

# The Block Bootstrap for Parameter Estimation Error In Recursive Estimation Schemes, With Applications to Predictive Evaluation\*

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## Abstract

This paper introduces a new block bootstrap which is valid for recursive  $m$ -estimators, in the sense that its use suffices to mimic the limiting distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t - \theta^\dagger)$ , where  $R$  denotes the length of the estimation period,  $P$  the number of recursively estimated parameters,  $\widehat{\theta}_t$  is a recursive  $m$ -estimator constructed using the first  $t$  observations, and  $\theta^\dagger$  is its probability limit. In the recursive case, earlier observations are used more frequently than temporally subsequent observations. This introduces a bias to the usual block bootstrap. We circumvent this problem by first resampling  $R$  observations from the initial  $R$  sample observations, and then concatenating onto this vector an additional  $P$  resampled observations from the remaining sample. Thereafter,  $\widehat{\theta}_t^*$  is constructed using the resampled series, and an adjustment term is added to  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t^* - \widehat{\theta}_t)$  in order to ensure that the distribution of this sum is the same as the distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t - \theta^\dagger)$ . This parameter estimation error bootstrap for recursive estimation schemes can be used to provide valid critical values in a variety of interesting testing contexts, and three such leading applications are developed. The first is a generalization of the reality check test of White (2000) that allows for non vanishing parameter estimation error. The second is an out-of-sample version of the integrated conditional moment (ICM) test of Bierens (1982,1990) and Bierens and Ploberger (1997) which provides out of sample tests consistent against generic (nonlinear) alternatives. Finally, the third is a procedure assessing the relative out-of-sample predictive accuracy of multiple conditional distribution models. This procedure is based on an extension of the Andrews (1997) conditional Kolmogorov test. The main findings from a small Monte Carlo experiment indicate that: (i) the adjustment term used in the suggested bootstrap substantially improve coverage rates relative to a bootstrap without adjustment, and (ii) the suggested bootstrap is as reliable as the standard block bootstrap within the context of full sample estimation.

*JEL classification:* C22, C51.

*Keywords:* block bootstrap, recursive estimation scheme, nonlinear causality, parameter estimation error, predictive density.

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# 1 Introduction

The main objective of this paper is to suggest a new block bootstrap which is valid for recursive  $m$ -estimators, in the sense that its use suffices to mimic the limiting distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t - \theta^\dagger)$ , where  $R$  denotes the length of the estimation period,  $P$  the number of recursively estimated parameters,  $\widehat{\theta}_t$  is a recursive  $m$ -estimator constructed using the first  $t$  observations, and  $\theta^\dagger$  is its probability limit. We call this bootstrap the parameter estimation error (PEE) bootstrap for recursive estimation schemes (or alternatively, the recursive PEE bootstrap). For fixed sampling schemes, the properties of the block bootstrap for  $m$ -estimators and/or GMM estimators with dependent observations have been studied by several authors. For example, Hall and Horowitz (1996) and Andrews (2002a) show that the block bootstrap provides improved critical values, in the sense of asymptotic refinements, for “studentized” GMM estimators and for tests of overidentifying restrictions, in the case where the covariance across moment conditions is zero after a given number of lags.<sup>1,2</sup> In addition, Inoue and Shintani (2001) show that the block bootstrap provides asymptotic refinements for linear overidentified GMM estimators for general mixing processes. A recent contribution which is useful in our context is that of Goncalves and White (2002a and 2002b), who show that for  $m$ -estimators, the limiting distribution of  $\sqrt{T}(\widehat{\theta}_T^* - \widehat{\theta}_T)$  provides a valid first order approximation to that of  $\sqrt{T}(\widehat{\theta}_T - \theta^\dagger)$  for heterogeneous and near epoch dependent series, where  $\widehat{\theta}_T^*$  is a resampled estimator, and  $T$  denotes the length of the entire sample. Based on the results mentioned above, one might expect  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t^* - \widehat{\theta}_t)$  to have the same limiting distribution as  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_t - \theta^\dagger)$ . However, in the recursive case, earlier observations are used more frequently than temporally subsequent observations. This introduces a bias to the usual block bootstrap, as under standard resampling with replacement schemes, any block from the original sample has the same probability of being selected. We circumvent this problem by first forming bootstrap samples as follows. Resample  $R$  observations from the initial  $R$  sample observations, and then concatenate onto this vector an additional  $P$  resampled observations from the remaining sample. Thereafter,

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<sup>1</sup>More recently, Andrews (2002b) suggests a new block bootstrap, called the block-block bootstrap, which ensures better higher order refinements than the standard block bootstrap. The key element in his approach is an adjustment of the original statistic in such a way that both the original and the bootstrap statistics share the same dependence structure.

<sup>2</sup>For the case of Markov processes, Horowitz (2002) introduces a new bootstrap procedure, based on a nonparametric estimator of the transition density, which provides “better” higher order refinements than the block bootstrap.

construct  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t^* - \hat{\theta}_t)$  and add an adjustment term in order to ensure that the distribution of the sum of both components is the same as the distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t - \theta^\dagger)$ , conditional on the sample, and for all samples except a set of probability measure approaching zero. The adjustment term compensates for the fact that, when resampling from the last  $P$  observations, each block (of length  $l$ ) has the same chance of being drawn, while in the construction of the actual  $m$ -estimator, earlier observations are more heavily used.

In principle, one could also devise a resampling scheme in which blocks are more heavily drawn from the beginning of the sample and, within each block, “earlier” observations are more heavily weighted. However, in practice, it is in general not feasible to implement a weighted resampling which exactly mimics the long run covariance of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t - \theta^\dagger)$ .

The recursive PEE bootstrap can be used to provide valid critical values in a variety of interesting testing contexts, and three such leading applications are developed. The first is a generalization of the reality check test of White (2000) that allows for non vanishing parameter estimation error. The second is an out-of-sample version of the integrated conditional moment (ICM) test of Bierens (1982,1990) and Bierens and Ploberger (1997) which provides out of sample tests consistent against generic (nonlinear) alternatives. Finally, the third is a procedure assessing the relative out-of-sample predictive accuracy of multiple conditional distribution models. This procedure is based on an extension of the Andrews (1997) conditional Kolmogorov test. There are two key links between these applications. First, all applications are made operational via use of the recursive PEE bootstrap. Second, all applications allow for misspecification among all models being estimated and compared, as opposed to the usual practice of assuming correct (dynamic) specification under the null hypothesis.<sup>3,4</sup>

To be more specific, the first application concerns the reality check of White (2000), which extends the Diebold and Mariano (1995) and West (1996) test for the relative predictive accuracy of two models by allowing for the joint comparison of multiple misspecified models against a given

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<sup>3</sup>This second feature is important when constructing predictive models, for example, as it is natural not to impose correct specification of any of the competing models; and if one were to impose correct specification under the null hypothesis, then inference would not be valid in general (e.g. when the assumption of martingale difference sequence errors is violated - see Corradi and Swanson (2003a)).

<sup>4</sup>Motivation for comparing misspecified models can be taken from the finance literature. For example, Hansen, Heaton and Luttmer (1995) and Hansen and Jeganathan (1997) consider the problem of comparing multiple misspecified models from the perspective of asset pricing and stochastic discount factor models.

benchmark. White obtains valid asymptotic critical values for his test via use of the Politis and Romano (1994) stationary bootstrap for the case in which parameter estimation error is asymptotically negligible. Using the recursive PEE bootstrap, we generalize the reality check to the case in which parameter estimation error does not vanish asymptotically.

The objective of the second application is to test the predictive accuracy of a given (non)linear model against generic (non)linear alternatives. The ICM type test used for this purpose differs from those developed by Bierens (1982,1990) and Bierens and Ploberger (1997) because parameters are estimated recursively, out-of-sample prediction models are analyzed, and the null hypothesis is that the reference model is the best “loss function specific” predictor, for a given information set. This application builds on previous work by Corradi and Swanson (2002) who use a conditional p-value method for constructing critical values in this context, extending earlier work by Hansen (1996) and Inoue (2001). However, the conditional p-value approach they use suffers from the fact that under the alternative, the simulated statistics diverge (at rate as high as  $\sqrt{\tilde{l}}$ ), conditional on the sample, where  $\tilde{l}$  plays a role analogous to the block length in the block bootstrap. As this feature may lead to reduced power in finite samples, we establish in the second application that the recursive PEE bootstrap can yields a  $\sqrt{P}$ -consistent test.

The third application builds on much of the recent literature on predictive evaluation, which has focused on out of sample comparison of models for the conditional mean (see among others Diebold and Mariano (1995), West (1996), Clark and McCracken (2001), McCracken (2000), White (2000), Corradi, Swanson and Olivetti (2001), and Corradi and Swanson (2002)). However, there are several instances in which a “good” model of the conditional mean does not suffice. For example, financial risk management involves tracking the entire distribution of a portfolio; or measuring certain distributional aspects, such as value at risk (see e.g. Duffie and Pan (1997)). In these cases, the choice of the best loss function specific model for the conditional mean may not be of too much help. Important contributions that go beyond the examination of models of conditional mean include assessing the correctness of conditional interval prediction (Christoffersen (1998)) and assessing volatility predictability by comparing unconditional and conditional interval forecasts (Christoffersen and Diebold (2000)).<sup>5</sup> A natural extension of these papers that allows for the joint evaluation of all conditional aspects, and hence the evaluation of the underly-

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<sup>5</sup>Prediction confidence intervals are also discussed in Granger, White and Kamstra (1989) Chatfield (1993), Diebold, Tay and Wallis (1998), Clements and Taylor (2001), and the references cited therein.

ing predictive density, is outlined in Diebold, Gunther and Tay (DGT: 1998). By making use of the probability integral transform, DGT suggest a simple and effective means by which predictive densities can be evaluated (see also Bai (2001), Diebold, Hahn and Tay (1999), and Hong (2001)). A feature common to the papers cited above is that they test the null hypothesis of (dynamic) correct specification. In the third application we outline a different approach to predictive density evaluation, which is based in part on ideas developed in Andrews (1997) and Corradi and Swanson (2003b). The main question that we address is the following: Given an information set  $X_t$ , which distributional model from amongst a set of misspecified models yields the more “accurate” out-of-sample prediction of the “true” distribution of  $y_{t+1}$ , given  $X_t$ ? The most noteworthy departure of our approach from others is that we compare distributional models, all of which may be misspecified, either dynamically or otherwise. More specifically, our objective is to form parametric conditional distributions for a scalar random variable,  $y_{t+1}$ , given some collection of conditioning variables,  $Z^t$ , and to select among these. Define the group of conditional distribution models from which we want to make a selection as  $F_1(u|Z^t, \theta_1^\dagger), \dots, F_n(u|Z^t, \theta_n^\dagger)$ , and define the true conditional distribution as  $F_0(u|Z^t, \theta_0) = \Pr(y_{t+1} \leq u|Z^t)$ . Relative accuracy among these models is measured using a distributional analog of mean square error (i.e. the average over  $U$  of  $E\left(\left(F_i(u|Z^t, \theta_i^\dagger) - F_0(u|Z^t, \theta_0)\right)^2\right)$ , where  $u \in U$ , and  $U$  is a possibly unbounded set on the real line). For a given benchmark model, application of the recursive PEE bootstrap then enables us to construct appropriate critical values when testing whether any competing model is “better” than the benchmark.

In order to assess the finite sample performance of the PEE recursive bootstrap, we carry out a set of Monte Carlo experiments in which we compare the new bootstrap with a version thereof that does not contain adjustment terms as well as with a standard version of the block bootstrap in which the entire sample of data is used to estimate the parameters of a model, instead of using subsamples with recursive estimation schemes. Interestingly, our bootstrap method appears to yield coverage probabilities approximately as good as when the standard block bootstrap is used, and there is clear evidence that the adjustment terms yield substantive improvements relative to the version of the recursive PEE bootstrap that does not contain the adjustment terms.

The rest of the paper is organized as follows. Section 2 explains the block bootstrap for recursive  $m$ -estimators and establishes its first order validity. Sections 3, 4 and 5 outline the three applications of the recursive block bootstrap: White’s reality check, out of sample integrated conditional

moment tests and predictive density evaluation. Monte Carlo findings are discussed in Section 6. Finally, concluding remarks are given in Section 7. All proofs are collected in an Appendix. Hereafter,  $P^*$  denotes the probability law governing the resampled series, conditional on the sample,  $E^*$  and  $Var^*$  the mean and variance operators associated with  $P^*$ ,  $o_P^*(1)$   $\Pr - P$  denotes a term converging to zero in  $P^*$ -probability, conditional on the sample except a subset of probability measure approaching zero, and finally  $O_P^*(1)$   $\Pr - P$  denotes a term which is bounded in  $P^*$ -probability, conditional on the sample except a subset of probability measure approaching zero.

## 2 The Block Bootstrap for Parameter Estimation Error in Recursive Schemes

In this section, we establish the first order validity of the block bootstrap when parameters are estimated using the *recursive m*-estimator defined as follows.

Let  $Z^t = (y_t, \dots, y_{t-s_1+1}, X_t, \dots, X_{t-s_2+1})$ ,  $t = 1, \dots, T$ , and let  $s = \max\{s_1, s_2\}$ . Additionally, we henceforth assume that  $i = 1, \dots, n$  models are estimated, as in the applications outlined in Section 3 and 5 below. Now, define the *recursive m*-estimator for the parameter vector associated with model  $i$  as:

$$\hat{\theta}_{i,t} = \arg \min_{\theta_i \in \Theta_i} \frac{1}{t} \sum_{j=s}^t q_i(y_j, Z^{j-1}, \theta_i), \quad R \leq t \leq T-1, \quad i = 1, \dots, n \quad (1)$$

and

$$\theta_i^\dagger = \arg \min_{\theta_i \in \Theta_i} E(q_i(y_j, Z^{j-1}, \theta_i)), \quad (2)$$

where  $q_i$  denotes the objective function for model  $i$ . Following standard practice (such as in the real-time forecasting literature), this estimator is first computed using  $R$  observations. In our applications we focus on 1-step ahead prediction, and the recursive estimators are thus computed first using  $R + 1$  observations, and then  $R + 2$  observations, and so on until the last estimator is constructed using  $T - 1$  observations; resulting in a sequence of  $P = T - R$  estimators. These estimators are then used to construct sequences of  $P$  1-step ahead forecasts and associated forecast errors, for example.

In order to properly capture the contribution of parameter estimation error given the recursive sampling scheme, it suffices to form bootstrap samples by first resampling  $R$  observations from

the initial  $R$  sample observations, and then concatenating onto this an additional  $P$  observations resampled from the  $P$  remaining sample observations. More specifically, let  $b_1 l_1 + b_2 l_2 = T - s + 1$ , with  $b_1 l_1 = R - s + 1$  and  $b_2 l_2 = P$ . Also, let  $W_t = (y_t, Z^{t-1})$ . First, draw  $b_1$  overlapping blocks, of length  $l_1$ , from  $s, \dots, R$  and then draw  $b_2$  overlapping blocks, of length  $l_2$ , from data indexed by  $R + 1, \dots, R + P$ , with replacement. The first  $R - s + 1$  pseudo observations,  $W_s^*, W_{s+1}^*, \dots, W_{s+l-1}^*, \dots, W_R^*$ , are equal to  $W_{I_1^R}, W_{I_1^R+1}, \dots, W_{I_1^R+l_1-1}, \dots, W_{I_{b_1}^R+l_1-1}$ , where  $I_i^R, i = 1, \dots, b_1$  are independent uniform random draws on the interval  $s, \dots, R - l_1 + 1$ ; and the remaining  $P$  pseudo observations,  $W_{R+1}^*, W_{R+2}^*, \dots, W_{R+l}^*, \dots, W_{R+P}^*$ , are equal to  $W_{I_1^P}, W_{I_1^P+1}, \dots, W_{I_1^P+l_2-1}, \dots, W_{I_{b_2}^P+l_2-1}$ , where  $I_i^P, i = 1, \dots, b_2$  are independent uniform random draws from data indexed by  $R + 1, R + 2, \dots, R + P - l - 1$ . Thus, conditional on the (entire) sample, the pseudo time series  $W_t^*, t = s, \dots, R, R + 1, \dots, R + P$ , consists of  $b = b_1 + b_2$  asymptotically independent, but non identically distributed blocks of length  $l_1$  and  $l_2$  respectively.<sup>6</sup> Also, conditionally on the sample, all of the moments of each block are asymptotically homogeneous, under the assumption that the underlying sample is strictly stationary. Therefore, the rescaled (partial) sums of the blocks satisfy the law of large numbers and the central limit theorem for asymptotically independent and homogeneous random variables, conditional on the (entire) sample. Now, define the recursive PEE bootstrap  $m$ -estimator as,

$$\widehat{\theta}_{i,t}^* = \arg \min_{\theta_i \in \Theta_i} \frac{1}{t} \sum_{j=s}^t q_i(y_j^*, Z^{*,j-1}, \theta_i), \quad R \leq t \leq T - 1, \quad i = 1, \dots, n.$$

In order to establish the asymptotic validity of this version of the block bootstrap, we require the following assumptions.

**Assumption A1:**  $(y_t, X_t)$ , with  $y_t$  scalar and  $X_t$  an  $R^\zeta$ -valued ( $0 < \zeta < \infty$ ) vector, is a strictly stationary and absolutely regular  $\beta$ -mixing process with size  $-4(4 + \psi)/\psi$ ,  $\psi > 0$ .

**Assumption A2:** (i)  $\theta_i^\dagger$  is uniquely identified (i.e.  $E(q_i(y_t, Z^{t-1}, \theta_i)) > E(q_i(y_t, Z^{t-1}, \theta_i^\dagger))$  for any  $\theta_i \neq \theta_i^\dagger$ ); (ii)  $q_i$  is twice continuously differentiable on the interior of  $\Theta_i$ , for  $i = 1, \dots, n$ , and for  $\Theta_i$  a compact subset of  $R^{\varrho(i)}$ ; (iii) the elements of  $\nabla_{\theta_i} q_i$  and  $\nabla_{\theta_i}^2 q_i$  are  $p$ -dominated on  $\Theta_i$ , with  $p > 2(2 + \psi)$ , where  $\psi$  is the same positive constant as defined in Assumption A1; and (iii)  $E(-\nabla_{\theta_i}^2 q_i(\theta_i))$  is negative definite uniformly on  $\Theta_i$ .<sup>7</sup>

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<sup>6</sup>More precisely, each block from  $R + 1, \dots, R + P - l - 1$  may overlap with any block from  $s, \dots, R$  for at most  $s$  observations, where  $s$  is finite.

<sup>7</sup>We say that  $\nabla_{\theta_i} q_i(y_t, Z^{t-1}, \theta_i)$  is  $2r$ -dominated on  $\Theta_i$  if its  $j$ -th element,  $j = 1, \dots, \varrho(i)$ , is such that  $|\nabla_{\theta_i} q_i(y_t, Z^{t-1}, \theta_i)|_j \leq D_t$ , and  $E(|D_t|^{2r}) < \infty$ . For more details on domination conditions, see Gallant and White

**Assumption A3:**  $T = R + P$ , and as  $T \rightarrow \infty$ ,  $P/R \rightarrow \pi$ , with  $0 < \pi < \infty$ .

Assumptions A1 and A2 are standard memory, moment, smoothness and identifiability conditions. A1 requires  $(y_t, X_t)$  to be strictly stationary and absolutely regular. The memory condition is stronger than  $\alpha$ -mixing, but weaker than (uniform)  $\phi$ -mixing. Assumption A3 requires that  $R$  and  $P$  grow at the same rate. Of course, if  $R$  grows faster than  $P$ , then  $\Psi_{R,P}^*$  (as defined below) vanishes in probability, and there is no need to capture the contribution of parameter estimation error when constructing bootstrap critical values for predictive accuracy tests such as those discussed in the sequel.

Define:

$$\begin{aligned} \Psi_{R,P}^* &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t}^* - \widehat{\theta}_{i,t}) + \left( -\frac{1}{T} \sum_{t=s}^T \nabla_{\theta_i}^2 q_i(y_t, Z^{t-1}, \widehat{\theta}_{i,T}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \left( \nabla_{\theta_i} q_i(y_{R+j}, Z^{R+j-1}, \widehat{\theta}_{i,T}) - \frac{1}{P} \sum_{j=1}^P \nabla_{\theta_i} q_i(y_{R+j}, Z^{R+j-1}, \widehat{\theta}_{i,T}) \right), \end{aligned} \quad (3)$$

where  $a_{R,j} = \frac{1}{R+j} + \frac{1}{R+j+1} + \dots + \frac{1}{R+P-1}$ .

**Theorem 1:** Let A1-A3 hold. Also, assume that as  $P, R \rightarrow \infty$ ,  $l_1, l_2 \rightarrow \infty$ , and that  $\frac{l_2}{P^{1/4}} \rightarrow 0$  and  $\frac{l_1}{R^{1/4}} \rightarrow 0$ . Then, as  $P$  and  $R \rightarrow \infty$ ,

$$P \left( \omega : \sup_{v \in \Re^{\varrho(i)}} \left| P_{R,P}^* (\Psi_{R,P}^* \leq v) - P \left( \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger) \leq v \right) \right| > \varepsilon \right) \rightarrow 0,$$

where  $P_{R,P}^*$  denotes the probability law of the resampled series, conditional on the (entire) sample.

Broadly speaking, Theorem 1 states that  $\Psi_{R,P}^*$  has the same limiting distribution as  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger)$ , conditional on sample, and for all samples except a set with probability measure approaching zero. As outlined in the following sections, application of Theorem 1 allows us to capture the contribution of (recursive) parameter estimation error to the covariance kernel of the limiting distribution of various statistics.

Though a detailed proof of Theorem 1 is given in the appendix, it is worthwhile giving an intuitive explanation of why there is an adjustment term in  $\Psi_{R,P}^*$ , as one might expect that  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t}^* - \widehat{\theta}_{i,t})$  has the same limiting distribution as  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger)$ . For notational simplicity, let  $h_{i,t} = \nabla_{\theta_i} q_i(y_t, Z^{t-1}, \theta_i^\dagger)$  and  $h_{i,t}^* = \nabla_{\theta_i} q_i(y_t^*, Z^{*,t-1}, \theta_i^\dagger)$ . Via a mean value expansion

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(1988, pp. 33).

around  $\theta^\dagger$ , using a similar argument as in Lemma A5 in West (1996), we have

$$\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger) = B_i^\dagger \frac{a_{R,0}}{\sqrt{P}} \sum_{t=s}^R h_{i,t} + B_i^\dagger \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} h_{i,R+j} + o_P(1), \quad (4)$$

where  $B_i^\dagger = \left( E \left( -\nabla_{\theta_i}^2 q_i(y_t, Z^{t-1}, \theta_i^\dagger) \right) \right)^{-1}$ . Also,

$$\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t}^* - \widehat{\theta}_{i,t}) = B_i^\dagger \frac{a_{R,0}}{\sqrt{P}} \sum_{t=s}^R (h_{i,t}^* - h_{i,t}) + B_i^\dagger \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} (h_{i,R+j}^* - h_{i,R+j}) + o_P^*(1), \quad \text{Pr}-P. \quad (5)$$

Now, the first term on the RHS of (5) has the same limiting distribution as the first term on the RHS of (4), conditional on sample. However, the second term on the RHS of (5) does not have the same limiting distribution as the second term on the RHS of (4), conditional on sample. The reason for this is that, up to a term of order  $O_P^*(l/\sqrt{P})$ ,  $E^* \left( \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} h_{i,R+j}^* \right) = \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \frac{1}{P} \sum_{j=1}^{P-1} h_{i,R+j} \neq \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} h_{i,R+j}$ . Now, rewrite (5) as,

$$\begin{aligned} \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t}^* - \widehat{\theta}_{i,t}) &= \left[ B_i^\dagger \frac{a_{R,0}}{\sqrt{P}} \sum_{t=s}^R (h_{i,t}^* - h_{i,t}) + B_i^\dagger \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \left( h_{i,R+j}^* - \frac{1}{P} \sum_{j=1}^{P-1} h_{i,j} \right) \right] \\ &\quad - B_i^\dagger \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \left( h_{i,R+j} - \frac{1}{P} \sum_{j=1}^{P-1} h_{i,j} \right) + o_P^*(1), \quad \text{Pr}-P. \end{aligned} \quad (6)$$

As shown in the proof of the theorem, the term in square brackets in (6) mimics the limiting distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^*)$ , conditional on sample. Also, the difference between the second term on the RHS of (3) and  $B_i^\dagger \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \left( h_{i,R+j} - \frac{1}{P} \sum_{j=1}^{P-1} h_{i,j} \right)$ , vanishes asymptotically. Therefore, the adjustment term completely offsets the second term on the RHS of (6), as  $R, P$  go to infinity. In the following sections we outline three applications in which valid bootstrap critical values can be constructed using the above approach for mimicing the limiting distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger)$ .

### 3 White's Reality Check for Data Snooping

In this section, we extend the White (2000) reality check to the case in which the effect of parameter estimation error does not vanish asymptotically. In particular, we show that the block bootstrap for recursive  $m$ -estimators properly mimics the contribution of parameter estimation error to the covariance kernel of the limiting distribution of the original reality check test. Although we focus

our attention in this paper on the recursive PEE bootstrap, which is based on resampling blocks of deterministic length, we conjecture that the same approach can be used to extend the stationary bootstrap employed by White (2000) to the case of nonvanishing parameter estimation error.

Let the generic forecast error be  $u_{i,t+1} = y_{t+1} - \kappa_i(Z^t, \theta_i^\dagger)$ , and let  $\widehat{u}_{i,t+1} = y_{t+1} - \kappa_i(Z^t, \widehat{\theta}_{i,t})$ , where  $\kappa_i(Z^t, \widehat{\theta}_{i,t})$  is the conditional mean function under model  $i$ . As above, we assume that the set of regressors may vary across different models, so that  $Z^t$  is meant to denote the collection of all potential regressors. Following White (2000), define the statistic

$$S_P = \max_{k=2,\dots,n} S_P(1,k),$$

where

$$S_P(1,k) = \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (g(\widehat{u}_{1,t+1}) - g(\widehat{u}_{k,t+1})) , \quad k = 2, \dots, n,$$

with  $g$  a given loss function. Recall that in this test, parameter estimation error need not be accounted for in the covariance kernel of the limiting distribution unless  $g \neq q_i$  for some  $i$ . This follows upon examination of the results of both West (1996) and White (2000). In particular, in West (1996), the parameter estimation error components that enter into the covariance kernel of the limiting distribution of his test statistic are zero whenever the same loss function is used for both predictive evaluation and in-sample estimation. The same argument holds for the reality check test. This means that as long as  $g = q_i \forall i$ , the White test can be applied regardless of the rate of growth of  $P$  and  $R$ . When we write the covariance kernel of the limiting distribution of the statistic (see below), however, we include terms capturing the contribution of parameter estimation error, thus implicitly assuming that  $g \neq q_i$  for some  $i$ . In practice, one reason why we allow for cases where  $g \neq q_i$  is that least squares is sometimes better behaved in finite samples and/or easier to implement than more generic  $m$ -estimators, so that practitioners sometimes use least squares for estimation and more complicated (possibly asymmetric) loss functions for predictive evaluation.<sup>8</sup> Of course, there are also applications for which parameter estimation error does not vanish, even if the same loss function is used for parameter estimation and predictive evaluation. One such

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<sup>8</sup>Consider linex loss, where  $g(u) = e^{au} - au - 1$ , so that for  $a > 0$  ( $a < 0$ ) positive (negative) errors are more (less) costly than negative (positive) errors. Here, the errors are exponentiated, so that in this particular case, laws of large numbers and central limit theorems may require a large number of observations before providing satisfactory approximations. This feature of linex loss is illustrated in the Monte Carlo findings of Corradi and Swanson (2002). (Linex loss is studied in Zellner (1993), Christoffersen and Diebold (1996, 1997) and Granger (1999), for example.)

application is discussed in the next section.

For a given loss function, the reality check tests the null hypothesis that a benchmark model (defined as model 1) performs equal to or better than all competitor models (i.e. models 2,...,n). The alternative is that at least one competitor performs better than the benchmark. Formally, the hypotheses are:

$$H_0 : \max_{k=2,\dots,n} E(g(u_{1,t+1}) - g(u_{k,t+1})) \leq 0$$

and

$$H_A : \max_{k=2,\dots,n} E(g(u_{1,t+1}) - g(u_{k,t+1})) > 0.$$

In order to derive the limiting distribution of  $S_P$  we require the following additional assumption.

**Assumption A4:** (i)  $\kappa_i$  is twice continuously differentiable on the interior of  $\Theta_i$  and the elements of  $\nabla_{\theta_i} \kappa_i(Z^t, \theta_i)$  and  $\nabla_{\theta_i}^2 \kappa_i(Z^t, \theta_i)$  are  $p$ -dominated on  $\Theta_i$ , for  $i = 2, \dots, n$ , with  $p > 2(2 + \psi)$ , where  $\psi$  is the same positive constant as that defined in Assumption A1; (ii)  $g$  is positive valued, twice continuously differentiable on  $\Theta_i$ , and  $g, g'$  and  $g''$  are  $p$ -dominated on  $\Theta_i$  with  $p$  defined as in (i); and (iii) let  $c_{kk} = \lim_{T \rightarrow \infty} \text{Var} \left( \frac{1}{\sqrt{T}} \sum_{t=s}^T (g(u_{1,t+1}) - g(u_{k,t+1})) \right)$ ,  $k = 2, \dots, n$ , define analogous covariance terms,  $c_{jk}$ ,  $j, k = 2, \dots, n$ , and assume that  $[c_{jk}]$  is positive semi-definite.

Assumptions A4(i)-(ii) are standard smoothness and domination conditions imposed on the conditional mean functions of the models. Assumption A4(iii) is standard in the literature that uses DM type tests (see e.g. West (1996) and White (2000)), and states that at least one of the competing models has to be nonnested with (and not nesting) the benchmark.

**Proposition 2:** Let assumptions A1-A4 hold, then

$$\max_{k=2,\dots,n} \left( S_P(1, k) - \sqrt{P} E(g(u_{1,t+1}) - g(u_{k,t+1})) \right) \xrightarrow{d} \max_{k=2,\dots,n} S(1, k),$$

where  $S = (S(1, 2), \dots, S(1, n))$  is a zero mean Gaussian process with covariance kernel given by  $V$ , with  $V$  a  $n \times n$  matrix with  $i, i$  element

$$v_{i,i} = S_{g_i g_i} + 2\Pi\mu'_1 B_1^\dagger C_{11} B_1^\dagger \mu_1 + 2\Pi\mu'_i B_i^\dagger C_{ii} B_i^\dagger \mu_i - 4\Pi\mu'_1 B_1^\dagger C_{1i} B_i^\dagger \mu_i + 2\Pi S_{g_i q_1} B_1^\dagger \mu_1 - 2\Pi S_{g_i q_i} B_i^\dagger \mu_i,$$

where  $S_{g_i g_i} = \sum_{\tau=-\infty}^{\infty} E((g(u_{1,1}) - g(u_{i,1})) (g(u_{1,1+\tau}) - g(u_{i,1+\tau})))$ ,

$$C_{ii} = \sum_{\tau=-\infty}^{\infty} E \left( \left( \nabla_{\theta_i} q_i(y_{1+s}, Z^s, \theta_i^\dagger) \right) \left( \nabla_{\theta_i} q_i(y_{1+s+\tau}, Z^{s+\tau}, \theta_i^\dagger) \right)' \right),$$

$$S_{g_i q_i} = \sum_{\tau=-\infty}^{\infty} E \left( (g(u_{1,1}) - g(u_{i,1})) \left( \nabla_{\theta_i} q_i(y_{1+s+\tau}, Z^{s+\tau}, \theta_i^\dagger) \right)' \right),$$

$$B_i^\dagger = \left( E \left( -\nabla_{\theta_i}^2 q_i(y_t, Z^{t-1}, \theta_i^\dagger) \right) \right)^{-1}, \quad \mu_i = E(\nabla_{\theta_i} g(u_{i,t+1})), \quad \text{and } \Pi = 1 - \pi^{-1} \ln(1 + \pi).$$

Just as in White (2000), note that under the null, the least favorable case arises when  $E(g(u_{1,t+1}) - g(u_{k,t+1})) = 0, \forall k$ . In this case, the distribution of  $S_P$  coincides with that of  $\max_{k=2,\dots,n} (S_P(1,k) - \sqrt{P}E(g(u_{1,t+1}) - g(u_{k,t+1})))$ , so that  $S_P$  has the above limiting distribution, which is a functional of a Gaussian process with a covariance kernel that reflects uncertainty due to parameter estimation error and dynamic mispecification. Additionally, when all competitor models are worse than the benchmark, the statistic diverges to minus infinity at rate  $\sqrt{P}$ . Finally, when only some competitor models are worse than the benchmark, the limiting distribution provides a conservative test, as  $S_P$  will always be smaller than  $\max_{k=2,\dots,n} (S_P(1,k) - \sqrt{P}E(g(u_{1,t+1}) - g(u_{k,t+1})))$ , asymptotically. Of course, when  $H_A$  holds, the statistic diverges to plus infinity at rate  $\sqrt{P}$ .

In a recent paper, Hansen (2001) explores the point made by White (2000) that the reality check test can have level going to zero at the same time that power goes to unity (i.e. that the test is conservative unless  $E(g(u_{1,t+1}) - g(u_{k,t+1})) = 0, \forall k$ ), and suggests a mean correction for  $S_P$  in order to address this feature of the test. Our version of the reality check has the same features and can also be modified using the method proposed by Hansen.<sup>9</sup>

Recall that the maximum of a Gaussian process is not Gaussian in general, so that standard critical values cannot be used to conduct inference on  $S_P$ . As pointed out by White (2000), one possibility in this case is to first estimate the covariance structure and then draw 1 realization from an  $(n - 1)$ -dimensional normal with covariance equal to the estimated covariance structure. From this realization, pick the maximum value over  $k = 2, \dots, n$ . Repeat this a large number of times, form an empirical distribution using the maximum values over  $k = 2, \dots, n$ , and obtain critical values in the usual way. A drawback to this approach is that we need to rely on an estimator of the covariance structure based on the available sample of observations, which in many cases may be small relative to the number of models being compared. Furthermore, whenever the

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<sup>9</sup>An alternative to the bootstrap critical values, may the construction of critical values based on subsampling (e.g. Politis, Romano and Wolf (1999), Ch.3). Heuristically, we construct  $T - 2b_T$  statistics using subsamples of length  $b_T$ , where  $b_T/T \rightarrow 0$ ; the empirical distribution of the statistics computed over the various subsamples, properly mimics the distribution of the statistic. Thus, it provides valid critical values even for the case of  $\max_{k=2,\dots,m} E(g(u_{1,t+1}) - g(u_{k,t+1})) = 0$ , but  $E(g(u_{1,t+1}) - g(u_{k,t+1})) < 0$  for some  $k$ . Needless to say, the problem is that unless the sample is very large, the empirical distribution of the subsampled statistics provides a poor approximation to the limiting distribution of the statistic. The subsampling approach has been followed by Linton, Maasoumi and Whang (2003), in the context of testing for stochastic dominance.

forecasting errors are not martingale difference sequences (as in our context), heteroskedasticity and autocorrelation consistent covariance matrices should be estimated, and thus a lag truncation parameter must be chosen. Another approach which avoids these problems involves using the stationary bootstrap of Politis and Romano (1994). This is the approach used by White (2000), and in general, bootstrap procedures have been shown to perform well in a variety of finite sample contexts (see e.g. Diebold and Chen (1996)). Our approach is to apply the recursive PEE bootstrap outlined above.

Hereafter, let  $u_{i,t+1}^* = y_{t+1}^* - \kappa_i(Z^{*,t}, \theta_i^\dagger)$  and  $\widehat{u}_{i,t+1}^* = y_{t+1}^* - \kappa_i(Z^{*,t}, \widehat{\theta}_{i,t}^*)$ , where

$$\widehat{\theta}_{i,t}^* = \arg \min_{\theta_i \in \Theta_i} \frac{1}{t} \sum_{j=s}^t q_i(y_j^*, Z^{*,j-1}, \theta_i), \quad R \leq t \leq T-1, \quad i = 1, \dots, n, \quad (7)$$

and let  $\widetilde{u}_{i,t+1} = y_{t+1} - \kappa_i(Z^{*,t}, \widehat{\theta}_{i,T})$ . Recall that  $y_t^*, Z^{*,t}$  has been obtained via the resampling procedure described in Section 2. Define

$$S_P^* = \max_{k=2, \dots, n} S_P^*(1, k),$$

where

$$\begin{aligned} S_P^*(1, k) &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} ((g(\widehat{u}_{1,t+1}^*) - g(\widehat{u}_{1,t+1})) - (g(\widehat{u}_{k,t+1}^*) - g(\widehat{u}_{k,t+1}))) \\ &\quad + \frac{1}{T} \sum_{t=s}^T \nabla_{\theta_1} g(\widehat{u}_{1,t+1}) \left( -\frac{1}{T} \sum_{t=s}^T \nabla_{\theta_1}^2 q_1(y_t, Z^{t-1}, \widehat{\theta}_{1,T}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla_{\theta_1} q_1(y_{R+i}, Z^{R+i-1}, \widehat{\theta}_{1,T}) - \frac{1}{P} \sum_{i=1}^{P-1} \nabla_{\theta_1} q_1(y_{R+i}, Z^{R+i-1}, \widehat{\theta}_{1,T}) \right) \\ &\quad - \frac{1}{T} \sum_{t=s}^T \nabla_{\theta_k} g(\widehat{u}_{k,t+1}) \left( -\frac{1}{T} \sum_{t=s}^T \nabla_{\theta_k}^2 q_k(y_t, Z^{t-1}, \widehat{\theta}_{k,T}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla_{\theta_k} q_k(y_{R+i}, Z^{R+i-1}, \widehat{\theta}_{k,T}) - \frac{1}{P} \sum_{i=1}^{P-1} \nabla_{\theta_k} q_k(y_{R+i}, Z^{R+i-1}, \widehat{\theta}_{k,T}) \right) \end{aligned} \quad (8)$$

where  $a_{R,i} = \frac{1}{R+i} + \frac{1}{R+i+1} + \dots + \frac{1}{R+P-1}$ ,  $\nabla_{\theta_1} g(\widehat{u}_{1,t+1}) = g'(y_{t+1} - \kappa_1(Z^t, \widehat{\theta}_{1,T})) \nabla_{\theta_1} \kappa_1(Z^t, \widehat{\theta}_{1,T})$ ,  $\nabla_{\theta_k} g(\widetilde{u}_{k,t+1})$  is defined analogously, and with the notation  $g'$  and  $g''$  we mean the first and second derivatives of  $g$  with respect to its argument.

**Proposition 3:** Let assumptions A1-A4 hold. Also, assume that as  $P, R \rightarrow \infty$ ,  $l_1, l_2 \rightarrow \infty$ , and that  $\frac{l_2}{P^{1/4}} \rightarrow 0$  and  $\frac{l_1}{R^{1/4}} \rightarrow 0$ . Then, as  $P$  and  $R \rightarrow \infty$ ,

$$P \left( \omega : \sup_{v \in \Re} \left| P_{R,P}^* \left( \max_{k=2, \dots, n} S_P^*(1, k) \leq v \right) - P \left( \max_{k=2, \dots, n} S_P^*(1, k) \leq v \right) \right| > \varepsilon \right) \rightarrow 0,$$

and

$$S_P^\mu(1, k) = S_P(1, k) - \sqrt{P}E(g(u_{1,t+1}) - g(u_{k,t+1})),$$

The above result suggests proceeding in the following manner. For any bootstrap replication, compute the bootstrap statistic,  $S_P^*$ . Perform  $B$  bootstrap replications ( $B$  large) and compute the quantiles of the empirical distribution of the  $B$  bootstrap statistics. Reject  $H_0$ , if  $S_P$  is greater than the  $(1 - \alpha)$ th-percentile. Otherwise, do not reject. Now, for all samples except a set with probability measure approaching zero,  $S_P$  has the same limiting distribution as the corresponding bootstrapped statistic when  $E(g(u_{1,t+1}) - g(u_{k,t+1})) = 0 \forall k$ , ensuring asymptotic size equal to  $\alpha$ . On the other hand, when one or more competitor models are strictly dominated by the benchmark, the rule provides a test with asymptotic size between 0 and  $\alpha$ . Under the alternative,  $S_P$  diverges to (plus) infinity, while the corresponding bootstrap statistic has a well defined limiting distribution, ensuring unit asymptotic power.

## 4 Out-of-Sample Integrated Conditional Moment Tests

Corradi and Swanson (CS: 2002) draw on both the consistent specification and predictive ability testing literatures and propose a test for predictive accuracy which is consistent against generic nonlinear alternatives, and which is designed for comparing nested models. The test is based on an out-of-sample version of the integrated conditional moment (ICM) test of Bierens (1982,1990) and Bierens and Ploberger (1997). One reason why using an ICM type test is more intuitively appealing in our context than, say, a Diebold and Mariano (DM: 1995), West (1996) or reality check type test, is that in addition to comparing nested models, we also use the same loss function for estimation and for predictive evaluation. To explain the problem that can arise when not using an ICM type test, consider the DM test. Note first that the difference between functionals of the “true” forecast errors is identically zero when the null model is nested. Additionally, as discussed above, parameter estimation error vanishes whenever the same loss function is used both in- and out-of-sample, regardless of the value of  $\pi$ . Thus, when the same loss function is used throughout, and the null model is nested, the numerator of DM type tests vanishes in probability under the null.<sup>10</sup> It turns out that the limiting distribution of the ICM type test statistic proposed by CS

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<sup>10</sup>Note that McCracken (1999) shows that a particular version of the DM test in which a null model is compared against a fixed alternative and the numerator is multiplied by  $\sqrt{P}$  has a nonstandard limiting distribution. However,

is a functional of a Gaussian process with a covariance kernel that reflects both the time series structure of the data as well as the contribution of parameter estimation error. As a consequence, critical values are data dependent and cannot be directly tabulated. CS establish the validity of the conditional p-value method for constructing critical values in this context, thus extending earlier work by Hansen (1996) and Inoue (2001). However, the conditional p-value approach suffers from the fact that under the alternative, the simulated statistics diverge (at rate as high as  $\sqrt{\tilde{l}}$ ), conditional on the sample and for all samples except a set of measure zero, where  $\tilde{l}$  plays a role analogous to  $l$  in the block bootstrap. As this feature may lead to reduced power in finite samples, we establish in this application that the recursive PEE bootstrap can also be used.

Summarizing the testing approach used in this application, assume that the objective is to test whether there exists any unknown alternative model that has better predictive accuracy than a given benchmark model, for a given loss function. A typical example is the case in which the benchmark model is a simple autoregressive model and we want to check whether a more accurate forecasting model can be constructed by including possibly unknown (non)linear functions of the past of the process or of the past of some other process(es).<sup>11</sup> Although this is the case that we focus on, the benchmark model can in general be any (non)linear model. As mentioned above, one important feature of this application is that the same loss function is used for in-sample estimation and out-of-sample prediction (see Granger (1993) and Weiss (1996)). In contrast to the previous application, however, this does not ensure that parameter estimation error vanishes asymptotically.

Let the benchmark model be:

$$y_t = \theta_{1,1}^\dagger + \theta_{1,2}^\dagger y_{t-1} + u_{1,t}, \quad (9)$$

where  $\theta_1^\dagger = (\theta_{1,1}^\dagger, \theta_{1,2}^\dagger)' = \arg \min_{\theta_1 \in \Theta_1} E(q_1(y_t - \theta_{1,1} - \theta_{1,2}y_{t-1}))$ ,  $\theta_1 = (\theta_{1,1}, \theta_{1,2})'$ ,  $y_t$  is a scalar,  $q_1 = g$ , as the same loss function is used both for in-sample estimation and out-of-sample predictive evaluation, and everything else is defined above. The generic alternative model is:

$$y_t = \theta_{2,1}^\dagger(\gamma) + \theta_{2,2}^\dagger(\gamma)y_{t-1} + \theta_{2,3}^\dagger(\gamma)w(Z^{t-1}, \gamma) + u_{2,t}(\gamma), \quad (10)$$

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his approach does not apply in the current context, as we consider generic alternatives, and as we allow for misspecification of all models being compared rather than assuming correct specification under the null. On the other hand, Giacomini (2002b) provides a test for conditional predictive ability valid for both nested and nonnested models. The key ingredient of her test is the fact that parameters are estimated using a fixed rolling window.

<sup>11</sup>For example, Swanson and White (1997) compare the predictive accuracy of various linear models against neural network models using both in-sample and out-of-sample model selection criteria.

where  $\theta_2^\dagger(\gamma) = (\theta_{2,1}^\dagger(\gamma), \theta_{2,2}^\dagger(\gamma), \theta_{2,3}^\dagger(\gamma))' = \arg \min_{\theta_2 \in \Theta_2} E(q_1(y_t - \theta_{2,1} - \theta_{2,2}y_{t-1} - \theta_{2,3}w(Z^{t-1}, \gamma)))$ ,  $\theta_2(\gamma) = (\theta_{2,1}(\gamma), \theta_{2,2}(\gamma), \theta_{2,3}(\gamma))'$ ,  $\theta_2 \in \Theta_2$ , where  $\Gamma$  is a compact subset of  $\Re^d$ , for some finite  $d$ . The alternative model is called “generic” because of the presence of  $w(Z^{t-1}, \gamma)$ , which is a generically comprehensive function, such as Bierens’ exponential, a logistic, or a cumulative distribution function (see e.g. Stinchcombe and White (1998) for a detailed explanation of generic comprehensiveness). One example has  $w(Z^{t-1}, \gamma) = \exp(\sum_{i=1}^{s_2} \gamma_i \Phi(X_{t-i}))$ , where  $\Phi$  is a measurable one to one mapping from  $\Re$  to a bounded subset of  $\Re$ , so that here  $Z^t = (X_t, \dots, X_{t-s_2+1})$ , and we are thus testing for nonlinear Granger causality. The hypotheses of interest are:

$$H_0 : E(g(u_{1,t+1}) - g(u_{2,t+1}(\gamma))) = 0 \text{ versus } H_A : E(g(u_{1,t+1}) - g(u_{2,t+1}(\gamma))) > 0. \quad (11)$$

Clearly, the reference model is nested within the alternative model, and given the definitions of  $\theta_1^\dagger$  and  $\theta_2^\dagger(\gamma)$ , the null model can never outperform the alternative. For this reason,  $H_0$  corresponds to equal predictive accuracy, while  $H_A$  corresponds to the case where the alternative model outperforms the reference model, as long as the errors above are loss function specific forecast errors. It follows that  $H_0$  and  $H_A$  can be restated as:

$$H_0 : \theta_{2,3}^\dagger(\gamma) = 0 \text{ versus } H_A : \theta_{2,3}^\dagger(\gamma) \neq 0,$$

for  $\forall \gamma \in \Gamma$ , except for a subset with zero Lebesgue measure. Now, given the definition of  $\theta_2^\dagger(\gamma)$ , note that

$$E \left( g'(y_{t+1} - \theta_{2,1}^\dagger(\gamma) - \theta_{2,2}^\dagger(\gamma)y_t - \theta_{2,3}^\dagger(\gamma)w(Z^t, \gamma)) \times \begin{pmatrix} -1 \\ -y_t \\ -w(Z^t, \gamma) \end{pmatrix} \right) = 0,$$

where  $g'$  is defined as above. Thus, under  $H_0$  we have that  $\theta_{2,3}^\dagger(\gamma) = 0$ ,  $\theta_{2,1}^\dagger(\gamma) = \theta_{1,1}^\dagger$ ,  $\theta_{2,2}^\dagger(\gamma) = \theta_{1,2}^\dagger$ , and  $E(g'(u_{1,t+1})w(Z^t, \gamma)) = 0$ . Thus, we can once again restate  $H_0$  and  $H_A$  as:

$$H_0 : E(g'(u_{1,t+1})w(Z^t, \gamma)) = 0 \text{ versus } H_A : E(g'(u_{1,t+1})w(Z^t, \gamma)) \neq 0, \quad (12)$$

for  $\forall \gamma \in \Gamma$ , except for a subset with zero Lebesgue measure. Finally, define  $\widehat{u}_{1,t+1} = y_{t+1} - (1 \ y_t) \widehat{\theta}_{1,t}$ . Following CS, the test statistic is:

$$M_P = \int_{\Gamma} m_P(\gamma)^2 \phi(\gamma) d\gamma, \quad (13)$$

and

$$m_P(\gamma) = \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} g'(\widehat{u}_{1,t+1})w(Z^t, \gamma), \quad (14)$$

where  $\int_{\Gamma} \phi(\gamma) d\gamma = 1$ ,  $\phi(\gamma) \geq 0$ , and  $\phi(\gamma)$  is absolutely continuous with respect to Lebesgue measure.

In the sequel, we need:

**Assumption A5:** (i)  $w$  is a bounded, twice continuously differentiable function on the interior of  $\Gamma$  and  $\nabla_{\gamma} w(Z^t, \gamma)$  is bounded uniformly in  $\Gamma$ ; and (ii)  $\nabla_{\gamma} \nabla_{\theta_1} q'_{1,t}(\theta_1)w(Z^{t-1}, \gamma)$  is continuous on  $\Theta_1 \times \Gamma$ , where  $q'_{1,t}(\theta_1) = q'_1(y_t - \theta_{1,1} - \theta_{1,2}y_{t-1})$ ,  $\Gamma$  a compact subset of  $R^d$  and is  $2r$ -dominated uniformly in  $\Theta_1 \times \Gamma$ , with  $r \geq 2(2 + \psi)$  where  $\psi$  is the same positive constant as that defined in Assumption A1.

Assumption A5 requires the function  $w$  to be bounded and twice continuously differentiable; such a requirement is satisfied by logistic or exponential functions, for example.

**Proposition 4:** Let assumptions A1-A3 and A5 hold. Then, the following results hold: (i) Under  $H_0$ ,

$$M_P = \int_{\Gamma} m_P(\gamma)^2 \phi(\gamma) d\gamma \xrightarrow{d} \int_{\Gamma} Z(\gamma)^2 \phi(\gamma) d\gamma,$$

where  $m_P(\gamma)$  is defined in equation (14) and  $Z$  is a Gaussian process with covariance kernel given by:

$$\begin{aligned} K(\gamma_1, \gamma_2) &= S_{gg}(\gamma_1, \gamma_2) + 2\Pi \mu'_{\gamma_1} B^\dagger S_{hh} B^\dagger \mu_{\gamma_2} + \Pi B^\dagger \mu'_{\gamma_1} S_{gh}(\gamma_2) \\ &\quad + \Pi \mu'_{\gamma_2} B^\dagger S_{gh}(\gamma_1), \end{aligned}$$

with  $\mu_{\gamma_1} = E(\nabla_{\theta_1}(g'_{t+1}(u_{1,t+1})w(Z^t, \gamma_1)))$ ,  $B^\dagger = (-E(\nabla_{\theta_1}^2 q_1(u_{1,t})))^{-1}$ ,

$S_{gg}(\gamma_1, \gamma_2) = \sum_{j=-\infty}^{\infty} E(g'(u_{1,s+1})w(Z^s, \gamma_1)g'(u_{1,s+j+1})w(Z^{s+j}, \gamma_2))$ ,

$S_{hh} = \sum_{j=-\infty}^{\infty} E(\nabla_{\theta_1} q_1(u_{1,s}) \nabla_{\theta_1} q_1(u_{1,s+j}))'$ ,

$S_{gh}(\gamma_1) = \sum_{j=-\infty}^{\infty} E(g'(u_{1,s+1})w(Z^s, \gamma_1) \nabla_{\theta_1} q_1(u_{1,s+j}))'$ , and  $\gamma$ ,  $\gamma_1$ , and  $\gamma_2$  are generic elements of  $\Gamma$ .

(ii) Under  $H_A$ , for  $\varepsilon > 0$ ,  $\lim_{P \rightarrow \infty} \Pr\left(\frac{1}{P} \int_{\Gamma} m_P(\gamma)^2 \phi(\gamma) d\gamma > \varepsilon\right) = 1$ .

As in the previous application, the limiting distribution under  $H_0$  is a Gaussian process with a covariance kernel that reflects both the dependence structure of the data and, for  $\pi > 0$ , the effect of parameter estimation error. Hence, critical values are data dependent and cannot be tabulated.

In order to implement this statistic using the recursive PEE bootstrap, define<sup>12</sup>

$$\widehat{\theta}_{1,t}^* = (\widehat{\theta}_{1,1,t}^*, \widehat{\theta}_{1,2,t}^*)' = \arg \min_{\theta_1 \in \Theta_1} \frac{1}{t} \sum_{j=2}^t q_1(y_j^* - \theta_{1,1} - \theta_{1,2}y_{j-1}^*). \quad (15)$$

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<sup>12</sup>Recall that  $y_t^*, Z^{*,t}$  has been obtained via the resampling procedure described in Section 2

Also, define  $\widehat{u}_{1,t+1}^* = y_{t+1}^* - (1 \ y_t^*) \widehat{\theta}_{1,t}^*$ . The bootstrap test statistic is:

$$M_P^* = \int_{\Gamma} m_P^*(\gamma)^2 \phi(\gamma) d\gamma,$$

where, recalling that  $g = q_1$ ,

$$\begin{aligned} m_P^*(\gamma) &= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(\widehat{u}_{1,t+1}^*) w(Z^{*,t}, \gamma) - g'(\widehat{u}_{1,t+1}) w(Z^t, \gamma)) \\ &\quad + \frac{1}{T} \sum_{t=2}^T \nabla_{\theta_1} q_1(y_t - \widehat{\theta}_{1,1,T} - \widehat{\theta}_{1,2,T} y_{t-1}) w(Z^t, \gamma) \left( -\frac{1}{T} \sum_{t=2}^T \nabla_{\theta_1}^2 q_1(y_t - \widehat{\theta}_{1,1,T} - \widehat{\theta}_{1,2,T} y_{t-1}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla_{\theta_1} q_1(y_{R+i} - \widehat{\theta}_{1,1,T} - \widehat{\theta}_{1,2,T} y_{R+i-1}) \right. \\ &\quad \left. - \frac{1}{P} \sum_{i=1}^{P-1} \nabla_{\theta_1} q_1(y_{R+i} - \widehat{\theta}_{1,1,T} - \widehat{\theta}_{1,2,T} y_{R+i-1}) \right) \end{aligned}$$

**Proposition 5:** Let assumptions A1-A3 and A5 hold. Also, assume that as  $P, R \rightarrow \infty$ ,  $l_1, l_2 \rightarrow \infty$ , and that  $\frac{l_2}{P^{1/4}} \rightarrow 0$  and  $\frac{l_1}{R^{1/4}} \rightarrow 0$ . Then, as  $P$  and  $R \rightarrow \infty$ ,

$$P \left( \omega : \sup_{v \in \Re} \left| P_{R,P}^* \left( \int_{\Gamma} m_P^*(\gamma)^2 \phi(\gamma) d\gamma \leq v \right) - P \left( \int_{\Gamma} m_P^*(\gamma)^2 \phi(\gamma) d\gamma \leq v \right) \right| > \varepsilon \right) \rightarrow 0,$$

where  $a_{R,i} = \frac{1}{R+i} + \frac{1}{R+i+1} + \dots + \frac{1}{R+P-1}$  and  $m_P^*(\gamma) = m_P(\gamma) - \sqrt{P} E(g'(u_{1,t+1}) w(Z^t, \gamma))$ .

The above result suggests proceeding the same way as in the first application. For any bootstrap replication, compute the bootstrap statistic,  $M_P^*$ . Perform  $B$  bootstrap replications ( $B$  large) and compute the percentiles of the empirical distribution of the  $B$  bootstrap statistics. Reject  $H_0$  if  $M_P$  is greater than the  $(1 - \alpha)$ th-percentile. Otherwise, do not reject. Now, for all samples except a set with probability measure approaching zero,  $M_P$  has the same limiting distribution as the corresponding bootstrap statistic under  $H_0$ , thus ensuring asymptotic size equal to  $\alpha$ . Under the alternative,  $M_P$  diverges to (plus) infinity, while the corresponding bootstrap statistic has a well defined limiting distribution, ensuring unit asymptotic power.

## 5 Predictive Density Evaluation

In several instances, such as financial risk management, for example, one is interested in predicting either a particular confidence interval (e.g. Value at Risk) or the entire conditional distribution of a variable of interest. Hence, over the last few years, a new strand of literature addressing the issue of

predictive density evaluation has arisen (see e.g. Diebold, Gunther and Tay (DGT: 1998), Christoffersen (1998), Bai (2001), Diebold, Hahn and Tay (1999), Hong (2001) and Christoffersen, Hahn and Inoue (2001)). The literature on the evaluation of predictive densities is largely concerned with testing the null of correct dynamic specification of an individual conditional distribution model. On the other hand, in the literature on the evaluation of point forecast models it is acknowledged that all models in a group that is being evaluated may be misspecified (see e.g. White (2000) and Corradi and Swanson (2002)). In this application, we draw on elements of these two literatures in order to provide a test for choosing among a group of misspecified out-of-sample predictive density models. Reiterating our above point, the focus of most of the papers cited above is that the density associated with the true conditional distribution is clearly the best predictive density. Therefore, evaluation of predictive densities is usually performed via tests for the correct (dynamic) specification of the conditional distribution. Along these lines, by making use of the probability integral transform, DGT suggest a simple and effective means by which predictive densities can be evaluated. Using the DGT terminology, if  $p_t(y_t|\Omega_{t-1})$  is the “true” conditional distribution of  $y_t|\Omega_{t-1}$ , then  $p_t(y_t|\Omega_{t-1})$  is an identically and independently distributed uniform random variable on  $[0, 1]$ ; so that the difference between an empirical version of  $p_t(y_t|\Omega_{t-1})$  constructed using estimated parameters and the 45 degree line can be used as measure of goodness of fit.<sup>13</sup> A feature common to the papers cited above is that the null hypothesis is that of (dynamic) correct specification. Our approach differs from these as we do not assume that any of the competing models (including the benchmark) are correctly specified. Thus, we posit that *all* models should be viewed as approximations of some true unknown underlying data generating process.

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<sup>13</sup>Using the same approach, Bai (2001) proposes a Kolmogorov type test based on the comparison of  $p_t(y_t|\Omega_{t-1}, \hat{\theta}_T)$  with the CDF of a uniform on  $[0, 1]$ . As a consequence of using estimated parameters, the limiting distribution of his test reflects the contribution of parameter estimation error and is not nuisance parameter free. To overcome this problem, Bai (2001) uses a novel device based on a martingalization argument to construct a modified Kolmogorov test which has a nuisance parameter free limiting distribution. His test has power against violations of uniformity but not against violations of independence. Hong (2001) proposes an interesting test, based on the generalized spectrum, which has power against both uniformity and independence violations, for the case in which the contribution of parameter estimation error vanishes asymptotically. If the null is rejected, Hong (2001) also proposes a test for uniformity robust to non independence, which is based on the comparison between a kernel density estimator and the uniform density. Diebold, Hahn and Tay (1999) propose a nonparametric correction for improving the density forecast when the uniform (but not the independence) assumption is violated.

In this application, our objective is to “choose” a conditional distribution model that provides the most accurate out-of-sample approximation of the true conditional distribution, given multiple predictive densities, and allowing for misspecification under both the null and the alternative hypotheses. At first blush, it may seem that the probability integral transform approach will yield a solution to this problem. However, it is difficult to map the degree of deviation from uniformity and independence into a meaningful measure of the degree of misspecification of a model, so that the probability integral transform method does not easily extend to the evaluation of multiple misspecified models.

Another strategy that yields tests of the null of correct specification that are equally as useful as those discussed above is the conditional Kolmogorov test approach of Andrews (1997), which is based on a direct comparison of empirical joint distributions with the product of parametric conditional and nonparametric marginal distributions. Corradi and Swanson (2003b) extend Andrews (1997) in order to allow for the in-sample comparison of multiple misspecified models. Our focus in this application is to extend their results to out-of-sample predictive density evaluation via use of the block recursive bootstrap. More specifically, and using the notation outlined above, our objective is to form parametric conditional distributions for a scalar random variable,  $y_{t+1}$ , given  $Z^t$ , and to select among these. Define the group of conditional distribution models from which we want to make a selection as  $F_1(u|Z^t, \theta_1^\dagger), \dots, F_n(u|Z^t, \theta_n^\dagger)$ , and define the true conditional distribution as  $F_0(u|Z^t, \theta_0) = \Pr(y_{t+1} \leq u|Z^t)$ . Hereafter, assume that  $q_i(y_t, Z^{t-1}, \theta_i) = -\ln f_i(y_t|Z^{t-1}, \theta_i)$ , where  $f_i(\cdot|\cdot, \theta_i)$  is the conditional density associated with  $F_i$ ,  $i = 1, \dots, n$ , so that in this application,  $\theta_i^\dagger$  is the probability limit of a quasi maximum likelihood estimator (QMLE). If model  $i$  is correctly specified, then  $\theta_i^\dagger = \theta_0$ . In the sequel,  $F_1(\cdot|\cdot, \theta_1^\dagger)$  is taken as the benchmark model, and the objective is to test whether some competitor model can provide a more accurate approximation of  $F_0(\cdot|\cdot, \theta_0)$  than the benchmark.<sup>14</sup>

We begin by assuming that accuracy is measured using a distributional analog of mean square error. More precisely, the squared (approximation) error associated with model  $i$ ,  $i = 1, \dots, n$ , is measured in terms of the average over  $U$  of  $E \left( \left( F_i(u|Z^t, \theta_i^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 \right)$ , where  $u \in U$ , and

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<sup>14</sup>In this test, the competing models are known. This is different than the probability integral transform approach where only the null model is explicitly stated.

$U$  is a possibly unbounded set on the real line. The hypotheses of interest are:

$$H_0 : \max_{k=2,\dots,n} \int_U E \left( \left( F_1(u|Z^t, \theta_1^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 - \left( F_k(u|Z^t, \theta_k^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 \right) \phi(u) du \leq 0 \quad (16)$$

versus

$$H_A : \max_{k=2,\dots,n} \int_U E \left( \left( F_1(u|Z^t, \theta_1^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 - \left( F_k(u|Z^t, \theta_k^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 \right) \phi(u) du > 0, \quad (17)$$

where  $\phi(u) \geq 0$  and  $\int_U \phi(u) = 1$ ,  $u \in U \subset \Re$ ,  $U$  possibly unbounded. Note that for a given  $u$ , we compare conditional distributions in terms of their (mean square) distance from the true distribution. We then average over  $U$ .<sup>15,16</sup> The statistic is:

$$Z_P = \max_{k=2,\dots,n} \int_U Z_{P,u}(1,k) \phi(u) du, \quad (18)$$

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<sup>15</sup>Kitamura (2002) proposes a comparison among misspecified conditional models, subject to given moment restrictions, in terms of the conditional entropy. Giacomini (2002a) proposed an evaluation method for predictive densities based on a weighted likelihood ratio, accuracy is measured in terms of the Kullback Leibler Information Criterion (KLIC).

<sup>16</sup>If interest focuses on predictive conditional confidence intervals (see e.g. Christoffersen (1998)), so that the objective is to “approximate”  $\Pr(\underline{u} \leq y_{t+1} \leq \bar{u}|Z^t)$ , then the null and alternative hypotheses can be stated as:

$$\begin{aligned} H'_0 : \max_{k=2,\dots,n} E & \left( \left( F_1(\bar{u}|Z^t, \theta_1^\dagger) - F_1(\underline{u}|Z^t, \theta_1^\dagger) \right)^2 - \left( F_0(\bar{u}|Z^t, \theta_0) - F_0(\underline{u}|Z^t, \theta_0) \right)^2 \right. \\ & \left. - \left( \left( F_k(\bar{u}|Z^t, \theta_k^\dagger) - F_k(\underline{u}|Z^t, \theta_k^\dagger) \right)^2 - \left( F_0(\bar{u}|Z^t, \theta_0) - F_0(\underline{u}|Z^t, \theta_0) \right)^2 \right) \leq 0. \end{aligned}$$

versus

$$\begin{aligned} H'_A : \max_{k=2,\dots,n} E & \left( \left( F_1(\bar{u}|Z^t, \theta_1^\dagger) - F_1(\underline{u}|Z^t, \theta_1^\dagger) \right)^2 - \left( F_0(\bar{u}|Z^t, \theta_0) - F_0(\underline{u}|Z^t, \theta_0) \right)^2 \right. \\ & \left. - \left( \left( F_k(\bar{u}|Z^t, \theta_k^\dagger) - F_k(\underline{u}|Z^t, \theta_k^\dagger) \right)^2 - \left( F_0(\bar{u}|Z^t, \theta_0) - F_0(\underline{u}|Z^t, \theta_0) \right)^2 \right) > 0. \end{aligned}$$

Analogously, if interest focuses on testing the null of equal accuracy of only two predictive conditional distribution models, say  $F_1$  and  $F_k$ , we can simply state the hypotheses as:

$$H''_0 : \int_U E \left( \left( F_1(u|Z^t, \theta_1^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 - \left( F_k(u|Z^t, \theta_k^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 \right) \phi(u) du = 0$$

versus

$$H''_A : \int_U E \left( \left( F_1(u|Z^t, \theta_1^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 - \left( F_k(u|Z^t, \theta_k^\dagger) - F_0(u|Z^t, \theta_0) \right)^2 \right) \phi(u) du \neq 0.$$

where

$$Z_{P,u}(1,k) = \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \hat{\theta}_{1,t}) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \hat{\theta}_{k,t}) \right)^2 \right). \quad (19)$$

In this application we find more "natural" to estimate each model by QMLE, so that in terms of equation (1)  $q_i = -\ln f_i$ , where  $f_i$  is the conditional density associated with model  $i$ , and  $\hat{\theta}_{i,t}$  is defined as  $\hat{\theta}_{i,t} = \arg \max_{\theta_i \in \Theta_i} \frac{1}{t} \sum_{j=s}^t \ln f_i(y_j, Z^{j-1}, \theta_i)$ ,  $R \leq t \leq T-1$ ,  $i = 1, \dots, n$ .

In Corradi and Swanson (2003b) we show how the hypotheses above can be restated as

$$H_0 : \max_{k=2,\dots,n} \int_U (\mu_1^2(u) - \mu_k^2(u)) \phi(u) du \leq 0$$

versus

$$H_A : \max_{k=2,\dots,n} \int_U (\mu_1^2(u) - \mu_k^2(u)) \phi(u) du > 0,$$

where  $\mu_i^2(u) = E \left( \left( 1\{y_t \leq u\} - F_i(u|Z^t, \theta_i^\dagger) \right)^2 \right)$ . In the sequel, we require the following additional assumption.

**Assumption A6:** (i)  $F_i(u|Z^t, \theta_i)$  is continuously differentiable on the interior of  $\Theta_i$  and  $\nabla_{\theta_i} F_i(u|Z^t, \theta_i^\dagger)$  is  $2r$ -dominated on  $\Theta_i$ , uniformly in  $u$ ,  $r > 2$ ,  $i = 1, \dots, n$ ;<sup>17</sup> and (ii) let  $v_{kk}(u) = \text{plim}_{T \rightarrow \infty} Var \left( \frac{1}{\sqrt{T}} \sum_{t=s}^T \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) - \left( \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \theta_k^\dagger) \right)^2 - \mu_k^2(u) \right) \right)$ ,  $k = 2, \dots, n$ , define analogous covariance terms,  $v_{j,k}(u)$ ,  $j, k = 2, \dots, n$ , and assume that  $[v_{j,k}(u)]$  is positive semi-definite, uniformly in  $u$ .

Analogous to assumption A4, assumptions A6(i)-(ii) are standard smoothness and domination conditions imposed on the conditional distributions of the models, and assumption A6(iii) states that at least one of the competing models,  $F_2(\cdot|\cdot, \theta_1^\dagger), \dots, F_n(\cdot|\cdot, \theta_n^\dagger)$ , has to be nonnested with (and non nesting) the benchmark.

**Proposition 6:** Let assumptions A1-A3 and A6 hold.<sup>18</sup> Then:

$$\max_{k=2,\dots,n} \int_U \left( Z_{P,u}(1,k) - \sqrt{P} (\mu_1^2(u) - \mu_k^2(u)) \right) \phi_U(u) du \xrightarrow{d} \max_{k=2,\dots,n} \int_U Z_{1,k}(u) \phi_U(u) du,$$

where  $Z_{1,k}(u)$  is a zero mean Gaussian process with covariance  $C_k(u, u')$  equal to:

$$E \left( \sum_{j=-\infty}^{\infty} \left( \left( 1\{y_{s+1} \leq u\} - F_1(u|Z^s, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) \left( \left( 1\{y_{s+j+1} \leq u'\} - F_1(u'|Z^{s+j}, \theta_1^\dagger) \right)^2 - \mu_1^2(u') \right) \right)$$

<sup>17</sup>We require that for  $j = 1, \dots, p_i$ ,  $\left( E \left( \nabla_{\theta} F_i(u|Z^t, \theta_i^\dagger) \right) \right)_j \leq D_t(u)$ , with  $\sup_t \sup_{u \in \mathfrak{R}} E(D_t(u)^{2r}) < \infty$ .

<sup>18</sup>Note that A2 should hold with  $q_i = -\ln f_i$ , and A2(ii) should read as  $E(f_i(y_{t+1}|Z^t, \theta_i)) < (f_i(y_{t+1}|Z^t, \theta_i^\dagger))$ , for all  $\theta_i \neq \theta_i^\dagger$ .

$$\begin{aligned}
& + E \left( \sum_{j=-\infty}^{\infty} \left( \left( 1\{y_{s+1} \leq u\} - F_k(u|Z^s, \theta_k^\dagger) \right)^2 - \mu_k^2(u) \right) \left( \left( 1\{y_{s+j+1} \leq u'\} - F_k(u'|Z^{s+j}, \theta_k^\dagger) \right)^2 - \mu_k^2(u') \right) \right) \\
& - 2E \left( \sum_{j=-\infty}^{\infty} \left( \left( 1\{y_{s+1} \leq u\} - F_1(u|Z^s, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) \left( \left( 1\{y_{s+j+1} \leq u'\} - F_k(u'|Z^{s+j}, \theta_k^\dagger) \right)^2 - \mu_k^2(u') \right) \right) \\
& + 8\Pi m_{\theta_1^\dagger}(u)' A(\theta_1^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_1} \ln f_1(y_{s+1}|Z^s, \theta_1^\dagger) \nabla_{\theta_1} \ln f_1(y_{s+j+1}|Z^{s+j}, \theta_1^\dagger)' \right) A(\theta_1^\dagger) m_{\theta_1^\dagger}(u') \\
& + 8\Pi m_{\theta_k^\dagger}(u)' A(\theta_k^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_k} \ln f_k(y_{s+1}|Z^s, \theta_k^\dagger) \nabla_{\theta_k} \ln f_k(y_{s+j+1}|Z^{s+j}, \theta_k^\dagger)' \right) A(\theta_k^\dagger) m_{\theta_k^\dagger}(u') \\
& - 8\Pi m_{\theta_1^\dagger}(u,)' A(\theta_1^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_1} \ln f_1(y_{s+1}|Z^s, \theta_1^\dagger) \nabla_{\theta_k} \ln f_k(y_{s+j+1}|Z^{s+j}, \theta_k^\dagger)' \right) A(\theta_k^\dagger) m_{\theta_k^\dagger}(u') \\
& - 4\Pi m_{\theta_1^\dagger}(u)' A(\theta_1^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_1} \ln f_1(y_{s+1}|Z^s, \theta_1^\dagger) \left( \left( 1\{y_{s+j+1} \leq u\} - F_1(u|Z^{s+j}, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) \right) \\
& + 4\Pi m_{\theta_1^\dagger}(u)' A(\theta_1^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_1} \ln f_1(y_{s+1}|Z^s, \theta_1^\dagger) \left( \left( 1\{y_{s+j+1} \leq u\} - F_k(u|Z^{s+j}, \theta_k^\dagger) \right)^2 - \mu_k^2(u) \right) \right) \\
& - 4\Pi m_{\theta_k^\dagger}(u)' A(\theta_k^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_k} \ln f_k(y_{s+1}|Z^s, \theta_k^\dagger)' \left( \left( 1\{y_{s+j+1} \leq u\} - F_k(u|Z^{s+j}, \theta_k^\dagger) \right)^2 - \mu_k^2(u) \right) \right) \\
& + 4\Pi m_{\theta_k^\dagger}(u)' A(\theta_k^\dagger) E \left( \sum_{j=-\infty}^{\infty} \nabla_{\theta_k} \ln f_k(y_{s+1}|Z^s, \theta_k^\dagger)' \left( \left( 1\{y_{s+j+1} \leq u\} - F_1(u|Z^{s+j}, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) \right) \quad (20)
\end{aligned}$$

with  $m_{\theta_i^\dagger}(u)' = E \left( \nabla_{\theta_i} F_i(u|Z^t, \theta_i^\dagger)' \left( 1\{y_{t+1} \leq u\} - F_i(u|Z^t, \theta_i^\dagger) \right) \right)$  and  $A(\theta_i^\dagger) = \left( E \left( -\nabla_{\theta_i}^2 \ln f_i(y_{t+1}|Z^t, \theta_i^\dagger) \right) \right)^{-1}$ .

From this proposition, we see that when all competing models provide an approximation to the true conditional distribution that is as (mean square) accurate as that provided by the benchmark (i.e. when  $\int_U (\mu_1^2(u) - \mu_k^2(u)) \phi(u) du = 0, \forall k$ ), then the limiting distribution is a zero mean Gaussian process with a covariance kernel which is not nuisance parameters free. Additionally, when all competitor models are worse than the benchmark, the statistic diverges to minus infinity at rate  $\sqrt{P}$ . Finally, when only some competitor models are worse than the benchmark, the limiting distribution provides a conservative test, as  $Z_P$  will always be smaller than

$\max_{k=2,\dots,n} \int_U \left( Z_{P,u}(1,k) - \sqrt{P} (\mu_1^2(u) - \mu_k^2(u)) \right) \phi(u) du$ , asymptotically. Of course, when  $H_A$  holds, the statistic diverges to plus infinity at rate  $\sqrt{P}$ .

Now, define the bootstrap statistic as:<sup>19</sup>

$$Z_P^* = \max_{k=2,\dots,n} \int_U Z_{P,u}^*(1,k) \phi(u) du,$$

where

$$\begin{aligned} Z_{P,u}^*(1,k) &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( \left( 1\{y_{t+1}^* \leq u\} - F_1(u|Z^{*,t}, \hat{\theta}_{1,t}^*) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \hat{\theta}_{1,t}) \right)^2 \right) \right. \\ &\quad \left. - \left( \left( 1\{y_{t+1}^* \leq u\} - F_k(u|Z^{*,t}, \hat{\theta}_{k,t}^*) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \hat{\theta}_{k,t}) \right)^2 \right) \right) \\ &\quad - \frac{2}{T} \sum_{t=s}^{T-1} \left( \nabla_{\theta_1} F_1(u|Z^t, \hat{\theta}_{1,T})' \left( 1\{y_{t+1}^* \leq u\} - F_1(u|Z^t, \hat{\theta}_{1,T}) \right) \right)' \left( -\frac{1}{T} \sum_{t=s}^{T-1} \nabla_{\theta_1}^2 f_1(y_t|Z^{t-1}, \hat{\theta}_{1,T}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla_{\theta_1} f_1(y_{R+i}|Z^{R+i-1}, \hat{\theta}_{1,T}) - \frac{1}{P} \sum_{i=1}^P \nabla_{\theta_1} f_1(y_{R+i}|Z^{R+i-1}, \hat{\theta}_{1,T}) \right) \\ &\quad + \frac{2}{T} \sum_{t=s}^T \nabla_{\theta_k} F_k(u|Z^t, \hat{\theta}_{k,T}) \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \hat{\theta}_{k,T}) \right)' \left( -\frac{1}{T} \sum_{t=s}^{T-1} \nabla_{\theta_k}^2 f_k(y_t|Z^{t-1}, \hat{\theta}_{k,T}) \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla_{\theta_k} f_k(y_{R+i}|Z^{R+i-1}, \hat{\theta}_{k,T}) - \frac{1}{P} \sum_{i=1}^P \nabla_{\theta_k} f_k(y_{R+i}|Z^{R+i-1}, \hat{\theta}_{k,T}) \right). \end{aligned} \quad (21)$$

**Proposition 7:** Let assumptions A1-A3 and A6 hold. Also, assume that as  $P, R \rightarrow \infty$ ,  $l_1, l_2 \rightarrow \infty$ , and that  $\frac{l_2}{P^{1/4}} \rightarrow 0$  and  $\frac{l_1}{R^{1/4}} \rightarrow 0$ . Then, as  $P$  and  $R \rightarrow \infty$ ,

$$P \left( \omega : \sup_{v \in \Re} \left| P_{R,P}^* \left( \max_{k=2,\dots,n} \int_U Z_{P,u}^*(1,k) \phi(u) du \leq v \right) - P \left( \max_{k=2,\dots,n} \int_U Z_{P,u}^\mu(1,k) \phi(u) du \leq v \right) \right| > \varepsilon \right) \rightarrow 0,$$

where  $a_{R,i} = \frac{1}{R+i} + \frac{1}{R+i+1} + \dots + \frac{1}{R+P-1}$  and  $Z_{P,u}^\mu(1,k) = Z_{P,u}(1,k) - \sqrt{P} (\mu_1^2(u) - \mu_k^2(u))$ .

The above result suggests proceeding as outlined in the reality check application discussed above.

## 6 Monte Carlo

Proposition 1 establishes the first order validity of the recursive PEE bootstrap. In this section we study its finite sample behavior via a small Monte Carlo experiment. In particular our objective is

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<sup>19</sup>As in the previous two applications,  $y_t^*$  and  $Z^{*,t}$  have been obtained via the resampling procedure described in Section 2

to answer the following two questions: (i) Does the inclusion of the adjustment term lead to (substantially) improved coverage probabilities? (ii) How does the recursive PEE bootstrap perform, relative to the “full sample” standard block bootstrap?

Two data generating processes are specified, namely  $y_t = c + \rho y_{t-1} + \varepsilon_t$  and  $y_t = c + \rho_1 y_{t-1} + \rho_2 y_{t-1} + \varepsilon_t$ , with  $\varepsilon_t \sim IN(0, 1)$ ,  $c = 0.1$ ,  $\rho = \{0.2, 0.4, 0.6, 0.8\}$  and  $\rho_1 = \rho_2 = \{0.1, 0.2, 0.3, 0.4\}$ . Given this setup, we proceed to estimate both AR(1) and AR(2) models for each of the two alternative DGPs. Thus, in the present context, when we estimate (via OLS) an AR(1) (or an AR(2)) model,  $\hat{\theta}_{i,t} = (\hat{c}_{i,t}, \hat{\rho}_{i,t})'$  (or  $\hat{\theta}_{i,t} = (\hat{c}_{i,t}, \hat{\rho}_{1,i,t}, \hat{\rho}_{2,i,t})'$ ), with  $i = 1, 2$  denoting the estimate models (AR(1) and AR(2), respectively), and  $\theta_i^\dagger = (c_i^\dagger, \rho_i^\dagger)'$  (or  $\theta_i^\dagger = (c_i^\dagger, \rho_{1,i}^\dagger, \rho_{2,i}^\dagger)'$ ), where  $\theta_i^\dagger$  denotes the probability limit of  $\hat{\theta}_{i,t}$ . Needless to say, in the case of correct dynamic specification,  $\theta_i^\dagger$  represents the parameters characterizing the conditional expectation, while in the case of dynamic misspecification (e.g. the DGP is AR(2) and we estimate an AR(1)),  $\theta_i^\dagger$  represents pseudo true values, which can be explicitly computed. We confine our attention on the slope parameters. For notational simplicity, consider the case in which we estimate a AR(1) and the DGP is also AR(1), so that we compute a  $P$ -sequence of estimators  $\hat{\rho}_t$ , bootstrap estimators  $\hat{\rho}_t^*$ , and we know that  $\rho^\dagger = \{0.2, 0.4, 0.6, 0.8\}$ . In this case the bootstrap statistic with adjustment is given by:<sup>20</sup>

$$\begin{aligned}\Psi_{1,R,P}^* &= \frac{1}{\sqrt{P}} \sum_{t=R}^{P-1} (\hat{\rho}_t^* - \hat{\rho}_t) + \left( \frac{1}{T} \sum_{t=2}^T (y_{t-1} - \bar{y})^2 \right)^{-1} \\ &\quad \times \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} \left( \hat{e}_{R+j} (y_{R+j-1} - \bar{y}) - \frac{1}{P} \sum_{j=1}^{P-1} \hat{e}_{R+j} (y_{R+j-1} - \bar{y}) \right),\end{aligned}$$

where  $\hat{e}_{R+j} = (y_{R+j} - \bar{y}) - \hat{\rho}_T (y_{R+j-1} - \bar{y})$ . Also, define the bootstrap statistic without adjustment as  $\Psi_{2,R,P}^{**} = \frac{1}{\sqrt{P}} \sum_{t=R}^{P-1} (\hat{\rho}_t^* - \hat{\rho}_t)$ , and the (full sample) standard bootstrap statistic as  $\Psi_{3,T}^{***} = \sqrt{T} (\hat{\rho}_T^* - \hat{\rho}_T)$ . Hereafter, let  $z_\alpha^*$  be the  $(1 - \alpha)$  quantile of the distribution of  $\Psi_{1,R,P}^*$ ,  $z_\alpha^{**}$  the  $(1 - \alpha)$  quantile of the distribution of  $\Psi_{2,R,P}^{**}$  and  $z_\alpha^{***}$  the  $(1 - \alpha)$  quantile of the distribution of  $\Psi_{3,T}^{***}$ . We now define  $100(1 - \alpha)\%$ , equal-tailed, two-sided confidence intervals corresponding to the recursive bootstrap with adjustment, the recursive bootstrap without adjustment and the full sample, standard block bootstrap, that is

$$CI^* : \left\{ \frac{1}{P} \sum_{t=R}^{P-1} \hat{\rho}_t - \frac{z_{\alpha/2}^*}{\sqrt{P}}, \frac{1}{P} \sum_{t=R}^{P-1} \hat{\rho}_t + \frac{z_{(1-\alpha/2)}^*}{\sqrt{P}} \right\} \quad (22)$$

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<sup>20</sup>Note that  $\hat{\rho}_t^*$  is computed using the pseudo time series obtained by first resampling  $b_1$  blocks from the first  $R$  observations and then concatenating  $b_2$  blocks resampled from the last  $P$  observations, as described in Section 2.

$$CI^{**} : \left\{ \frac{1}{P} \sum_{t=R}^{P-1} \hat{\rho}_t - \frac{z_{\alpha/2}^{**}}{\sqrt{P}}, \frac{1}{P} \sum_{t=R}^{P-1} \hat{\rho}_t + \frac{z_{(1-\alpha/2)}^{**}}{\sqrt{P}} \right\} \quad (23)$$

$$CI^{***} : \left\{ \hat{\rho}_T - \frac{z_{\alpha/2}^{***}}{\sqrt{T}}, \hat{\rho}_T + \frac{z_{(1-\alpha/2)}^{***}}{\sqrt{T}} \right\} \quad (24)$$

The coverage probabilities for  $CI^*$ ,  $CI^{**}$ ,  $CI^{***}$  are then obtained by computing the proportion of times, across simulation replications, for which  $\rho^\dagger$  falls into the respective interval. By comparing the coverage probabilities for  $CI^*$  and  $CI^{**}$  we have a direct measure of the impact of the adjustment term. Broadly speaking, if the difference between the actual and nominal coverage is smaller for  $CI^*$  than for  $CI^{**}$ , then it is definitely worthwhile to construct bootstrap critical values based on the recursive bootstrap with adjustment. The coverage probabilities for  $CI^*$  and  $CI^{***}$  are useful in the sense that their comparison tells us whether the recursive PEE bootstrap provides critical values for the statistic  $\frac{1}{\sqrt{P}} \sum_{t=R}^{P-1} (\hat{\rho}_t - \rho^\dagger)$  which are as reliable as those provided by the standard block bootstrap for the statistic  $\sqrt{T} (\hat{\rho}_T - \rho^\dagger)$ . Note that we compute the coverage probability for  $CI^{***}$  using  $T$  observations, this is done purposely to give an advantage to the standard block bootstrap.

All bootstrap empirical distributions are based on 200 bootstrap replications, and all tabulated results are based on 500 Monte Carlo simulations. In addition, samples of  $T = \{600, 1200, 2400\}$  observations are used, and the number of estimators constructed in the context of the PEE recursive scheme bootstrap is  $P = 0.5T$ , with the first estimator constructed using  $T - P$  observations, the second with  $T - P + 1$  observations, etc. The nominal coverage probability, across all experiments, is set equal to 0.95. We have tried a variety of values of  $\alpha$  in the construction of the confidence intervals. However, as the results are qualitatively the same, we report results only for  $\alpha = 0.05$ .

Our findings are reported in Tables 1-4, and are organized as follows. The second column lists the bootstrap used to mimic the distribution of PEE associated with either the AR(1) autoregressive parameter (denoted  $\hat{\rho}$  in the tables) or the autoregressive parameters from the AR(2) model (denoted  $\hat{\rho}_1$  and  $\hat{\rho}_2$  in the table). Entries corresponding to *rec* correspond to coverage probabilities associated with  $CI^*$ . Entries corresponding to *full* are the rejection probabilities associated with  $CI^{***}$ . Tables 1-4 is broken into two panels, depending upon whether data were generated according to an AR(1) process (Panel A) or an AR(2) process (Panel B), and the autoregressive parameters of the DGPs are given in the header line for each panel. In addition, block lengths used are denoted

by the various values of  $l_1 = l_2$ .<sup>21</sup> In Tables 5 and 6, results are gathered for the recursive PEE bootstrap with no adjustment terms, so that entries denote the coverage probabilities associated with  $CI^{**}$ .

Turning now to the results, a number of clear-cut conclusions emerge. First, inspection of Tables 1-4 suggests that the recursive PEE bootstrap (*rec*) performs approximately as well as the standard full sample block bootstrap (*full*). As expected, both types of bootstraps have good coverage when the autoregressive parameters in the models are smaller, with performance worsening as these parameters increase from 0.2 to 0.8 in the AR(1) case (see Panel A of Tables 1-4) and from 0.1 to 0.4 in the AR(2) case (see Panel B of the same tables). In addition, performance improvement is rather similar across the two bootstraps when the sample size is increased. These findings suggest that the recursive PEE bootstrap can be applied with the same level of confidence as the standard block bootstrap within the context of full sample estimation. Second, when  $\rho_2^\dagger = 0$ , which is the case in Panel A of Tables 1-4, as the true DGP in these cases is an AR(1) process, coverage is often better than in the case in which we estimate an AR(2). Interestingly, the converse does not always hold. In particular, when the true DGP is an AR(2) and an AR(1) is estimated then coverage associated with  $\hat{\rho}$  is often as good as that associated with  $\hat{\rho}_1$  and  $\hat{\rho}_2$ , and is often better than that associated with  $\hat{\rho}_2$ . This is perhaps surprising, given that we always set  $\rho_1 = \rho_2$ , and suggests a complicated interaction between parsimony and model specification. Third, notice in Tables 1 that the best coverage usually occurs with *rec* for block lengths around one half as long as for *full*. This is not surprising, given that  $P = 0.5T$ , so that, at least loosely speaking, samples used in the construction of *rec* are around 1/2 as long as those used in the *full*, so that the relative length of the blocks is the same. However, notice that this finding does not hold up when the autoregressive parameters are increased (see Tables 2-4), and in particular when  $\rho$  is largest (Table 4), in this case both the *full* and the *rec* bootstraps prefer the longest block that we tried. Finally, upon comparison of the results reported in Tables 5 and 6 with the *rec* bootstrap results from Tables 1-4, it is clear that there is everywhere an improvement in coverage when the PEE recursive scheme bootstrap is constructed with the adjustment term.

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<sup>21</sup>As  $P = R = 0.5T$ , we use the same block length when resampling from the first  $R$  observations and from the last  $P$  observations.

## 7 Concluding Remarks

In this paper we have introduced a parameter estimation error (PEE) bootstrap for recursive estimation schemes (named the recursive PEE bootstrap) and have shown its first order validity, that is its ability of mimicing the limiting distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t - \theta^\dagger)$ , where  $R$  denotes the length of the estimation period,  $P$  the number of recursively estimated parameters,  $\hat{\theta}_t$  is a recursive  $m$ -estimator constructed using the first  $t$  observations, and  $\theta^\dagger$  is its probability limit. In recursive estimation schemes, earlier observations are used more frequently than temporally subsequent observations. This introduces a bias to the usual block bootstrap, as under standard resampling with replacement schemes, any block from the original sample has the same probability of being selected. We circumvent this problem by first forming bootstrap samples as follows. Resample  $R$  observations from the initial  $R$  sample observations, and then concatenate onto this vector an additional  $P$  resampled observations from the remaining sample. Thereafter, construct  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t^* - \hat{\theta}_t)$  and add an adjustment term in order to ensure that the distribution of the sum of both components is the same as the distribution of  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\hat{\theta}_t - \theta^\dagger)$ , conditional on the sample, and for all samples except a set of probability measure approaching zero. The recursive PEE bootstrap can be used to provide valid critical values in a variety of interesting testing contexts, and three such leading applications are developed. The first is a generalization of the reality check test of White (2000) that allows for non vanishing parameter estimation error. The second is an out-of-sample version of the integrated conditional moment (ICM) test of Bierens (1982,1990) and Bierens and Ploberger (1997) which provides out of sample tests consistent against generic (nonlinear) alternatives. Finally, the third is a procedure assessing the relative out-of-sample predictive accuracy of multiple conditional distribution models. This procedure is based on an extension of the Andrews (1997) conditional Kolmogorov test. Finally, the finite sample behavior of the recursive PEE bootstrap has been analyzed via a small Monte Carlo study, from which two main findings emerge. First, across different DGPs and sample sizes, the recursive PEE bootstrap is overall as reliable as the standard (full sample) block bootstrap. Second, across all experiments, the recursive PEE bootstrap ensures a much better coverage than the corresponding bootstrap procedure which neglects the adjustment term.

## 8 Appendix

The proof of Theorem 1 requires the following three lemmas. As the statement below holds for  $i = 1, \dots, n$  and the proof is the same regardless which model we consider, for notational simplicity we drop the subscript  $i$ . Also, for notational simplicity, in the sequel we set  $l_1 = l_2 = l$ .

**Lemma A1:** Let A1-A3 hold. If as  $R \rightarrow \infty$  and  $P \rightarrow \infty$ ,  $l \rightarrow \infty$ ,  $l/R \rightarrow 0$  and  $l/P \rightarrow 0$ , then (i)  $\sup_{t \geq R} |\hat{\theta}_t^* - \hat{\theta}| = o_{P^*}(1)$ ,  $\text{Pr}-P$ , and (ii)  $\sup_{t \geq R} |\hat{\theta}_t^* - \theta^\dagger| = o_{P^*}(1)$ ,  $\text{Pr}-P$ .

**Lemma A2:** Let A1-A3 hold. If as  $R \rightarrow \infty$  and  $P \rightarrow \infty$ ,  $l \rightarrow \infty$ ,  $l/R^4 \rightarrow 0$  and  $l/P^4 \rightarrow 0$ , then  $\sup_{t \geq R} t^\vartheta |(\hat{\theta}_t^* - \theta^\dagger)| = o_{P^*}(1)$ ,  $\text{Pr}-P$ , for all  $\vartheta < 0.5$ .

**Lemma A3:** Let A1-A3 hold. If as  $R \rightarrow \infty$  and  $P \rightarrow \infty$ ,  $l \rightarrow \infty$ ,  $l/R^4 \rightarrow 0$  and  $l/P^4 \rightarrow 0$ , then if  $P/R \rightarrow \pi > 0$ , then

$$\text{Var}^* \left( \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \frac{1}{t} \sum_{j=s}^t (\nabla_\theta q(y_j^*, Z^{*,j-1}, \theta^\dagger)) \right) = 2\Pi C_{00}, \quad \text{Pr}-P,$$

where  $C_{00} = \sum_{j=-\infty}^{\infty} E \left( (\nabla_\theta q(y_{1+s}, Z^s, \theta^\dagger)) (\nabla_\theta q(y_{1+s+j}, Z^{s+j}, \theta^\dagger))' \right)$  and  $\Pi = 1 - \pi^{-1} \ln(1 + \pi)$ .

**Proof of Lemma A1:** (i) We need to extend the consistency results for bootstrap  $m$ -estimators of Goncalves and White (2002b, Theorem 2.1), to the case of recursive  $m$ -estimators. Recalling that for  $t \geq R$ ,

$$\hat{\theta}_t = \arg \min_{\theta \in \Theta} \frac{1}{t} \sum_{j=s}^t q(y_j, Z^{j-1}, \theta) \text{ and } \hat{\theta}_t^* = \arg \min_{\theta \in \Theta} \frac{1}{t} \sum_{j=s}^t q(y_j^*, Z^{*,j-1}, \theta),$$

and given that the argmin is a measurable function, and because of the unique identifiability conditions in A2(ii), it suffices to show that

$$\sup_{t \geq R} \sup_{\theta \in \Theta} \left| \frac{1}{t} \sum_{j=s}^t (q(y_j^*, Z^{*,j-1}, \theta) - q(y_j, Z^{j-1}, \theta)) \right| = o_{P^*}(1), \quad \text{Pr}-P.$$

Hereafter, for notational simplicity let  $q(y_j^*, Z^{*,j-1}, \theta) = q_j^*(\theta)$  and  $q(y_j, Z^{j-1}, \theta) = q_j(\theta)$ , and let  $\mu = E(q_j(\theta))$ ,  $\forall \theta \in \Theta$ . Now,

$$\sup_{t \geq R} \sup_{\theta \in \Theta} \left| \frac{1}{t} \sum_{j=s}^t (q_j^*(\theta) - q_j(\theta)) \right| \leq \sup_{t \geq R} \sup_{\theta \in \Theta} \left| \frac{1}{t} \sum_{j=s}^t (q_j^*(\theta) - E^*(q_j^*(\theta))) \right| \quad (25)$$

$$+ \sup_{t \geq R} \sup_{\theta \in \Theta} \left| \frac{1}{t} \sum_{j=s}^t (q_j(\theta) - \mu) \right| + \sup_{t \geq R} \sup_{\theta \in \Theta} \left| \frac{1}{t} \sum_{j=s}^t (E^*(q_j^*(\theta)) - \mu) \right|. \quad (26)$$

Now, recalling that  $R = b_1 l$  and  $P = b_2 l$ , let  $t = (b_1 + k)l$  for any generic  $k = 1, \dots, b_2$ ,

$$\frac{1}{t} \sum_{j=s}^t E^*(q_j^*(\theta)) = \frac{1}{(b_1 + k)l} \sum_{j=s}^R E^*(q_j^*(\theta)) + \frac{1}{(b_1 + k)l} \sum_{j=R+1}^{R+kl} E^*(q_j^*(\theta)) + O\left(\frac{l}{R}\right), \quad \text{Pr}-P,$$

uniformly in  $\theta$ , as under A3,  $P$  and  $R$  grow at the same rate, as the sample size increases. Now,

$$\begin{aligned} \frac{1}{t} \sum_{j=s}^R E^*(q_j^*(\theta)) &= \frac{b_1}{t} \left( \frac{q_{s+1}(\theta) + \dots + q_{l+s}(\theta)}{R-l+1} + \frac{q_{2+s}(\theta) + \dots + q_{l+1+s}(\theta)}{R-l+1} + \dots + \frac{q_{R-l+1}(\theta) + \dots + q_R(\theta)}{R-l+1} \right) \\ &= \frac{b_1}{b_1 + k} \frac{1}{R} \sum_{j=1}^R q_j(\theta) + O\left(\frac{l}{R}\right), \quad \text{Pr}-P, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{t} \sum_{j=R+1}^{R+kl} E^*(q_j^*(\theta)) &= \frac{k}{t} \left( \frac{q_{R+1}(\theta) + \dots + q_{R+l}(\theta)}{P-l+1} + \dots + \frac{q_{R+P-l+1}(\theta) + \dots + q_{R+P-1}(\theta)}{P-l+1} \right) \\ &= \frac{k}{k+b_1} \frac{1}{P} \sum_{j=R+1}^{R+P} q_j(\theta) + O\left(\frac{l}{R}\right), \quad \text{Pr}-P. \end{aligned}$$

Thus

$$\frac{1}{t} \sum_{j=1}^t E^*(q_j^*(\theta)) = \frac{b_1}{(b_1 + k)} \frac{1}{R} \sum_{j=1}^R q_j(\theta) + \frac{k}{(b_1 + k)} \frac{1}{P} \sum_{j=R+1}^{R+P} q_j(\theta) + O\left(\frac{l}{R}\right), \quad \text{Pr}-P,$$

and so the second term on the right hand side of (26), by the uniform strong law of large number approaches zero in probability. Analogously, the first term on the right hand side of (26), by the uniform strong law of large number is  $o_P(1)$ . As for the first term on the RHS of (25), it is majorized by:

$$\sup_{\theta \in \Theta} \left| \frac{1}{b_1} \sum_{j=s}^{b_1} \left( \frac{U_j(\theta) - E^*(U_j(\theta))}{l} \right) \right| + \sup_{k \geq 1} \sup_{\theta \in \Theta} \left| \frac{k}{b_1 + k} \frac{1}{k} \sum_{j=s}^k \left( \frac{U_j(\theta) - E^*(U_j(\theta))}{l} \right) \right|, \quad (27)$$

where for  $j = s-1, \dots, R-l$ ,  $U_i$  are independent discrete uniform taking value  $(q_{j+1}(\theta) + \dots + q_{j+l}(\theta))$  with probability  $1/(R-l+1)$ , and for  $j = R, \dots, R+P-l$ ,  $U_i$  are independent discrete uniform taking value  $(q_{j+1+R}(\theta) + \dots + q_{j+l+R}(\theta))$  with probability  $1/(P-l+1)$ . By the uniform law of large numbers for asymptotically independent and homogeneous observations, the sum of the two term in (27) approaches zero in  $P^*$ -probability.

(ii) Immediate as,

$$\sup_{t \geq R} |\widehat{\theta}_t^* - \theta^\dagger| \leq \sup_{t \geq R} |\widehat{\theta}_t^* - \widehat{\theta}_t| + \sup_{t \geq R} |\widehat{\theta}_t - \theta^\dagger|,$$

and the first term is  $o_P^*(1)$ ,  $\Pr - P$  by part (i), while the second term is  $o(1) \Pr - P$ .

**Proof of Lemma A2:** First note that,

$$t^\vartheta \left( \widehat{\theta}_t^* - \theta^\dagger \right) = \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta^2 q(y_j^*, Z^{*,j-1}, \overline{\theta}_t^*) \right)^{-1} \left( \frac{1}{t^{1-\vartheta}} \sum_{j=s}^t \nabla_\theta q(y_j^*, Z^{*,j-1}, \theta^\dagger) \right),$$

with  $\overline{\theta}_t^* \in (\widehat{\theta}_t^*, \theta^\dagger)$ . Hereafter, for notational simplicity let  $\nabla_\theta^2 q(y_j^*, Z^{*,j-1}, \theta) = \nabla^2 q_j^*(\theta)$ ,  $\nabla_\theta^2 q(y_j, Z^{j-1}, \theta) = \nabla^2 q_j(\theta)$ , and let  $B^\dagger = (E(-\nabla_\theta^2 q_t(\theta^\dagger)))^{-1}$ ,

$$\sup_{t \geq R} \left| \frac{1}{t} \sum_{j=s}^t \left( \nabla^2 q_j^*(\overline{\theta}_t^*) - B^{\dagger-1} \right) \right| \leq \sup_{t \geq R} \left| \frac{1}{t} \sum_{j=s}^t \left( \nabla^2 q_j^*(\overline{\theta}_t^*) - E^* \left( \nabla^2 q_j^*(\overline{\theta}_t^*) \right) \right) \right| \quad (28)$$

$$+ \sup_{t \geq R} \left| \frac{1}{t} \sum_{j=s}^t \left( \nabla^2 q_j(\overline{\theta}_t) - B^{\dagger-1} \right) \right| + \sup_{t \geq R} \left| \frac{1}{t} \sum_{j=s}^t \left( \nabla^2 q_j(\overline{\theta}_t) - E^* \left( \nabla^2 q_j^*(\overline{\theta}_t^*) \right) \right) \right|, \quad (29)$$

as  $\overline{\theta}_t^* \in (\widehat{\theta}_t^*, \theta^\dagger)$  and  $\overline{\theta}_t \in (\widehat{\theta}_t, \theta^\dagger)$ , given Lemma A1,  $\sup_{t \geq R} |\overline{\theta}_t^* - \overline{\theta}_t| = o_{P^*}(1) \Pr - P$ , thus the right hand side of (28) and the sum of the two term in (29) are  $o_{P^*}(1) \Pr - P$ , by the same argument used in the proof of Lemma A1. Given A3(ii), it follows immediately that

$$\sup_{t \geq R} \left| \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta^2 q(y_j^*, Z^{*,j-1}, \overline{\theta}_t^*) \right)^{-1} - B^\dagger \right| = o_{P^*}(1), \quad \Pr - P. \quad (30)$$

Let  $n_t = (2t \log \log t)^{1/2}$ , and let  $\nabla_\theta q(y_j^*, Z^{*,j-1}, \theta) = h_j^*(\theta)$ , and  $\nabla_\theta q(y_j, Z^{j-1}, \theta) = h_j(\theta)$ ,

$$\sup_{t \geq R} \left| \frac{1}{n_t} \sum_{j=s}^t h_j^*(\theta^\dagger) \right| \leq \sup_{t \geq R} \left| \frac{1}{n_t} \sum_{j=s}^t \left( h_j^*(\theta^\dagger) - E^* \left( h_j^*(\theta^\dagger) \right) \right) \right| + \sup_{t \geq R} \left| \frac{1}{n_t} \sum_{j=2}^t E^* \left( h_j^*(\theta^\dagger) \right) \right|, \quad (31)$$

and noting that, by the same argument as in the proof of Lemma A1, up to a term of order  $O(l/P^{1/2})$ ,  $\Pr - P$ ,

$$\begin{aligned} \sup_{t \geq R} \left| \frac{1}{n_t} \sum_{j=s}^t E^* \left( h_j^*(\theta^\dagger) \right) \right| &\leq \left| \frac{1}{\sqrt{2R \log \log R}} \sum_{j=s}^R h_j(\theta^\dagger) \right| \\ &+ \sup_{k \geq 1} \left| \frac{1}{\sqrt{2kl \log \log(kl)}} \sum_{j=R}^{R+kl} h_j(\theta^\dagger) \right|, \end{aligned} \quad (32)$$

and both the terms on the RHS of (32) are  $O(1)$ ,  $a.s. - P$ , as, given A1 and A3 each component of both terms satisfies the conditions for the functional law of the iterated logarithm (e.g. Theorem 2 in Eberlain (1986)).

It remains to show that the first term on the RHS of (31) is  $O_{P^*}(1)$ ,  $\Pr - P$ . To further simplify the notation, we denote  $h_j^*(\theta^\dagger)$  and  $h_j(\theta^\dagger)$  as  $h_j^*$  and  $h_j$ , respectively. First, recalling that, except for an overlapping of at most  $s$  observations, all blocks are independent each other, conditionally on the sample,

$$\begin{aligned} Var^* \left( \frac{1}{\sqrt{T}} \sum_{t=s}^{T-1} h_t^* \right) &= Var^* \left( \frac{1}{\sqrt{T}} \sum_{t=s}^R h_t^* \right) + Var^* \left( \frac{1}{\sqrt{T}} \sum_{t=R+1}^{R+P-1} h_t^* \right) + O \left( \frac{l}{T^{1/2}} \right), \quad \Pr - P \\ &= Var^* \left( \frac{1}{\sqrt{T}} \sum_{i=1}^{b_1} \sum_{j=1}^l h_{I_i^R+j} \right) + Var^* \left( \frac{1}{\sqrt{T}} \sum_{i=1}^{b_2} \sum_{j=1}^l h_{I_i^P+j} \right) + O \left( \frac{l}{T^{1/2}} \right), \quad \Pr - P, \\ &= E^* \left( \frac{R}{T} \frac{1}{R} \sum_{i=1}^{b_1} \sum_{j=1}^l \sum_{k=1}^l h_{I_i^R+j} h_{I_i^R+k} \right) + E^* \left( \frac{P}{T} \frac{1}{P} \sum_{i=1}^{b_2} \sum_{j=1}^l \sum_{k=1}^l h_{I_i^P+j} h_{I_i^P+k} \right) + O \left( \frac{l}{T^{1/2}} \right), \quad \Pr \text{ (3B)} \end{aligned}$$

where for  $i = 1, \dots, b_1$   $I_i^R$  are independent discrete uniform on  $s, \dots, R-l$ , while for  $i = 1, \dots, b_2$   $I_i^P$  are independent discrete uniform on  $R, R+1, \dots, R+P-l$ , thus after few simple manipulations, the equality on (33) can be rewritten as

$$\frac{R}{T} \frac{1}{R} \sum_{k=l}^{R-l} \sum_{j=-l}^l h_k h'_{k+j} + \frac{P}{T} \frac{1}{P} \sum_{k=R+l}^{R+P-l} \sum_{j=-l}^l h_k h'_{k+j} + O \left( \frac{l}{T^{1/2}} \right), \quad \Pr - P, \quad (34)$$

also given A(1) and A(3) and the growth conditions on the parameter  $l$ , the sum of the first to two terms in (34), up to term approaching zero  $\Pr - P$ , is equal to  $C_{00}$ , and so

$V^* = \lim_{T \rightarrow \infty} Var^* \left( \frac{1}{\sqrt{T}} \sum_{t=1}^T h_j^*(\theta^\dagger) \right)$  is  $O(1)$ ,  $\Pr - P$ . Now,

$$\begin{aligned} \sup_{t \geq R} \left\| V^{*-1/2} \frac{1}{n_t} \sum_{j=s}^t (h_j^* - E^*(h_j^*)) \right\| &\leq \left\| V^{*-1/2} \frac{1}{\sqrt{2b_1 \log \log b_1}} \sum_{i=1}^{b_1} \left( \frac{U_i - E^*(U_i)}{l} \right) \right\| \\ &\quad + \sup_{k \geq 1} \left\| V^{*-1/2} \frac{1}{\sqrt{2k \log \log b_1}} \sum_{i=1}^{b_1} \left( \frac{U_i - E^*(U_i)}{l} \right) \right\|, \quad (35) \end{aligned}$$

where for  $i = s, \dots, R-l$ ,  $U_i$  is independent discrete uniform taking value  $(h_{i+1}(\theta) + \dots + h_{i+l}(\theta))$  with probability  $1/(R-l+1)$ , and for  $i = R, \dots, R+P-l$ ,  $U_i$  is independent discrete uniform taking value  $(h_{i+1+R}(\theta) + \dots + h_{i+l+R}(\theta))$  with probability  $1/(P-l+1)$ . Therefore the assumptions of Theorem 1 in Eberlain (1986) are satisfied and so the right hand side of (35) is  $O_{a.s.}^*(1)$   $\Pr - P$ , thus  $\sup_{t \geq R} \left| \frac{1}{b_t} \sum_{j=s}^t (h_j^*(\theta^\dagger) - E^*(h_j^*)) \right|$  is also  $O_{a.s.}^*(1)$   $\Pr - P$ . Recalling (30), the desired statement then follows.

**Proof of Lemma A3:**

As in the proof of Lemma A2, let  $\nabla_\theta q(y_j^*, Z^{*,j-1}, \theta) = h_j^*(\theta)$ , and  $\nabla_\theta q(y_j, Z^{j-1}, \theta) = h_j(\theta)$ , also let  $\nabla_\theta q(y_j^*, Z^{*,j-1}, \theta^\dagger) = h_j^*$ , and  $\nabla_\theta q(y_j, Z^{j-1}, \theta^\dagger) = h_j$ . Along the lines of West (1996, proof of Lemma A5),

$$\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \frac{1}{t} \sum_{j=s}^t h_j^* = \frac{a_{R,0}}{\sqrt{P}} \sum_{j=s}^R h_j^* + \frac{1}{\sqrt{P}} (a_{R,1} h_{R+1}^* + \dots + a_{R,P-1} h_{R+P-1}^*) + o_{P^*}(1), \quad \text{Pr}-P \quad (36)$$

where  $a_{R,i} = (R+i)^{-1} + \dots + (R+P-1)^{-1}$ , for  $0 \leq i < P-1$ .<sup>22</sup> Thus,

$$\begin{aligned} Var^* \left( \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \frac{1}{t} \sum_{j=1}^t h_j^* \right) &= \frac{R}{P} Var^* \left( a_{R,0} \frac{1}{\sqrt{R}} \sum_{j=1}^R h_j^* \right) \\ &\quad + \frac{1}{P} Var^* \left( \sum_{j=1}^{P-1} a_{R,j} h_{R+j}^* \right) + \frac{1}{P} Cov^* \left( a_{R,0} \sum_{j=1}^R h_j^*, \sum_{j=1}^{P-1} a_{R,j} h_{R+j}^* \right) \end{aligned}$$

Given that any of the last  $b_2$  blocks can be correlated with any of the first  $b_1$  blocks for at most  $s$  observations,  $s$  finite, up to a term approaching zero conditionally on the sample, the covariance term is equal to zero. Now, for  $t = s, \dots, R$ ,  $E^*(h_t^*) = R^{-1} \sum_{t=s}^R h_t + O(l/P) = \bar{h}_R + O(l/P)$ , thus up to a term of order  $O(l/R^{1/2})$ ,

$$\begin{aligned} Var^* \left( a_{R,0} \frac{1}{\sqrt{R}} \sum_{j=1}^R h_j^* \right) &= a_{R,0}^2 Var^* \left( \frac{1}{\sqrt{R}} \sum_{k=1}^{b_1} \sum_{i=1}^l h_{I_k+i} \right) \\ &= a_{R,0}^2 E^* \left( \frac{1}{R} \sum_{k=1}^{b_1} \sum_{i=1}^l \sum_{k=1}^l (h_{I_k+i} - \bar{h}_R)(h_{I_k+j} - \bar{h}_R)' \right) \\ &= a_{R,0}^2 \left( \frac{1}{R} \sum_{t=l}^{R-l} \sum_{j=-l}^l (h_t - \bar{h}_R)(h_{t+j} - \bar{h}_R)' \right) + O(l/R^{1/2}) \quad \text{Pr}-P. \end{aligned}$$

So,

$$\begin{aligned} \frac{R}{P} Var^* \left( a_{R,0} \frac{1}{\sqrt{R}} \sum_{j=1}^R h_j^* \right) &= \frac{Ra_{R,0}^2}{P} \sum_{j=-l}^l \gamma_j + \frac{Ra_{R,0}^2}{P} \left( \frac{1}{R} \sum_{t=l}^{R-l} \sum_{j=-l}^l ((h_t - \bar{h}_R)(h_{t+j} - \bar{h}_R)' - \gamma_j) \right) + O\left(\frac{l^2}{R}\right), \quad (37) \end{aligned}$$

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<sup>22</sup>The  $o_{P^*}(1)$ ,  $\text{Pr}-P$  term on the RHS of (36) comes from the fact that the summation run from  $j = s$  instead of  $j = 1$ .

where  $\gamma_j = \text{Cov}(h_1, h_{1+j})$ . By West (1996, proof of Lemma A5), it follows that  $\frac{Ra_{R,0}^2}{P} \sum_{j=-l}^l \gamma_j \rightarrow \pi^{-1} \ln^2(1+\pi) C_{00}$ , while the second term on the RHS above goes to zero  $\text{Pr} - P$  (see e.g. Theorem 2 in Newey and West (1987)). Now, for  $j = 1, \dots, P$ ,  $E^*(a_{R,j} h_{R+j}^*) = a_{R,j} P^{-1} \sum_{j=1}^{P-1} h_{R+j} + O(l/P) = a_{R,j} \bar{h}_P + O(l/P)$ , thus up to a term of order  $O(l/P^{1/2})$   $\text{Pr} - P$ ,

$$\begin{aligned}
& \text{Var}^* \left( \frac{1}{\sqrt{P}} \sum_{j=1}^{P-1} a_{R,j} h_{R+j}^* \right) = \text{Var}^* \left( \frac{1}{\sqrt{P}} \sum_{k=1}^{b_2} \sum_{i=1}^l a_{R,((k-1)l+i)} h_{R+I_k+i} \right) \\
&= \frac{1}{P} E^* \left( \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} (h_{R+I_k+i} - \bar{h}_P) (h_{R+I_k+j} - \bar{h}_P)' \right) \\
&= \frac{1}{P} \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} E^* ((h_{R+I_k+i} - \bar{h}_P) (h_{R+I_k+j} - \bar{h}_P)') \\
&= \frac{1}{P} \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} \left( \frac{1}{P} \sum_{t=l}^{P-l} (h_{R+t+i} - \bar{h}_P) (h_{R+t+j} - \bar{h}_P)' \right) + O(l/P^{1/2}) \text{ Pr} - P \\
&= \frac{1}{P} \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} \gamma_{i-j} \\
&\quad + \frac{1}{P} \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} \left( \frac{1}{P} \sum_{t=l}^{P-l} ((h_{R+t+i} - \bar{h}_P) (h_{R+t+j} - \bar{h}_P)' - \gamma_{i-j}) \right) \\
&\quad + O(l/P^{1/2}) \text{ Pr} - P
\end{aligned} \tag{38}$$

We need to show that the last term on the last equality in (38) is  $o(1)$   $\text{Pr} - P$ . First note that it is majorized by

$$\begin{aligned}
& \left| \frac{b_2}{P} \sum_{i=1}^l \sum_{j=1}^l \left( \frac{1}{P} \sum_{t=l}^{P-l} ((h_{R+t+i} - \bar{h}_P) (h_{R+t+j} - \bar{h}_P)' - \gamma_{i-j}) \right) \right| \\
&= \left| \frac{1}{P} \sum_{t=l}^{P-l} \sum_{j=-l}^l ((h_{R+t} - \bar{h}_P) (h_{R+t+j} - \bar{h}_P)' - \gamma_j) \right| + O(l/P^{1/2}) \text{ Pr} - P
\end{aligned} \tag{39}$$

The first term on the RHS of (39) goes to zero in probability, by the same argument as in Lemma 2 in Corradi (1999)<sup>23</sup>. As for the first term on the RHS of the last equality in (38),

$$\frac{1}{P} \sum_{k=1}^{b_2} \sum_{i=1}^l \sum_{j=1}^l a_{R,((k-1)l+i)} a_{R,((k-1)l+j)} \gamma_{i-j} = \frac{1}{P} \sum_{t=l}^{P-l} \sum_{j=-l}^l a_{R,t} a_{R,t+j} \gamma_j + O(l/P^{1/2}) \text{ Pr} - P$$

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<sup>23</sup>The domination condition here are weaker than those in Lemma 2 in Corradi (1999) as we require only convergence to zero in probability and not almost surely.

$$= \frac{1}{P} \sum_{t=l}^{P-l} a_{R,t}^2 \sum_{j=-l}^l \gamma_j + \frac{1}{P} \sum_{t=l}^{P-l} \sum_{j=-l}^l (a_{R,t} a_{R,t+j} - a_{R,t}^2) \gamma_j + O(l/P^{1/2}) \text{ Pr } - P$$

By the same argument as in Lemma A5 in West (1996), the second term on the RHS above approaches zero, while

$$\frac{1}{P} \sum_{t=l}^{P-l} a_{R,t}^2 \sum_{j=-l}^l \gamma_j \rightarrow (2[1 - \pi^{-1} \ln(1 + \pi)] - \pi^{-1} \ln^2(1 + \pi)) C_{00}.$$

As the first term on the RHS of (37) converges to  $\pi^{-1} \ln^2(1 + \pi) C_{00}$  (see West (1996), p.1082), the desired outcome then follows.

### Proof of Theorem 1:

$$\begin{aligned} \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\widehat{\theta}_t^* - \widehat{\theta}_t) &= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\widehat{\theta}_t^* - \theta^\dagger) - \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\widehat{\theta}_t - \theta^\dagger) \\ &= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \left( -\frac{1}{t} \sum_{j=s}^t \nabla_\theta^2 q(y_j^*, Z^{*,j-1}, \bar{\theta}_t^*) \right)^{-1} \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j^*, Z^{*,j-1}, \theta^\dagger) \right) \\ &\quad - \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \left( -\frac{1}{t} \sum_{j=s}^t \nabla_\theta^2 q(y_j, Z^{j-1}, \bar{\theta}_t) \right)^{-1} \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j, Z^{j-1}, \theta^\dagger) \right), \end{aligned} \quad (40)$$

where  $\bar{\theta}_t^* \in (\widehat{\theta}_t^*, \theta^\dagger)$  and  $\bar{\theta}_t \in (\widehat{\theta}_t, \theta^\dagger)$ .

Given Lemma A1 and A2 and given A1-A3,

$$\sup_{t \geq R} \left( \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j^*, Z^{*,j-1}, \bar{\theta}_t^*) \right)^{-1} - \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j, Z^{j-1}, \bar{\theta}_t) \right)^{-1} \right) = o_P^*(1), \text{ Pr } - P,$$

and also

$$\sup_{t \geq R} \left( \left( -\frac{1}{t} \sum_{j=s}^t \nabla_\theta^2 q(y_j^*, Z^{*,j-1}, \bar{\theta}_t^*) \right)^{-1} - B^\dagger \right) = o_P^*(1), \text{ Pr } - P, \quad (41)$$

so the RHS of (40) can be written as:

$$\begin{aligned} &\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} B^\dagger \left( \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j^*, Z^{*,j-1}, \theta^\dagger) - \frac{1}{t} \sum_{j=s}^t \nabla_\theta q(y_j, Z^{j-1}, \theta^\dagger) \right) + o_P^*(1), \text{ Pr } - P \\ &= B^\dagger \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \left( \frac{1}{t} \sum_{j=s}^t h_j^* - \frac{1}{t} \sum_{j=s}^t h_t \right) + o_P^*(1), \text{ Pr } - P, \end{aligned} \quad (42)$$

by letting  $\nabla_{\theta}q(y_j^*, Z^{*,j-1}, \theta^\dagger) = h_t^*$ ,  $\nabla_{\theta}q(y_j, Z^{j-1}, \theta^\dagger) = h_t$ . Recalling that  $a_{R,s} = (R+s)^{-1} + \dots + (R+P-1)$ ,  $0 \leq s \leq P-1$ , and that for  $t = 1, \dots, R$ ,  $E^*(h_t^*) = R^{-1} \sum_{t=1}^R h_t + O(l/P) = \bar{h}_R + O(l/P)$ , and for  $j = 1, \dots, P$ ,  $E^*(a_{R,j}h_{R+j}^*) = a_{R,j}P^{-1} \sum_{j=1}^{P-1} h_{R+j} + O(l/P) = a_{R,j}\bar{h}_P + O(l/P)$ , the RHS of (42) writes as,

$$\begin{aligned} & B^\dagger a_{R,0} \frac{1}{\sqrt{P}} \sum_{t=1}^R (h_t^* - h_t) + B^\dagger \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} (h_{R+i}^* - \bar{h}_P) \\ & - B^\dagger \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} (h_{R+i} - \bar{h}_P) + o_P^*(1), \quad \text{Pr } P. \end{aligned} \quad (43)$$

The sum of the first two terms in (43) satisfies a central limit theorem for mixing triangular arrays (Wooldridge and White (1988)) and, by Lemma A3, has asymptotic variance equal to  $2\Pi C_{00}$ , which is the same as the asymptotic variance of  $P^{-1/2} \sum_{t=R}^{T-1} (\hat{\theta}_t - \theta_t^\dagger)$  (see Lemma A5, in West (1996)), conditionally on the samples and for all samples but a subset of measure approaching zero. Therefore, it suffices to show that the last term on the RHS of (3), i.e. the adjustment term, is equal to  $B^\dagger \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} (h_{R+i} - \bar{h}_P)$ , up to a term vanishing asymptotically. Given A1 and A2,  $\left( -\frac{1}{T} \sum_{t=s}^{T-1} \nabla_{\theta_i}^2 q_i(y_t, Z^{t-1}, \hat{\theta}_T) \right)^{-1} - B^\dagger = o(1) \text{ Pr } P$  (i.e.  $o_P(1)$ ), where  $\hat{\theta}_T$  is the estimator constructed using all  $T$  observations.

Now let  $h_{R+i}(\hat{\theta}_T) = \nabla_{\theta}q(y_{R+i}, Z^{R+i-1}, \hat{\theta}_T)$ , and  $\bar{h}_P(\hat{\theta}_T) = P^{-1} \sum_{i=1}^P \nabla_{\theta}q(y_{R+i}, Z^{R+i-1}, \hat{\theta}_T)$ , and let  $\nabla^2 h_{R+i}(\hat{\theta}_T) = \nabla_{\theta}^2 q(y_{R+i}, Z^{R+i-1}, \hat{\theta}_T)$ . Now,

$$\begin{aligned} & B^\dagger \frac{1}{\sqrt{P}} \sum_{i=1}^{P-1} a_{R,i} \left( (h_{R+i}(\hat{\theta}_T) - \bar{h}_P(\hat{\theta}_T)) - (h_{R+i} - \bar{h}_P) \right) \\ & = B^\dagger \frac{1}{P} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla^2 h_{R+i}(\bar{\theta}_T) - \bar{\nabla}^2 h_P(\bar{\theta}_T) \right) \sqrt{P} (\hat{\theta}_T - \theta^\dagger) = o(1), \quad \text{Pr } P, \end{aligned} \quad (44)$$

as  $\sqrt{P} (\hat{\theta}_T - \theta_0^\dagger) = O(1) \text{ Pr } P$ , and by the uniform law of large numbers for mixing triangular arrays,  $\frac{1}{P} \sum_{i=1}^{P-1} a_{R,i} \left( \nabla^2 h_{R+i}(\bar{\theta}_T) - \bar{\nabla}^2 h_P(\bar{\theta}_T) \right) = o(1) \text{ Pr } P$ . The desired result then follows.

**Proof of Proposition 2:** Via a mean value expansion, and given A1,A2

$$\begin{aligned}
S_P(1, k) &= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g(\hat{u}_{1,t+1}) - g(\hat{u}_{k,t+1})) \\
&= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g(u_{1,t+1}) - g(u_{k,t+1})) \\
&\quad + \frac{1}{P} \sum_{t=R}^{T-1} g'(\bar{u}_{1,t+1}) \nabla_{\theta_1} \kappa_1(Z^t, \bar{\theta}_{1,t}) P^{1/2} (\hat{\theta}_{1,t} - \theta_1^\dagger) \\
&\quad - \frac{1}{P} \sum_{t=R}^{T-1} g'(\bar{u}_{k,t+1}) \nabla_{\theta_k} \kappa_k(Z^t, \bar{\theta}_{k,t}) P^{1/2} (\hat{\theta}_{k,t} - \theta_k^\dagger) \\
&= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g(u_{1,t+1}) - g(u_{k,t+1})) \\
&\quad + \mu_1 \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\hat{\theta}_{1,t} - \theta_1^\dagger) - \mu_k \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\hat{\theta}_{k,t} - \theta_k^\dagger) + o_P(1),
\end{aligned}$$

where  $\mu_1 = E(g'(u_{1,t+1}) \nabla_{\theta_1} \kappa_1(Z^t, \theta_1^\dagger))$ , and  $\mu_k$  is defined analogously. Now, when all competitors have the same predictive accuracy as the benchmark model, by the same argument as in Theorem 4.1 in West (1996),

$$(S_P(1, 2), \dots, S_P(1, n)) \xrightarrow{d} N(0, V),$$

where  $V$  is a  $n \times n$  matrix with  $i, j$  element  $v_{i,j} = S_{g_i g_j} + 2\Pi C_{ii} + 2\Pi C_{jj} - 2\Pi C_{ij}$ , where  $C_{ij} = \sum_{j=-\infty}^{\infty} E((\nabla_{\theta} q_i(y_{1+s}, Z^s, \theta_i^\dagger)) (\nabla_{\theta} q_j(y_{1+s+j}, Z^{s+j}, \theta_j^\dagger))')$ ,  $\Pi = 1 - \pi^{-1} \ln(1 + \pi)$ . The distribution of  $S_P$  then follows straightforwardly from the continuous mapping theorem.

**Proof of Proposition 3:** For brevity, we just analyze model 1.

$$\begin{aligned}
\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g(\hat{u}_{1,t+1}^*) - g(\hat{u}_{1,t+1})) &= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g(u_{1,t+1}^*) - g(u_{1,t+1})) \\
&\quad + \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\nabla_{\theta_1} g(\bar{u}_{1,t+1}^*) (\hat{\theta}_{1,t}^* - \theta_1^\dagger) - \nabla_{\theta_1} g(\bar{u}_{1,t+1}) (\hat{\theta}_{1,t} - \theta_1^\dagger)),
\end{aligned} \tag{45}$$

where  $\bar{u}_{1,t+1}^* = y_{t+1} - \kappa_1(Z^{*,t}, \bar{\theta}_{1,t}^*)$ ,  $\bar{u}_{1,t+1} = y_{t+1} - \kappa_1(Z^t, \bar{\theta}_{1,t})$ ,  $\bar{\theta}_{1,t}^* \in (\hat{\theta}_{1,t}^*, \theta_1^\dagger)$  and  $\bar{\theta}_{1,t} \in (\hat{\theta}_{1,t}, \theta_1^\dagger)$ .

As an almost straightforward consequence of Theorem 3.5 in Kunsch (1989), the first term on the RHS of (45) has the same limiting distribution as  $P^{-1/2} \sum_{t=R}^{T-1} (g(u_{1,t+1}) - E(g(u_{1,t+1})))$ . As for

the second line in (45), it writes as:

$$\begin{aligned}
& \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \nabla_{\theta_1} g(\bar{u}_{1,t+1}^*) (\hat{\theta}_{1,t}^* - \hat{\theta}_{1,t}) - \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\nabla_{\theta_1} g(\bar{u}_{1,t+1}^*) - \nabla_{\theta_1} g(\bar{u}_{1,t+1})) (\hat{\theta}_{1,t} - \theta_1^\dagger) \\
&= \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \nabla_{\theta_1} g(\bar{u}_{1,t+1}^*) (\hat{\theta}_{1,t}^* - \hat{\theta}_{1,t}) + o_P^*(1), \quad \text{Pr}-P \\
&= \mu_1 B_1^\dagger \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (h_{1,t}^* - h_{1,t}) + o_P^*(1), \quad \text{Pr}-P,
\end{aligned} \tag{46}$$

where  $h_{1,t+1}^* = \nabla_{\theta_1} q_1(y_{t+1}^*, Z^{*,t}, \theta_1^\dagger)$ ,  $h_{1,t+1} = \nabla_{\theta_1} q_1(y_{t+1}, Z^t, \theta_1^\dagger)$ . Also, the last line in (46) writes as:

$$\begin{aligned}
& \mu_1 B_1^\dagger \left( a_{R,0}^2 \frac{1}{P^{1/2}} \sum_{t=1}^R (h_{1,t}^* - h_{1,t}) + \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i}^* - \bar{h}_{1,P}) \right) \\
& - \mu_1 B_1^\dagger \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i} - \bar{h}_{1,P}) + o_P^*(1), \quad \text{Pr}-P,
\end{aligned} \tag{47}$$

where  $\bar{h}_{1,P}$  is the sample average of  $h_{1,t}$  computed over the last  $P$  observations. Given Lemma A3, by the same argument used in the proof of Theorem 1, the first line in (47) has the same limiting distribution as  $\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\hat{\theta}_{1,t} - \theta_1^\dagger)$ , conditional on sample. Therefore we need to show that the correction term for model 1 (i.e. second and third, fourth line on the RHS of (8)) offsets the second line in (47), up to a  $o(1)$   $\text{Pr}-P$  term. Let  $h_{1,t+1}(\hat{\theta}_{1,T}) = \nabla_{\theta_1} q_1(y_{t+1}, Z^t, \hat{\theta}_{1,T})$  and  $\bar{h}_{1,P}(\hat{\theta}_{1,T})$  is the sample average of  $h_{1,t+1}(\hat{\theta}_{1,T})$ , over the last  $P$  observations. Now, by the uniform law of large numbers

$$\frac{1}{T} \sum_{t=s}^{T-1} \nabla_{\theta_1} g(\bar{u}_{1,t+1}^*) \left( \frac{1}{T} \sum_{t=s}^{T-1} \nabla_{\theta_1}^2 q_1(y_t^*, Z^{*,t-1}, \hat{\theta}_{1,T}) \right)^{-1} - \mu_1 B_1^\dagger = o_P^*(1), \quad \text{Pr}-P.$$

Also, by the same argument used in the proof of Theorem 1, it follows that,

$$\frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i} - \bar{h}_{1,P}) - \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i}(\hat{\theta}_{1,T}) - \bar{h}_{1,P}(\hat{\theta}_{1,T})) = o(1), \quad \text{Pr}-P.$$

**Proof of Proposition 4:** From Theorem 1 in Corradi and Swanson (2002).

**Proof of Proposition 5:** Recall that  $g = q_1$ , also let  $\bar{u}_{1,t+1}^* = y_{t+1}^* - (1 \ y_t^*) \bar{\theta}_{1,t}^*$ , where  $\bar{\theta}_{1,t}^* \in (\hat{\theta}_{1,t}^*, \theta_1^\dagger)$ , and  $\hat{\theta}_{1,t}^*$  is defined in (15). Then,

$$\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(\hat{u}_{1,t+1}^*) w(Z^{*,t}, \gamma) - g'(\hat{u}_{1,t+1}) w(Z^t, \gamma)) = \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(u_{1,t+1}^*) w(Z^{*,t}, \gamma) - g'(u_{1,t+1}) g(Z^t, \gamma))$$

$$\begin{aligned}
& + \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\nabla_{\theta_1} g'(\bar{u}_{1,t+1}^*))' w(Z^{*,t}, \gamma) (\hat{\theta}_{1,t}^* - \theta_1^\dagger) - \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\nabla_{\theta_1} g'(\bar{u}_{1,t+1}))' w(Z^t, \gamma) (\hat{\theta}_{1,t} - \theta_1^\dagger) \\
= & \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(u_{1,t+1}^*) w(Z^{*,t}, \gamma) - g'(u_{1,t+1}) w(Z^t, \gamma)) + \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (\nabla_{\theta_1} g'(\bar{u}_{1,t+1}^*))' w(Z^{*,t}, \gamma) (\hat{\theta}_{1,t}^* - \hat{\theta}_{1,t}) \\
& + \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} ((\nabla_{\theta_1} g'(\bar{u}_{1,t+1}^*))' w(Z^{*,t}, \gamma) - (\nabla_{\theta_1} g'(\bar{u}_{1,t+1}))' w(Z^t, \gamma)) (\hat{\theta}_{1,t} - \theta_1^\dagger), \tag{48}
\end{aligned}$$

the last term in (48) is  $o_P^*(1)$   $\text{Pr}-P$ , as given A3,  $\sup_{t \geq R} P^{1/2} (\hat{\theta}_{1,t} - \theta_1^\dagger) = O_P(1)$ , and

$$\sup_{\theta_1 \in \Theta_1} \frac{1}{P} \sum_{t=R}^{T-1} ((\nabla_{\theta_1} g'(u_{1,t+1}^*(\theta_1)))' w(Z^{*,t}, \gamma) - (\nabla_{\theta_1} g'(u_{1,t+1}(\theta_1)))' w(Z^t, \gamma)) = o_{P^*}(1), \quad \text{Pr}-P,$$

where  $u_{1,t+1}^*(\theta_1) = y_{t+1}^* - (1 \ y_t^*) \theta_1$ , and  $u_{1,t+1}(\theta_1) = y_{t+1} - (1 \ y_t) \theta_1$ . Now, pointwise in  $\gamma$ , the first term on the RHS of the last equality in (48), has the same limiting distribution as  $\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(u_{1,t+1}) w(Z^t, \gamma) - E(g'(u_{1,t+1}) w(Z^t, \gamma)))$ . Stochastic equicontinuity on  $\Gamma$  can be shown along the lines of Theorem 2 in Corradi and Swanson (2002). Therefore, under  $H_0$ , any continuous functional over  $\Gamma$  of  $\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(u_{1,t+1}^*) w(Z^{*,t}, \gamma) - g'(u_{1,t+1}) w(Z^t, \gamma))$  has the same limiting distribution of the same functional of  $\frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (g'(u_{1,t+1}) w(Z^t, \gamma) - E(g'(u_{1,t+1}) w(Z^t, \gamma)))$ , conditional on the sample and for all samples but a subset of measure approaching zero. By the same argument used in the proof of Proposition 3, the last term on the RHS of the last equality in (48) is  $o_P^*(1)$   $\text{Pr}-P$ . It now remains to consider the second term on the RHS of the last equality in (48). Hereafter, let  $\bar{\nabla}_{\theta_1} g_P = P^{-1} \sum_{i=1}^{P-1} \nabla_{\theta_1} g(u_{1,i})$ , and  $\mu_\gamma = E(\nabla_{\theta_1} g'(u_{1,t+1}) w(Z^t, \gamma))$ . Recalling that  $q_1 = g$ , by the same argument used above, it writes as,

$$\begin{aligned}
& \mu'_\gamma \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \left( \left( -\frac{1}{t} \sum_{j=s}^t \nabla_{\theta_1}^2 g(\bar{u}_{1,j}^*) \right)^{-1} \left( \frac{1}{t} \sum_{j=s}^t \nabla_{\theta_1} g(u_{1,j}^*) \right) \right. \\
& \quad \left. - \left( -\frac{1}{t} \sum_{j=s}^t \nabla_{\theta_1}^2 g(\bar{u}_{1,j}) \right)^{-1} \left( \frac{1}{t} \sum_{j=s}^t \nabla_{\theta_1} g(u_{1,j}) \right) \right) + o_P^*(1), \quad \text{Pr}-P \\
= & \mu'_\gamma B^\dagger \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} \left( \frac{1}{t} \sum_{j=s}^t (\nabla_{\theta_1} g(u_{1,j}^*) - \nabla_{\theta_1} g(u_{1,j})) \right) + o_P^*(1), \quad \text{Pr}-P \\
= & \mu'_\gamma B^\dagger \frac{a_{R,0}}{P^{1/2}} \sum_{t=s}^R (\nabla_{\theta_1} g(u_{1,j}^*) - \nabla_{\theta_1} g(u_{1,j})) + \mu'_\gamma B^\dagger \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (\nabla_{\theta_1} g(u_{1,R+i}^*) - \bar{\nabla}_{\theta_1} g_P) \\
& - \mu'_\gamma B^\dagger \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (\nabla_{\theta_1} g(u_{1,R+i}) - \bar{\nabla}_{\theta_1} g_P) + o_P^*(1), \quad \text{Pr}-P.
\end{aligned}$$

Now, by the same argument as in the proof of Theorem 1, the sum of the first two terms on the RHS of the last equality, mimic the contribution of parameter estimation error, while the last term is offset by the adjustment term. The desired outcome then follows.

**Proof of Proposition 6:** This proof requires just a small modification to the proof of Theorem 1 in Corradi and Swanson (2003b). In fact, the only difference is that in the current context parameters are estimated recursively. Also, recall that parameters are estimated by QMLE (using the density associated with the candidate conditional model), so that  $q_i = -\ln f_i$ ,  $i = 1, \dots, n$  with  $f_i$  being the density associated with  $F_i$ . Let  $\mu_i^2(u) = E \left( \left( 1\{y_{t+1} \leq u\} - F_i(u|Z^t, \theta_i^\dagger) \right)^2 \right)$   
 $= E \left( \left( 1\{y_{t+1} \leq u\} - F_0(u|Z^t, \theta_0) \right)^2 \right) + E \left( \left( F_0(u|Z^t, \theta_0) - F_i(u|Z^t, \theta_i^\dagger) \right)^2 \right)$ . Thus,

$$\begin{aligned}
Z_{P,u}(1, k) &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \hat{\theta}_{1,t}) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \hat{\theta}_{k,t}) \right)^2 \right) \\
&= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \hat{\theta}_{1,t}) \right)^2 - \mu_1^2(u) \right) \\
&\quad - \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \hat{\theta}_{k,t}) \right)^2 - \mu_k^2(u) \right) + \sqrt{P}(\mu_1^2(u) - \mu_k^2(u)) \\
&= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right)^2 - \mu_1^2(u) \right) - \frac{1}{\sqrt{P}} \sum_{t=s}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \theta_k^\dagger) \right)^2 - \mu_k^2(u) \right) \\
&\quad - \frac{2}{P} \sum_{t=R}^{T-1} \nabla_{\theta_1} F_1(u|Z^t, \bar{\theta}_{1,t})' \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right) \sqrt{P} (\hat{\theta}_{1,t} - \theta_1^\dagger) \\
&\quad + \frac{2}{P} \sum_{t=R}^{T-1} \nabla_{\theta_k} F_k(u|Z^t, \bar{\theta}_{k,t})' \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \theta_k^\dagger) \right) \sqrt{P} (\hat{\theta}_{k,t} - \theta_k^\dagger) \\
&\quad + \sqrt{P}(\mu_1^2(u) - \mu_k^2(u)) + o_P(1)
\end{aligned} \tag{49}$$

where  $\bar{\theta}_{i,t} \in (\hat{\theta}_{i,t}, \theta_i^\dagger)$ ,  $i = 1, \dots, n$ , and where the  $o_P(1)$  term holds uniformly in  $u \in U$ . Now, given A1,A2 and A6, by the uniform law of large numbers for  $\beta$ -mixing processes,

$$\sup_{\theta_i \in \Theta_i} \left| \frac{1}{P} \sum_{t=R}^{T-1} \nabla_{\theta_i} F_i(u|Z^t, \theta_i)' \left( 1\{y_{t+1} \leq u\} - F_i(u|Z^t, \theta_i) \right) - \mu_{\theta_i} \right| = o_P(1).$$

Now,

$$\begin{aligned} \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} (\widehat{\theta}_{i,t} - \theta_i^\dagger) &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( -\frac{1}{t} \sum_{j=s}^t \nabla_{\theta_i}^2 \ln f_i(y_j | Z^{j-1}, \theta_i^\dagger) \right)^{-1} \left( -\frac{1}{t} \sum_{j=s}^t \nabla_{\theta_i} \ln f_i(y_j | Z^{j-1}, \theta_i^\dagger) \right) \\ &= B_i^\dagger \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( -\frac{1}{t} \sum_{j=s}^t \nabla_{\theta_i} \ln f_i(y_j | Z^{j-1}, \theta_i^\dagger) \right) + o_P(1), \end{aligned}$$

where  $B_i^\dagger = \left( E \left( -\nabla_{\theta_i}^2 \ln f_i(y_{t+1} | Z^t, \theta_i^\dagger) \right) \right)^{-1}$  and  $\mu_{\theta_i} = E \left( \nabla_{\theta_i} F_i(u | Z^t, \theta_i)' (1\{y_{t+1} \leq u\} - F_i(u | Z^t, \theta_i)) \right)$ .

For any given  $u, u'$  convergence in distribution to a normal with the same covariance as in the statement of the theorem follows by the same argument as in Proposition 6 and by the Cramer Wold device. Finally stochastic equicontinuity in  $u$  follows by the same argument as in the proof of Theorem 1 in Corradi and Swanson (2003b). Convergence of finite dimension distribution and stochastic equicontinuity ensure weak convergence to a process on  $U$ .

**Proof of Proposition 7:** As for the first two lines on the RHS of the bootstrap statistic, equation (21):

$$\begin{aligned} &\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( \left( 1\{y_{t+1}^* \leq u\} - F_1(u | Z^{*,t}, \widehat{\theta}_{1,t}^*) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_1(u | Z^t, \widehat{\theta}_{1,t}) \right)^2 \right) \right. \\ &\quad \left. - \left( \left( 1\{y_{t+1}^* \leq u\} - F_k(u | Z^{*,t}, \widehat{\theta}_{k,t}^*) \right)^2 - \left( 1\{y_{t+1} \leq u\} - F_k(u | Z^t, \widehat{\theta}_{k,t}) \right)^2 \right) \right) \\ &= \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \left( \left( \left( 1\{y_{t+1}^* \leq u\} - F_1(u | Z^{*,t}, \theta_1^\dagger) \right) - \nabla_{\theta_1} F_1(u | Z^{*,t}, \bar{\theta}_{1,t}) (\widehat{\theta}_{1,t}^* - \theta_1^\dagger) \right)^2 \right. \right. \\ &\quad \left. \left. - \left( \left( 1\{y_{t+1} \leq u\} - F_1(u | Z^t, \theta_1^\dagger) \right) - \nabla_{\theta_1} F_1(u | Z^t, \bar{\theta}_{1,t}) (\widehat{\theta}_{1,t} - \theta_1^\dagger) \right)^2 \right) \right. \\ &\quad \left. - \left( \left( \left( 1\{y_{t+1}^* \leq u\} - F_k(u | Z^{*,t}, \theta_k^\dagger) \right) - \nabla_{\theta_k} F_k(u | Z^{*,t}, \bar{\theta}_{k,t}) (\widehat{\theta}_{k,t}^* - \theta_k^\dagger) \right)^2 \right. \right. \\ &\quad \left. \left. - \left( \left( 1\{y_{t+1} \leq u\} - F_k(u | Z^t, \theta_k^\dagger) \right) - \nabla_{\theta_k} F_k(u | Z^t, \bar{\theta}_{k,t}) (\widehat{\theta}_{k,t} - \theta_k^\dagger) \right)^2 \right) \right), \quad (50) \end{aligned}$$

where  $\bar{\theta}_{i,t}^* \in (\widehat{\theta}_{i,t}^*, \theta_i^\dagger)$ ,  $\bar{\theta}_{i,t} \in (\widehat{\theta}_{i,t}, \theta_i^\dagger)$ . Now, by a similar argument as that used in the proof of

Theorem 2 in Corradi and Swanson (2003b), (50) writes as:

$$\begin{aligned}
& \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( F_1^2(u|Z^{*,t}, \theta_1^\dagger) - F_1^2(u|Z^t, \theta_1^\dagger) \right) \\
& - \frac{2}{\sqrt{P}} \sum_{t=R}^{T-1} \left( F_1(u|Z^{*,t}, \theta_1^\dagger) 1\{y_{t+1}^* \leq u\} - F_1(u|Z^t, \theta_1^\dagger) 1\{y_{t+1} \leq u\} \right) \\
& - \frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1}^* \leq u\} - F_1(u|Z^{*,t}, \theta_1^\dagger) \right) \nabla_{\theta_1} F_1(u|Z^{*,t}, \bar{\theta}_{1,t}^*)' \right) \sqrt{P} \left( \hat{\theta}_{1,t}^* - \theta_1^\dagger \right) \\
& + \frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right) \nabla_{\theta_1} F_1(u|Z^t, \bar{\theta}_{1,t})' \right) \sqrt{P} \left( \hat{\theta}_{1,t} - \theta_1^\dagger \right) \\
& - \frac{1}{\sqrt{P}} \sum_{t=s}^{T-1} \left( F_k^2(u|Z^{*,t}, \theta_k^\dagger) - F_k^2(u|Z^t, \theta_k^\dagger) \right) \\
& + \frac{2}{\sqrt{P}} \sum_{t=R}^{T-1} \left( F_k(u|Z^{*,t}, \theta_k^\dagger) 1\{y_{t+1}^* \leq u\} - F_k(u|Z^t, \theta_k^\dagger) 1\{y_{t+1} \leq u\} \right) \\
& + \frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1}^* \leq u\} - F_k(u|Z^{*,t}, \theta_k^\dagger) \right) \nabla_{\theta_k} F_k(u|Z^{*,t}, \bar{\theta}_{k,t}^*)' \right) \sqrt{P} \left( \hat{\theta}_{k,t}^* - \theta_k^\dagger \right) \\
& - \frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_k(u|Z^t, \theta_k^\dagger) \right) \nabla_{\theta_k} F_k(u|Z^t, \bar{\theta}_{k,t})' \right) \sqrt{P} \left( \hat{\theta}_{k,t} - \theta_k^\dagger \right) + o_{P^*}(1), \Pr - (B1)
\end{aligned}$$

As shown in the proof of Theorem 2 in Corradi and Swanson (2003b),  $\frac{1}{\sqrt{T}} \sum_{t=s}^T \left( F_i^2(u|Z^{*,t}, \theta_i^\dagger) - F_i^2(u|Z^t, \theta_i^\dagger) \right)$  has the same limiting distribution as  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( F_i^2(u|Z^t, \theta_i^\dagger) - E(F_i^2(u|Z^t, \theta_1^\dagger)) \right)$ , as a process over  $U$ , and  $\frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( F_i(u|Z^{*,t}, \theta_i^\dagger) 1\{y_{t+1}^* \leq u\} - F_i(u|Z^t, \theta_i^\dagger) 1\{y_{t+1} \leq u\} \right)$  has the same limiting distribution as  $\frac{1}{\sqrt{P}} \sum_{t=s}^{T-1} \left( F_i(u|Z^t, \theta_i^\dagger) 1\{y_{t+1} \leq u\} - E(F_i(u|Z^t, \theta_i^\dagger) 1\{y_{t+1} \leq u\}) \right)$ , as a process over  $U$ .

For sake of simplicity we just analyze the parameter estimation error component of model 1.

Now, by a similar argument as that used in the proof of Proposition 3,

$$\begin{aligned}
& -\frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1}^* \leq u\} - F_1(u|Z^{*,t}, \theta_1^\dagger) \right) \nabla_{\theta_1} F_1(u|Z^{*,t}, \bar{\theta}_{1,t}^*)' \right) \sqrt{P} \left( \hat{\theta}_{1,t}^* - \theta_1^\dagger \right) \\
& + \frac{2}{P} \sum_{t=R}^{T-1} \left( \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right) \nabla_{\theta_1} F_1(u|Z^t, \bar{\theta}_{1,t})' \right) \sqrt{P} \left( \hat{\theta}_{1,t} - \theta_1^\dagger \right) \\
= & -2\mu_{\theta_1^\dagger} A(\theta_1^\dagger) \sqrt{P} \left( \hat{\theta}_{1,t}^* - \hat{\theta}_{1,t} \right) + o_p^*(1), \quad \text{Pr}-P \\
= & -2\mu_{\theta_1^\dagger} A(\theta_1^\dagger) \frac{1}{\sqrt{P}} \sum_{t=R}^{T-1} \left( \frac{1}{t} \sum_{j=s}^t \left( \ln f_1(y_j^*|Z^{*,j-1}, \theta_1^\dagger) - \ln f_1(y_j|Z^{j-1}, \theta_1^\dagger) \right) \right) + o_p^*(1), \quad \text{Pr}-R52
\end{aligned}$$

where  $m_{\theta_1^\dagger}(u)' = E \left( \nabla_{\theta_1} F_1(u|Z^t, \theta_1^\dagger)' \left( 1\{y_{t+1} \leq u\} - F_1(u|Z^t, \theta_1^\dagger) \right) \right)$  and  $A(\theta_1^\dagger) = \left( E \left( -\nabla_{\theta_1}^2 f_1(y_{t+1}|Z^t, \theta_1^\dagger) \right) \right)^{-1}$ .

Let  $h_{1,t+1}^* = \nabla_{\theta_1} \ln f_1(y_{t+1}^*|Z^{*,t}, \theta_1^\dagger)$ ,  $h_{1,t+1} = \nabla_{\theta_1} \ln f_1(y_{t+1}|Z^t, \theta_1^\dagger)$ . Also, the last line in (52)

writes as:

$$\begin{aligned}
& -2\mu_{\theta_1^\dagger} A(\theta_1^\dagger) \left( a_{R,0}^2 \frac{1}{P^{1/2}} \sum_{t=R}^{T-1} (h_{1,t}^* - h_{1,t}) + \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i}^* - \bar{h}_{1,P}) \right) \\
& + 2\mu_{\theta_1^\dagger} A(\theta_1^\dagger) \frac{1}{P^{1/2}} \sum_{i=1}^{P-1} a_{R,i} (h_{1,R+i} - \bar{h}_{1,P}) + o_P^*(1), \quad \text{Pr}-P.
\end{aligned}$$

The desired outcome then follows by the same argument as in the proof of Theorem 1 and of Proposition 3.

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Table 1: Finite Sample Properties of the Bootstrap for Parameter Estimation Error: Recursive Scheme with Adjustment Terms versus Full Sample Scheme: Part I<sup>(\*)</sup>

Panel A: DGP is an AR(1) Process - $\rho = 0.2$													
<i>smpl</i>	<i>boot</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$	
600	<i>rec</i>	$\hat{\rho}$	0.732	0.800	0.896	0.870	0.912	0.904	0.894	0.870	0.864	0.844	
		$\hat{\rho}_1$	0.730	0.812	0.904	0.890	0.910	0.898	0.900	0.884	0.878	0.850	
		$\hat{\rho}_2$	0.898	0.898	0.908	0.916	0.938	0.910	0.896	0.908	0.862	0.856	
	<i>full</i>	$\hat{\rho}$	0.655	0.875	0.870	0.850	0.890	0.905	0.900	0.940	0.895	0.915	
		$\hat{\rho}_1$	0.665	0.875	0.865	0.880	0.885	0.920	0.900	0.925	0.895	0.900	
		$\hat{\rho}_2$	0.865	0.905	0.905	0.940	0.910	0.940	0.920	0.935	0.885	0.890	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$													
1200	<i>rec</i>	$\hat{\rho}$	0.612	0.868	0.902	0.900	0.924	0.926	0.908	0.880	0.904	0.882	
		$\hat{\rho}_1$	0.624	0.874	0.900	0.904	0.934	0.918	0.900	0.896	0.914	0.890	
		$\hat{\rho}_2$	0.874	0.924	0.924	0.918	0.918	0.888	0.940	0.912	0.884	0.866	
	<i>full</i>	$\hat{\rho}$	0.560	0.855	0.925	0.930	0.905	0.910	0.940	0.905	0.915	0.885	
		$\hat{\rho}_1$	0.595	0.880	0.920	0.915	0.925	0.925	0.940	0.900	0.915	0.885	
		$\hat{\rho}_2$	0.880	0.910	0.915	0.930	0.915	0.880	0.925	0.925	0.925	0.910	
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.436	0.780	0.898	0.934	0.928	0.920	0.914	0.910	0.934	0.888	
		$\hat{\rho}_1$	0.452	0.796	0.908	0.934	0.924	0.920	0.918	0.920	0.928	0.888	
		$\hat{\rho}_2$	0.890	0.904	0.930	0.932	0.914	0.924	0.930	0.938	0.892	0.894	
	<i>full</i>	$\hat{\rho}$	0.315	0.820	0.890	0.920	0.930	0.930	0.930	0.950	0.930	0.880	
		$\hat{\rho}_1$	0.350	0.815	0.880	0.940	0.930	0.930	0.935	0.935	0.940	0.900	
		$\hat{\rho}_2$	0.890	0.920	0.955	0.925	0.935	0.955	0.935	0.950	0.925	0.935	
Panel B: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.1$													
$l = 4 \quad l = 6 \quad l = 10 \quad l = 12 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 50 \quad l = 60$													
600	<i>rec</i>	$\hat{\rho}$	0.828	0.850	0.898	0.892	0.910	0.894	0.894	0.864	0.868	0.834	
		$\hat{\rho}_1$	0.862	0.872	0.914	0.902	0.924	0.908	0.912	0.878	0.880	0.834	
		$\hat{\rho}_2$	0.832	0.886	0.900	0.920	0.912	0.896	0.904	0.890	0.886	0.840	
	<i>full</i>	$\hat{\rho}$	0.805	0.830	0.900	0.860	0.910	0.875	0.910	0.920	0.870	0.900	
		$\hat{\rho}_1$	0.825	0.865	0.905	0.885	0.930	0.880	0.950	0.925	0.885	0.920	
		$\hat{\rho}_2$	0.835	0.860	0.905	0.870	0.920	0.895	0.925	0.875	0.905	0.935	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
1200	<i>rec</i>	$\hat{\rho}$	0.794	0.892	0.916	0.918	0.914	0.906	0.914	0.898	0.886	0.872	
		$\hat{\rho}_1$	0.816	0.906	0.928	0.926	0.912	0.912	0.916	0.900	0.896	0.882	
		$\hat{\rho}_2$	0.786	0.912	0.920	0.940	0.932	0.902	0.930	0.892	0.912	0.856	
	<i>full</i>	$\hat{\rho}$	0.750	0.875	0.915	0.890	0.910	0.905	0.925	0.920	0.885	0.870	
		$\hat{\rho}_1$	0.765	0.890	0.925	0.895	0.915	0.920	0.925	0.920	0.880	0.865	
		$\hat{\rho}_2$	0.815	0.905	0.895	0.910	0.880	0.925	0.905	0.935	0.905	0.940	
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.702	0.882	0.894	0.938	0.914	0.936	0.910	0.920	0.922	0.896	
		$\hat{\rho}_1$	0.734	0.892	0.898	0.936	0.912	0.942	0.912	0.920	0.926	0.902	
		$\hat{\rho}_2$	0.738	0.890	0.916	0.940	0.912	0.910	0.938	0.906	0.904	0.916	
	<i>full</i>	$\hat{\rho}$	0.650	0.795	0.885	0.905	0.905	0.930	0.940	0.915	0.915	0.885	
		$\hat{\rho}_1$	0.690	0.825	0.910	0.905	0.915	0.925	0.945	0.925	0.920	0.890	
		$\hat{\rho}_2$	0.695	0.900	0.945	0.950	0.930	0.935	0.910	0.925	0.905	0.905	

(\*) Notes: The second column lists the bootstrap used to examine parameter estimation error (PEE) associated with either an AR(1) autoregressive parameter ( $\hat{\rho}$ ) or two autoregressive parameters from an AR(2) model ( $\hat{\rho}_1$  and  $\hat{\rho}_2$ ). Entries corresponding to *rec* correspond to coverage probabilities based on the recursive PEE bootstrap with adjustment terms, in the context of a recursive estimation scheme, so that in our framework they denote the proportion of times that the (pseudo) true parameter  $\rho$  falls into  $CI^{**}$ , with  $\alpha$  equal to 0.05. Results based on different values of  $\alpha$  are available upon request. In the context of the recursive scheme,  $P$  recursive estimators are constructed, where  $P=0.5T$  and  $T$  is the *full* sample, set at either 600, 1200, or 2400 observations. Entries corresponding to *full* are coverage probabilities for the case where no recursive estimation is carried out, the entire sample is used for *one time* estimation of the parameters, and the standard block bootstrap is used to mimic the distribution of PEE. Data are generated according to either an AR(1) process (Panel A) or an AR(2) process (Panel B), and the autoregressive parameters of the DGPs are given in the header line for each panel. Block lengths used are denoted by the various values of  $l$ . In all experiments, 500 Monte Carlo iterations were carried out (see above for further details).

Table 2: Finite Sample Properites of the Bootstrap for Parameter Estimation Error: Recursive Scheme with Adjustment Terms versus Full Sample Scheme: Part II<sup>(\*)</sup>

Panel A: DGP is an AR(1) Process - $\rho = 0.4$													
<i>smpl</i>	<i>boot</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$	
600	<i>rec</i>	$\hat{\rho}$	0.340	0.590	0.830	0.815	0.815	0.880	0.855	0.840	0.870	0.840	
		$\hat{\rho}_1$	0.420	0.635	0.850	0.825	0.860	0.905	0.890	0.870	0.905	0.870	
		$\hat{\rho}_2$	0.930	0.920	0.880	0.945	0.925	0.910	0.905	0.885	0.865	0.885	
	<i>full</i>	$\hat{\rho}$	0.220	0.585	0.750	0.800	0.835	0.880	0.905	0.895	0.880	0.880	
		$\hat{\rho}_1$	0.330	0.610	0.770	0.850	0.820	0.900	0.900	0.910	0.905	0.890	
		$\hat{\rho}_2$	0.855	0.870	0.950	0.895	0.945	0.865	0.930	0.890	0.890	0.925	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$													
1200	<i>rec</i>	$\hat{\rho}$	0.120	0.660	0.835	0.900	0.890	0.890	0.890	0.895	0.880	0.845	
		$\hat{\rho}_1$	0.195	0.720	0.845	0.910	0.895	0.890	0.910	0.910	0.900	0.860	
		$\hat{\rho}_2$	0.910	0.915	0.930	0.940	0.935	0.945	0.930	0.905	0.945	0.900	
	<i>full</i>	$\hat{\rho}$	0.065	0.605	0.730	0.865	0.875	0.870	0.895	0.905	0.885	0.885	
		$\hat{\rho}_1$	0.120	0.660	0.785	0.880	0.885	0.880	0.910	0.910	0.905	0.900	
		$\hat{\rho}_2$	0.885	0.920	0.905	0.930	0.915	0.910	0.940	0.900	0.935		
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.015	0.495	0.810	0.840	0.900	0.910	0.890	0.910	0.895	0.895	
		$\hat{\rho}_1$	0.030	0.605	0.825	0.860	0.925	0.915	0.880	0.900	0.925	0.900	
		$\hat{\rho}_2$	0.905	0.920	0.960	0.925	0.920	0.910	0.920	0.895	0.925	0.875	
	<i>full</i>	$\hat{\rho}$	0.000	0.415	0.770	0.850	0.885	0.905	0.905	0.915	0.945	0.905	
		$\hat{\rho}_1$	0.010	0.485	0.820	0.900	0.890	0.900	0.915	0.935	0.935	0.915	
		$\hat{\rho}_2$	0.890	0.905	0.935	0.920	0.910	0.920	0.945	0.940	0.905	0.915	
Panel B: DGP is an AR(2) Process $\rho_1 = \rho_2 = 0.2$													
$l = 4 \quad l = 6 \quad l = 10 \quad l = 12 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 50 \quad l = 60$													
600	<i>rec</i>	$\hat{\rho}$	0.710	0.825	0.895	0.885	0.915	0.870	0.885	0.845	0.885	0.845	
		$\hat{\rho}_1$	0.790	0.880	0.925	0.885	0.905	0.875	0.895	0.885	0.900	0.865	
		$\hat{\rho}_2$	0.745	0.815	0.880	0.845	0.885	0.895	0.890	0.880	0.870	0.845	
	<i>full</i>	$\hat{\rho}$	0.540	0.805	0.840	0.890	0.875	0.910	0.900	0.870	0.905	0.885	
		$\hat{\rho}_1$	0.620	0.860	0.870	0.925	0.925	0.940	0.890	0.885	0.890	0.905	
		$\hat{\rho}_2$	0.690	0.840	0.895	0.865	0.890	0.910	0.890	0.905	0.930	0.900	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$													
1200	<i>rec</i>	$\hat{\rho}$	0.530	0.790	0.880	0.905	0.900	0.880	0.935	0.895	0.940	0.900	
		$\hat{\rho}_1$	0.605	0.855	0.910	0.940	0.930	0.885	0.945	0.895	0.960	0.895	
		$\hat{\rho}_2$	0.660	0.860	0.880	0.910	0.910	0.915	0.905	0.875	0.925	0.870	
	<i>full</i>	$\hat{\rho}$	0.400	0.815	0.870	0.900	0.880	0.925	0.920	0.895	0.905	0.860	
		$\hat{\rho}_1$	0.525	0.875	0.900	0.935	0.895	0.945	0.920	0.885	0.915	0.875	
		$\hat{\rho}_2$	0.590	0.820	0.860	0.925	0.895	0.890	0.930	0.930	0.915	0.925	
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.300	0.775	0.875	0.915	0.920	0.925	0.920	0.925	0.890	0.890	
		$\hat{\rho}_1$	0.395	0.790	0.905	0.945	0.945	0.940	0.915	0.930	0.890	0.880	
		$\hat{\rho}_2$	0.430	0.810	0.925	0.860	0.925	0.920	0.905	0.895	0.930	0.915	
	<i>full</i>	$\hat{\rho}$	0.285	0.720	0.885	0.930	0.935	0.920	0.935	0.935	0.905	0.870	
		$\hat{\rho}_1$	0.360	0.760	0.890	0.925	0.955	0.925	0.945	0.925	0.905	0.900	
		$\hat{\rho}_2$	0.310	0.775	0.860	0.915	0.920	0.910	0.875	0.895	0.945	0.925	

(\*) Notes: See notes to Table 1.

Table 3: Finite Sample Properites of the Bootstrap for Parameter Estimation Error: Recursive Scheme with Adjustment Terms versus Full Sample Scheme: Part III(\*)

			Panel A: DGP is an AR(1) Process - $\rho = 0.6$										
<i>smpl</i>	<i>boot</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$	
600	<i>rec</i>	$\hat{\rho}$	0.945	0.730	0.385	0.360	0.290	0.150	0.180	0.145	0.140	0.150	
		$\hat{\rho}_1$	0.805	0.510	0.240	0.250	0.175	0.095	0.110	0.120	0.120	0.115	
		$\hat{\rho}_2$	0.130	0.070	0.110	0.065	0.115	0.085	0.075	0.110	0.110	0.130	
	<i>full</i>	$\hat{\rho}$	0.945	0.815	0.435	0.355	0.290	0.225	0.175	0.160	0.120	0.155	
		$\hat{\rho}_1$	0.865	0.630	0.325	0.260	0.210	0.130	0.135	0.095	0.080	0.170	
		$\hat{\rho}_2$	0.060	0.070	0.060	0.055	0.105	0.075	0.075	0.075	0.115	0.090	
			$l = 4$	$l = 10$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 40$	$l = 50$	$l = 60$	$l = 100$	
1200	<i>rec</i>	$\hat{\rho}$	1.000	0.630	0.350	0.290	0.195	0.235	0.150	0.120	0.130	0.095	
		$\hat{\rho}_1$	0.975	0.515	0.235	0.170	0.140	0.100	0.090	0.105	0.105	0.145	
		$\hat{\rho}_2$	0.085	0.055	0.075	0.050	0.075	0.085	0.065	0.075	0.085	0.125	
	<i>full</i>	$\hat{\rho}$	1.000	0.675	0.405	0.310	0.240	0.195	0.155	0.150	0.110	0.135	
		$\hat{\rho}_1$	1.000	0.485	0.295	0.200	0.160	0.135	0.070	0.130	0.095	0.080	
		$\hat{\rho}_2$	0.085	0.060	0.055	0.045	0.085	0.090	0.050	0.075	0.095	0.100	
			$l = 4$	$l = 10$	$l = 20$	$l = 30$	$l = 40$	$l = 50$	$l = 60$	$l = 80$	$l = 100$	$l = 120$	
2400	<i>rec</i>	$\hat{\rho}$	1.000	0.880	0.440	0.245	0.185	0.115	0.145	0.105	0.090	0.095	
		$\hat{\rho}_1$	1.000	0.735	0.300	0.210	0.130	0.110	0.110	0.080	0.100	0.100	
		$\hat{\rho}_2$	0.070	0.035	0.060	0.105	0.095	0.070	0.090	0.075	0.105	0.105	
	<i>full</i>	$\hat{\rho}$	1.000	0.920	0.495	0.320	0.220	0.190	0.145	0.105	0.135	0.140	
		$\hat{\rho}_1$	1.000	0.750	0.290	0.210	0.120	0.140	0.110	0.070	0.090	0.090	
		$\hat{\rho}_2$	0.080	0.050	0.060	0.100	0.060	0.065	0.090	0.065	0.120	0.080	
			Panel B: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.3$										
			$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$	
600	<i>rec</i>	$\hat{\rho}$	0.580	0.350	0.225	0.165	0.170	0.140	0.155	0.160	0.130	0.165	
		$\hat{\rho}_1$	0.460	0.205	0.095	0.105	0.070	0.095	0.120	0.125	0.125	0.125	
		$\hat{\rho}_2$	0.310	0.220	0.150	0.105	0.105	0.100	0.080	0.115	0.095	0.145	
	<i>full</i>	$\hat{\rho}$	0.640	0.450	0.225	0.180	0.145	0.160	0.195	0.135	0.120	0.105	
		$\hat{\rho}_1$	0.445	0.290	0.160	0.100	0.095	0.145	0.125	0.115	0.090	0.130	
		$\hat{\rho}_2$	0.420	0.295	0.200	0.120	0.130	0.090	0.090	0.080	0.110	0.155	
			$l = 4$	$l = 10$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 40$	$l = 50$	$l = 60$	$l = 100$	
1200	<i>rec</i>	$\hat{\rho}$	0.785	0.345	0.110	0.145	0.145	0.150	0.115	0.130	0.130	0.120	
		$\hat{\rho}_1$	0.645	0.235	0.075	0.095	0.110	0.090	0.100	0.090	0.135	0.135	
		$\hat{\rho}_2$	0.605	0.195	0.090	0.100	0.100	0.130	0.130	0.090	0.095	0.125	
	<i>full</i>	$\hat{\rho}$	0.870	0.270	0.180	0.160	0.145	0.085	0.095	0.080	0.130	0.090	
		$\hat{\rho}_1$	0.710	0.190	0.120	0.110	0.080	0.050	0.060	0.080	0.105	0.120	
		$\hat{\rho}_2$	0.620	0.235	0.140	0.100	0.110	0.070	0.095	0.065	0.080	0.150	
			$l = 4$	$l = 10$	$l = 20$	$l = 30$	$l = 40$	$l = 50$	$l = 60$	$l = 80$	$l = 100$	$l = 120$	
2400	<i>rec</i>	$\hat{\rho}$	0.950	0.370	0.225	0.175	0.095	0.130	0.130	0.075	0.080	0.145	
		$\hat{\rho}_1$	0.875	0.280	0.155	0.115	0.065	0.085	0.090	0.065	0.055	0.095	
		$\hat{\rho}_2$	0.880	0.310	0.110	0.065	0.085	0.125	0.020	0.070	0.095	0.115	
	<i>full</i>	$\hat{\rho}$	0.970	0.435	0.180	0.130	0.105	0.130	0.070	0.055	0.085	0.115	
		$\hat{\rho}_1$	0.935	0.345	0.140	0.110	0.090	0.090	0.075	0.030	0.070	0.085	
		$\hat{\rho}_2$	0.925	0.365	0.170	0.120	0.095	0.060	0.120	0.050	0.050	0.100	

(\*) Notes: See notes to Table 1.

Table 4: Finite Sample Properites of the Bootstrap for Parameter Estimation Error: Recursive Scheme with Adjustment Terms versus Full Sample Scheme: Part IV(\*)

Panel A: DGP is an AR(1) Process - $\rho = 0.8$													
<i>smpl</i>	<i>boot</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$	
600	<i>rec</i>	$\hat{\rho}$	0.000	0.025	0.285	0.420	0.545	0.675	0.695	0.775	0.780	0.830	
		$\hat{\rho}_1$	0.095	0.345	0.710	0.765	0.800	0.895	0.860	0.850	0.870	0.855	
		$\hat{\rho}_2$	0.940	0.975	0.960	0.945	0.925	0.960	0.925	0.905	0.920	0.850	
	<i>full</i>	$\hat{\rho}$	0.000	0.010	0.165	0.295	0.415	0.620	0.695	0.735	0.795	0.830	
		$\hat{\rho}_1$	0.040	0.240	0.615	0.685	0.815	0.835	0.895	0.900	0.880	0.880	
		$\hat{\rho}_2$	0.940	0.960	0.910	0.945	0.950	0.970	0.960	0.955	0.920	0.915	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$													
1200	<i>rec</i>	$\hat{\rho}$	0.000	0.040	0.305	0.460	0.595	0.695	0.730	0.785	0.815	0.860	
		$\hat{\rho}_1$	0.000	0.405	0.635	0.770	0.770	0.875	0.885	0.905	0.885	0.885	
		$\hat{\rho}_2$	0.960	0.910	0.920	0.935	0.955	0.950	0.930	0.905	0.880	0.875	
	<i>full</i>	$\hat{\rho}$	0.000	0.050	0.190	0.390	0.530	0.620	0.680	0.790	0.795	0.840	
		$\hat{\rho}_1$	0.000	0.325	0.630	0.735	0.795	0.845	0.855	0.900	0.915	0.925	
		$\hat{\rho}_2$	0.965	0.970	0.925	0.940	0.960	0.935	0.915	0.920	0.960	0.945	
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.000	0.000	0.220	0.545	0.635	0.710	0.805	0.835	0.840	0.845	
		$\hat{\rho}_1$	0.000	0.175	0.615	0.800	0.825	0.885	0.935	0.905	0.900	0.910	
		$\hat{\rho}_2$	0.950	0.955	0.945	0.960	0.950	0.940	0.945	0.920	0.950	0.930	
	<i>full</i>	$\hat{\rho}$	0.000	0.000	0.190	0.480	0.570	0.750	0.795	0.865	0.805	0.895	
		$\hat{\rho}_1$	0.000	0.085	0.515	0.755	0.830	0.890	0.930	0.920	0.885	0.930	
		$\hat{\rho}_2$	0.960	0.950	0.945	0.940	0.930	0.960	0.955	0.940	0.915	0.940	
Panel B: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.4$													
$l = 4 \quad l = 6 \quad l = 10 \quad l = 12 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 50 \quad l = 60$													
600	<i>rec</i>	$\hat{\rho}$	0.155	0.360	0.650	0.665	0.755	0.830	0.785	0.795	0.840	0.830	
		$\hat{\rho}_1$	0.580	0.695	0.855	0.840	0.895	0.920	0.865	0.865	0.875	0.890	
		$\hat{\rho}_2$	0.515	0.690	0.855	0.885	0.890	0.905	0.900	0.865	0.840	0.855	
	<i>full</i>	$\hat{\rho}$	0.075	0.250	0.580	0.620	0.655	0.765	0.825	0.845	0.865	0.855	
		$\hat{\rho}_1$	0.395	0.645	0.845	0.850	0.895	0.905	0.875	0.920	0.900	0.905	
		$\hat{\rho}_2$	0.430	0.670	0.770	0.805	0.870	0.915	0.915	0.870	0.900	0.880	
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$													
1200	<i>rec</i>	$\hat{\rho}$	0.035	0.520	0.670	0.755	0.800	0.810	0.810	0.905	0.865	0.840	
		$\hat{\rho}_1$	0.215	0.780	0.880	0.895	0.895	0.895	0.905	0.940	0.915	0.845	
		$\hat{\rho}_2$	0.220	0.775	0.830	0.830	0.915	0.930	0.890	0.875	0.910	0.860	
	<i>full</i>	$\hat{\rho}$	0.015	0.410	0.635	0.735	0.820	0.820	0.850	0.860	0.885	0.875	
		$\hat{\rho}_1$	0.125	0.755	0.820	0.870	0.900	0.935	0.940	0.905	0.945	0.925	
		$\hat{\rho}_2$	0.110	0.665	0.810	0.870	0.885	0.915	0.895	0.890	0.905	0.930	
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$													
2400	<i>rec</i>	$\hat{\rho}$	0.000	0.310	0.670	0.780	0.840	0.890	0.855	0.910	0.885	0.905	
		$\hat{\rho}_1$	0.025	0.645	0.860	0.870	0.885	0.950	0.890	0.925	0.935	0.920	
		$\hat{\rho}_2$	0.020	0.480	0.825	0.855	0.930	0.880	0.890	0.890	0.870	0.900	
	<i>full</i>	$\hat{\rho}$	0.000	0.185	0.665	0.780	0.810	0.870	0.865	0.915	0.890	0.900	
		$\hat{\rho}_1$	0.010	0.520	0.850	0.875	0.915	0.860	0.900	0.945	0.915	0.940	
		$\hat{\rho}_2$	0.005	0.485	0.765	0.860	0.905	0.900	0.915	0.905	0.915	0.930	

(\*) Notes: See notes to Table 1.

Table 5: Finite Sample Properites of the Bootstrap for Parameter Estimation Error: Recursive Scheme without Adjustment Terms: Part I<sup>(\*)</sup>

		Panel A: DGP is an AR(1) Process - $\rho = 0.2$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.710	0.766	0.876	0.842	0.882	0.868	0.874	0.844	0.822	0.816
	$\hat{\rho}_1$	0.710	0.784	0.894	0.860	0.882	0.882	0.872	0.852	0.832	0.818
	$\hat{\rho}_2$	0.870	0.882	0.880	0.888	0.902	0.894	0.868	0.888	0.838	0.814
$l = 4 \ l = 10 \ l = 15 \ l = 20 \ l = 25 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 100$											
1200	$\hat{\rho}$	0.602	0.846	0.874	0.862	0.912	0.892	0.888	0.854	0.876	0.842
	$\hat{\rho}_1$	0.636	0.860	0.880	0.862	0.918	0.896	0.886	0.880	0.876	0.854
	$\hat{\rho}_2$	0.848	0.898	0.894	0.886	0.902	0.878	0.912	0.892	0.870	0.842
$l = 4 \ l = 10 \ l = 20 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 80 \ l = 100 \ l = 120$											
2400	$\hat{\rho}$	0.446	0.770	0.874	0.916	0.896	0.886	0.882	0.880	0.892	0.868
	$\hat{\rho}_1$	0.456	0.768	0.884	0.916	0.906	0.890	0.890	0.890	0.898	0.854
	$\hat{\rho}_2$	0.856	0.878	0.898	0.912	0.898	0.902	0.914	0.898	0.876	0.866
		Panel B: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.1$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.350	0.580	0.825	0.795	0.795	0.860	0.845	0.830	0.850	0.800
	$\hat{\rho}_1$	0.435	0.610	0.830	0.815	0.835	0.885	0.870	0.835	0.880	0.815
	$\hat{\rho}_2$	0.920	0.910	0.850	0.930	0.895	0.895	0.855	0.880	0.835	0.860
$l = 4 \ l = 10 \ l = 15 \ l = 20 \ l = 25 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 100$											
1200	$\hat{\rho}$	0.135	0.670	0.820	0.865	0.865	0.840	0.880	0.870	0.860	0.815
	$\hat{\rho}_1$	0.200	0.720	0.845	0.900	0.875	0.880	0.895	0.885	0.885	0.840
	$\hat{\rho}_2$	0.875	0.880	0.925	0.905	0.895	0.920	0.890	0.900	0.920	0.860
$l = 4 \ l = 10 \ l = 20 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 80 \ l = 100 \ l = 120$											
2400	$\hat{\rho}$	0.015	0.530	0.790	0.830	0.870	0.885	0.865	0.920	0.875	0.860
	$\hat{\rho}_1$	0.040	0.615	0.810	0.830	0.910	0.875	0.875	0.890	0.875	0.860
	$\hat{\rho}_2$	0.885	0.895	0.965	0.875	0.875	0.880	0.900	0.885	0.900	0.875
		Panel C: DGP is an AR(1) Process - $\rho = 0.4$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.804	0.794	0.876	0.856	0.890	0.854	0.890	0.846	0.834	0.802
	$\hat{\rho}_1$	0.838	0.838	0.886	0.868	0.894	0.876	0.890	0.864	0.842	0.806
	$\hat{\rho}_2$	0.802	0.864	0.866	0.896	0.878	0.874	0.878	0.868	0.852	0.814
$l = 4 \ l = 10 \ l = 15 \ l = 20 \ l = 25 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 100$											
1200	$\hat{\rho}$	0.770	0.864	0.884	0.880	0.876	0.894	0.894	0.872	0.860	0.854
	$\hat{\rho}_1$	0.802	0.882	0.894	0.900	0.888	0.898	0.894	0.868	0.866	0.866
	$\hat{\rho}_2$	0.780	0.884	0.890	0.906	0.884	0.882	0.906	0.874	0.894	0.828
$l = 4 \ l = 10 \ l = 20 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 80 \ l = 100 \ l = 120$											
2400	$\hat{\rho}$	0.706	0.866	0.876	0.904	0.888	0.910	0.884	0.892	0.898	0.862
	$\hat{\rho}_1$	0.726	0.870	0.872	0.898	0.890	0.910	0.892	0.878	0.904	0.868
	$\hat{\rho}_2$	0.732	0.862	0.890	0.908	0.892	0.898	0.912	0.880	0.872	0.886
		Panel D: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.2$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.680	0.780	0.855	0.845	0.860	0.865	0.845	0.840	0.865	0.800
	$\hat{\rho}_1$	0.765	0.850	0.890	0.890	0.875	0.880	0.880	0.845	0.875	0.825
	$\hat{\rho}_2$	0.710	0.825	0.845	0.840	0.850	0.860	0.875	0.835	0.855	0.805
$l = 4 \ l = 10 \ l = 15 \ l = 20 \ l = 25 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 100$											
1200	$\hat{\rho}$	0.515	0.765	0.850	0.905	0.900	0.840	0.900	0.880	0.900	0.880
	$\hat{\rho}_1$	0.605	0.820	0.900	0.910	0.925	0.870	0.915	0.900	0.935	0.850
	$\hat{\rho}_2$	0.640	0.855	0.850	0.885	0.885	0.885	0.885	0.865	0.915	0.840
$l = 4 \ l = 10 \ l = 20 \ l = 30 \ l = 40 \ l = 50 \ l = 60 \ l = 80 \ l = 100 \ l = 120$											
2400	$\hat{\rho}$	0.315	0.755	0.870	0.920	0.895	0.900	0.885	0.910	0.875	0.850
	$\hat{\rho}_1$	0.400	0.780	0.895	0.905	0.885	0.925	0.875	0.925	0.880	0.840
	$\hat{\rho}_2$	0.445	0.775	0.915	0.860	0.905	0.880	0.895	0.850	0.865	0.880

(\*) Notes: See notes to Table 1. Entries correspond to those reported in Table 1 under the *rec* bootstrap, except that no adjustment term is included. See above for further details.

Table 6: Finite Sample Properites of the Bootstrap for Parameter Estimation Error: Recursive Scheme without Adjustment Terms: Part I(\*)

		Panel A: DGP is an AR(1) Process - $\rho = 0.6$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.055	0.310	0.605	0.645	0.690	0.800	0.805	0.825	0.830	0.830
	$\hat{\rho}_1$	0.190	0.505	0.720	0.730	0.800	0.875	0.860	0.840	0.850	0.830
	$\hat{\rho}_2$	0.845	0.905	0.880	0.910	0.840	0.905	0.915	0.850	0.840	0.845
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100$											
1200	$\hat{\rho}$	0.000	0.400	0.630	0.675	0.795	0.765	0.815	0.825	0.860	0.870
	$\hat{\rho}_1$	0.045	0.505	0.750	0.795	0.855	0.840	0.890	0.870	0.865	0.840
	$\hat{\rho}_2$	0.895	0.930	0.900	0.910	0.920	0.890	0.925	0.880	0.895	0.845
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$											
2400	$\hat{\rho}$	0.000	0.155	0.540	0.750	0.810	0.870	0.815	0.855	0.875	0.865
	$\hat{\rho}_1$	0.000	0.290	0.670	0.770	0.875	0.855	0.865	0.865	0.890	0.860
	$\hat{\rho}_2$	0.925	0.950	0.935	0.880	0.885	0.920	0.890	0.890	0.865	0.875
		Panel B: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.3$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.000	0.035	0.280	0.465	0.540	0.640	0.660	0.765	0.760	0.800
	$\hat{\rho}_1$	0.100	0.365	0.675	0.755	0.815	0.890	0.850	0.840	0.850	0.780
	$\hat{\rho}_2$	0.925	0.975	0.920	0.920	0.910	0.920	0.885	0.885	0.905	0.800
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100 \quad l = 100$											
1200	$\hat{\rho}$	0.000	0.045	0.315	0.455	0.575	0.680	0.725	0.790	0.820	0.830
	$\hat{\rho}_1$	0.000	0.405	0.620	0.755	0.775	0.860	0.860	0.870	0.870	0.870
	$\hat{\rho}_2$	0.975	0.915	0.895	0.905	0.945	0.895	0.895	0.870	0.875	0.835
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$											
2400	$\hat{\rho}$	0.000	0.000	0.235	0.545	0.580	0.700	0.790	0.815	0.815	0.815
	$\hat{\rho}_1$	0.000	0.190	0.610	0.775	0.805	0.865	0.905	0.855	0.890	0.900
	$\hat{\rho}_2$	0.950	0.935	0.925	0.940	0.935	0.935	0.935	0.870	0.915	0.915
		Panel C: DGP is an AR(1) Process - $\rho = 0.8$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.405	0.635	0.775	0.800	0.790	0.850	0.825	0.815	0.850	0.825
	$\hat{\rho}_1$	0.535	0.760	0.885	0.855	0.885	0.895	0.855	0.870	0.840	0.840
	$\hat{\rho}_2$	0.665	0.755	0.855	0.880	0.870	0.885	0.880	0.865	0.850	0.825
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100 \quad l = 100$											
1200	$\hat{\rho}$	0.195	0.630	0.855	0.830	0.825	0.865	0.850	0.835	0.850	0.845
	$\hat{\rho}_1$	0.345	0.750	0.885	0.895	0.855	0.880	0.865	0.905	0.835	0.835
	$\hat{\rho}_2$	0.395	0.800	0.875	0.870	0.880	0.865	0.850	0.865	0.895	0.855
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$											
2400	$\hat{\rho}$	0.055	0.630	0.785	0.810	0.875	0.865	0.840	0.880	0.865	0.835
	$\hat{\rho}_1$	0.145	0.720	0.820	0.835	0.915	0.885	0.855	0.890	0.920	0.855
	$\hat{\rho}_2$	0.140	0.685	0.870	0.920	0.890	0.870	0.935	0.905	0.870	0.870
		Panel D: DGP is an AR(2) Process - $\rho_1 = \rho_2 = 0.4$									
<i>smp</i>	<i>coeff</i>	$l = 4$	$l = 6$	$l = 10$	$l = 12$	$l = 15$	$l = 20$	$l = 25$	$l = 30$	$l = 50$	$l = 60$
600	$\hat{\rho}$	0.165	0.350	0.660	0.620	0.750	0.780	0.770	0.775	0.800	0.820
	$\hat{\rho}_1$	0.570	0.695	0.840	0.830	0.870	0.885	0.845	0.835	0.835	0.865
	$\hat{\rho}_2$	0.510	0.725	0.825	0.840	0.875	0.880	0.865	0.845	0.805	0.830
$l = 4 \quad l = 10 \quad l = 15 \quad l = 20 \quad l = 25 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 100 \quad l = 100$											
1200	$\hat{\rho}$	0.035	0.515	0.670	0.720	0.770	0.795	0.770	0.885	0.870	0.810
	$\hat{\rho}_1$	0.215	0.775	0.855	0.870	0.880	0.875	0.880	0.925	0.895	0.815
	$\hat{\rho}_2$	0.225	0.740	0.780	0.790	0.875	0.910	0.850	0.845	0.880	0.820
$l = 4 \quad l = 10 \quad l = 20 \quad l = 30 \quad l = 40 \quad l = 50 \quad l = 60 \quad l = 80 \quad l = 100 \quad l = 120$											
2400	$\hat{\rho}$	0.000	0.290	0.635	0.770	0.800	0.865	0.840	0.855	0.865	0.890
	$\hat{\rho}_1$	0.030	0.665	0.815	0.850	0.850	0.920	0.875	0.880	0.920	0.890
	$\hat{\rho}_2$	0.035	0.465	0.805	0.860	0.905	0.885	0.855	0.850	0.835	0.875

(\*) Notes: See notes to Table 5.