Quiz 3

Given the following set of vectors:

$$v1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$v2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Apply the Gram-Schmidt orthogonalisation process to these vectors to obtain a set of orthonormal vectors u1, u2.

$$u1 = v1u1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$u2 = v2 - \frac{v2 \cdot u1}{u1 \cdot u1} u1$$

$$v2 \cdot u1 = 2 \times 1 + 1 \times 3 = 5$$

$$u1 \cdot u1 = 1 \times 1 + 3 \times 3 = 10$$

$$u2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{5}{10} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$$

The matrix A is given by:

$$A = \begin{pmatrix} 16 & -8 \\ -8 & 5 \end{pmatrix}$$

is a positive definite matrix.

True or False?

$$\det(A) = 16 \times 5 - (-8) \times (-8) = 80 - 64 = 16$$

$16 > 0$ so the matrix is positive definite. True.

The matrix C is given by:

$$C = \begin{pmatrix} 16 & -8 \\ -8 & 5 \end{pmatrix}$$

can be decomposed into

$$C = LL^T$$

where L is a lower triangular matrix.
Use Cholesky decomposition to work out the matrix L

$L = \begin{pmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{pmatrix}$

$l_{11} = \sqrt{a_{11}} = \sqrt{16} = 4$
$l_{21} = \frac{a_{12}}{l_{11}}$
$a_{12} = -8$
$l_{11} = 4$
$l_{21} = \frac{-8}{4} = -2$
$l_{22} = \sqrt{a_{22} - l_{21}^2}$
$a_{22} = 5$
$l_{21} = -2$
$l_{22} = \sqrt{5 - (-2)^2} = \sqrt{5 - 4} = \sqrt{1} = 1$

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a = 4$, $b = 0$, $c = -2$, $d = 1$ so

$$L = \begin{pmatrix} 4 & 0 \\ -2 & 1 \end{pmatrix}$$

Given the matrix;

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

the LU decomposition of A can be written as the product of a lower unit triangular matrix L and an upper triangular matrix U.

Fill in the blanks for L and U.

$$L = \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & 10 \end{pmatrix}$$

Given that a matrix A can be decomposed into LU, Where

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

Solve the system of equations represented by $Ax = b$ where

$$b = \begin{pmatrix} 20 \\ 61 \\ 104 \end{pmatrix}$$

The solution vector:

$$x = \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 20 \\ 61 \\ 104 \end{pmatrix}$$

$y_1 = 20$
$2y_1 + y_2 = 61$
$4y_1 + 3y_2 + y_3 = 104$

$$y_1 = 20$$

$$2 \cdot 20 + y_2 = 61 \implies y_2 = 61 - 40 = 21$$

$$4 \cdot 20 + 3 \cdot 21 + y_3 = 104 \implies 80 + 63 + y_3 = 104 \implies y_3 = 104 - 143 = -39$$

$y = \begin{pmatrix} 20 \\ 21 \\ -39 \end{pmatrix}$

$$U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 20 \\ 21 \\ -39 \end{pmatrix}$$

$x_1 + 2x_2 + 4x_3 = 20$
$x_2 + 5x_3 = 21$
$x_3 = -39$

$$x_2 + 5(-39) = 21 \implies x_2 - 195 = 21 \implies x_2 = 216$$

$$x_1 + 2(216) + 4(-39) = 20 \implies x_1 + 432 - 156 = 20 \implies x_1 = 20 - 276 = -256$$

$$x = \begin{pmatrix} -256 \\ 216 \\ -39 \end{pmatrix}$$