Problem set 1

1. Prove theorem 1.1.2.

   Hint: Recall the axioms of a vector space from your Linear Algebra module and check each one.

2. Prove theorem 1.1.4.

   Hint: Recall the axioms of an inner product space from your Linear Algebra module and check each one.

3. Complete example 1.1.8.

   Hint: This is about the interaction between continuity and integrability. Look up the relevant theorem, if you don't know it.

4. Complete the proof of proposition 1.1.10.

   Hint: The lecture notes contain some ideas.

5. Complete example 1.1.11.

   Hint: Break ϕ into real and imaginary parts, then each into positive and negative parts, then dominate each by
   $$\begin{cases} \frac{M}{x^2} & \text{if } |x| > 1, \\ M & \text{if } |x| \leqslant 1, \end{cases}$$
   for some $M > 0$, and show these integrals converge.

6. Prove the first 3 cases in proposition 1.2.4.

   Hint: Check the definitions directly.

7. Express each of the following equations using a formal differential operator $\mathcal{L}$:
   $$y''(x) - 3x^2 y(x) = 0 \qquad y'''(x) = \lambda y(x)$$
   $$u_t(x,t) = 5u_{xx}(x,t) \qquad y''(x) + y(x)\,|y(x)|^2 = 0$$
   If the equation can't be expressed using a formal differential operator of the kind we study, then explain why not. Can any of these be conveniently expressed using the named differential operators studied in §1.2?

Optional extra problems

8. Prove theorem 1.1.1.

   Hint: Recall the axioms of a vector space from your Linear Algebra module and check each one.

9. Prove theorem 1.1.3.

   Hint: Recall the axioms of an inner product space from your Linear Algebra module and check each one.

10. Complete example 1.1.9.

    Hint: You can use example 1.1.8 to help here.

11. Complete example 1.1.12.

    Hint: You can use example 1.1.11 to help here.

12. Complete example 1.1.13.

    Hint: The difficult part is to show that the sum of two compactly supported functions is also compactly supported. You will need different a and b for each of the original two functions; what are the corresponding a and b for the sum function?

13. Prove the final 3 cases in proposition 1.2.4.

    Hint: Check the definitions directly.