infty$  a.s. for all  $(i,t) \in \{1, 2, \dots, m_n\}$  and for all  $n$  sufficiently large, so that  $\max_{(i,t)} |\phi_{(i,t)}| = \max_{(i,t)} E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \leq C$  a.s. Moreover, the upper bound in (12) implies that  $\max_{1 \leq (i,t) \leq m_n} E \left[ \underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] = \max_{1 \leq (i,t) \leq m_n} (\mu_n^{\min})^4 E \left[ \underline{u}_{(i,t),n}^4 | \mathcal{F}_n^Z \right] \leq C$  a.s. It follows by applying part (d) of Lemma S2-8 that  $\mathcal{TT}_6 \xrightarrow{p} 0$ .

It follows from the above calculations that the terms  $\mathcal{TT}_k \xrightarrow{p} 0$  for each  $k \in \{1, \dots, 6\}$ , which in light of equation (14) implies that  $\sum_{(i,t)=2}^{m_n} E \left[ \mathcal{J}_{(i,t),n}^2 | \mathcal{F}_{(i,t)-1,n} \right] - 1 = o_p(1)$ . This establishes condition (21) of Lemma S2-15. It now follows from Lemma S2-15 that  $\mathcal{J}_n = (\mu_n^{\min} / \sqrt{K_{2,n}}) a' \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0, 1)$ . Since this result holds for all  $a \in \mathbb{R}^d$  such that  $\|a\|_2 = 1$ , applying the Cramér-Wold device, we further deduce that

$$\left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) \left( \tilde{L}_n \Lambda_{II,n} \tilde{L}_n' \right)^{-1/2} \tilde{L}_n H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon \xrightarrow{d} N(0, I_d), \quad (16)$$

where  $\Lambda_{II,n} = (\mu_n^{\min})^2 H_n^{-1} \Sigma_{2,n} H_n^{-1} / K_{2,n}$  with  $H_n = \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n$ . Next, recall that  $\hat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1, Q)} (y - X\delta) / 2] [\partial \hat{Q}_{FELIM}(\delta) / \partial \delta]$ ; and note that, by Lemma S2-10, we have  $-D_\mu^{-1} \left( \partial \hat{\Delta}(\bar{\delta}_n) / \partial \delta' \right) D_\mu^{-1} = H_n + o_p(1)$ , with  $H_n$  being positive definite in light of Assumption 3(iii), so that upon inverting the expansion given in expression (5) above and multiplying by  $(\mu_n^{\min}) / \sqrt{K_{2,n}}$ , we obtain

$$\left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) D_\mu \left( \hat{\delta}_{L,n} - \delta_0 \right) = \left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) H_n^{-1} D_\mu^{-1} \hat{\Delta}(\delta_0) [1 + o_p(1)]$$

$$= \left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon [1 + o_p(1)],$$

where the last equality comes from applying expression (8). It follows by multiplying both sides of the equation above by  $\left( \tilde{L}_n \Lambda_{II,n} \tilde{L}_n \right)^{-1/2} \tilde{L}_n$  and applying the result given in expression (16) that  $\left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) \left( \tilde{L}_n \Lambda_{II,n} \tilde{L}_n \right)^{-1/2} \tilde{L}_n D_\mu \left( \hat{\delta}_{L,n} - \delta_0 \right) \xrightarrow{d} N(0, I_d)$ .

Turning our attention now to  $\hat{\delta}_{F,n}$ , note that, using expression (7) above, we can write

$$\begin{aligned} & \frac{\left( \mu_n^{\min} \right) D_\mu \left( \hat{\delta}_{F,n} - \delta_0 \right)}{\sqrt{K_{2,n}}} \\ = & \frac{\left( \mu_n^{\min} \right) \left( D_\mu^{-1} X' \left[ A - \hat{\ell}_{F,n} M^{(Z_1, Q)} \right] X D_\mu^{-1} \right)^{-1} D_\mu^{-1} X' \left[ A - \hat{\ell}_{F,n} M^{(Z_1, Q)} \right] (y - X \delta_0)}{\sqrt{K_{2,n}}} \end{aligned}$$

It follows by applying Lemmas S2-12 and S2-13 that

$$\frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} D_\mu \left( \hat{\delta}_{F,n} - \delta_0 \right) = \frac{\mu_n^{\min}}{\sqrt{K_{2,n}}} H_n^{-1} D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1), \quad (17)$$

noting that, in this case,  $\left( \mu_n^{\min} \right) / \sqrt{K_{2,n}} = o(1)$  but  $\sqrt{K_{2,n}} / \left( \mu_n^{\min} \right)^2 \rightarrow 0$ . It follows by multiplying both sides of equation (17) above by  $\left( \tilde{L}_n \Lambda_{II,n} \tilde{L}_n \right)^{-1/2} \tilde{L}_n$  and applying the result given in expression (16) that

$$\left( \mu_n^{\min} / \sqrt{K_{2,n}} \right) \left( \tilde{L}_n \Lambda_{II,n} \tilde{L}_n \right)^{-1/2} \tilde{L}_n D_\mu \left( \hat{\delta}_{F,n} - \delta_0 \right) \xrightarrow{d} N(0, I_d). \quad \square$$

## Appendix S2: Key Lemmas Used in Proving the Main Theorems

In this appendix, we state a number of lemmas that are used in the proofs of the main theorems of the paper. Proofs for these lemmas are available in a separate online appendix which can be viewed at the URL: [http://econweb.umd.edu/~chao/Research/research\\_files/Additional\\_Online\\_Appendix\\_Jackknife\\_Estimation\\_Cluster\\_Sample\\_IV\\_Model\\_December\\_20\\_2022.pdf](http://econweb.umd.edu/~chao/Research/research_files/Additional_Online_Appendix_Jackknife_Estimation_Cluster_Sample_IV_Model_December_20_2022.pdf)

**Lemma S2-1:** Let  $A = P^\perp - M^{(Z, Q)} D_{\hat{\vartheta}} M^{(Z, Q)}$ . Then, under Assumptions 2-6, the following statements hold as  $K_{2,n}$ ,  $n \rightarrow \infty$ .

- (a)  $\sum_{(i,t),(j,s)=1, (i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 = O_{a.s.}(K_{2,n})$ .
- (b)  $\sum_{(i,t),(j,s)=1, (i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^4 = O_{a.s.}(K_{2,n}^3/n^2)$ .
- (c)  $\sum_{(j,s)=1}^{m_n} \sum_{(i,t),(k,v)=1, (i,t) \neq (j,s), (k,v) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 A_{(j,s),(k,v)}^2 = O_{a.s.}(K_{2,n}^2/n)$ .
- (d)  $\max_{1 \leq (i,t) \leq m_n} \left( \sum_{(j,s)=1}^{m_n} A_{(i,t),(j,s)}^2 \right) = O_{a.s.}(K_{2,n}/n)$ .
- (e)  $\sum_{i_1, i_2=1}^n \sum_{j=1, j \neq i_1, i_2}^n \sum_{t_1=1}^{T_{i_1}} \sum_{t_2=1}^{T_{i_2}} \sum_{s_1, s_2=1}^{T_j} A_{(i_1, t_1), (j, s_1)}^2 A_{(i_2, t_2), (j, s_2)}^2 = O_{a.s.}(K_{2,n}^2/n)$  and  $\sum_{i=1}^n \sum_{j_1=1, j_1 \neq i}^n \sum_{j_2=1, j_2 \neq i}^n \sum_{t_1, t_2=1}^{T_i} \sum_{s_1=1}^{T_{j_1}} \sum_{s_2=1}^{T_{j_2}} A_{(i, t_1), (j_1, s_1)}^2 A_{(i, t_2), (j_2, s_2)}^2 = O_{a.s.}(K_{2,n}^2/n)$



- (f)  $\sum_{i=1}^n \sum_{t=1}^{T_i} \sum_{s=1}^{T_i} A_{(i,t),(i,s)}^2 = O_{a.s.} (K_{2,n}^2/n)$ .  
(g)  $\sum_{i=1}^n \sum_{s_1, t_1=1, s_1 \neq t_1}^{T_i} \sum_{s_2, t_2=1, s_2 \neq t_2}^{T_i} A_{(i,t_1),(i,s_1)}^2 A_{(i,t_2),(i,s_2)}^2 = O_{a.s.} (K_{2,n}^4/n^3)$ .

**Lemma S2-2:** Let Assumptions 1-6 be satisfied. Then, the following statements are true:

- (a)  $D_\mu^{-1} X' M^{(Z_1, Q)} X D_\mu^{-1} = O_p \left( n (\mu_n^{\min})^{-2} \right)$ ; (b)  $D_\mu^{-1} X' A X D_\mu^{-1} = H_n + o_p(1)$ , where  $H_n = \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n = O_p(1)$ .

**Lemma S2-3:** Let  $\underline{U} = U - \varepsilon \rho'$  and  $\underline{U}_{(i,t)} = U_{(i,t)} - \rho \varepsilon_{(i,t)}$  and let  $VC(X|\mathcal{F}_n^Z)$  denote the conditional covariance matrix of the random vector  $X$  given  $\mathcal{F}_n^Z$ . Under Assumptions 1-2, 5-6, and 8; there exists positive constants  $0 < \underline{C} \leq \overline{C} < \infty$  such that the following statements are true.

- (a)  $\lambda_{\max} [VC(\Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)] \leq \overline{C}$  a.s. and  $\lambda_{\min} [VC(\Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} | \mathcal{F}_n^Z)] \geq \underline{C}$  a.s. for all  $n$  sufficiently large.  
(b)  $VC(\underline{U}' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z) \geq \underline{C} I_d > \frac{0}{d \times d}$  a.s., for all  $n$  sufficiently large.  
(c)  $\lambda_{\max} (VC[\underline{U}' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z]) \leq \overline{C}$  a.s.,  $\lambda_{\max} (VC[\underline{U}' A \varepsilon / \sqrt{K_{2,n}}]) \leq \overline{C}$ ,  $\lambda_{\max} (VC[U' A \varepsilon / \sqrt{K_{2,n}} | \mathcal{F}_n^Z]) \leq \overline{C}$  a.s., and  $\lambda_{\max} (VC[U' A \varepsilon / \sqrt{K_{2,n}}]) \leq \overline{C}$ , for all  $n$  sufficiently large.  
(d) For any  $a \in \mathbb{R}^d$  with  $\|a\|_2 = 1$  and for all  $n$  sufficiently large,  $\lambda_{\min}(\Sigma_n) \geq \underline{C} > 0$  a.s. and  $a' \Sigma_n^{-1} a \leq \overline{C} < \infty$  a.s., where  $\Sigma_n = VC(\mathcal{Y}_n | \mathcal{F}_n^Z) = \Sigma_{1,n} + \Sigma_{2,n}$ , as defined in section 4 of the main paper, and where  $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$ .

**Lemma S2-4:** Under Assumptions 1-6,  $D_\mu^{-1} X' A \varepsilon = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} U' A \varepsilon = O_p(\max\{1, \sqrt{K_{2,n}} / (\mu_n^{\min})\})$

**Lemma S2-5:** Under Assumptions 1-6,  $D_\mu^{-1} X' M^{(Z_1, Q)} \varepsilon = O_p(n / \mu_n^{\min})$ .

**Lemma S2-6:** If Assumptions 2 and 8 are satisfied; then, for  $1 \leq p \leq 8$  and for all  $n$ , there exists a positive constant  $C$  such that  $\max_{1 \leq (i,t) \leq m_n} E \left[ \left\| \underline{U}_{(i,t)} \right\|_2^p | \mathcal{F}_n^Z \right] \leq C < \infty$  a.s., where  $\underline{U}_{(i,t)} = U_{(i,t)} - \rho \varepsilon_{(i,t)}$ .

**Lemma S2-7:** Under Assumptions 1-6, the following results hold: (a)  $\widehat{\ell}_{L,n} = o_p \left( [\mu_n^{\min}]^2 / n \right)$ ; (b)  $\widehat{\ell}_{F,n} = o_p \left( [\mu_n^{\min}]^2 / n \right)$ .

**Lemma S2-8:** Let  $A$  be as defined above. Assume that i)  $(u_{(1,1),n}, \varepsilon_{(1,1)}) , \dots, (u_{(1,T_1),n}, \varepsilon_{(1,T_1)}) , (u_{(2,1),n}, \varepsilon_{(2,1),n}) , \dots, (u_{(2,T_2),n}, \varepsilon_{(2,T_2),n}) , \dots, (u_{(n,1),n}, \varepsilon_{(n,1),n}) , \dots, (u_{(n,T_n),n}, \varepsilon_{(n,T_n),n})$  are independent conditional on  $\mathcal{F}_n^Z = \sigma(Z)$ ; ii) there exists a constant  $C$  such that, almost surely for all  $n$  sufficiently large,  $\max_{1 \leq (i,t) \leq m_n} E \left( u_{(i,t),n}^4 | \mathcal{F}_n^Z \right) \leq C$ ,  $\max_{1 \leq (i,t) \leq m_n} E \left( \varepsilon_{(i,t),n}^4 | \mathcal{F}_n^Z \right) \leq C$ , and

$\max_{1 \leq (i,t) \leq m_n} |\phi_{(i,t),n}| \leq C$ . In addition, define  $\bar{\psi}_{(j,s),n} = E[u_{(j,s),n} \varepsilon_{(j,s),n} | \mathcal{F}_n^Z]$  for  $(j,s) = 1, \dots, m_n$ . Then, under Assumptions 5 and 6, the following statements are true:

- (a)  $K_{2,n}^{-1} \sum_{1 \leq (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)}^2 \phi_{(i,t),n} \left\{ u_{(j,s),n} \varepsilon_{(j,s),n} - \bar{\psi}_{(j,s),n} \right\} \xrightarrow{p} 0$ ;  
(b)  $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \left\{ u_{(j,s),n} \varepsilon_{(k,v),n} + \varepsilon_{(j,s),n} u_{(k,v),n} \right\} \xrightarrow{p} 0$ ;  
(c)  $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} \varepsilon_{(j,s),n} \varepsilon_{(k,v),n} \xrightarrow{p} 0$ ;  
(d)  $K_{2,n}^{-1} \sum_{1 \leq (k,v) < (j,s) < (i,t) \leq m_n} A_{(i,t),(j,s)} A_{(i,t),(k,v)} \phi_{(i,t),n} u_{(j,s),n} u_{(k,v),n} \xrightarrow{p} 0$ .

**Lemma S2-9:** Let

$$\widehat{\Delta}(\delta_0) = -\frac{(y - X\delta_0)' M^{(Z_1, Q)}(y - X\delta_0)}{2} \frac{\partial}{\partial \delta} \left\{ \frac{(y - X\delta)' A(y - X\delta)}{(y - X\delta)' M^{(Z_1, Q)}(y - X\delta)} \right\} \Big|_{\delta=\delta_0}.$$

If Assumptions 1-6 and 8 are satisfied; then,  $D_\mu^{-1} \widehat{\Delta}(\delta_0) = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon + o_p(1)$ , where  $\underline{U} = U - \varepsilon \rho'$  and where  $\rho = \lim_{n \rightarrow \infty} E[U' M^Q \varepsilon] / E[\varepsilon' M^Q \varepsilon]$ .

**Lemma S2-10:** Let Assumptions 1-6 be satisfied, and let  $\bar{\delta}_n$  be any estimator such that, as  $n \rightarrow \infty$ ,  $D_\mu(\bar{\delta}_n - \delta_0) / \mu_n^{\min} = o_p(1)$ . Then,  $-D_\mu^{-1}(\partial \widehat{\Delta}(\bar{\delta}_n) / \partial \delta') D_\mu^{-1} = H_n + o_p(1)$ , where  $H_n = \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n$  and where

$$\begin{aligned} \widehat{\Delta}(\delta) &= -[(y - X\delta)' M^{(Z_1, Q)}(y - X\delta) / 2] [\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta] \\ &= X' A(y - X\delta) - \widehat{\ell}(\delta) X' M^{(Z_1, Q)}(y - X\delta), \text{ with} \\ \widehat{\ell}(\delta) &= (y - X\delta)' A(y - X\delta) / [(y - X\delta)' M^{(Z_1, Q)}(y - X\delta)]. \end{aligned}$$

In addition, we also have

$$D_\mu^{-1} X' [A - \widehat{\ell}(\bar{\delta}_n) M^{(Z_1, Q)}] X D_\mu^{-1} = H_n + o_p(1). \quad (18)$$

**Lemma S2-11:** Let  $\widehat{\ell}_L = Q(\widetilde{\beta}) = \min_{\beta \in \overline{B}} Q(\beta)$ , where  $Q(\beta)$  is as defined in Assumption 9.

Then,  $\widehat{\ell}_L$  is also the smallest root of the determinantal equation  $\det[\overline{X}' A \overline{X} - \widehat{\ell}_L \overline{X}' M^{(Z_1, Q)} \overline{X}] = 0$ , where  $\overline{X} = [y, X]$ . Assume in addition that condition (13) in Assumption 9 is satisfied; then,  $\widehat{\ell}_L$  has the representation

$$\widehat{\ell}_L = \frac{(y - X\widehat{\delta}_L)' A(y - X\widehat{\delta}_L)}{(y - X\widehat{\delta}_L)' M^{(Z_1, Q)}(y - X\widehat{\delta}_L)}, \quad (19)$$

where  $\widehat{\delta}_L$  denotes the FELIM estimator. Moreover,  $\overline{X}' A(y - X\widehat{\delta}_L) - \widehat{\ell}_L \overline{X}' M^{(Z_1, Q)}(y - X\widehat{\delta}_L) = 0$ . In particular, this implies that  $\widehat{\Delta}(\widehat{\delta}_L) = 0$ , where

$$\widehat{\Delta}(\delta) = -[(y - X\delta)' M^{(Z_1, Q)}(y - X\delta) / 2] (\partial \widehat{Q}_{FELIM}(\delta) / \partial \delta), \text{ so that } \widehat{\delta}_L \text{ satisfies the set of (normalized) first-order conditions for minimizing the variance ratio objective function } \widehat{Q}_{FELIM}(\delta) = (y - X\delta)' A(y - X\delta) / [(y - X\delta)' M^{(Z_1, Q)}(y - X\delta)].$$

**Lemma S2-12:** If Assumptions 1-6 are satisfied; then,

$$\begin{aligned} D_\mu^{-1} X' [A - \widehat{\ell}_{F,n} M^{(Z_1, Q)}] X D_\mu^{-1} &= H_n + o_p(1), \text{ where } H_n = \Upsilon' Z_2' M^{(Z_1, Q)} Z_2 \Upsilon / n, \\ \widehat{\ell}_{F,n} &= [\widehat{\ell}_{L,n} - (1 - \widehat{\ell}_{L,n})(C/m_n)] / [1 - (1 - \widehat{\ell}_{L,n})(C/m_n)], \text{ and } \widehat{\ell}_{L,n} \text{ is smallest root of the determinantal equation } \det\{\overline{X}' A \overline{X} - \widehat{\ell}_{L,n} \overline{X}' M^{(Z_1, Q)} \overline{X}\} = 0, \text{ with } \overline{X} = [y \quad X]. \end{aligned}$$

**Lemma S2-13:** If Assumptions 1-6 and 8-9 are satisfied; then,  $D_\mu^{-1} X' [A - \widehat{\ell}_{F,n} M^{(Z_1, Q)}] (y - X\delta_0) = \mathcal{Y}_n [1 + o_p(1)]$ , where  $\mathcal{Y}_n = \Upsilon' Z_2' M^{(Z_1, Q)} \varepsilon / \sqrt{n} + D_\mu^{-1} \underline{U}' A \varepsilon$  with  $\underline{U} = U - \varepsilon \rho'$  and  $\rho = \lim_{n \rightarrow \infty} E[U' M^Q \varepsilon] / E[\varepsilon' M^Q \varepsilon]$ .

**Lemma S2-14:** For any  $a \in \mathbb{R}^d$  such that  $\|a\| = 1$ , define  $b_{1n} = \Sigma_n^{-1/2} a$ ,  $b_{2n} = \sqrt{K_{2,n}} D_\mu^{-1} \Sigma_n^{-1/2} a$ ,  $\underline{u}_{2,(i,t),n} = b_{2n}' \underline{U}_{(i,t)}$

$$\begin{aligned}
&= \sqrt{K_{2,n}} a' \Sigma_n^{-1/2} D_\mu^{-1} \underline{U}_{(i,t)}, \quad \sigma_{(i,t),n}^2 = E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right], \quad \tilde{\psi}_{(i,t),n} = E \left[ \underline{u}_{2,(i,t),n} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right], \quad \text{and} \quad \tilde{\omega}_{(i,t),n}^2 = \\
&E \left[ \underline{u}_{2,(i,t),n}^2 | \mathcal{F}_n^Z \right]. \text{ If Assumptions 1-2 and 5-6 are satisfied; then, the following statements are true.} \\
\text{(a)} \quad &\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left[ b'_{1n} \Upsilon' Z'_2 M^{(Z_1, Q)} e_{(i,t)} / \sqrt{n} \right] \left( A_{(i,t),(j,s)} / \sqrt{K_{2,n}} \right) \left\{ \varepsilon_{(j,s)} \tilde{\psi}_{(i,t),n} + \underline{u}_{2,(j,s)} \sigma_{(i,t),n}^2 \right\} \\
&= O_p \left( K_{2,n}^{1/4} / \mu_n^{\min} \right) = o_p(1). \\
\text{(b)} \quad &\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left( A_{(i,t),(j,s)}^2 / K_{2,n} \right) \left( \varepsilon_{(j,s)}^2 - \sigma_{(j,s),n}^2 \right) \tilde{\omega}_{(i,t),n}^2 = O_p \left( K_{2,n} (\mu_n^{\min})^{-2} n^{-1/2} \right) = o_p(1). \\
\text{(c)} \quad &\sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} \left( A_{(i,t),(j,s)}^2 / K_{2,n} \right) \left( \underline{u}_{2,(j,s),n}^2 - \tilde{\omega}_{(j,s),n}^2 \right) \sigma_{(i,t),n}^2 = O_p \left( K_{2,n} (\mu_n^{\min})^{-2} n^{-1/2} \right) = \\
&o_p(1).
\end{aligned}$$

**Lemma S2-15 (Gänsler and Stute, 1977):** Let  $\{X_{i,n}, \mathcal{F}_{i,n}, 1 \leq i \leq k_n, n \geq 1\}$  be a square integrable martingale difference array. Suppose that for all  $\epsilon > 0$

$$\sum_{i=1}^{k_n} E \left[ X_{i,n}^2 \mathbb{I} \{ |X_{i,n}| > \epsilon \} | \mathcal{F}_{i-1,n} \right] \xrightarrow{p} 0 \text{ and} \quad (20)$$

$$\sum_{i=1}^{k_n} E \left[ X_{i,n}^2 | \mathcal{F}_{i-1,n} \right] \xrightarrow{p} 1. \quad (21)$$

Then,  $\sum_{i=1}^{k_n} X_{i,n} \xrightarrow{d} N(0, 1)$ .

**Remark:** Note that a sufficient condition for condition (20), which we will verify in lieu of (20) in the proof of Theorems 2 and 3 in Appendix S1, is the following

$$\sum_{i=1}^{k_n} E \left[ |X_{i,n}|^{2+\delta} \right] \xrightarrow{p} 0, \text{ for some } \delta > 0. \quad (22)$$

**Lemma S2-16:** Let  $\tilde{L}_n$  be a sequence of  $l \times d$  nonrandom matrices (with  $l \leq d$ ) such that  $\|\tilde{L}_n\|_F^2 \leq \overline{C} < \infty$  for some constant  $\overline{C}$ , and let  $\Sigma_{2,n} = VC \left( D_\mu^{-1} \underline{U}' A \varepsilon | \mathcal{F}_n^Z \right) = D_\mu^{-1} VC \left( \underline{U}' A \varepsilon | \mathcal{F}_n^Z \right) D_\mu^{-1}$ . Assume that there exists a positive constant  $\underline{C}$  such that  $\lambda_{\min} \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right) \geq \underline{C} > 0$  a.s.n. Furthermore, let  $a \in \mathbb{R}^d$  such that  $\|a\|_2 = 1$  and let  $\underline{u}_{a,(i,t),n} = a' \left( (\mu_n^{\min})^2 \tilde{L}_n H_n^{-1} \Sigma_{2,n} H_n^{-1} \tilde{L}_n' / K_{2,n} \right)^{-1/2} \tilde{L}_n D_\mu^{-1} \underline{U}_{(i,t)}$ . Let Assumptions 1-2 and 5-6 be satisfied and assume that  $(\mu_n^{\min})^2 / K_{2,n} = o(1)$  but  $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$ . Under these conditions, the following statements are true:

$$\begin{aligned}
\text{(a)} \quad &\left[ (\mu_n^{\min})^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \underline{u}_{a,(j,s),n}^2 - E \left[ \underline{u}_{a,(j,s),n}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right] \\
&= O_p \left( n^{-1/2} \right) = o_p(1); \\
\text{(b)} \quad &\left[ (\mu_n^{\min})^2 / K_{2,n} \right] \sum_{(i,t)=2}^{m_n} \sum_{(j,s)=1}^{(i,t)-1} A_{(i,t),(j,s)}^2 \left( \varepsilon_{(j,s)}^2 - E \left[ \varepsilon_{(j,s)}^2 | \mathcal{F}_n^Z \right] \right) E \left[ \underline{u}_{a,(i,t),n}^2 | \mathcal{F}_n^Z \right] \\
&= O_p \left( n^{-1/2} \right) = o_p(1).
\end{aligned}$$

**Lemma S2-17** Under Assumptions 1-6,  $D_\mu^{-1}X'AD(\varepsilon \circ \varepsilon)AXD_\mu^{-1} = \Sigma_{1,n}$   
 $+ \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1} + o_p(1)$ , where  $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1, Q)} D_{\sigma^2} M^{(Z_1, Q)} Z_2 \Upsilon / n$ ,  
 $\sigma_{(i,t)}^2 = E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$ ,  $D_{\sigma^2} = \text{diag} \left( \sigma_{(1,1)}^2, \dots, \sigma_{(n, T_n)}^2 \right)$ , and  $\Psi_{(j,s)} = E \left[ U_{(j,s)} U_{(j,s)}' | \mathcal{F}_n^Z \right]$ .

**Lemma S2-18** Let Assumptions 1-6 and 8 be satisfied, and let  $\{\hat{\delta}_n\}$  be any sequence of estimators such that  $\|\hat{\delta}_n - \delta_0\|_2 \xrightarrow{p} 0$  as  $n \rightarrow \infty$ , as long as  $\sqrt{K_{2,n}} / (\mu_n^{\min})^2 \rightarrow 0$ . Also, define the following notations: let  $\hat{\varepsilon} = M^{(Z, Q)} (y - X \hat{\delta}_n)$ ,  $J = [M^Q \circ M^Q]^{-1}$ ,  $S_1 = X'AD(J[\hat{\varepsilon} \circ \hat{\varepsilon}])AX$ ,  
 $S_2 = (\hat{\varepsilon} \circ \hat{\varepsilon})' J(A \circ A) J(\hat{\varepsilon}'_d \circ M^{(Z, Q)} X)$ ,  
 $\underline{S}_2 = (\hat{\varepsilon} \circ \hat{\varepsilon})' J(A \circ A) J(\hat{\varepsilon}'_d \circ \underline{\hat{U}})$  with  $\underline{\hat{U}} = M^{(Z, Q)} X - \hat{\varepsilon}'_d$ ,  $S_3 = (\hat{\varepsilon} \circ \hat{\varepsilon})' J(A \circ A) J(\hat{\varepsilon} \circ \hat{\varepsilon})$ ,  
 $S_4 = (\hat{\varepsilon}'_d \circ M^{(Z, Q)} X)' J(A \circ A) J(\hat{\varepsilon}'_d \circ M^{(Z, Q)} X)$ ,  $\underline{S}_4 = (\hat{\varepsilon}'_d \circ \underline{\hat{U}})' J(A \circ A) J(\hat{\varepsilon}'_d \circ \underline{\hat{U}})$ , and  
 $\Sigma_{1,n} = \Upsilon' Z_2' M^{(Z_1, Q)} D_{\sigma^2} M^{(Z_1, Q)} Z_2 \Upsilon / n$ . In addition, define  $\sigma_{(i,t)}^2 = E \left[ \varepsilon_{(i,t)}^2 | \mathcal{F}_n^Z \right]$ ,  
 $D_{\sigma^2} = \text{diag} \left( \sigma_{(1,1)}^2, \dots, \sigma_{(n, T_n)}^2 \right)$ ,  $\phi_{(i,t)} = E \left[ U_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ ,  $\Psi_{(i,t)} = E \left[ U_{(i,t)} U_{(i,t)}' | \mathcal{F}_n^Z \right]$ ,  
 $\underline{\phi}_{(i,t)} = E \left[ \underline{U}_{(i,t)} \varepsilon_{(i,t)} | \mathcal{F}_n^Z \right]$ , and  $\underline{\Psi}_{(i,t)} = E \left[ \underline{U}_{(i,t)} \underline{U}_{(i,t)}' | \mathcal{F}_n^Z \right]$  where  $\underline{U}_{(i,t)} = U_{(i,t)} - \rho \varepsilon_{(i,t)}$  and where  
for notational convenience we suppress the dependence of  $\sigma_{(i,t)}^2$ ,  $\phi_{(i,t)}$ ,  $\Psi_{(i,t)}$ ,  $\underline{\phi}_{(i,t)}$ , and  $\underline{\Psi}_{(i,t)}$  on  $\mathcal{F}_n^Z = \sigma(Z)$ . Then, under the above conditions, the following statements are true.

- (a)  $D_\mu^{-1} S_1 D_\mu^{-1} = \Sigma_{1,n} + \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 D_\mu^{-1} \Psi_{(j,s)} D_\mu^{-1} + o_p \left( \max \left\{ 1, K_{2,n} (\mu_n^{\min})^{-2} \right\} \right)$ .
- (b)  $S_3 / K_{2,n} - K_{2,n}^{-1} \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \sigma_{(j,s)}^2 = o_p(1)$ .
- (c)  $D_\mu^{-1} S_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 D_\mu^{-1} \phi_{(i,t)} \phi_{(j,s)}' D_\mu^{-1} = o_p \left( K_{2,n} (\mu_n^{\min})^{-2} \right)$ .
- (d)  $(\mu_n^{\min} / K_{2,n}) S_2 D_\mu^{-1} - (\mu_n^{\min} / K_{2,n}) \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \phi_{(j,s)}' D_\mu^{-1} = o_p(1)$ .
- (e)  $D_\mu^{-1} \hat{\rho}_n = O_p \left( (\mu_n^{\min})^{-1} \right)$  and  $D_\mu^{-1} (\hat{\rho}_n - \rho) = o_p \left( (\mu_n^{\min})^{-1} \right)$ , where  $\rho = \lim_{n \rightarrow \infty} \rho_n = \lim_{n \rightarrow \infty} (E[U' M^Q \varepsilon] / n) / (E[\varepsilon' M^Q \varepsilon] / n)$ .
- (f)  $D_\mu^{-1} \underline{S}_4 D_\mu^{-1} - \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 D_\mu^{-1} \underline{\phi}_{(i,t)} \underline{\phi}_{(j,s)}' D_\mu^{-1} = o_p \left( K_{2,n} (\mu_n^{\min})^{-2} \right)$ .
- (g)  $(\mu_n^{\min} / K_{2,n}) - (\mu_n^{\min} / K_{2,n}) \sum_{(i,t),(j,s)=1,(i,t) \neq (j,s)}^{m_n} A_{(i,t),(j,s)}^2 \sigma_{(i,t)}^2 \underline{\phi}_{(j,s)}' D_\mu^{-1} = o_p(1)$ .

## References

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