Minicourse D to N maps

1. Differential operators

1.1 Inner product spaces of functions

A real interval is a set $(-\infty,\infty)$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, (a,b), $[a,b)$, $(a,b]$ or $[a,b]$, where $-\infty<a<b<\infty$. Usually denoted $I\subset\mathbb{R}$.

A real valued function on I is a function $f: I\to\mathbb{R}$.

A complex valued function on I is a function $\varphi: I\to\mathbb{C}$. Note that $\varphi = f+ig$ $\exists\, f,g$ real valued functions on I. Also the conjugate of φ is $\overline{\varphi} = f-ig$. f,g are the real part imaginary part of φ.

Theorem 1.1.1 The set of real valued functions on I forms a vector space over $\mathbb{R}$.

Proof Exercise. Check the axioms.

Theorem 1.1.2 The set of complex valued functions on I forms a vector space over $\mathbb{C}$.

Proof Exercise. Check the axioms. It may be quicker to break into real & imaginary parts, then use Theorem 1.1.1.

These vector spaces are too general to be interesting. Let's ask for integrability.

Theorem 1.1.3 $\langle\cdot,\cdot\rangle$ defined by $\langle f,g\rangle = \int_I f(x)g(x)\,dx$ is a bilinear (real) inner product, defined for all pairs of real valued functions on I for which the integral converges.

Proof Exercise. Check the axioms.

Theorem 1.1.4 $\langle \cdot, \cdot \rangle$ defined by $\langle \varphi, \psi \rangle = \int_I \varphi(x) \overline{\psi(x)}\, dx$ is a sesquilinear (complex) inner product, defined for all pairs of complex valued functions on I for which the integral converges

Proof Exercise.

Example 1.1.5 $C_{\mathbb{R}}[0,1] := \{ f : [0,1] \to \mathbb{R} \text{ s.t. } f \text{ is continuous} \}$ is an inner product space.

Example 1.1.6 $\forall n \in \mathbb{N}_0$, $C^n(0,1) := \{ \varphi : (0,1) \to \mathbb{C} \text{ s.t. } \forall k \in \{0,1,2,\dots,n\}$ the k^{th} derivative of φ, $\varphi^{(k)} : (0,1) \to \mathbb{C}$ is continuous$\}$

is not an inner product space, because $\varphi(x) = \frac{1}{x} \in C^n(0,1)\ \forall n \in \mathbb{N}_0$

Example 1.1.7 $C^\infty(0,1) := \bigcap_{n \in \mathbb{N}_0} C^n(0,1)$ is not an inner product space, using the same eg.

Example 1.1.8 $\forall n \in \mathbb{N}_0$, $C^n[0,1] = \{ \varphi \in C^n(0,1) \text{ s.t. } \forall k \in \{0,1,2,\dots,n\} \lim_{x \to 0^+} \varphi^{(k)}(x)$ and $\lim_{x \to 1^-} \varphi^{(k)}(x)$ exist and are finite$\}$

is an inner product space over $\mathbb{C}$. Proof: Exercise.

Example 1.1.9 $C^\infty[0,1] := \bigcap_{n \in \mathbb{N}_0} C^n[0,1]$ is an inner product space. Proof: Exercise.

Proposition 1.1.10 None of $C^n(0,1)$ nor $C^n[0,1]$, for $n \in \mathbb{N}_0 \cup \{\infty\}$ is (sequentially) closed.

Proof For $j \in \mathbb{N}$, consider φ_j defined by $\varphi_j(x) = \begin{cases} 2jx + 1 - j & x \in \left(\frac{1}{2} - \frac{1}{2j}, \frac{1}{2}\right], \\ 1 + j - 2jx & x \in \left(\frac{1}{2}, \frac{1}{2} + \frac{1}{2j}\right), \\ 0 & \text{otherwise}. \end{cases}$

Exercise: (a) Show $\varphi_j \in C^0(0,1)$, $\varphi_j \in C^0[0,1]$.

(b) Show $\varphi_\infty(x) := \lim_{j \to \infty} (\varphi_j(x)) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{otherwise}. \end{cases}$

(c) Explain why $\varphi_\infty \notin C^0(0,1)$, $\varphi_\infty \in C^0[0,1]$.

But this example is not differentiable, so certainly not continuously differentiable, so it does not work for skip

$n \geq 1$. We use the same idea for $n \geq 1$ but with smooth mollifiers instead of linear functions.

$$\mu(x) = \begin{cases} \exp\left(1 - \frac{1}{1-x^2}\right) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note $\mu(0) = 1$.

μ is C^∞ on $(-1,0), (0,1)$, because it is a composition of smooth functions.
μ is C^∞ on $(-\infty,-1), (1,\infty)$, because is a constant function.

We need the derivatives at $-1, 0, 1$. We calculate

$$1 - \frac{1}{1-h^2} = \frac{1-h^2-1}{1-h^2} = \frac{-h^2}{1-h^2}, \text{ so } \mu(0) = \exp(0) = 1$$

$$\mu'(0) = \lim_{h \to 0} \left(\frac{\exp\left(1 - \frac{1}{1-h^2}\right) - \exp\left(1 - \frac{1}{1-0^2}\right)}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h} \left[-1 + \exp\left(\frac{-h^2}{1-h^2} \right) \right] \right)$$

$$= \lim_{h \to 0} \left(\frac{1}{h} \left[-1 + 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{-h^2}{1-h^2} \right)^k \right] \right)$$

$$= \lim_{h \to 0} \left(\cancel{\frac{1}{h}} \left(\frac{-h^{\cancel{2}}}{1-h^2} \right) \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{-h^2}{1-h^2} \right)^{k-1} \right)$$

$$= 0$$

$$\mu'(1) = \underbrace{\lim_{h \to 0^+} \left(\frac{1}{h} \left(\mu(1+h) - \mu(1) \right) \right)}_{=:A} = \underbrace{\lim_{h \to 0^-} \left(\frac{1}{h} \left(\mu(1+h) - \mu(1) \right) \right)}_{=:B}, \text{ provided they}$$

exist & are equal. $A = \lim_{h \to 0^+} \left(\frac{1}{h} (0 - 0) \right) = 0$

$$B = \lim_{h \to 0^+} \left(\frac{-1}{h} \left(\exp\left(1 - \frac{1}{1-(1-h)^2}\right) - 0 \right) \right)$$

$$= \lim_{h \to 0^+} \left(\frac{-1}{h} \exp\left(\frac{-(1-h)^2}{1-(1-h)^2} \right) \right)$$

$$= \lim_{h \to 0^+} \left(-\exp(-\log(h)) \exp\left(-\frac{1}{2h} + 1 - \frac{1}{4-2h} \right) \right) \qquad \text{(Exercise)}$$

$$= -\lim_{h \to 0^+} \left(\exp\left(\underbrace{-\log(h)}_{\to -\infty} \underbrace{- \frac{1}{2h}}_{\to -\infty} + \underbrace{1}_{\to 1} - \underbrace{\frac{1}{4-2h}}_{\to -\frac{1}{4}} \right) \right)$$

$$= 0$$

Similarly $\mu'(-1) = 0$.

$$\mu'(x) = \begin{cases} -\frac{d}{dx}\left(\frac{1}{1-x^2}\right) \exp\left(1 - \frac{1}{1-x^2}\right) & \text{if } x \in (-1,0) \cup (0,1) \\ 0 & \text{otherwise} \end{cases}$$

$$-\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{2x}{(1-x^2)^2}$$

$$= \begin{cases} \frac{2x}{(1-x^2)^2} \exp\left(1 - \frac{1}{1-x^2}\right) & \text{if } x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

Skip all of this. It is too much effort

Example 1.1.11 $\mathcal{S} := \{ \varphi : (-\infty, \infty) \to \mathbb{C}$ such that $\forall\, k \in \mathbb{N}_0,\ \forall\, j \in \mathbb{N}_0,\ \lim_{x \to \pm\infty} \left(x^j \varphi^{(k)}(x) \right) = 0 \}$

is an inner product space over $\mathbb{C}$. Proof: Exercise. Break φ into real & imaginary parts, then each into positive & negative parts, then dominate each by $\frac{M}{x^2}$ $\exists\, M > 0$, and show those integrals converge.

Example 1.1.12 $\mathcal{S}[0,\infty) := \{ \varphi : [0,\infty) \to \mathbb{C}$ s.t. $\exists\, \psi \in \mathcal{S}$ for which $\psi|_{[0,\infty)} = \varphi \}$ is an inner product space over $\mathbb{C}$.

Example 1.1.13 $C_0^\infty[0,1] := \{\varphi \in C^\infty[0,1] : \exists\, a,b$ with $0 < a < b < 1$ and $\varphi(x) = 0\ \forall x \in [0,a] \cup [b,1]\}$

is an inner product space called the space of smooth functions compactly supported on $[0,1]$.

1.2 Formal differential operators

A formal (linear) differential operator of order n is an operator that accepts complex valued functions of "sufficient smoothness" and returns another function that is a linear combination of its 0^{th} to n^{th} derivatives. The coefficients in this linear combination may depend on x, and c_n is not the zero function.

$$\mathcal{L} = \sum_{j=0}^{n} c_j(x) \frac{d^j}{dx^j} \qquad \text{so} \qquad \mathcal{L}[\varphi](x) = \sum_{j=0}^{n} c_j(x)\, \varphi^{(j)}(x)$$

If all c_j are constants, $\mathcal{L}$ is called a formal constant coefficient differential operator.

Nonlinear differential operators also exist, but we will not study them. Hence we omit "linear".

power in the sense of composition of operators just like we are used to have

Example 1.2.1 $\mathcal{D} := -i \frac{d}{dx}$, $\Delta_{formal} := -\frac{d^2}{dx^2}$, $\mathcal{D}^n$, $\frac{d^2}{dx^2} + 3\frac{d}{dx}$ are all constant coefficient differential operators. Note that $\Delta_{formal} = \mathcal{D}^2$.

Example 1.2.2

For $m \in \mathbb{C}$ (usually $m \in \mathbb{N}_0$),

$$f \mapsto -\frac{d}{dx}\left(x f'(x)\right) + \frac{m^2}{x} f(x)$$

is a variable coefficient differentiable operator known as the formal Bessel operator. Here $\mathcal{L} = x\Delta_{formal} - \frac{d}{dx} + \frac{m^2}{x}$.

Example 1.2.3

For any (smooth enough) function v, $\mathcal{L} = \Delta_{formal} + v(x)$ is the formal Schrödinger operator.

Note that we are being imprecise about the domain of $\mathcal{L}$ & interval on which it acts. This is intentional. Really, because it does not have a fixed vector space as its domain, it is not a linear operator at all. To be precise, it is a "formula", a kind of abstract mathematical object that produces outputs from inputs "whenever the input makes sense." We call it a formal linear differential operator because, provided the inputs are all drawn from an appropriate vector space of functions, it is a linear operator on that space, and it is related to derivatives. Let's make that more concrete.

Proposition 1.2.4

Suppose						
If $f \in$	$C^\infty[0,1]$	$C^n[0,1]$	$\mathcal{S}$	$\mathcal{S}[0,\infty)$	$C_0^\infty[0,1]$	$C^\infty[0,1]$ s.t. $f(0)=0$
then $\mathcal{D}f \in$	$C^\infty[0,1]$	$C^{n-1}[0,1]$	$\mathcal{S}$	$\mathcal{S}[0,\infty)$	$C_0^\infty[0,1]$	$C^\infty[0,1]$

1.3 <u>Linear differential operators</u> Given suitable inner product spaces of functions Φ_1, Φ_2 a <u>(linear) differential operator</u> on Φ is an operator $L: \Phi_1 \to \Phi_2$ such that, wherever it is defined, $L = \mathcal{L}$, for some formal (linear) differential operator $\mathcal{L}$.

Note that a formal differential operator is <u>not</u> a special kind of differential operator, but rather a different kind of object, which is not even truly a linear operator.

We say two differential operators $L_1: \Phi_1 \to \Phi_2$, $L_2: \Psi_1 \to \Psi_2$ <u>share a formal differential operator</u> $\mathcal{L}$ if Φ_1, Ψ_1 are vector spaces of functions on I_1, I_2 respectively, $I := I_1 \cap I_2$ is nontrivial (ie more than a single point) and $\forall\, \varphi \in \{f|_I : f \in \Phi_1\} \cap \{g|_I : g \in \Psi_1\}$,

$$L_1 \varphi = \mathcal{L} \varphi = L_2 \varphi.$$

<u>Example 1.3.1</u> On $C_0^\infty[0,1], C^\infty[0,1], \mathcal{S}, \mathcal{S}[0,\infty)$, any formal constant coefficient differential operator $\mathcal{L}$ yields a constant coefficient differential operator L. If $\mathcal{L}$ on $C^\infty[0,1] = L_1$, $\mathcal{L}$ on $\mathcal{S}$ is L_2, $\mathcal{L}$ on $C_0^\infty[0,5]$ is L_3, then L_1, L_2, L_3 all share the formal differential operator $\mathcal{L}$.

<u>Example 1.3.2</u> $D_1: C^3[0,1] \to C^2[0,1]$ given by $D_1 = \mathcal{D}$ shares the formal operator $\mathcal{D}$ with $D_2: \{\varphi \in C^\infty[-\pi,\pi] \text{ s.t. } \varphi(-\pi) = \varphi(\pi)\} \to C^\infty[-\pi,\pi]$ given by $D_2 = \mathcal{D}$.

1.4 Adjoints of differential operators

Suppose $\mathcal{L}$ is a formal differential operator whose coefficients are "nice enough". Then the adjoint of $\mathcal{L}$ with respect to inner product $\langle \cdot, \cdot \rangle$, $\mathcal{L}^*$, is the formal differential operator for which, for all appropriate functions u, v, $\langle \mathcal{L} u, v \rangle = \langle u, \mathcal{L}^* v \rangle$. The formal adjoint of a differential operator is the adjoint of its corresponding formal differential operator.

This definition is very imprecise, because we did not say what "appropriate functions" are. This is a bit complicated if $\mathcal{L}$ has variable coefficients. If $\mathcal{L}$ is constant coefficient, we can work (equivalently) on $C_0^\infty[0,1]$ or $\mathcal{S}$. So, with $\langle \cdot, \cdot \rangle$ the usual sesquilinear inner product,

Example 1.4.1 Choosing $\mathcal{L} = \mathcal{D}$, $\forall\, u \in C_0^\infty[0,1]$,

$$\langle \mathcal{D} u, v \rangle = -i \int_0^1 u'(x)\, \overline{v(x)}\, dx \overset{\text{integrate by parts}}{=} -i \Big[u(x)\, v(x) \Big]_{x=0}^{x=1} + i \int_0^1 u(x)\, \overline{v'(x)}\, dx$$

$$\underset{u,v \text{ compactly supported}}{=} 0 + \int_0^1 u(x)\, \overline{[-i v'(x)]}\, dx = \langle u, \mathcal{D} v \rangle,$$

so $\mathcal{D}$ is self-adjoint.

Exercise: Check this argument also works on $\mathcal{S}$.

Example 1.4.2 The adjoint of $\frac{d}{dx} = -\frac{d}{dx}$ in a complex inner product space. (Exercise)

Example 1.4.3 The adjoint of $\mathcal{D}^3$ is $\mathcal{D}^3$. (Exercise. Use proposition 1.2.4.)

If Φ is an inner product space of functions on the interval I, and $L : \Phi \to \Phi$ is a differential operator, then (classical, or Lagrange) adjoint of L is the differential operator $L^* : \Phi \to \Phi$ for which $\forall\, u, v \in \Phi$, $\langle L u, v \rangle = \langle u, L v \rangle$.

Example 1.4.4 Let D be $\mathcal{D}$ on $C_0^\infty[0,1]$. Then, by the argument in example 1.4.1, $D^* = D$.

Note that Δ is formally self adjoint

Example 1.4.5 Let Δ be Δ_{formal} on $C^\infty[0,1]$. Then, $\forall u,v \in C^\infty[0,1]$

$$\langle \Delta u, v\rangle = -\int_0^1 u''(x)\overline{v(x)}\,dx = \Big[-u'(x)\overline{v(x)} + u(x)\overline{v'(x)}\Big]_{x=0}^{x=1} - \int_0^1 u(x)\overline{v''(x)}\,dx$$

$$= -u'(1)\overline{v(1)} + u(1)\overline{v'(1)} + u'(0)\overline{v(0)} - u(0)\overline{v'(0)} + \langle u, \Delta v\rangle \quad \text{Good}$$

Boundary terms. Problem!

So it feels like Δ is self adjoint, but there are boundary terms getting in the way. We can remove the boundary terms by restricting the domain of Δ. We could use $C_0^\infty[0,1]$, but this is unnecessarily (and, as we shall see, unhelpfully) restrictive.

Let $\Phi = \{\varphi \in C^\infty[0,1] \text{ s.t. } \varphi(0)=0,\ \varphi(1)=0\}$.

Then, if $u,v\in\Phi$, the boundary terms all vanish. So, if we change the definition of Δ to be Δ_{formal} on Φ instead of on $C^\infty[0,1]$, then Δ is self adjoint. But the original Δ was only formally self adjoint.

Example 1.4.6 Let Δ be Δ_{formal} on $\Phi = \{\varphi\in C^\infty[0,1] \text{ s.t. } \varphi'(0)=0,\ \varphi(0)=\varphi(1)\}$. Then Δ is still formally self adjoint. Is it self adjoint? If $u,v\in\Phi$, then the boundary terms are the same as in Example 1.4.5, but now

$$-u'(1)\overline{v(1)} + u(1)\underbrace{\overline{v'(1)}}_{=0} + \underbrace{u'(0)}_{=0}\overline{v(0)} - u(0)\overline{v'(0)} = -u'(1)\underbrace{\overline{v(1)}}_{=\overline{v(0)}} - \underbrace{u(0)}_{=u(1)}\overline{v'(0)}$$

and this need not be 0. Instead, suppose $u\in\Phi$, but $v\in\Phi^*$, with

$$\Phi^* = \{\psi\in C^\infty[0,1] \text{ s.t. } \psi(1)=0,\ \psi'(0)=\psi'(1)\}.\text{ Then}$$

because we can't apply Δ to v; it's not in Φ.

$$\langle\Delta u, v\rangle = -u'(1)\underbrace{\overline{v(1)}}_{=0} + u(1)\overline{v'(1)} + \underbrace{u'(0)}_{=0}\overline{v(0)} - \underbrace{u(0)\overline{v'(0)}}_{=u(1)\overline{v'(1)}} + \langle u, \Delta_{formal} v\rangle$$

$$= \langle u, \Delta_{formal} v\rangle$$

So we want to say Δ^* is the operator on Φ^* given by $\Delta_{formal}{}^* = \Delta_{formal}$. This motivates the definition:

Suppose Ψ is an inner product space of functions, $\Phi \subset \Psi$ is a subspace, and $L : \Phi \to \Psi$ is a differential operator. Then Φ^* is the maximal subspace of Φ for which $\exists$ differential operator L^* such that $\forall\, u \in \Phi$, $v \in \Phi^*$, $\langle Lu, v\rangle = \langle u, L^* v\rangle$. L^* is called the <u>adjoint</u> of L, and Φ^* is the adjoint domain to Φ. If Ψ is a space of smooth functions, then these are the <u>classical or Lagrange adjoints</u>.

<u>Exercise</u> L is given by $\mathcal{D}^3$ on $\Phi = \{\varphi \in C^\infty[0,1] : \varphi(0) = 0,\ \varphi(1) = 0,\ \varphi'(0) = 0\}$. Find L^*, Φ^* if $\Psi = C^\infty[0,1]$.

If Ψ is an inner product space of functions on an interval I, and $L : \Psi \to$ some space and $x \in I$, then $B : \Psi \to \mathbb{C}$ given by $B\varphi = L\varphi(x)$ is a <u>differential form</u>. If $I = [a,b]$ and $x \in \{a,b\}$, then B is a <u>boundary form</u>.

<u>Example 1.4.7</u> In the above exercise, $\Psi = C^\infty[0,1]$, $B_1\varphi := \varphi(0)$, $B_2\varphi := \varphi(1)$, $B_3\varphi := \varphi'(0)$ are boundary forms.

Typically, we restrict Ψ to Φ and Φ^* for L and L^* by imposing a few equations $B_k\varphi = 0$ for each of Φ, Φ^*, where B_k are boundary forms.

must be $= 0$. If $= 5$ then Φ is not a vector space as 0 is excluded & not closed under addition.

<u>Exercises</u>! Let $\Psi = S[0,\infty)$, $\Phi = \{\varphi \in \Psi : \varphi(0) = 0\}$. $\Delta : \Phi \to \Psi$ given by $\Delta\varphi = \Delta_{formal}$. Show Δ is self adjoint.

2. As above, let $\varphi'(0) = 0$ instead of $\varphi(0) = 0$.

3. $\Psi = C^\infty[0,1]$, $\Phi = \{\varphi \in \Psi : \varphi(0) = 0,\ \varphi(1) = 0,\ \varphi'(0) = \beta\varphi'(1)\}$, $L : \Phi \to \Psi$ given by $\mathcal{D}^3$, $\beta \in \mathbb{C}$. Find L^*, Φ^*.

1.5 <u>Eigenfunctions / eigenvalues</u> If $\mathcal{L}$ is a formal differential operator and $\lambda \in \mathbb{C}$ and φ is a nonzero function for which $\mathcal{L}\varphi = \lambda\varphi$, then φ is a <u>formal eigenfunction</u> of $\mathcal{L}$ and λ is the <u>formal eigenvalue</u>.

If $L : \Phi \to \Psi$ is a differential operator given by formal differential operator $\mathcal{L}$, then nonzero $\varphi \in \Phi$, $\lambda \in \mathbb{C}$ are an <u>eigenfunction</u> & <u>eigenvalue</u> of L iff $L\varphi = \lambda\varphi$. ψ, μ are a <u>formal eigenfunction</u> and <u>formal eigenvalue</u> of L if $\mathcal{L}\psi = \lambda\psi$.

<u>Example 1.5.1</u> $e^{i\lambda x}$ is a formal eigenfunction of $\mathcal{D}$ with formal eigenvalue λ.

<u>Example 1.5.2</u> $\Delta : \{C^\infty[0,1] : \varphi(0) = 0, \varphi(1) = 0\} \to C^\infty[0,1]$ given by Δ_{formal} has eigenfunctions $\sin(n\pi x)$ $\forall n \in \mathbb{N}$, with eigenvalues $n^2\pi^2$.

<u>Theorem 1.5.3</u> If u is an eigenfunction of L with e.v. λ and v is an eigenfunction of L^* with e.v. μ, then $\lambda = \bar{\mu}$ or $\langle u, v\rangle = 0$.
<u>Proof</u> $\lambda\langle u,v\rangle = \langle Lu, v\rangle = \langle u, L^*v\rangle = \bar{\mu}\langle u, v\rangle$.

2. Initial boundary value problems

2.1 What are IBVP?

An evolution partial differential equation in 2 variables is an equation of the form

$$\frac{\partial}{\partial t} q(x,t) + \mathcal{L} q(x,t) = 0,$$

for $\mathcal{L}$ a formal differential operator acting on the x variable. This does not give enough information to specify a solution, so we have to equip it with extra equations.
In practice, because such PDE describe "evolution" into the future, it makes sense to prescribe also an initial condition of the form

$t=0$, hence "initial"

$$q(x, 0) = Q(x),$$

where Q is some known function, is smooth enough, etc.

This is still not enough to get a single solution, for the problem to be "well posed". For that, we also need boundary conditions like

$$q(0,t) = 0,$$
$$q_x(3,t) = 0.$$

Note: q_x is shorthand for $\frac{\partial q}{\partial x}$. $q_{xx} = \frac{\partial^2 q}{\partial x^2}$, etc.

These arise from physical considerations. Eg, q is temperature of metal rod of length 3, with $x=0$ end held at constant temperature $0°C$ and $x=3$ end insulated.

Putting these together, we get

IBVP
$$\begin{cases} \text{PDE} & \frac{\partial}{\partial t} q(x,t) + \mathcal{L} q(x,t) = 0 \\ \text{BC} & q(0,t) = 0 \\ \text{BC} & q_x(3,t) = 0 \\ \text{IC} & q(x,0) = Q(x) \end{cases}$$

Can be combined into $q_t(x,t) + L q(x,t) = 0$ by specifying $\Psi = C^\infty[0,3]$, $\Phi = \{\varphi \in \Psi \text{ s.t. } \varphi(0) = 0,\ \varphi'(3) = 0\}$, $L : \Phi \to \Psi$ given by $L = \mathcal{L}$.

If $\mathcal{L} = \Delta_{\text{formal}}$, this IBVP has a unique solution, is well posed.

2.2 Principle of linear superposition

The IBVP is made up of the operators L, $\frac{\partial}{\partial t}$ and "evaluate at $t=0$", all of which are linear. This can be very helpful for solving one problem using another, or breaking IBVP into manageable chunks.

Example 2.2.1 Suppose we know a function v which satisfies

$$\left[\frac{\partial}{\partial t} + L\right] v(x,t) = R(x,t)$$

and we want to solve the problem

$$\left[\frac{\partial}{\partial t} + L\right] q(x,t) = R(x,t)$$
$$q(x,0) = Q(x)$$

for a particular specified function Q. Let $u = q - v$. Then

$$\left[\frac{\partial}{\partial t} + L\right] u = \left[\frac{\partial}{\partial t} + L\right](q - v) = R - R = 0,$$
$$u(x,0) = Q(x) - v(x,0),$$

and we can easily calculate $v(x,0)$ by evaluating v at $t=0$. Now the problem for u might be easier to solve, because there is not $R(x,t)$ in the PDE.

2.3 Solution by Fourier series transforms: Finite interval

For φ a suitable complex valued function on $[0,1]$, we define the Fourier sine series transform by

$$F_{s\,ser}[\varphi](j) = 2\int_0^1 \varphi(x)\sin(j\pi x)\,dx, \qquad j \in \mathbb{N},$$

domain: functions of x
codomain: functions of j

and, for $(C_j)_{j\in\mathbb{N}}$ a suitable sequence of complex numbers the inverse Fourier sine series transform by

$$F^{-1}_{s\,ser}\left[(C_j)_{j\in\mathbb{N}}\right](x) = \sum_{j=1}^{\infty} C_j \sin(j\pi x), \qquad x \in [0,1].$$

Similarly, $F_{c\,ser}[\varphi](j) = 2\int_0^1 \varphi(x)\cos(j\pi x)\,dx$, $j \in \mathbb{N}$, $F_{c\,ser}[\varphi](0) = \int_0^1 \varphi(x)\underbrace{\cos(0\pi x)}_{=1}dx$ (no 2)

$$F^{-1}_{c\,ser}\left[(C_j)_{j\in\mathbb{N}_0}\right](x) = \sum_{j=0}^{\infty} C_j \cos(j\pi x)\,dx$$

Theorem 2.3.1 (Fourier inversion theorem) If $\varphi \in C^\infty[0,1]$, then, $\forall\, x \in (0,1)$,

$$F^{-1}_{s\,ser}\left[\left(F_{s\,ser}[\varphi](j)\right)_{j\in\mathbb{N}}\right](x) = \varphi(x),$$

$$F^{-1}_{c\,ser}\left[\left(F_{c\,ser}[\varphi](j)\right)_{j\in\mathbb{N}_0}\right](x) = \varphi(x),$$

and, taking the limits of LHS as $x \to 0^+$ and $x \to 1^-$ yield $\varphi(0)$, $\varphi(1)$, respectively.

With slightly more careful statement, this holds for much less smooth φ, but C^∞ is enough for us.

Theorem 2.3.2 (Fourier diagonalisation)

For $k=0,1$, let $\Delta_k : \Phi_k \longrightarrow C^\infty[0,1]$ be given by Δ_{formal}, where
$\Phi_0 = \{\varphi \in C^\infty[0,1] \text{ s.t. } \varphi(0)=0,\ \varphi(1)=0\}$, $\Phi_1 = \{\varphi \in C^\infty[0,1] \text{ s.t. } \varphi'(0)=0,\ \varphi'(1)=0\}$.

Then, $\forall\, \varphi \in \Phi_0$, $\quad F_{s\,ser}[\Delta_0 \varphi](j) = (j\pi)^2\ F_{s\,ser}[\varphi](j)$,
$\forall\, \varphi \in \Phi_1$, $\quad F_{c\,ser}[\Delta_1 \varphi](j) = (j\pi)^2\ F_{c\,ser}[\varphi](j)$.

Proof Exercise

Example 2.3.3 IBVP $\left[\frac{\partial}{\partial t} + K\Delta_0\right] q(x,t) = 0$, $\quad q(x,0) = Q(x)$ can be solved using $F_{s\,ser}$.

PDE $\Rightarrow$ $\quad F_{s\,ser}\left[q_t(x,t)\right](j) + K\, F_{s\,ser}\left[-q_{xx}(x,t)\right](j) = 0$

But $\quad F_{s\,ser}[q_t](j) = \int_0^1 \frac{\partial}{\partial t} q(x,t)\sin(j\pi x)\,dx = \frac{d}{dt}\int_0^1 q(x,t)\sin(j\pi x)\,dx$

$$\Rightarrow \left[\frac{\partial}{\partial t} + K(j\pi)^2\right] F_{s\,ser}[q](j;t) = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(e^{K(j\pi)^2 t}\, F_{s\,ser}[q](j;t)\right) = 0$$

$$\Rightarrow \int_0^t \frac{\partial}{\partial \tau}\left(e^{K(j\pi)^2 \tau}\, F_{s\,ser}[q](j;\tau)\right)d\tau = \int_0^t 0\, d\tau = 0$$

$$\Rightarrow e^{K(j\pi)^2 t}\, F_{s\,ser}[q](j;t) - \underbrace{F_{s\,ser}[q](j;0)}_{=F_{s\,ser}[Q](j)} = 0$$

$$\Rightarrow \quad q(x,t) = F_{s\,ser}^{-1}\left[e^{-K(j\pi)^2 t}\, F_{s\,ser}[Q](j)\right](x)$$

2.4 Solution by Fourier transforms: Half line

For φ a suitable complex valued function on $[0,\infty)$, we define the Fourier sine transform by

$$F_s[\varphi](\lambda) = \frac{2}{\pi}\int_0^\infty \varphi(x)\sin(\lambda x)\,dx, \qquad \lambda \geq 0,$$

domain: functions of x
codomain: functions of λ

and, for $F:[0,\infty)\to\mathbb{C}$ a suitable function the inverse Fourier sine transform by

$$F_s^{-1}[F](x) = \int_0^\infty F(\lambda)\sin(\lambda x), \qquad x \in [0,\infty).$$

For φ a suitable complex valued function on $[0,\infty)$, we define the Fourier cosine transform by

$$F_c[\varphi](\lambda) = \frac{2}{\pi}\int_0^\infty \varphi(x)\cos(\lambda x)\,dx, \qquad \lambda \geq 0,$$

and, for $F:[0,\infty)\to\mathbb{C}$ a suitable function the inverse Fourier cosine transform by

$$F_c^{-1}[F](x) = \int_0^\infty F(\lambda)\cos(\lambda x), \qquad x \in [0,\infty).$$

Theorem 2.4.1 (Fourier inversion theorem) If $\varphi \in \mathcal{S}[0,\infty)$, then, $\forall\, x \in (0,\infty)$,

$$F_s^{-1}\big[\, F_s[\varphi](\lambda) \,\big](x) = \varphi(x),$$

$$F_c^{-1}\big[\, F_c[\varphi](\lambda) \,\big](x) = \varphi(x),$$

and, taking the limits of LHS as $x\to 0^+$ yields $\varphi(0)$

Theorem 2.4.2 (Fourier diagonalisation)

For $k = 0, 1$, let $\Delta_k : \Phi_k \longrightarrow \mathcal{S}[0,\infty)$ be given by Δ_{formal}, where

$\Phi_0 = \{\varphi \in \mathcal{S}[0,\infty) \text{ s.t. } \varphi(0) = 0\}$, $\Phi_1 = \{\varphi \in \mathcal{S}[0,\infty) \text{ s.t. } \varphi'(0) = 0\}$.

Then, $\forall\, \varphi \in \Phi_0$, $\mathcal{F}_s[\Delta_0 \varphi](\lambda) = \lambda^2 \mathcal{F}_s[\varphi](\lambda)$,

$\forall\, \varphi \in \Phi_1$, $\mathcal{F}_c[\Delta_1 \varphi](\lambda) = \lambda^2 \mathcal{F}_c[\varphi](\lambda)$.

Proof Exercise

2.5 Exponential Fourier transforms

For φ a suitable complex valued function on $(-\infty,\infty)$, we define the Fourier (exponential) transform by

$$F[\varphi](\lambda) = \int_{-\infty}^{\infty} \varphi(x)\, e^{-i\lambda x}\, dx, \qquad \lambda \in (-\infty,\infty),$$

domain: functions of x
codomain: functions of λ

and, for F a suitable complex valued function on $(-\infty,\infty)$, we define the inverse Fourier (exponential) transform by

$$F^{-1}[F](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)\, e^{-i\lambda x}, \qquad x \in (-\infty,\infty).$$

For φ a suitable complex valued function on $[-b,b]$ (ie $a=-b$), we define the Fourier (exponential) series transform by

$$F_{ser}[\varphi](j) = \frac{1}{2b}\int_{-b}^{b} \varphi(x)\, e^{-ij\pi x/b}\, dx, \qquad j \in \mathbb{Z},$$

and, for $(C_j)_{j\in\mathbb{Z}}$ a suitable sequence of complex numbers, we define the inverse Fourier (exponential) series transform by

$$F_{ser}^{-1}[(C_j)_{j\in\mathbb{Z}}](x) = \sum_{j=-\infty}^{\infty} C_j\, e^{ij\pi x/b}, \qquad x \in [-b,b].$$

Theorem 2.5.1 (Fourier inversion theorem) If $\varphi \in \mathcal{S}$, then, $\forall x \in (-\infty,\infty)$,

$$F^{-1}[F[\varphi](\lambda)](x) = \varphi(x).$$

If $\varphi \in C^{\infty}[-b,b]$, then $\forall x \in (-b,b)$ $\quad F_{ser}^{-1}[(F[\varphi](j))_{j\in\mathbb{Z}}](x) = \varphi(x)$,

and the limits $x \to -b^{+}$, $x \to b^{-}$ of LHS evaluate to $\varphi(-b)$, $\varphi(b)$, respectively.

Theorem 2.5.2 (Fourier diagonalisation)

For $n \in \mathbb{N}$, let $D_n : \Phi_n \longrightarrow S$ be given by $\mathcal{D}^n$, where
$\Phi_n = \{\varphi \in C^\infty[-b,b]$ s.t., $\forall j \in \{0,1,\dots,n-1\}$, $\varphi^{(j)}(-b) = \varphi^{(j)}(b)\}$

Then, $\forall \varphi \in \Phi_n$, $F_{ser}[D_n \varphi](j) =$?? $F_{ser}[\varphi](j)$

Let $D : S \to S$ be given by $\mathcal{D}$. Then, $\forall \varphi \in S$, $F[D\varphi](\lambda) =$?? $F[\varphi](\lambda)$

Proof: Exercise.

Theorem 2.5.3 (Stronger Fourier inversion theorem) If $\varphi \in C^1[a,b]$ and $\varphi(x) = 0$ for $x<a, x>b$ (but φ need not be continuous at a or b), then, $\forall x \in (a,b)$

$$\frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} F[\varphi](\lambda)\, e^{i\lambda x}\, dx = \varphi(x),$$

Almost the usual inverse transform, but this is the principle value of that integral.

and taking $\lim_{x\to a^+}$, $\lim_{x\to b^-}$ of LHS yields $\varphi(a)$, $\varphi(b)$, respectively.

Now we can take a function defined on a finite interval, take its Fourier transform, and transform back

Theorem 2.5.4 (Orthogonality of exponential Fourier basis) Suppose $b > 0$. Let $E_j : [-b, b] \to \mathbb{C}$ be given by $E_j(x) = e^{ij\pi x/b}$. Then, $\forall\, j, k \in \mathbb{Z}$,

① If $j = k$, $\langle E_j, E_k \rangle \neq 0$,
② If $j \neq k$, $\langle E_j, E_k \rangle = 0$.

Proof ① This inner product is a squared norm, which is positive definite, and $E_j \neq 0$, so $\| E_j \|^2 > 0$.

② $$\langle E_j, E_k \rangle = \int_{-b}^{b} e^{ij\pi x/b} e^{-ik\pi x/b}\, dx = \int_{-b}^{b} e^{i(j-k)\pi x/b}\, dx$$

$$= \frac{b}{i(j-k)\pi}\left(e^{i(j-k)\pi} - e^{-i(j-k)\pi}\right) = \frac{2b}{(j-k)\pi}\sin\left((j-k)\pi\right) = 0,$$

because $j - k \in \mathbb{Z}$.

Corollary 2.5.5 If, $\forall x \in [-b, b]$ $\sum_{j \in \mathbb{Z}} e^{ij\pi x/b} \alpha_j = \sum_{j \in \mathbb{Z}} e^{ij\pi x/b} \beta_j$ then, $\forall\, j \in \mathbb{Z}$, $\alpha_j = \beta_j$.

Proof Using the notation from theorem 2.5.4, we know $\sum_{j \in \mathbb{Z}} E_j(x)\alpha_j = \sum_{j \in \mathbb{Z}} E_j(x)\beta_j$. But then, $\forall\, k \in \mathbb{Z}$, $\langle \sum_{j \in \mathbb{Z}} \alpha_j E_j, E_k \rangle = \langle \sum_{j \in \mathbb{Z}} \beta_j E_j, E_k \rangle$

$$\Rightarrow \sum_{j \in \mathbb{Z}} \alpha_j \langle E_j, E_k \rangle = \sum_{j \in \mathbb{Z}} \beta_j \langle E_j, E_k \rangle.$$

By theorem 2.5.4, all terms but the $j = k$ term in each of these series is 0. So

$$\alpha_k \langle E_k, E_k \rangle = \beta_k \langle E_k, E_k \rangle.$$

Also by theorem 2.5.4, $\langle E_k, E_k \rangle \neq 0$. So $\alpha_k = \beta_k$. $\square$

Theorem 2.5.6 Suppose that $(C_j)_{j\in\mathbb{Z}}$ is a sequence of complex numbers for which, $\forall x\in[-b,b]$, the series

$$F_{ser}^{-1}\left[(C_j)_{j\in\mathbb{Z}}\right](x) = \sum_{j=-\infty}^{\infty} C_j e^{ij\pi x/b} =: \varphi(x)$$

converges. Then $\varphi : \mathbb{R} \to \mathbb{C}$ defined using the above formula is a periodic function, with period $2b$.

Proof We have to show that the series converges $\forall x\in\mathbb{R}$ and $\varphi(x+2b)=\varphi(x)$. By hypothesis, we have convergence for $x\in[-b,b)$. By the Archimedean property of $\mathbb{R}$, we have that $\forall y\in\mathbb{R}$, $\exists n\in\mathbb{N}$ s.t. $y\in[-nb,nb)$. So it is enough to prove an inductive step in both directions. Specifically, we prove:

If $\varphi(x)$ converges then $\varphi(x\pm 2b)$ converges and $\varphi(x\pm 2b)=\varphi(x)$.

$$\varphi(x\pm 2b) = \sum_{j\in\mathbb{Z}} C_j e^{ij\pi(x\pm 2b)/b}$$

$$= \sum_{j\in\mathbb{Z}} C_j e^{ij\pi x/b} e^{ij\pi 2}$$

$$= \sum_{j\in\mathbb{Z}} C_j e^{ij\pi x/b} \left(e^{i2\pi}\right)^j \qquad \text{and } e^{i2\pi}=1$$

$$= \sum_{j\in\mathbb{Z}} C_j e^{ij\pi x/b} \qquad = \varphi(x). \qquad \square$$

not very smooth. Continuously differentiable is certainly enough

Corollary 2.5.7 Suppose $f:[0,T]\to\mathbb{C}$ is smooth enough. Extend domain of f to $[-\frac{T}{2},T]$ by $f(t)=f(t+T)$ $\forall\ t\in[-\frac{T}{2},0)$. Then, $\forall t\in(0,T)$,

$$f(t) = F_{ser}^{-1}\left[F_{ser}[f]\right](t).$$

Proof For $t\in(0,\frac{T}{2})$, this is theorem 2.5.1. For $t\in[\frac{T}{2},T)$, this is theorems 2.5.1 and theorem 2.5.6.

2.6 Fourier Transforms as inner products with eigenfunctions of adjoint differential operators

Proof of theorem 2.5.2 for $D : S \to S$ given by $\mathcal{D}$ uses integration by parts:

$$F[D\varphi](\lambda) = \int_{-\infty}^{\infty} -i\varphi'(x)\, e^{-i\lambda x}\, dx = \Big[-i\varphi(x)e^{-i\lambda x}\Big]_{x\to-\infty}^{x\to+\infty} - \int_{-\infty}^{\infty}(-i)^2\lambda\,\varphi(x)\,e^{-i\lambda x}\,dx$$

$$= 0 - 0 + \lambda\, F[\varphi](\lambda).$$

The boundary terms evaluate to 0 because of the domain of D, and the remaining integral is the transform we seek.

We performed similar calculations in §1.4 when finding the adjoints of differential operators. Indeed, defining $y(x;\lambda) := e^{i\lambda x}$ note: not $-$, and conjugate of λ we find

$$F[D\varphi](\lambda) = \langle D\varphi, y(\cdot;\lambda)\rangle.$$

So theorem 2.5.2 is $\langle D\varphi, y(\cdot;\lambda)\rangle = \langle \varphi, \bar{\lambda} y(\cdot;\lambda)\rangle = \lambda\langle \varphi, y(\cdot;\lambda)\rangle.$

For each $\lambda \in \mathbb{R}$ $y(\cdot;\lambda)$ is a formal eigenfunction of the adjoint of D (which is just D itself) with eigenvalue $\bar{\lambda}$ ($=\lambda$, because $\lambda \in \mathbb{R}$).

real, so conjugate is unnecessary, but it might be necessary later

Now consider $F_{j,\,\mathrm{ser}}[\varphi](j) = \int_0^1 \varphi(x)\, 2\sin(j\pi x)\, dx = \langle \varphi, \overline{2\sin(j\pi\,\cdot)}\rangle$

Theorem 2.3.2 says $\langle \Delta_0\varphi, \overline{2\sin(j\pi\,\cdot)}\rangle = (j\pi)^2\langle \varphi, \overline{2\sin(j\pi\,\cdot)}\rangle.$

So $\boxed{\text{Fourier transforms are defined as inner products with (formal) eigenfunctions of the adjoint.}}$

We can use this idea to find the "right" Fourier transform for a particular diff. op. / IBVP.

Example 2.6.1 Suppose $\Delta : \Phi \to C^\infty[0,1]$ is given by Δ_{formal} and

$$\Phi = \{\varphi \in C^\infty[0,1] \text{ s.t. } B_1\varphi = 0, \ B_2\varphi = 0\}.$$

We know Δ is formally self adjoint & $\exists$ boundary forms B_1^*, B_2^* s.t. $\Delta^* : \Phi^* \to C^\infty[0,1]$ is given by Δ_{formal} and $\Phi^* = \{\varphi \in C^\infty[0,1] \text{ s.t. } B_1^*\varphi = 0, \ B_2^*\varphi = 0\}$.

We also know that the formal eigenfunctions of Δ^* are $y(\cdot;\lambda)$, $y(\cdot;-\lambda)$ with formal eigenvalues $\bar{\lambda}^2$.

So the eigenfunctions of Δ^* must be linear combinations of $y(\cdot;\lambda)$, $y(\cdot;-\lambda)$.

So the Fourier series type transform that diagonalizes Δ must be of the form

$$F_\Delta[\varphi](j) = \int_0^1 \varphi(x)\left(A_j e^{i\lambda_j x} + B_j e^{-i\lambda_j x}\right) dx$$

where $(\lambda_j)_{j\in\mathbb{N}}$ is some sequence for which $\bar{\lambda}_j^2$ are the eigenvalues of Δ^*.

3. Bounds on integrals

3.1 Triangle inequalities

Recall from Proof:

Theorem 3.1.1 (Real triangle inequality) If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$.

Proof See module Proof.

Corollary: $\forall a, b \in \mathbb{R}$, $|a-b| \leq |a| + |b|$, using $b \mapsto -b$

Theorem 3.1.2 (Complex modulus triangle inequality) If $a, b \in \mathbb{R}$, then $\underbrace{|a+ib|}_{\text{complex modulus}} \leq \underbrace{|a| + |b|}_{\text{real absolute value.}}$

Proof $|a+ib|^2 = \left(\sqrt{a^2+b^2}\right)^2 = a^2+b^2 \leq a^2 + 2|ab| + b^2 = (|a|+|b|)^2$, and the square function is monotonic for positive domain. $\square$

Theorem 3.1.3 (Integral triangle inequality) If $f : I \to \mathbb{R}$ is an integrable real valued function, then

$$\left| \int_I f(x)\, dx \right| \leq \int_I |f(x)|\, dx.$$

Proof

$$\left| \int_I f(x)\, dx \right| = \left| \int_I f^+(x)\, dx - \int_I f^-(x)\, dx \right|$$

for functions $f^+, f^- : I \to [0, \infty)$ s.t. $\forall x \in I$, $f(x) = f^+(x) - f^-(x)$

$$\overset{\text{Theorem 3.1.1}}{\leq} \left| \int_I f^+(x)\, dx \right| + \left| \int_I f^-(x)\, dx \right|$$

$$= \int_I f^+(x)\, dx + \int_I f^-(x)\, dx = \int_I |f(x)|\, dx \qquad \square$$

Theorem 3.1.4 (Complex integral triangle inequality) If $\varphi : I \to \mathbb{C}$ is an integrable complex valued function, then

$$\left| \int_I \varphi(x)\, dx \right| \leq \int_I |\varphi(x)|\, dx.$$

Proof Omitted.

3.2 Cauchy - Schwarz

Theorem 3.2.1 Provided all the integrals exist, for complex valued functions on I,

$$\left|\langle f, g\rangle\right|^2 \leq \|f\|^2 \; \|g\|^2$$

ie
$$\left| \int_I f(x)\, \overline{g(x)}\, dx \right|^2 \leq \int_I |f(x)|^2\, dx \int_I |g(x)|^2\, dx$$

$$= \int_I f(x)\, \overline{f(x)}\, dx \int_I g(x)\, \overline{g(x)}\, dx .$$

Proof Omitted.

3.3 Geometric

Theorem 3.3.1 Suppose $-\infty < a < b < \infty$ and $f:[a,b] \to \mathbb{R}^+$ is such that $\sup_{x\in[a,b]} f(x)$ exists. Then

$$\int_a^b f(x)\, dx \leq \sup_{x\in[a,b]}\big(f(x)\big)\; (b-a).$$

Proof The integral is the area between the lines $x=a$, $x=b$, $y=0$, and the curve $y=f(x)$. But $y=f(x)$ lies nowhere above the line $y=\sup_{x\in[a,b]} f(x)$. So the area between the lines $x=a$, $x=b$, $y=0$, $y=\sup_{x\in[a,b]} f(x)$ is greater. That rectangle has area given by the formula on the right. $\square$

4 D to N maps

4.1 What is a D to N map?

An IBVP is a problem:

Find $q : I \times [0,T] \to \mathbb{C}$ s.t. PDE, BC, IC.

A D to N map is a problem:

Find $q : I \times [0,T] \to \mathbb{C}$ and its spatial derivatives $\boxed{\text{evaluated at the boundaries}}$ s.t PDE, BC, IC.

Example 4.1.1 Suppose $q(x,t)$ satisfies $\left[\frac{\partial}{\partial t} + K\frac{\partial^2}{\partial x^2}\right] q(x,t) = 0$, $\underbrace{q(0,t)=0,\ q(1,t)=0}_{\text{Dirichlet values specified}}$, $q(x,0) = q_0(x)$. Find $\underbrace{q_x(0,t),\ q_x(1,t)}_{\text{Neumann values sought}}$.

Solution: We already have a formula for $q(x,t)$, obtained in example 2.3.3:

$$q(x,t) = \sum_{j=1}^{\infty} e^{-K(j\pi)^2 t} F_{s\,ser}[Q](j) \sin(j\pi x)$$

$$\Rightarrow \quad q_x(0,t) = \pi \sum_{j=1}^{\infty} j e^{-K(j\pi)^2 t} F_{s\,ser}[Q](j), \qquad q_x(1,t) = \pi \sum_{j=1}^{\infty} j(-1)^j e^{-K(j\pi)^2 t} F_{s\,ser}[Q](j)$$

This is why it is called a D to N map: "Dirichlet to Neumann." But the problem could go the other way: Neumann to Dirichlet, or be more complicated if higher order derivatives are involved. So I prefer to think of it as a "Data to UNknown" map instead.

4.2 Why D to N maps

In some situations, you only care about the value of q close to (or at) the boundary. For example, in marine engineering it does not matter about the waves far out to sea, but it does matter how the waves behave at the harbour wall you are designing.

In some situations, it can be that solving the D to N map makes it much easier to solve the full IBVP.

Example 4.2.1 (Stage 1 of the Unified transform method / Fokas transform method)

Similar to example 4.1.1, suppose $q(x,t)$ satisfies $\left[\frac{\partial}{\partial t} - K\frac{\partial^2}{\partial x^2}\right] q(x,t) = 0$, $\underbrace{q(0,t) = f(t),\ q(1,t) = g(t)}_{\text{inhomogeneous; cf example 4.1.1}}$

$q(x,0) = Q(x)$. Find $q : [0,1]\times[0,T] \to \mathbb{C}$

Extend q to spatial domain $\mathbb{R}$, by the rule

$q(x,t) = 0$ if $x \notin [0,1]$. Then we can apply the exponential Fourier transform to the PDE:

$$F[q_t(x,t)](\lambda) + K\,F[-q_{xx}(x,t)](\lambda) = 0$$

$$\Rightarrow \frac{d}{dt}F[q](\lambda;t) + K\lambda^2 F[q](\lambda;t) = \Big(i\lambda \overset{f(t)}{q(0,t)} - q_x(0,t)\Big) - e^{-i\lambda}\Big(i\lambda \overset{g(t)}{q(1,t)} - q_x(1,t)\Big) \qquad (*)$$

$$\Rightarrow F[q](\lambda;t) = e^{-K\lambda^2 t}\Bigg[\underbrace{F[Q](\lambda;t) + i\lambda\int_0^t e^{K\lambda^2\tau}\left(g(\tau) - e^{-i\lambda}f(\tau)\right)d\tau}_{\text{data of the IBVP}}$$

$$+ \underbrace{\int_0^t e^{K\lambda^2\tau}\left(e^{-i\lambda}q_x(1,\tau) - q_x(0,\tau)\right)d\tau}_{\text{unknown.}}\Bigg]$$

$$\Rightarrow q(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x} \dots \, d\lambda$$

So solving IBVP is reduced to the problem of finding $q_x(1,\tau)$ and $q_x(0,\tau)$ in terms of the known data Q, f, g.

If we can find a D to N map, the IBVP is solved.

5 The "Q equation" method

5.1 Derivation of the "Q equation"

For a given PDE & boundary conditions (or diff op), the Q equation is an equation relating the Fourier transform of q to the time derivative of the Fourier transform of q. ie, it is exactly equation (*) from example 4.2.1.

It is called the "Q equation" for historic reasons. The original derivation was much longer, and defined an object Q which was only later understood to be $-F[q](\lambda;t)$ (note the negative).

Henceforth, because we will not use the other Fourier transforms, we will use a simpler notation:

$$\hat{q}(\lambda;t) := F[q](\lambda;t) = \int_{-\infty}^{\infty} q(x,t)\, e^{-i\lambda x}\, dx,$$

where the domain of $q(\cdot,t)$ has been extended from I to $(-\infty,\infty)$ as 0 everywhere outside I.

Example 5.1.1 Consider the IBVP $[\frac{\partial}{\partial t} - i\mathcal{D}^3]q(x,t) = 0$, $q(x,0) = Q(x)$, $q(0,t) = f(t)$, $q(1,t) = g(t)$, $q_x(0,t) = h(t)$. See problem 2.5 & problem 3.8. Obtain the Q equation.

PDE $\Rightarrow \frac{\partial}{\partial t}\hat{q}(\lambda;t) - i\, F[\mathcal{D}^3 q](\lambda;t) = 0$

$$\Rightarrow \frac{\partial}{\partial t}\hat{q}(\lambda;t) + \Big[q_{xx}(x,t)e^{-i\lambda x} + i\lambda q_x(x,t)e^{-i\lambda x} - \lambda^2 q(x,t)e^{-i\lambda x} \Big]_{x=0}^{x=1} - i\lambda^3 \hat{q}(\lambda;t) = 0$$

$$\Rightarrow \Big[\frac{\partial}{\partial t} - i\lambda^3\Big]\hat{q}(\lambda;t) = \Big(q_{xx}(0,t) + i\lambda q_x(0,t) - \lambda^2 q(0,t) \Big)$$

$$- e^{-i\lambda}\Big(q_{xx}(1,t) + i\lambda q_x(0,t) - \lambda^2 q(0,t) \Big).$$

5.2 Reduction via temporal Fourier series

Next, we will use the transform F_{ser} on $[-\frac{T}{2}, \frac{T}{2}]$ in the Q equation, but we will apply it in time, not in space. Of course, none of these functions are defined for $t<0$, but corollary 2.5.7 shows this is not a problem, as long as we extend the definitions from $[0,T]$ to $[-\frac{T}{2}, T]$ appropriately.

The Q equation is equation ($*$) from example 4.2.1:

$$\left[\frac{\partial}{\partial t} + K\lambda^2\right]\hat{q}(\lambda;t) = \left(i\lambda f(t) - a(t)\right) - e^{-i\lambda}\left(i\lambda g(t) - b(t)\right),$$

in which f, g are the boundary data of the problem, and the unknown Neumann boundary values are denoted by $a(t) := q_x(0,t)$, $b(t) := q_x(1,t)$. So we seek a, b.

We denote $F_j := F_{ser}[f](j) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-ij\omega t}\,dt$, where $\omega := \frac{2\pi}{T}$,

$$G_j := F_{ser}[g](j), \qquad A_j := F_{ser}[a](j), \qquad B_j := F_{ser}[b](j),$$

$$q_j(\lambda) := F_{ser}[\hat{q}(\lambda;\cdot)](j).$$

Then, by corollary 2.5.7,

$$f(t) = \sum_{j\in\mathbb{Z}} F_j e^{ij\omega t}, \qquad a(t) = \sum_{j\in\mathbb{Z}} A_j e^{ij\omega t},$$

$$g(t) = \sum_{j\in\mathbb{Z}} G_j e^{ij\omega t}, \qquad b(t) = \sum_{j\in\mathbb{Z}} B_j e^{ij\omega t},$$

So to find a, b (which is our aim), it is enough to find $(A_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$.

$$\hat{q}(\lambda;t) = \sum_{j\in\mathbb{Z}} q_j(\lambda)e^{ij\omega t} \implies \frac{\partial}{\partial t}\hat{q}(\lambda;t) = \sum_{j\in\mathbb{Z}} i\omega j\, q_j(\lambda)e^{ij\omega t}.$$

Substituting these into the Q equation, we find

$$\sum_{j\in\mathbb{Z}} e^{ij\omega t}\left(i\omega j + K\lambda^2\right)q_j(\lambda) = \sum_{j\in\mathbb{Z}} e^{ij\omega t}\left(i\lambda F_j - A_j - e^{-i\lambda}i\lambda G_j + e^{-i\lambda}B_j\right)$$

Hence, by Corollary 2.5.5, $\forall\, j \in \mathbb{Z}$,

$$\left(i\omega_j + K\lambda^2\right) q_j(\lambda) = i\lambda\left(F_j - e^{-i\lambda} G_j\right) - \left(A_j - e^{-i\lambda} B_j\right). \qquad (**)$$

Recall, we are seeking A_j and B_j.

5.3 Solution of D to N map

Equation $(**)$ holds $\forall\, j \in \mathbb{Z}$, as argued above. But what about λ? Tracing the origin of λ, it holds $\forall\, \lambda$ s.t. $\hat{q}(\lambda; t)$ makes sense. But

$$\hat{q}(\lambda; t) = \int_{-\infty}^{\infty} q(x,t) e^{-i\lambda x}\, dx = \int_0^1 q(x,t) e^{-i\lambda x}\, dx.$$

The integrand is continuous, so the only way this could fail to converge is if it converges to something infinite. Let's check.

Suppose $\lambda = u + iv$ $\exists\, u, v \in \mathbb{R}$. Then

$$\left|\hat{q}(\lambda; t)\right| = \left| \int_0^1 q(x,t)\, e^{-iux} e^{vx}\, dx \right|$$

$$\leq \int_0^1 |q(x,t)|\, |e^{-iux}|\, |e^{vx}|\, dx \qquad \text{by theorem 3.1.4}$$

$$= \int_0^1 |q(x,t)|\, e^{vx}\, dx$$

$$\leq e^{v} \max_{x \in [0,1]} |q(x,t)| \, (1-0) \qquad \text{by theorem 3.3.1}$$

$< \infty$, because q is continuous.

This argument holds $\forall\, \lambda \in \mathbb{C}$, so $(\star\star)$ is true $\forall\, \lambda \in \mathbb{C}$. In particular, it is true for those λ for which $i\omega j + K\lambda^2 = 0$. For each $j \in \mathbb{Z}\setminus\{0\}$, there are two such λ, λ_j and $-\lambda_j$, where

$j > 0$	$j < 0$
$i\omega j + K\lambda_j^2 = 0$	$i\omega j + K\lambda_j^2 = 0$
$\Leftrightarrow \quad \lambda_j^2 = -i\omega j / K$	$\Leftrightarrow \quad \lambda_j^2 = i\omega(-j)/K$
$= e^{-i\pi/2}\,\frac{\omega j}{K}$	$= e^{i\frac{\pi}{2}}\,\frac{\omega(-j)}{K}$
$\Leftarrow \quad \lambda_j = e^{-i\pi/4}\sqrt{\frac{\omega j}{K}}$	$\Leftarrow \quad \lambda_j = e^{i\pi/4}\sqrt{\frac{\omega(-j)}{K}}$

In each case, we were careful to only take the square root $\sqrt{\ }$ of a positive number, and the point $e^{\pm i\pi/2}$ lies on the unit circle with angle expressed as a number in $(-\pi, \pi)$.

$$\boxed{\text{So, } \forall\, j \in \mathbb{Z}\setminus\{0\}, \qquad \lambda_j = e^{-\operatorname{sgn}(j)\, i\pi/4}\sqrt{\frac{\omega|j|}{K}}\,.}$$

We want to say LHS of $(\star\star)$ is zero for $\lambda = \pm\lambda_j$. But we should first check that $q_j(\pm\lambda_j)$ is finite.

$$\left|q_j(\lambda)\right| = \left|\frac{1}{T}\int_{-T/2}^{T/2} \hat{q}(\lambda; t)\, e^{-ij\omega t}\, dt\right|$$

$$\leq \frac{1}{T}\int_{-T/2}^{T/2} |\hat{q}(\lambda;t)|\, \underbrace{|e^{-ij\omega t}|}_{=1}\, dt \qquad \text{by theorem 3.1.4}$$

$$\leq \underbrace{\frac{1}{T}\left(\frac{T}{2} - \left(-\frac{T}{2}\right)\right)}_{=1} \max_{t \in [-\frac{T}{2}, \frac{T}{2}]} |\hat{q}(\lambda;t)| \qquad \text{by theorem 3.3.1}$$

$$\leq e^{\operatorname{Im}(\lambda)} \max_{\substack{x \in [0,1],\\ t \in [-\frac{T}{2}, \frac{T}{2}]}} |q(x,t)| \qquad \text{by earlier argument}$$

$< \infty$, because q is continuous in both x and t.

So yes, LHS of $(\star\star)$ is zero at $\lambda = \pm \lambda_j$, $\forall j \in \mathbb{Z} \setminus \{0\}$.

This yields two linear equations:

$$\begin{pmatrix} 1 & -e^{-i\lambda_j} \\ 1 & -e^{i\lambda_j} \end{pmatrix} \begin{pmatrix} A_j \\ B_j \end{pmatrix} = \begin{pmatrix} i\lambda_j (F_j - e^{-i\lambda_j} G_j) \\ -i\lambda_j (F_j - e^{i\lambda_j} G_j) \end{pmatrix}$$

which may be solved for A_j, B_j, as long as the system is full rank.

<u>Exercise:</u> Check the system is full rank.

<u>Exercise:</u> Solve the linear system.

But this only works for $j \in \mathbb{Z} \setminus \{0\}$. What about $j = 0$?

If $j = 0$, then $(\star\star)$ simplifies to

$$K\lambda^2 q_0(\lambda) = i\lambda (F_0 - e^{-i\lambda} G_0) - (A_0 - e^{-i\lambda} B_0).$$

Clearly $\lambda_0 = 0$, but that only gives one equation for two unknowns: A_0, B_0. To get the other, differentiate $(\star\star)$ with respect to λ:

$$K\lambda^2 q_0'(\lambda) + 2K\lambda q_0(\lambda) = i\lambda\, i\, e^{-i\lambda} G_0 + i(F_0 - e^{-i\lambda} G_0) - i e^{-i\lambda} B_0$$

$$\Leftrightarrow \quad K\lambda\left(\lambda q_0'(\lambda) + 2q_0(\lambda)\right) = iF_0 - e^{-i\lambda}(i+\lambda) G_0 - i e^{-i\lambda} B_0$$

We know $q_0(\lambda)$ is finite.

<u>Exercise</u> Check $q_0'(\lambda)$ is finite. Hint: First get a formula for $\frac{\partial}{\partial\lambda}\hat{q}(\lambda;t)$ by differentiating

under the integral.

So, at $\lambda = 0$, the left side of this equation evaluates to 0. Together with (**) at 0, we get the linear system

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 0 \\ F_0 - G_0 \end{pmatrix}.$$

This system has full rank (because it has 1s along the diagonal and is upper-triangular, so determinant is 1). So it can be solved for A_0, B_0.

But now we have $(A_j)_{j\in\mathbb{Z}}$ and $(B_j)_{j\in\mathbb{Z}}$. From those, we can reconstruct functions $a(t) = q_x(0,t)$ and $b(t) = q_x(1,t)$, using the formulae

$$a(t) = \sum_{j\in\mathbb{Z}} A_j e^{ij\omega t} \qquad \text{and} \qquad b(t) = \sum_{j\in\mathbb{Z}} B_j e^{ij\omega t}.$$

This completes the D to N map.

5.4 Induced initial value

Our Data to uNknown map produces the unknown boundary values from the boundary data only; initial datum is not used in the Q equation method. This seems strange, because the initial datum should affect the boundary values. In certain cases, this can be argued physically. In example 4.1.1 we saw a mathematical demonstration.

It turns out that by using temporal exponential Fourier series expansions of $f, g, \hat{q}$ etc, we assumed the solution of the IBVP is periodic in time. That might be counterintuitive, but it is really just a restatement of theorem 2.5.6. But there is only one initial datum that gives a time periodic solution, and we can't expect it to be Q. So what is that initial datum? It must be the initial value of the q that has the boundary values we calculated using our D to N map.

<u>Exercise</u> Calculate $q(x,0)$, using f, g known, and a, b derived above. Hint: Use the Q equation and/or stage 1 of the unified transform method.

<u>Exercise</u> Use the Q equation method and the principle of linear superposition to decompose the IBVP of example 4.2.1 into two easier problems that we have already solved.

<u>Exercise</u> Perform an analogous decomposition for the IBVP in example 5.1.1.

<u>Exercise</u> Implement the Q equation method, continuing example 5.1.1.