Problem set 3

1. Rewrite IBVP
$$\left[\frac{\partial}{\partial t} + \Delta_{\text{formal}} + V(x)\right] q(x,t) = 0,$$
$$q(0,t) = 0,$$
$$q(\pi,t) = 0,$$
$$q(x,0) = Q(x).$$
into one using a (nonformal) differential operator.

2. Suppose we know a function v which satisfies
$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] v(x,t) = 0,$$
$$v(0,t) = f(t),$$
$$v(1,t) = g(t),$$
and we want to solve the problem
$$\left[\frac{\partial}{\partial t} + \mathcal{L}\right] q(x,t) = 0,$$
$$q(0,t) = f(t),$$
$$q(1,t) = g(t),$$
$$q(x,0) = Q(x).$$
How can we use v to make the problem for q simpler?

3. Prove theorem 2.3.2.

   Hint: Use integration by parts.

4. In example 2.3.3 we used the Fourier sine series transform. Explain in detail (demonstrate exactly what goes wrong when you try) why we could not have used the Fourier cosine series transform to solve this problem.

5. Let Δ_1 be the operator considered in theorem 2.3.2. Solve the IBVP $[\frac{\partial}{\partial t} + K\Delta_1]q(x,t) = 0$, $q(x,0) = Q(x)$.

6. Prove theorem 2.4.2.

   Hint: Use integration by parts.

7. Let Δ_0 be the operator considered in theorem 2.4.2. Solve the IBVP $[\frac{\partial}{\partial t} + K\Delta_0]q(x,t) = 0$, $q(x,0) = Q(x)$.

   Explain your choice of transform, and why another choice would not have been appropriate.

8. Explain why none of the transforms we have met are appropriate for solving IBVP $[\frac{\partial}{\partial t} - \mathrm{i}L]q(x,t) = 0$, $q(x,0) = Q(x)$, where $L : \Phi \to C^\infty[0,1]$ is given by $\mathcal{D}^3$, for
$$\Phi = \{\phi \in C^\infty[0,1] \text{ s.t. } \phi(0) = 0,\ \phi(1) = 0,\ \phi'(0) = 0\}.$$

9. Complete the statement of theorem 2.5.2 and prove it.
10. For $b = \pi$, use the Fourier exponential series transform to solve an IBVP. You should choose the IBVP so that it is appropriate for solution via this particular transform.

    Explain why the other transforms would not be appropriate for solving this IBVP.
11. The Fourier exponential transform F is defined for $\lambda \in (-\infty, \infty)$, but it may be possible to define it for some more $\lambda \in \mathbb{C}$.

    If $\phi(x) = \mathrm{e}^{-x^2}$, for what $\lambda \in \mathbb{C}$ is $\mathrm{F}[\phi](\lambda)$ defined?

    If $\phi(x) = x^2 + 5$ for $x \in [0, 1]$ and 0 otherwise, for what $\lambda \in \mathbb{C}$ is $\mathrm{F}[\phi](\lambda)$ defined?
12. Let $\Delta : \mathcal{S} \to \mathcal{S}$ be the differential operator given by Δ_{formal}. Solve the IBVP $[\frac{\partial}{\partial t} + K\Delta]q(x,t) = 0$, $q(x,0) = Q(x)$.

Optional extra problems

13. Give an appropriate forward transform for solving the IBVP $[\frac{\partial}{\partial t} + K\Delta]q(x,t) = 0$, $q(x,0) = Q(x)$, where $\Delta : \Phi \to C^\infty[0,\pi]$ is given by Δ_{formal}, for

$$\Phi = \{\phi \in C^\infty[0,\pi] \text{ s.t. } \phi(0) = 0,\ \phi(\pi) = 0\}.$$

    State and prove the equivalent of theorem 2.3.2 for your transform. Use theorem 2.3.1 to state and derive the equivalent of theorem 2.3.1 (i.e. give the inverse transform) for your transform. Solve the IBVP.
14. In example 2.3.3, what values of K are possible?
15. Let Δ_1 be the operator considered in theorem 2.4.2. Solve the IBVP $[\frac{\partial}{\partial t} + K\Delta_1]q(x,t) = 0$, $q(x,0) = Q(x)$.
16. The Fourier sine transform $\mathrm{F_s}$ is defined for $\lambda \geqslant 0$, but it could be defined for some more $\lambda \in \mathbb{C}$. For what $\lambda \in \mathbb{C}$ could the Fourier sine transform be defined?

    For what $j \in \mathbb{C}$ could $\mathrm{F_{s\,ser}}$ be defined?
17. Fix some $n \in \mathbb{N}$. Let $L : \mathcal{S} \to \mathcal{S}$ be the differential operator given by $\mathcal{D}^n$. Solve the IBVP $[\frac{\partial}{\partial t} + L]q(x,t) = 0$, $q(x,0) = Q(x)$.