Problem set 2

1. Look up Euler's equidimensional equation of second order, and its solutions, on Wikipedia.
   (a) Write Euler's equidimensional equation using an operator.
   (b) Think carefully about the domain and codomain of the operator. (How) do the solutions of Euler's equidimensional equation change depending on the domain?

2. Complete example 1.4.2.
   Hint: Follow example 1.4.1.

3. Complete example 1.4.3. You can use either $\mathcal{S}$ or $C_0^\infty[0,1]$.
   Hint: Just as in the previous examples, you have to integrate by parts.

4. Let
   $$\Phi = \{\phi \in C^\infty[0,1] \text{ s.t. } \phi(0) = 0,\ \phi(1) = 0\}$$
   and let $L : \Phi \to C^\infty[0,1]$ be the Schrödinger operator with real valued potential $V \in C_{\mathbb{R}}^\infty[0,1]$.
   (a) Show that L is formally selfadjoint.
   What if V is not real valued?
   (b) Is L selfadjoint?
   What if boundary condition $\phi(1) = 0$ is replaced with $\phi'(1) = 0$.
   (c) What are the boundary forms in this example?

5. The operator L is given by $\mathcal{D}^3$ on
   $$\Phi = \{\phi \in C^\infty[0,1] \text{ s.t. } \phi(0) = 0,\ \phi(1) = 0,\ \phi'(0) = 0\}.$$
   Find $L^\star$, $\Phi^\star$, if $\Psi = C^\infty[0,1]$.

6. Let $\Psi = \mathcal{S}[0,\infty)$, $\Phi = \{\phi \in \Psi \text{ s.t. } \phi(0) = 0\}$, and Δ be given by Δ_{formal}. Show that Δ is selfadjoint.

7. Let $\Psi = C^\infty[0,1]$, $\beta \in \mathbb{C}$, $\Phi = \{\phi \in \Psi \text{ s.t. } \phi(0) = 0,\ \phi(1) = 0,\ \phi'(0) = \beta\phi'(1)\}$, and Δ be given by $\mathcal{D}^3$. Find $L^\star$, $\Phi^\star$, if $\Psi = C^\infty[0,1]$.

8. Let $n \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\mathrm{e}^{\mathrm{i}\lambda x}$ is a formal eigenfunction of the formal differential operator $\mathcal{D}^n$ with formal eigenvalue λ^n.
   (a) Find all the other formal eigenfunctions with the same eigenvalue; hence show that the eigenspace has dimension n.
   (b) Explain how imposing a boundary form on the domain of the operator reduces the dimension of the eigenspace by one.
   (c) Find all n eigenfunctions of this operator with eigenvalue 0.

Optional extra problems

9. Implement example 1.4.1 in $\mathcal{S}$.

10. Let $\Psi = \mathcal{S}[0,\infty)$, $\Phi = \{\phi \in \Psi \text{ s.t. } \phi'(0) = 0\}$, and Δ be given by Δ_{formal}. Show that Δ is selfadjoint.